Analysis of Algorithms
CS 477/677

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Lecture 17

Matrix-Chain Multiplication

- Given a chain of matrices $\langle A_1, A_2, \ldots, A_n \rangle$, where for $i = 1, 2, \ldots, n$ matrix A_i has dimensions $p_{i-1} \times p_i$, fully parenthesize the product $A_1 \cdot A_2 \cdots A_n$ in a way that minimizes the number of scalar multiplications.

$$A_1 \quad \cdot \quad A_2 \quad \cdots \quad A_i \quad \cdot \quad A_{i+1} \quad \cdots \quad A_n$$

$$p_0 \times p_1 \quad p_1 \times p_2 \qquad p_{i-1} \times p_i \quad p_i \times p_{i+1} \quad p_{n-1} \times p_n$$

1. The Structure of an Optimal Parenthesization

- Notation:

$$A_{i\ldots j} = A_i \; A_{i+1} \cdots A_j, \; i \le j$$

- For $i < j$:

$$\begin{aligned} A_{i\ldots j} &= A_i \; A_{i+1} \cdots A_j \\ &= A_i \; A_{i+1} \cdots A_k \; A_{k+1} \cdots A_j \\ &= A_{i\ldots k} \; A_{k+1\ldots j} \end{aligned}$$

- Suppose that an optimal parenthesization of $A_{i\ldots j}$ splits the product between A_k and A_{k+1}, where $i \le k < j$

2. A Recursive Solution

$$m[i, j] = m[i, k] + m[k+1, j] + p_{i-1}p_kp_j$$

- We do not know the value of k
  - There are j – i possible values for k: k = i, i+1, …, j-1
- Minimizing the cost of parenthesizing the product $A_i\ A_{i+1} \cdots A_j$ becomes:

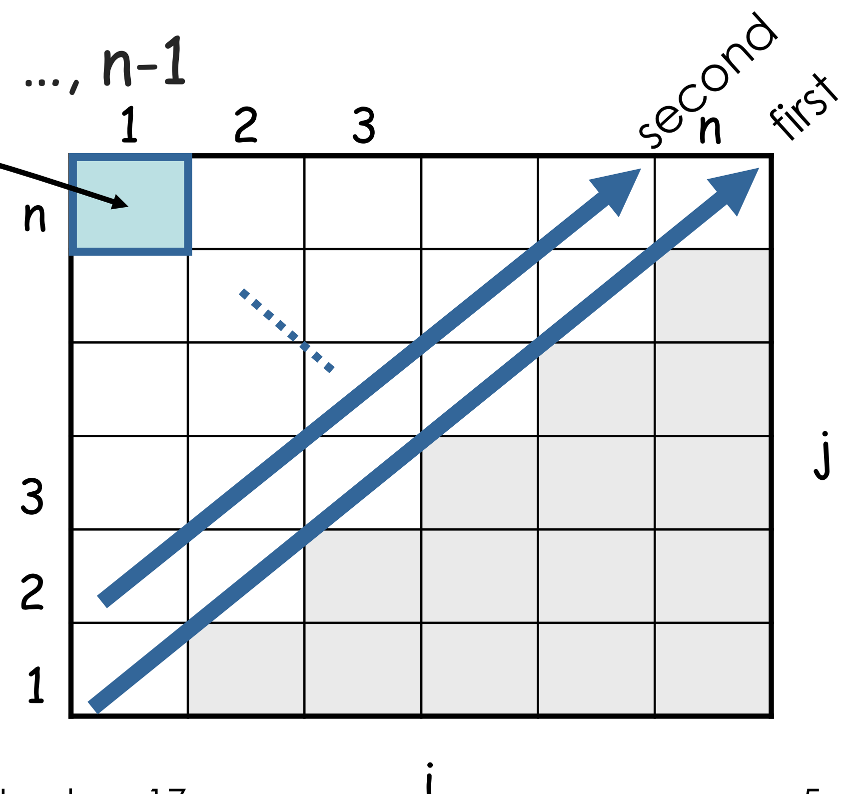
$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

3. Computing the Optimal Costs

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

- Length = 1: $i = j$, $i = 1, 2, \ldots, n$
- Length = 2: $j = i + 1$, $i = 1, 2, \ldots, n-1$

$m[1, n]$ gives the optimal solution to the problem



Compute elements on each diagonal, starting with the longest diagonal.
In a similar matrix **s** we keep the optimal values of **k**.

Memoization

- Top-down approach with the efficiency of typical bottom-up dynamic programming approach
- Maintains an entry in a table for the solution to each subproblem
  - **memoize** the inefficient recursive top-down algorithm
- When a subproblem is first encountered its solution is computed and stored in that table
- Subsequent “calls” to the subproblem simply look up that value

Memoized Matrix-Chain

Alg.: MEMOIZED-MATRIX-CHAIN(**p**)

1. $n \leftarrow$ length[p]
2. **for** i $\leftarrow$ 1 **to** n
3. **do for** j $\leftarrow$ i **to** n
4. **do** m[i, j] $\leftarrow \infty$
5. **return** LOOKUP-CHAIN(p, 1, n) ← Top-down approach

Initialize the **m** table with large values that indicate whether the values of **m[i, j]** have been computed

Memoized Matrix-Chain

Alg.: LOOKUP-CHAIN(p, i, j)

Running time is $O(n^3)$

```
if m[i, j] < ∞
        then return m[i, j]
if i = j
    then m[i, j] ← 0
    else for k ← i to j - 1
                do q ← LOOKUP-CHAIN(p, i, k) +
                       LOOKUP-CHAIN(p, k+1, j) + p_{i-1}p_k p_j
                   if q < m[i, j]
                      then m[i, j] ← q
```

$$m[i, j] = \min_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_k p_j\}$$

Dynamic Progamming vs. Memoization

- Advantages of dynamic programming vs. memoized algorithms
  - No overhead for recursion
  - The regular pattern of table accesses may be used to reduce time or space requirements
- Advantages of memoized algorithms vs. dynamic programming
  - More intuitive

Optimal Substructure - Examples

- Assembly line
  - Fastest way of going through a station **j** contains: the fastest way of going through station **j-1** on either line
- Matrix multiplication
  - Optimal parenthesization of $A_i \cdot A_{i+1} \cdots A_j$ that splits the product between A_k and A_{k+1} contains:
    - an optimal solution to the problem of parenthesizing $A_{i..k}$
    - an optimal solution to the problem of parenthesizing $A_{k+1..j}$

Parameters of Optimal Substructure

- Intuitively, the running time of a dynamic programming algorithm depends on two factors:
  - Number of subproblems overall
  - How many choices we examine for each subproblem
- Assembly line
  - $\Theta(n)$ subproblems (n stations)
  - 2 choices for each subproblem

  $\Theta(n)$ overall
- Matrix multiplication:
  - $\Theta(n^2)$ subproblems ($1 \le i \le j \le n$)
  - At most $n-1$ choices

  $\Theta(n^3)$ overall

Longest Common Subsequence

- Given two sequences

  $X = \langle x_1, x_2, \ldots, x_m \rangle$

  $Y = \langle y_1, y_2, \ldots, y_n \rangle$

  find a maximum length common subsequence (LCS) of X and Y
- *E.g.:*

  $X = \langle A, B, C, B, D, A, B \rangle$
- Subsequence of X:
  - A subset of elements in the sequence taken in order (but not necessarily consecutive)

  $\langle A, B, D \rangle$, $\langle B, C, D, B \rangle$, etc.

Example

$X = \langle A, B, C, B, D, A, B \rangle$ $X = \langle A, B, C, B, D, A, B \rangle$

$Y = \langle B, D, C, A, B, A \rangle$ $Y = \langle B, D, C, A, B, A \rangle$

- $\langle B, C, B, A \rangle$ and $\langle B, D, A, B \rangle$ are longest common subsequences of X and Y (length = 4)
- $\langle B, C, A \rangle$, however is not a LCS of X and Y

Brute-Force Solution

- For every subsequence of X, check whether it's a subsequence of Y
- There are 2^m subsequences of X to check
- Each subsequence takes $\Theta(n)$ time to check
  - scan Y for first letter, from there scan for second, and so on
- Running time: $\Theta(n2^m)$

1. Making the choice

$X = \langle A, B, D, E \rangle$

$Y = \langle Z, B, E \rangle$

- Choice: include one element into the common sequence (E) and solve the resulting subproblem

$X = \langle A, B, \cancel{D}, G \rangle$ $\quad\quad$ $X = \langle A, B, D, G \rangle$

$Y = \langle Z, B, D \rangle$ $\quad\quad$ $Y = \langle Z, B, \cancel{D} \rangle$

- Choice: exclude an element from a string and solve the resulting subproblem

Notations

- Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$ we define the i-th prefix of X, for $i = 0, 1, 2, \ldots, m$

  $$X_i = \langle x_1, x_2, \ldots, x_i \rangle$$

- **c[i, j]** = the length of a LCS of the sequences

  $$X_i = \langle x_1, x_2, \ldots, x_i \rangle \text{ and } Y_j = \langle y_1, y_2, \ldots, y_j \rangle$$

2. A Recursive Solution

Case 1: $x_i = y_j$

e.g.: $X_i = \langle A, B, D, E \rangle$

$Y_j = \langle Z, B, E \rangle$

$$c[i, j] = c[i - 1, j - 1] + 1$$

– Append $x_i = y_j$ to the LCS of X_{i-1} and Y_{j-1}

– Must find a LCS of X_{i-1} and Y_{j-1} $\Rightarrow$ optimal solution to a problem includes optimal solutions to subproblems

2. A Recursive Solution

Case 2: $x_i \neq y_j$

e.g.: $X_i = \langle A, B, D, G \rangle$

$Y_j = \langle Z, B, D \rangle$

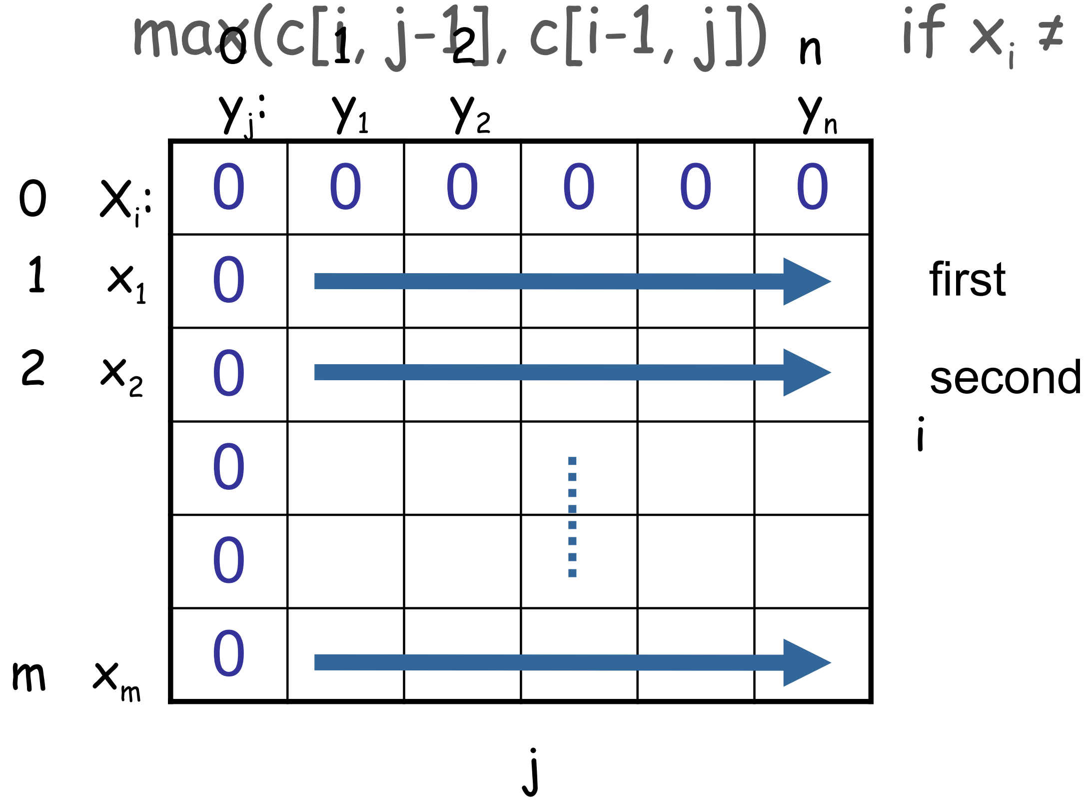
$$c[i, j] = \max \{ c[i - 1, j], c[i, j-1] \}$$

- Must solve two problems
  - find a LCS of X_{i-1} and Y_j: $X_{i-1} = \langle A, B, D \rangle$ and $Y_j = \langle Z, B, D \rangle$
  - find a LCS of X_i and Y_{j-1}: $X_i = \langle A, B, D, G \rangle$ and $Y_j = \langle Z, B \rangle$

- Optimal solution to a problem includes optimal solutions to subproblems

3. Computing the Length of the LCS

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$



4. Additional Information

$$
c[i, j] = \begin{cases} 0 & \text{if } i,j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}
$$

b & c:

		0	1	2	3		n
		y_j	A	C	D		F
0	x_i	0	0	0	0	0	0
1	A	0					
2	B	0			c[i-1,j]		
3	C	0		c[i,j-1]	↑		
		0					
m	D	0					

i

j

A matrix **b[i, j]**:

- For a subproblem [i, j] it tells us what choice was made to obtain the optimal value
- If $x_i = y_j$
  b[i, j] = " ↖ "
- Else, if
  c[i - 1, j] ≥ c[i, j-1]
  b[i, j] = " ↑ "
  else
  b[i, j] = " ← "

LCS-LENGTH(X, Y, m, n)

```
for i ← 1 to m
    do c[i, 0] ← 0
for j ← 0 to n
    do c[0, j] ← 0
for i ← 1 to m
    do for j ← 1 to n
        do if xi = yj
            then c[i, j] ← c[i - 1, j - 1] + 1
                 b[i, j ] ← "↖"
            else if c[i - 1, j] ≥ c[i, j - 1]
                then c[i, j] ← c[i - 1, j]
                     b[i, j] ← "↑"
                else c[i, j] ← c[i, j - 1]
                     b[i, j] ← "←"
return c and b
```

The length of the LCS is zero if one of the sequences is empty

Case 1: $x_i = y_j$

Case 2: $x_i \neq y_j$

Running time: **Θ(mn)**

Example

$X = \langle A, B, C, B, D, A, B \rangle$
$Y = \langle B, D, C, A, B, A \rangle$

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

If $x_i = y_j$

$b[i, j] =$ " ↖ "

Else if

$c[i - 1, j] \geq c[i, j-1]$

$b[i, j] =$ " ↑ "

else

$b[i, j] =$ " ← "

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

4. Constructing a LCS

- Start at **b[m, n]** and follow the arrows
- When we encounter a "↖" in **b[i, j]** $\Rightarrow x_i = y_j$ is an element of the LCS

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

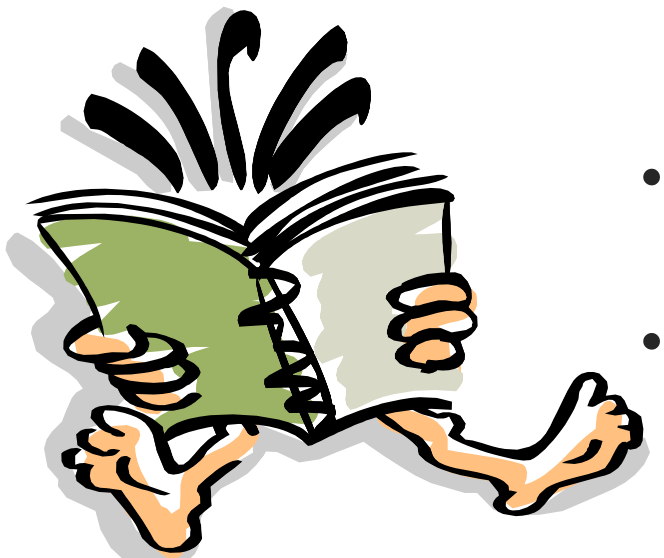
PRINT-LCS(b, X, i, j)

Running time: $\Theta(m + n)$

```
if i = 0 or j = 0
  then return
if b[i, j] = "↖"
      then PRINT-LCS(b, X, i - 1, j - 1)
           print x_i
else if b[i, j] = "↑"
           then PRINT-LCS(b, X, i - 1, j)
           else PRINT-LCS(b, X, i, j - 1)
```

Initial call: PRINT-LCS(b, X, length[X], length[Y])

Readings



- For this lecture
  - Chapter 14
- Coming next
  - Chapter 14