Analysis of Algorithms
CS 477/677

Instructor: Monica Nicolescu

Lecture 14

Augmenting Data Structures

- Let’s look at two new problems:
  - Dynamic order statistic
  - Interval search
- It is unusual to have to design all-new data structures from scratch
  - Typically: store additional information in an already known data structure
  - The augmented data structure can support new operations
- We need to correctly maintain the new information without loss of efficiency

Dynamic Order Statistics

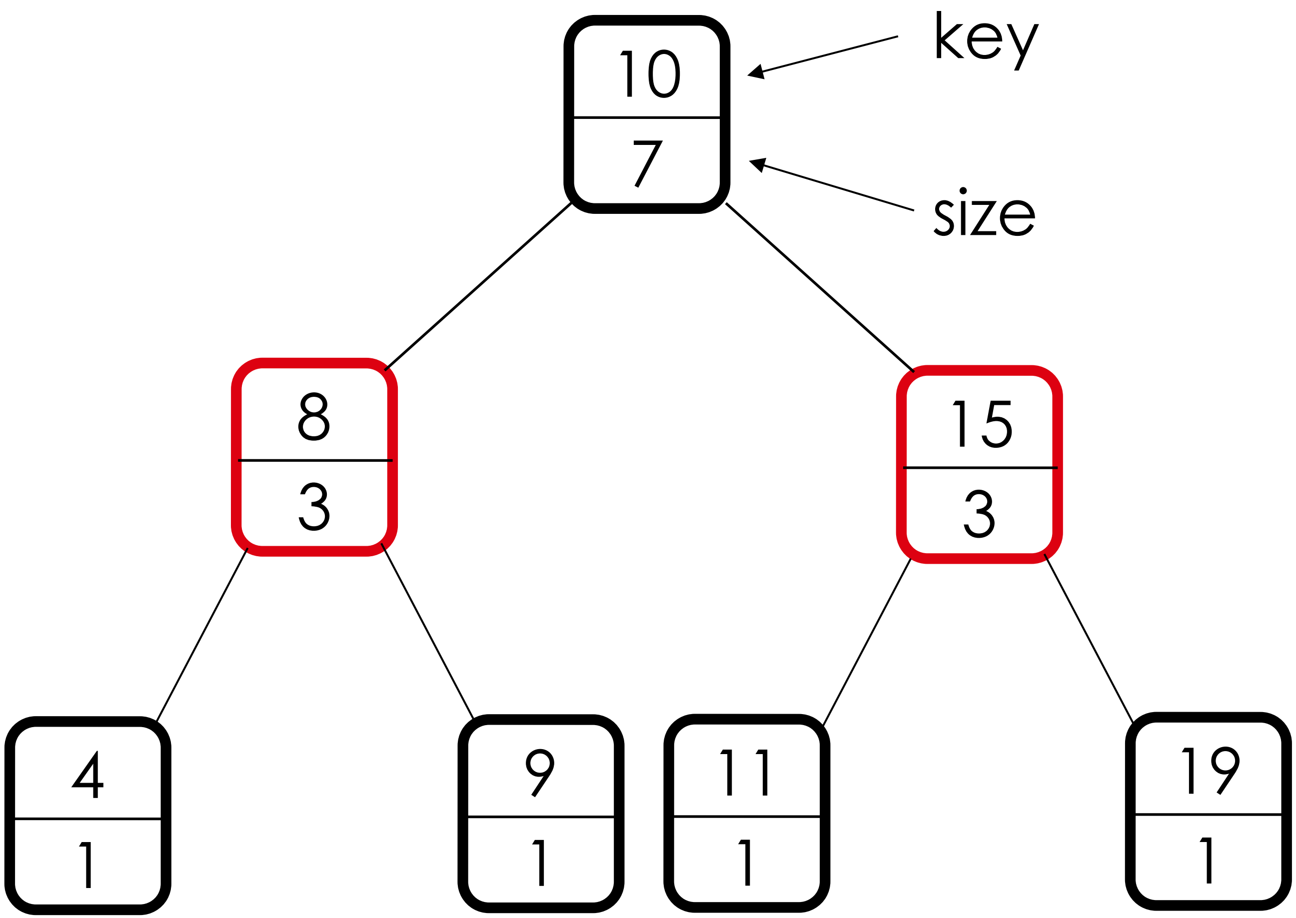
- *Def.:* the **i**-th order statistic of a set of **n** elements, where i ∈ {1, 2, …, n} is the element with the **i**-th smallest key.
- We can retrieve an order statistic from an unordered set:
  - Using: RANDOMIZED-SELECT
  - In: O(n) time
- We will show that:
  - With red-black trees we can achieve this in ***O(lgn)***
  - Finding the **rank** of an element takes also ***O(lgn)***

Order-Statistic Tree

- *Def.:* **Order-statistic tree:** a red-black tree with additional information stored in each node
- Node representation:
  - Usual fields: key[x], color[x], p[x], left[x], right[x]
  - Additional field: size[x] that contains the number of (internal) nodes in the subtree rooted at x (including x itself)
- For any internal node of the tree:

$$size[x] = size[left[x]] + size[right[x]] + 1$$

Example: Order-Statistic Tree



OS-SELECT

Goal:

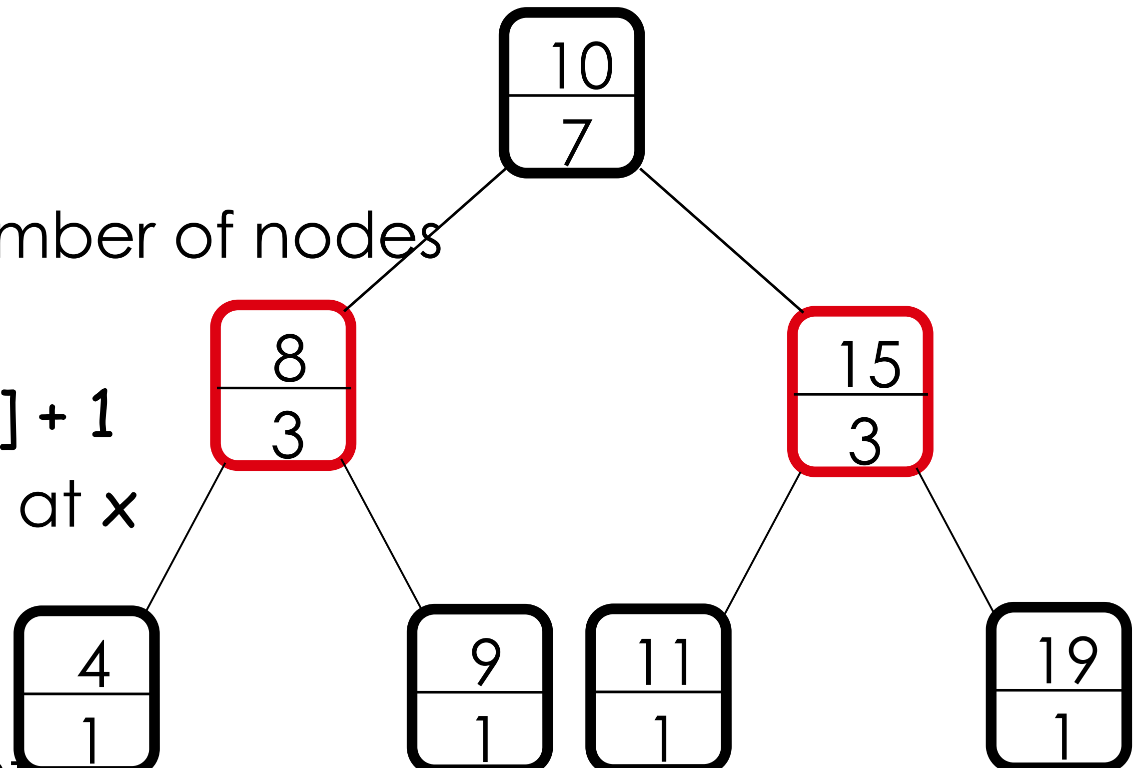
- Given an order-statistic tree, return a pointer to the node containing the **i**-th smallest key in the subtree rooted at **x**

Idea:

- **size[left[x]]** = the number of nodes

that are smaller than **x**

- **rank'[x] = size[left[x]] + 1** in the subtree rooted at **x**
- If **i = rank'[x]** Done!
- If **i < rank'[x]**: look left
- If **i > rank'[x]**: look right



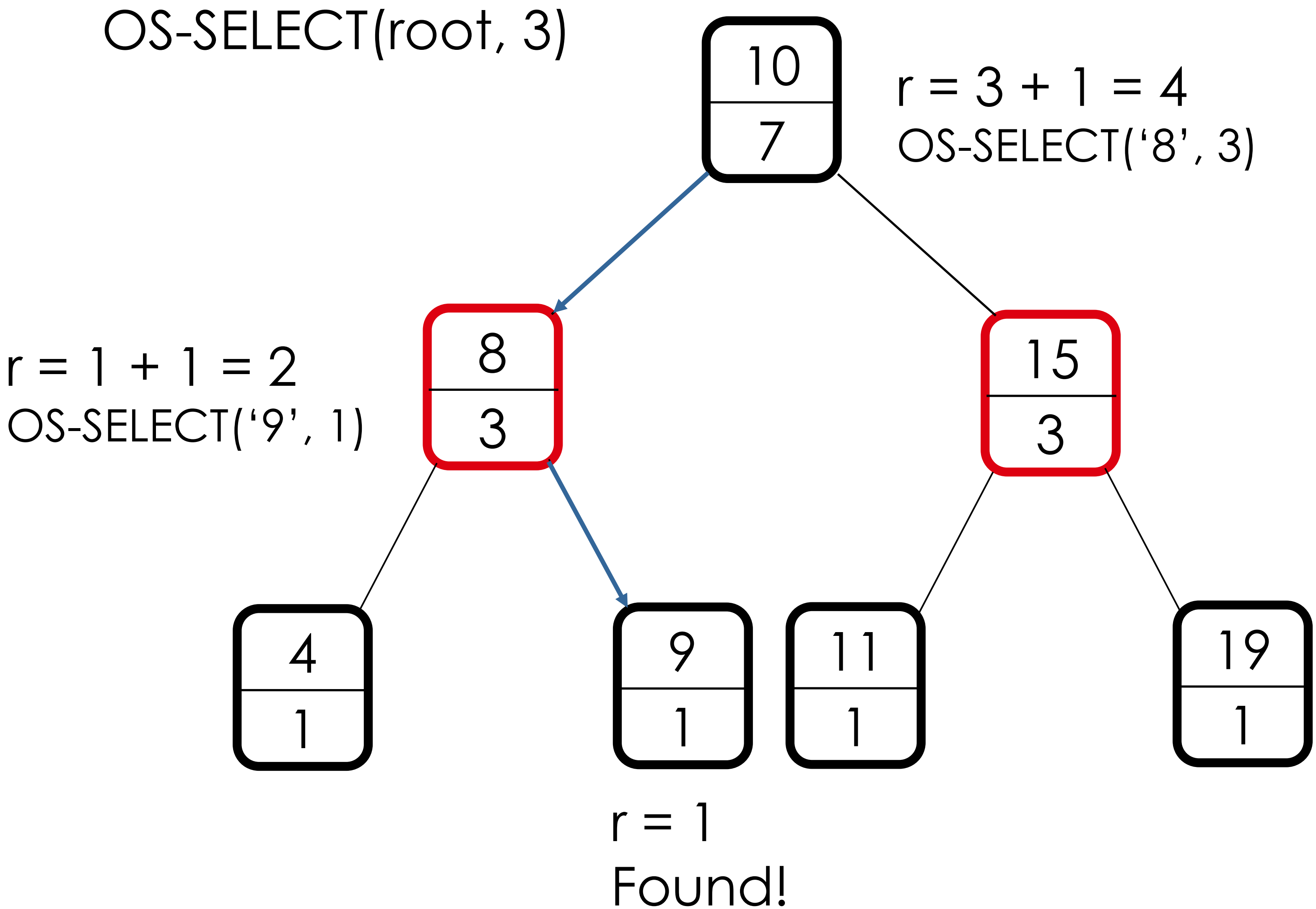
OS-SELECT(*x*, *i*)

1. $r \leftarrow size[left[x]] + 1$ ► compute the rank of x within the subtree rooted at x
2. **if** $i = r$
3. **then return** x
4. **elseif** $i < r$
5. **then return** OS-SELECT($left[x]$, i)
6. **else return** OS-SELECT($right[x]$, $i - r$)

Initial call: OS-SELECT($root[T]$, i)

Running time: $O(\lg n)$

Example: OS-SELECT



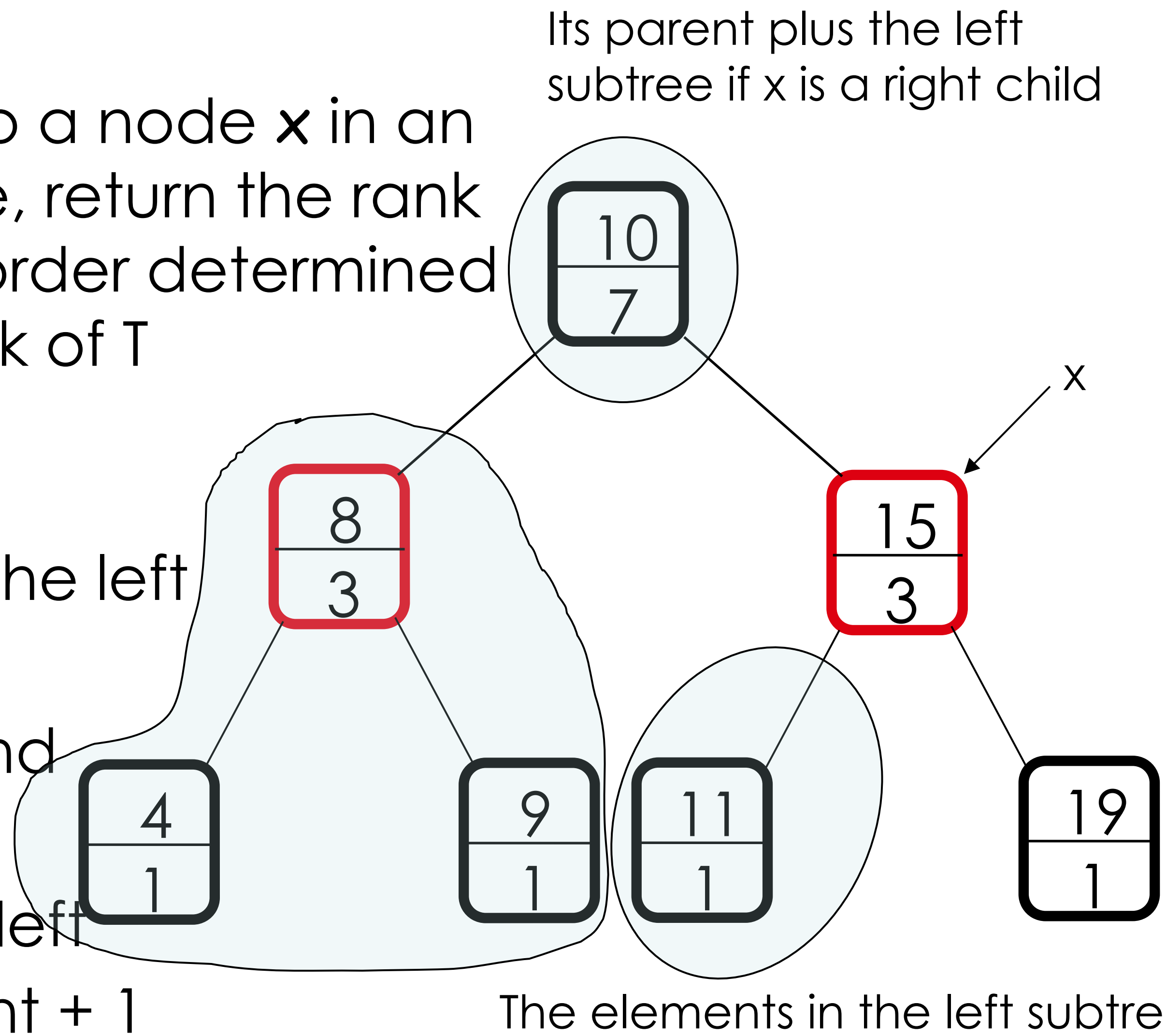
OS-RANK

Goal:

- Given a pointer to a node **x** in an order-statistic tree, return the rank of **x** in the linear order determined by an inorder walk of T

Idea:

- Add elements in the left subtree
- Go up the tree and if a right child: add the elements in the left subtree of the parent + 1



OS-RANK(T, x)

1. $r \leftarrow size[left[x]] + 1$

   Add to the rank the elements in its left subtree + 1 for itself

2. $y \leftarrow x$

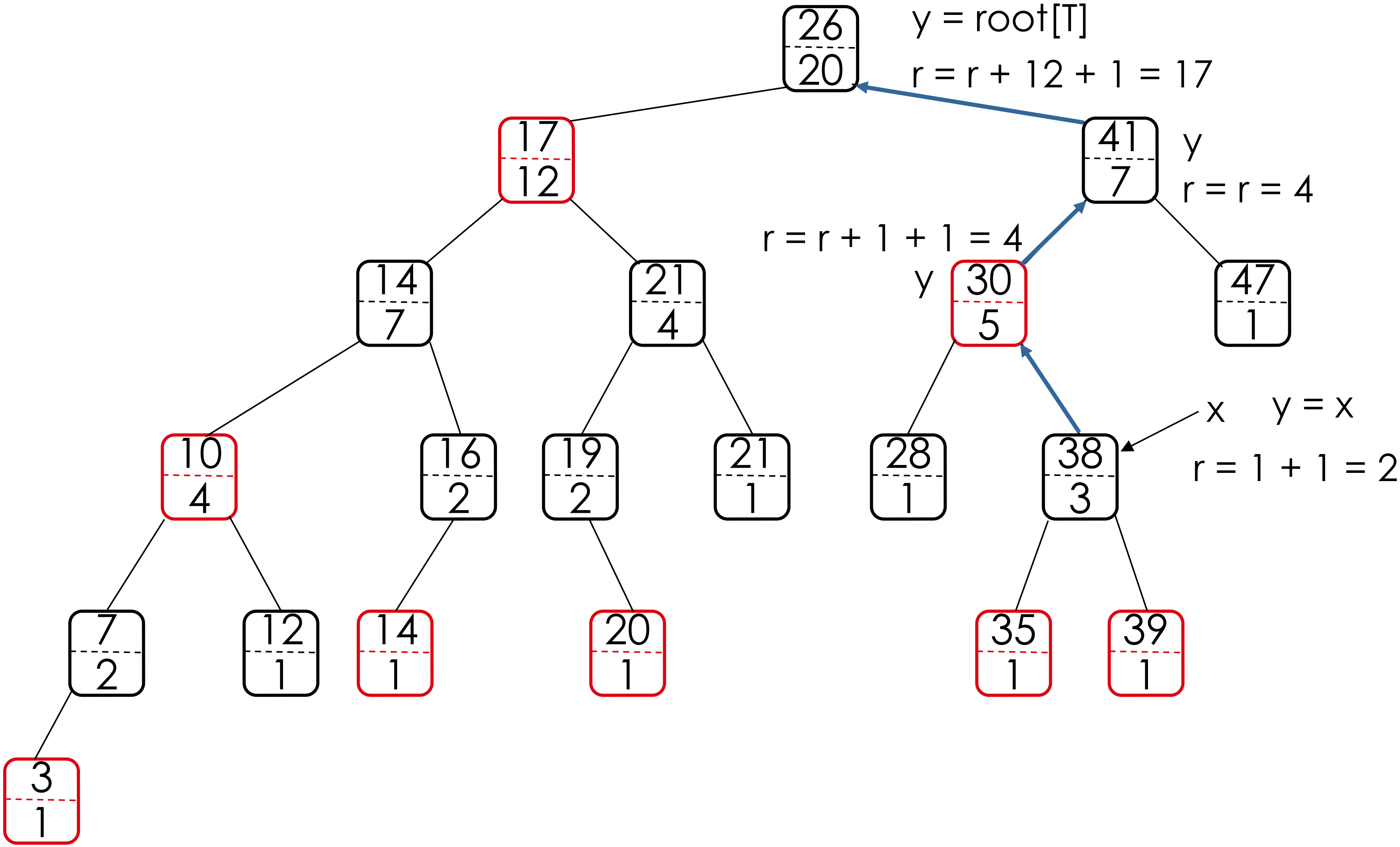
   Set y as a pointer that will traverse the tree

3. **while** $y \neq root[T]$
4. **do if** $y = right[p[y]]$
5. **then** $r \leftarrow r + size[left[p[y]]] + 1$
6. $y \leftarrow p[y]$
7. **return** r

If a right child add the size of the parent's left subtree + 1 for the parent

Running time: O(lgn)

Example: OS-RANK



Maintaining Subtree Sizes

- We need to maintain the ***size*** field during INSERT and DELETE operations
- Need to maintain them efficiently
- Otherwise, might have to recompute all ***size*** fields, at a cost of $\Omega(n)$

Maintaining **Size** for OS-INSERT

- Insert in a red-black tree has two stages

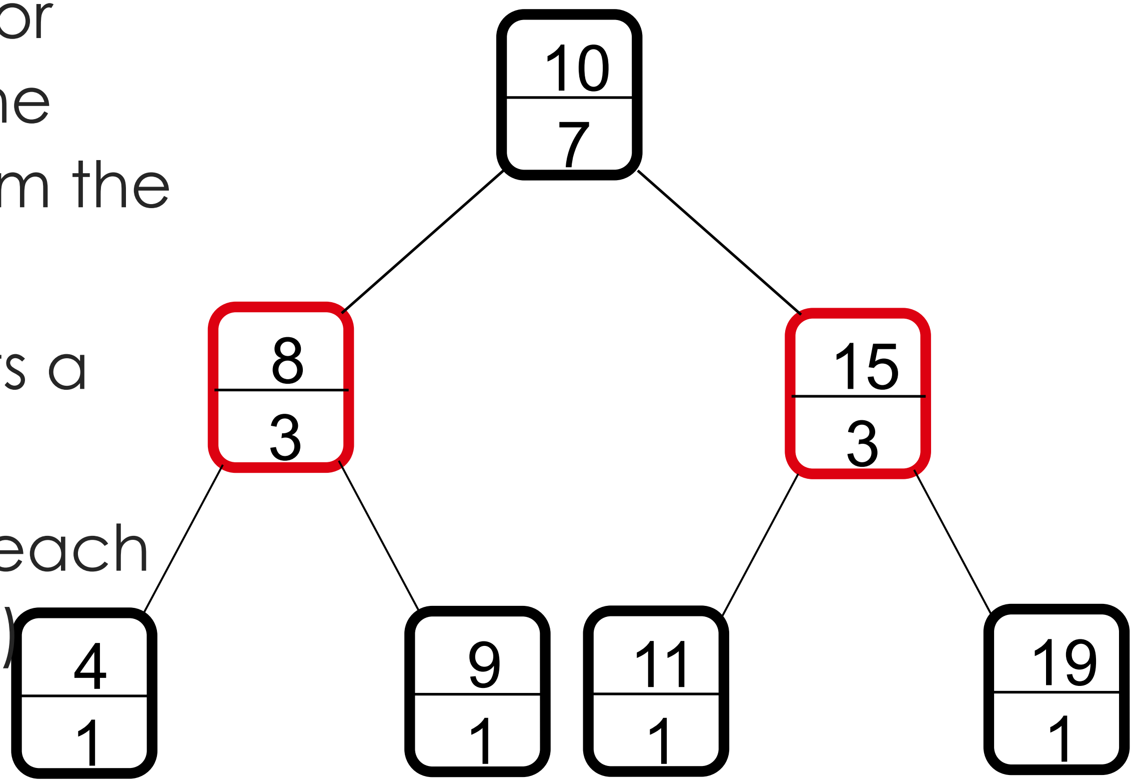
  1. Perform a binary-search tree insert
  2. Perform rotations and change node colors to restore red-black tree properties

OS-INSERT

Idea for maintaining the ***size*** field during insert

Phase 1 (going down):

- Increment size[x] for each node x on the traversed path from the root to the leaves
- The new node gets a size of 1
- Constant work at each node, so still O(lgn)

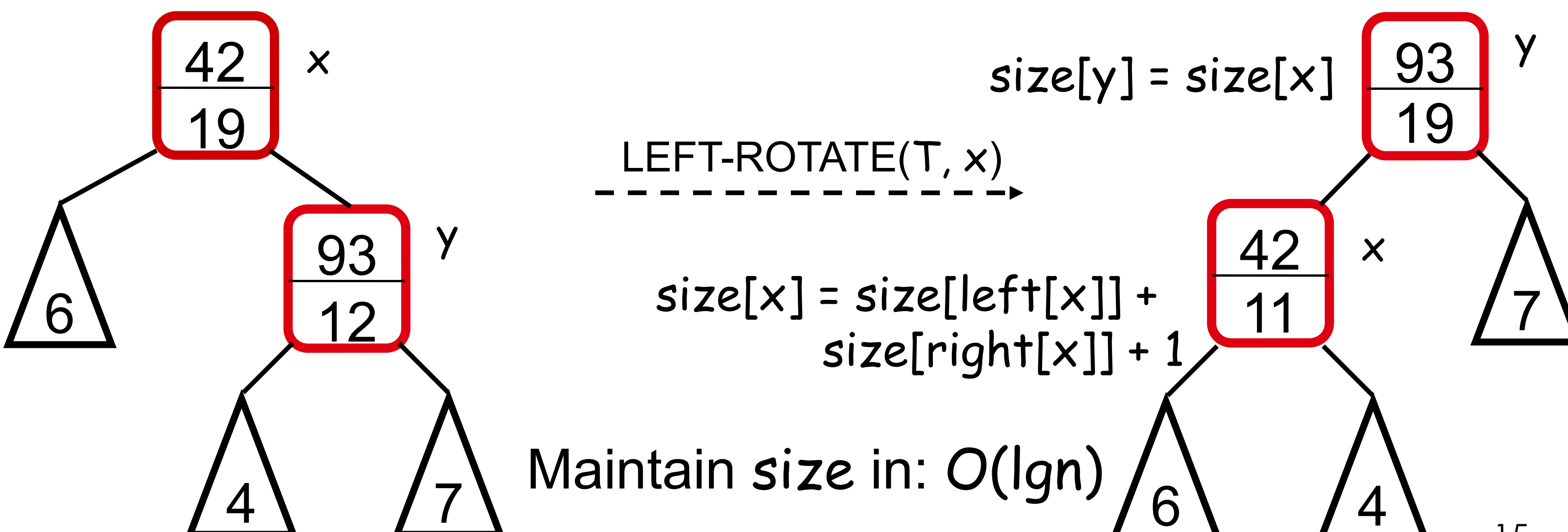


OS-INSERT

Idea for maintaining the ***size*** field during insert

Phase 2 (going up):

- During RB-INSERT-FIXUP there are:
  - $O(\lg n)$ changes in node colors
  - At most two rotations   Rotations affect the subtree sizes !!



Maintain ***size*** in: $O(\lg n)$

Augmenting a Data Structure

1. Choose an underlying data structure

   ⇒ Red-black trees

2. Determine additional information to maintain

   ⇒ size[x]

3. Verify that we can maintain additional information for existing data structure operations

   ⇒ Shown how to maintain size during modifying operations

4. Develop new operations

   ⇒ Developed OS-RANK and OS-SELECT

Augmenting Red-Black Trees

Theorem: Let **f** be a field that augments a red-black tree. If the contents of **f** for a node can be computed using only the information in **x**, **left[x]**, **right[x]** ⇒ we can maintain the values of **f** in all nodes during insertion and deletion, without affecting their ***O(lgn)*** running time.

Examples

1. Can we augment a RBT with **size[x]**?

   Yes: **size[x] = size[left[x]] + size[right[x]] + 1**

2. Can we augment a RBT with **height[x]**?

   Yes: **height[x] = 1 + max(height[left[x]], height[right[x]])**

3. Can we augment a RBT with **rank[x]**?

   No, inserting a new minimum will cause all **n** rank values to change

Interval Trees

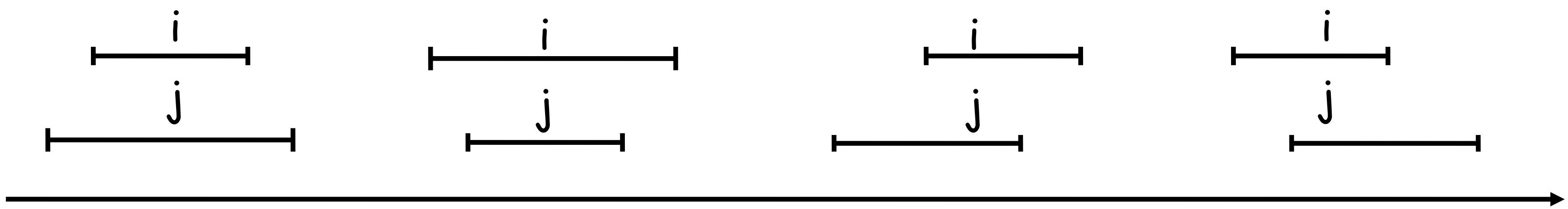
Def.: **Interval tree** = a red-black tree that maintains a dynamic set of elements, each element **x** having associated an interval $int[x]$.

- Operations on interval trees:
  - INTERVAL-INSERT(T, x)
  - INTERVAL-DELETE(T, x)
  - INTERVAL-SEARCH(T, i)

Interval Properties

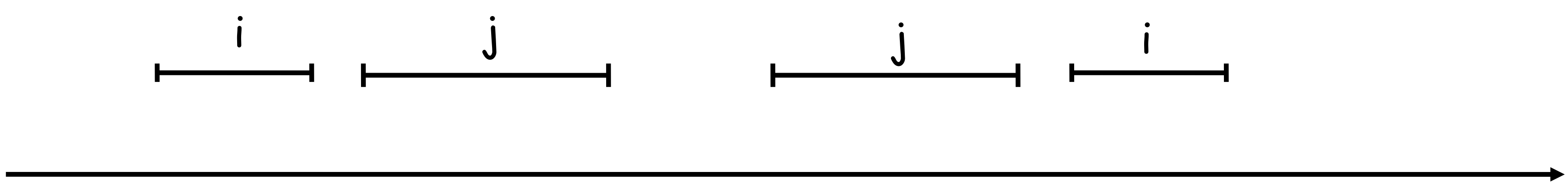
- Intervals i and j overlap iff:

  low[i] ≤ high[j] and low[j] ≤ high[i]



- Intervals i and j do not overlap iff:

  high[i] < low[j] or high[j] < low[i]



Interval Trichotomy

- Any two intervals **i** and **j** satisfy the **interval trichotomy**: exactly one of the following three properties holds:

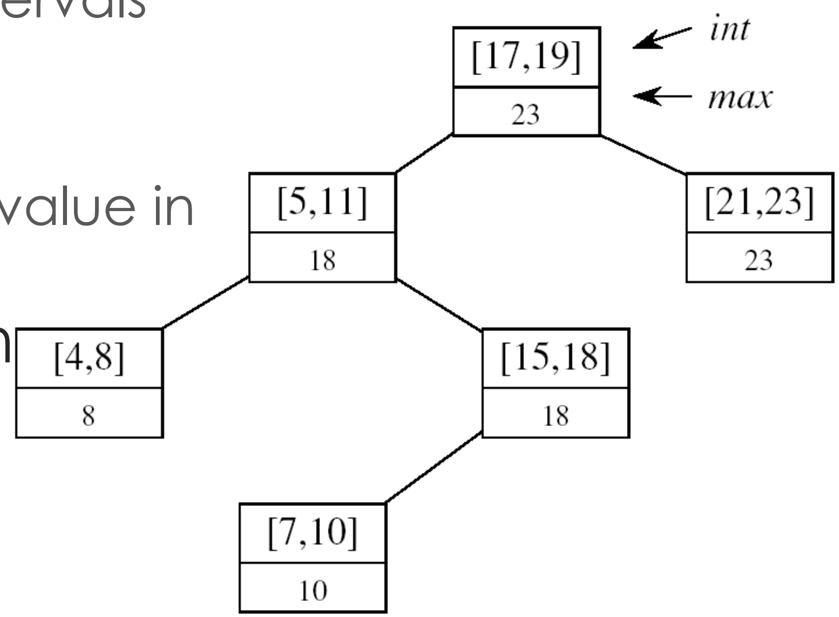
a) **i** and **j** overlap,

b) **i** is to the left of **j** (**high[i] < low[j]**)

c) **i** is to the right of **j** (**high[j] < low[i]**)

Designing Interval Trees

1. Underlying data structure
   – Red-black trees
   – Each node **x** contains: an interval int[x], and the key: low[int[x]]
   – An inorder tree walk will list intervals sorted by their low endpoint
2. Additional information
   – max[x] = maximum endpoint value in subtree rooted at **x**
3. Maintaining the information



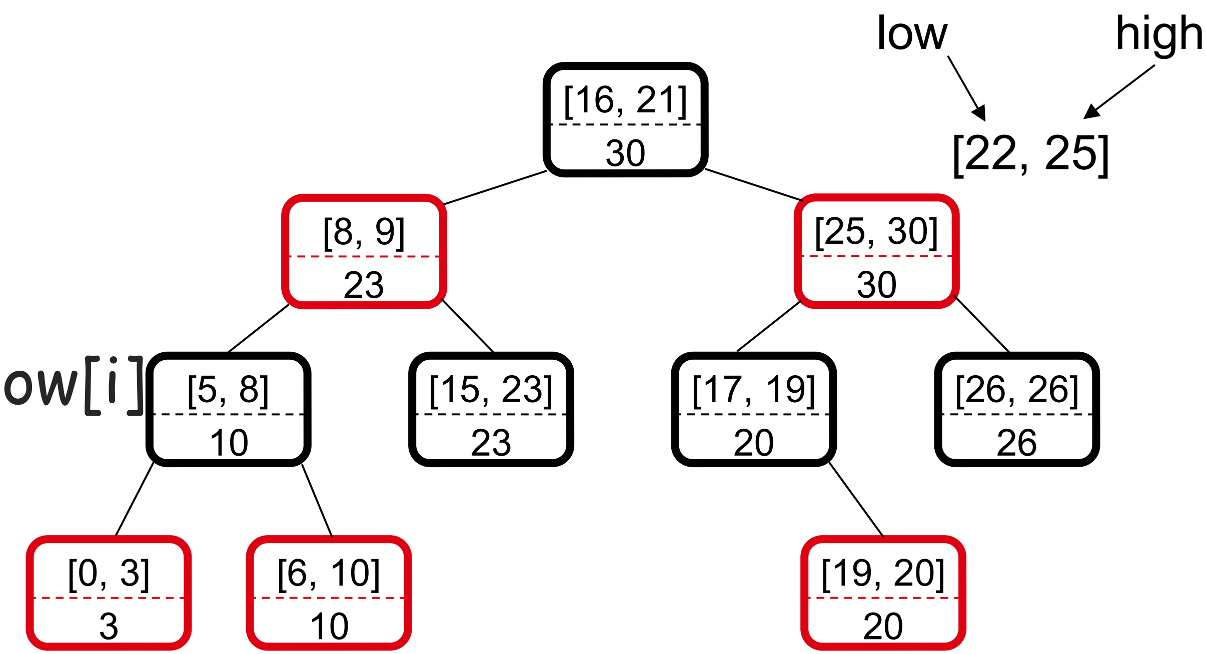
$$\text{max}[x] = \max \begin{cases} \text{high}[\text{int}[x]] \\ \text{max}[\text{left}[x]] \\ \text{max}[\text{right}[x]] \end{cases}$$

Constant work at each node, so still $O(\lg n)$ time

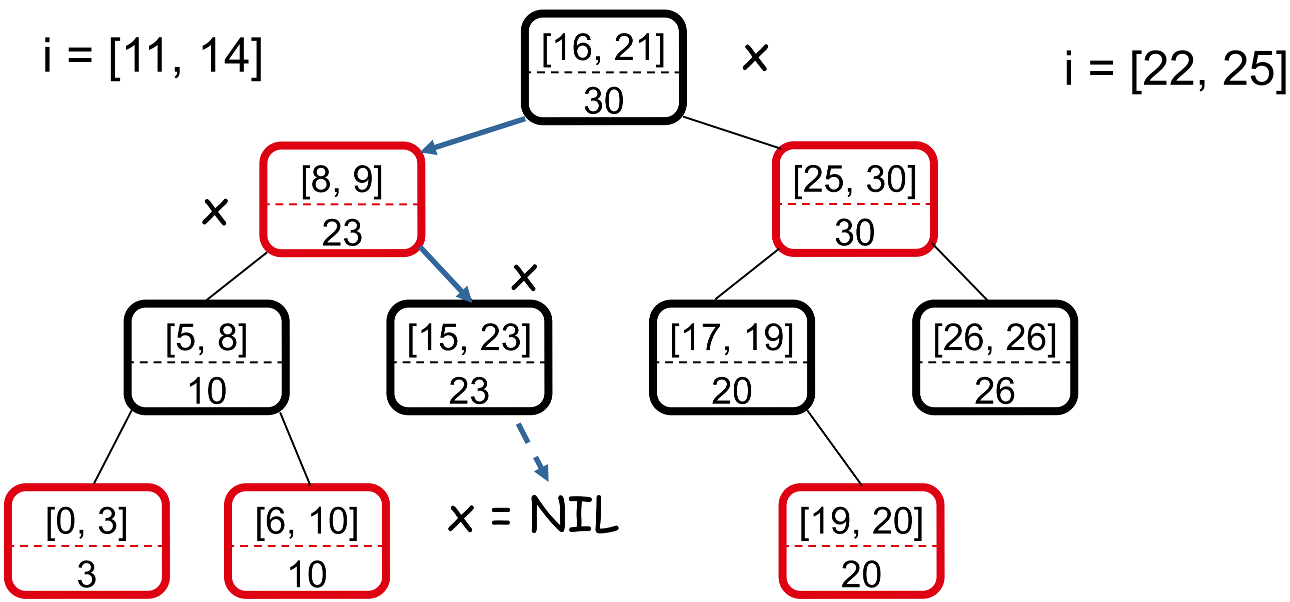
Designing Interval Trees

4. Develop new operations

- INTERVAL-SEARCH(T, i):
  - Returns a pointer to an element **x** in the interval tree T, such that **int[x]** overlaps with **i**, or **NIL** otherwise
- Idea:
- Check if **int[x]** overlaps with i
- **Max[left[x]] ≥ low[i]**
  - Go left
- Otherwise, go right



Example



i = [11, 14]
i = [22, 25]
[16, 21]
30
x
[8, 9]
23
x
[25, 30]
30
x
[5, 8]
10
[15, 23]
23
[17, 19]
20
[26, 26]
26
[0, 3]
3
[6, 10]
10
x = NIL
[19, 20]
20

INTERVAL-SEARCH(T, i)

1. $x \leftarrow root[T]$
2. **while** $x \neq nil[T]$ and i does not overlap $int[x]$
3. **do if** $left[x] \neq nil[T]$ and
   $max[left[x]] \geq low[i]$
4. **then** $x \leftarrow left[x]$
5. **else** $x \leftarrow right[x]$
6. **return** x

Theorem

At the execution of interval search: if the search goes right, then either:
- There is an overlap in right subtree, or
- There is no overlap in either subtree

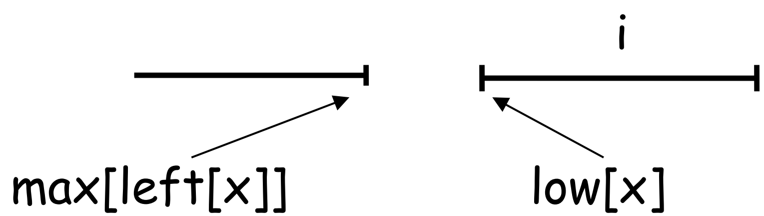
- Similar when the search goes left
- It is safe to always proceed in only one direction

Theorem

- **Proof:** If search goes right:
  - If there is an overlap in right subtree, done
  - If there is no overlap in right ⇒ show there is no overlap in left
  - Went right because:

left[x] = nil[T] ⇒ no overlap in left, or

max[left[x]] < low[i] ⇒ no overlap in left

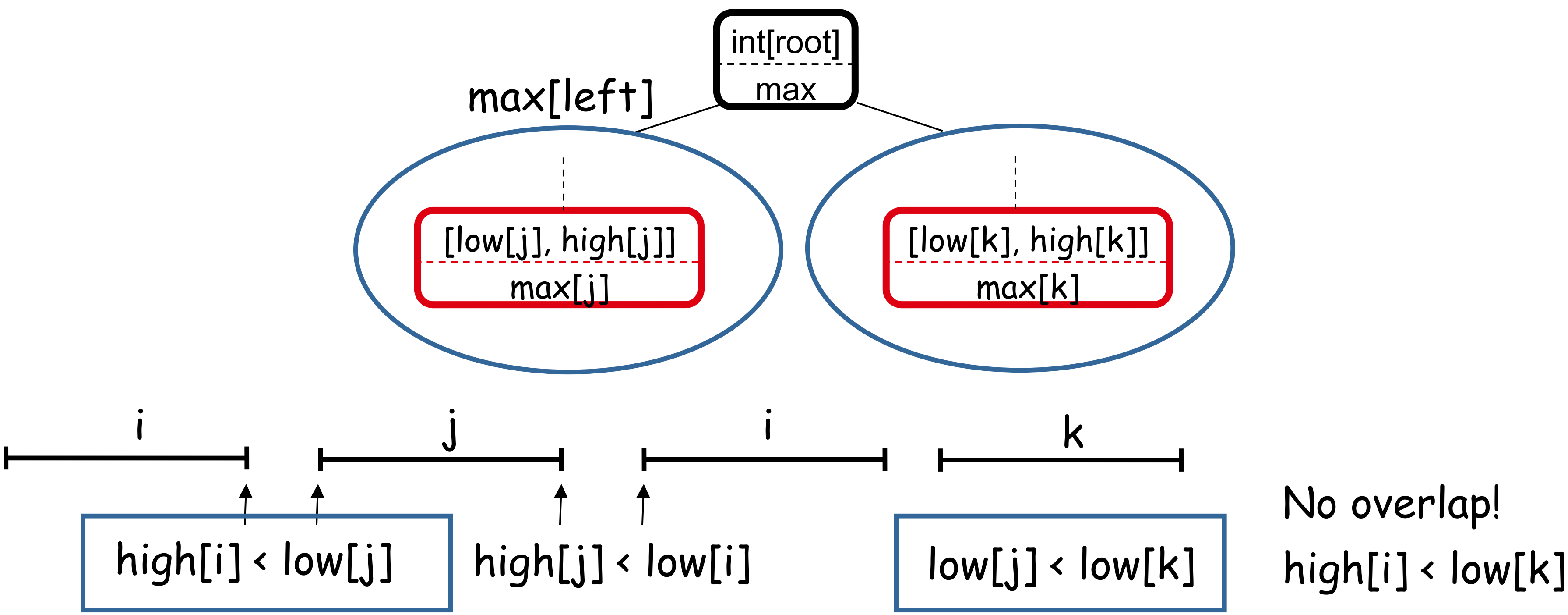


Theorem - Proof

If search goes left:

- If there is an overlap in left subtree, done
- If there is no overlap in left, show there is no overlap in right
- Went left because:

  low[i] ≤ max[left[x]] = high[j] for some j in left subtree



Material up to this point included in the second midterm

MID-TERM 2

Second Midterm Exam

- Tuesday, **April 4** in class (note the day change)
- 75 minutes
- Exam structure:
  - TRUE/FALSE questions
  - short questions on the topics discussed in class
  - homework-like problems

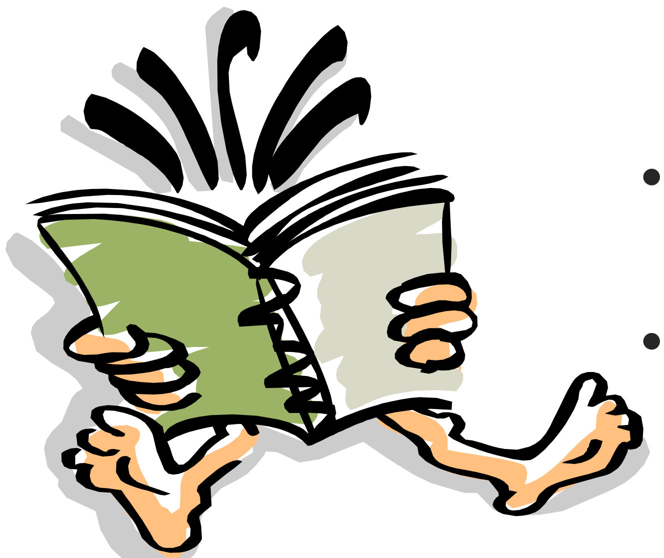
Topics

- All topics from midterm 1 up to dynamic programming
  - Randomized quicksort
  - Probability background
  - The selection problem
  - Sorting in linear time
  - Heaps
  - Augmenting data structures (RBT, OS-Trees, interval trees)

General Advice for Study

- **Understand** how the algorithms are working
  - Work through the examples we did in class
  - "Narrate" for yourselves the main steps of the algorithms in a few sentences
- Know **when** or **for what problems** the algorithms are applicable
- **Do not memorize** algorithms

Readings



- For this lecture
  - Chapter 17, 14
- Coming next
  - Sections 14.2-14.4