0.1 Finite Differences

Poisson's equation is given by,

$$-\nabla^2 u(x,y) = \rho(x,y) \tag{0.1.1}$$

The laplacian in euclidean geometry is defined as the sum of second derivatives of the function $u(x,y)$ with each of the independent variables. In 2D we have,

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} \tag{0.1.2}$$

Difference Quotient

Forward

$$u_x = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

Backward

$$u_x = \lim_{h\to 0} \frac{u(x,y) - u(x-h,y)}{h}$$

Centered

$$u_x = \lim_{h\to 0} \frac{u(x+\frac{h}{2},y) - u(x-\frac{h}{2},y)}{h}$$

Second Derivative

$$\begin{aligned} u_{xx} &= \lim_{h\to 0} \frac{u_x(x+\frac{h}{2},y) - u_x(x-\frac{h}{2},y)}{h} \\ &= \lim_{h\to 0} \frac{\frac{u(x+h,y)-u(x,y)}{h} - \frac{(u(x,y)-u(x-h,y))}{h}}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} \end{aligned}$$

Considering h to be sufficiently small we obtain the finite Difference approximation introducing the error term that will be ascertained later,

$$u_{xx} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} \tag{0.1.3}$$

If Domain of the function is confined to a Region R and discritized by considering a set of points lying on the intersections of the square mesh given below,

$$u_{xx}(jh, kh)$$

the Finite Difference approximations are then applied to each and every point on the grid to obtain a system of linear equations which can then be solved explicity for less number of points or implicitly using iterative relaxation methods.

Error