Finite Difference Method For Poisson Equation

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Contents

Introduction

- Poisson Equation is an elliptic partial differential Equation.having general form as -:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(r)$$

- Differential Equation can be converted into a system of linear equation by approxiamting them as derivatives by method of finite differences.
- There are many iterative schemes which can be employed to solve this system of linear equation such as Gauss Seidel, SOR and Jacobi Method.

Theory

The below figure shows an interleaved capacitor -:

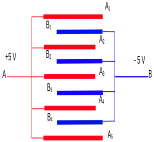


Figure: Diagram dipciting the arrangement of plates in a interleaved fashion.

After non -dimensionalising we had following poisson equation-:

$$\frac{\partial^2 U'(x', y')}{\partial x'^2} + \frac{\partial^2 U'(x', y')}{\partial y'^2} = -\frac{\rho'(x', y')s^2}{\epsilon_0 \nu}$$

Problem

$$\rho'(x', y') = \begin{cases} -2\epsilon_0 \times 10^5 & : \text{if } (x', y') \in B_i \text{ where } i = 1, 2, 3, 4 \\ 2\epsilon_0 \times 10^5 & : \text{if } (x', y') \in A_i \text{ where } i = 2, 3, 4 \\ 0 & : \ \text{elsewhere} \end{cases} \tag{1}$$

With boundary conditions as -:

$$U(0, y) = +5/\nu \qquad U(4, y) = +5/\nu \tag{2}$$

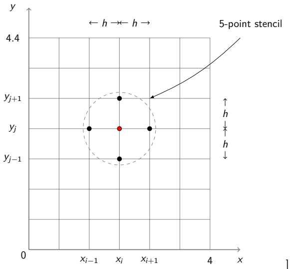
$$U_y(x, 0) = 0 \qquad U_y(x, 4.4) = 0 \tag{3}$$

Methodology

- Finite Difference converts PDE into difference equation.
- Domain is converted into a mesh of equidistant grid points.
- Taylor series is used to approximate the value at these grid points.
- After using Finite Difference operators we get the following stencil for poisson equation

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j+1} - h^2 \frac{\rho'_{i,j} s^2}{\epsilon_0 \nu}\right] \tag{4}$$

This stencil can be represented with the help of following diagram -:



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