Derivation for Reflection Coefficients of a Truncated Slab Waveguide

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1 Problem Geometry

Consider a dielectric slab waveguide such that waves propagate along the z-axis. The slab extends infinitely along y, and has interfaces at $|x| = d$. At the plane $z = 0$, the slab terminates abruptly.

The slab has refractive index n, and relative permittivity $\epsilon_r = n^2$.

In this model, we use the convention $e^{i(\omega t - \beta z)}$
We will make frequent use of the following wave vectors:

$$k = \omega/c$$

(continuum modes)

$$\rho = \sqrt{k^2 - \beta_c^2}$$
$$\sigma = \sqrt{n^2k^2 - \beta_c^2}$$

(guided modes)

$$\gamma = \sqrt{\beta_n^2 - k^2}$$
$$\kappa = \sqrt{n^2k^2 - \beta_m^2}$$

2 Relationship between Primary and Complimentary fields

For each polarization, the y-directed field component is referred to as dominant. Since we will be exploiting orthogonality using the z-component of power flow, we care about the corresponding x-oriented field component of the complimentary field.

2.1 TM

We employ Maxwell's Equations to reduce the problem to the H_y field components:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t} = i\omega\epsilon\mathbf{E} \tag{1}$$

Also,

$$\nabla \times \mathbf{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & H_y & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial H_y}{\partial z}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x}\right) \tag{2}$$

$$= \hat{x}\left(i\beta H_y\right) + \hat{z}\left(-i\sqrt{k^2-\beta^2}H_y\right) \tag{3}$$

where it is important to note that k changes inside and outside the slab. For the transverse Electric field, then

$$E_x \cdot (i\omega\epsilon) = i\beta H_y \tag{4}$$

$$E_x = \frac{\beta}{\omega\epsilon} H_y \tag{5}$$

2.2 TE

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} = -i\omega\mu\mathbf{H} \tag{6}$$

Also,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial E_y}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x}\right) \tag{7}$$

$$= \hat{x}\left(i\beta E_y\right) + \hat{z}\left(-i\sqrt{k^2-\beta^2}E_y\right) \tag{8}$$

where it is important to note that k changes inside and outside the slab. For the transverse Electric field, then

$$H_x \cdot (-i\omega\mu) = i\beta E_y \tag{9}$$

$$H_x = -\frac{\beta}{\omega\mu} E_y \tag{10}$$

3 Orthogonality

The orthogonality between modes is expressed using the power carried by the fields

$$P_z\delta_{m,n} = \frac{1}{2}\int_{-\infty}^{\infty} \mathbf{E}_m \times \mathbf{H}_n^* \cdot \hat{z}\, dx \tag{11}$$

where m, n refer to different modes. For TM modes ($H_x, H_z = 0$) this reduces to

$$P_z\delta_{m,n} = \frac{1}{2}\int_{-\infty}^{\infty} E_{x,m}H_{y,n}^* dx = \int_{0}^{\infty} \frac{\beta_m}{\omega\epsilon} H_{y,m}H_{y,n} dx \tag{12}$$

For TE modes, orthogonality is expressed as

$$P_z\delta_{m,n} = \frac{1}{2}\int_{-\infty}^{\infty} E_{y,m}H_{x,n}^* dx = \int_{0}^{\infty} \frac{\beta_m}{\omega\mu} E_{y,m}E_{y,n} dx \tag{13}$$

(Note: for the TE case, the value of beta is negated from the complex conjugation)

For continuum modes, Gelin states that P_z should be replaced by $P_z\frac{\beta}{|\beta|}$ to account for evanescence, or complex power flow. (should there be a minus sign in the TE result?)

4 TM Even Modes

4.1 Primary Equations

Begin with a statement of the continuity of transverse fields across the end facet boundary:

$$H^i_{y,m} + \sum_n a_n H^r_{y,n} + \int_0^\infty q^r(\rho) H^r_y(\rho) d\rho = \int_0^\infty q^t(\rho) H^t_y(\rho) d\rho \tag{14}$$

$$E^i_{x,m} + \sum_n a_n E^r_{x,n} + \int_0^\infty q^r(\rho) E^r_x(\rho) d\rho = \int_0^\infty q^t(\rho) E^t_x(\rho) d\rho \tag{15}$$

Substituting (5) into (15) and canceling the ωs and ϵ_os, we now have the primary equations:

$$H^i_{y,m} + \sum_n a_n H^r_{y,n} + \int_0^\infty q^r(\rho) H^r_y(\rho) d\rho = \int_0^\infty q^t(\rho) H^t_y(\rho) d\rho \tag{16}$$

$$\frac{\beta_m}{\epsilon_r} H^i_{y,m} - \sum_n a_n \frac{\beta_n}{\epsilon_r} H^r_{y,n} - \int_0^\infty q^r(\rho) \frac{\beta(\rho)}{\epsilon_r} H^r_y(\rho) d\rho = \int_0^\infty q^t(\rho) \beta(\rho) H^t_y(\rho) d\rho \tag{17}$$

4.2 Solving for q^t

We're going to isolate $q^t(\rho)$. Dummy variables of integration will be indicated as ρ' for distinction. Begin by multiplying each term in (16) by $H^{t*}_y(\rho)$ and integrating over all space:

$$\begin{aligned}
&\int_0^\infty H^i_{y,m} H^{t*}_y(\rho)\, dx + \sum_n a_n \int_0^\infty H^r_{y,n} H^{t*}_y(\rho)\, dx + \int_0^\infty q^r(\rho') \int_0^\infty H^r_y(\rho') H^{t*}_y(\rho)\, dx\, d\rho' \\
&= \int_0^\infty q^t(\rho') \int_0^\infty H^t_y(\rho') H^{t*}_y(\rho)\, dx\, d\rho'
\end{aligned} \tag{18}$$

From (12), we can eliminate all but our target mode with wave vector ρ:

$$\begin{aligned}
\kappa_m(\rho) + \sum_n a_n \kappa_n(\rho) + \int_0^\infty q^r(\rho') \zeta(\rho', \rho)\, d\rho' &= \int_0^\infty q^t(\rho') P \frac{\omega \epsilon_o}{|\beta(\rho)|} \delta(\rho' - \rho)\, d\rho' \\
&= q^t(\rho) \cdot P \frac{\omega \epsilon_o}{|\beta(\rho)|}
\end{aligned} \tag{19}$$

where we have introduced the functions

$$\kappa_n(\rho) = \int_0^\infty H^i_{y,n} H^{t*}_y(\rho)\, dx \tag{20}$$

$$\zeta(\rho', \rho) = \int_0^\infty H^r_y(\rho') H^{t*}_y(\rho)\, dx \tag{21}$$

We can repeat this procedure for (17), producing

$$\beta_m \nu_m(\rho) - \sum_n a_n \beta_n \nu_n(\rho) - \int_0^\infty q^r(\rho')\beta(\rho') f(\rho',\rho)\, d\rho' = q^t(\rho)\beta(\rho) \cdot P \frac{\omega \epsilon_o}{|\beta(\rho)|} \tag{22}$$

here introducing

$$\nu_n(\rho) = \int_0^\infty \frac{H^i_{y,n} H^{t*}_y(\rho)}{\epsilon_r}\, dx \tag{23}$$

$$f(\rho',\rho) = \int_0^\infty \frac{H^r_y(\rho') H^{t*}_y(\rho)}{\epsilon_r}\, dx \tag{24}$$

Combining the results as $\beta_m \nu_m(\rho)$ (19) + $\kappa_m(\rho)$ (22) gives a single equation which may be solved for q^t:

$$\begin{aligned}
2\beta_m \nu_m(\rho)\kappa_m(\rho) &+ \sum_n a_n(\beta_m \nu_m(\rho)\kappa_n(\rho) - \beta_n \nu_n(\rho)\kappa_m(\rho)) \\
&+ \int_0^\infty q^r(\rho')(\beta_m \nu_m(\rho)\zeta(\rho',\rho) - \beta(\rho')\kappa_m(\rho) f(\rho',\rho))\, d\rho' \\
&= q^t(\rho)(\beta_m \nu_m(\rho) P \frac{\omega\epsilon_o}{|\beta(\rho)|} + \kappa_m(\rho)\beta(\rho) P \frac{\omega\epsilon_o}{|\beta(\rho)|}) \\
&= q^t(\rho)(\beta_m \nu_m(\rho) + \kappa_m(\rho)\beta(\rho)) P \frac{\omega\epsilon_o}{|\beta(\rho)|}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow q^t(\rho) = \frac{1}{\omega\epsilon_o P} \frac{|\beta(\rho)|}{\beta_m \nu_m(\rho) + \kappa_m(\rho)\beta(\rho)} \Bigg\{ & 2\beta_m \nu_m(\rho)\kappa_m(\rho) \\
& + \sum_n a_n(\beta_m \nu_m(\rho)\kappa_n(\rho) - \beta_n \nu_n(\rho)\kappa_m(\rho)) \\
+ \int_0^\infty q^r(\rho')(\beta_m \nu_m(\rho)\zeta(\rho',\rho) - \beta(\rho')\kappa_m(\rho) f(\rho',\rho))\, d\rho' \Bigg\} &
\end{aligned} \tag{25}$$

This formula matches that in the Appendix of Gelin 1981, except that I have substituted f and ζ in place of the explicit overlap integrals.

4.3 Solving for a_n

As above, we will combine the primary equations using orthogonality to extract the coefficients. This time we will multiply each term in (16) by $H^{r*}_{y,n}/\epsilon_r$ (note that the summation index will be changed to k - it will disappear almost immediately so don't worry about it!):

$$\int_0^\infty \frac{H^i_{y,m} H^{r*}_{y,n}}{\epsilon_r}\,dx + \sum_k a_k \int_0^\infty \frac{H^r_{y,k} H^{r*}_{y,n}}{\epsilon_r}\,dx + \int_0^\infty q^r(\rho') \int_0^\infty \frac{H^r_y(\rho') H^{r*}_{y,n}}{\epsilon_r}\,dx\,d\rho'$$
$$= \int_0^\infty q^t(\rho') \int_0^\infty \frac{H^t_y(\rho') H^{r*}_{y,n}}{\epsilon_r}\,dx\,d\rho' \tag{26}$$

Orthogonality eliminates virtually all the terms on the left side.

$$P\frac{\omega\epsilon_o}{\beta_m}\delta_{m,n} + \sum_k a_k P\frac{\omega\epsilon_o}{\beta_n}\delta_{k,n} = \int_0^\infty q^t(\rho')\nu_n(\rho')\,d\rho'$$
$$a_n P\frac{\omega\epsilon_o}{\beta_n} = -P\frac{\omega\epsilon_o}{\beta_n}\delta_{m,n} + \int_0^\infty q^t(\rho')\nu_n(\rho')\,d\rho'$$
$$a_n = \frac{\beta_n}{\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')\nu_n(\rho')\,d\rho' - P\frac{\omega\epsilon_o}{\beta_n}\delta_{m,n}\right\}$$
$$a_n = \frac{1}{\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')\beta_n\nu_n(\rho')\,d\rho' - P\omega\epsilon_o\delta_{m,n}\right\} \tag{27}$$

Repeating the process for (17), but excluding the factor of $1/\epsilon_r$:

$$\beta_m \int_0^\infty \frac{H^i_{y,m} H^{r*}_{y,n}}{\epsilon_r}\,dx - \sum_k a_k\beta_k \int_0^\infty \frac{H^r_{y,k} H^{r*}_{y,n}}{\epsilon_r}\,dx - \int_0^\infty q^r(\rho')\beta(\rho') \int_0^\infty \frac{H^r_y(\rho') H^{r*}_{y,n}}{\epsilon_r}\,dx\,d\rho'$$
$$= \int_0^\infty q^t(\rho')\beta(\rho') \int_0^\infty H^t_y(\rho') H^{r*}_{y,n}\,dx\,d\rho' \tag{28}$$

$$\beta_m P\frac{\omega\epsilon_o}{\beta_m}\delta_{m,n} - \sum_k a_k\beta_k P\frac{\omega\epsilon_o}{\beta_n}\delta_{k,n} = \int_0^\infty q^t(\rho')\beta(\rho')\kappa_n(\rho')\,d\rho'$$
$$-\,a_n P\omega\epsilon_o = -P\omega\epsilon_o\delta_{m,n} + \int_0^\infty q^t(\rho')\beta(\rho')\kappa_n(\rho')\,d\rho'$$
$$a_n = \frac{1}{\omega\epsilon_o P}\left\{-\int_0^\infty q^t(\rho')\beta(\rho')\kappa_n(\rho')\,d\rho' + P\omega\epsilon_o\delta_{m,n}\right\} \tag{29}$$

Combining (27) + (29) yields:

$$2a_n = \frac{1}{\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')[\beta_n\nu_n(\rho') - \beta(\rho')\kappa_n(\rho')] + P\omega\epsilon_o\delta_{m,n} - P\omega\epsilon_o\delta_{m,n}\;d\rho'\right\}$$
$$a_n = \frac{1}{2\omega\epsilon_o P}\left\{\int_0^\infty q^t(\rho')[\beta_n\nu_n(\rho') - \beta(\rho')\kappa_n(\rho')]\,d\rho'\right\} \tag{30}$$

4.4 Solving for q^r

Once again, we will combine the primary equations using orthogonality to extract the coefficients. This time we will multiply each term in (16) by $H_y^{r*}(\rho)/\epsilon_r$:

$$\int_0^\infty \frac{H^i_{y,m} H_y^{r*}(\rho)}{\epsilon_r}\,dx + \sum_k a_k \int_0^\infty \frac{H^r_{y,k} H_y^{r*}(\rho)}{\epsilon_r}\,dx + \int_0^\infty q^r(\rho') \int_0^\infty \frac{H^r_y(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho'$$
$$= \int_0^\infty q^t(\rho') \int_0^\infty \frac{H^t_y(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho' \tag{31}$$

Orthogonality eliminates virtually all the terms on the left side.

$$\begin{aligned}
\int_0^\infty q^r(\rho') \int_0^\infty \frac{H^r_y(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho' &= \int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
\int_0^\infty q^r(\rho') \frac{P\omega\epsilon_o}{|\beta(\rho)|}\delta(\rho'-\rho)\,d\rho' &= \int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
q^r(\rho) \frac{P\omega\epsilon_o}{|\beta(\rho)|} &= \int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
q^r(\rho) &= \frac{|\beta(\rho)|}{\omega\epsilon_o P} \int_0^\infty q^t(\rho') f(\rho,\rho')\,d\rho' \\
q^r(\rho) &= \frac{1}{\omega\epsilon_o P}\frac{|\beta(\rho)|}{\beta(\rho)} \int_0^\infty q^t(\rho')\beta(\rho) f(\rho,\rho')\,d\rho'
\end{aligned} \tag{32}$$

Repeating the process for (17), but excluding the factor of $1/\epsilon_r$:

$$\beta_m \int_0^\infty \frac{H^i_{y,m} H_y^{r*}(\rho)}{\epsilon_r}\,dx - \sum_k a_k \beta_k \int_0^\infty \frac{H^r_{y,k} H_y^{r*}(\rho)}{\epsilon_r}\,dx - \int_0^\infty q^r(\rho')\beta(\rho') \int_0^\infty \frac{H^r_y(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho'$$
$$= \int_0^\infty q^t(\rho')\beta(\rho') \int_0^\infty H^t_y(\rho') H_y^{r*}(\rho)\,dx\,d\rho' \tag{33}$$

$$\begin{aligned}
-\int_0^\infty q^r(\rho')\beta(\rho') \int_0^\infty \frac{H^r_y(\rho') H_y^{r*}(\rho)}{\epsilon_r}\,dx\,d\rho' &= \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\,d\rho' \\
-\int_0^\infty q^r(\rho')\beta(\rho') \frac{P\omega\epsilon_o}{|\beta(\rho)|}\delta(\rho'-\rho)\,d\rho' &= \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\,d\rho' \\
-q^r(\rho)\beta(\rho') \frac{P\omega\epsilon_o}{|\beta(\rho)|} &= \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\,d\rho' \\
q^r(\rho) &= -\frac{1}{\omega\epsilon_o P}\frac{|\beta(\rho)|}{\beta(\rho)} \int_0^\infty q^t(\rho')\beta(\rho')\zeta(\rho,\rho')\,d\rho'
\end{aligned} \tag{34}$$

Combining (32) + (34) yields:

$$
\begin{aligned}
2q^r(\rho) &= \frac{1}{\omega\epsilon_o P}\frac{|\beta(\rho)|}{\beta(\rho)}\int_0^\infty q^t(\rho')[\beta(\rho)f(\rho,\rho') - \beta(\rho')\zeta(\rho,\rho')]\,d\rho' \\
q^r(\rho) &= \frac{1}{2\omega\epsilon_o P}\frac{|\beta(\rho)|}{\beta(\rho)}\int_0^\infty q^t(\rho')[\beta(\rho)f(\rho,\rho') - \beta(\rho')\zeta(\rho,\rho')]\,d\rho' \qquad (35)
\end{aligned}
$$

5 TE Even Modes

5.1 Primary Equations

Begin with a statement of the continuity of transverse fields across the end facet boundary:

$$E^i_{y,m} + \sum_n a_n E^r_{y,n} + \int_0^\infty q^r(\rho) E^r_y(\rho)\, d\rho = \int_0^\infty q^t(\rho) E^t_y(\rho)\, d\rho \tag{36}$$

$$H^i_{x,m} + \sum_n a_n H^r_{x,n} + \int_0^\infty q^r(\rho) H^r_x(\rho)\, d\rho = \int_0^\infty q^t(\rho) H^t_x(\rho)\, dp \tag{37}$$

Substituting (10) into (37) and canceling the ωs and μs, we now have the primary equations:

$$E^i_{y,m} + \sum_n a_n E^r_{y,n} + \int_0^\infty q^r(\rho) E^r_y(\rho)\, d\rho = \int_0^\infty q^t(\rho) E^t_y(\rho)\, d\rho \tag{38}$$

$$-\beta_m E^i_{y,m} + \sum_n a_n \beta_n E^r_{y,n} + \int_0^\infty q^r(\rho)\beta(\rho) E^r_y(\rho)\, d\rho = -\int_0^\infty q^t(\rho)\beta(\rho) E^t_y(\rho)\, d\rho \tag{39}$$

$$\Rightarrow \beta_m E^i_{y,m} - \sum_n a_n \beta_n E^r_{y,n} - \int_0^\infty q^r(\rho)\beta(\rho) E^r_y(\rho)\, d\rho = \int_0^\infty q^t(\rho)\beta(\rho) E^t_y(\rho)\, d\rho \tag{40}$$

5.2 Solving for q^t

We're going to isolate $q^t(\rho)$. Dummy variables of integration will be indicated as ρ' for distinction. Begin by multiplying each term in (38) by $E^{t*}_y(\rho)$ and integrating over all space:

$$\begin{aligned}
&\int_0^\infty E^i_{y,m} E^{t*}_y(\rho)\, dx + \sum_n a_n \int_0^\infty E^r_{y,n} E^{t*}_y(\rho)\, dx + \int_0^\infty q^r(\rho') \int_0^\infty E^r_y(\rho') E^{t*}_y(\rho)\, dx\, d\rho' \\
&= \int_0^\infty q^t(\rho') \int_0^\infty E^t_y(\rho') E^{t*}_y(\rho)\, dx\, d\rho'
\end{aligned} \tag{41}$$

From (13), we can eliminate all but our target mode with wave vector ρ:

$$\begin{aligned}
\frac{1}{2} G_m(\rho) + \frac{1}{2}\sum_n a_n G_n(\rho) + \frac{1}{2}\int_0^\infty q^r(\rho') F(\rho',\rho)\, d\rho' &= \int_0^\infty q^t(\rho') \left(-P \frac{\omega\mu}{|\beta(\rho)|} \delta(\rho' - \rho) \right) d\rho' \\
&= q^t(\rho) \cdot P \frac{\omega\mu}{|\beta(\rho)|}
\end{aligned} \tag{42}$$

where we have introduced the functions

$$G_n(\rho) = \int_{-\infty}^{\infty} E^i_{y,n} E^{t*}_y(\rho)\, dx \tag{43}$$

$$F(\rho', \rho) = \int_{-\infty}^{\infty} E^r_y(\rho') E^{t*}_y(\rho)\, dx \tag{44}$$

We can repeat this procedure for (40), producing

$$\frac{1}{2}\beta_m G_m(\rho) - \frac{1}{2}\sum_n a_n \beta_n G_n(\rho) - \frac{1}{2}\int_0^{\infty} q^r(\rho')\beta(\rho')F(\rho', \rho)\, d\rho' = -q^t(\rho)\beta(\rho) \cdot P\frac{\omega\mu}{|\beta(\rho)|} \tag{45}$$

Combining the results as $\beta_m(\rho)$ (42) + (45) gives a single equation which may be solved for q^t:

$$\begin{aligned} G_m\beta_m + \frac{1}{2}\sum_n a_n(\beta_m - \beta_n)G_n + \frac{1}{2}\int_0^{\infty} q^r(\rho')(\beta_m - \beta(\rho'))F(\rho', \rho)\, d\rho' \\ = q^t(\rho)(\beta_m + \beta(\rho)) \cdot P\frac{\omega\mu}{|\beta(\rho)|} \end{aligned} \tag{46}$$

$$\begin{aligned} \Rightarrow q^t(\rho) = \frac{1}{2\omega\mu P}\frac{|\beta(\rho)|}{\beta_m + \beta(\rho)}\Big\{ 2\beta_m G_m(\rho) + \sum_n a_n(\beta_m - \beta_n)G_n(\rho) \\ + \int_0^{\infty} q^r(\rho')(\beta_m - \beta(\rho'))F(\rho', \rho)\, d\rho' \Big\} \end{aligned} \tag{47}$$

This formula matches (5) in Gelin 1981.

5.3 a_n

Begin with the primary equations. Multiply each term by $E^{r*}_{y,n}$ and integrate x from 0 to infinity. Equation (38) becomes:

$$0 + \sum_k a_k \int_0^{\infty} E^r_{y,k} E^{r*}_{y,n}\, dx + 0 = \int_0^{\infty} q^t(\rho') \int_0^{\infty} E^t_y(\rho') E^{r*}_{y,n}\, dx\, d\rho'$$

$$a_n \cdot \left(P\frac{\omega\mu}{\beta_n}\right) = \frac{1}{2}\int_0^{\infty} q^t(\rho') G_n(\rho')\, d\rho'$$

Similarly, (40) becomes:

$$0 - \sum_k a_k \beta_k \int_0^\infty E^r_{y,k} E^{r*}_{y,n} \, dx + 0 = \int_0^\infty q^t(\rho')\beta(\rho') \int_0^\infty E^t_y(\rho') E^{r*}_{y,n} \, dx \, d\rho'$$
$$a_n \beta_n \cdot \left(-P \frac{\omega\mu}{\beta_n} \right) = \frac{1}{2} \int_0^\infty q^t(\rho')\beta(\rho') G_n(\rho') \, d\rho'$$
$$a_n \cdot (P\omega\mu) = -\frac{1}{2} \int_0^\infty q^t(\rho')\beta(\rho') G_n(\rho') \, d\rho'$$

Adding β_n times the former result to the latter yields, thus:

$$2a_n \cdot (P\omega\mu) = \frac{1}{2} \int_0^\infty q^t(\rho')(\beta_n - \beta(\rho')) G_n(\rho') \, d\rho'$$
$$\Rightarrow a_n = \frac{1}{4\omega\mu P} \int_0^\infty q^t(\rho')(\beta_n - \beta(\rho')) G_n(\rho') \, d\rho' \tag{48}$$
$$\tag{49}$$

5.4 q^r

Begin with the primary equations. Multiply each term by $E^{r*}_y(\rho)$ and integrate x from 0 to infinity. Equation (38) becomes:

$$0 + 0 + \int_0^\infty q^r(\rho) \int_0^\infty E^r_y(\rho') E^{r*}_y(\rho) \, dx \, d\rho' = \int_0^\infty q^t(\rho') \int_0^\infty E^t_y(\rho') E^{r*}_y(\rho) \, dx \, d\rho'$$
$$q^r(\rho) \cdot \left(P \frac{\omega\mu}{|\beta(\rho)|} \right) = \frac{1}{2} \int_0^\infty q^t(\rho') F^*(\rho, \rho') \, d\rho'$$

Similarly, (40) becomes:

$$0 + 0 - \int_0^\infty q^r(\rho)\beta(\rho') \int_0^\infty E^r_y(\rho') E^{r*}_y(\rho) \, dx \, d\rho' = \int_0^\infty q^t(\rho')\beta(\rho') \int_0^\infty E^t_y(\rho') E^{r*}_y(\rho) \, dx \, d\rho'$$
$$\beta(\rho) q^r(\rho) \cdot \left(P \frac{\omega\mu}{|\beta(\rho)|} \right) = -\frac{1}{2} \int_0^\infty q^t(\rho')\beta(\rho') F^*(\rho, \rho') \, d\rho'$$

Adding $\beta(\rho)$ times the former result to the latter yields, thus:

$$2\beta(\rho) q^r \cdot \left(\frac{P\omega\mu}{|\beta(\rho)|} \right) = \frac{1}{2} \int_0^\infty q^t(\rho')(\beta(\rho) - \beta(\rho')) F^*(\rho, \rho') \, d\rho'$$
$$\Rightarrow q^r = \frac{1}{4\omega\mu P} \frac{|\beta(\rho)|}{\beta(\rho)} \int_0^\infty q^t(\rho')(\beta(\rho) - \beta(\rho')) F^*(\rho, \rho') \, d\rho' \tag{50}$$

6 Derivation of the Mode Amplitude Coefficients

6.1 TM Even - Guided Modes

$$
\begin{aligned}
P &= \int_0^d \frac{\beta_n}{\omega\epsilon}|A_n|^2\cos^2(\kappa_n x)\,dx + \int_d^\infty \frac{\beta_n}{\omega\epsilon_o}|A_n|^2\cos^2(\kappa_n d)e^{2\gamma d}e^{-2\gamma x}\,dx \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\int_0^d \frac{1}{\epsilon_r}\cos^2(\kappa_n x)\,dx + \cos^2(\kappa_n d)e^{2\gamma d}\int_d^\infty e^{-2\gamma x}\,dx\right] \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \cos^2(\kappa_n d)e^{2\gamma d}\left(\frac{1}{-2\gamma}e^{-2\gamma x}\right)\Big|_d^\infty\right] \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \cos^2(\kappa_n d)e^{2\gamma d}\frac{1}{2\gamma}e^{-2\gamma d}\right] \\
&= \frac{\beta_n|A|^2}{\omega\epsilon_o}\left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \frac{\cos^2(\kappa_n d)}{2\gamma}\right] \\
\Rightarrow A &= \sqrt{\frac{\omega\epsilon_o P}{\beta_n\cdot\psi}}
\end{aligned}
$$

where $\psi = \left[\frac{2\kappa_n d+\sin(2\kappa_n d)}{4\kappa_n\epsilon_r} + \frac{\cos^2(\kappa_n d)}{2\gamma}\right]$.

6.2 Source Code

Note that this formula is not the one used by Marcuse, but the values seem to match within 10^{-14}.

```
psi = (2*Km*d + sin(2*Km*d))/(4*Km*eps) + cos(Km*d)**2/(2*gm)
Am = sqrt(w*eo*P/(Bm*psi))
```

6.3 TM Even - Radiation Modes

The derivation of the coefficients for Radiation Modes is provided in Marcuse's book, "Light Transmission Optics", pp. 316-318. The result is reprinted in his 1969 paper "Radiation Losses of Tapered Dielectric Slab Waveguides", and the formulae match. Gelin reprints the formula, although he includes a factor of ϵ^2 where Marcuse would have just ϵ.

6.4 TE Even - Guided Modes

$$
\begin{aligned}
P &= \int_0^d \frac{\beta_n}{\omega\mu}|A_n|^2 \cos^2(\kappa_n x)\ dx + \int_d^\infty \frac{\beta_n}{\omega\mu}|A_n|^2 \cos^2(\kappa_n d) e^{2\gamma d} e^{-2\gamma x}\ dx \\
&= \frac{\beta_n|A|^2}{\omega\mu}\left[\int_0^d \cos^2(\kappa_n x)\ dx + \cos^2(\kappa_n d) e^{2\gamma d}\int_d^\infty e^{-2\gamma x}\ dx\right] \\
&= \frac{\beta_n|A|^2}{\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{4\kappa_n} + \cos^2(\kappa_n d) e^{2\gamma d}\left(\frac{1}{-2\gamma}e^{-2\gamma x}\right)\Bigg|_d^\infty\right] \\
&= \frac{\beta_n|A|^2}{\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{4\kappa_n} + \cos^2(\kappa_n d) e^{2\gamma d}\frac{1}{2\gamma}e^{-2\gamma d}\right] \\
&= \frac{\beta_n|A|^2}{\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{4\kappa_n} + \frac{\cos^2(\kappa_n d)}{2\gamma}\right] \\
&= \frac{\beta_n|A|^2}{2\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{2\kappa_n} + \frac{\cos^2(\kappa_n d)}{\gamma}\right] \\
&= \frac{\beta_n|A|^2}{2\omega\mu}\left[\frac{2\kappa_n d + \sin(2\kappa_n d)}{2\kappa_n} + \frac{1+\cos(2\kappa_n d)}{2\gamma}\right]
\end{aligned}
$$

7 Derivation of the Overlap Integral Functions

7.1 Fields of the Even Modes

For even guided modes, the fields are given by

$$\phi_{y,n}(x) = \begin{cases} A_n \cos(\kappa x) & : |x| < d \\ A_n \cos(\kappa d) e^{\gamma d} e^{-\gamma x} & : |x| \geq d \end{cases}$$

where ϕ stands for either E or H, depending on the polarization. The radiation/continuum modes are

$$\phi_y(x, \rho) = \begin{cases} B_n^r \cos(\sigma x) & : |x| < d \\ B_n^r \left[D e^{-i\rho|x|} + D^* e^{-i\rho|x|} \right] & : |x| \geq d \end{cases}$$

Note that A, B, and D depend on the polarization. (See section 6). Finally, the free-space modes in the right half-space are described by:

$$\phi_y(x, \rho) = B^t \cos(\rho x)$$

7.2 $\kappa_n(\rho)$

Recalling the definition from section 4.2:

$$\kappa_n(\rho) = \int_0^\infty H_{y,n}^i H_y^{t*}(\rho)\, dx$$

Using the definition of the fields for even TM modes, we get the following:

$$
\begin{aligned}
\kappa_n(\rho) &= \int_0^d A_n \cos(\kappa x) B^t \cos(\rho x)\, dx + \int_d^\infty A_n \cos(\kappa d) e^{\gamma d} e^{-\gamma x} B^t \cos(\rho x)\, dx \\
&= A_n B^t \left\{ \int_0^d \cos(\kappa x)\cos(\rho x)\, dx + \cos(\kappa d) e^{\gamma d} \int_d^\infty e^{-\gamma x} \cos(\rho x)\, dx \right\} \\
&= A_n B^t \left\{ \frac{\kappa \sin(\kappa x)\cos(\rho x) - \rho \sin(\rho x)\cos(\kappa x)}{\kappa^2 - \rho^2} \Big|_0^d \right. \\
&\qquad\qquad \left. + \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma x}(\rho \sin(\rho x) - \gamma \cos(\rho x))}{\gamma^2 + \rho^2} \Big|_d^\infty \right\} \\
&= A_n B^t \left\{ \frac{\kappa \sin(\kappa d)\cos(\rho d) - \rho \sin(\rho d)\cos(\kappa d)}{\kappa^2 - \rho^2} \right. \\
&\qquad\qquad \left. - \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma d}(\rho \sin(\rho d) - \gamma \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= A_n B^t \cos(\kappa d) \left\{ \frac{\kappa \tan(\kappa d)\cos(\rho d) - \rho \sin(\rho d)}{\kappa^2 - \rho^2} - \frac{\rho \sin(\rho d) - \gamma \cos(\rho d)}{\gamma^2 + \rho^2} \right\}
\end{aligned}
$$

7.3 Source Code

```
kp[i,:,:] = Am[i] * Bt * cos(Km[i]*d) * ((Km[i]*cos(p*d)*tan(Km[i]*d) - p*sin(p*d))/(     Km[i]**2 - p**2) - (p*sin(p*d) - gm[i]*cos(p*d))/(gm[i]**2 + p**2))
```

7.4 $G_n(\rho)$

$$
G_n(\rho) = \int_{-\infty}^{\infty} E^i_{y,n} E^{t*}_y(\rho)\, dx \tag{51}
$$

$$
\begin{aligned}
G_n(\rho) &= 2\int_0^d A_n\cos(\kappa x)B^t\cos(\rho x)\,dx + 2\int_d^\infty A_n\cos(\kappa d)e^{\gamma d}e^{-\gamma x}B^t\cos(\rho x)\,dx \\
&= 2A_nB^t\left\{\int_0^d \cos(\kappa x)\cos(\rho x)\,dx + \cos(\kappa d)e^{\gamma d}\int_d^\infty e^{-\gamma x}\cos(\rho x)\,dx\right\} \\
&= 2A_nB^t\left\{\frac{\kappa\sin(\kappa x)\cos(\rho x)-\rho\sin(\rho x)\cos(\kappa x)}{\kappa^2-\rho^2}\Big|_0^d\right. \\
&\qquad\qquad\left. + \cos(\kappa d)e^{\gamma d}\frac{e^{-\gamma x}(\rho\sin(\rho x)-\gamma\cos(\rho x))}{\gamma^2+\rho^2}\Big|_d^\infty\right\} \\
&= 2A_nB^t\left\{\frac{\kappa\sin(\kappa d)\cos(\rho d)-\rho\sin(\rho d)\cos(\kappa d)}{\kappa^2-\rho^2}\right. \\
&\qquad\qquad\left. - \cos(\kappa d)e^{\gamma d}\frac{e^{-\gamma d}(\rho\sin(\rho d)-\gamma\cos(\rho d))}{\gamma^2+\rho^2}\right\} \\
&= 2A_nB^t\cos(\kappa d)\left\{\frac{\kappa\tan(\kappa d)\cos(\rho d)-\rho\sin(\rho d)}{\kappa^2-\rho^2} - \frac{\rho\sin(\rho d)-\gamma\cos(\rho d)}{\gamma^2+\rho^2}\right\}
\end{aligned}
$$

Since this function occurs for TE cases only, we can substitute the identity $\kappa\tan(\kappa d) = \gamma$:

$$
\begin{aligned}
&= 2A_nB^t\cos(\kappa d)\left\{\frac{\gamma\cos(\rho d)-\rho\sin(\rho d)}{\kappa^2-\rho^2} - \frac{\rho\sin(\rho d)-\gamma\cos(\rho d)}{\gamma^2+\rho^2}\right\} \\
&= 2A_nB^t\cos(\kappa d)\left\{\frac{(\kappa^2-\rho^2)(\gamma\cos(\rho d)-\rho\sin(\rho d)) - (\gamma^2+\rho^2)(\rho\sin(\rho d)-\gamma\cos(\rho d))}{(\kappa^2-\rho^2)(\gamma^2+\rho^2)}\right\} \\
&= 2A_nB^t\cos(\kappa d)\left\{\frac{(\kappa^2-\rho^2+\gamma^2+\rho^2)(\gamma\cos(\rho d)) - (\kappa^2-\rho^2+\gamma^2+\rho^2)(\rho\sin(\rho d))}{(\kappa^2-\rho^2)(\gamma^2+\rho^2)}\right\} \\
&= 2A_nB^t(\kappa^2+\gamma^2)\cos(\kappa d)\left\{\frac{\gamma\cos(\rho d)-\rho\sin(\rho d)}{(\kappa^2-\rho^2)(\gamma^2+\rho^2)}\right\} \\
\Rightarrow G_n(\rho) &= 2A_nB^tk^2(\epsilon_r-1)\cos(\kappa d)\left\{\frac{\gamma\cos(\rho d)-\rho\sin(\rho d)}{(\kappa^2-\rho^2)(\gamma^2+\rho^2)}\right\}
\end{aligned}
$$

This formula matches (7) in Gelin 1981.

7.5 Source Code

```
G[i,:,:] = 2 * k**2 * (eps-1) * Am[i] * Bt * cos(Km[i]*d) * (gm[i
    ]*cos(p*d) - p*sin(p*d)) / ((Km[i]**2 - p**2)*(gm[i]**2 + p
    **2))
```

7.6 $\nu_n(\rho)$

$$\nu_n(\rho) = \frac{\int_0^\infty H_{y,n}^i H_y^{t*}(\rho)}{\epsilon_r}\,dx$$

Using the definition of the fields for even TM modes, we get the following:

$$\begin{aligned}
\nu_n(\rho) &= \int_0^d \frac{A_n \cos(\kappa x) B^t \cos(\rho x)}{\epsilon_r}\,dx + \int_d^\infty A_n \cos(\kappa d) e^{\gamma d} e^{-\gamma x} B^t \cos(\rho x)\,dx \\
&= A_n B^t \left\{ \int_0^d \frac{\cos(\kappa x)\cos(\rho x)}{\epsilon_r}\,dx + \cos(\kappa d) e^{\gamma d} \int_d^\infty e^{-\gamma x} \cos(\rho x)\,dx \right\} \\
&= A_n B^t \Bigg\{ \frac{\kappa \sin(\kappa x)\cos(\rho x) - \rho \sin(\rho x)\cos(\kappa x)}{\epsilon_r(\kappa^2 - \rho^2)} \Bigg|_0^d \\
&\qquad\qquad + \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma x}(\rho \sin(\rho x) - \gamma \cos(\rho x))}{\gamma^2 + \rho^2} \Bigg|_d^\infty \Bigg\} \\
&= A_n B^t \Bigg\{ \frac{\kappa \sin(\kappa d)\cos(\rho d) - \rho \sin(\rho d)\cos(\kappa d)}{\epsilon_r(\kappa^2 - \rho^2)} \\
&\qquad\qquad - \cos(\kappa d) e^{\gamma d} \frac{e^{-\gamma d}(\rho \sin(\rho d) - \gamma \cos(\rho d))}{\gamma^2 + \rho^2} \Bigg\} \\
&= A_n B^t \cos(\kappa d) \left\{ \frac{\kappa \cos(\rho d)\tan(\kappa d) - \rho \sin(\rho d)}{\epsilon_r(\kappa^2 - \rho^2)} - \frac{\rho \sin(\rho d) - \gamma \cos(\rho d)}{\gamma^2 + \rho^2} \right\}
\end{aligned}$$

7.7 Source Code

```
V[i,:,:]  = Am[i] * Bt * cos(Km[i]*d) * ((Km[i]*cos(p*d)*tan(Km[i]*d) - p*sin(p*d))/(eps*(Km[i]**2 - p**2)) - (p*sin(p*d) - gm[i]*cos(p*d))/(gm[i]**2 + p**2))
```

7.8 $\zeta(\rho', \rho)$

$$\zeta(\rho', \rho) = \int_0^\infty H_y^r(\rho') H_y^{t*}(\rho)\, dx$$

$$\begin{aligned}
\zeta(\rho', \rho) &= \int_0^d B^r(\rho') \cos(\sigma' x) B^t(\rho) \cos(\rho x)\, dx + \int_d^\infty B^r(\rho')[De^{-i\rho' x} + D^* e^{i\rho' x}] B^t(\rho) \cos(\rho x)\, dx \\
&= B^r(\rho') B^t(\rho) \left\{ \int_0^d \cos(\sigma' x) \cos(\rho x)\, dx + \int_d^\infty \cos(\rho x) \cdot 2 \cdot \mathrm{Re}[De^{-i\rho' x}]\, dx \right\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' x) \cos(\rho x) - \rho \cos(\sigma' x) \sin(\rho x)}{\sigma'^2 - \rho^2} \Bigg|_0^d \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\left[\int_d^\infty \cos(\rho x) \cdot De^{-i\rho' x}\, dx \right] \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\left[D \frac{e^{-ix\rho'}(-\rho \sin(\rho x) + i\rho' \cos(\rho x))}{\rho'^2 - \rho^2}\, dx \right] \Bigg|_d^\infty \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad - 2 \cdot \mathrm{Re}\left[D \frac{e^{-ix\rho'}(-\rho \sin(\rho x) + i\rho' \cos(\rho x))}{\rho'^2 - \rho^2} \right] \Bigg|_0^d \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad - 2 \cdot \mathrm{Re}\left[D \frac{e^{-i\rho' d}(-\rho \sin(\rho d) + i\rho' \cos(\rho d)) - i\rho'}{\rho'^2 - \rho^2} \right] \Bigg\} \\
&= B^r(\rho') B^t(\rho) \Bigg\{ \frac{\sigma' \sin(\sigma' d) \cos(\rho d) - \rho \cos(\sigma' d) \sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad\qquad + 2 \cdot \mathrm{Re}\left[D \frac{e^{-i\rho' d}(\rho \sin(\rho d) - i\rho' \cos(\rho d)) + i\rho'}{\rho'^2 - \rho^2} \right] \Bigg\}
\end{aligned}$$

7.9 Source Code

```
z = Br1*Bt2 * ((o1*sin(od)*cos(pd) - p2*cos(od)*sin(pd))/
    (o1**2-p2**2)  + 2*(D1 * (exp(-1j*p1*d) * (p2*sin(pd)-1j*p1*
    cos(pd))) + 1j*p1 ).real / (p1**2-p2**2) )
```

7.10 $f(\rho', \rho)$

$$f(\rho', \rho) = \int_0^\infty \frac{H_y^r(\rho')H_y^{t*}(\rho)}{\epsilon_r}\,dx$$

$$\begin{aligned}
f(\rho', \rho) &= \int_0^d \frac{B^r(\rho')\cos(\sigma' x)B^t(\rho)\cos(\rho x)}{\epsilon_r}\,dx + \int_d^\infty B^r(\rho')[De^{-i\rho' x} + D^* e^{i\rho' x}]B^t(\rho)\cos(\rho x)\,dx \\
&= B^r(\rho')B^t(\rho)\left\{\int_0^d \frac{\cos(\sigma' x)\cos(\rho x)}{\epsilon_r}\,dx + \int_d^\infty \cos(\rho x)\cdot 2\cdot \mathrm{Re}[De^{-i\rho' x}]\,dx\right\} \\
&= B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' x)\cos(\rho x) - \rho\cos(\sigma' x)\sin(\rho x)}{\epsilon_r(\sigma'^2 - \rho^2)}\Bigg|_0^d \\
&\qquad\qquad + 2\cdot\mathrm{Re}\left[\int_d^\infty \cos(\rho x)\cdot De^{-i\rho' x}\,dx\right]\Bigg\} \\
&= B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\epsilon_r(\sigma'^2 - \rho^2)} \\
&\qquad\qquad + 2\cdot\mathrm{Re}\left[D\frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2}\,dx\right]\Bigg|_d^\infty\Bigg\} \\
&= B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\epsilon_r(\sigma'^2 - \rho^2)} \\
&\qquad\qquad - 2\cdot\mathrm{Re}\left[D\frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2}\right]\Bigg|_0^d\Bigg\} \\
&= B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\epsilon_r(\sigma'^2 - \rho^2)} \\
&\qquad\qquad - 2\cdot\mathrm{Re}\left[D\frac{e^{-i\rho' d}(-\rho\sin(\rho d) + i\rho'\cos(\rho d)) - i\rho'}{\rho'^2 - \rho^2}\right]\Bigg\} \\
&= B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\epsilon_r(\sigma'^2 - \rho^2)} \\
&\qquad\qquad + 2\cdot\mathrm{Re}\left[D\frac{e^{-i\rho' d}(\rho\sin(\rho d) - i\rho'\cos(\rho d)) + i\rho'}{\rho'^2 - \rho^2}\right]\Bigg\}
\end{aligned}$$

7.11 Source Code

```
f = Br1*Bt2 * ((o1*sin(od)*cos(pd) - p2*cos(od)*sin(pd))/(n**2 *
    (o1**2-p2**2)) + 2*(D1 * (exp(-1j*p1*d) * (p2*sin(pd)-1j*p1*
    cos(pd))) + 1j*p1 ).real / (p1**2-p2**2) )
```

7.12 $F(\rho', \rho)$

$$F(\rho',\rho) = \int_{-\infty}^{\infty} E_y^r(\rho') E_y^{t*}(\rho)\, dx$$

$$\begin{aligned}
F(\rho',\rho) &= 2\int_0^d B^r(\rho')\cos(\sigma' x) B^t(\rho)\cos(\rho x)\, dx + 2\int_d^\infty B^r(\rho')[De^{-i\rho' x} + D^* e^{i\rho' x}] B^t(\rho)\cos(\rho x)\, dx \\
&= 2B^r(\rho')B^t(\rho)\left\{\int_0^d \cos(\sigma' x)\cos(\rho x)\, dx + \int_d^\infty \cos(\rho x)\cdot 2\cdot \mathrm{Re}[De^{-i\rho' x}]\, dx\right\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' x)\cos(\rho x) - \rho\cos(\sigma' x)\sin(\rho x)}{\sigma'^2 - \rho^2}\Bigg|_0^d \\
&\qquad + 2\cdot \mathrm{Re}\Bigg[\int_d^\infty \cos(\rho x)\cdot De^{-i\rho' x}\, dx\Bigg]\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad + 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2}\, dx\Bigg]\Bigg|_d^\infty\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad - 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-ix\rho'}(-\rho\sin(\rho x) + i\rho'\cos(\rho x))}{\rho'^2 - \rho^2}\Bigg]\Bigg|_0^d\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad - 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-i\rho' d}(-\rho\sin(\rho d) + i\rho'\cos(\rho d)) - i\rho'}{\rho'^2 - \rho^2}\Bigg]\Bigg\} \\
&= 2B^r(\rho')B^t(\rho)\Bigg\{\frac{\sigma'\sin(\sigma' d)\cos(\rho d) - \rho\cos(\sigma' d)\sin(\rho d)}{\sigma'^2 - \rho^2} \\
&\qquad + 2\cdot \mathrm{Re}\Bigg[D\frac{e^{-i\rho' d}(\rho\sin(\rho d) - i\rho'\cos(\rho d)) + i\rho'}{\rho'^2 - \rho^2}\Bigg]\Bigg\}
\end{aligned}$$

7.13 Source Code

```
od = o1*d; pd = p2*d.transpose()
F = 2*Br1*Bt2 * ((o1*sin(od)*cos(pd) - p2*cos(od)*sin(pd))/(o1
    **2-p2**2) + 2*(D1 * (exp(-1j*p1*d) * (p2*sin(pd)-1j*p1*cos(pd
    ))) + 1j*p1 ).real / (p1**2-p2**2) )
```

7.14 Fields of the Odd Modes

For odd guided modes, the fields are given by

$$\phi_{y,n}(x) = \begin{cases} A_n \sin(\kappa x) & : |x| < d \\ \frac{x}{|x|} A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma |x|} & : |x| \geq d \end{cases}$$

where ϕ stands for either E or H, depending on the polarization. The radiation/continuum modes are

$$\phi_y(x, \rho) = \begin{cases} B_n^r \sin(\sigma x) & : |x| < d \\ \frac{x}{|x|} B_n^r \left[D e^{-i\rho |x|} + D^* e^{-i\rho |x|} \right] & : |x| \geq d \end{cases}$$

Note that A, B, and D depend on the polarization. (See section 6). Finally, the free-space modes in the right half-space are described by:

$$\phi_y(x, \rho) = B^t \sin(\rho x)$$

7.15 $\Xi_n(\rho)$

$$
\begin{aligned}
\Xi_n(\rho) &= \int_{-\infty}^{\infty} E^r_{y,n} E^{t*}_y(\rho)\ dx \\
&= \int_{-d}^{d} A_n \sin(\kappa x) B^t(\rho) \sin(\rho x)\ dx \\
&\qquad\qquad + \int_{-\infty}^{-d} -A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma |x|} B^t(\rho) \sin(\rho x)\ dx \\
&\qquad\qquad + \int_{d}^{\infty} A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma |x|} B^t(\rho) \sin(\rho x)\ dx \\
&= 2\int_{0}^{d} A_n \sin(\kappa x) B^t(\rho) \sin(\rho d)\ dx + 2\int_{d}^{\infty} A_n \sin(\kappa d) e^{\gamma d} e^{-\gamma x} B^t(\rho) \sin(\rho x)\ dx \\
&= 2A_n B^t(\rho) \left\{ \int_0^d \sin(\kappa x)\sin(\rho d)\ dx + \sin(\kappa d) e^{\gamma d} \int_d^{\infty} e^{-\gamma x} \sin(\rho x)\ dx \right\} \\
&= 2A_n B^t(\rho) \left\{ \left. \frac{\rho \sin(\kappa x)\cos(\rho x) - \kappa \cos(\kappa x) \sin(\rho x)}{\kappa^2 - \rho^2} \right|_0^d \right. \\
&\qquad\qquad \left. - \sin(\kappa d) e^{\gamma d} \left. \frac{e^{-\gamma x}(\gamma \sin(\rho x) + \rho \cos(\rho x))}{\gamma^2 + \rho^2} \right|_d^{\infty} \right\} \\
&= 2A_n B^t(\rho) \left\{ \frac{\rho \sin(\kappa d)\cos(\rho d) - \kappa \cos(\kappa d) \sin(\rho d)}{\kappa^2 - \rho^2} \right. \\
&\qquad\qquad \left. + \sin(\kappa d) e^{\gamma d} \frac{e^{-\gamma d}(\gamma \sin(\rho d) + \rho \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= 2A_n B^t(\rho) \left\{ \frac{\rho \sin(\kappa d)\cos(\rho d) - \kappa \cos(\kappa d) \sin(\rho d)}{\kappa^2 - \rho^2} + \frac{\sin(\kappa d)(\gamma \sin(\rho d) + \rho \cos(\rho d))}{\gamma^2 + \rho^2} \right\} \\
&= 2A_n B^t(\rho) \sin(\kappa d) \left\{ \frac{\rho \cos(\rho d) - \kappa \cot(\kappa d) \sin(\rho d)}{\kappa^2 - \rho^2} + \frac{\gamma \sin(\rho d) + \rho \cos(\rho d)}{\gamma^2 + \rho^2} \right\}
\end{aligned}
$$

7.16 Source Code

```
xi[i,:,:] = 2* Am[i] * Bt * sin(Km[i]*d) * ((p*cos(p*d) - Km[i]*
    cot(Km[i]*d)*sin(p*d))/(Km[i]**2-p**2) + (gm[i]*sin(p*d) + p*
    cos(p*d))/(gm[i]**2 + p**2))
```

7.17 $\Phi_n(\rho', \rho)$

$$\begin{aligned}
\Phi_n(\rho', \rho) &= \int_{-\infty}^{\infty} E_y^r(\rho') E_y^{t*}(\rho)\; dx \\
&= \int_{-d}^{d} B_y^r(\rho') \sin(\sigma x) B^t(\rho) \sin(\rho x)\; dx \\
&\qquad\qquad + \int_{-\infty}^{-d} -B_y^r(\rho')[De^{-i\rho'|x|} + D^* e^{i\rho'|x|}] B^t(\rho) \sin(\rho x)\; dx \\
&\qquad\qquad + \int_{d}^{\infty} B_y^r(\rho')[De^{-i\rho'|x|} + D^* e^{i\rho'|x|}] B^t(\rho) \sin(\rho x)\; dx \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \int_0^d \sin(\sigma x) \sin(\rho x)\; dx + \int_d^{\infty} [De^{-i\rho'|x|} + D^* e^{i\rho'|x|}] \sin(\rho x)\; dx \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma x)\cos(\rho x) - \sigma \cos(\sigma x) \sin(\rho x)}{\sigma^2 - \rho^2} \Bigg|_0^d + 2\mathrm{Re}\left[\int_d^{\infty} De^{-i\rho' x} \sin(\rho x)\; dx \right] \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma d)\cos(\rho d) - \sigma \cos(\sigma d) \sin(\rho d)}{\sigma^2 - \rho^2} + 2\mathrm{Re}\left[D \frac{e^{-i\rho' x}(\rho\cos(\rho x) + i\rho' \sin(\rho x))}{\rho'^2 - \rho^2} \Bigg|_d^{\infty} \right] \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma d)\cos(\rho d) - \sigma \cos(\sigma d) \sin(\rho d)}{\sigma^2 - \rho^2} - 2\mathrm{Re}\left[D \frac{e^{-i\rho' x}(\rho\cos(\rho x) + i\rho' \sin(\rho x))}{\rho'^2 - \rho^2} \Bigg|_0^{d} \right] \right\} \\
&= 2B_y^r(\rho') B^t(\rho) \left\{ \frac{\rho \sin(\sigma d)\cos(\rho d) - \sigma \cos(\sigma d) \sin(\rho d)}{\sigma^2 - \rho^2} - 2\mathrm{Re}\left[D \frac{e^{-i\rho' d}(\rho\cos(\rho d) + i\rho' \sin(\rho d)) - \rho}{\rho'^2 - \rho^2} \right] \right\}
\end{aligned}$$

8 Hamid's Derivations

8.1 Derivation of A:

Let's start with the form of ψ in chapter 6.1 of your notes:

$$\psi = \frac{2\kappa_n d + sin\,(2\kappa_n d)}{4\epsilon_r \kappa_n} + \frac{cos^2\,(\kappa_n d)}{2\gamma_n}$$

We know that:

$$tan\,(\kappa_n d) = \frac{\epsilon_r \gamma_n}{\kappa_n}$$

Therefore:

$$cos^2\left(\kappa_n d\right) = \left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1}$$

$$sin\left(2\kappa_n d\right) = \frac{2\epsilon_r\gamma_n}{\kappa_n}\left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1}$$

So that:

$$\begin{aligned}
\psi =& \frac{d}{2\epsilon_r} + \frac{sin\left(2\kappa_n d\right)}{4\epsilon_r\kappa_n} + \frac{cos^2\left(\kappa_n d\right)}{2\gamma_n} \\
=& \frac{d}{2\epsilon_r} + \frac{1}{2\kappa_n}\frac{\gamma_n}{\kappa_n}\left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1} + \frac{1}{2\gamma_n}\left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1} \\
=& \frac{d}{2\epsilon_r} + \left(1 + \frac{\epsilon_r^2\gamma_n^2}{\kappa_n^2}\right)^{-1}\left(\frac{\gamma_n}{2\kappa_n^2} + \frac{1}{2\gamma_n}\right) \\
=& \frac{d}{2\epsilon_r} + \frac{1}{2\gamma_n}\left(\kappa_n^2 + \epsilon_r^2\gamma_n^2\right)^{-1}\left(\gamma_n^2 + \kappa_n^2\right)
\end{aligned}$$

We have:

$$\begin{aligned}
\kappa_n^2 + \epsilon_r^2\gamma_n^2 &= \left(\epsilon_r^2 - 1\right)\beta_n^2 + \epsilon_r\left(1 - \epsilon_r\right)k_0^2 \\
&= \left(\epsilon_r - 1\right)\left(\left(\epsilon_r + 1\right)\beta_n^2 - \epsilon_r k_0^2\right) \\
&= \left(\epsilon_r - 1\right)\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)
\end{aligned}$$

$$\kappa_n^2 + \gamma_n^2 = \left(\epsilon_r - 1\right)k_0^2$$

Therefore:

$$
\begin{aligned}
A =& \sqrt{\frac{\omega\epsilon_0 P}{\beta_n \psi}} \\
=& \sqrt{\frac{\omega\epsilon_0 P}{\beta_n \left(\frac{d}{2\epsilon_r} + \left(\kappa_n^2 + \epsilon_r^2\gamma_n^2\right)^{-1} \frac{1}{2\gamma_n}\left(\gamma_n^2 + \kappa_n^2\right)\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_0\epsilon_r P}{\beta_n \left(d + \frac{\epsilon_r}{\gamma_n}\left(\kappa_n^2 + \epsilon_r^2\gamma_n^2\right)^{-1}\left(\gamma_n^2 + \kappa_n^2\right)\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_r\epsilon_0 P}{\beta_n \left(d + \frac{\epsilon_r}{\gamma_n}\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)^{-1} k_0^2\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_r\epsilon_0 P\gamma_n}{\beta_n \left(d\gamma_n + \epsilon_r\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)^{-1} k_0^2\right)}} \\
=& \sqrt{\frac{2\omega\epsilon_0\epsilon_r P\gamma_n\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)}{\beta_n\left(\epsilon_r k_0^2 + d\gamma_n\left(\epsilon_r\gamma_n^2 + \beta_n^2\right)\right)}}
\end{aligned}
$$

This shows that the form of A in chapter 6.1 of your notes is consistent with the equation 8.3-40 of the book and also equation A-4 of the appendix of the main article.

8.2 Derivation of $F\left(\rho,\rho'\right)$:

$$\begin{aligned}
F\left(\rho,\rho'\right) =&2\int_0^d B^r\left(\rho'\right) cos\left(\sigma' x\right) B^t\left(\rho\right) cos\left(\rho x\right) dx\\
&+2\int_d^\infty B^r\left(\rho'\right)\left[D\left(\rho'\right)e^{-i\rho' x}+D^*\left(\rho'\right)e^{i\rho' x}\right] B^t\left(\rho\right) cos\left(\rho x\right) dx\\
&=2B^r\left(\rho'\right) B^t\left(\rho\right)\int_0^d cos\left(\sigma' x\right) cos\left(\rho x\right) dx\\
&+2B^r\left(\rho'\right) B^t\left(\rho\right)\int_d^\infty\left[D\left(\rho'\right)e^{-i\rho' x}+D^*\left(\rho'\right)e^{i\rho' x}\right] cos\left(\rho x\right) dx\\
&=B^r\left(\rho'\right) B^t\left(\rho\right)\int_0^d\left(cos\left(\left(\sigma'+\rho\right)x\right)+cos\left(\left(\sigma'-\rho\right)x\right)\right) dx\\
&+2B^r\left(\rho'\right) B^t\left(\rho\right)\int_d^\infty\left[D\left(\rho'\right)e^{-i\rho' x}+D^*\left(\rho'\right)e^{i\rho' x}\right] cos\left(\rho x\right) dx\\
&=B^r\left(\rho'\right) B^t\left(\rho\right)\left\{\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\}\\
&+2B^r\left(\rho'\right) B^t\left(\rho\right)\int_d^\infty\left[D\left(\rho'\right)e^{-i\rho' x}+D^*\left(\rho'\right)e^{i\rho' x}\right] cos\left(\rho x\right) dx
\end{aligned}$$

But we know that:

$$D\left(\rho\right)=\frac{1}{2}\left[cos\left(\sigma d\right)-\frac{i\sigma}{\epsilon_r\rho}sin\left(\sigma d\right)\right]e^{i\rho d}$$

So that:

$$\begin{aligned}
&2\int_d^\infty\left[D\left(\rho'\right)e^{-i\rho' x}+D^*\left(\rho'\right)e^{i\rho' x}\right] cos\left(\rho x\right) dx\\
&=\int_d^\infty\left[cos\left(\sigma' d\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{i\rho' d}e^{-i\rho' x}cos\left(\rho x\right) dx\\
&+\int_d^\infty\left[cos\left(\sigma' d\right)+\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{-i\rho' d}e^{i\rho' x}cos\left(\rho x\right) dx\\
&=\left[cos\left(\sigma' d\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{i\rho' d}\int_d^\infty e^{-i\rho' x}cos\left(\rho x\right) dx\\
&+\left[cos\left(\sigma' d\right)+\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{-i\rho' d}\int_d^\infty e^{i\rho' x}cos\left(\rho x\right) dx
\end{aligned}$$

But:

$$
\begin{aligned}
&\int_d^{\infty} e^{-i\rho' x} cos\left(\rho x\right) dx \\
&= \frac{1}{2}\int_d^{\infty} e^{-i(\rho'+\rho)x} dx + \frac{1}{2}\int_d^{\infty} e^{-i(\rho'-\rho)x} dx \\
&= \frac{1}{2}\int_0^{\infty} e^{-i(\rho'+\rho)x} dx + \frac{1}{2}\int_0^{\infty} e^{-i(\rho'-\rho)x} dx \\
&- \frac{1}{2}\int_0^{d} e^{-i(\rho'+\rho)x} dx - \frac{1}{2}\int_0^{d} e^{-i(\rho'-\rho)x} dx \\
&= \frac{\pi}{2}\delta\left(\rho+\rho'\right) + \frac{\pi}{2}\delta\left(\rho-\rho'\right) + \frac{1}{2i\left(\rho+\rho'\right)} + \frac{1}{2i\left(\rho'-\rho\right)} \\
&+ \frac{e^{-i(\rho'+\rho)d}-1}{2i\left(\rho'+\rho\right)} + \frac{e^{-i(\rho'-\rho)d}-1}{2i\left(\rho'-\rho\right)} \\
&= \frac{\pi}{2}\delta\left(\rho+\rho'\right) + \frac{\pi}{2}\delta\left(\rho-\rho'\right) + \frac{e^{-i(\rho'+\rho)d}}{2i\left(\rho'+\rho\right)} + \frac{e^{-i(\rho'-\rho)d}}{2i\left(\rho'-\rho\right)}
\end{aligned}
$$

Similarly:

$$
\int_d^{\infty} e^{i\rho' x} cos\left(\rho x\right) dx = \frac{\pi}{2}\delta\left(\rho-\rho'\right) + \frac{\pi}{2}\delta\left(\rho+\rho'\right) - \frac{e^{i(\rho'+\rho)d}}{2i\left(\rho'+\rho\right)} - \frac{e^{i(\rho'-\rho)d}}{2i\left(\rho'-\rho\right)}
$$

Therefore:

$$
\begin{aligned}
&2\int_d^\infty \left[D\left(\rho'\right)e^{-i\rho' x}+D^*\left(\rho'\right)e^{i\rho' x}\right]cos\left(\rho x\right)dx\\
&=\left[cos\left(\sigma' d\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{i\rho' d}\int_d^\infty e^{-i\rho' x}cos\left(\rho x\right)dx\\
&+\left[cos\left(\sigma' d\right)+\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{-i\rho' d}\int_d^\infty e^{i\rho' x}cos\left(\rho x\right)dx\\
&=\frac{1}{2}\left[cos\left(\sigma' d\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{i\rho' d}\left[\pi\delta\left(\rho-\rho'\right)+\frac{e^{-i(\rho'+\rho)d}}{i\left(\rho'+\rho\right)}+\frac{e^{-i(\rho'-\rho)d}}{i\left(\rho'-\rho\right)}\right]\\
&+\frac{1}{2}\left[cos\left(\sigma' d\right)+\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]e^{-i\rho' d}\left[\pi\delta\left(\rho-\rho'\right)-\frac{e^{i(\rho'+\rho)d}}{i\left(\rho'+\rho\right)}-\frac{e^{i(\rho'-\rho)d}}{i\left(\rho'-\rho\right)}\right]\\
&=\frac{1}{2}\left[cos\left(\sigma' d\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]\left[\pi e^{i\rho' d}\delta\left(\rho-\rho'\right)+\frac{e^{-i\rho d}}{i\left(\rho'+\rho\right)}+\frac{e^{i\rho d}}{i\left(\rho'-\rho\right)}\right]\\
&+\frac{1}{2}\left[cos\left(\sigma' d\right)+\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right]\left[\pi e^{-i\rho' d}\delta\left(\rho-\rho'\right)-\frac{e^{i\rho d}}{i\left(\rho'+\rho\right)}-\frac{e^{-i\rho d}}{i\left(\rho'-\rho\right)}\right]\\
&=2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right)\\
&+\frac{1}{2i\left(\rho'-\rho\right)}\left(cos\left(\sigma' d\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right)e^{i\rho d}\\
&-\frac{1}{2i\left(\rho'-\rho\right)}\left(cos\left(\sigma' d\right)+\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right)e^{-i\rho d}\\
&+\frac{1}{2i\left(\rho'+\rho\right)}\left(cos\left(\sigma' d\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right)e^{-i\rho d}\\
&-\frac{1}{2i\left(\rho'+\rho\right)}\left(cos\left(\sigma' d\right)+\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\right)e^{i\rho d}\\
&=2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right)\\
&+\frac{1}{2i\left(\rho'-\rho\right)}\left[cos\left(\sigma' d\right)\left(e^{i\rho d}-e^{-i\rho d}\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\left(e^{i\rho d}+e^{-i\rho d}\right)\right]\\
&+\frac{1}{2i\left(\rho'+\rho\right)}\left[cos\left(\sigma' d\right)\left(e^{-i\rho d}-e^{i\rho d}\right)-\frac{i\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)\left(e^{-i\rho d}+e^{i\rho d}\right)\right]\\
&=2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right)\\
&+\frac{1}{\left(\rho'-\rho\right)}\left[cos\left(\sigma' d\right)sin\left(\rho d\right)-\frac{\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)cos\left(\rho d\right)\right]\\
&+\frac{1}{\left(\rho'+\rho\right)}\left[-cos\left(\sigma' d\right)sin\left(\rho d\right)-\frac{\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)cos\left(\rho d\right)\right]\\
&=2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right)\\
&+\left[\frac{1}{\left(\rho'-\rho\right)}-\frac{1}{\left(\rho'+\rho\right)}\right]cos\left(\sigma' d\right)sin\left(\rho d\right)\\
&-\left[\frac{1}{\left(\rho'-\rho\right)}+\frac{1}{\left(\rho'+\rho\right)}\right]\frac{\sigma'}{\epsilon_r\rho'}sin\left(\sigma' d\right)cos\left(\rho d\right)\\
&=2\Re\left(D\left(\rho'\right)\right)\pi\delta\left(\rho-\rho'\right)\\
&+\frac{2\rho}{\left(\rho'^2-\rho^2\right)}cos\left(\sigma' d\right)sin\left(\rho d\right)\\
&-\frac{2\sigma'}{\left(\rho'^2-\rho^2\right)\epsilon_r}sin\left(\sigma' d\right)cos\left(\rho d\right)
\end{aligned}
$$

Therefore:

$$\begin{aligned}
F\left(\rho,\rho'\right) =& B^r\left(\rho'\right)B^t\left(\rho\right)\left\{2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\left\{\frac{2\rho}{\left(\rho'^2-\rho^2\right)}cos\left(\sigma'd\right)sin\left(\rho d\right)-\frac{2\sigma'}{\left(\rho'^2-\rho^2\right)\epsilon_r}sin\left(\sigma'd\right)cos\left(\rho d\right)\right\}\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)\left\{2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{\rho}{\left(\rho'^2-\rho^2\right)}\left(sin\left(\left(\rho+\sigma'\right)d\right)+sin\left(\left(\rho-\sigma'\right)d\right)\right)\\
&-B^r\left(\rho'\right)B^t\left(\rho\right)\frac{\sigma'}{\left(\rho'^2-\rho^2\right)\epsilon_r}\left(sin\left(\left(\rho+\sigma'\right)d\right)-sin\left(\left(\rho-\sigma'\right)d\right)\right)\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)\left\{2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\sigma'+\rho}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\sigma'-\rho}\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{1}{\left(\rho'^2-\rho^2\right)}\left(\rho-\frac{\sigma'}{\epsilon_r}\right)sin\left(\left(\rho+\sigma'\right)d\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{1}{\left(\rho'^2-\rho^2\right)}\left(\rho+\frac{\sigma'}{\epsilon_r}\right)sin\left(\left(\rho-\sigma'\right)d\right)\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)sin\left(\left(\sigma'+\rho\right)d\right)\left(\frac{1}{\sigma'+\rho}+\frac{\rho-\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)sin\left(\left(\sigma'-\rho\right)d\right)\left(\frac{1}{\sigma'-\rho}-\frac{\rho+\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}\right)
\end{aligned}$$

But We know:

$$\frac{1}{\sigma'+\rho}+\frac{\rho-\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}=\frac{1}{\left(\sigma'+\rho\right)\left(\rho'^2-\rho^2\right)}\left(\rho'^2-\rho^2+\left(\rho+\sigma'\right)\left(\rho-\frac{\sigma'}{\epsilon_r}\right)\right)$$

$$\frac{1}{\sigma'-\rho}-\frac{\rho+\frac{\sigma'}{\epsilon_r}}{\rho'^2-\rho^2}=\frac{1}{\left(\sigma'-\rho\right)\left(\rho'^2-\rho^2\right)}\left(\rho'^2-\rho^2-\left(\sigma'-\rho\right)\left(\rho+\frac{\sigma'}{\epsilon_r}\right)\right)$$

$$
\begin{aligned}
\rho'^2 - \rho^2 + \left(\rho + \sigma'\right)\left(\rho - \frac{\sigma'}{\epsilon_r}\right) &= \rho'^2 - \rho^2 + \left(\rho + \sigma'\right)\rho - \left(\rho + \sigma'\right)\frac{\sigma'}{\epsilon_r} \\
&= \rho'^2 - \rho^2 + \left(\rho^2 + \sigma'\rho\right) - \left(\frac{\rho\sigma'}{\epsilon_r} + \frac{\sigma'^2}{\epsilon_r}\right) \\
&= \rho'^2 + \sigma'\rho\left(1 - \frac{1}{\epsilon_r}\right) - \frac{\sigma'^2}{\epsilon_r} \\
&= k_0^2\left(1 - \epsilon_r\right) + \sigma'\left(\sigma' + \rho\right)\left(1 - \frac{1}{\epsilon_r}\right)
\end{aligned}
$$

$$
\begin{aligned}
\rho'^2 - \rho^2 + \left(\rho - \sigma'\right)\left(\rho + \frac{\sigma'}{\epsilon_r}\right) &= \rho'^2 - \rho^2 + \left(\rho - \sigma'\right)\rho + \left(\rho - \sigma'\right)\frac{\sigma'}{\epsilon_r} \\
&= \rho'^2 - \rho^2 + \left(\rho^2 - \sigma'\rho\right) + \left(\frac{\rho\sigma'}{\epsilon_r} - \frac{\sigma'^2}{\epsilon_r}\right) \\
&= \rho'^2 - \sigma'\rho\left(1 - \frac{1}{\epsilon_r}\right) - \frac{\sigma'^2}{\epsilon_r} \\
&= k_0^2\left(1 - \epsilon_r\right) + \sigma'\left(\sigma' - \rho\right)\left(1 - \frac{1}{\epsilon_r}\right)
\end{aligned}
$$

Since:

$$
\rho'^2 = k_0^2\left(1 - \epsilon_r\right) + \sigma'^2
$$

Therefore:

$$
\begin{aligned}
F\left(\rho,\rho'\right) =& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\sigma'+\rho\right)\left(\rho'^2-\rho^2\right)}\left(k_0^2\left(1-\epsilon_r\right)+\sigma'\left(\sigma'+\rho\right)\left(1-\frac{1}{\epsilon_r}\right)\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\sigma'-\rho\right)\left(\rho'^2-\rho^2\right)}\left(k_0^2\left(1-\epsilon_r\right)+\sigma'\left(\sigma'-\rho\right)\left(1-\frac{1}{\epsilon_r}\right)\right)\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\left\{\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\sigma'+\rho\right)}\frac{k_0^2\left(1-\epsilon_r\right)}{\left(\rho'^2-\rho^2\right)}+\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\rho'^2-\rho^2\right)}\sigma'\left(1-\frac{1}{\epsilon_r}\right)\right\}\\
&+B^r\left(\rho'\right)B^t\left(\rho\right)\left\{\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\sigma'-\rho\right)}\frac{k_0^2\left(1-\epsilon_r\right)}{\left(\rho'^2-\rho^2\right)}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\rho'^2-\rho^2\right)}\sigma'\left(1-\frac{1}{\epsilon_r}\right)\right\}\\
=& B^r\left(\rho'\right)B^t\left(\rho\right)2\pi\Re\left(D\left(\rho'\right)\right)\delta\left(\rho-\rho'\right)\\
&-B^r\left(\rho'\right)B^t\left(\rho\right)\left(\frac{sin\left(\left(\sigma'+\rho\right)d\right)}{\left(\sigma'+\rho\right)}+\frac{sin\left(\left(\sigma'-\rho\right)d\right)}{\left(\sigma'-\rho\right)}\right)\frac{k_0^2\left(\epsilon_r-1\right)}{\left(\rho'^2-\rho^2\right)}\\
&+2B^r\left(\rho'\right)B^t\left(\rho\right)\sigma'\left(\epsilon_r-1\right)\frac{sin\left(\sigma'd\right)cos\left(\rho d\right)}{\left(\rho'^2-\rho^2\right)\epsilon_r}
\end{aligned}
$$

8.3 Integration Identity to remove singularities

Let's consider the following integral to evaluate:

$$
\begin{aligned}
\int_0^\infty \frac{H(\rho,\rho')}{\rho'^2-\rho^2}d\rho' =& \int_0^\infty \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'\\
&+H(\rho,\rho)\int_0^\infty \frac{1}{\rho'^2-\rho^2}d\rho'\\
=& \int_0^\infty \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'\\
&+H(\rho,\rho)\int_0^\infty \frac{1}{\rho'^2-\rho^2}d\rho'\\
=& \int_0^\infty \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'\\
&+H(\rho,\rho)\int_{-\rho}^\infty \frac{dx}{x\left(x+2\rho\right)}
\end{aligned}
$$

Since:

$$
\begin{aligned}
\int_{-\rho}^{\infty} \frac{dx}{x\,(x+2\rho)} &= \frac{1}{2\rho}\int_{-\rho}^{\infty}\left[\frac{1}{x}-\frac{1}{x+2\rho}\right]dx \\
&= \frac{1}{2\rho}\int_{-\rho}^{\rho}\left[\frac{1}{x}-\frac{1}{x+2\rho}\right]dx + \frac{1}{2\rho}\int_{\rho}^{\infty}\left[\frac{1}{x}-\frac{1}{x+2\rho}\right]dx \\
&= \frac{-1}{2\rho}\left[ln\,(x+2\rho)\right]_{-\rho}^{\rho} + \frac{1}{2\rho}\left[ln\left(\frac{x}{x+2\rho}\right)\right]_{\rho}^{\infty} \\
&= -\frac{1}{2\rho}ln\,(3) - \frac{1}{2\rho}ln\left(\frac{1}{3}\right) = 0
\end{aligned}
$$

Therefore:

$$
\int_0^{\infty} \frac{H(\rho,\rho')}{\rho'^2-\rho^2}d\rho' = \int_0^{\infty} \frac{H(\rho,\rho')-H(\rho,\rho)}{\rho'^2-\rho^2}d\rho'
$$

In this way the integrand becomes finite at the singularity. So that, doing the integration numerically, becomes feasible.

9 Recasting q^t and q^r as b and d

We have rederived the formulas in Gelin's paper for the TE Even modes:

$$
\begin{aligned}
q^t &= \frac{1}{2\omega\mu P}\frac{|\beta(\rho)|}{\beta_m+\beta(\rho)}\left\{2\beta_m G_m(\rho) + \sum_n(\beta_m-\beta_n)a_n G_n(\rho) + \int_0^{\infty} q^r(\rho')[\beta_m-\beta(\rho')]F(\rho',\rho)\,d\rho'\right\} \\
a_n &= \frac{1}{4\omega\mu P}\int_0^{\infty} q^t(\rho')[\beta_n-\beta(\rho')]G_n(\rho')\,d\rho' \\
q^r &= \frac{1}{4\omega\mu P}\frac{|\beta(\rho)|}{\beta(\rho)}\int_0^{\infty} q^r(\rho')[\beta(\rho)-\beta(\rho')]F^*(\rho,\rho')\,d\rho'
\end{aligned}
$$

The integrals in these functions have two singularities: one at $\rho' = k_o$ and one at $\rho' = \rho$, both contained inside the function F. The latter of these will ultimately be removed by using Hamid's identity and subtracting the singularity out. Here, we will focus on dealing with the singularity at $\rho = k_o$.

Gelin identifies a method for doing this in his paper, suggesting that the q functions can be rewritten as

$$
\begin{aligned}
q^t(\rho) &= b(\rho)Z(\rho);\text{ and} \\
q^r(\rho) &= d(\rho)Z(\rho) \\
\text{such that } Z(\rho) &= \alpha\frac{\omega\mu}{|\beta(\rho)|}
\end{aligned}
$$

where $\alpha = 1$ for $\rho \leq k_o$ and $\alpha = j$ for $\rho > k_o$. If we solve for $b(\rho)$, we have

$$b(\rho) = \frac{q^t}{Z(\rho)} = \frac{|\beta(\rho)|}{\alpha\omega\mu} \frac{1}{2\omega\mu P} \frac{|\beta(\rho)|}{\beta_m + \beta(\rho)} \Big\{ 2\beta_m G_m(\rho) + \sum_n (\beta_m - \beta_n) a_n G_n(\rho) + \int_0^\infty q^r(\rho')[\beta_m - \beta(\rho')] F(\rho', \rho)\, d\rho' \Big\}$$

We're going to bring the factor of $|\beta(\rho)|^2$ inside the brackets, and attempt to cancel terms inside the overlap integral functions. To this end, let's first focus on the product

$$|\beta(\rho)|^2 G_n(\rho) = |\beta(\rho)|^2 \cdot 2A_n B^t k^2(\epsilon_r - 1) \left\{ \frac{\gamma\cos(\rho d) - \rho\sin(\rho d)}{(\kappa^2 - \rho^2)(\gamma^2 + \rho^2)} \right\}$$