1 Non -Dimensionalisation

1.1 Laplace Equation

In this section we would like to non-dimensionalise Laplace equation. The general form of Laplace equation is:

$$\vec{\nabla}^2 \varphi(\vec{r}) = 0$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or

$$u_{xx} + u_{yy} = 0$$

where u_{xx} represents the second order partial derivative of u with respect to x. and u_{yy} represents second order partial derivative of u with respect to y. In our case $u = V$ so we need to non -dimensionalise the following equation:

$$V_{xx} + V_{yy} = 0$$

Now we will replace the dimensional variable with non-dimensional variables. Let

$$\hat{V} = \frac{V}{V_s}, \hat{x} = \frac{x}{x_s}, \hat{y} = \frac{y}{y_s}$$

here the $\hat{V}, \hat{x}, \hat{y}$ are dimensionless variables and V_s, x_s, y_s are the scaling variables having the same unit as their counterparts.

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial(\hat{V}V_s)}{\partial(\hat{x}x_s)}\right)$$

$$= \frac{V_s}{x_s}\left[\frac{\partial}{\partial(x_s\hat{x})}\left(\frac{\partial \hat{V}}{\partial \hat{x}}\right)\right]$$

$$= \frac{V_s}{x_s^2}\left[\frac{\partial^2 \hat{V}}{\partial \hat{x}^2}\right]$$

similarly, we can write for y as :

$$\frac{V_s}{y_s^2}\left[\frac{\partial^2 \hat{V}}{\partial \hat{y}^2}\right]$$

Now we can write our equation as:

$$= \frac{V_s}{y_s^2}\left[\frac{\partial^2 \hat{V}}{\partial \hat{y}^2}\right] + \frac{V_s}{x_s^2}\left[\frac{\partial^2 \hat{V}}{\partial \hat{x}^2}\right] = 0$$

So now if choose same magnitude for our scaling factor then we write our equation as:

$$= \frac{V_s}{x_s^2}\left[\frac{\partial^2 \hat{V}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{x}^2}\right] = 0$$

$$= \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{x}^2} = 0$$

Hence so if we just use the same sacling factors for both x and y variable then we don't need to non-dimensionalise the Laplace equation.

1.2 Poisson Equation

Now since the general form of Poisson Equation is as follows:

$$\vec{\nabla}^2 \varphi(\vec{r}) = f(\vec{r})$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(r)$$

So for our case we can write it as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\rho}{\epsilon_0}$$

Now following the same procedure for LHS as done in case of Laplace equation we get:

$$\frac{V_s}{x_s^2}\left[\frac{\partial^2 \hat{V}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{x}^2}\right] = \frac{\rho}{\epsilon_0}$$

or

$$\frac{\partial^2 \hat{V}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{x}^2} = \frac{\rho}{\epsilon_0}\frac{x_s^2}{V_s}$$

Now we will choose the scaling factors such that the value of RHS becomes unity to ease our computation part and it will depend upon the question.

2 Error Analysis

2.1 Model Problem

In this we will analyse a model problem and perform analysis on it so that we can get a genaral idea about the error induced in our methods.