Finite Difference Method For Poisson Equation

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Talk

Plam

- We will start the talk by introducing our topic.
- After introdudcing the topic, we would like to exaplain our question.
- Then we will proceed to describe the various methods we have used in the project.
- After discussion on methods, we would like to discuss our results.
- We will conclude the talk with conclusion of results and by sharing our experience in completing this project.

Introduction

- Poisson Equation is an elliptic partial differential Equation.having general form as -:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(r)}$$

- Differetial Equation can be solved by variety of methods such as spectral methods, finite element methods, FDI.
- Method of finite differences converts differential Equation into a system of linear equation by approxiamting the derivatives using taylor series.
- This system of linear equation obtained can be solved in two ways to obtain the solution of differential equation-:
    - Implicit Methods
    - Explicit Methods
- Implicit methods include many iterative schemes which can be employed to solve this system of linear equation such as Gauss Seidel, SOR and Jacobi Method.

Theory

The below figure shows an interleaved capacitor -:

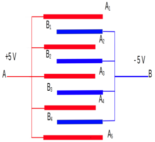


Figure: Diagram dipciting the arrangement of plates in a interleaved fashion.

After non -dimensionalising we had following poisson equation-:

$$\frac{\partial^2 U'(x',y')}{\partial x'^2} + \frac{\partial^2 U'(x',y')}{\partial y'^2} = -\frac{\rho'(x',y')s^2}{\epsilon_0 \nu}$$

Problem

$$\rho'(x',y') = \begin{cases} -2\epsilon_0 \times 10^5 & : \text{if } (x',y') \in B_i \text{ where } i = 1,2,3,4 \\ 2\epsilon_0 \times 10^5 & : \text{if } (x',y') \in A_i \text{ where } i = 2,3,4 \\ 0 & : \text{ elsewhere} \end{cases} \tag{1}$$

With boundary conditions as -:

$$U(0,y) = +5/\nu \qquad U(4,y) = +5/\nu \tag{2}$$

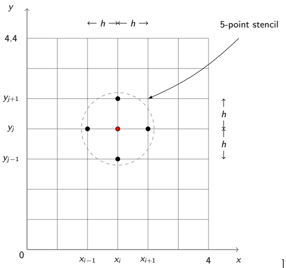
$$U_y(x,0) = 0 \qquad U_y(x,4.4) = 0 \tag{3}$$

Methodology

- Finite Difference converts PDE into difference equation.
- Domain is converted into a mesh of equidistant grid points.
- Taylor series is used to approximate the value at these grid points.
- After using Finite Difference operators we get the following stencil for poisson equation

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j+1} + h^2 \frac{\rho'_{i,j} s^2}{\epsilon_0 \nu}\right] \tag{4}$$

This stencil can be represented with the help of following diagram -:



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Truncation Error

- Truncation Error arises due to truncation of Taylor series used for approximating the value of derivative to form a difference equation.
- First order derivative can be approximated in three ways-:
  - Forward Difference Method
  - Backward Difference Method
  - Central Difference Method

Forward Difference Method	$\mathcal{O}(\Delta x)$
Backward Difference Method	$\mathcal{O}(\Delta x)$
Central Difference Method	$\mathcal{O}((\Delta x)^2)$

- Above table shows that central order approximation are more accurate than one sided differences.
- Second Order derivative is also second order accurate.

Iterative Method

- Iterative methods are techniques that exploit the properites of system to solve it implicitly to make computation faster.
- We have used following iteratiion schemes-:

Jacobi Method

This method starts with a guess value and with each iteration, it replace guess values with new obtained values from iteration.

$$x_{i,j}^{(n+1)} = S(X^{(n)}, P, h, i, j)$$

Gauss Seidel Method

This method uses the obtained value in the same iteration for other unknowns.

$$x_{i,j}^{(n+1)} = S(X^{(n+1)}, P, h, i, j)$$

SOR

This method involves a relaxation factor (greater than 1) which is multiplied to values obtained from Gauss Seidel before replacing old values.

$$x_{i,j}^{(n+1)} = x_{i,j}^{(n)} + \omega(S(X^{(n+1)}, P, h, i, j) - x_{i,j}^{(n+1)})$$

Expectation

We expect the following graph-:

- α_1 should decrease from positive to negative potential
- α_2 should have negatively charged potential
- α_3 should increase from negative to positive potential
- α_4 should have positively charged potential

Numerical Result

We obtained following graphs as solution -:

Conclusion

Result

- After computing thte problem of interleaved capacitor from three different iterative schemes, we have arrived at the conclusion that SOR is the fastest among three.

Experiences

- We have gained more intuition and it has helped us to grasp the concept of electrostatic potential.
- We have learnt a lot of new techniques and methods related to computational solution of differential equation.
- There are some techniques and methods which we could not implemnet but surely they have given us some things to ponder upon in future.
- This project has also introduced us to delight of finding out new things as most of things we did in this project were completely new to us.

References

[1] K. Morton and D. Mayers, *Numerical Solution of Partial Differential Equations: An Introduction*. Cambridge University Press, 2005, ISBN: 9781139443203. [Online]. Available: https://books.google.co.in/books?id=GW6%5C_AwAAQBAJ.