

Name: \_\_\_\_\_

Entry number: \_\_\_\_\_

There are 2 questions for a total of 10 points.

1. (6 points) Prove or disprove: Given any 17 natural numbers, it is possible to choose 5 whose sum is divisible by 5.

**Solution:** We will prove the above statement. For  $i \in \{0, 1, 2, 3, 4\}$  let  $S_i$  denote the sub-collection of the numbers that leave a remainder  $i$  when divided by 5. We break the proof into the following two cases:

Case 1 : At least one of  $S_0, S_1, S_2, S_3, S_4$  is empty.

In this case, there is one sub-collection that has at least 5 numbers (otherwise the sum of numbers in all the sub-collections cannot exceed 16). WLOG, let this sub-collection be  $S_i$ . Consider any five numbers from  $S_i$ . These numbers can be written as:  $5k_1 + i, 5k_2 + i, 5k_3 + i, 5k_4 + i, 4k_5 + i$  for some natural numbers  $k_1, \dots, k_5$  since  $S_i$  contains numbers that leave remainder  $i$  when divided by  $i$ . The sum of these numbers is  $5(k_1 + k_2 + k_3 + k_4 + k_5) + 5i$  which is divisible by 5.

Case 2 None of the sets are empty.

In this case, consider one number from each of the sub-collections  $S_0, S_1, S_2, S_3, S_4$ . These numbers can be written as  $5k_0, 5k_1+1, 5k_2+2, 5k_3+3, 5k_4+4$  for some natural numbers  $k_0, \dots, k_4$  from the definition of  $S_0, \dots, S_4$ . The sum of these numbers is  $5(k_0 + k_1 + k_2 + k_3 + k_4) + 10$  which is divisible by 5.

2. (4 points) Prove or disprove: For any positive integer  $n$ ,  $n^5 - 5n^3 + 4n$  is always divisible by 5.

**Solution:** We will prove the above statement. Note that  $n^5 - 5n^3 + 4n$  can be factored as follows:

$$n^5 - 5n^3 + 4n = n(n^2 - 1)(n^2 - 4) = (n - 2)(n - 1)n(n + 1)(n + 2).$$

We can now consider the following cases:

1.  $n$  leaves remainder 0 when divided by 5: In this case, the number is divisible by 5.
2.  $n$  leaves remainder 1 when divided by 5: In this case,  $(n - 1)$  is divisible by 5 and hence the number is divisible by 5.
3.  $n$  leaves remainder 2 when divided by 5: In this case,  $(n - 2)$  is divisible by 5 and hence the number is divisible by 5.
4.  $n$  leaves remainder 3 when divided by 5: In this case,  $(n + 2)$  is divisible by 5 and hence the number is divisible by 5.
5.  $n$  leaves remainder 4 when divided by 5: In this case,  $(n + 1)$  is divisible by 5 and hence the number is divisible by 5.

Since  $n^5 - 5n^3 + 4n$  is divisible by 5 in all the above cases (representing all possibilities for number  $n$ ), we conclude that  $n^5 - 5n^3 + 4n$  is divisible by 5 for all positive integers  $n$ .