Contents

1   Introduction

2   Finite Difference Method

Finite Difference Methods(FDM) are used for approximating the solution of partial differential equations over a set of finite points, arranged in a geometrical structure called a **mesh**[1], in the continous domain of solution. The methods involve the idea of reducing the given PDE, by means of truncated taylor series approximation of the derivatives, to a difference equation which is much easier to digest numerically.

2.1   Finite Difference Approximations

The quality of the solution depends on the quality of approximations made to the derivatives. Consider this one-dimensional structured mesh of nodes $(x_0, x_1, x_2, .., x_i, .., x_n)$ at which the solution $U(x_i)$ is to be found, such that the difference $h = x_{i+1} - x_i$ is constant throughout the mesh and $x_i \equiv x_0 + ih$.
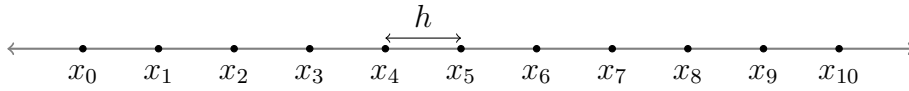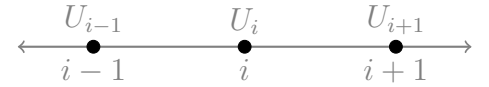
Figure 1: 1D mesh with 11 nodes and a meshsize h

Let U_i represent the solution at the i-th node and

$$\frac{\partial U}{\partial x}\bigg|_{x_i} = U_x(x_0 + ih) \equiv U_x|_i$$

$$\frac{\partial^2 U}{\partial x^2}\bigg|_{x_i} = U_{xx}(x_0 + ih) \equiv U_{xx}|_i$$

The first order derivative can be defined as,

$$U_x|_i = \lim_{h \to 0} \frac{U_{i+1} - U_i}{h}$$

$$\text{or,} \qquad U_x|_i = \lim_{h \to 0} \frac{U_i - U_{i-1}}{h}$$

$$\text{or,} \qquad U_x|_i = \lim_{h \to 0} \frac{U_{i+1} - U_{i-1}}{2h}$$
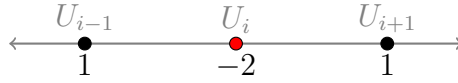
Finite difference approximations are obtained by dropping the limit and can be written as,

$$\text{Forward Difference} \qquad U_x|_i \approx \frac{U_{i+1} - U_i}{h} \equiv \delta_x^+ U_i$$

$$\text{Backward Difference} \qquad U_x|_i \approx \frac{U_i - U_{i-1}}{h} \equiv \delta_x^- U_i$$

$$\text{Central Difference} \qquad U_x|_i \approx \frac{U_{i+1} - U_{i-1}}{2h} \equiv \delta_{2x} U_i$$

---

[1]An object which consists of points which are spaced in a specific geometrical pattern is referred to as a **mesh** and each point in this mesh is called a **node**. The distance between any two adjacent nodes in a mesh with uniform spacing is called its **meshsize**

Where $\delta_x^+, \delta_x^-, \delta_{2x}$ are called the **finite diference operators** for approximating **first-order derivatives** and their expansion is called the **finite difference quotient**, each representing forward, backward and centered respectively. Second and Higher order finite difference Quotients can also be obtained,

$$\begin{aligned}
U_{xx}|_i &= \lim_{h \to 0} \frac{U_x(x_i + \frac{h}{2}) - U_x(x_i - \frac{h}{2})}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left[\frac{U(x+h) - U(x)}{h} - \frac{(U(x) - U(x-h))}{h} \right] \\
&= \lim_{h \to 0} \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} \\
&\approx \boxed{\delta_x^2 U_i \equiv \frac{1}{h^2}(U_{i+1} - 2U_i + U_{i-1})} \qquad \text{[Central second-order Difference]}
\end{aligned}$$

U_{i-1} U_i U_{i+1}

1   -2 1

2.2 Local Truncation Error of Finite Difference Approximations

The *'error'* that accompanies *'approximations'* in the method must be acccounted for. In this section, the truncation error in the derivative approximations is ascertained which will later help us deduce the error in PDE's solved using these approximations.

The local truncation error for derivative approximations is defined here as the difference between the exact value of the derivate and the approximated value at node i, it can be calculated using Taylor series expansions about i,

For Forward difference operator,

$$
\begin{aligned}
\tau &\equiv \delta_x^+ U_i - U_x|_i \\
&= \frac{1}{\Delta x} \left(U_{i+1} - U_i \right) - U_x|_i \\
&= \frac{1}{\Delta x} \left[\left(U_i + \Delta x U_x|_i + \frac{1}{2} \Delta x^2 U_{xx}|_i + \mathcal{O}(\Delta x^3) \right) - U_i \right] - U_x|_i \\
&= \frac{1}{2} \Delta x U_{xx}|_i + \mathcal{O}(\Delta x^2) = \mathcal{O}(\Delta x)
\end{aligned}
$$

For Backward difference operator,

$$
\begin{aligned}
\tau &\equiv \delta_x^- U_i - U_x|_i \\
&= \frac{1}{\Delta x} \left(U_i - U_{i-1} \right) - U_x|_i \\
&= \frac{1}{\Delta x} \left[U_i - \left(U_i - \Delta x U_x|_i + \frac{1}{2} \Delta x^2 U_{xx}|_i + \mathcal{O}(\Delta x^3) \right) \right] - U_x|_i \\
&= -\frac{1}{2} \Delta x U_{xx}|_i + \mathcal{O}(\Delta x^2) = \mathcal{O}(\Delta x)
\end{aligned}
$$

For Central difference operator,

$$
\begin{aligned}
\tau &\equiv \delta_{2x} U_i - U_x|_i \\
&= \frac{1}{2\Delta x} \left(U_{i+1} - U_{i-1} \right) - U_x|_i \\
&= \frac{1}{2\Delta x} \left[\left(U_i + \Delta x U_x|_i + \frac{1}{2} \Delta x^2 U_{xx}|_i + \frac{1}{6} \Delta x^3 U_{xxx}|_i + \frac{1}{12} \Delta x^4 U_{xxxx}|_i + \mathcal{O}(\Delta x^5) \right) \right. \\
&\quad \left. - \left(U_i - \Delta x U_x|_i + \frac{1}{2} \Delta x^2 U_{xx}|_i - \frac{1}{6} \Delta x^3 U_{xxx}|_i + \frac{1}{12} \Delta x^4 U_{xxxx}|_i + \mathcal{O}(\Delta x^5) \right) \right] - U_x|_i \\
&= -\frac{1}{6} \Delta x^2 U_{xxx_i} + \mathcal{O}(\Delta x^4) = \mathcal{O}(\Delta x^2)
\end{aligned}
$$

where in the above expressions we assume that the Higher order derivatives of U at i are well defined. For a fairly small Δx (less than 1) we can confidently say that $\mathcal{O}(\Delta x^2)$ is samller than $\mathcal{O}(\Delta x)$[1]. Thus we note that the centered difference approximation (second-order accurate) approximates the derivative more accurately than either of the *one-sided diferences* which are first-order accurate.[2]

---

[1] The definition of the "big $\mathcal{O}$" notation says that if for given functions $f(x)$ and $g(x)$ for $x \in S$ where S is some subset of $\mathbf{R}$, there exists a positive constant A such that $|f(x)| \leq A|g(x)| \ \forall \ x \in S$, we say that $f(x)$ is the "big $\mathcal{O}$" of $g(x)$ or that $f(x)$ is of order of $g(x)$, mathematically given by $f(x) = \mathcal{O}(g(x))$

[2] Forward and Backward differences are also called one-sided differences

2.3 Reducing PDE to a discretised Difference equation

2.3.1 Truncation error analysis of discretised PDE

3 Iterative methods to solve linear algebraic equations

In the last section we have discussed alternative methods for approximating the solution of linear elliptic equations, the finite difference method, using either the differential equation or an integral form, Each of these methods gives rise to a system of linear algebraic equations, which may be very large. A two-dimensional problem may lead to a system of several thousand unknowns, and three-dimensional problems involving several hundred thousand unknowns are common in real engineering situations. The solution of such a system is a major problem in itself as traditional methods like Gaussian-elimination result in large computation times we are forced to employ faster methods. As we have seen above, the system of equations produced by a discretisation has many special features and an efficient solution procedure must exploit these. The most obvious property of the system is that it is extremely sparse. Even when there are many thousand unknowns, each equation will involve one unknown and the unknowns at its immediate neighbourhood. In particular, if we write the equations in the conventional notation,

$$A\vec{x} = \vec{b}$$

where A is an N × N matrix, b the given data vector and x the vector of N unknown interior mesh values, there is an implied one-dimensional ordering of these values which is somewhat unnatural and obscures the important property that only immediate neighbours are involved. Each row of the matrix A involves only a very small number of non-zero elements, commonly five or seven; moreover in many problems a suitable ordering of the unknowns will lead to a matrix in which these non-zero elements occur in a regular pattern. In devising efficient methods for the solution of the system these structural properties will be important,

3.1 Jacobi Method

3.2 Gauss-Seidel Method

3.3 Relaxation methods and Successive Over-Relaxation(SOR)

4 Model Problem

Consider the simplest one-dimensional model problem,

$$\frac{d^2U}{dx^2} = 0 \quad \forall \quad x \in (a, b) \tag{4.1}$$
$$U(a) = 1 \quad \text{and} \quad U(b) = -1$$

Nondimensionalising

$$U = \frac{u}{5V} \tag{4.2}$$

$$\tag{4.3}$$

5 Main Problem

Analysis