Successive Over Relaxation Method

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1 Introduction

This method is a variant of **Guass- siedel law** for solving a linear system of equation by iterative method , resulting in faster convergence. The main purpose of this method is to solve the linear systems automatically on the digital systems.These methods were designed for computation by **human calculators**.

*This method was introduced by **David M. Young Jr.** and by **Stanley P. Frankel** in* 1950

2 Mathematic Form

Suppose, we have n numbers of linear equations with x as unknown. So , it can also be written as:

$$AX = b \tag{1}$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & . & . & a_{3n} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & . & . & a_{nn} \end{bmatrix},$$

X= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{bmatrix}$ and b= $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ . \\ b_n \end{bmatrix}$

Now breaking matrix A in D (DIAGONAL COMPONENT), L (LOWER TRIANGULAR MATRIX) and U (UPPER TRIANGULAR MATRIX)

Therefore, A can be written as-

$$A = D + L + U \tag{2}$$

where, $D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$

and $U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.

Now, equation (1) can be written as -

$$(D + \omega L)\mathbf{x} = \omega\mathbf{b} - [\omega U + (\omega - 1)D]\mathbf{x} \tag{3}$$

Here , the constant ω is called the relaxation factor. As, we already mentioned that it is a iterative method so we can solve the left hand side of the above equation for $\mathbf{x}$ using the previous value of $\mathbf{x}$ from the right hand side.

Now using **iteration** it can be written as :
$\mathbf{x}^{(k+1)} = (D + \omega L)^{-1}\big(\omega\mathbf{b} - [\omega U + (\omega - 1)D]\mathbf{x}^{(k)}\big) = L_w\mathbf{x}^{(k)} + \mathbf{c}$
where,
X^k is the n^{th} approximation.
X^{k+1} is the next approximation to n^{th}.

Now,by using **forward substitution** it can be written as:

$$x_i^{k+1} = (1 - \omega)x_i^k + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right)$$

$$where, \quad i = 1, 2, \ldots, n.$$