A model of optimal extraction and site rehabilitation*

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Abstract

Environmental issues during and after production are a major issue in contemporary exhaustible resource production. Production deteriorates the state of the environment and is a source of possibly harmful emissions. After the production has ceased, the production site is in need of rehabilitation and clean-up. This paper analyzes the last two stages of exhaustible resource production: extraction and site rehabilitation decisions. The socially optimal regulation is investigated, and it is found that a pollution tax, a shut-down date and a requirement for the firm to deposit funds for costly rehabilitation with a profit tax are enough for the socially optimal extraction profile. It is also found that it is irrelevant from the social point of view when the firm pays the funds.

Keywords: Dynamic optimization; Emission tax; Environmental policy; Exhaustible resources; Reclamation; Rehabilitation; Stock pollution; Waste

JEL codes: Q30, Q38, Q50, Q58.

1 Introduction

Exhaustible resource producers are in principle required to pay the costs related to site rehabilitation that occurs after the production has shut-down. The purpose of site rehabilitation operations is to clean-up the the site and return it to alternative uses. The current problem is that the required payment is often set too low, and therefore the incentives to rehabilitate the site are small. This has not gone unnoticed in newspapers around the world, including the United Kingdom (Monbiot, 2015), Australia (Seccombe, 2014), the United States (Preston, 2017) and Canada (Hoekstra, 2017).[1] Regarding the

***Acknowledgements:** I would like to thank Olli Tahvonen, Markku Ollikainen, Margaret Insley, Robert Cairns, Nahid Masoudi and the participants of WCERE 2018 conference, NAERE 2018 workshop, CIREQ workshop and Ca' Foscari University economics department seminar for useful comments and discussions.

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‡This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant Agreement N. 748066

[1] Different types of instruments, such as bonds and trust payments, are used in an attempt to incentivize rehabilitation. The required payments are often insufficient to cover the estimated costs. For

low bonding requirements in Canada, the Energy and Mines Minister of British Columbia Bill Bennett stated in 2017 that

> *"We are going to have to establish some concrete, specific, measurable objectives and principles or parameters - whatever you want to call it - on how do we actually assess the amount of financial security we need to have full assurance for the taxpayer."*

The objective of this paper is to formalize this need as an optimization model with resource extraction decision followed by site rehabilitation and clean-up.

Site rehabilitation and externalities related to the environment are relevant for almost any exhaustible resource production. Extracting an exhaustible resource such as oil, gas or minerals from the ground requires moving large quantities of material, and the refinement of the end product produces wastes. In Canada, most of the oil reserves are in tar sands (oil sands) where the oil is often obtained by extracting and refining the bitumen found in sandstone. The resource is extracted by surface mining, which causes loss of land and biodiversity. In addition, refining oil generates wastes, which are stored in the constructed tailings bonds. After the production site has been shut-down, these tailings bonds remain unless suitable rehabilitation operations are conducted at the site. The problem is that the tailings bonds contain contaminants and represent a hazard to the environment and to the public health (Heyes *et al.*, 2018).

In the U.S, shale oil and gas (tight oil and gas) extraction using hydraulic fracturing and horizontal drilling is a source of contaminants such as uranium, lead, salt and methane. The environmental problems remain after shut-down unless rehabilitation is conducted. This is also true for hard rock mining, in which the main environmental problem is acid mine drainage (AMD), that is, the flow of toxic substances from the site that have been released by acidic waters (Dold, 2014). In modern mining, large quantities of waste rocks and tailings are produced, and this is a costly challenge for eventual mine area rehabilitation (Mudd, 2010). A particularly harmful method of surface mining is the so-called mountaintop mining: Coal seams are extracted by removing the top of

example in British Columbia, the cost estimate is over one billion dollars higher than the collected securities (Hoekstra, 2017).

a mountain or a hill and by dumping of spoils to adjacent streams and rivers, which results in environmental impacts and possible health issues (Palmer *et al.*, 2010). According to Palmer *et al.* (2010), the problems related to mountaintop mining are the loss of ecosystems, release of toxins (for example SO_4 and Se, of which some persist and some bio-accumulate), groundwater problems, release of airborne toxins and the loss of CO_2 storage.

Research questions and the model. In some of the above examples the environmental problems occur during and after the extraction of the resource. In others, as with AMD, the environmental problems manifest mainly after the production has been shut-down (Dold, 2014). The purpose of this study is to model the socially optimal production of the resource and the costly rehabilitation and clean-up of the site, and to analyze the regulation needed to internalize the externality. The research questions are: 1. If the rehabilitation operation is financed using a rehabilitation trust, how large is the optimal deposit by the producer to the rehabilitation trust? 2. What is the optimal tax on pollution generation? 3. When should the production operation be shut-down? 4. Should there be any regulation regarding the producer's payments, that is, does it matter when the producer pays the monies?

These questions are investigated in a two-stage model. During the production or extraction stage, the socially optimal extraction rate and the shut-down date are chosen. At this stage, the model is analyzed under multiple cost structures, which all have been applied in the literature. The production model itself is used to describe AMD, in which the pollution stock is built during the production stage, but which causes damages only after the production shut-down. The production model is also analyzed in the more general setting with a more general pollution stock dynamics including natural cleaning processes and with a stock that causes damages during the production stage.[2] After the production stage, the rehabilitation stage begins and the site is optimally rehabilitated. The pollution stock causes damages during this stage and the rehabilitation is costly. To implement the socially optimal allocation, the regulator sets a time-dependent pollution tax, a shut-down date and a requirement that the producer pays the eventual rehabilita-

[2] These natural cleaning processes include for example immobilization of the pollutants over time.

tion costs by depositing money to a rehabilitation trust.[3] These monies the producer is assumed to transfer to the trust from its operational profits.

<u>Contribution to the literature.</u> The study contributes to the literature of polluting exhaustible resources extraction by analyzing optimal regulation in a two-stage model. The model is "a full description" of the last two stages of contemporary exhaustible resource exploitation, namely of extraction and rehabilitation. Exploration and discovery are the first stages of the exploitation process, but as environmental issues and site rehabilitation are currently important, and because of the logical process of thinking optimality from the final stage backwards, this study aims to partly complete the picture of optimal use of polluting exhaustible resources by investigating the last two stages of exploitation. Hence, the study is related to the literature on optimal use of exhaustible resources and to the literature on optimal environmental policy applied to polluting exhaustible resources. The theoretical literature on exhaustible resource production is vast. It includes the early studies on the optimal use of exhaustible resources (Dasgupta and Heal, 1974; Hartwick, 1978), on exploration (Pindyck, 1978) and on imperfect competition (Stiglitz, 1976; Salant, 1976; Pindyck, 1987). The literature has expanded along these lines and in many others. For example the literature on mining has investigated the optimal capacity choice (Campbell, 1980; Lozada, 1993; Cairns, 2001; Holland, 2003). All of this is important, but in contemporary exhaustible resource production environmental externalities play a major role. One strand of literature along this line includes studies investigating the optimal carbon tax on a polluting exhaustible resource (Ulph and Ulph, 1994; Hoel and Kverndokk, 1996; Tahvonen, 1997).

Studies related specifically to mining and environmental policy include Stollery (1985), Roan and Martin (1996), Cairns (2004), Sullivan and Amacher (2009), Farzin (1996) and White *et al.* (2012). As mentioned in the beginning, the production sites are often in need of rehabilitation when they are shut-down.[4] Therefore the paper by Roan and Martin is

[3]Therefore, the regulation includes two additional instruments on top of the pollution tax. The first one is the reclamation payment paid by the firm. It is included since without it the regulator must pay the reclamation using tax payer money. The second one is the date at which the firm must shut-down the production facility, and it must be included since without it, when a fixed shut-down date is applied, the firm's choice may not produce the social optimum. However, as the tax is defined on the socially optimal production interval and the rehabilitation payment is lump-sum, it can also be said that there is only one instrument: a time-dependent tax with a lump-sum component.

[4]In this study rehabilitation, reclamation and clean-up of the production site are used as synonyms.

important for the current study.[5] They analyze the reclamation and production decision of a mine under command and control regulation with a model, where the reclamation is conducted during the production process by reclaiming some of the waste. However, the reclamation is usually performed after the site has been shut-down.[6] This is the reason why this study uses a two-stage model, in which the production stage is followed by the rehabilitation stage. As the description of the rehabilitation stage, two models are applied: first, the clean-up model of Lappi (2018), which allows waiting with the clean-up; and second, the clean-up model of Caputo and Wilen (1995) in which the pollution or waste stock is cleaned continuously as time goes on.[7]

Policy implications. Optimal regulation of a polluting exhaustible resource firm must guide the producer to exploit the resource optimally, to shut-down the site at the optimal date, to leave optimal amount of pollution to the site after production shut-down and to deposit a sufficient amount of money to a rehabilitation trust. In principle, the results give a formula to calculate the socially optimal deposit to the rehabilitation trust. Mitchell and Casman (2011) state that the payment to the so-called trust account in Pennsylvania's mining sector is based on the present value reclamation costs, and it is unclear how the damages from the pollution stock are taken into account.[8] Similarly, in Canada and Australia the bond payment by the coal producer is based on the reclamation costs (Cheng and Skousen, 2017). Social optimality, however, calls for reclamation trust payments that are also based on damages and not just costs.

More importantly, the socially optimal reclamation payment must take into account the socially optimal production stage choices that yield the pollution stock or the state of the environment which is to be rehabilitated. It is not sufficient from the social optimum point of view to simply use a pollution tax during the production stage, since then the

[5]Sullivan and Amacher (2009) analyze the socially optimal reclamation decision between forestry and grassland options, but they do not consider pollution generation or the optimal production of the resource. In a recent working paper, Aghakazemjourabbaf and Insley (2018) compare environmental bond and strict liability rule, when the extracting firm must clean-up the production site, but they do not consider the social optimum like done here.

[6]It is sometimes possible to reclaim waste during the production operation. In the paper's model this would mean adding an "abatement" option for the firm. This extension is not modeled here in order to focus on the reclamation operation that occurs at the end.

[7]A third rehabilitation model is briefly analyzed in Appendix A.4.

[8]According to Mitchell and Casman (2011), at the beginning of the mining operation, the producer pays the present value of the reclamation cost multiplied with a volatility premium.

site is not rehabilitated. Neither is it sufficient to require only a rehabilitation payment from the firm, since then extraction rate and pollution stock level are distorted relative to the social optimum. Therefore both a tax with an optimal shut-down date and a rehabilitation payment are needed to internalize the externality.

The study is organized as follows. Section 2 develops the notation and the main assumptions. In Section 3 the two rehabilitation models, and in particular the properties of the problems' value functions, are investigated. The optimal production decision is analyzed in Section 4. Section 5 analyzes the producer's incentives when it is confronted with the optimal regulation. Section 6 concludes the study. Most of the proofs are relocated in the appendices.

2 Notation and assumptions

This section introduces the notation and assumptions for the production stage models and for the rehabilitation stage model that allows waiting with the rehabilitation decision, that is, Lappi's model. The notation for Caputo's and Wilen's model is introduced in Section 3.2. Figure 1 depicts the time-line of the model.

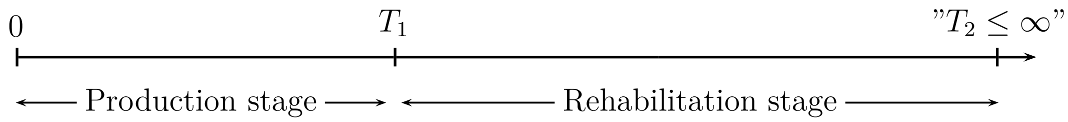

Figure 1: An illustration of the model's time-line. The production stage is followed by the rehabilitation stage. The production stage ends and the rehabilitation stage begins at time T_1. The rehabilitation stage ends at T_2, which may be finite or infinite.

Extraction begins at time zero and continues until the optimally chosen shut-down date T_1. After that the rehabilitation stage commences and continues possibly forever. At the production stage, the optimal extraction rate $q(t)$ and the optimal shut-down date T_1 are chosen to maximize the net benefits while taking into account that the extraction depletes the resource stock $X(t)$ and creates a stock externality, say pollution, which is essentially captured by the state variable $N(t)$ and by the damage function. This stock is interpreted as the amount of pollution at time t, but other interpretations are of course possible. For example, variable N can be interpreted simply as an index for the state of the environment. A higher index value means lower environmental state. At the beginning of the production stage $N(0) = 0$, and the environmental is at its best condition. As extraction commences, the index begins to increase and the state to worsen. At time T_1 the index obtains a relatively high value, and as the production ceases the state of the environment begins to improve by natural processes such as re-vegetation and pollutant decay and immobilization. In what follows the stock variable $N(t)$ is referred to as the (size of the) pollution stock. The price of the resource is assumed to be a constant p. The extraction costs are captured by cost function G, whose value depends on the extraction rate and possibly on the amount of resource left on the ground. In Section 4 the production stage is analyzed under different extraction cost function specifications, and the respective assumptions are developed there.

When choosing the extraction rate and the shut-down date of the production stage, the regulator must take into account the effect of these choices on the rehabilitation stage. These effects are conveyed through the discounted scrap value, denoted with $S(N(T_1), T_1)$, which is included in the production stage objective, and which is the central target for investigation in Section 3 where the optimal rehabilitation is analyzed. Function S gives the discounted rehabilitation stage value. The initial value of the pollution stock at the rehabilitation stage, denoted with n_{T_1}, is the terminal value of the pollution stock at the production stage, $N(T_1)$. That is, $n_{T_1} = N(T_1)$. The rehabilitation cost function is C. In the first rehabilitation model to be considered, this cost depends on the size of the rehabilitation, $v \in [0, n_{T_1}]$, and satisfies properties $C' > 0$ and $C'' \geq 0$. Hence the rehabilitation cost is strictly increasing and convex in the rehabilitation size, and there are no fixed costs related to rehabilitation process. The pollution stock causes damages and the damage function is denoted with D. This function satisfies properties $D(0) = 0$, $D' > 0$ and $D'' \geq 0$.

The pollution stock dynamics during the production stage is given by the following initial value problem:

$$\dot{N}(t) = \alpha q(t) - f(N(t)), \quad N(0) = 0. \tag{1}$$

Here, parameter $\alpha > 0$ describes the accumulation of the pollution stock as the resource is extracted. Function f, which describes the decrease of the stock through natural processes, is explained shortly. The pollution stock dynamics during the rehabilitation stage is given by the following initial value problem:

$$\dot{N}(t) = -f(N(t)), \quad N(T_1^-) = n_{T_1} > 0, \tag{2}$$

where symbols $N(T_1^-)$ denote the left-sided limit of $N(T_1)$, that is, $N(T_1^-) = \lim_{t \to T_1^-} N(t)$. Symbols $N(T_1^+)$ are used to denote the right-sided limit. The extraction rate and hence the pollution accumulation is zero during the rehabilitation stage.

Two important assumptions are made regarding the stock dynamics. First, it is assumed that f is twice continuously differentiable, $f > 0$ for $N > 0$, $f(0) = 0$ and that there exists a constant B such that $|f'(N)| \leq B$. This guarantees that there exists a unique global solution to Equation (2) (and to Equation (1) for a given $q(t)$). This solution is denoted with $N(t; n_{T_1}, T_1)$. Second, it is supposed that the solution satisfies inequality $N(t; n_{T_1}, T_1) > 0$ for all t, that is, the stock never disappears other than through rehabilitation. For future reference, denote with $N_n(t; n_{T_1}, T_1)$ the partial derivative with respect to the second variable evaluated at the point (n_{T_1}, T_1).[9]

3 Rehabilitation stage

In this section the optimal rehabilitation decision is analyzed by considering two different rehabilitation models. First, in Section 3.1, a model is presented in which the regulator is allowed to postpone the rehabilitation, if it is optimal to do so. Second, in Section 3.2, the regulator's rehabilitation decision is modeled as a continuous rehabilitation process which begins when the production stage ends.[10] Whichever model is used as a description for the rehabilitation stage, the whole model is analyzed starting from the rehabilitation stage after which the optimal production stage decision and its properties are investigated. The main link between the stages is the rehabilitation stage value function, which acts as the scrap value in the production stage decision. The analysis of the production stage turns out to be the same for both models.

[9] Similar notation will be used in what follows without further explanations. For example, notation $N_T(t; n_{T_1}, T_1)$ means the partial derivative of $N(t; n_{T_1}, T_1)$ with respect to third variable.

[10] Third model is analyzed in Appendix A.4.

3.1 Waiting is allowed: Rehabilitation as a jump in the stock

The regulator takes as given the inherited pollution stock from the production stage and minimizes the sum of the total discounted damages from the pollution stock and the discounted rehabilitation costs by choosing the date and the size of the rehabilitation operation, or equivalently

$$\max_{\{\tau,v\}} \int_{T_1}^{T_2} -D(N(t))e^{-r(t-T_1)}\ \mathrm{d}t - C(v)e^{-r(\tau-T_1)} \tag{3}$$

$$\text{s.t} \quad \dot{N}(t) = -f(N(t)), \quad N(T_1^-) = n_{T_1} > 0, \quad N(T_2^+) = 0, \tag{4}$$

$$N(\tau^+) - N(\tau^-) = -v. \tag{5}$$

Here $\tau \in [T_1, T_2]$ and $v \in [0, n_{T_1}]$. The terminal date T_2 is taken here to be fixed and finite. At the end of this section the infinite horizon case is analyzed. The model is illustrated in the following figure:

Figure 2: An illustration of the rehabilitation model, in which rehabilitation date τ and rehabilitation size v are chosen from the set $[T_1, T_2] \times [0, n_{T_1}]$.

This problem with $T_1 = 0$ has been analyzed in Lappi (2018).[11] Since there is only one rehabilitation operation, and since the solution, $N(t; n_{T_1}, T_1)$, exists for the initial value problem in (4), the above problem is equivalent to the problem

$$\max_{\tau\in[T_1,T_2]} \left\{ \int_{T_1}^{\tau} -D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)}\ \mathrm{d}t - C(N(\tau; n_{T_1}, T_1))e^{-r(\tau-T_1)} \right\}. \tag{6}$$

This follows from the assumption that $N(t; n_{T_1}, T_1) > 0$ for all $t \in [T_1, T_2]$ and from the restriction $N(T_2^+) = 0$, which imply that there must exist a rehabilitation operation in which the remaining pollution stock is cleaned. This allows to plug the solution $N(t; n_{T_1}, T_1)$ and $v = N(\tau; n_{T_1}, T_1)$ into the objective function in (3), which results in the maximization problem (6) after applying the assumption $D(0) = 0$. Note that after τ

[11]Some of the presented results are replications. Namely Lemma A.1, and Part (i) of Lemma A.2 and Part (i) of Proposition 1 are replications of the results in Lappi (2018).

the damages are zero by the assumption $D(0) = 0$, that is,

$$\int_{\tau}^{T_2} -D(0)e^{-r(t-T_1)}\,\mathrm{d}t = 0. \tag{7}$$

The Lagrangian related to problem (6) is

$$\begin{aligned} L = \lambda_0 \bigg(\int_{T_1}^{\tau} -D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)}\,\mathrm{d}t - C(N(\tau; n_{T_1}, T_1))e^{-r(\tau-T_1)} \bigg) \\ - \lambda_1(T_1 - \tau) - \lambda_2(\tau - T_2), \end{aligned} \tag{8}$$

in which λ_0, λ_1 and λ_2 are multipliers (constants). Fritz John Theorem tells that there exists multipliers $(\lambda_0, \lambda_1, \lambda_2) \neq (0, 0, 0)$ such that $\lambda_0 \in \{0, 1\}$, and that the following conditions hold at the optimal rehabilitation date τ^*:

$$\begin{aligned} \lambda_0 \bigg(&- D(N(\tau; n_{T_1}, T_1))e^{-r(\tau-T_1)} + rC(N(\tau; n_{T_1}, T_1))e^{-r(\tau-T_1)} \\ &- C'(N(\tau; n_{T_1}, T_1))\dot{N}(\tau; n_{T_1}, T_1)e^{-r(\tau-T_1)} \bigg) + \lambda_1 - \lambda_2 = 0, \end{aligned} \tag{9}$$

$$\lambda_1 \geq 0, \quad \tau - T_1 \geq 0, \quad \lambda_1(\tau - T_1) = 0, \tag{10}$$

$$\lambda_2 \geq 0, \quad \tau - T_2 \leq 0, \quad \lambda_2(\tau - T_2) = 0. \tag{11}$$

Clearly $\lambda_0 = 1$. Hence the optimal rehabilitation date satisfies the following condition:

$$\begin{aligned} &D(N(\tau^*; n_{T_1}, T_1)) - rC(N(\tau^*; n_{T_1}, T_1)) \\ &\qquad - C'(N(\tau^*; n_{T_1}, T_1))f(N(\tau^*; n_{T_1}, T_1)) \begin{cases} \geq 0, & \text{if } \tau^* = T_1, \\ = 0, & \text{if } \tau^* \in (T_1, T_2), \\ \leq 0, & \text{if } \tau^* = T_2. \end{cases} \end{aligned} \tag{12}$$

In words, in an interior optimum, the rehabilitation is postponed until the cost of waiting one more unit of time equals the benefit of waiting. The cost is the additional damage and the benefit is the sum of the interest on the unused rehabilitation funds and of the decrease in the rehabilitation costs due to decrease in the pollution stock through natural processes. The related optimal rehabilitation size is

$$v^* = N(\tau^*; n_{T_1}, T_1), \tag{13}$$

that is, at the optimal rehabilitation date the production site is fully rehabilitated. Note that this is a consequence of the assumption $N(T^+) = 0$.

Analysis is continued by investigating the properties of the rehabilitation stage program's value function and of the discounted rehabilitation cost. It is assumed throughout that for an optimal interior rehabilitation date the following inequality holds:

$$
\begin{aligned}
D'(N(\tau^*;n_{T_1},T_1)) - C'(N(\tau^*;n_{T_1},T_1))r - C''(N(\tau^*;n_{T_1},T_1))f(N(\tau^*;n_{T_1},T_1)) \\
- C'(N(\tau^*;n_{T_1},T_1))f'(N(\tau^*;n_{T_1},T_1)) < 0. \qquad (14)
\end{aligned}
$$

It can be calculated that if the second derivative of the objective function in (6) is strictly negative (which implies that the function is strictly concave), then the inequality in (14) holds for an optimal interior rehabilitation date.[12]

The optimal rehabilitation date τ^* and the optimal rehabilitation size v^* depend on the parameters n_{T_1}, T_1, T_2 and r. Since only n_{T_1} and T_1 are endogenous variables in the whole model, the optimal rehabilitation date and size are denoted with $\tau^* = \tau(n_{T_1},T_1)$ and $v^* = v(n_{T_1},T_1)$, and the explicit dependency of the optimal values on r and T_2 is left out of the notation. The value function of the maximization problem (3)–(5) is defined as

$$
\begin{aligned}
V(n_{T_1},T_1) := \int_{T_1}^{\tau(n_{T_1},T_1)} -D(N(t;n_{T_1},T_1))e^{-r(t-T_1)}\,\mathrm{d}t \\
- C(v(n_{T_1},T_1))e^{-r(\tau(n_{T_1},T_1)-T_1)}. \qquad (15)
\end{aligned}
$$

The necessary amount of money needed for the rehabilitation at the end of the production stage is given by $C(v(n_{T_1},T_1))e^{-r(\tau(n_{T_1},T_1)-T_1)}$, since this sum will increase to $C(v(n_{T_1},T_1))$ as time progresses from T_1 to τ^*. To simplify the notation define two variable functions K and S with

$$
K(n,T) := C(v(n,T))e^{-r(\tau(n,T)-T)} \quad \text{and} \quad S(n,T) := V(n,T)e^{-rT}. \qquad (16)
$$

Function K measures the necessary amount of money in the rehabilitation trust at the end of the production stage (the cost of rehabilitation, which has been discounted to T). This function is needed to define the amount of money the firm is required to deposit to the trust by the end of the production stage. Function S measures the rehabilitation stage value, which is discounted to time $t = 0$. Function S acts as the discounted scrap value function in the production stage problem.

[12] [See Supplementary material at the end of the manuscript; Section B.]

Dependency of S and K on the initial pollution stock level. Before analyzing the properties of S and K, the dependency of the rehabilitation date and size on the initial pollution stock is investigated.[13]

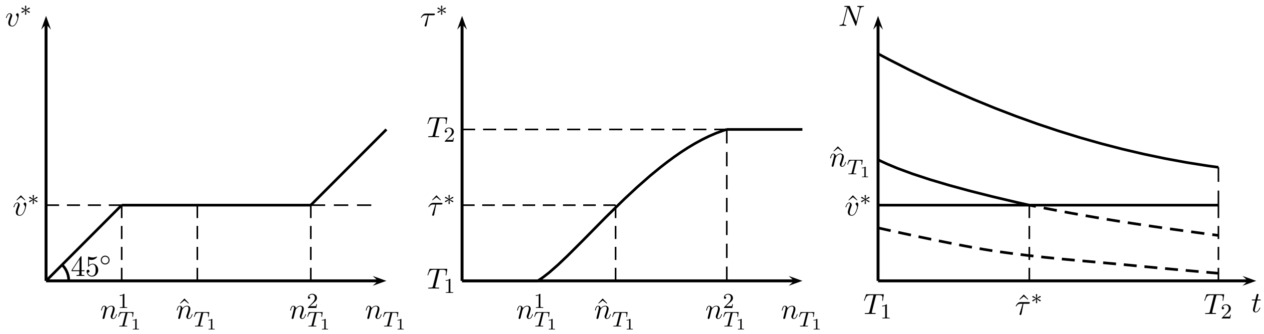

Figure 3: Illustration of the optimal rehabilitation decision for different initial pollution stocks. Here $(n^1_{T_1}, n^2_{T_1})$ is the interval on which τ^* is an interior point. Figure on the left depicts the optimal rehabilitation size as function of the initial pollution stock. In the middle, the optimal rehabilitation date is plotted as a function of the initial pollution stock. The last figure contains pollution stock paths for different initial pollution stocks and illustrates the different possibilities for the rehabilitation (the dashed curves represent to time paths of the pollution stock if rehabilitation is sub-optimally not conducted).

The optimal rehabilitation decision is illustrated in Figure 3, when τ^* is an interior point of $[T_1, T_2]$ on some open n-interval $(n^1_{T_1}, n^2_{T_1})$. The figure on the left depicts the optimal rehabilitation size as a function of the initial pollution stock, and the figure on the middle depicts the optimal date as a function of the initial pollution stock. As illustrated in these figures, as the initial pollution stock increases from zero towards $n^1_{T_1}$, the optimal rehabilitation date remains at the initial date T_1 and consequently the optimal rehabilitation size increases. The initial pollution stock value $n^1_{T_1}$ is the largest initial value at which the optimal date is T_1. After $n^1_{T_1}$ an increase in the initial pollution stock increases the optimal rehabilitation date, but keeps the size fixed, until the value reaches $n^2_{T_1}$. After $n^2_{T_1}$ the date remains at T_2 and the size of the rehabilitation increases. The figure on the right illustrates the evolution of the three possible paths of the pollution stock in time. The path starting from $\hat{n}_{T_1}$ decreases in time as the natural processes clean the stock until at time $\hat{\tau}^*$ the site is rehabilitated. The amount of pollution cleaned is in this case $\hat{v}^*$. This figure also illustrates two other possibilities at which the rehabilitation

[13]Lemmas A.1 and A.2 in Appendix A.1 provide the mathematical details of the dependency of τ^* and v^* on these endogenous parameters. These lemmas are part of the proof of Proposition 1.

is either instantaneous (the dashed line over the interval $[T_1, T_2]$) or is delayed until the terminal date (the solid upper line).

Next proposition shows how a change in the initial pollution stock affects the discounted scrap value and the discounted rehabilitation cost.

Proposition 1. *Let* $(n^1_{T_1}, n^2_{T_1})$ *be an interval on which* $\tau^* \in (T_1, T_2)$. *Then*

(i) $K_n(n_{T_1}, T_1) < 0$ *for all* $n_{T_1} \in (n^1_{T_1}, n^2_{T_1})$,

(ii) $K_n(n_{T_1}, T_1) > 0$ *for all* $n_{T_1} \in (0, n^1_{T_1}) \cup (n^2_{T_1}, \infty)$,

(iii) $S_n(n_{T_1}, T_1) < 0$ *for all* $n_{T_1} \in (0, \infty)$.

Proof. See Appendix A.1. □

The result regarding the the discounted rehabilitation cost in Part (*i*) of this proposition replicates the results of Lappi (2018). The discounted rehabilitation cost is either strictly increasing or strictly decreasing in the initial pollution stock depending whether the rehabilitation occurs at the boundary or at the interior of the interval $[T_1, T_2]$. When the rehabilitation occurs at an interior point, the size of the rehabilitation is constant with respect to the initial pollution stock and the rehabilitation date is increasing in the initial pollution stock (Part (*i*) of Lemma A.2; see also Figure 3). This implies, that the discounted cost of rehabilitation decreases since the current cost is thus unaffected and the discount factor is decreased by the increase in the initial pollution stock. When the rehabilitation occurs on the boundary of $[T_1, T_2]$ as in Part (*ii*) (and n_{T_1} is not equal to $n^1_{T_1}$ or $n^2_{T_1}$), a small increase in the initial pollution stock causes no change in the rehabilitation date, but increases the size of the rehabilitation. Hence the rehabilitation cost and also the discounted rehabilitation cost increase.

Part (*iii*) says that the discounted scrap value is decreasing in the initial pollution stock. This is very intuitive at least when the rehabilitation occurs at T_1, since in that case the initial pollution stock has no effect on the future damages (they are zero) but increases the rehabilitation cost. When the rehabilitation occurs at T_2, an increase in the initial pollution stock increases the overall damages and also the discounted cost of rehabilitation. For an interior rehabilitation date, two opposing effects exist. First, as

Part (i) of Proposition 1 shows, the discounted rehabilitation cost is strictly decreasing in the initial pollution stock. Second, the overall damages are increasing in the initial pollution stock. Part (iii) shows that the discounted scrap value is strictly decreasing in the initial pollution stock, and that an increase in the overall damages is strictly greater than the decrease in the rehabilitation cost.

Dependency of S and K on the initial time. A change in the initial time of the rehabilitation stage has also an effect on the optimal rehabilitation date and size.[14] When the rehabilitation occurs at the beginning of the rehabilitation stage, and the shut-down date T_1 of the production stage increases by a small amount, the rehabilitation date also increases by approximately the same amount, but the size of the rehabilitation operations stays approximately unchanged. Similarly, if the rehabilitation occurs at the interior of the rehabilitation stage, then only the rehabilitation date adjusts upwards as the shut-down date T_1 is increased. But when rehabilitation occurs at T_2, a unit increase in the shut-down date increases the rehabilitation size approximately by amount $f(N(T_2; n_{T_1}, T_1))$. This is the pollution stock amount at T_2, which is not cleaned by natural processes due to shorter rehabilitation interval.

Next the dependency of the discounted rehabilitation costs and the discounted scrap value on the initial time T_1 (the shut-down date of the production stage) is investigated.

Proposition 2.

(i) *If* $\tau^* = T_1$, *then* $K_T(n_{T_1}, T_1) = 0$ *and* $S_T(n_{T_1}, T_1) = -rV(n_{T_1}, T_1)e^{-rT_1}$.

(ii) *If* $\tau^* \in (T_1, T_2)$, *then* $K_T(n_{T_1}, T_1) = 0$ *and* $S_T(n_{T_1}, T_1) = -rV(n_{T_1}, T_1)e^{-rT_1}$.

(iii) *If* $\tau^* = T_2$, *then*

$$K_T(n_{T_1}, T_1) = C'(v(n_{T_1}, T_1))e^{-r(T_2-T_1)}f(N(T_2; n_{T_1}, T_1)) \tag{17}$$

$$+ rC(v(n_{T_1}, T_1))e^{-r(T_2-T_1)} \tag{18}$$

[14] In the appendix, Lemma A.3 offers the mathematical details for these effects.

and

$$S_T(n_{T_1}, T_1) = e^{-rT_1}\Bigg(D(N(T_2; n_{T_1}, T_1))e^{-r(T_2-T_1)} - C'(v(n_{T_1}, T_1))f(N(T_2; n_{T_1}, T_1))e^{-r(T_2-T_1)} + r\int_{T_1}^{T_2} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)}\,\mathrm{d}t\Bigg) \quad (19)$$

Proof. See Appendix A.2. □

Parts (i) and (ii) show that if $\tau^* \in [T_1, T_2)$, then an increase in the shut-down date has no effect on the discounted rehabilitation cost. There are two reasons for this. First, the rehabilitation size and hence the rehabilitation cost remains unchanged, when the shut-down date increases by one unit. Second, the discount factor remains the same because the rehabilitation date increases by one unit, when the shut-down date increases by one unit. However, Part (iii) shows that if $\tau^* = T_2$, then an increase in the shut-down date increases the discounted rehabilitation cost. There are two channels for this. First, as the shut-down date increases by one unit, the eventual rehabilitation comes one unit of time sooner, and the regulator suffers a cost, which is the interest on the discounted rehabilitation cost. Second, since the rehabilitation date comes one unit of time sooner, the pollution stock is not cleaned by the natural processes during that time unit, which means that the rehabilitation cost is higher with this shorter rehabilitation interval.

The results regarding the discounted scrap value are central for the optimal shut-down date decision of the production site, which is analyzed in Section 4. Parts (i) and (ii) show that if $\tau^* < T_2$, then the discounted scrap value is strictly increasing in the shut-down date of the production stage. In this case equation $S_T(n_{T_1}, T_1) = -rV(n_{T_1}, T_1)e^{-rT_1}$ holds, which means that a unit increase in the shut-down date increases the discounted scrap value approximately by the amount $-rV(n_{T_1}, T_1)e^{-rT_1}$. This is the discounted interest on the avoided monetary value of the rehabilitation stage program, and therefore, if one begins the rehabilitation stage one unit of time later, one receives a benefit which is in absolute value terms the interest on the discounted monetary value of the rehabilitation stage.

This section concludes with a consideration of the case with $T_2 = \infty$. First, and not very rigorously speaking, if one lets T_2 approach infinity in Equation (19), then the partial derivative $S_T(n_{T_1}, T_1)$ becomes

$$S_T(n_{T_1}, T_1) = re^{-rT_1} \int_{T_1}^{\infty} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} = -rV(n_{T_1}, T_1)e^{-rT_1}. \tag{20}$$

Second, if the rehabilitation stage optimization program is over the interval $[T_1, \infty)$, that is if $T_2 = \infty$ at the outset, and if its never optimal to rehabilitate, then similar calculations as in the proof of Proposition 2 yield Equation (20).[15] Hence, in the limiting case and in the case with $T_2 = \infty$ and no rehabilitation, one obtains the equation $S_T(n_{T_1}, T_1) = -rV(n_{T_1}, T_1)e^{-rT_1}$, which has the clear economic intuition explained above.

3.2 Waiting is not allowed: Rehabilitation as a continuous stock depreciation

In Caputo and Wilen (1995) the site is continuously rehabilitated starting from the initial time T_1 and continuing until the fixed time T_2 at which the rehabilitation operation is stopped.[16] In their model the pollution stock N depreciates exponentially at the rate δ and causes damages during interval $[T_1, T_2]$ according to damage function D. The rehabilitation process is continuous and the rate of rehabilitation is R. If a strictly positive pollution stock is left after T_2, then a residual damage is realized. The rehabilitation cost function is denoted with C, which depends on the pollution stock N and on the rate of rehabilitation. In Caputo and Wilen this cost function has the properties $C_N < 0$ and $C_R > 0$ (note the correspondence of the latter with the assumption $C' > 0$ used in the previous rehabilitation model). Following the previously applied notation, denote the optimal rehabilitation rate with $R(t; n_{T_1}, T_1)$ and the corresponding optimal pollution stock with $N(t; n_{T_1}, T_1)$. Then the value function of Caputo and Wilen (1995) evaluated at (n_{T_1}, T_1) can be written in the current context as

$$\begin{aligned} \Phi(n_{T_1}, T_1) = &\int_{T_1}^{T_2} -\Big(C(N(t; n_{T_1}, T_1), R(t; n_{T_1}, T_1)) + D(N(t; n_{T_1}, T_1))\Big)e^{-r(t-T_1)} \, \mathrm{d}t \\ &+ \int_{T_2}^{\infty} -D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} \, \mathrm{d}t, \end{aligned} \tag{21}$$

where $n_{T_1} = N(T_1)$ is again the pollution stock from the production stage. Denote the discounted value function (or the discounted scrap value) evaluated at (n_{T_1}, T_1) with

[15] [See Supplementary material at the end of the manuscript; Section B.]

[16] In Caputo and Wilen $T_1 = 0$, since in their waste pile clean-up model no production stage precedes the rehabilitation stage.

$\Psi(n_{T_1}, T_1) = e^{-rT_1}\Phi(n_{T_1}, T_1)$. The difference to the previous model that allows waiting is in the terminal pollution stock at time T_2, which is not restricted to be zero and in the reclamation cost function, which depends on the size of the pollution stock. Note also, that in Caputo's and Wilen's model the depreciation of the pollution stock through natural processes is assumed to be exponential whereas in the model of Section 3.1 a more general formulation is applied.

Next it is analyzed how the discounted scrap value depends on the parameter values n_{T_1} and T_1. To investigate the dependence on the initial pollution stock (Part (i) of Proposition 3), it is assumed that

$$N_n(t; n_{T_1}, T_1) > 0 \quad \text{and} \quad R_n(t; n_{T_1}, T_1) > 0, \tag{22}$$

that is, both the path of the optimal pollution stock and the path of the optimal rehabilitation rate increase for every t as the initial pollution stock increases.[17]

Proposition 3.

(i) *If* $D'(N) \geq -C_N(N,R)$ *for all* (N,R), *then* $\Psi_n(n_{T_1}, T_1) < 0$.

(ii)

$$\Psi_T(n_{T_1}, T_1) = -r\Phi(n_{T_1}, T_1)e^{-rT_1} + C(N(T_2; n_{T_1}, T_1), R(T_2; n_{T_1}, T_1))e^{-rT_2} > 0. \tag{23}$$

Proof. See Appendix A.3. □

That is, the discounted scrap value decreases as the pollution stock at the shut-down date of the production stage increases, and increases as the shut-down date increases, just like in the previous model, in which waiting is allowed.[18]

4 Production stage

As explained above, the initial value of the pollution stock at the rehabilitation stage, n_{T_1}, is the terminal value of the pollution stock at the production stage, $N(T_1)$. When

[17]It seems to be a difficult task to verify these properties in Caputo's and Wilen's model.

[18]Appendix A.4 presents a third rehabilitation model based on the standard "abatement model", in which a fraction of the pollution stock is removed at the shut-down date. The same properties of the value function as in the other models hold.

deciding the socially optimal extraction rate and the shut-down date, the regulator must take into account the effect of choices on the rehabilitation stage. These effects are conveyed through the discounted scrap value $S(N(T_1), T_1)$ (or $\Psi(N(T_1), T_1)$). In what follows, only S is used to denote the discounted scrap value. The results would remain the same, if Ψ was used.

General case. Let the stock dynamics for the pollution stock be given by Equation (1) and suppose that this stock causes damages during the production stage. Suppose that the extraction cost depends on the amount of resource on the ground X and on the rate of extraction q. The cost function is denoted with G. A typical assumption about the cross-partial derivative G_{qX} is that it is strictly negative (as for example in Pindyck (1978), Caputo (1990), Pesaran (1990), Tahvonen (1997), Krautkraemer (1998) and Cairns (2014)), which reflects a situation where the marginal extraction cost increases as the resource stock diminishes. Fixed operating costs are allowed in the sense that $G(0, X) \geq 0$ for any resource stock level. In particular, if $G(0, X) = 0$, then there are no fixed costs and if $G(0, X) > 0$, then there are fixed costs. These costs include any operation costs that are borne even if the extraction rate is zero, but the site is not shut-down. After the production site is shut-down these costs are also zero. Then the regulator's maximization problem is given as

$$\max_{\{q(t), T_1\}} \int_0^{T_1} \Big(pq(t) - G(q(t), X(t)) - D(N(t))\Big)e^{-rt}\, \mathrm{d}t + S(N(T_1), T_1) \tag{24}$$

$$\text{s.t} \quad \dot{X}(t) = -q(t), \quad X(0) = x_0, \quad X(T_1) \geq 0, \tag{25}$$

$$\dot{N}(t) = \alpha q(t) - f(N(t)), \quad N(0) = 0, \quad N(T_1) \geq 0, \tag{26}$$

$$q(t) \geq 0, \tag{27}$$

where S is given by (16). Variable p is the constant price of the resource and x_0 is the initial amount of the resource on the ground.

The present value Hamiltonian related to this problem is

$$H(q, X, N, \mu, \gamma, t) = \mu_0\Big(pq - G(q, X) - D(N)\Big)e^{-rt} - \mu q + \gamma(\alpha q - f(N)). \tag{28}$$

Theorem 16 in Chapter 6 of Seierstad and Sydsæter (1987) applied.[19] The necessary

[19]It is assumed that $\mu_0 = 1$.

conditions include the following:

$$
\begin{aligned}
H_q &= \big(p - G_q(q, X)\big)e^{-rt} - \mu + \gamma\alpha \le 0, \quad q \ge 0, \quad qH_q = 0, && (29)\\
\dot{X} &= -q, && (30)\\
\dot{N} &= \alpha q - f(N), && (31)\\
\dot{\mu} &= G_X(q, X)e^{-rt}, && (32)\\
\dot{\gamma} &= D'(N)e^{-rt} + \gamma f'(N), && (33)
\end{aligned}
$$

$$\mu(T_1) \ge 0, \quad X(T_1) \ge 0, \quad \mu(T_1)X(T_1) = 0, \tag{34}$$

$$\gamma(T_1) - S_n(N(T_1), T_1) \ge 0, \quad N(T_1) \ge 0, \quad N(T_1)[\gamma(T_1) - S_n(N(T_1), T_1)] = 0, \tag{35}$$

$$
\begin{aligned}
&H(q(T_1), X(T_1), N(T_1), \mu(T_1), \gamma(T_1), T_1)\\
&+ S_T(N(T_1), T_1) \begin{cases} \le 0, & \text{if } T_1 = 0,\\ = 0, & \text{if } T_1 \in (0, T_2),\\ \ge 0, & \text{if } T_1 = T_2. \end{cases}
\end{aligned}
\tag{36}
$$

Next proposition presents the optimal shut-down rule for the case with increasing and convex extraction cost.[20]

Proposition 4. *Assume that $\tau^* \in [T_1, T_2)$ or $T_2 = \infty$. Suppose that the extraction cost function G satisfies properties $G_q > 0$ and $G_{qq} > 0$ for all q and X.*

(i) If $q(T_1) > 0$, then

$$
\begin{aligned}
q(T_1)\left[\frac{G(q(T_1), X(T_1))}{q(T_1)} - G_q(q(T_1), X(T_1))\right] + D(N(T_1))&\\
= -rV(N(T_1), T_1) - \gamma(T_1)e^{rT_1}f(N(T_1)).&
\end{aligned}
\tag{37}
$$

(ii) If $q(T_1) = 0$, then

$$G(0, X(T_1)) + D(N(T_1)) = -rV(N(T_1), T_1) - \gamma(T_1)e^{rT_1}f(N(T_1)). \tag{38}$$

[20] It is assumed that either $\tau^* \in [T_1, T_2)$ or $T_2 = \infty$. The reason for this assumption is three-fold: first, the analysis of the case $\tau^* = T_2 < \infty$ is cumbersome (see Part (*iii*) of Proposition 2); second, T_2 is exogenous to the model and there is no non-technical reason to fix it (a finite T_2 does guarantee the existence of an optimal solution at the rehabilitation stage); third and most importantly, the scrap value function has similar properties with $T_2 = \infty$ and with $\tau^* < T_2 < \infty$, as discussed at the end of Section 3.

Proof. (i) Suppose that $q(T_1) > 0$. Then $\Big(p - G_q(q(T_1, X(T_1)))\Big)e^{-rT_1} - \mu(T_1) + \gamma(T_1)\alpha = 0$ by (29), and

$$\begin{aligned}\Big(pq(T_1) - G(q(T_1), X(T_1)) - D(N(T_1))\Big)e^{-rT_1} - \mu(T_1)q(T_1) + \gamma(T_1)(\alpha q(T_1) - f(N(T_1))) \\ = -S_T(N(T_1), T_1). \qquad (39)\end{aligned}$$

by (36). Combining these equations and simplifying yields the following equation:

$$\begin{aligned}q(T_1)\left[\frac{G(q(T_1), X(T_1))}{q(T_1)} - G_q(q(T_1), X(T_1))\right]e^{-rT_1} + e^{-rT_1}D(N(T_1)) + \gamma(T_1)f(N(T_1)) \\ = -rV(N(T_1), T_1)e^{-rT_1}. \qquad (40)\end{aligned}$$

The desired equation follows from this.

(ii) Suppose that $q(T_1) = 0$. The desired equation is obtained readily from condition (36). □

On the left-side of these equations is the cost of waiting one more unit of time and on the right-side is the benefit of waiting one unit of time. At the optimal shut-down date these values must be the same. The cost of waiting consists of two parts. In the shut-down rule (37), the first is the difference between the average and the marginal costs multiplied by the terminal production value, and the second is the damages from the pollution stock at the shut-down date. The benefit of waiting is also a sum of two parts. The first is the interest on the avoided rehabilitation stage value, and the second, namely the term $-\gamma(T_1)e^{rT_1}f(N(T_1))$, is the value of the removed pollution stock by the natural process during the unit of time spent on waiting, where the value is measured with the current shadow value of the pollution stock.

A typical assumption in the literature about the extraction cost function, found for example in Pindyck (1978), Pindyck (1987) and Tahvonen (1997), is $G(q, X) = qc(X)$. The function c is assumed to satisfy property $c' < 0$.[21] With this specification the shut-down rule becomes

$$D(N(T_1)) = -rV(N(T_1), T_1) - \gamma(T_1)e^{rT_1}f(N(T_1)), \qquad (41)$$

since the marginal and average costs are equal. Hence the shut-down rule is characterized by the pollution stock dynamics and damages at the terminal pollution stock.

[21]For this function $G_{qq} = 0$, but assumption $G_{qq} > 0$ was not used in the proof.

Acid mine drainage. The model can capture acid mine drainage (AMD). AMD takes a long time to develop, and it causes damages mainly after the production stage (Dold, 2014). As long as the tailings are kept water saturated during the production stage the oxidization process is slow and AMD is limited. After the production stage the mine maintenance operations cease and the AMD is amplified and begins to cause problems. The amount of tailings grows as the resource is exploited. Hence, to capture AMD, it is now assumed that the stock variable N causes damages only during the rehabilitation stage. In this case the fixed production costs are important as the following result shows:

Proposition 5. *Assume that $\tau^* \in [T_1, T_2)$ or $T_2 = \infty$. Suppose that the extraction cost function G satisfies properties $G_q > 0$ and $G_{qq} > 0$ for all q and X.*

(*i*) *If $G(0, X) > 0$ for all X, then the production stage ends at the time instant $T_1 > 0$, which satisfies either condition*

$$q(T_1)\left[\frac{G(q(T_1), X(T_1))}{q(T_1)} - G_q(q(T_1), X(T_1))\right] = -rV(N(T_1), T_1) - \gamma(T_1)e^{rT_1}f(N(T_1)) \tag{42}$$

or conditions

$$q(T_1) = 0 \quad \text{and} \quad G(0, X(T_1)) = -rV(N(T_1), T_1) - \gamma(T_1)e^{rT_1}f(N(T_1)). \tag{43}$$

(*ii*) *However, if $G(0, X) = 0$ for all X, then no optimal solution exists with $T_1 > 0$ and $q(t) > 0$ for any t.*

Proof. (i) Equations (42) and (43) follow from Part (i) of Proposition 4 since pollution damage is absent (set $D(N(T_1)) = 0$).

(ii) Suppose $G(0, X) = 0$ for all X. Assume $q(T_1) > 0$. Then

$$\begin{aligned} e^{-rT_1}\Big(G_q(q(T_1), X(T_1))q(T_1) - G(q(T_1)X(T_1))\Big) &= \\ - S_T(N(T_1), T_1) + \gamma(T_1)f(N(T_1)) &< 0, \end{aligned} \tag{44}$$

This is not possible with the assumed cost structure. Assume $q(T_1) = 0$. Then (36) implies that $0 = -S_T(N(T_1), T_1) + \gamma(T_1)f(N(T_1))$, since pollution damages are absent. But

$$- S_T(N(T_1), T_1) + \gamma(T_1)f(N(T_1)) = rV(N(T_1), T_1)e^{-rT_1} + \gamma(T_1)f(N(T_1)) \leq 0, \tag{45}$$

since $V \leq 0$, $\gamma(T_1) < 0$ and $f \geq 0$. Therefore $f(N(T_1)) = 0$ and $V(N(T_1), T_1) = 0$. These equations are true only if $N(T_1) = 0$. This implies $q = 0$ for all t. $\square$

This result also holds if it is assumed, like Gaudet *et al.* (1995), Roan and Martin (1996) and Cairns (2004) do, that the cost function is independent of the resource left on the ground. When $q(T_1) > 0$, the average extraction cost is above the marginal extraction cost function at the end of production operation. Therefore it is socially optimal during some interval at the end of the production stage for the production operator to make a loss compared to the case in which the extraction is stopped when average cost equals the marginal cost. The amount of loss at the shut-down date is given by the left-side of (42), and it is balanced with the benefit of postponing the rehabilitation stage by one time unit.

The second part of this result says that if there are no fixed costs, then there is no economically interesting solution to the problem with acid mine drainage. Intuitively, zero fixed costs imply that the regulator can increase the value of the objective with any increase in the shut-down date, since this action postpones the (negative) rehabilitation stage value without causing any cost. However, from the practical point of view, fixed costs are positive in mining. For example, the tailings dams containing the wastes must be kept (and are kept in practice) water saturated in order to prevent AMD (Dold, 2014). This is costly even if extraction has stopped but the site is not shut-down. Another example is the compensation to the land owner as in oil production in Alberta (Muehlenbachs, 2015). However, note that no solution exists with the often applied cost function $G(q, X) = qc(X)$, since it does not involve fixed costs.

Of course, there are other possible assumptions about the extraction costs. For example, Eswaran *et al.* (1983), Mumy (1984) and Cairns (2008) assume a concave-convex extraction cost function (marginal cost is U-shaped) that does not depend on X. Recall that with this cost specification a standard exhaustible resource model without rehabilitation stage produces as the optimal shut-down date an instant t, which satisfies equation $G(q(t))/q(t) = G_q(q(t))$. This does not hold when the optimal decision of the rehabilitation stage is taken into account. Instead, Equation (42) holds. The reason is that q cannot decrease below the value that minimizes G_q, since the net price increases at the rate of interest by (29) (cost is independent of X). Therefore $q(T_1) > 0$, and Equation (42) characterizes the shut-down date.

5 Producer's response to the optimal regulation

In this section investigates producer's response to the optimal regulation, when the payments are collected using a profit tax. The socially optimal values are denoted with the $*$-symbol. The regulation consists of the socially optimal shut-down date for the production stage, ${T_1}^*$, of the amount of money the producer must have deposited to the rehabilitation trust before the production ends, $K(N^*({T_1}^*), {T_1}^*)$, and of the pollution tax $\Gamma(t) := -\gamma(t)e^{rt}$ on the pollution generation αq. In addition, the producer is required to collect the monies to the rehabilitation trust from its profits. Does it matter from the social welfare point of view how the producer collects the funds from its profits, and is any additional regulation needed?

Additional regulation, if any is needed, consists of rules that govern the payments. Suppose that for every time instant the producer has to decide a fraction of the net profit that goes to the trust. This fraction is denoted with $\theta(t)$, and the size of the rehabilitation trust at time t is denoted with $B(t)$. When production commences the trust is empty and the deposited monies grow at the rate of interest r. It is assumed that the producer's discount rate equals the regulator's discount rate r, but at the end of this section the other possibility that the producer's rate of time preference differs from r is investigated. For example, it can be greater than r, which is often the case in practice.

The producer's problem is to choose the extraction rate and the fraction of profits going to the trust to maximize the total discounted net profit while taking into account the state constraints. Mathematically the problem is to

$$\max_{\{q(t),\theta(t)\}} \int_0^{{T_1}^*} \Big(pq(t) - G(q(t), X(t)) - \Gamma(t)\alpha q(t)\Big)\Big(1 - \theta(t)\Big)e^{-rt}\,\mathrm{d}t \tag{46}$$

$$\text{s.t}\quad \dot{X}(t) = -q(t), \quad X(0) = x_0, \quad X(T_1) \geq 0, \tag{47}$$

$$\dot{B}(t) = rB(t) + \theta(t)\Big(pq(t) - G(q(t), X(t)) - \Gamma(t)\alpha q(t)\Big), \tag{48}$$

$$B(0) = 0, \quad B({T_1}^*) = K(N^*({T_1}^*), {T_1}^*), \tag{49}$$

$$q(t) \geq 0, \quad \theta(t) \in [0, 1]. \tag{50}$$

The present value Hamiltonian related to this problem is

$$H(q,\theta,X,B,\mu,\eta,t) = \Big(pq - G(q,X) - \Gamma\alpha q\Big)(1-\theta)e^{-rt} \tag{51}$$
$$- \mu q + \eta\Big(rB + \theta(pq - G(q,X) - \Gamma\alpha q)\Big). \tag{52}$$

To simplify the notation, let

$$Z(q) := pq - G(q,X) - \Gamma\alpha q, \tag{53}$$

which is the instantaneous net profit.

The necessary conditions include the following:

$$H_q = \Big(p - G_q(q,X) - \Gamma\alpha\Big)(1-\theta)e^{-rt} - \mu + \eta\theta(p - G_q(q,X) - \Gamma\alpha) \leq 0, \tag{54}$$

$$q \geq 0, \quad qH_q = 0, \tag{55}$$

$$\theta = \begin{cases} 0 & \text{if } -Z(q)e^{-rt} + \eta Z(q) < 0, \\ \text{any } \theta \in [0,1] & \text{if } -Z(q)e^{-rt} + \eta Z(q) = 0, \\ 1 & \text{if } -Z(q)e^{-rt} + \eta Z(q) > 0, \end{cases} \tag{56}$$

$$\dot{X} = -q, \tag{57}$$

$$\dot{B} = rB + \theta\Big(pq - G(q,X) - \Gamma\alpha q\Big), \tag{58}$$

$$\dot{\mu} = G_X(q,X)e^{-rt}, \tag{59}$$

$$\dot{\eta} = -r\eta, \tag{60}$$

$$\mu({T_1}^*) \geq 0, \quad X({T_1}^*) \geq 0, \quad \mu({T_1}^*)X({T_1}^*) = 0, \tag{61}$$

$$\eta({T_1}^*) \text{ "has no condition".} \tag{62}$$

Define production as profitable at time t, if it satisfies the inequality $Z(q(t))(1-\theta(t)) > 0$, and define the project as profitable on the production interval $[0, {T_1}^*]$, if

$$\int_0^{{T_1}^*} Z(q(t))\Big(1-\theta(t)\Big)\ \mathrm{d}t > 0. \tag{63}$$

That is, the project is profitable on the production stage, if the total discounted net profit is strictly positive. These definitions are applied in the next lemma, which characterizes the shadow value of the rehabilitation funds.

Lemma 1. *Equation $\eta(t) = e^{-rt}$ holds for a profitable project that collects sufficient amount of money to cover the rehabilitation costs.*

Proof. See Appendix A.5. □

This lemma implies that the shut-down date and the pollution tax together with the requirement that the rehabilitation funds must be collected from the operative profits with a profit tax gives the producer incentives to extract the resource according to the social optimum as shown in the next result.

Proposition 6. *Suppose that the producer's time preference matches regulator's preference. A profitable project that collects sufficient amount of money to cover the rehabilitation costs produces the socially optimal extraction and state variable paths, which is independent of the collection of the rehabilitation funds.*

Proof. Lemma 1 implies that $\eta(t) = e^{-rt}$. This together with the optimal regulation ${T_1}^*$, $K(N^*({T_1}^*), {T_1}^*)$ and $\Gamma(t) = -\gamma(t)e^{rt}$ implies that conditions (54), (55), (57), (59) and (61) become

$$H_q = \big(p - G_q(q, X)\big)e^{-rt} - \mu + \gamma\alpha \leq 0, \tag{64}$$

$$q \geq 0, \quad qH_q = 0, \tag{65}$$

$$\dot{X} = -q, \tag{66}$$

$$\dot{\mu} = G_X(q, X)e^{-rt}, \tag{67}$$

$$\mu({T_1}^*) \geq 0, \quad X({T_1}^*) \geq 0, \quad \mu({T_1}^*)X({T_1}^*) = 0. \tag{68}$$

These conditions match conditions (29), (30), (32) and (34) of the social optimum. These are independent of η and B. In addition, the regulation (Γ and ${T_1}^*$) has been chosen such that it satisfies conditions (31), (33), (35) and (36). Hence, the producer's choice of extraction coincides with the socially optimal extraction and also produces the socially optimal paths for the state variables X and N. Condition (56) guarantees that any path with $\theta \in [0, 1]$ such that the end-point constraint in (49) holds is sufficient for social optimum. □

This result says, that the given regulation yields the social optimum independently of when the producer collects the monies for the rehabilitation. The producer is free from the social point of view to collect the monies for example during some interval at the beginning of the production stage, or during some interval at the end of the production stage. This is essentially due to Lemma 1, which states that the shadow value of money in the rehabilitation trust decreases at the rate of interest. The producer is therefore indifferent between putting one unit of money to the trust and keeping (saving) it.

In practice the producer's rate of time preference is higher than the regulator's. To see what kind of tax rule is needed for social optimum, let $\sigma(t)$ be any time dependent rate of time preference for the producer and suppose that it differs from r. The above regulation is not sufficient to yield the socially optimal allocation. However, a slight adjustment to the profit tax rule recovers the social optimum.

Proposition 7. *Suppose that the producer's rate of time preference $\sigma(t)$ is different from the regulator's time preference r. Then a profitable project that collects sufficient amount of money to cover the rehabilitation costs produces the socially optimal extraction and state variable paths, if the fractional profit tax is replaced with a profit tax defined with*

$$\chi(t) := (1 - \theta(t))e^{(\sigma(t)-r)t}.$$

The social optimum is obtained independently of the collection of the rehabilitation funds.

Proof. The Hamiltonian in the producer's problem with time preference σ and tax rule $\chi(t) = (1 - \theta(t))e^{(\sigma-r)t}$ equals the Hamiltonian H in (51)-(52). This implies that the results in Lemma 1 and in Proposition 6 hold. □

The new tax rule effectively changes the producer's rate of time preference to match the social rate of time preference and therefore the present value of profit tax for the producer is the same as for the regulator. Again, if this rule is used, the producer is free to make the payments at any time intervals it desires and the social optimum is achieved.

6 Conclusions

This study analyzes the final stages of polluting exhaustible resource production, namely, the production and rehabilitation stages. The socially optimal extraction and rehabilitation is characterized, and the regulation that induces the producer to behave in the socially optimal way is investigated. The key to analyze the model is to study the value function of the rehabilitation stage problem. Doing this enables to add this value function as the scrap value function for the production stage problem to find out what is the optimal regulation. The optimal regulation calls for a pollution tax (although other instruments may also be feasible), an optimal shut-down date and a requirement that the producer must deposit sufficient funds to cover the eventual rehabilitation costs. There is no need (in this model) to require the producer to pay the funds at the beginning of the production stage – indeed, from the social optimum point of view, the producer is free to choose when to pay the funds. In practice, though, it must be remembered that the producer may have incentives to postpone the deposits and try to drive the operation to bankruptcy before the socially optimal shut-down date in order to avoid paying the rehabilitation costs. Still, in principle the results dictate a formula for the size of the payment by the producer to enable the socially optimal rehabilitation operations.

A Appendix

A.1 Proof of Proposition 1

First, helpful lemmas are developed.

Lemma A.1. $N_n(t; n_{T_1}, T_1) > 0$.

Proof. (The proof is the same as in Lappi (2018) apart from the notation and is shown here for completeness.) The qualitative theory of differential equations is applied. Consider the differential equation $\dot{N} = -f(N)$ on an interval $[T_1 - \epsilon, T_2]$ for small $\epsilon > 0$. The solution exists on this interval due to $|f'(N)| \leq B$ by Theorem 7.6.2 on page 278 in Sydsæter *et al.* (2008). Note that

$$N_n(t; n_{T_1}, T_1) = p(t; \tilde{t}_0), \quad \tilde{t}_0 = T_1, \tag{A.1}$$

in which the function $p(t; \tilde{t}_0)$ solves the differential equation

$$\dot{z} = -f'(N(t; n_{T_1}, T_1))z, \tag{A.2}$$

and satisfies $p(T_1; T_1) = 1$ (see Sydsæter *et al.* (2008) Theorem 7.6.3 on page 279; the function p is the resolvent of (A.2)). Hence the partial derivative $N_n(t; n_{T_1}, T_1)$ has the same sign as the function p at t. By the linearity of the differential equation (A.2) and the initial condition $p(T_1; T_1) = 1$, the solution p, which is C^1, is always strictly positive: If it is negative somewhere, say at $\hat{t}$, it would have to be zero for some $t_1 < \hat{t}$ with $\dot{p}(t_1; T_1) = 0$ by (A.2). But once zero, the solution is always zero, which contradicts for example the definition of $\hat{t}$. Hence $N_n(t; n_{T_1}, T_1) > 0$. ☐

Lemma A.2.

(i) If $\tau^* \in (T_1, T_2)$ on some open n-interval $(n^1_{T_1}, n^2_{T_1})$, then $\tau_n(n_{T_1}, T_1) > 0$ and $v_n(n_{T_1}, T_1) = 0$ on that interval.

(ii) If $\tau^* = T_1$ on some open n-interval, then $\tau_n(n_{T_1}, T_1) = 0$ and $v_n(n_{T_1}, T_1) = 1$ on that interval. If $\tau^* = T_2$ on some open n-interval, then $\tau_n(n_{T_1}, T_1) = 0$ and $v_n(n_{T_1}, T_1) > 0$ on that interval.

Proof. (i) (The proof is the same as in Lappi (2018) apart from the notation and is shown here for completeness.) The optimal interior rehabilitation date satisfies the equation

$$D(N(\tau^*; n_{T_1}, T_1)) - C(N(\tau^*; n_{T_1}, T_1))r - C'(N(\tau^*; n_{T_1}, T_1))f(N(\tau^*; n_{T_1}, T_1)) = 0, \quad \text{(A.3)}$$

and the optimal size of the rehabilitation is given by

$$v^* = N(\tau^*; n_{T_1}, T_1). \quad \text{(A.4)}$$

To shorten the notation, denote

$$\begin{aligned} \Delta :=& D'(N(\tau; n_{T_1}, T_1)) - C'(N(\tau; n_{T_1}, T_1))r \\ & - C''(N(\tau; n_{T_1}, T_1))f(N(\tau; n_{T_1}, T_1)) - C'(N(\tau; n_{T_1}, T_1))f'(N(\tau; n_{T_1}, T_1)). \end{aligned} \quad \text{(A.5)}$$

By Equation (14) $\Delta < 0$ at $\tau = \tau^*$. Apply then the Implicit Function Theorem to Equation (A.3) to obtain

$$\tau_n(n_{T_1}, T_1) = -\frac{\Delta N_n(\tau^*; n_{T_1}, T_1)}{\Delta \dot{N}(\tau^*; n_{T_1}, T_1)} = -\frac{N_n(\tau^*; n_{T_1}, T_1)}{\dot{N}(\tau^*; n_{T_1}, T_1)}, \quad \text{(A.6)}$$

and implicitly differentiate Equation (A.4) with respect to n to obtain

$$v_n(n_{T_1}, T_1) = \dot{N}(\tau^*; n_{T_1}, T_1)\tau_n(n_{T_1}, T_1) + N_n(\tau^*; n_{T_1}, T_1). \quad \text{(A.7)}$$

Plug (A.6) into (A.7) to obtain $v_n(n_{T_1}, T_1) = 0$. The second result, that $\tau_n(n_{T_1}, T_1) > 0$, follows from Lemma A.1 and from (A.6).

(ii) Suppose that $\tau^* = T_1$ or $\tau^* = T_2$ on some open n-interval. Clearly $\tau_n(n_{T_1}, T_1) = 0$. Furthermore, since $v(n_{T_1}, T_1) = n_{T_1}$ for $\tau^* = T_1$, $v_n(n_{T_1}, T_1) = 1$ for $\tau^* = T_1$. Consider then the case with $\tau^* = T_2$, and note that it follows from the identity $v(n_{T_1}, T_1) = N(\tau(n_{T_1}, T_1); n_{T_1}, T_1)$ that

$$v_n(n_{T_1}, T_1) = \dot{N}(\tau^*; n_{T_1}, T_1)\tau_n(n_{T_1}, T_1) + N_n(\tau^*; n_{T_1}, T_1). \tag{A.8}$$

Combining this with $\tau^* = T_2$ gives

$$v_n(n_{T_1}, T_1) = \dot{N}(T_2; n_{T_1}, T_1)\tau_n(n_{T_1}, T_1) + N_n(T_2; n_{T_1}, T_1) = N_n(T_2; n_{T_1}, T_1), \tag{A.9}$$

which is strictly positive by Lemma A.1. □

Proof of Part (i) of Proposition 1:

Proof. The partial derivative of K with respect to n evaluated at (n_{T_1}, T_1) is

$$\begin{aligned} K_n(n_{T_1}, T_1) = {} & C'(v(n_{T_1}, T_1))v_n(n_{T_1}, T_1)e^{-r(\tau(n_{T_1}, T_1) - T_1)} \\ & - r\tau_n(n_{T_1}, T_1)C(v(n_{T_1}, T_1))e^{-r(\tau(n_{T_1}, T_1) - T_1)}. \end{aligned} \tag{A.10}$$

Note that for all $n_{T_1} \in (n^1_{T_1}, n^2_{T_1})$, $\tau^* \in (T_1, T_2)$, which implies by Part (i) of Lemma A.2 that $v_n(n_{T_1}, T_1) = 0$. Hence Equation (A.10) simplifies to

$$K_n(n_{T_1}, T_1) = -r\tau_n(n_{T_1}, T_1)C(v(n_{T_1}, T_1))e^{-r(\tau(n_{T_1}, T_1) - T_1)}. \tag{A.11}$$

This is strictly negative since $\tau_n(n_{T_1}, T_1) > 0$ by Part (i) of Lemma A.2. □

Proof of Part (ii) of Proposition 1:

Proof. The rehabilitation date is on the boundary of the interval $[T_1, T_2]$ for all $n_{T_1} \in (0, n^1_{T_1}) \cup (n^2_{T_1}, \infty)$. This implies by Lemma A.2 that $\tau_n(n_{T_1}, T_1) = 0$ and $v_n(n_{T_1}, T_1) > 0$. Then

$$K_n(n_{T_1}, T_1) = C'(v(n_{T_1}, T_1))v_n(n_{T_1}, T_1)e^{-r(\tau(n_{T_1}, T_1) - T_1)} > 0 \tag{A.12}$$

by Equation (A.10). □

Proof of Part (iii) of Proposition 1:

Proof. The envelope theorem (Theorem 6.2 in Carter (2001); Milgrom and Segal (2002)) is applied to optimization problem (6) with fixed T_1. This gives

$$
\begin{aligned}
S_n(n_{T_1}, T_1) &= V_n(n_{T_1}, T_1)e^{-rT_1} && \text{(A.13)}\\
&= \Bigg(\int_{T_1}^{\tau(n_{T_1}, T_1)} -D'(N(t; n_{T_1}, T_1))N_n(t; n_{T_1}, T_1)e^{-r(t-T_1)}\, \mathrm{d}t && \text{(A.14)}\\
&- C'(N(t; n_{T_1}, T_1))N_n(t; n_{T_1}, T_1)e^{-r(\tau(n_{T_1}, T_1)-T_1)}\Bigg)e^{-rT_1}. && \text{(A.15)}
\end{aligned}
$$

This is strictly negative by Lemma A.1. ☐

A.2 Proof of Proposition 2

Note that it follows from the identity $v(n, T) = N(\tau(n, T); n, T)$ (recall Equation (13)) that

$$v_T(n, T) = \dot{N}(\tau^*; n, T)\tau_T(n, T) + N_T(\tau^*; n, T). \tag{A.16}$$

This is used in the next result.

Lemma A.3.

(*i*) *Suppose that* $n_{T_1} \notin \{n^1_{T_1}, n^2_{T_1}\}$. *If* $\tau^* = T_1$, *then*

$$\tau_T(n_{T_1}, T_1) = 1 \quad \textit{and} \quad v_T(n_{T_1}, T_1) = 0. \tag{A.17}$$

(*ii*) *If* $\tau^* \in (T_1, T_2)$, *then*

$$\tau_T(n_{T_1}, T_1) = 1 \quad \textit{and} \quad v_T(n_{T_1}, T_1) = 0. \tag{A.18}$$

(*iii*) *Suppose that* $n_{T_1} \notin \{n^1_{T_1}, n^2_{T_1}\}$. *If* $\tau^* = T_2$, *then*

$$\tau_T(n_{T_1}, T_1) = 0 \quad \textit{and} \quad v_T(n_{T_1}, T_1) = f(N(T_2; n_{T_1}, T_1)). \tag{A.19}$$

Proof. (*i*) Let $\tau^* = T_1$. Clearly $\tau_T(n_{T_1}, T_1) = 1$, since $\tau^* = \tau(n_{T_1}, T_1) = T_1$ by assumption. Furthermore, since the differential equation in Equation (2) is autonomous, it holds that

$$N(t; n_{T_1}, T_1) = N(t - T_1; n_{T_1}, 0). \tag{A.20}$$

Therefore

$$N_T(t; n_{T_1}, T_1) = \dot{N}(t - T_1; n_{T_1}, 0) \cdot (-1) = -\dot{N}(t; n_{T_1}, T_1) \tag{A.21}$$

for all $t \in [T_1, T_2]$. This, Equation (A.16) and $\tau_T(n_{T_1}, T_1) = 1$ imply that $v_T(n_{T_1}, T_1) = 0$.

(ii) Let $\tau^* \in (T_1, T_2)$. Condition (12) holds as an equality. Differentiating it with respect to variable T and evaluating at (n_{T_1}, T_1) one obtains the following equation:

$$D'(N(\tau^*; n_{T_1}, T_1))\Big[\dot{N}(\tau^*; n_{T_1}, T_1)\tau_T(n_{T_1}, T_1) + N_T(\tau^*; n_{T_1}, T_1)\Big] \tag{A.22}$$
$$- rC'(N(\tau^*; n_{T_1}, T_1))\Big[\dot{N}(\tau^*; n_{T_1}, T_1)\tau_T(n_{T_1}, T_1) + N_T(\tau^*; n_{T_1}, T_1)\Big] \tag{A.23}$$
$$- C''(N(\tau^*; n_{T_1}, T_1))f(N(\tau^*; n_{T_1}, T_1))\Big[\dot{N}(\tau^*; n_{T_1}, T_1)\tau_T(n_{T_1}, T_1) + N_T(\tau^*; n_{T_1}, T_1)\Big] \tag{A.24}$$
$$- C'(N(\tau^*; n_{T_1}, T_1))f'(N(\tau^*; n_{T_1}, T_1))\Big[\dot{N}(\tau^*; n_{T_1}, T_1)\tau_T(n_{T_1}, T_1) \tag{A.25}$$
$$+ N_T(\tau^*; n_{T_1}, T_1)\Big] = 0. \tag{A.26}$$

The term $\dot{N}(\tau^*; n_{T_1}, T_1)\tau_T(n_{T_1}, T_1) + N_T(\tau^*; n_{T_1}, T_1)$ forms a common term, and the term, which multiplies it, is strictly negative by Equation (14). Hence,

$$\tau_T(n_{T_1}, T_1) = -\frac{N_T(\tau^*; n_{T_1}, T_1)}{\dot{N}(\tau^*; n_{T_1}, T_1)}, \tag{A.27}$$

which implies that $\tau_T(n_{T_1}, T_1) = 1$ since $N_T(t; n_{T_1}, T_1) = -\dot{N}(t; n_{T_1}, T_1)$ also at $t = \tau^*$. The proof that $v_T(n_{T_1}, T_1) = 0$ is the same as in the proof of Part (i).

(iii) Let $\tau^* = T_2$. Equation $\tau_T(n_{T_1}, T_1) = 0$ follows from the equations $\tau^* = \tau(n_{T_1}, T_1) = T_2$. To show that the latter equation, $v_T(n_{T_1}, T_1) = f(N(T_2; n_{T_1}, T_1))$, holds, note that it follows from (A.16) that $v_T(n_{T_1}, T_1) = N_T(T_2; n_{T_1}, T_1)$. Combining this with equation

$$N_T(T_2; n_{T_1}, T_1) = -\dot{N}(T_2; n_{T_1}, T_1) = f(N(T_2; n_{T_1}, T_1)) \tag{A.28}$$

yields the result. □

Proof of Part (i) of Proposition 2:

Proof. Note that by Equation (16) the formula for $K_T(n_{T_1}, T_1)$ is

$$K_T(n_{T_1}, T_1) = C'(v(n_{T_1}, T_1))e^{-r(\tau(n_{T_1}, T_1) - T_1)}v_T(n_{T_1}, T_1) \tag{A.29}$$
$$+ C(v(n_{T_1}, T_1))\Big(- r(\tau_T(n_{T_1}, T_1) - 1)\Big)e^{-r(\tau(n_{T_1}, T_1) - T_1)}. \tag{A.30}$$

Also, differentiating function S in (16) with respect to T gives

$$S_T(n,T) = V_T(n,T)e^{-rT} - rV(n,T)e^{-rT} \tag{A.31}$$

$$= e^{-rT}\Big[V_T(n,T) - rV(n,T)\Big] \tag{A.32}$$

$$= e^{-rT}\Bigg[D(N(T;n,T)) - D(N(\tau(n,T);n,T))e^{-r(\tau(n,T)-T)}\tau_T(n,T) \tag{A.33}$$

$$+ \int_T^{\tau(n,T)} -D'(N(t;n,T))N_T(t;n,T)e^{-r(t-T)} - D(N(t;n,T))re^{-r(t-T)} \,\mathrm{d}t \tag{A.34}$$

$$- C'(v(n,T))e^{-r(\tau(n,T)-T)}v_T(n,T) \tag{A.35}$$

$$- C(v(n,T))\Big(-r(\tau_T(n,T)-1)\Big)e^{-r(\tau(n,T)-T)} \tag{A.36}$$

$$- r\left(\int_T^{\tau(n,T)} -D(N(t;n,T))e^{-r(t-T)} \,\mathrm{d}t - C(v(n,T))e^{-r(\tau(n,T)-T)}\right)\Bigg]. \tag{A.37}$$

After canceling terms, one obtains that

$$S_T(n,T) = e^{-rT}\Bigg(D(N(T;n,T)) - D(N(\tau(n,T);n,T))e^{-r(\tau(n,T)-T)}\tau_T(n,T) \tag{A.38}$$

$$+ \int_T^{\tau(n,T)} -D'(N(t;n,T))N_T(t;n,T)e^{-r(t-T)} \,\mathrm{d}t \tag{A.39}$$

$$- C'(v(n,T))e^{-r(\tau(n,T)-T)}v_T(n,T) \tag{A.40}$$

$$+ rC(v(n,T))e^{-r(\tau(n,T)-T)}\tau_T(n,T)\Bigg). \tag{A.41}$$

Part (i) of Lemma A.3 says that $\tau_T(n_{T_1},T_1) = 1$ and $v_T(n_{T_1},T_1) = 0$. Applying these to Equation (A.29)-(A.30) implies that $K_T(n_{T_1},T_1) = 0$.

Let $\tau^* = T_1$. Then the formula for $S_T(n,T)$, (A.38)-(A.41), evaluated at (n_{T_1},T_1) gives after using Part (i) of Lemma A.3

$$S_T(n_{T_1},T_1) = e^{-rT_1}\Bigg(D(n_{T_1}) - D(n_{T_1}) \tag{A.42}$$

$$+ \int_{T_1}^{T_1} -D'(N(t;n_{T_1},T_1))N_{T_1}(t;n_{T_1},T_1)e^{-r(t-T_1)} \,\mathrm{d}t \tag{A.43}$$

$$+ rC(v(n_{T_1},T_1))\Bigg). \tag{A.44}$$

Since $v(n_{T_1},T_1) = n_{T_1}$,

$$S_T(n_{T_1},T_1) = rC(n_{T_1})e^{-rT_1} = -rV(n_{T_1},T_1)e^{-rT_1}. \tag{A.45}$$

□

Proof of Part *(ii)* of Proposition 2:

Proof. Part *(ii)* of Lemma A.3 says that $\tau_T(n_{T_1}, T_1) = 1$ and $v_T(n_{T_1}, T_1) = 0$. Applying these to Equation (A.29)-(A.30) implies that $K_T(n_{T_1}, T_1) = 0$.

Let $\tau^* \in (T_1, T_2)$. By Part *(ii)* of Lemma A.3 $\tau_T(n_{T_1}, T_1) = 1$ and $v_T(n_{T_1}, T_1) = 0$. Using these, equation $N_T(t; n_{T_1}, T_1) = -\dot{N}(t; n_{T_1}, T_1)$ and the formula for $S_T(n, T)$ gives

$$S_T(n_{T_1}, T_1) = e^{-rT_1}\Bigg(D(n_{T_1}) \tag{A.46}$$

$$- D(N(\tau(n_{T_1}, T_1); n, T_1))e^{-r(\tau(n_{T_1}, T_1)-T_1)} \tag{A.47}$$

$$+ \int_{T_1}^{\tau(n_{T_1}, T_1)} -D'(N(t; n_{T_1}, T_1))(-\dot{N}(t; n_{T_1}, T_1))e^{-r(t-T_1)}\,\mathrm{d}t \tag{A.48}$$

$$+ rC(v(n_{T_1}, T_1))e^{-r(\tau(n_{T_1}, T_1)-T_1)}\Bigg). \tag{A.49}$$

Integrating by parts gives

$$S_T(n_{T_1}, T_1) = e^{-rT_1}\Bigg(D(n_{T_1}) \tag{A.50}$$

$$- D(N(\tau(n_{T_1}, T_1); n, T_1))e^{-r(\tau(n_{T_1}, T_1)-T_1)} \tag{A.51}$$

$$+ \Big/_{T_1}^{\tau(n_{T_1}, T_1)} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} \tag{A.52}$$

$$- \int_{T_1}^{\tau(n_{T_1}, T_1)} D(N(t; n_{T_1}, T_1))(-r)e^{-r(t-T_1)}\,\mathrm{d}t \tag{A.53}$$

$$+ rC(v(n_{T_1}, T_1))e^{-r(\tau(n_{T_1}, T_1)-T_1)}\Bigg) \tag{A.54}$$

$$= e^{-rT_1}\Bigg(r\int_{T_1}^{\tau(n_{T_1}, T_1)} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)}\,\mathrm{d}t \tag{A.55}$$

$$+ rC(v(n_{T_1}, T_1))e^{-r(\tau(n_{T_1}, T_1)-T_1)}\Bigg) \tag{A.56}$$

$$= -rV(n_{T_1}, T_1)e^{-rT_1}. \tag{A.57}$$

□

Proof of Part *(iii)* of Proposition 2:

Proof. Part (iii) of Lemma A.3 says that $\tau_T(n_{T_1}, T_1) = 0$ and $v_T(n_{T_1}, T_1) = f(N(T_2; n_{T_1}, T_1))$. Applying these to Equation (A.29)-(A.30) implies that

$$K_T(n_{T_1}, T_1) = C'(v(n_{T_1}, T_1))e^{-r(T_2-T_1)} f(N(T_2; n_{T_1}, T_1)) \tag{A.58}$$
$$+ rC(v(n_{T_1}, T_1))e^{-r(T_2-T_1)}. \tag{A.59}$$

Let $\tau^* = T_2$. Using Part (iii) of Lemma A.3, equation $N_T(t; n_{T_1}, T_1) = -\dot{N}(t; n_{T_1}, T_1)$ and the formula for $S_T(n, T)$ gives, after integration by parts,

$$S_T(n_{T_1}, T_1) = e^{-rT_1}\Bigg(D(n_{T_1}) \tag{A.60}$$
$$+ \int_{T_1}^{T_2} -D'(N(t; n_{T_1}, T_1))(-\dot{N}(t; n_{T_1}, T_1))e^{-r(t-T_1)} \text{ d}t \tag{A.61}$$
$$- C'(v(n_{T_1}, T_1))e^{-r(T_2-T)} f(N(T_2; n_{T_1}, T_1))\Bigg) \tag{A.62}$$
$$= e^{-rT_1}\Bigg(D(n_{T_1}) + \Big/_{T_1}^{T_2} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} \tag{A.63}$$
$$+ r\int_{T_1}^{T_2} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} \text{ d}t \tag{A.64}$$
$$- C'(v(n_{T_1}, T_1)) f(N(T_2; n_{T_1}, T_1))e^{-r(T_2-T_1)}\Bigg) \tag{A.65}$$
$$= e^{-rT_1}\Bigg(D(N(T_2; n_{T_1}, T_1))e^{-r(T_2-T_1)} \tag{A.66}$$
$$- C'(v(n_{T_1}, T_1)) f(N(T_2; n_{T_1}, T_1))e^{-r(T_2-T_1)} \tag{A.67}$$
$$+ r\int_{T_1}^{T_2} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} \text{ d}t\Bigg). \tag{A.68}$$

□

A.3 Proof of Proposition 3

The following short-hands are used to simplify the notation: $C(t; \cdot) := C(N(t; n_{T_1}, T_1), R(t; n_{T_1}, T_1))$ and $D(t; \cdot) := D(N(t; n_{T_1}, T_1))$.

(i) The partial derivative of Ψ with respect to n evaluated at (n_{T_1}, T_1) is

$$\Psi_n(n_{T_1}, T_1) = \int_{T_1}^{T_2} -\Big(C_N(t; \cdot)N_n(t; \cdot) + C_R(t; \cdot)R_n(t; \cdot) + D'(t; \cdot)N_n(t; \cdot)\Big)e^{-rt} \text{ d}t \tag{A.69}$$
$$+ \int_{T_1}^{\infty} -D'(t; \cdot)N_n(t; \cdot)e^{-rt} \text{ d}t, \tag{A.70}$$

The required result follows from this, from assumptions in (22) and from assumption $D'(N) \geq -C_N(N,R)$.

(ii) The partial derivative of Ψ with respect to T evaluated at (n_{T_1}, T_1) is

$$
\begin{aligned}
\Psi_T(n_{T_1}, T_1) &= \Big(\Phi_T(n_{T_1}, T_1) - r\Phi(n_{T_1}, T_1)\Big)e^{-rT_1} && \text{(A.71)}\\
&= \Bigg(C(N(T_1; n_{T_1}, T_1), R(T_1; n_{T_1}, T_1)) + D(N(T_1; n_{T_1}, T_1)) && \text{(A.72)}\\
&\quad + \int_{T_1}^{T_2} -\Big(C_N(t;\cdot)N_T(t;\cdot) + C_R(t;\cdot)R_T(t;\cdot) + D'(t;\cdot)N_T(t;\cdot)\Big)e^{-r(t-T_1)} && \text{(A.73)}\\
&\quad + r\Big(-C(t;\cdot) - D(t;\cdot)\Big)e^{-r(t-T_1)}\,\mathrm{d}t && \text{(A.74)}\\
&\quad + \int_{T_2}^{\infty} -D'(t;\cdot)N_T(t;\cdot)e^{-r(t-T_1)} - rD(t;\cdot)e^{-r(t-T_1)}\,\mathrm{d}t && \text{(A.75)}\\
&\quad - r\int_{T_1}^{T_2} -\Big(C(t;\cdot) + D(t;\cdot)\Big)e^{-r(t-T_1)}\,\mathrm{d}t && \text{(A.76)}\\
&\quad - r\int_{T_2}^{\infty} -D(t;\cdot)e^{-r(t-T_1)}\,\mathrm{d}t\Bigg)e^{-rT_1}. && \text{(A.77)}
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
\Psi_T(n_{T_1}, T_1) = \Bigg(&C(N(T_1; n_{T_1}, T_1), R(T_1; n_{T_1}, T_1)) + D(N(T_1; n_{T_1}, T_1)) && \text{(A.78)}\\
&+ \int_{T_1}^{T_2} -\Big(C_N(t;\cdot)N_T(t;\cdot) + C_R(t;\cdot)R_T(t;\cdot) + D'(t;\cdot)N_T(t;\cdot)\Big)e^{-r(t-T_1)}\,\mathrm{d}t && \text{(A.79)}\\
&+ \int_{T_2}^{\infty} -D'(t;\cdot)N_T(t;\cdot)e^{-r(t-T_1)}\,\mathrm{d}t\Bigg)e^{-rT_1}. && \text{(A.80)}
\end{aligned}
$$

Following Caputo and Wilen (1995), one can form a three-dimensional differential equation system with state variables N, R and λ (the costate variable in Caputo's and Wilen's model), which governs the dynamics of the system. The system, consisting of equations (7a), (7c) and (9) given in Caputo and Wilen, is autonomous. Hence

$$
y_T(t; n_{T_1}, T_1) = \dot{y}(t - T_1; n_{T_1}, 0)\cdot(-1) = -\dot{y}(t; n_{T_1}, T_1) \tag{A.81}
$$

for all $y = N, R, \lambda$. This implies that

$$
\begin{aligned}
-\Big(C_N(t;\cdot)N_T(t;\cdot) + C_R(t;\cdot)R_T(t;\cdot) + D'(t;\cdot)N_T(t;\cdot)\Big) &= && \text{(A.82)}\\
&= C_N(t;\cdot)\dot{N}(t;\cdot) + C_R(t;\cdot)\dot{R}(t;\cdot) + D'(t;\cdot)\dot{N}(t;\cdot) && \text{(A.83)}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(C(t;\cdot) + D(t;\cdot)\Big) && \text{(A.84)}
\end{aligned}
$$

and

$$-D'(t;\cdot)N_T(t;\cdot) = \frac{\mathrm{d}}{\mathrm{d}t}D(t;\cdot). \tag{A.85}$$

Therefore, integrating (A.79) and (A.80) by parts, gives

$$\Psi_T(n_{T_1}, T_1) = \Bigg(C(N(T_1; n_{T_1}, T_1), R(T_1; n_{T_1}, T_1)) + D(N(T_1; n_{T_1}, T_1)) \tag{A.86}$$

$$+ \Big/_{T_1}^{T_2} \Big(C(N(t;\cdot), R(t;\cdot)) + D(N(t;\cdot))\Big)e^{-r(t-T_1)} \tag{A.87}$$

$$- \int_{T_1}^{T_2} \Big(C(N(t;\cdot), R(t;\cdot) + D(N(t;\cdot))\Big)(-r)e^{-r(t-T_1)}\ \mathrm{d}t \tag{A.88}$$

$$+ \Big/_{T_2}^{\infty} D(N(t;\cdot))e^{-r(t-T_1)} - \int_{T_2}^{\infty} D(N(t;\cdot))(-r)e^{-r(t-T_1)}\ \mathrm{d}t\Bigg)e^{-rT_1} \tag{A.89}$$

Simplifying yields

$$\Psi_T(n_{T_1}, T_1) = \Bigg(C(N(T_2; n_{T_1}, T_1), R(T_2; n_{T_1}, T_1))e^{-r(T_2-T_1)} \tag{A.90}$$

$$+ r\int_{T_1}^{T_2} \Big(C(N(t;\cdot), R(t;\cdot)) + D(N(t;\cdot))\Big)e^{-r(t-T_1)}\ \mathrm{d}t \tag{A.91}$$

$$+ r\int_{T_2}^{\infty} D(N(t;\cdot))e^{-r(t-T_1)}\ \mathrm{d}t\Bigg)e^{-rT_1} \tag{A.92}$$

$$= -r\Phi(n_{T_1}, T_1)e^{-rT_1} + C(N(T_2; n_{T_1}, T_1), R(T_2; n_{T_1}, T_1))e^{-rT_2} \tag{A.93}$$

$$> 0. \tag{A.94}$$

A.4 A third rehabilitation model

The socially optimal rehabilitation level can be modeled as the solution to problem

$$\max_{\{R\}} \Big\{ -D(n_{T_1} - R) - C(R)\Big\}. \tag{A.95}$$

An interior solution satisfies equation $D'(n_{T_1} - R) = C'(R)$, and therefore

$$R_n(n_{T_1}) = -\frac{D''(n_{T_1} - R(n_{T_1}))}{-D''(n_{T_1} - R(n_{T_1})) - C''(R(n_{T_1}))} > 0. \tag{A.96}$$

Define the discounted value of the program with $\Lambda(n, T) := \Big(-D(n-R(n)) - C(R(n))\Big)e^{-rT}$. Then

$$\Lambda_T(n_{T_1}, T_1) = -r\Big(-D(n_{T_1} - R(n_{T_1})) - C(R(n_{T_1}))\Big)e^{-rT_1} > 0, \tag{A.97}$$

$$\Lambda_n(n_{T_1}, T_1) = -D'(n_{T_1} - R(n_{T_1}))e^{-rT_1} < 0. \tag{A.98}$$

Note that these results are similar to the ones found in the first rehabilitation model (compare to propositions 1 and 2, if rehabilitation occurs there at time $t = T_1$. The discounted rehabilitation costs, $\Omega(n_{T_1}, T_1) := C(R(n_{T_1}))e^{-rT_1}$, satisfies properties $\Omega_T(n_{T_1}, T_1) = -rC(R(n_{T_1}))e^{-rT_1} < 0$ and $\Omega_n(n_{T_1}, T_1) = C'(R(n_{T_1}))R_n(n_{T_1})e^{-rT_1} > 0$.

A.5 Proof of Lemma 1

Let the project be profitable and suppose that the terminal condition in (49) is met. Equation (60) implies that $\eta(t) = ke^{-rt}$ for some real number k. If $\theta \in (0, 1)$ maximizes the Hamiltonian, then $k = 1$ by (56).

Suppose that the maximizer is on the boundary of $[0, 1]$. Since the project has to collect sufficient amount on money to the bank account, $\theta = 1$, $q > 0$ and $Z(q) > 0$ on some interval. Since $\theta = 1$, inequality $Z(q)(ke^{-rt} - e^{-rt}) \geq 0$ holds on that interval. This implies that $ke^{-rt} - e^{-rt} \geq 0$ since $Z(q) > 0$. Hence $k \geq 1$. Similarly, since the project is profitable, there exists an interval on which $\theta = 0$, $q > 0$ and $Z(q) > 0$. On that interval inequality $Z(q)(ke^{-rt} - e^{-rt}) \leq 0$ holds and therefore $k \leq 1$. Hence $k = 1$.

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B Supplementary material for reviewers

This is supplementary material to manuscript "A model of optimal extraction and rehabilitation".

Supplementary material for page 10: In the paper, the following maximization problem (Equation (6)) is analyzed:

$$\max_{\tau\in[T_1,T_2]} \left\{ \int_{T_1}^{\tau} -D(N(t;n_{T_1},T_1))e^{-r(t-T_1)}\ \mathrm{d}t - C(N(\tau;n_{T_1},T_1))e^{-r(\tau-T_1)} \right\}, \tag{B.1}$$

and it is assumed that (Equation (14))

$$\begin{aligned} D'(N(\tau^*;n_{T_1},T_1)) - C'(N(\tau^*;n_{T_1},T_1))r - C''(N(\tau^*;n_{T_1},T_1))f(N(\tau^*;n_{T_1},T_1)) \\ - C'(N(\tau^*;n_{T_1},T_1))f'(N(\tau^*;n_{T_1},T_1)) < 0. \end{aligned} \tag{B.2}$$

Denote the left-side of this inequality with Y. The following claim is made on page 10:

"*It can be calculated that if the second derivative of the objective function in* (B.1) *is strictly negative (which implies that the function is strictly concave), then the inequality in* (14) *holds for an optimal interior rehabilitation date.*"

This claim is verified here. The first derivative of the objective function in Equation (B.1) is

$$- D(N(\tau;n_{T_1},T_1))e^{-r(\tau-T_1)} - C'(N(\tau;n_{T_1},T_1))\dot{N}(\tau;n_{T_1},T_1)e^{-r(\tau-T_1)} \tag{B.3}$$

$$+ rC(N(\tau;n_{T_1},T_1))e^{-r(\tau-T_1)}, \tag{B.4}$$

and the second derivative is[22]

$$- D'(N(\tau))\dot{N}(\tau)e^{-r(\tau-T_1)} + rD(N(\tau))e^{-r(\tau-T_1)} \tag{B.5}$$

$$- C''(N(\tau))\dot{N}(\tau)^2e^{-r(\tau-T_1)} - C'(N(\tau))\ddot{N}(\tau)e^{-r(\tau-T_1)} \tag{B.6}$$

$$+ rC'(N(\tau))\dot{N}(\tau)e^{-r(\tau-T_1)} + rC'(N(\tau))\dot{N}(\tau)e^{-r(\tau-T_1)} \tag{B.7}$$

$$- r^2C(N(\tau))e^{-r(\tau-T_1)}. \tag{B.8}$$

This is strictly negative by the assumption in the claim. Note that the first derivative (B.3)-(B.4) equals zero at $\tau^* \in (T_1,T_2)$. Hence, evaluating the second derivative at

[22]The explicit dependency of N on the parameters is left-out from the notation.

$\tau = \tau^* \in (T_1, T_2)$ gives

$$
\begin{aligned}
&- D'(N(\tau^*))\dot{N}(\tau^*)e^{-r(\tau^*-T_1)} && \text{(B.9)}\\
&- C''(N(\tau^*))\dot{N}(\tau^*)^2e^{-r(\tau^*-T_1)} - C'(N(\tau^*))\ddot{N}(\tau^*)e^{-r(\tau^*-T_1)} && \text{(B.10)}\\
&+ rC'(N(\tau^*))\dot{N}(\tau^*)e^{-r(\tau^*-T_1)}. && \text{(B.11)}
\end{aligned}
$$

Using equation $\ddot{N}(\tau^*) = -f'(N(\tau^*))\dot{N}(\tau^*)$ and the definition of Y, the second derivative becomes

$$
\begin{aligned}
&- D'(N(\tau^*))\dot{N}(\tau^*)e^{-r(\tau^*-T_1)} - C''(N(\tau^*))\dot{N}(\tau^*)^2e^{-r(\tau^*-T_1)} && \text{(B.12)}\\
&+ C'(N(\tau^*))f'(N(\tau^*))\dot{N}(\tau^*)e^{-r(\tau^*-T_1)} + rC'(N(\tau^*))\dot{N}(\tau^*)e^{-r(\tau^*-T_1)} && \text{(B.13)}\\
&= -\dot{N}(\tau^*)e^{-r(\tau^*-T_1)}\Big[D'(N(\tau^*)) - C''(N(\tau^*))f(N(\tau^*)) \\
&- C'(N(\tau^*))f'(N(\tau^*)) - C'(N(\tau^*))r\Big] && \text{(B.14)}\\
&= -\dot{N}(\tau^*)e^{-r(\tau^*-T_1)}Y, && \text{(B.15)}
\end{aligned}
$$

which is strictly negative by assumption. Since $-\dot{N}(\tau^*) > 0$, this implies that $Y < 0$ as claimed.

Supplementary material for page 15: It is claimed that if $T_2 = \infty$ at the outset, and if its never optimal to rehabilitate, then equation

$$S_T(n_{T_1}, T_1) = -rV(n_{T_1}, T_1)e^{-rT_1} \tag{B.16}$$

holds. The calculations related to this are presented here. Note first that in this case

$$V(n_{T_1}, T_1) = \int_{T_1}^{\infty} -D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)}, \tag{B.17}$$

and $S_T(n_{T_1}, T_1) = V_T(n_{T_1}, T_1)e^{-rT_1} - rV(n_{T_1}, T_1)e^{-rT_1}$. Using these and integration by parts yields

$$S_T(n_{T_1}, T_1) = \Bigg(\int_{T_1}^{\infty} -D'(N(t; n_{T_1}, T_1))N_T(t; n_{T_1}, T_1)e^{-r(t-T_1)} \tag{B.18}$$

$$- rD(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} \, \mathrm{d}t + D(N(T_1; n_{T_1}, T_1)) - rV(n_{T_1}, T_1)\Bigg)e^{-rT_1} \tag{B.19}$$

$$= \Bigg(\Big/_{T_1}^{\infty} D(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} + \int_{T_1}^{\infty} D(N(t; n_{T_1}, T_1))re^{-r(t-T_1)} \, \mathrm{d}t \tag{B.20}$$

$$- \int_{T_1}^{\infty} rD(N(t; n_{T_1}, T_1))e^{-r(t-T_1)} \, \mathrm{d}t + D(N(T_1; n_{T_1}, T_1)) \tag{B.21}$$

$$- rV(n_{T_1}, T_1)\Bigg)e^{-rT_1} \tag{B.22}$$

$$= -rV(n_{T_1}, T_1)e^{-rT_1}. \tag{B.23}$$