Major Exam

Graded

Student

Har Ashish Arora

Total Points

36 / 40 pts

Question 1

Question 1

5 / 6 pts

- ✔ **+ 2 pts** Part (a) is correct
  - **+ 1 pt** To compute solution $y(x) = e^{-5x} + e^{-5x} \int_0^x e^{5t} q(t) dt$
  - **+ 0.5 pts** To observe $q(t)$ is bounded.
  - **+ 0.5 pts** To conclude $y(x)$ is bounded.
- **+ 4 pts** Part (b) is correct
  - ✔ **+ 1.5 pts** To compute the general solution of the homogeneous part, that is , $y_h = c_1 1/x + c_2 log x/x$
  - ✔ **+ 0.5 pts** To compute the Wronskian correctly.
  - **+ 1.5 pts** To compute u_1, u_2 corrcetly in the particular solution $y_p = (1/x)u_1 + (log x/x)u_2$
  - **+ 0.5 pts** To obtain the final solution correctly.
- **+ 0 pts** Part (a) or Part (b) or both not attempted/incorrect.
- **+ 1 pt** Point adjustment

1 Calculation error

Question 2

Question 2

6 / 6 pts

+ 0 pts Not correct/not attempted

Part(a)

+ 3 pts For the correct solution

+ 1.5 pts For showing Wronskian is zero at a common point

+ 1 pt Showing Y1 and Y2 are linearly dependent

+ 0.5 pts For showing y1 and y2 have common zero

+ 1 pt For partially correct approach

Part(b)

+ 3 pts For the correct solutions

+ 1 pt Step-1: for finding two solutions y_1 and y_2 in terms of a_1 and a_2

+ 0.5 pts for partially correct step-1

+ 1 pt step-2: for writing Wronskian in terms of a_1 and a_2

+ 0.5 pts for partially correct step-2

+ 1 pt Step-3: for concluding Wronskian is constant if and only if a_1 is zero with justification

+ 0.5 pts for partially correct step-3

+ 0.5 pts For a partially correct solution

✔ **+ 6 pts** Correct solution

Question 3

Question 3

5 / 5 pts

✔ **+ 2 pts** Part (a) is correct

✔ **+ 3 pts** Part (b) is correct

+ 0 pts Part (a) is incorrect/not answered

+ 0 pts Part (b) is incorrect/not answered

+ 1 pt Part (a): correctly showing $|F(s)| \leq \frac{M}{s}$

+ 1 pt Part(a): Correct justification while showing $\lim_{s \to \infty} F(s) = 0$

+ 0.5 pts Part(b): Writing correctly the following formulas: i. $\mathcal{L}(f'(t))$ and ii. $\mathcal{L}(\int_0^t f(\tau)d\tau)$

+ 0.5 pts Part(b): Getting the correct algebraic equation using Laplace transform to the ode.

+ 1 pt Part(b): Correctly solved $F(s)$

+ 1 pt Part(b): Correctly solved the inverse of Laplace transform to get $y(t)$.

+ 1.5 pts Part(b): Algebraic equation is correct but expression of $F(s)$ is incorrect.

+ 2.5 pts Part(b): If the answer is correct till the inverse but $y(t)$ is incorrect

+ 0.5 pts Part(a): Partially correct justification while showing $\lim_{s \to \infty} F(s) = 0$

+ 0.5 pts Part (a): Partial correct while showing $|F(s)| \leq \frac{M}{s}$

Question 4

Question 4

5 / 5 pts

✔ **+ 5 pts** Correct.

+ 0 pts Incorrect or nothing is written.

+ 0.5 pts For correct expression of $\mathcal{L}(x)$ and for applying correct Laplace transform of $x'(t)$.

+ 0.5 pts For correct expression of $\mathcal{L}(y)$ and for applying correct Laplace transform of $y'(t)$.

+ 1 pt Finding correct $\mathcal{L}(x)$, solving the system of equations.

+ 1 pt Finding correct $\mathcal{L}(y)$, solving the system of equations.

+ 1 pt Finding correct $x(t)$, using the inverse laplace transform.

+ 1 pt Finding correct $y(t)$, using the inverse laplace transform.

Question 5

Question 5

4 / 4 pts

✔ **+ 4 pts** Correct

+ 0 pts Incorrect/ Not attempting

+ 2 pts Finding A(y) with justification

+ 0.5 pts Writing the condition for exact equation

+ 0.5 pts Writing A(y)=y^2 +c or A(y)=y^2 without justification

+ 2 pts Correct solution with justification

+ 0.5 pts Correct solution without justification

+ 0.5 pts Writing F(x,y)= x^2 + y^2x + f(y) or F(x,y)=xy^2 + g(x)

Question 6

Question 6

3 / 4 pts

+ 0.5 pts For calculating correct eigenvalues $\sqrt{2}, \sqrt{2}$.

+ 0.5 pts For computing eigenvector $v = (v_1, v_2)^T$ satisfying $v_1 + \sqrt{2}v_2 = 0$.

+ 0.5 pts For finding $u = (u_1, u_2)^T$ satisfying $u_1 + \sqrt{2}u_2 = v_2$.

+ 0.5 pts For finding the fundamental solutions $y^{(1)} = ve^{\sqrt{2}t}$ and $y^{(2)} = vte^{\sqrt{2}t} + ue^{\sqrt{2}t}$.

✔ **+ 2 pts** For correctly finding the solution of the corresponding homogeneous equation y_h (including correct eigen values, an eigen vector and a generalized eigen vector).

✔ **+ 0.5 pts** For writing the Fundamental matrix $X = [y^{(1)}, y^{(2)}]$.

+ 0.5 pts For correctly finding X^{-1} E.g., $e^{-\sqrt{2}t}\begin{pmatrix} -t & 1-\sqrt{2}t \\ 1 & \sqrt{2} \end{pmatrix}$

✔ **+ 0.5 pts** Writing that the particular solution is of the form $y^{(p)} = Xw(t)$, where $w'(t) = X^{-1}G(t)$.

+ 0.5 pts For determining w. E.g., $(-t^2/2, t)^T$ and hence $y^{(p)} = (e^{\sqrt{2}t}(t - \frac{1}{\sqrt{2}}t^2), e^{\sqrt{2}t}(\frac{t^2}{2}))^T$.

+ 4 pts Completely correct.

+ 0 pts Not attempted or completely incorrect.

Question 7

Question 7

4 / 4 pts

✔ **+ 4 pts** Correct

+ 0 pts Incorrect/Not done

Showing LT

+ 1 pt Correct (either in one step with linear combination or in two steps first for sum and then scalar multiplication)

+ 0.5 pts Partial correct but missed some part in showing LT or just mention that T(alpha X + beta Y) = alpha T(X) + beta T(Y)

Basis of Ker T

+ 1.5 pts correct (skew symmetric matrices [0, 1; -1, 0] , basis in matrix form or 4-tuple form are acceptable)

+ 0.5 pts Part marking

Bases of range T

+ 1.5 pts Correct ([1, 0; 0, 0], [0, 0; 0, 1], [0, 1; 1, 0]). Note that the dimension is 3, which equals the set of symmetric matrices. There may be other valid bases as well. Both matrix forms and 4-tuple forms for basis are acceptable.

+ 0.5 pts Part marking

Question 8

Question 8

4 / 6 pts

+ 0 pts i) Not Attempted or Completely Wrong

+ 1 pt i) for $\lambda = a$, $\frac{a+b\pm\sqrt{(a+b)^2-4ab+8}}{2}$ and we need $a > 0$.

+ 1 pt i) Proving that $(a+b)^2 - 4ab + 8 > 0$ for all the values of a and b.

+ 1 pt i) Showing that $ab > 2$ ensures all the roots are positive.

✔ **+ 3 pts** I) All the steps of I) are correct.

+ 0 pts ii) Not Attempted or Completely Wrong

+ 1 pt ii) Let $\alpha = x + yi$, where x and y are real numbers. Then the image of $M_\alpha(1) = (x-y)1 + y(1+i)$ and $M_\alpha(1+i) = -2y1 + (x+y)(1+i)$

+ 1 pt ii) For writing the matrix using rubrics 7 such that the first row is $(x-y, -2y)$ and second row is $(y, x+y)$

✔ **+ 1 pt** ii) $Ker(M_\alpha) \neq \{0\}$ implies that $x = y = 0$.

+ 3 pts ii) All the steps of ii) are correct.

DEPARTMENT OF MATHEMATICS, IIT DELHI
SEMESTER II 2024 – 25
MTL 101 (Linear Algebra and Differential Equations) - MAJOR EXAM

Date: 01/05/2025 (Thursday) Time: 3:30 pm- 5:30 pm.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

Name:	HAR ASHISH ARORA.		BLOCK LETTER ONLY
Entry Number:	2024EE10904.	**Group:**	13
Gradescope Id:	EE1240904	**Lecture Hall:**	108

Question 1:

a) Let q be a continuous function on $[0,\infty)$ and

$$\lim_{x\to\infty} q(x) = 10.$$

Prove that the solution $y(x)$ of the IVP

$$y'(x) + 5y(x) = q(x), \quad y(0) = 1$$

is bounded on $[0,\infty)$.

b) Find the general solution of the non-homogeneous equation:

$$x^2y'' + 3xy' + y = \frac{1}{x}, \quad x > 0.$$

[2+4=6]

(a) We have $y'(x) + 5y(x) = q(x)$. $y(0) = 1$.

This is a first order linear equation. Integrating factor: $e^{\int 5\,dx} = e^{5x}$

$\Rightarrow y'(x)e^{5(x)} + 5y(x)e^{5x} = q(x)e^{5x}$

$\Rightarrow \dfrac{d(ye^{5x})}{dx} = q(x)e^{5x} \Rightarrow \int d(ye^{5x}) = \int q(x)e^{5x}dx$

$ye^{5x} = \cancel{\frac{1}{e^{5x}}} \int q(x)e^{5x}dx + C$

$$y = \frac{1}{e^{5x}} \int_0^x q(x)\, e^{5x}\, dx + \frac{c}{e^{5x}}$$

If q is continuous function on $[0, \infty)$, then $q\, e^{5x}$ is a continuous function on $[0,\infty)$ and that indicates that the integral of this function exists and is finite.

Furthermore, $\because$ $y(0) = 1$, we can show that $c = 1$.

$$\Rightarrow y = \frac{1}{e^{5x}} \left(\int_0^x q(x) e^{5x} dx + 1 \right).$$

If q is a continuous f^n and $\lim_{x\to\infty} q(x) = 10$ is a finite number, that implies q itself is finite over $[0,\infty)$ and thus $q(x)$ is bounded and thus we can say $|q(x)| \leq M$ for some $M \in \mathbb{R}^+$.

Thus $|y| = \frac{1}{e^{5x}} \left| \int_0^x q(x)\, e^{5x} dx + 1 \right| \leq \frac{1}{e^{5x}} \left(1 + \left|\int_0^x q(x) e^{5x} dx\right|\right)$

$\leq \frac{1}{e^{5x}} \left(1 + \int_0^x |q(x)|\, e^{5x} dx\right) \leq \frac{1}{e^{5x}}\left(1 + M\int_0^x e^{5x} dx\right)$

$= \frac{1}{e^{5x}} \left(Me^{5x} - M + 1\right) = M - \frac{M-1}{e^{5x}}$ and this is bounded in $[0,\infty)$ by some number, say K. Thus the sol^n is bounded on $[0,\infty)$.

(b) We have $x^2 y'' + 3xy' + y = \frac{1}{x}$, $x > 0$.

First, find sol^n of homogenous part, $x^2y'' + 3xy' + y = 0$. This is the euler-cauchy eq^n. Let $y = x^m$ be a sol^n. Then

$$m(m-1)x^m + 3mx^m + x^m = 0 \Rightarrow m(m-1) + 3m + 1 = 0 \quad \because x>0,\ x \neq 0.$$

$$\Rightarrow m^2 - m + 3m + 1 = 0 \Rightarrow m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1.$$

Thus $\frac{1}{x}$ is a sol^n of the homogenous part. To find the other L.I. soln, use reduction of order formula. Let $y_2(x) = v(x)y_1(x)$, where $y_1(x) = \frac{1}{x}$. Then $v'(x) = \int \frac{e^{-\int \frac{3x}{x^2}dx}}{(y_1)^2}dx$

$$= \int \frac{e^{-3\ln x}}{\frac{1}{x^2}}dx = \int \frac{\frac{1}{x^3}}{\frac{1}{x^2}}dx = \int \frac{1}{x}dx = \ln x \Rightarrow \quad \because x>0$$

$y_2(x) = \frac{1}{x}\ln(x)$. Thus general sol^n of homogenous part is $y_h(x) = c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\right)\ln x$. Now, solve for the non-homogenous part. The forcing f^n in standard form is $\frac{1}{x^3}$. ~~This is a part of the homognous sol^n, so try for undetermined coefficients $A(x)\left(\frac{1}{x}\right) = A = y_p$. This will not yield a suitable value of A, so try $A(x)^2\left(\frac{1}{x}\right) = Ax$. You will get For undetermined coefficients, try $\frac{A}{x^3}$~~. Try variation of parameters.

$$\begin{bmatrix} \frac{1}{x} & \frac{1}{x}\ln x \\ -\frac{1}{x^2} & -\frac{\ln x}{x^2} + \frac{1}{x^2} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{x^3} \end{bmatrix}. \quad W(x) = \cancel{-\frac{\ln x}{x^3}} + \frac{1}{x^3} + \cancel{\frac{\ln x}{x^3}} = \frac{1}{x^3}$$

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$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \frac{1}{x}\ln x \\ \frac{1}{x^3} & -\frac{\ln x}{x^2} + \frac{1}{x^2} \end{vmatrix}}{} = x^3\left(-\frac{1}{x^4}\ln x\right) = -\frac{1}{x}\ln x$$

$$v_2' = x^3\left(\frac{1}{x^4}\right) = \frac{1}{x} \Rightarrow \ln x. \qquad v_2 =$$

$$\Rightarrow v_1 = \int -\frac{1}{x}\ln x\, dx = -\frac{(\ln x)^2}{2}.$$

$$\Rightarrow y(x) = c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\right)\ln x - \frac{1}{x}\ln x\left(\frac{1}{x}\right) + \frac{(\ln x)^2}{x}. \text{ (Ans)}$$

$$\cancel{W' = y_1'y_2' - y_1'y_2' + y_1y_2'' - y_1''y_2}$$
$$\cancel{= -y_1(py_2' + qy_2) + y_2(py_1' + qy_1)}$$
$$\cancel{= -p(y_1y_2' - y_2y_1')}$$

Question 2:

a) Show that if two solutions of a second order homogeneous differential equation with continuous coefficients on an interval, $I \subset \mathbb{R}$, have a common zero, then all their zeros are in common.

b) Suppose y_1 and y_2 are two linearly independent solutions of the ODE

$$y'' + a_1y' + a_2y = 0, \quad t \in I$$

where $a_1, a_2 \in \mathbb{R}$.
Show that $W(y_1, y_2)(t)$ is constant if and only if $a_1 = 0$.

[3+3=6]

(a) We will use the Wronskian of the two solutions to show this. We know that for a second order homogenous differential equation with continous coefficients as:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

the wronskian for two solutions $y_1(x), y_2(x)$ is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2,$$

Differentiate the wronskian:

$$W'(x) = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'}$$

$$= y_1y_2'' - y_1''y_2 \quad$$ Use the differental equation to put y_i'' in terms of y_i':

$$W'(x) = -y_1(py_2' + qy_2) + y_2(py_1' + qy_1)$$

$$= -p(y_1y_2' - y_1'y_2) = -p(x)W(x).$$

Thus $W'(x) = -p(x)\,W(x)$

$$\Rightarrow W(x) = Ce^{-\int p(x)\,dx'} \quad \text{where } C \in \mathbb{R}.$$

This implies, importantly, that if $W(x) = 0$ at a particular point, say x_0, Then $W(x) = 0$ at all points in the interval for this.

That means that if $y_1(x), y_2(x)$ have a common zero, say at $x = x_0$, them $W(x_0) = \cancel{y_1(x_0)}^{0}\, y_2'(x_0) - y_1'(x_0)\cdot \cancel{y_2(x_0)}^{0}$
$$\equiv 0.$$

Thus, this necessitates for the conditions applied, that $W(x) = 0$ in that interval $I \subset \mathbb{R}$. If that is true, then

$$y_1(x)\,y_2'(x) = y_1'(x)\,y_2(x)$$

$$\frac{y_1'(x)}{y_1(x)} = \frac{y_2'(x)}{y_2(x)} \Rightarrow \frac{d(y_1)}{\cancel{(dx)}y_1} = \frac{d(y_2)}{\cancel{(dx)}y_2}$$

$$\Rightarrow \ln(y_1) = \ln(y_2) + c \Rightarrow \boxed{y_1 = Ky_2}.$$ (This is true when they have a common zero!).

Thus, $y_1(x) = K\,y_2(x)$, $K \in \mathbb{R}$ when all the cond^{n}s are met. This implies that if they have a common zero, then all their zeroes are common.

(b) I have already derived in part (a) that

$W(t) = Ce^{-\int p(t)\,dt}$, if $p(t)$, $q(t)$ are continuous.
(for homogenous 2nd order diff eq^n).

Here $p(t) = a_1$

($\Rightarrow$) $W(y_1, y_2)(t)$ is constant (does not depend on (t). Thus

$$e^{-\int a_1 dt} = e^{-a_1 t} \text{ must be constant.}$$

Possible only when $a_1 = 0$.

($\Leftarrow$) $a_1 = 0$. $W(y_1, y_2)(t) = Ce^{-\int 0 dt} = C$

$= \text{constant}$.

Thus proved.

Question 3:

(a) Let $F(s) = \mathcal{L}(f(t))(s)$ be defined for $s > 0$. Suppose that $|f(t)| \leq M$ for all $t \geq 0$. Show that

$$\lim_{s \to +\infty} F(s) = 0.$$

(b) Using Laplace transform, solve the IVP

$$y'(t) - 4\int_0^t y(\tau)\, d\tau = 1, \quad y(0) = 0.$$

[2+3=5]

(a) Definition of Laplace transform:

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

We have already applied a significant growth condition stronger than the growth condition normally required for the Laplace transform to exist, which is $|f(t)| \leq M' e^{Kt}$ where M' is a constant and $K < s$. Here, we have just $|f(t)| \leq M$, where $M \in \mathbb{R}$, $M < \infty$. Implies that $f(t)$ is bounded.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \leq \int_0^\infty e^{-st} |f(t)|\, dt$$

$$\leq \int_0^\infty e^{-st} M\, dt = M \int_0^\infty e^{-st}\, dt.$$

Also, $F(s) = \int_0^\infty e^{-st} f(t)\, dt \geq -\int_0^\infty e^{-st} M\, dt$

$$\geq -M \int_0^\infty e^{-st}\, dt.$$

Thus $-M\int_0^\infty e^{-st}\, dt \leq F(s) \leq M\int_0^\infty e^{-st}\, dt.$

~~Let time~~ Apply $\lim_{s \to +\infty}$ on both sides. $e^{-st} \to 0$, then. By sandwich thm,

$\cancel{0 \leq F(s)}$ $0 \leq \lim_{s \to \infty} F(s) \leq 0 \Rightarrow \boxed{\lim_{s \to \infty} F(s) = 0}$.

(b) We wish to solve

$$y'(t) - 4\int_0^t y(\tau)\,d\tau = 1, \quad y(0) = 0.$$

Note that $\int_0^t y(\tau)\,d\tau = y(t) * 1$ (convolution),

And $\mathcal{L}\{f*g\} = \mathcal{L}\{f\} \times \mathcal{L}\{g\}$.

Also, $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$. Applying laplacian to both sides, then: (Let $\mathcal{L}\{y(t)\} = Y(s)$):

$$\mathcal{L}\{y'(t)\} - 4\,\mathcal{L}\{y(t) * 1\} = \mathcal{L}\{1\}$$

$$\Rightarrow s\,Y(s) - 4\,Y(s) \times \mathcal{L}\{1\} = \mathcal{L}\{1\}$$

$$\Rightarrow s\,Y(s) - 4\,Y(s)\left(\frac{1}{s}\right) = \frac{1}{s}$$

$$\Rightarrow Y(s)\left(s - \frac{4}{s}\right) = \frac{1}{s} \Rightarrow Y(s)\left(\frac{s^2-4}{\cancel{s}}\right) = \frac{1}{\cancel{s}}$$

$$\Rightarrow Y(s) = \frac{1}{s^2-4} \Rightarrow \mathcal{L}\{y(t)\} = \frac{1}{s^2-4}$$

We know that $\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}$. Thus, applying inverse laplace to both sides,

$$\boxed{y(t) = \frac{1}{2}\sinh(2t)} = \frac{e^{2t} - e^{-2t}}{4} \quad \text{(Ans.)}$$

Question 4: Using the Laplace transform, solve the following system:

$$x'(t) = 3x(t) + 4y(t) + \delta(t),$$
$$y'(t) = -2x(t) + y(t) + 5\delta(t),$$

with initial conditions

$$x(0) = 1, \quad y(0) = -2,$$

where $\delta(t)$ is the Dirac delta function at $t = 0$.

[5]

Apply the laplace transform on both equations. We get,

$$\mathcal{L}\{x'(t)\} = 3\mathcal{L}\{x(t)\} + 4\mathcal{L}\{y(t)\} + \mathcal{L}\{\delta(t)\}$$

$$\mathcal{L}\{y'(t)\} = -2\mathcal{L}\{x(t)\} + \mathcal{L}\{y(t)\} + 5\mathcal{L}\{\delta(t)\}.$$

Now, $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$. Using this property,

(Let $\mathcal{L}\{x(t)\} = X(s)$, $\mathcal{L}\{y(t)\} = Y(s)$). Also,

$\mathcal{L}\{\delta(t-a)\} = e^{-as} \Rightarrow \mathcal{L}\{\delta(t)\} = e^{-(0)s} = e^0 = 1$

$$\Rightarrow \quad sX(s) - 1 = 3X(s) + 4Y(s) + 1$$

$$sY(s) + 2 = -2X(s) + Y(s) + 5$$

$$\Rightarrow (s-3)X(s) - 4Y(s) = 2 \quad \text{— ①}$$

$$(s-1)Y(s) + 2X(s) = 3 \quad \text{— ②}$$

Multiply eq^n ① by $\frac{2}{s-3}$:

$$2X(s) - \frac{8}{s-3}Y(s) = \frac{4}{s-3}$$

Now, subtract eq^n 2 from this to get

$$-\frac{8Y(s)}{s-3} - (s-1)Y(s) = \frac{4}{s-3} - 3$$

$$\Rightarrow Y(s)\left(\frac{8}{s-3} + s-1\right) = 3 - \frac{4}{s-3}$$

$$Y(s)\left(\frac{8}{s-3} + s-1\right) = 3 - \frac{4}{s-3}$$

$$\Rightarrow Y(s)\left(\frac{8+s^2-4s+3}{\cancel{s-3}}\right) = \frac{3s-9-4}{\cancel{s-3}}$$

$$Y(s) = \frac{3s-13}{s^2-4s+11} = \frac{3s-13}{(s-2)^2+7}$$

$$= \frac{3(s-2)}{(s-2)^2+7} - \frac{7}{(s-2)^2+7}$$

using Laplace inverse, $y(t) = \mathcal{L}^{-1}\left\{\frac{3(s-2)}{(s-2)^2+7} - \frac{7}{(s-2)^2+7}\right\}$

$$\boxed{y(t) = 3e^{2t}\cos\sqrt{7}\,t - \sqrt{7}\,e^{2t}\sin\sqrt{7}\,t}$$

$\because$ $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$, $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$.

(using first shifting theorem).

Similarly, we can solve for $X(s)$:

Multiply equation ② by $\frac{4}{s-1}$ to get:

$$4Y(s) + \frac{8}{s-1}X(s) = \frac{12}{s-1}$$ Add to equation

① to get

$$\frac{8}{s-1}X(s) + (s-3)X(s) = 2 + \frac{12}{s-1}$$

$$\Rightarrow X(s)\left(\frac{8}{s-1} + s-3\right) = 2 + \frac{12}{s-1}$$

$$X(s) \times \left(\frac{8+s^2-4s+3}{\cancel{s-1}} \right) = \frac{2s-2+12}{\cancel{s-1}}$$

$$\Rightarrow X(s) = \frac{2s+10}{s^2-4s+11} = \frac{2s+10}{(s-2)^2+7}$$

$$= \frac{2(s-2)+14}{(s-2)^2+7} = 2\,\frac{(s-2)}{(s-2)^2+7} + \frac{14}{(s-2)^2+7}$$

using Laplace inverse,

$$x(t) = \mathcal{L}^{-1}\left\{ 2\,\frac{(s-2)}{(s-2)^2+7} + \frac{14}{(s-2)^2+7} \right\}$$

$$\boxed{x(t) = 2e^{2t}\cos\sqrt{7}\,t + 2\sqrt{7}\,e^{2t}\sin\sqrt{7}\,t}$$

Using the same formulae I had written earlier when finding $y(t)$.

Question 5: Find $A(y)$ such that equation

$$(2x + A(y))dx + 2xydy = 0$$

is an exact equation and solve it. [4]

Sol^n: For an exact equation of the type

$$Mdx + Ndy = 0,$$

we need $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Here $M = 2x + A(y)$, $N = 2xy$.

$\Rightarrow \dfrac{\partial M}{\partial y} = A'(y)$; $\dfrac{\partial N}{\partial x} = 2y$.

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow A'(y) = 2y \Rightarrow \boxed{A(y) = y^2} + c,\ c \in \mathbb{R}$

(after integrating both sides).

Choose the $A(y)$ to be $y^2 + c$ $\Rightarrow M = 2x + y^2 + c$

To solve, we assume a sol^n of the form

$$u(x,y) = \int M dx + K(y)$$
$$= \int (2x + y^2 + c)dx + K(y) = x^2 + y^2x + cx + K(y).$$

Now, we have, for an exact differential equation,

$\dfrac{\partial u}{\partial y} = N$. Thus, $\dfrac{\partial u}{\partial y} = 2yx + K'(y) = N = 2xy$

$\Rightarrow K'(y) = 0 \Rightarrow K(y) = c_1$ (after integrating).

Thus, the sol^n becomes

$$u(x,y) = x^2 + xy^2 + cx + c_1 = c_0 \quad (\text{we set } u(x,y) = c_0)$$

$$\Rightarrow \quad \boxed{u(x,y) = x^2 + xy^2 + cx = c'}$$

is a sol^n (in the implicit form).

where $c', c \in \mathbb{R}$.

Question 6: Solve the following system of first-order linear ordinary differential equations using the method of variation of parameters:

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{G}(t),$$

where

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 2\sqrt{2} \end{pmatrix}, \quad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{G}(t) = \begin{pmatrix} e^{\sqrt{2}t} \\ 0 \end{pmatrix}.$$

[4]

We have $y'(t) = Ay(t) + G(t)$.

First, solve the homogenous system

$y'(t) = Ay(t)$.

We do this by finding the eigenvalues & eigenvectors of the matrix A.

Using characteristic eq^n to find eigenvalues of A:

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & -2 \\ 1 & 2\sqrt{2}-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(2\sqrt{2}-\lambda) - (-2) = 0$

$\Rightarrow \lambda(\lambda - 2\sqrt{2}) + 2 = 0 \Rightarrow \lambda^2 - 2\sqrt{2}\lambda + 2 = 0$

$\lambda = \dfrac{2\sqrt{2} \pm \sqrt{(2\sqrt{2})^2 - 4(2)}}{2} = \sqrt{2}$.

So, we have a repeated root / eigenvalue, $\lambda = \sqrt{2}$.

We get the eigenvector X_1 corresponding to this eigenvalue, $\lambda = \sqrt{2}$, by solving $AX_1 = \lambda X_1 \Rightarrow (A - \lambda I)X_1 = 0$. Let $X_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

Thus, we have to reduce $(A - \lambda I)$ to RRE.

$A - \lambda I = \begin{bmatrix} -\sqrt{2} & -2 \\ 1 & \sqrt{2} \end{bmatrix} \xrightarrow{R_1 \to -\frac{R_1}{\sqrt{2}}} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$.

This is in RRE. let y_1 be free variables, then

$x_1 = -\sqrt{2}s \Rightarrow X_1 = \begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}$. (An eigenvector corresponding to $\lambda = \sqrt{2}$). Since repeated roots, to get X_2, we use generalised eigenvector notion & solve

$$(A-\lambda I)X_2 = X_1 \Rightarrow \left[\begin{array}{cc|c} -\sqrt{2} & -2 & -\sqrt{2}\\ 1 & \sqrt{2} & 1\end{array}\right] \xrightarrow{R_1 \to -\frac{R_1}{\sqrt{2}}}$$

$$\left[\begin{array}{cc|c} 1 & \sqrt{2} & 1\\ 1 & \sqrt{2} & 1\end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & \sqrt{2} & 1\\ 0 & 0 & 0\end{array}\right]$$ Again let $y_2 = s'$

$\Rightarrow x_2 = 1 - \sqrt{2}s'$. Choose $s' = 0$, $X_2 = \begin{pmatrix}1\\ 0\end{pmatrix}$.

General sol^n $y_h(t)$ for homogenous part is given by

$$y_h(t) = c_1 X_1 e^{\lambda t} + c_2(tX_1 + X_2)e^{\lambda t}$$

$$= c_1 \begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix} e^{\sqrt{2}t} + c_2 \begin{pmatrix}-\sqrt{2}t+1\\ t\end{pmatrix} e^{\sqrt{2}t}.$$

~~The sol^n~~ $= \Phi(t)\, c$, where $\Phi(x)$ is the fundamental matrix $\Phi(t) = \begin{bmatrix}-\sqrt{2} & -\sqrt{2}t+1\\ 1 & t\end{bmatrix} e^{\sqrt{2}t}$.

$y_h(t) = \phi(t)c$. Using variation of parameters, let $c = v(t) \Rightarrow y_p = \phi(t)v(t) \Rightarrow y_p' = \phi' v + v'\phi$. Put in eq^n, we get $\cancel{\phi' v + \phi}\ \phi v' = \cancel{\phi} v + G(t) \Rightarrow \cancel{v = \phi}$

$v' = \phi^{-1}G(t) \Rightarrow v = \int \phi^{-1}G(t)\,dt$.

$\Phi^{-1} = \frac{1}{|\Phi|}\begin{bmatrix} t & \sqrt{2}t-1\\ -1 & -\sqrt{2}\end{bmatrix} e^{\sqrt{2}t}$. $|\phi| = -\sqrt{2}e^{\sqrt{2}t} \times te^{\sqrt{2}t} - e^{\sqrt{2}t}(-\sqrt{2}t+1)e^{\sqrt{2}t}$

$= \cancel{-\sqrt{2}te^{2\sqrt{2}t}} + \cancel{\sqrt{2}te^{2\sqrt{2}t}} - e^{2\sqrt{2}t}$

$$\Rightarrow \Phi^{-1} = -\frac{1}{e^{2\sqrt{2}t}}\begin{bmatrix} t & \sqrt{2}t-1 \\ -1 & -\sqrt{2} \end{bmatrix}$$

Now, $\Phi^{-1} G = -\frac{1}{e^{2\sqrt{2}t}}\begin{bmatrix} t & \sqrt{2}t-1 \\ -1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} e^{\sqrt{2}t} \\ 0 \end{bmatrix}$

$$= -\frac{1}{e^{\sqrt{2}t}}\begin{bmatrix} t & \sqrt{2}t-1 \\ -1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{e^{\sqrt{2}t}}\begin{bmatrix} t \\ -1 \end{bmatrix}$$

$= \begin{bmatrix} -te^{-\sqrt{2}t} \\ e^{-\sqrt{2}t} \end{bmatrix}$ Integrating this wrt t,

$$I_1 = \int -te^{-\sqrt{2}t}dt = -\left(\frac{te^{-\sqrt{2}t}}{-\sqrt{2}} + \frac{1}{\sqrt{2}}\int e^{-\sqrt{2}t}dt\right)$$

$$= \frac{te^{-\sqrt{2}t}}{\sqrt{2}} + \frac{1}{2}e^{-\sqrt{2}t}$$

$I_2 = \int e^{-\sqrt{2}t}dt = -\frac{e^{-\sqrt{2}t}}{\sqrt{2}}$ $\Rightarrow v = \begin{bmatrix} \frac{t}{\sqrt{2}} + \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} e^{-\sqrt{2}t}$

Thus particular sol^n $y_p = \Phi v = \begin{bmatrix} -\sqrt{2} & -\sqrt{2}t+1 \\ 1 & t \end{bmatrix}\begin{bmatrix} \frac{t}{\sqrt{2}} + \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \begin{bmatrix} -\cancel{t} - \frac{1}{\sqrt{2}} + \cancel{t} - \frac{1}{\sqrt{2}} \\ \cancel{\frac{t}{\sqrt{2}}} + \frac{1}{2} - \cancel{\frac{t}{\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \frac{1}{2} \end{bmatrix}$$

Thus, the sol^n for this system of equations is

$$y(t) = c_1\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{\sqrt{2}t} + c_2\begin{pmatrix} -\sqrt{2}t+1 \\ t \end{pmatrix} e^{\sqrt{2}t} + \begin{pmatrix} -\sqrt{2} \\ \frac{1}{2} \end{pmatrix}$$

— (Ans)

Question 7: Consider the vector space of 2×2 real matrices with entrywise addition and entrywise scalar multiplication:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}.$$

Consider $T : V \to V$ defined by

$$T(X) = X + X^T.$$

(a) Show that T is a linear transformation.

(b) Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

[4]

(a) In order for T to be a linear transformation, take two vectors v & u from V and show that

$$T(\alpha u + v) = \alpha T(u) + T(v) \quad \text{where } \alpha \in \mathbb{R} \text{ (field)}.$$

Then $T(\alpha u + v) = (\alpha u + v) + (\alpha u + v)^T$

$= \alpha u + (\alpha u)^T + v + v^T = \alpha u + \alpha u^T + (v + v^T)$

$= \alpha (u + u^T) + (v + v^T) = \alpha T(u) + v.$

Thus T is a linear transformation (rather linear operator).

(b). $\mathrm{Ker}(T) = \{ v \in V : T(v) = 0 \}$..

$T(v) = 0 \Rightarrow v + v^T = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = 0$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = 0$

$a = 0$, $b + c = 0$, $d = 0$. We need the $\boxed{b+c=0}$ to get the ~~Kernel of~~ basis of the Kernel space.

let c be a free variable s, then $b = -c = -s$. Thus, all vectors in the Kernel space satisfy $a = 0$, $d = 0$, $b = -s$, $c = s$.

and are thus written as $\begin{bmatrix} 0 & -s \\ s & 0 \end{bmatrix}$

$$= \cancel{0\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + s} + s\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$\because$ All vectors can be written as linear combination of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, is L.I. (w/ itself).

the basis set for kernel space is $B = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$.

<u>Basis set for range space:</u>

$\mathrm{Im}(T) = \{ T(v) : v \in V \}$

$$T(v) = v + v^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$= \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}.$$ These vectors can all be written as $2a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b+c)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$\because$ $2a$, $(b+c)$, $2d$ can vary freely, call them α_1, α_2, α_3 respectively. Then, all vectors $T(v)$ are written as

$$\alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

which is span of $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = B'$.

These vectors are also all L.I. To prove,

take $\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0.$

$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{\alpha_1 = \alpha_2 = \alpha_3 = 0}$

$\because$ All of them are zero $(\alpha_1, \alpha_2, \alpha_3)$, the vectors are lineaely dependent.

Thus B' spans the image space and is L.I.

Thus B' is the basis set of Im(T).

Question 8:

i) Consider the matrix

$$A = \begin{pmatrix} a & -1 & 0 \\ -1 & b & -1 \\ 0 & -1 & a \end{pmatrix}$$

where a, b are real numbers. Determine the conditions on the pair (a, b) under which all the eigenvalues of A are positive.

ii) Consider $V = \mathbb{C}$, the 2-dimensional vector space of complex numbers over the field of real numbers, $\mathbb{R}$. Let the basis $\mathcal{B} = \{1, 1+i\}$ be ordered as it is displayed. For $\alpha \in \mathbb{C}$, define the linear transformation $M_\alpha : V \to V$ given by

$$M_\alpha(\beta) = \alpha\beta.$$

Compute the matrix $[M_\alpha]_{\mathcal{B}}$. Also, find α for which $\mathrm{Ker}(M_\alpha) \neq 0$.

[3+3=6]

(i) Characteristic equation:

$$0 = |A - \lambda I| = \begin{vmatrix} a-\lambda & -1 & 0 \\ -1 & b-\lambda & -1 \\ 0 & -1 & a-\lambda \end{vmatrix} = (a-\lambda)\left((b-\lambda)(a-\lambda) - 1\right) + (a-\lambda)$$

$$= (a-\lambda)\left((b-\lambda)(a-\lambda) - 2\right) = 0$$

$$= (\lambda-a)\left((\lambda-a)(\lambda-b) - 2\right) = 0$$

$$= (\lambda-a)\left(\lambda^2 - (a+b)\lambda + (ab-2)\right) = 0.$$

For positive eigenvalues,

$$\lambda = a, \ \frac{a+b \pm \sqrt{(a+b)^2 - 4(ab-2)}}{2}$$

$$= a, \ \frac{(a+b) \pm \sqrt{(a-b)^2 + 8}}{2}$$

We need the cond^{n}s $a>0$, $\sqrt{(a-b)^2+8} < a+b$ In order to keep it positive. These are the required cond^{n}s on a & b. $\because$ LHS is +ve, Then RHS also +ve $\Rightarrow$ $\boxed{a+b>0}$.

And squaring,

$$\cancel{a+b}\ a^2+b^2-2ab+8 < a^2+b^2+2ab$$

$$4ab > 8 \Rightarrow ab > 2.$$

$\Rightarrow$ $\boxed{a>0,\ a+b>0,\ ab>2}$ required cond^{n}s.

(ii) $M_\alpha(1) = \alpha$

$M_\alpha(1+i) = \alpha + i\alpha.$

Writing these in terms of the basis values,

$M_\alpha(1) = \alpha = \alpha(1) + 0(1+i)$

$M_\alpha(1+i) = \alpha + i\alpha = 0(1) + \alpha(1+i).$

Thus, the matrix $[M_\alpha]_B$ would be

$[M_\alpha]_B = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}.$ We need $\mathrm{Ker}(M_\alpha) \neq 0$, i.e.

$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, (x,y) should not only be the non-trivial sol^n. ~~There is no such α for which $\mathrm{Ker}(M_\alpha) \neq 0$.~~ The only α for which $\mathrm{Ker}(M_\alpha) \neq 0$ is $\alpha = 0$ $\because$ we have

$x\alpha = 0,\ y\alpha = 0.$ Thus,

$\boxed{\alpha = 0}.$