Finite Difference Method For Poisson Equation

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Contents

Introduction

- Poisson Equation is an elliptic partial differential Equation.having general form as -:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(r)$$

- Differential Equation can be converted into a system of linear equation by approxiamting them as derivatives by method of finite differences.
- There are many iterative schemes which can be employed to solve this system of linear equation such as Gauss Seidel, SOR and Jacobi Method.

Theory

The below figure shows an interleaved capacitor -:

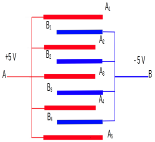


Figure: Diagram dipciting the arrangement of plates in a interleaved fashion.

After non -dimensionalising we had following poisson equation-:

$$\frac{\partial^2 U'(x',y')}{\partial x'^2} + \frac{\partial^2 U'(x',y')}{\partial y'^2} = -\frac{\rho'(x',y')s^2}{\epsilon_0 \nu}$$

Problem

$$\rho'(x', y') = \begin{cases} -2\epsilon_0 \times 10^5 & : \text{if } (x', y') \in B_i \text{ where } i = 1,2,3,4 \\ 2\epsilon_0 \times 10^5 & : \text{if } (x', y') \in A_i \text{ where } i = 2,3,4 \\ 0 & : \text{ elsewhere} \end{cases} \tag{1}$$

With boundary conditions as -:

$$U(0, y) = +5/\nu \qquad U(4, y) = +5/\nu \tag{2}$$

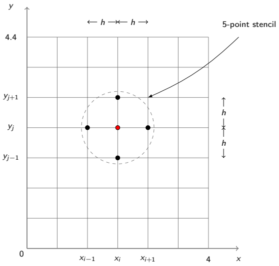
$$U_y(x, 0) = 0 \qquad U_y(x, 4.4) = 0 \tag{3}$$

Methodology

- Finite Difference converts PDE into difference equation.
- Domain is converted into a mesh of equidistant grid points.
- Taylor series is used to approximate the value at these grid points.
- After using Finite Difference operators we get the following stencil for poisson equation

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j+1} + h^2 \frac{\rho'_{i,j} s^2}{\epsilon_0 \nu}\right] \tag{4}$$

This stencil can be represented with the help of following diagram -:

[]

Iterative Method

- Iterative methods are techniques that exploit the properites of system to solve it implicitly to make computation faster.
- We have used following iteratiion schemes-:

Jacobi Method

This method starts with a guess value and with each iteration, it replace guess values with new obtained values from iteration.

$$x_{i,j}^{(n+1)} = S(X^{(n)}, P, h, i, j)$$

Gauss Seidel Method

This method uses the obtained value in the same iteration for other unknowns.

$$x_{i,j}^{(n+1)} = S(X^{(n+1)}, P, h, i, j)$$

SOR

This method involves a relaxation factor (greater than 1) which is multiplied to values obtained from Gauss Seidel before replacing old values.

$$x_{i,j}^{(n+1)} = x_{i,j}^{(n)} + \omega(S(X^{(n+1)}, P, h, i, j) - x_{i,j}^{(n+1)})$$

Expectation

We expect the following graph-:

- α_1 should decrease from positive to negative potential
- α_2 should have negatively charged potential
- α_3 should increase from negative to positive potential
- α_4 should have positively charged potential

Numerical Result

We obtained following graphs as solution -: