0.1 Finite Difference Methods

Finite Difference Methods(FDM) are used for approximating the solution of partial differential equations over a set of finite points, arranged in a geometrical structure called a **mesh**[1], in the continous domain of solution. The methods involve the idea of reducing the given PDE, by means of truncated taylor series approximation of the derivatives, to a difference equation which is much easier to digest numerically.

0.1.1 Finite Difference Approximations

The quality of the solution depends on the quality of approximations made to the derivatives. Consider this one-dimensional structured mesh of nodes $(x_0, x_1, x_2, .., x_i, .., x_n)$ at which the solution $U(x_i)$ is to be found, such that the difference $h = x_{i+1} - x_i$ is constant throughout the mesh and $x_i = x_0 + ih$
Let U_i represent the solution at the i-th node and

$$\left.\frac{\partial U}{\partial x}\right|_{x_i} = U_x(x_0 + ih) \equiv U_x|_i$$

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x_i} = U_{xx}(x_0 + ih) \equiv U_{xx}|_i$$

The first order derivative can be defined as,

$$U_x|_i = \lim_{h\to 0} \frac{U_{i+1} - U_i}{h}$$

$$\text{or,} \qquad U_x|_i = \lim_{h\to 0} \frac{U_i - U_{i-1}}{h}$$

$$\text{or,} \qquad U_x|_i = \lim_{h\to 0} \frac{U_{i+1} - U_{i-1}}{2h}$$

Finite difference approximations are obtained by dropping the limit and can be written as,

$$\text{Forward Difference} \qquad U_x|_i \approx \frac{U_{i+1} - U_i}{h} \equiv \delta_h^+ U_i$$

$$\text{Backward Difference} \qquad U_x|_i \approx \frac{U_i - U_{i-1}}{h} \equiv \delta_h^- U_i$$

$$\text{Central Difference} \qquad U_x|_i \approx \frac{U_{i+1} - U_{i-1}}{2h} \equiv \delta_{2h} U_i$$

Where $\delta_h^+, \delta_h^-, \delta_{2h}$ are called the **first-order finite diference operators** and the expansion is called the **finite difference quotient**, each representing forward,backward and centered respectively. Second and Higher order finite difference Quotients can also be obtained,

$$\begin{aligned}
U_{xx}|_i &= \lim_{h\to 0} \frac{U_x(x_i + \frac{h}{2}, y) - U_x(x_i - \frac{h}{2}, y)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{U(x+h,y) - U(x,y)}{h} - \frac{(U(x,y) - u(x-h,y))}{h}\right] \\
&= \lim_{h\to 0} \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} \\
&\approx \boxed{\delta_h^2 U_i \equiv \frac{1}{h^2}(U_{i+1} - 2U_i + U_{i-1})} \qquad \text{[Central second-order Difference]}
\end{aligned}$$

[1]An object which consists of points which are spaced in a specific geometrical pattern is referred to as a **mesh** and each point in this mesh is called a **node**. The distance between any two adjacent nodes in a mesh with uniform spacing is called its **meshsize**

0.2 Basic Trucation Error Analysis

0.3 Iterative Methods

0.4 Problems

Analysis