Computational techniques for solving the Poisson's equation

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Content I

Introduction

- Poisson Equation is an elliptic partial differential equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(r)}$$

- Differetial Equation can be solved by variety of methods - spectral methods, finite element methods, finite volume methods.
- Method of finite differences converts differential Equation into a system of linear equations by approximating the derivatives using taylor series.
- This system of linear equations obtained can be solved in two ways to obtain the solution of differential equation-:
  - Implicit Methods
  - Explicit Methods
- Implicit methods include many iterative schemes which can be employed to solve this system of linear equation such as Gauss Seidel, SOR and Jacobi Method.

Theory

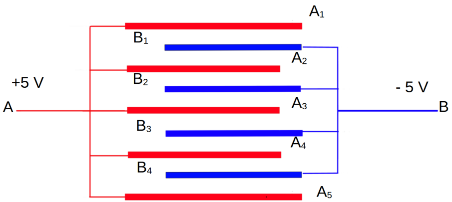


Figure: Diagram dipciting the arrangement of plates in a interleaved fashion.

After non-dimensionalizing we had following poisson's equation :-

$$\frac{\partial^2 U'(x',y')}{\partial x'^2} + \frac{\partial^2 U'(x',y')}{\partial y'^2} = -\frac{\rho'(x',y')s^2}{\epsilon_0 \nu}$$

Problem

$$\rho'(x', y') = \begin{cases} -2\epsilon_0 \times 10^5 & : \text{if } (x', y') \in B_i \text{ where } i = 1, 2, 3, 4 \\ 2\epsilon_0 \times 10^5 & : \text{if } (x', y') \in A_i \text{ where } i = 2, 3, 4 \\ 0 & : \text{ elsewhere} \end{cases} \tag{1}$$

With boundary conditions as -:

$$U(0, y) = +5/\nu \qquad U(4, y) = +5/\nu \tag{2}$$

$$U_y(x, 0) = 0 \qquad U_y(x, 4.4) = 0 \tag{3}$$

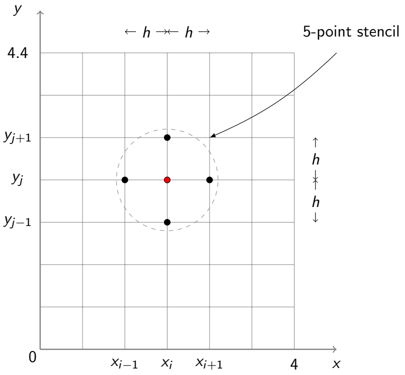
Methodology I

- Finite Difference converts PDE into difference equation.
- Domain is converted into a mesh of equidistant grid points.
- Taylor series is used to approximate the value at these grid points.
- After using Finite Difference operators we get the following stencil for poisson equation

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} + h^2 \frac{\rho'_{i,j} s^2}{\epsilon_0 \nu}\right] \tag{4}$$

Methodology II

This stencil can be represented with the help of following diagram



Truncation Error

- Truncation Error arises due to truncation of Taylor series used for approximating the value of derivative to form a difference equation.
- First order derivative can be approximated in three ways-:
  - Forward Difference Method
  - Backward Difference Method
  - Central Difference Method

Forward Difference Method	$\mathcal{O}(\Delta x)$
Backward Difference Method	$\mathcal{O}(\Delta x)$
Central Difference Method	$\mathcal{O}((\Delta x)^2)$

- Above table shows that central order approximation are more accurate than one sided differences.
- Second Order derivative is also second order accurate.

Iterative Method I

- Iterative methods are techniques that exploit the properites of system to solve it implicitly to make computation faster.
- We have used following iteratiion schemes-:

Jacobi Method

This method starts with a guess value and with each iteration, it replace guess values with new obtained values from iteration.

$$x_{i,j}^{(n+1)} = S(X^{(n)}, P, h, i, j)$$

Gauss Seidel Method

This method uses the obtained value in the same iteration for other unknowns.

$$x_{i,j}^{(n+1)} = S(X^{(n+1)}, P, h, i, j)$$

Iterative Method II

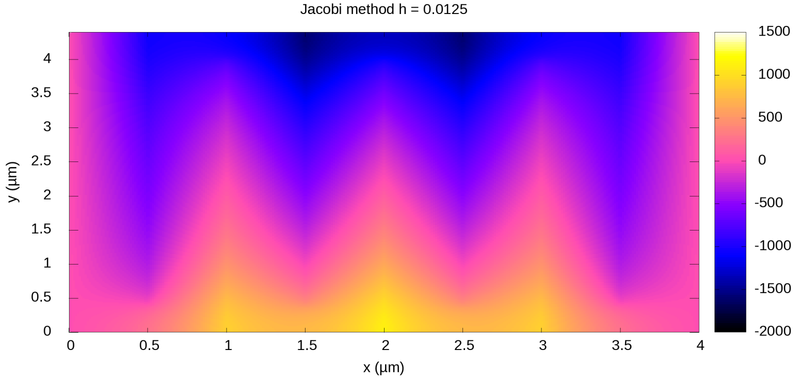
SOR

This method involves a relaxation factor (greater than 1) which is multiplied to values obtained from Gauss Seidel before replacing old values.

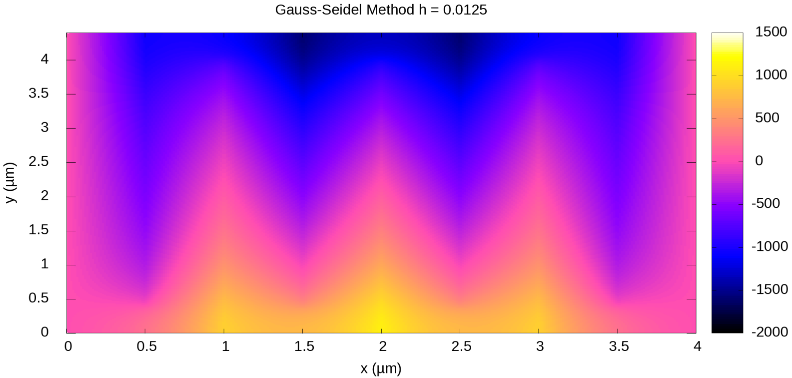
$$x_{i,j}^{(n+1)} = x_{i,j}^{(n)} + \omega(S(X^{(n+1)}, P, h, i, j) - x_{i,j}^{(n+1)})$$

Numerical Result I

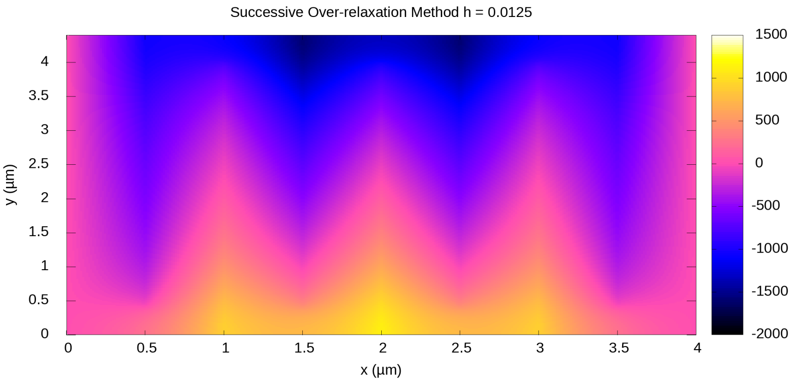
We obtained following graphs as solution



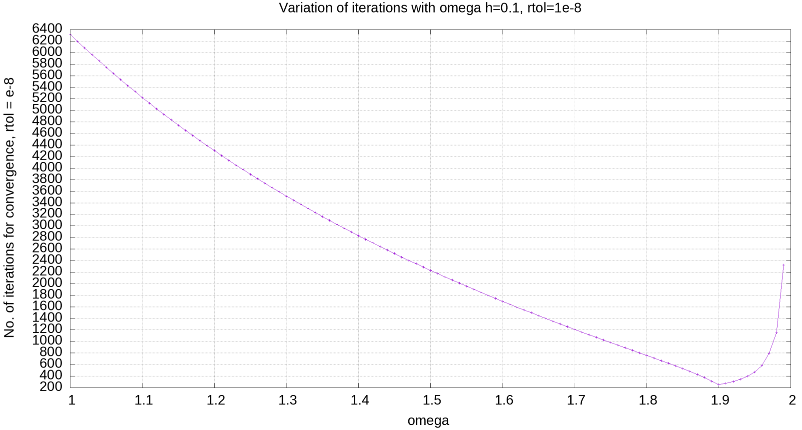
Numerical Result II



Numerical Result III



Numerical Result IV



Conclusion

Result

- After computing thte problem of interleaved capacitor from three different iterative schemes, we have arrived at the conclusion that SOR is the fastest among three.

Experiences

- We gained better intuition and it has helped us to gain a better grasp on the concept of electrostatic potential.
- We learnt the method of finite differences which lies at the core of many computational methods for differential equation.
- This project has also introduced us to delight of finding out new things, as most of things we did in this project were completely new to us.

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