Derivation for Reflection Coefficients of a Truncated Slab Waveguide

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1 Problem Geometry

Consider a dielectric slab waveguide such that waves propagate along the z-axis. The slab extends infinitely along y, and has interfaces at $|x| = d$. At the plane $z = 0$, the slab terminates abruptly.

The slab has refractive index n, and relative permittivity $\epsilon_r = n^2$.

In this model, we use the convention $e^{i(\omega t - \beta z)}$
We will make frequent use of the following wave vectors:

$$k = \omega/c$$

(continuum modes)

$$\rho = \sqrt{k^2 - \beta_c^2}$$
$$\sigma = \sqrt{n^2k^2 - \beta_c^2}$$

(guided modes)

$$\gamma = \sqrt{\beta_n^2 - k^2}$$
$$\kappa = \sqrt{n^2k^2 - \beta_m^2}$$

2 Relationship between Primary and Complimentary fields

For each polarization, the y-directed field component is referred to as dominant. Since we will be exploiting orthogonality using the z-component of power flow, we care about the corresponding x-oriented field component of the complimentary field.

2.1 TM

We employ Maxwell's Equations to reduce the problem to the H_y field components:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t} = i\omega\epsilon \mathbf{E} \tag{1}$$

Also,

$$\nabla \times \mathbf{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & H_y & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial H_y}{\partial z}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x}\right) \tag{2}$$

$$= \hat{x}\left(i\beta H_y\right) + \hat{z}\left(-i\sqrt{k^2 - \beta^2} H_y\right) \tag{3}$$

where it is important to note that k changes inside and outside the slab. For the transverse Electric field, then

$$E_x \cdot (i\omega\epsilon) = i\beta H_y \tag{4}$$

$$E_x = \frac{\beta}{\omega\epsilon} H_y \tag{5}$$

2.2 TE

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} = -i\omega\mu \mathbf{H} \tag{6}$$

Also,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial E_y}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x}\right) \tag{7}$$

$$= \hat{x}\left(i\beta E_y\right) + \hat{z}\left(-i\sqrt{k^2 - \beta^2} E_y\right) \tag{8}$$

where it is important to note that k changes inside and outside the slab. For the transverse Electric field, then

$$H_x \cdot (-i\omega\mu) = i\beta E_y \tag{9}$$

$$H_x = -\frac{\beta}{\omega\mu} E_y \tag{10}$$

3 Orthogonality

The orthogonality between modes is expressed using the power carried by the fields

$$P_z \delta_{m,n} = \frac{1}{2} \int_{-\infty}^{\infty} \mathbf{E}_m \times \mathbf{H}_n^* \cdot \hat{z} \, dx \tag{11}$$

where m, n refer to different modes. For TM modes ($H_x, H_z = 0$) this reduces to

$$P_z \delta_{m,n} = \frac{1}{2} \int_{-\infty}^{\infty} E_{x,m} H_{y,n}^* dx = \int_{0}^{\infty} \frac{\beta_m}{\omega \epsilon} H_{y,m} H_{y,n} dx \tag{12}$$

For TE modes, orthogonality is expressed as

$$P_z \delta_{m,n} = \frac{1}{2} \int_{-\infty}^{\infty} E_{y,m} H_{x,n}^* dx = \int_{0}^{\infty} \frac{\beta_m}{\omega \mu} E_{y,m} E_{y,n} dx \tag{13}$$

(Note: for the TE case, the value of beta is negated from the complex conjugation)

For continuum modes, Gelin states that P_z should be replaced by $P_z \frac{\beta}{|\beta|}$ to account for evanescence, or complex power flow. (should there be a minus sign in the TE result?)

4 TM Even Modes

4.1 Primary Equations

Begin with a statement of the continuity of transverse fields across the end facet boundary:

$$H^i_{y,m} + \sum_n a_n H^r_{y,n} + \int_0^\infty q^r(\rho) H^r_y(\rho) d\rho = \int_0^\infty q^t(\rho) H^t_y(\rho) d\rho \tag{14}$$

$$E^i_{x,m} + \sum_n a_n E^r_{x,n} + \int_0^\infty q^r(\rho) E^r_x(\rho) d\rho = \int_0^\infty q^t(\rho) E^t_x(\rho) d\rho \tag{15}$$

Substituting (5) into (15) and canceling the ωs and ϵ_os, we now have the primary equations:

$$H^i_{y,m} + \sum_n a_n H^r_{y,n} + \int_0^\infty q^r(\rho) H^r_y(\rho) d\rho = \int_0^\infty q^t(\rho) H^t_y(\rho) d\rho \tag{16}$$

$$\frac{\beta_m}{\epsilon_r} H^i_{y,m} - \sum_n a_n \frac{\beta_n}{\epsilon_r} H^r_{y,n} - \int_0^\infty q^r(\rho) \frac{\beta(\rho)}{\epsilon_r} H^r_y(\rho) d\rho = \int_0^\infty q^t(\rho) \beta(\rho) H^t_y(\rho) d\rho \tag{17}$$

4.2 Solving for q^t

We're going to isolate $q^t(\rho)$. Dummy variables of integration will be indicated as ρ' for distinction. Begin by multiplying each term in (16) by $H^{t*}_y(\rho)$ and integrating over all space:

$$\begin{aligned} &\int_0^\infty H^i_{y,m} H^{t*}_y(\rho)\, dx + \sum_n a_n \int_0^\infty H^r_{y,n} H^{t*}_y(\rho)\, dx + \int_0^\infty q^r(\rho') \int_0^\infty H^r_y(\rho') H^{t*}_y(\rho)\, dx\, d\rho' \\ &= \int_0^\infty q^t(\rho') \int_0^\infty H^t_y(\rho') H^{t*}_y(\rho)\, dx\, d\rho' \end{aligned} \tag{18}$$

From (12), we can eliminate all but our target mode with wave vector ρ:

$$\begin{aligned} \kappa_m(\rho) + \sum_n a_n \kappa_n(\rho) + \int_0^\infty q^r(\rho') \zeta(\rho', \rho)\, d\rho' &= \int_0^\infty q^t(\rho') P \frac{\omega \epsilon_o}{|\beta(\rho)|} \delta(\rho' - \rho)\, d\rho' \\ &= q^t(\rho) \cdot P \frac{\omega \epsilon_o}{|\beta(\rho)|} \end{aligned} \tag{19}$$

where we have introduced the functions

$$\kappa_n(\rho) = \int_0^\infty H^i_{y,n} H^{t*}_y(\rho)\, dx \tag{20}$$

$$\zeta(\rho', \rho) = \int_0^\infty H^r_y(\rho') H^{t*}_y(\rho)\, dx \tag{21}$$

We can repeat this procedure for (17), producing

$$\beta_m \nu_m(\rho) - \sum_n a_n \beta_n \nu_n(\rho) - \int_0^\infty q^r(\rho')\beta(\rho')f(\rho',\rho)\,d\rho' = q^t(\rho)\beta(\rho)\cdot P\frac{\omega\epsilon_o}{|\beta(\rho)|} \tag{22}$$

here introducing

$$\nu_n(\rho) = \int_0^\infty \frac{H^i_{y,n}H^{t*}_y(\rho)}{\epsilon_r}\,dx \tag{23}$$

$$f(\rho',\rho) = \int_0^\infty \frac{H^r_y(\rho')H^{t*}_y(\rho)}{\epsilon_r}\,dx \tag{24}$$

Combining the results as $\beta_m\nu_m(\rho)$ (19) + $\kappa_m(\rho)$ (22) gives a single equation which may be solved for q^t:

$$\begin{aligned}
2\beta_m\nu_m(\rho)\kappa_m(\rho) &+ \sum_n a_n(\beta_m\nu_m(\rho)\kappa_n(\rho) - \beta_n\nu_n(\rho)\kappa_m(\rho)) \\
&+ \int_0^\infty q^r(\rho')(\beta_m\nu_m(\rho)\zeta(\rho',\rho) - \beta(\rho')\kappa_m(\rho)f(\rho',\rho))\,d\rho' \\
&= q^t(\rho)(\beta_m\nu_m(\rho)P\frac{\omega\epsilon_o}{|\beta(\rho)|} + \kappa_m(\rho)\beta(\rho)P\frac{\omega\epsilon_o}{|\beta(\rho)|}) \\
&= q^t(\rho)(\beta_m\nu_m(\rho) + \kappa_m(\rho)\beta(\rho))P\frac{\omega\epsilon_o}{|\beta(\rho)|}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow q^t(\rho) = \frac{1}{\omega\epsilon_o P}\frac{|\beta(\rho)|}{\beta_m\nu_m(\rho)+\kappa_m(\rho)\beta(\rho)}\Big\{&2\beta_m\nu_m(\rho)\kappa_m(\rho) \\
&+\sum_n a_n(\beta_m\nu_m(\rho)\kappa_n(\rho) - \beta_n\nu_n(\rho)\kappa_m(\rho)) \\
+\int_0^\infty q^r(\rho')(\beta_m\nu_m(\rho)\zeta(\rho',\rho) - \beta(\rho')\kappa_m(\rho)f(\rho',\rho))\,d\rho'\Big\}&
\end{aligned} \tag{25}$$

This formula matches that in the Appendix of Gelin 1981, except that I have substituted f and ζ in place of the explicit overlap integrals.

4.3 Source Code

```
integral = np.trapz(qr1 * (Bo*vm*z - Bc1*kpm*f), x=p1, axis=1)
sigma = np.sum([(Bo*vm*kp[n] - Bm[n]*V[n]*kpm) * am[n] for n in
    range(N)], axis=0)
```

```
qt = 1/(w*eo*P) * abs(Bc) / (Bo*vm+Bc*kpm) * (2*Bo*vm*kpm +
    integral + sigma)
```

4.4 Solving for a_n

As above, we will combine the primary equations using orthogonality to extract the coefficients. This time we will multiply each term in (16) by $H^{r*}_{y,n}/\epsilon_r$ (note that the summation index will be changed to k - it will disappear almost immediately so don't worry about it!):

$$\int_0^\infty \frac{H^i_{y,m} H^{r*}_{y,n}}{\epsilon_r}\,dx + \sum_k a_k \int_0^\infty \frac{H^r_{y,k} H^{r*}_{y,n}}{\epsilon_r}\,dx + \int_0^\infty q^r(\rho') \int_0^\infty \frac{H^r_y(\rho') H^{r*}_{y,n}}{\epsilon_r}\,dx\,d\rho'$$
$$= \int_0^\infty q^t(\rho') \int_0^\infty \frac{H^t_y(\rho') H^{r*}_{y,n}}{\epsilon_r}\,dx\,d\rho' \tag{26}$$

Orthogonality eliminates virtually all the terms on the left side.

$$\begin{aligned}
P\frac{\omega\epsilon_o}{\beta_m}\delta_{m,n} + \sum_k a_k P\frac{\omega\epsilon_o}{\beta_n}\delta_{k,n} &= \int_0^\infty q^t(\rho')\nu_n(\rho')\,d\rho' \\
a_n P\frac{\omega\epsilon_o}{\beta_n} &= -P\frac{\omega\epsilon_o}{\beta_n}\delta_{m,n} + \int_0^\infty q^t(\rho')\nu_n(\rho')\,d\rho' \\
a_n &= \frac{\beta_n}{\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')\nu_n(\rho')\,d\rho' - P\frac{\omega\epsilon_o}{\beta_n}\delta_{m,n}\right\} \\
a_n &= \frac{1}{\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')\beta_n\nu_n(\rho')\,d\rho' - P\omega\epsilon_o\delta_{m,n}\right\}
\end{aligned} \tag{27}$$

Repeating the process for (17), but excluding the factor of $1/\epsilon_r$:

$$\beta_m \int_0^\infty \frac{H^i_{y,m} H^{r*}_{y,n}}{\epsilon_r}\,dx - \sum_k a_k\beta_k \int_0^\infty \frac{H^r_{y,k} H^{r*}_{y,n}}{\epsilon_r}\,dx - \int_0^\infty q^r(\rho')\beta(\rho') \int_0^\infty \frac{H^r_y(\rho') H^{r*}_{y,n}}{\epsilon_r}\,dx\,d\rho'$$
$$= \int_0^\infty q^t(\rho')\beta(\rho') \int_0^\infty H^t_y(\rho') H^{r*}_{y,n}\,dx\,d\rho' \tag{28}$$

$$\begin{aligned}
\beta_m P\frac{\omega\epsilon_o}{\beta_m}\delta_{m,n} - \sum_k a_k\beta_k P\frac{\omega\epsilon_o}{\beta_n}\delta_{k,n} &= \int_0^\infty q^t(\rho')\beta(\rho')\kappa_n(\rho')\,d\rho' \\
-a_n P\omega\epsilon_o &= -P\omega\epsilon_o\delta_{m,n} + \int_0^\infty q^t(\rho')\beta(\rho')\kappa_n(\rho')\,d\rho' \\
a_n &= \frac{1}{\omega\epsilon_o P}\left\{-\int_0^\infty q^t(\rho')\beta(\rho')\kappa_n(\rho')\,d\rho' + P\omega\epsilon_o\delta_{m,n}\right\}
\end{aligned} \tag{29}$$

Combining (27) + (29) yields:

$$
\begin{aligned}
2a_n &= \frac{1}{\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')[\beta_n\nu_n(\rho') - \beta(\rho')\kappa_n(\rho')] + P\omega\epsilon_o\delta_{m,n} - P\omega\epsilon_o\delta_{m,n}\,d\rho'\right\} \\
a_n &= \frac{1}{2\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')[\beta_n\nu_n(\rho') - \beta(\rho')\kappa_n(\rho')]\,d\rho'\right\} \qquad (30)
\end{aligned}
$$

4.5 Source Code

```
am = np.array([ (1/(2*w*eo*P) * np.trapz(qt * (Bm[n]*V[n] - Bc*kp
    [n]), x=p, axis=0)) for n in range(N) ])
```

4.6 Solving for q^r

Once again, we will combine the primary equations using orthogonality to extract the coefficients. This time we will multiply each term in (16) by $H_y^{r*}(\rho)/\epsilon_r$:

$$
\begin{aligned}
&\int_0^\infty \frac{H^i_{y,m} H_y^{r*}(\rho)}{\epsilon_r}\,dx + \sum_k a_k \int_0^\infty \frac{H^r_{y,k} H_y^{r*}(\rho)}{\epsilon_r}\,dx + \int_0^\infty q^r(\rho') \int_0^\infty \frac{H_y^r(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho' \\
&= \int_0^\infty q^t(\rho') \int_0^\infty \frac{H_y^t(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho' \qquad (31)
\end{aligned}
$$

Orthogonality eliminates virtually all the terms on the left side.

$$
\begin{aligned}
\int_0^\infty q^r(\rho') \int_0^\infty \frac{H_y^r(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho' &= \int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
\int_0^\infty q^r(\rho') \frac{P\omega\epsilon_o}{|\beta(\rho)|}\delta(\rho'-\rho)\,d\rho' &= \int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
q^r(\rho)\frac{P\omega\epsilon_o}{|\beta(\rho)|} &= \int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
q^r(\rho) &= \frac{|\beta(\rho)|}{\omega\epsilon_o P}\int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
q^r(\rho) = \frac{1}{\omega\epsilon_o P}\frac{|\beta(\rho)|}{\beta(\rho)} &\int_0^\infty q^t(\rho')\beta(\rho) f(\rho,\rho')\,d\rho' \qquad (32)
\end{aligned}
$$

Repeating the process for (17), but excluding the factor of $1/\epsilon_r$:

$$
\beta_m \int_0^\infty \frac{H^i_{y,m} H^{r*}_y(\rho)}{\epsilon_r}\, dx - \sum_k a_k \beta_k \int_0^\infty \frac{H^r_{y,k} H^{r*}_y(\rho)}{\epsilon_r}\, dx - \int_0^\infty q^r(\rho')\beta(\rho') \int_0^\infty \frac{H^r_y(\rho') H^{r*}_y(\rho)}{\epsilon_r}\, dx\, d\rho'
$$

$$
= \int_0^\infty q^t(\rho')\beta(\rho') \int_0^\infty H^t_y(\rho') H^{r*}_y(\rho)\, dx\, d\rho' \tag{33}
$$

$$
\begin{aligned}
-\int_0^\infty q^r(\rho')\beta(\rho') \int_0^\infty \frac{H^r_y(\rho') H^{r*}_y(\rho)}{\epsilon_r}\, dx\, d\rho' &= \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\, d\rho' \\
-\int_0^\infty q^r(\rho')\beta(\rho') \frac{P\omega\epsilon_o}{|\beta(\rho)|} \delta(\rho' - \rho)\, d\rho' &= \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\, d\rho' \\
-q^r(\rho)\beta(\rho') \frac{P\omega\epsilon_o}{|\beta(\rho)|} &= \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\, d\rho' \\
q^r(\rho) &= -\frac{1}{\omega\epsilon_o P} \frac{|\beta(\rho)|}{\beta(\rho)} \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\, d\rho' \qquad (34)
\end{aligned}
$$

Combining (32) + (34) yields:

$$
\begin{aligned}
2q^r(\rho) &= \frac{1}{\omega\epsilon_o P} \frac{|\beta(\rho)|}{\beta(\rho)} \int_0^\infty q^t(\rho')[\beta(\rho) f(\rho,\rho') - \beta(\rho')\zeta(\rho,\rho')]\, d\rho' \\
q^r(\rho) &= \frac{1}{2\omega\epsilon_o P} \frac{|\beta(\rho)|}{\beta(\rho)} \int_0^\infty q^t(\rho')[\beta(\rho) f(\rho,\rho') - \beta(\rho')\zeta(\rho,\rho')]\, d\rho' \qquad (35)
\end{aligned}
$$

4.7 Source Code

```
integral = np.trapz(qt1 * (Bc2 * f.transpose(1,0,2) - Bc1 * z.
    transpose(1,0,2)), x=p1, axis=1)
qr = 1/(2*w*eo*P) * (abs(Bc)/Bc) * integral
```

5 TE Even Modes

5.1 Primary Equations

Begin with a statement of the continuity of transverse fields across the end facet boundary:

$$E^i_{y,m} + \sum_n a_n E^r_{y,n} + \int_0^\infty q^r(\rho) E^r_y(\rho)\, d\rho = \int_0^\infty q^t(\rho) E^t_y(\rho)\, d\rho \tag{36}$$

$$H^i_{x,m} + \sum_n a_n H^r_{x,n} + \int_0^\infty q^r(\rho) H^r_x(\rho)\, d\rho = \int_0^\infty q^t(\rho) H^t_x(\rho)\, dp \tag{37}$$

Substituting (10) into (37) and canceling the ωs and μs, we now have the primary equations:

$$E^i_{y,m} + \sum_n a_n E^r_{y,n} + \int_0^\infty q^r(\rho) E^r_y(\rho)\, d\rho = \int_0^\infty q^t(\rho) E^t_y(\rho)\, d\rho \tag{38}$$

$$-\beta_m E^i_{y,m} + \sum_n a_n \beta_n E^r_{y,n} + \int_0^\infty q^r(\rho)\beta(\rho) E^r_y(\rho)\, d\rho = -\int_0^\infty q^t(\rho)\beta(\rho) E^t_y(\rho)\, d\rho \tag{39}$$

$$\Rightarrow \beta_m E^i_{y,m} - \sum_n a_n \beta_n E^r_{y,n} - \int_0^\infty q^r(\rho)\beta(\rho) E^r_y(\rho)\, d\rho = \int_0^\infty q^t(\rho)\beta(\rho) E^t_y(\rho)\, d\rho \tag{40}$$

5.2 Solving for q^t

We're going to isolate $q^t(\rho)$. Dummy variables of integration will be indicated as ρ' for distinction. Begin by multiplying each term in (38) by $E^{t*}_y(\rho)$ and integrating over all space:

$$\int_0^\infty E^i_{y,m} E^{t*}_y(\rho)\, dx + \sum_n a_n \int_0^\infty E^r_{y,n} E^{t*}_y(\rho)\, dx + \int_0^\infty q^r(\rho') \int_0^\infty E^r_y(\rho') E^{t*}_y(\rho)\, dx\, d\rho'$$
$$= \int_0^\infty q^t(\rho') \int_0^\infty E^t_y(\rho') E^{t*}_y(\rho)\, dx\, d\rho' \tag{41}$$

From (13), we can eliminate all but our target mode with wave vector ρ:

$$\begin{aligned}\frac{1}{2} G_m(\rho) + \frac{1}{2}\sum_n a_n G_n(\rho) + \frac{1}{2}\int_0^\infty q^r(\rho') F(\rho', \rho)\, d\rho' &= \int_0^\infty q^t(\rho') \left(-P\frac{\omega\mu}{|\beta(\rho)|}\delta(\rho' - \rho)\right) d\rho' \\ &= q^t(\rho) \cdot P\frac{\omega\mu}{|\beta(\rho)|}\end{aligned} \tag{42}$$

where we have introduced the functions

$$G_n(\rho) = \int_{-\infty}^{\infty} E^i_{y,n} E^{t*}_y(\rho)\, dx \tag{43}$$

$$F(\rho', \rho) = \int_{-\infty}^{\infty} E^r_y(\rho') E^{t*}_y(\rho)\, dx \tag{44}$$

We can repeat this procedure for (40), producing

$$\frac{1}{2}\beta_m G_m(\rho) - \frac{1}{2}\sum_n a_n \beta_n G_n(\rho) - \frac{1}{2}\int_0^{\infty} q^r(\rho')\beta(\rho')F(\rho', \rho)\, d\rho' = -q^t(\rho)\beta(\rho) \cdot P\frac{\omega\mu}{|\beta(\rho)|} \tag{45}$$

Combining the results as $\beta_m(\rho)$ (42) + (45) gives a single equation which may be solved for q^t:

$$\begin{aligned} G_m\beta_m + \frac{1}{2}\sum_n a_n(\beta_m - \beta_n)G_n + \frac{1}{2}\int_0^{\infty} q^r(\rho')(\beta_m - \beta(\rho'))F(\rho', \rho)\, d\rho' \\ = q^t(\rho)(\beta_m + \beta(\rho)) \cdot P\frac{\omega\mu}{|\beta(\rho)|} \end{aligned} \tag{46}$$

$$\begin{aligned} \Rightarrow q^t(\rho) = \frac{1}{2\omega\mu P}\frac{|\beta(\rho)|}{\beta_m + \beta(\rho)}\Big\{ 2\beta_m G_m(\rho) + \sum_n a_n(\beta_m - \beta_n)G_n(\rho) \\ + \int_0^{\infty} q^r(\rho')(\beta_m - \beta(\rho'))F(\rho', \rho)\, d\rho' \Big\} \end{aligned} \tag{47}$$

This formula matches (5) in Gelin 1981.

5.3 Source Code

```
qr1 = np.tile(qr,(len(p),1,1))
integral = np.trapz(qr1 * (Bo-Bc1) * F, x=p1, axis=1)
# integral = np.sum(qr1 * (Bo-Bc1) * F, axis=1) * dp
sigma = np.sum([(Bo-Bm[n]) * am[n] * G[n,:] for n in range(N)],
    axis=0)
qt = 1/(2*w*mu*P) * abs(Bc) / (Bo+Bc) * (2*Bo*G[incident_mode,:]
   + integral + sigma)
```

5.4 a_n

Begin with the primary equations. Multiply each term by $E^{r*}_{y,n}$ and integrate x from 0 to infinity. Equation (38) becomes:

$$0 + \sum_k a_k \int_0^\infty E^r_{y,k} E^{r*}_{y,n} \, dx + 0 = \int_0^\infty q^t(\rho') \int_0^\infty E^t_y(\rho') E^{r*}_{y,n} \, dx \, d\rho'$$

$$a_n \cdot \left(P \frac{\omega\mu}{\beta_n} \right) = \frac{1}{2} \int_0^\infty q^t(\rho') G_n(\rho') \, d\rho'$$

Similarly, (40) becomes:

$$0 - \sum_k a_k \beta_k \int_0^\infty E^r_{y,k} E^{r*}_{y,n} \, dx + 0 = \int_0^\infty q^t(\rho') \beta(\rho') \int_0^\infty E^t_y(\rho') E^{r*}_{y,n} \, dx \, d\rho'$$

$$a_n \beta_n \cdot \left(-P \frac{\omega\mu}{\beta_n} \right) = \frac{1}{2} \int_0^\infty q^t(\rho') \beta(\rho') G_n(\rho') \, d\rho'$$

$$a_n \cdot (P\omega\mu) = -\frac{1}{2} \int_0^\infty q^t(\rho') \beta(\rho') G_n(\rho') \, d\rho'$$

Adding β_n times the former result to the latter yields, thus:

$$2a_n \cdot (P\omega\mu) = \frac{1}{2} \int_0^\infty q^t(\rho')(\beta_n - \beta(\rho')) G_n(\rho') \, d\rho'$$

$$\Rightarrow a_n = \frac{1}{4\omega\mu P} \int_0^\infty q^t(\rho')(\beta_n - \beta(\rho')) G_n(\rho') \, d\rho' \tag{48}$$

$$\tag{49}$$

5.5 Source Code

```
am = np.array([ (1/(4*w*mu*P) * np.trapz(qt * (Bm[n]-Bc) * G[n
    ,:], x=p, axis=0)) for n in range(N) ])
```

5.6 q^r

Begin with the primary equations. Multiply each term by $E^{r*}_y(\rho)$ and integrate x from 0 to infinity. Equation (38) becomes:

$$0+0+\int_0^\infty q^r(\rho)\int_0^\infty E_y^r(\rho')E_y^{r*}(\rho)\;dx\;d\rho' = \int_0^\infty q^t(\rho')\int_0^\infty E_y^t(\rho')E_y^{r*}(\rho)\;dx\;d\rho'$$
$$q^r(\rho)\cdot\left(P\frac{\omega\mu}{|\beta(\rho)|}\right) = \frac{1}{2}\int_0^\infty q^t(\rho')F^*(\rho,\rho')\;d\rho'$$

Similarly, (40) becomes:

$$0+0-\int_0^\infty q^r(\rho)\beta(\rho')\int_0^\infty E_y^r(\rho')E_y^{r*}(\rho)\;dx\;d\rho' = \int_0^\infty q^t(\rho')\beta(\rho')\int_0^\infty E_y^t(\rho')E_y^{r*}(\rho)\;dx\;d\rho'$$
$$\beta(\rho)q^r(\rho)\cdot\left(P\frac{\omega\mu}{|\beta(\rho)|}\right) = -\frac{1}{2}\int_0^\infty q^t(\rho')\beta(\rho')F^*(\rho,\rho')\;d\rho'$$

Adding $\beta(\rho)$ times the former result to the latter yields, thus:

$$2\beta(\rho)q^r\cdot\left(\frac{P\omega\mu}{|\beta(\rho)|}\right) = \frac{1}{2}\int_0^\infty q^t(\rho')(\beta(\rho)-\beta(\rho'))F^*(\rho,\rho')\;d\rho'$$
$$\Rightarrow q^r = \frac{1}{4\omega\mu P}\frac{|\beta(\rho)|}{\beta(\rho)}\int_0^\infty q^t(\rho')(\beta(\rho)-\beta(\rho'))F^*(\rho,\rho')\;d\rho' \tag{50}$$

5.7 Source Code

```
integral = np.trapz(qt1 * (Bc2-Bc1) * F.transpose(1,0,2), x=p1,
    axis=1)
# integral = np.sum(qt1 * (Bc2-Bc1) * F.transpose(1,0,2), axis=1)
    * dp
qr = 1/(4*w*mu*P) * (abs(Bc)/Bc) * integral
```

6 Derivation of the Mode Amplitude Coefficients

6.1 TM Even - Guided Modes

$$
\begin{aligned}
P &= \int_0^d \frac{\beta_n}{\omega\epsilon}|A_n|^2\cos^2(\kappa_n x)\,dx + \int_d^\infty \frac{\beta_n}{\omega\epsilon_o}|A_n|^2\cos^2(\kappa_n d)e^{2\gamma d}e^{-2\gamma x}\,dx \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\int_0^d \frac{1}{\epsilon_r}\cos^2(\kappa_n x)\,dx + \cos^2(\kappa_n d)e^{2\gamma d}\int_d^\infty e^{-2\gamma x}\,dx\right] \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \cos^2(\kappa_n d)e^{2\gamma d}\left(\frac{1}{-2\gamma}e^{-2\gamma x}\right)\Bigg|_d^\infty\right] \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \cos^2(\kappa_n d)e^{2\gamma d}\frac{1}{2\gamma}e^{-2\gamma d}\right] \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \frac{\cos^2(\kappa_n d)}{2\gamma}\right] \\
\Rightarrow A &= \sqrt{\frac{\omega\epsilon_o P}{\beta_n\cdot\psi}}
\end{aligned}
$$

where $\psi = \left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \frac{\cos^2(\kappa_n d)}{2\gamma}\right]$.

6.2 Source Code

Note that this formula is not the one used by Marcuse, but the values seem to match within 10^{-14}.

```
psi = (2*Km*d + sin(2*Km*d))/(4*Km*eps) + cos(Km*d)**2/(2*gm)
Am = sqrt(w*eo*P/(Bm*psi))
```

6.3 TM Even - Radiation Modes

The derivation of the coefficients for Radiation Modes is provided in Marcuse's book, "Light Transmission Optics", pp. 316-318. The result is reprinted in his 1969 paper "Radiation Losses of Tapered Dielectric Slab Waveguides", and the formulae match. Gelin reprints the formula, although he includes a factor of ϵ^2 where Marcuse would have just ϵ.

6.4 TE Even - Guided Modes

$$
\begin{aligned}
P &= \int_0^d \frac{\beta_n}{\omega\mu}|A_n|^2 \cos^2(\kappa_n x)\ dx + \int_d^\infty \frac{\beta_n}{\omega\mu}|A_n|^2 \cos^2(\kappa_n d) e^{2\gamma d} e^{-2\gamma x}\ dx \\
&= \frac{\beta_n |A|^2}{\omega\mu}\left[\int_0^d \cos^2(\kappa_n x)\ dx + \cos^2(\kappa_n d) e^{2\gamma d}\int_d^\infty e^{-2\gamma x}\ dx\right] \\
&= \frac{\beta_n |A|^2}{\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{4\kappa_n} + \cos^2(\kappa_n d) e^{2\gamma d}\left(\frac{1}{-2\gamma}e^{-2\gamma x}\right)\Bigg|_d^\infty\right] \\
&= \frac{\beta_n |A|^2}{\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{4\kappa_n} + \cos^2(\kappa_n d) e^{2\gamma d}\frac{1}{2\gamma}e^{-2\gamma d}\right] \\
&= \frac{\beta_n |A|^2}{\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{4\kappa_n} + \frac{\cos^2(\kappa_n d)}{2\gamma}\right] \\
&= \frac{\beta_n |A|^2}{2\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{2\kappa_n} + \frac{\cos^2(\kappa_n d)}{\gamma}\right] \\
&= \frac{\beta_n |A|^2}{2\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{2\kappa_n} + \frac{1 + \cos(2\kappa_n d)}{2\gamma}\right]
\end{aligned}
$$

7 Derivation of the Overlap Integral Functions

7.1 Fields of the Even Modes

For even guided modes, the fields are given by

$$\phi_{y,n}(x) = \begin{cases} A_n \cos(\kappa x) & : |x| < d \\ A_n \cos(\kappa d) e^{\gamma d} e^{-\gamma x} & : |x| \geq d \end{cases}$$

where ϕ stands for either E or H, depending on the polarization. The radiation/continuum modes are

$$\phi_y(x, \rho) = \begin{cases} B_n^r \cos(\sigma x) & : |x| < d \\ B_n^r \left[D e^{-i\rho|x|} + D^* e^{-i\rho|x|} \right] & : |x| \geq d \end{cases}$$

Note that A, B, and D depend on the polarization. (See section 6). Finally, the free-space modes in the right half-space are described by:

$$\phi_y(x, \rho) = B^t \cos(\rho x)$$

7.2 $\kappa_n(\rho)$

Recalling the definition from section 4.2:

$$\kappa_n(\rho) = \int_0^\infty H_{y,n}^i H_y^{t*}(\rho)\, dx$$

Using the definition of the fields for even TM modes, we get the following:

$$
\begin{aligned}
\kappa_n(\rho) &= \int_0^d A_n \cos(\kappa x) B^t \cos(\rho x)\, dx + \int_d^\infty A_n \cos(\kappa d) e^{\gamma d} e^{-\gamma x} B^t \cos(\rho x)\, dx \\
&= A_n B^t \left\{ \int_0^d \cos(\kappa x)\cos(\rho x)\, dx + \cos(\kappa d) e^{\gamma d} \int_d^\infty e^{-\gamma x} \cos(\rho x)\, dx \right\} \\
&= A_n B^t \left\{ \frac{\kappa \sin(\kappa x)\cos(\rho x) - \rho \sin(\rho x)\cos(\kappa x)}{\kappa^2 - \rho^2} \Big|_0^d \right. \\
&\qquad\qquad\qquad \left. + \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma x}(\rho \sin(\rho x) - \gamma \cos(\rho x))}{\gamma^2 + \rho^2} \Big|_d^\infty \right\} \\
&= A_n B^t \left\{ \frac{\kappa \sin(\kappa d)\cos(\rho d) - \rho \sin(\rho d)\cos(\kappa d)}{\kappa^2 - \rho^2} \right. \\
&\qquad\qquad\qquad \left. - \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma d}(\rho \sin(\rho d) - \gamma \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= A_n B^t \cos(\kappa d) \left\{ \frac{\kappa \tan(\kappa d)\cos(\rho d) - \rho \sin(\rho d)}{\kappa^2 - \rho^2} - \frac{\rho \sin(\rho d) - \gamma \cos(\rho d)}{\gamma^2 + \rho^2} \right\}
\end{aligned}
$$

7.3 Source Code

```
kp[i,:,:] = Am[i] * Bt * cos(Km[i]*d) * ((Km[i]*cos(p*d)*tan(Km[i
    ]*d) - p*sin(p*d))/(     Km[i]**2 - p**2) - (p*sin(p*d) - gm[i
    ]*cos(p*d))/(gm[i]**2 + p**2))
```

7.4 $G_n(\rho)$

$$
G_n(\rho) = \int_{-\infty}^{\infty} E^i_{y,n} E^{t*}_y(\rho)\, dx \tag{51}
$$

$$
\begin{aligned}
G_n(\rho) &= 2\int_0^d A_n \cos(\kappa x) B^t \cos(\rho x)\, dx + 2\int_d^\infty A_n \cos(\kappa d) e^{\gamma d} e^{-\gamma x} B^t \cos(\rho x)\, dx \\
&= 2A_n B^t \left\{ \int_0^d \cos(\kappa x)\cos(\rho x)\, dx + \cos(\kappa d) e^{\gamma d} \int_d^\infty e^{-\gamma x} \cos(\rho x)\, dx \right\} \\
&= 2A_n B^t \left\{ \frac{\kappa \sin(\kappa x)\cos(\rho x) - \rho \sin(\rho x)\cos(\kappa x)}{\kappa^2 - \rho^2} \Big|_0^d \right. \\
&\qquad\qquad\qquad\qquad \left. + \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma x}(\rho \sin(\rho x) - \gamma \cos(\rho x))}{\gamma^2 + \rho^2} \Big|_d^\infty \right\} \\
&= 2A_n B^t \left\{ \frac{\kappa \sin(\kappa d)\cos(\rho d) - \rho \sin(\rho d)\cos(\kappa d)}{\kappa^2 - \rho^2} \right. \\
&\qquad\qquad\qquad\qquad \left. - \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma d}(\rho \sin(\rho d) - \gamma \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= 2A_n B^t \cos(\kappa d) \left\{ \frac{\kappa \tan(\kappa d)\cos(\rho d) - \rho \sin(\rho d)}{\kappa^2 - \rho^2} - \frac{\rho \sin(\rho d) - \gamma \cos(\rho d)}{\gamma^2 + \rho^2} \right\}
\end{aligned}
$$

Since this function occurs for TE cases only, we can substitute the identity $\kappa \tan(\kappa d) = \gamma$:

$$
\begin{aligned}
&= 2A_n B^t \cos(\kappa d) \left\{ \frac{\gamma \cos(\rho d) - \rho \sin(\rho d)}{\kappa^2 - \rho^2} - \frac{\rho \sin(\rho d) - \gamma \cos(\rho d)}{\gamma^2 + \rho^2} \right\} \\
&= 2A_n B^t \cos(\kappa d) \left\{ \frac{(\kappa^2 - \rho^2)(\gamma \cos(\rho d) - \rho \sin(\rho d)) - (\gamma^2 + \rho^2)(\rho \sin(\rho d) - \gamma \cos(\rho d))}{(\kappa^2 - \rho^2)(\gamma^2 + \rho^2)} \right\} \\
&= 2A_n B^t \cos(\kappa d) \left\{ \frac{(\kappa^2 - \rho^2 + \gamma^2 + \rho^2)(\gamma \cos(\rho d)) - (\kappa^2 - \rho^2 + \gamma^2 + \rho^2)(\rho \sin(\rho d))}{(\kappa^2 - \rho^2)(\gamma^2 + \rho^2)} \right\} \\
&= 2A_n B^t (\kappa^2 + \gamma^2) \cos(\kappa d) \left\{ \frac{\gamma \cos(\rho d) - \rho \sin(\rho d)}{(\kappa^2 - \rho^2)(\gamma^2 + \rho^2)} \right\} \\
\Rightarrow G_n(\rho) &= 2A_n B^t k^2 (\epsilon_r - 1) \cos(\kappa d) \left\{ \frac{\gamma \cos(\rho d) - \rho \sin(\rho d)}{(\kappa^2 - \rho^2)(\gamma^2 + \rho^2)} \right\}
\end{aligned}
$$

This formula matches (7) in Gelin 1981.

7.5 Source Code

```
G[i,:,:] = 2 * k**2 * (eps-1) * Am[i] * Bt * cos(Km[i]*d) * (gm[i
    ]*cos(p*d) - p*sin(p*d)) / ((Km[i]**2 - p**2)*(gm[i]**2 + p
    **2))
```

7.6 $\nu_n(\rho)$

$$\nu_n(\rho) = \frac{\int_0^\infty H^i_{y,n} H^{t*}_y(\rho)}{\epsilon_r}\, dx$$

Using the definition of the fields for even TM modes, we get the following:

$$\begin{aligned}
\nu_n(\rho) &= \int_0^d \frac{A_n \cos(\kappa x) B^t \cos(\rho x)}{\epsilon_r}\, dx + \int_d^\infty A_n \cos(\kappa d) e^{\gamma d} e^{-\gamma x} B^t \cos(\rho x)\, dx \\
&= A_n B^t \left\{ \int_0^d \frac{\cos(\kappa x)\cos(\rho x)}{\epsilon_r}\, dx + \cos(\kappa d) e^{\gamma d} \int_d^\infty e^{-\gamma x} \cos(\rho x)\, dx \right\} \\
&= A_n B^t \left\{ \frac{\kappa \sin(\kappa x)\cos(\rho x) - \rho \sin(\rho x)\cos(\kappa x)}{\epsilon_r(\kappa^2 - \rho^2)} \Big|_0^d \right. \\
&\qquad\qquad \left. + \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma x}(\rho \sin(\rho x) - \gamma \cos(\rho x))}{\gamma^2 + \rho^2} \Big|_d^\infty \right\} \\
&= A_n B^t \left\{ \frac{\kappa \sin(\kappa d)\cos(\rho d) - \rho \sin(\rho d)\cos(\kappa d)}{\epsilon_r(\kappa^2 - \rho^2)} \right. \\
&\qquad\qquad \left. - \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma d}(\rho \sin(\rho d) - \gamma \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= A_n B^t \cos(\kappa d) \left\{ \frac{\kappa \cos(\rho d)\tan(\kappa d) - \rho \sin(\rho d)}{\epsilon_r(\kappa^2 - \rho^2)} - \frac{\rho \sin(\rho d) - \gamma \cos(\rho d)}{\gamma^2 + \rho^2} \right\}
\end{aligned}$$

7.7 Source Code

```
V[i,:,:]  = Am[i] * Bt * cos(Km[i]*d) * ((Km[i]*cos(p*d)*tan(Km[i
    ]*d) - p*sin(p*d))/(eps*(Km[i]**2 - p**2)) - (p*sin(p*d) - gm[
    i]*cos(p*d))/(gm[i]**2 + p**2))
```

7.8 $\zeta(\rho', \rho)$

$$\zeta(\rho', \rho) = \int_0^\infty H_y^r(\rho') H_y^{t*}(\rho)\, dx$$

$$\begin{aligned}
\zeta(\rho', \rho) &= \int_0^d B^r(\rho')\cos(\sigma' x) B^t(\rho)\cos(\rho x)\, dx + \int_d^\infty B^r(\rho')[De^{-i\rho' x} + D^* e^{i\rho' x}] B^t(\rho)\cos(\rho x)\, dx \\
&= B^r(\rho') B^t(\rho) \left\{ \int_0^d \cos(\sigma' x)\cos(\rho x)\, dx + \int_d^\infty \cos(\rho x) \cdot 2 \cdot \mathrm{Re}[De^{-i\rho' x}]\, dx \right\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' x)\cos(\rho x) - \rho\cos(\sigma' x)\sin(\rho x)}{\sigma'^2 - \rho^2} \Bigg|_0^d \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\Bigg[\int_d^\infty \cos(\rho x) \cdot De^{-i\rho' x}\, dx \Bigg] \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2}\, dx \Bigg] \Bigg|_d^\infty \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad - 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2} \Bigg] \Bigg|_0^d \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad - 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-i\rho' d}(-\rho\sin(\rho d) + i\rho'\cos(\rho d)) - i\rho'}{\rho'^2 - \rho^2} \Bigg] \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-i\rho' d}(\rho\sin(\rho d) - i\rho'\cos(\rho d)) + i\rho'}{\rho'^2 - \rho^2} \Bigg] \Bigg\}
\end{aligned}$$

7.9 Source Code

```
z = Br1*Bt2 * ((o1*sin(od)*cos(pd) - p2*cos(od)*sin(pd))/
    (o1**2-p2**2)  + 2*(D1 * (exp(-1j*p1*d) * (p2*sin(pd)-1j*p1*
    cos(pd))) + 1j*p1 ).real / (p1**2-p2**2) )
```

7.10 $f(\rho', \rho)$

$$f(\rho', \rho) = \int_0^\infty \frac{H_y^r(\rho') H_y^{t*}(\rho)}{\epsilon_r} \, dx$$

$$\begin{aligned}
f(\rho', \rho) &= \int_0^d \frac{B^r(\rho') \cos(\sigma' x) B^t(\rho) \cos(\rho x)}{\epsilon_r} \, dx + \int_d^\infty B^r(\rho') [D e^{-i\rho' x} + D^* e^{i\rho' x}] B^t(\rho) \cos(\rho x) \, dx \\
&= B^r(\rho') B^t(\rho) \left\{ \int_0^d \frac{\cos(\sigma' x) \cos(\rho x)}{\epsilon_r} \, dx + \int_d^\infty \cos(\rho x) \cdot 2 \cdot \mathrm{Re}[D e^{-i\rho' x}] \, dx \right\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' x) \cos(\rho x) - \rho \cos(\sigma' x) \sin(\rho x)}{\epsilon_r (\sigma'^2 - \rho^2)} \Bigg|_0^d \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\Bigg[\int_d^\infty \cos(\rho x) \cdot D e^{-i\rho' x} \, dx \Bigg] \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\epsilon_r (\sigma'^2 - \rho^2)} \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-ix\rho'} (-\rho \sin(\rho x) + i\rho' \cos(\rho x))}{\rho'^2 - \rho^2} \, dx \Bigg] \Bigg|_d^\infty \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\epsilon_r (\sigma'^2 - \rho^2)} \\
&\qquad\qquad - 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-ix\rho'} (-\rho \sin(\rho x) + i\rho' \cos(\rho x))}{\rho'^2 - \rho^2} \Bigg] \Bigg|_0^d \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\epsilon_r (\sigma'^2 - \rho^2)} \\
&\qquad\qquad - 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-i\rho' d} (-\rho \sin(\rho d) + i\rho' \cos(\rho d)) - i\rho'}{\rho'^2 - \rho^2} \Bigg] \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\epsilon_r (\sigma'^2 - \rho^2)} \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\Bigg[D \frac{e^{-i\rho' d} (\rho \sin(\rho d) - i\rho' \cos(\rho d)) + i\rho'}{\rho'^2 - \rho^2} \Bigg] \Bigg\}
\end{aligned}$$

7.11 Source Code

```
f = Br1*Bt2 * ((o1*sin(od)*cos(pd) - p2*cos(od)*sin(pd))/(n**2 *
    (o1**2-p2**2)) + 2*(D1 * (exp(-1j*p1*d) * (p2*sin(pd)-1j*p1*
    cos(pd))) + 1j*p1 ).real / (p1**2-p2**2) )
```

7.12 $F(\rho', \rho)$

$$F(\rho', \rho) = \int_{-\infty}^{\infty} E_y^r(\rho') E_y^{t*}(\rho)\, dx$$

$$\begin{aligned}
F(\rho', \rho) &= 2\int_0^d B^r(\rho')\cos(\sigma' x) B^t(\rho)\cos(\rho x)\, dx + 2\int_d^\infty B^r(\rho')[De^{-i\rho' x} + D^* e^{i\rho' x}]B^t(\rho)\cos(\rho x)\, dx \\
&= 2B^r(\rho')B^t(\rho)\left\{\int_0^d \cos(\sigma' x)\cos(\rho x)\, dx + \int_d^\infty \cos(\rho x)\cdot 2\cdot \mathrm{Re}[De^{-i\rho' x}]\, dx\right\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' x)\cos(\rho x) - \rho\cos(\sigma' x)\sin(\rho x)}{\sigma'^2 - \rho^2}\Bigg|_0^d \\
&\qquad + 2\cdot \mathrm{Re}\Bigg[\int_d^\infty \cos(\rho x)\cdot De^{-i\rho' x}\, dx\Bigg]\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad + 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2}\, dx\Bigg]\Bigg|_d^\infty\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad - 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2}\Bigg]\Bigg|_0^d\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad - 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-i\rho' d}(-\rho\sin(\rho d) + i\rho'\cos(\rho d)) - i\rho'}{\rho'^2 - \rho^2}\Bigg]\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad + 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-i\rho' d}(\rho\sin(\rho d) - i\rho'\cos(\rho d)) + i\rho'}{\rho'^2 - \rho^2}\Bigg]\Bigg\}
\end{aligned}$$

7.13 Source Code

```
od = o1*d; pd = p2*d.transpose()
F = 2*Br1*Bt2 * ((o1*sin(od)*cos(pd) - p2*cos(od)*sin(pd))/(o1
    **2-p2**2) + 2*(D1 * (exp(-1j*p1*d) * (p2*sin(pd)-1j*p1*cos(pd
    ))) + 1j*p1 ).real / (p1**2-p2**2) )
```

7.14 Fields of the Odd Modes

For odd guided modes, the fields are given by

$$\phi_{y,n}(x) = \begin{cases} A_n \sin(\kappa x) & : |x| < d \\ \frac{x}{|x|} A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma |x|} & : |x| \geq d \end{cases}$$

where ϕ stands for either E or H, depending on the polarization. The radiation/continuum modes are

$$\phi_y(x, \rho) = \begin{cases} B_n^r \sin(\sigma x) & : |x| < d \\ \frac{x}{|x|} B_n^r \left[D e^{-i\rho |x|} + D^* e^{-i\rho |x|} \right] & : |x| \geq d \end{cases}$$

Note that A, B, and D depend on the polarization. (See section 6). Finally, the free-space modes in the right half-space are described by:

$$\phi_y(x, \rho) = B^t \sin(\rho x)$$

7.15 $\Xi_n(\rho)$

$$
\begin{aligned}
\Xi_n(\rho) &= \int_{-\infty}^{\infty} E^r_{y,n} E^{t*}_y(\rho)\, dx \\
&= \int_{-d}^{d} A_n \sin(\kappa x) B^t(\rho) \sin(\rho x)\, dx \\
&\qquad\qquad + \int_{-\infty}^{-d} -A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma |x|} B^t(\rho) \sin(\rho x)\, dx \\
&\qquad\qquad + \int_{d}^{\infty} A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma |x|} B^t(\rho) \sin(\rho x)\, dx \\
&= 2\int_{0}^{d} A_n \sin(\kappa x) B^t(\rho) \sin(\rho d)\, dx + 2\int_{d}^{\infty} A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma x} B^t(\rho) \sin(\rho x)\, dx \\
&= 2A_n B^t(\rho) \left\{ \int_{0}^{d} \sin(\kappa x) \sin(\rho d)\, dx + \sin(\kappa d) e^{\gamma d} \int_{d}^{\infty} e^{-\gamma x} \sin(\rho x)\, dx \right\} \\
&= 2A_n B^t(\rho) \left\{ \left. \frac{\rho \sin(\kappa x)\cos(\rho x) - \kappa \cos(\kappa x) \sin(\rho x)}{\kappa^2 - \rho^2} \right|_0^d \right. \\
&\qquad\qquad \left. - \sin(\kappa d) e^{\gamma d} \left. \frac{e^{-\gamma x}(\gamma \sin(\rho x) + \rho \cos(\rho x))}{\gamma^2 + \rho^2} \right|_d^\infty \right\} \\
&= 2A_n B^t(\rho) \left\{ \frac{\rho \sin(\kappa d)\cos(\rho d) - \kappa \cos(\kappa d) \sin(\rho d)}{\kappa^2 - \rho^2} \right. \\
&\qquad\qquad \left. + \sin(\kappa d) e^{\gamma d} \frac{e^{-\gamma d}(\gamma \sin(\rho d) + \rho \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= 2A_n B^t(\rho) \left\{ \frac{\rho \sin(\kappa d)\cos(\rho d) - \kappa \cos(\kappa d) \sin(\rho d)}{\kappa^2 - \rho^2} + \frac{\sin(\kappa d)(\gamma \sin(\rho d) + \rho \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= 2A_n B^t(\rho) \sin(\kappa d) \left\{ \frac{\rho \cos(\rho d) - \kappa \cot(\kappa d) \sin(\rho d)}{\kappa^2 - \rho^2} + \frac{\gamma \sin(\rho d) + \rho \cos(\rho d)}{\gamma^2 + \rho^2} \right\}
\end{aligned}
$$

7.16 Source Code

```
xi[i,:,:] = 2* Am[i] * Bt * sin(Km[i]*d) * ((p*cos(p*d) - Km[i]*
    cot(Km[i]*d)*sin(p*d))/(Km[i]**2-p**2) + (gm[i]*sin(p*d) + p*
    cos(p*d))/(gm[i]**2 + p**2))
```

7.17 $\Phi_n(\rho', \rho)$

$$
\begin{aligned}
\Phi_n(\rho', \rho) &= \int_{-\infty}^{\infty} E_y^r(\rho') E_y^{t*}(\rho)\ dx \\
&= \int_{-d}^{d} B_y^r(\rho') \sin(\sigma x) B^t(\rho) \sin(\rho x)\ dx \\
&\qquad\qquad\qquad + \int_{-\infty}^{-d} -B_y^r(\rho')[De^{-i\rho'|x|} + D^* e^{i\rho'|x|}] B^t(\rho) \sin(\rho x)\ dx \\
&\qquad\qquad\qquad + \int_{d}^{\infty} B_y^r(\rho')[De^{-i\rho'|x|} + D^* e^{i\rho'|x|}] B^t(\rho) \sin(\rho x)\ dx \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \int_0^d \sin(\sigma x) \sin(\rho x)\ dx + \int_d^{\infty} [De^{-i\rho'|x|} + D^* e^{i\rho'|x|}] \sin(\rho x)\ dx \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma x) \cos(\rho x) - \sigma \cos(\sigma x) \sin(\rho x)}{\sigma^2 - \rho^2} \Bigg|_0^d + 2\mathrm{Re}\left[\int_d^{\infty} De^{-i\rho' x} \sin(\rho x)\ dx \right] \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma d) \cos(\rho d) - \sigma \cos(\sigma d) \sin(\rho d)}{\sigma^2 - \rho^2} + 2\mathrm{Re}\left[D \frac{e^{-i\rho' x}(\rho \cos(\rho x) + i\rho' \sin(\rho x))}{\rho'^2 - \rho^2} \Bigg|_d^{\infty} \right] \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma d) \cos(\rho d) - \sigma \cos(\sigma d) \sin(\rho d)}{\sigma^2 - \rho^2} - 2\mathrm{Re}\left[D \frac{e^{-i\rho' x}(\rho \cos(\rho x) + i\rho' \sin(\rho x))}{\rho'^2 - \rho^2} \Bigg|_0^{d} \right] \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma d) \cos(\rho d) - \sigma \cos(\sigma d) \sin(\rho d)}{\sigma^2 - \rho^2} - 2\mathrm{Re}\left[D \frac{e^{-i\rho' d}(\rho \cos(\rho d) + i\rho' \sin(\rho d)) - \rho}{\rho'^2 - \rho^2} \right] \right\}
\end{aligned}
$$

8 Hamid's Derivations

8.1 Derivation of A:

Let's start with the form of ψ in chapter 6.1 of your notes:

$$\psi = \frac{2\kappa_n d + sin\,(2\kappa_n d)}{4\epsilon_r \kappa_n} + \frac{cos^2\,(\kappa_n d)}{2\gamma_n}$$

We know that:

$$tan\,(\kappa_n d) = \frac{\epsilon_r \gamma_n}{\kappa_n}$$

Therefore:

$$cos^2\left(\kappa_n d\right) = \left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1}$$

$$sin\left(2\kappa_n d\right) = \frac{2\epsilon_r\gamma_n}{\kappa_n}\left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1}$$

So that:

$$\begin{aligned}
\psi =& \frac{d}{2\epsilon_r} + \frac{sin\left(2\kappa_n d\right)}{4\epsilon_r\kappa_n} + \frac{cos^2\left(\kappa_n d\right)}{2\gamma_n} \\
=& \frac{d}{2\epsilon_r} + \frac{1}{2\kappa_n}\frac{\gamma_n}{\kappa_n}\left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1} + \frac{1}{2\gamma_n}\left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1} \\
=& \frac{d}{2\epsilon_r} + \left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1}\left(\frac{\gamma_n}{2\kappa_n^2} + \frac{1}{2\gamma_n}\right) \\
=& \frac{d}{2\epsilon_r} + \frac{1}{2\gamma_n}\left(\kappa_n^2 + \epsilon_r^2\gamma_n^2\right)^{-1}\left(\gamma_n^2 + \kappa_n^2\right)
\end{aligned}$$

We have:

$$\begin{aligned}
\kappa_n^2 + \epsilon_r^2\gamma_n^2 &= \left(\epsilon_r^2 - 1\right)\beta_n^2 + \epsilon_r\left(1 - \epsilon_r\right)k_0^2 \\
&= \left(\epsilon_r - 1\right)\left(\left(\epsilon_r + 1\right)\beta_n^2 - \epsilon_r k_0^2\right) \\
&= \left(\epsilon_r - 1\right)\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)
\end{aligned}$$

$$\kappa_n^2 + \gamma_n^2 = \left(\epsilon_r - 1\right)k_0^2$$

Therefore:

$$
\begin{aligned}
A =& \sqrt{\frac{\omega\epsilon_0 P}{\beta_n \psi}} \\
=& \sqrt{\frac{\omega\epsilon_0 P}{\beta_n \left(\frac{d}{2\epsilon_r} + \left(\kappa_n^2 + \epsilon_r^2\gamma_n^2\right)^{-1} \frac{1}{2\gamma_n}\left(\gamma_n^2 + \kappa_n^2\right)\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_0\epsilon_r P}{\beta_n \left(d + \frac{\epsilon_r}{\gamma_n}\left(\kappa_n^2 + \epsilon_r^2\gamma_n^2\right)^{-1}\left(\gamma_n^2 + \kappa_n^2\right)\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_r\epsilon_0 P}{\beta_n \left(d + \frac{\epsilon_r}{\gamma_n}\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)^{-1} k_0^2\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_r\epsilon_0 P\gamma_n}{\beta_n \left(d\gamma_n + \epsilon_r\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)^{-1} k_0^2\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_0\epsilon_r P\gamma_n\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)}{\beta_n\left(\epsilon_r k_0^2 + d\gamma_n\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)\right)}}
\end{aligned}
$$

This shows that the form of A in chapter 6.1 of your notes is consistent with the equation 8.3-40 of the book and also equation A-4 of the appendix of the main article.

8.2 Derivation of $F(\rho, \rho')$:

$$\begin{aligned}
F\left(\rho,\rho'\right) =& 2\int_0^d B^r\left(\rho'\right) cos\left(\sigma' x\right) B^t\left(\rho\right) cos\left(\rho x\right) dx \\
&+2\int_d^\infty B^r\left(\rho'\right)\left[D\left(\rho'\right) e^{-i\rho' x} + D^*\left(\rho'\right) e^{i\rho' x}\right] B^t\left(\rho\right) cos\left(\rho x\right) dx \\
&=2B^r\left(\rho'\right) B^t\left(\rho\right)\int_0^d cos\left(\sigma' x\right) cos\left(\rho x\right) dx \\
&+2B^r\left(\rho'\right) B^t\left(\rho\right)\int_d^\infty \left[D\left(\rho'\right) e^{-i\rho' x} + D^*\left(\rho'\right) e^{i\rho' x}\right] cos\left(\rho x\right) dx \\
&=B^r\left(\rho'\right) B^t\left(\rho\right)\int_0^d \left(cos\left(\left(\sigma'+\rho\right)x\right) + cos\left(\left(\sigma'-\rho\right)x\right)\right) dx \\
&+2B^r\left(\rho'\right) B^t\left(\rho\right)\int_d^\infty \left[D\left(\rho'\right) e^{-i\rho' x} + D^*\left(\rho'\right) e^{i\rho' x}\right] cos\left(\rho x\right) dx \\
&=B^r\left(\rho'\right) B^t\left(\rho\right)\left\{\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho} + \frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\} \\
&+2B^r\left(\rho'\right) B^t\left(\rho\right)\int_d^\infty \left[D\left(\rho'\right) e^{-i\rho' x} + D^*\left(\rho'\right) e^{i\rho' x}\right] cos\left(\rho x\right) dx
\end{aligned}$$

But we know that:

$$D\left(\rho\right) = \frac{1}{2}\left[cos\left(\sigma d\right) - \frac{i\sigma}{\epsilon_r \rho} sin\left(\sigma d\right)\right] e^{i\rho d}$$

So that:

$$\begin{aligned}
&2\int_d^\infty \left[D\left(\rho'\right) e^{-i\rho' x} + D^*\left(\rho'\right) e^{i\rho' x}\right] cos\left(\rho x\right) dx \\
&= \int_d^\infty \left[cos\left(\sigma' d\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{i\rho' d} e^{-i\rho' x} cos\left(\rho x\right) dx \\
&+ \int_d^\infty \left[cos\left(\sigma' d\right) + \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{-i\rho' d} e^{i\rho' x} cos\left(\rho x\right) dx \\
&= \left[cos\left(\sigma' d\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{i\rho' d}\int_d^\infty e^{-i\rho' x} cos\left(\rho x\right) dx \\
&+ \left[cos\left(\sigma' d\right) + \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{-i\rho' d}\int_d^\infty e^{i\rho' x} cos\left(\rho x\right) dx
\end{aligned}$$

But:

$$
\begin{aligned}
&\int_d^\infty e^{-i\rho' x} cos\left(\rho x\right) dx \\
&= \frac{1}{2}\int_d^\infty e^{-i(\rho'+\rho)x} dx + \frac{1}{2}\int_d^\infty e^{-i(\rho'-\rho)x} dx \\
&= \frac{1}{2}\int_0^\infty e^{-i(\rho'+\rho)x} dx + \frac{1}{2}\int_0^\infty e^{-i(\rho'-\rho)x} dx \\
&- \frac{1}{2}\int_0^d e^{-i(\rho'+\rho)x} dx - \frac{1}{2}\int_0^d e^{-i(\rho'-\rho)x} dx \\
&= \frac{\pi}{2}\delta\left(\rho+\rho'\right) + \frac{\pi}{2}\delta\left(\rho-\rho'\right) + \frac{1}{2i\left(\rho+\rho'\right)} + \frac{1}{2i\left(\rho'-\rho\right)} \\
&+ \frac{e^{-i(\rho'+\rho)d}-1}{2i\left(\rho'+\rho\right)} + \frac{e^{-i(\rho'-\rho)d}-1}{2i\left(\rho'-\rho\right)} \\
&= \frac{\pi}{2}\delta\left(\rho+\rho'\right) + \frac{\pi}{2}\delta\left(\rho-\rho'\right) + \frac{e^{-i(\rho'+\rho)d}}{2i\left(\rho'+\rho\right)} + \frac{e^{-i(\rho'-\rho)d}}{2i\left(\rho'-\rho\right)}
\end{aligned}
$$

Similarly:

$$
\int_d^\infty e^{i\rho' x} cos\left(\rho x\right) dx = \frac{\pi}{2}\delta\left(\rho-\rho'\right) + \frac{\pi}{2}\delta\left(\rho+\rho'\right) - \frac{e^{i(\rho'+\rho)d}}{2i\left(\rho'+\rho\right)} - \frac{e^{i(\rho'-\rho)d}}{2i\left(\rho'-\rho\right)}
$$

Therefore:

$$
\begin{aligned}
&2\int_d^\infty \left[D\left(\rho'\right)e^{-i\rho' x} + D^*\left(\rho'\right)e^{i\rho' x}\right] cos\left(\rho x\right)dx \\
&= \left[cos\left(\sigma' d\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{i\rho' d} \int_d^\infty e^{-i\rho' x} cos\left(\rho x\right)dx \\
&+ \left[cos\left(\sigma' d\right) + \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{-i\rho' d} \int_d^\infty e^{i\rho' x} cos\left(\rho x\right)dx \\
&= \frac{1}{2}\left[cos\left(\sigma' d\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{i\rho' d} \left[\pi\delta\left(\rho-\rho'\right) + \frac{e^{-i(\rho'+\rho)d}}{i\left(\rho'+\rho\right)} + \frac{e^{-i(\rho'-\rho)d}}{i\left(\rho'-\rho\right)}\right] \\
&+ \frac{1}{2}\left[cos\left(\sigma' d\right) + \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] e^{-i\rho' d} \left[\pi\delta\left(\rho-\rho'\right) - \frac{e^{i(\rho'+\rho)d}}{i\left(\rho'+\rho\right)} - \frac{e^{i(\rho'-\rho)d}}{i\left(\rho'-\rho\right)}\right] \\
&= \frac{1}{2}\left[cos\left(\sigma' d\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] \left[\pi e^{i\rho' d}\delta\left(\rho-\rho'\right) + \frac{e^{-i\rho d}}{i\left(\rho'+\rho\right)} + \frac{e^{i\rho d}}{i\left(\rho'-\rho\right)}\right] \\
&+ \frac{1}{2}\left[cos\left(\sigma' d\right) + \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right] \left[\pi e^{-i\rho' d}\delta\left(\rho-\rho'\right) - \frac{e^{i\rho d}}{i\left(\rho'+\rho\right)} - \frac{e^{-i\rho d}}{i\left(\rho'-\rho\right)}\right] \\
&= 2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right) \\
&+ \frac{1}{2i\left(\rho'-\rho\right)}\left(cos\left(\sigma' d\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right) e^{i\rho d} \\
&- \frac{1}{2i\left(\rho'-\rho\right)}\left(cos\left(\sigma' d\right) + \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right) e^{-i\rho d} \\
&+ \frac{1}{2i\left(\rho'+\rho\right)}\left(cos\left(\sigma' d\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right) e^{-i\rho d} \\
&- \frac{1}{2i\left(\rho'+\rho\right)}\left(cos\left(\sigma' d\right) + \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\right) e^{i\rho d} \\
&= 2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right) \\
&+ \frac{1}{2i\left(\rho'-\rho\right)}\left[cos\left(\sigma' d\right)\left(e^{i\rho d} - e^{-i\rho d}\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\left(e^{i\rho d} + e^{-i\rho d}\right)\right] \\
&+ \frac{1}{2i\left(\rho'+\rho\right)}\left[cos\left(\sigma' d\right)\left(e^{-i\rho d} - e^{i\rho d}\right) - \frac{i\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right)\left(e^{-i\rho d} + e^{i\rho d}\right)\right] \\
&= 2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right) \\
&+ \frac{1}{\left(\rho'-\rho\right)}\left[cos\left(\sigma' d\right) sin\left(\rho d\right) - \frac{\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right) cos\left(\rho d\right)\right] \\
&+ \frac{1}{\left(\rho'+\rho\right)}\left[-cos\left(\sigma' d\right) sin\left(\rho d\right) - \frac{\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right) cos\left(\rho d\right)\right] \\
&= 2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right) \\
&+ \left[\frac{1}{\left(\rho'-\rho\right)} - \frac{1}{\left(\rho'+\rho\right)}\right] cos\left(\sigma' d\right) sin\left(\rho d\right) \\
&- \left[\frac{1}{\left(\rho'-\rho\right)} + \frac{1}{\left(\rho'+\rho\right)}\right] \frac{\sigma'}{\epsilon_r \rho'} sin\left(\sigma' d\right) cos\left(\rho d\right) \\
&= 2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right) \\
&+ \frac{2\rho}{\left(\rho'^2-\rho^2\right)} cos\left(\sigma' d\right) sin\left(\rho d\right) \\
&- \frac{2\sigma'}{\left(\rho'^2-\rho^2\right)\epsilon_r} sin\left(\sigma' d\right) cos\left(\rho d\right)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
F\left(\rho,\rho'\right) =& B^r\left(\rho'\right)B^t\left(\rho\right)\left\{2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\left\{\frac{2\rho}{\left(\rho'^2-\rho^2\right)}cos\left(\sigma' d\right)sin\left(\rho d\right)-\frac{2\sigma'}{\left(\rho'^2-\rho^2\right)\epsilon_r}sin\left(\sigma' d\right)cos\left(\rho d\right)\right\}\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)\left\{2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{\rho}{\left(\rho'^2-\rho^2\right)}\left(sin\left(\left(\rho+\sigma'\right)d\right)+sin\left(\left(\rho-\sigma'\right)d\right)\right)\\
&-B^r\left(\rho'\right)B^t\left(\rho\right)\frac{\sigma'}{\left(\rho'^2-\rho^2\right)\epsilon_r}\left(sin\left(\left(\rho+\sigma'\right)d\right)-sin\left(\left(\rho-\sigma'\right)d\right)\right)\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)\left\{2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{1}{\left(\rho'^2-\rho^2\right)}\left(\rho-\frac{\sigma'}{\epsilon_r}\right)sin\left(\left(\rho+\sigma'\right)d\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{1}{\left(\rho'^2-\rho^2\right)}\left(\rho+\frac{\sigma'}{\epsilon_r}\right)sin\left(\left(\rho-\sigma'\right)d\right)\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)sin\left(\left(\sigma'+\rho\right)d\right)\left(\frac{1}{\sigma'+\rho}+\frac{\rho-\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)sin\left(\left(\sigma'-\rho\right)d\right)\left(\frac{1}{\sigma'-\rho}-\frac{\rho+\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}\right)
\end{aligned}
$$

But We know:

$$
\frac{1}{\sigma'+\rho}+\frac{\rho-\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}=\frac{1}{\left(\sigma'+\rho\right)\left(\rho'^2-\rho^2\right)}\left(\rho'^2-\rho^2+\left(\rho+\sigma'\right)\left(\rho-\frac{\sigma'}{\epsilon_r}\right)\right)
$$

$$
\frac{1}{\sigma'-\rho}-\frac{\rho+\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}=\frac{1}{\left(\sigma'-\rho\right)\left(\rho'^2-\rho^2\right)}\left(\rho'^2-\rho^2-\left(\sigma'-\rho\right)\left(\rho+\frac{\sigma'}{\epsilon_r}\right)\right)
$$

$$
\begin{aligned}
\rho'^2 - \rho^2 + \left(\rho + \sigma'\right)\left(\rho - \frac{\sigma'}{\epsilon_r}\right) &= \rho'^2 - \rho^2 + \left(\rho + \sigma'\right)\rho - \left(\rho + \sigma'\right)\frac{\sigma'}{\epsilon_r} \\
&= \rho'^2 - \rho^2 + \left(\rho^2 + \sigma'\rho\right) - \left(\frac{\rho\sigma'}{\epsilon_r} + \frac{\sigma'^2}{\epsilon_r}\right) \\
&= \rho'^2 + \sigma'\rho\left(1 - \frac{1}{\epsilon_r}\right) - \frac{\sigma'^2}{\epsilon_r} \\
&= k_0^2\left(1 - \epsilon_r\right) + \sigma'\left(\sigma' + \rho\right)\left(1 - \frac{1}{\epsilon_r}\right)
\end{aligned}
$$

$$
\begin{aligned}
\rho'^2 - \rho^2 + \left(\rho - \sigma'\right)\left(\rho + \frac{\sigma'}{\epsilon_r}\right) &= \rho'^2 - \rho^2 + \left(\rho - \sigma'\right)\rho + \left(\rho - \sigma'\right)\frac{\sigma'}{\epsilon_r} \\
&= \rho'^2 - \rho^2 + \left(\rho^2 - \sigma'\rho\right) + \left(\frac{\rho\sigma'}{\epsilon_r} - \frac{\sigma'^2}{\epsilon_r}\right) \\
&= \rho'^2 - \sigma'\rho\left(1 - \frac{1}{\epsilon_r}\right) - \frac{\sigma'^2}{\epsilon_r} \\
&= k_0^2\left(1 - \epsilon_r\right) + \sigma'\left(\sigma' - \rho\right)\left(1 - \frac{1}{\epsilon_r}\right)
\end{aligned}
$$

Since:

$$
\rho'^2 = k_0^2\left(1 - \epsilon_r\right) + \sigma'^2
$$

Therefore:

$$
\begin{aligned}
F\left(\rho,\rho'\right) =& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\sigma'+\rho\right)\left(\rho'^2-\rho^2\right)}\left(k_0^2\left(1-\epsilon_r\right)+\sigma'\left(\sigma'+\rho\right)\left(1-\frac{1}{\epsilon_r}\right)\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\sigma'-\rho\right)\left(\rho'^2-\rho^2\right)}\left(k_0^2\left(1-\epsilon_r\right)+\sigma'\left(\sigma'-\rho\right)\left(1-\frac{1}{\epsilon_r}\right)\right)\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\left\{\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\sigma'+\rho\right)}\frac{k_0^2\left(1-\epsilon_r\right)}{\left(\rho'^2-\rho^2\right)}+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\rho'^2-\rho^2\right)}\sigma'\left(1-\frac{1}{\epsilon_r}\right)\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\left\{\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\sigma'-\rho\right)}\frac{k_0^2\left(1-\epsilon_r\right)}{\left(\rho'^2-\rho^2\right)}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\rho'^2-\rho^2\right)}\sigma'\left(1-\frac{1}{\epsilon_r}\right)\right\}\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&-B^r\left(\rho'\right)B^t\left(\rho\right)\left(\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\sigma'+\rho\right)}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\sigma'-\rho\right)}\right)\frac{k_0^2\left(\epsilon_r-1\right)}{\left(\rho'^2-\rho^2\right)}\\
&+2B^r\left(\rho'\right)B^t\left(\rho\right)\sigma'\left(\epsilon_r-1\right)\frac{sin\left(\sigma'd\right)cos\left(\rho d\right)}{\left(\rho'^2-\rho^2\right)\epsilon_r}
\end{aligned}
$$

8.3 Integration Identity to remove singularities

Let's consider the following integral to evaluate:

$$
\begin{aligned}
\int_0^\infty \frac{H(\rho,\rho')}{\rho'^2-\rho^2}d\rho' &= \int_0^\infty \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'\\
&+H(\rho,\rho)\int_0^\infty \frac{1}{\rho'^2-\rho^2}d\rho'\\
&= \int_0^\infty \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'\\
&+H(\rho,\rho)\int_0^\infty \frac{1}{\rho'^2-\rho^2}d\rho'\\
&= \int_0^\infty \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'\\
&+H(\rho,\rho)\int_{-\rho}^\infty \frac{dx}{x\left(x+2\rho\right)}
\end{aligned}
$$

Since:

$$
\begin{aligned}
\int_{-\rho}^{\infty} \frac{dx}{x\,(x+2\rho)} &= \frac{1}{2\rho}\int_{-\rho}^{\infty}\left[\frac{1}{x}-\frac{1}{x+2\rho}\right]dx \\
&= \frac{1}{2\rho}\int_{-\rho}^{\rho}\left[\frac{1}{x}-\frac{1}{x+2\rho}\right]dx + \frac{1}{2\rho}\int_{\rho}^{\infty}\left[\frac{1}{x}-\frac{1}{x+2\rho}\right]dx \\
&= \frac{-1}{2\rho}\left[ln\,(x+2\rho)\right]_{-\rho}^{\rho} + \frac{1}{2\rho}\left[ln\left(\frac{x}{x+2\rho}\right)\right]_{\rho}^{\infty} \\
&= -\frac{1}{2\rho}ln\,(3) - \frac{1}{2\rho}ln\left(\frac{1}{3}\right) = 0
\end{aligned}
$$

Therefore:

$$
\int_{0}^{\infty} \frac{H(\rho,\rho')}{\rho'^2-\rho^2}d\rho' = \int_{0}^{\infty} \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'
$$

In this way the integrand becomes finite at the singularity. So that, doing the integration numerically, becomes feasible.