COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 6: Relations, Functions, Cardinality and Infinite Sets.
6 March 2025

Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM. Questions marked with a * may be a little harder and can be considered optional.

Problem 1 [LLM18, Prob. 4.14]
For sets A, B, C, D, prove that if $A \times B$ and $C \times D$ are disjoint then either A and C are disjoint or B and D are disjoint.

Problem 2
Given binary relations $\mathcal{R} \subseteq A \times B$ and $\mathcal{S} \subseteq B \times C$, the *composition* of $\mathcal{R}$ with $\mathcal{S}$, is defined by $\mathcal{R} \circ \mathcal{S} = \{(a, c) : \exists b \in B : a\mathcal{R}b \wedge b\mathcal{S}c\}$.

Problem 2.1 [LLM18, Prob. 4.24][1]
Given a set S, we define $\mathcal{I}_S = \{(a, a) : a \in S\}$. Given $\mathcal{R} \subseteq A \times B$, identify which of the properties of relations (e.g., function, surjective, total etc) implies each of the following properties

1. $\mathcal{R} \circ \mathcal{R}^{-1} \subseteq \mathcal{I}_A$
2. $\mathcal{R}^{-1} \circ \mathcal{R} \subseteq \mathcal{I}_B$
3. $\mathcal{R}^{-1} \circ \mathcal{R} \supseteq \mathcal{I}_B$
4. $\mathcal{R} \circ \mathcal{R}^{-1} \supseteq \mathcal{I}_A$

Problem 2.2 [Sak02]
Prove that for binary relations $\mathcal{R}, \mathcal{R}'$ from A to B and $\mathcal{S}, \mathcal{S}'$ from B to C, if $\mathcal{R} \subseteq \mathcal{R}'$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{R}' \circ \mathcal{S}'$.

Problem 2.3 [Sak02]
Given $\mathcal{R} \subseteq A \times B$ and $\mathcal{S}, \mathcal{T} \subseteq B \times C$, prove or find an example that disproves

1. $\mathcal{R} \circ (\mathcal{S} \cup \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \cup (\mathcal{R} \circ \mathcal{T})$
2. $\mathcal{R} \circ (\mathcal{S} \cap \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \cap (\mathcal{R} \circ \mathcal{T})$
3. $\mathcal{R} \circ (\mathcal{S} \setminus \mathcal{T}) = (\mathcal{R} \circ \mathcal{S}) \setminus (\mathcal{R} \circ \mathcal{T})$

Problem 3 [LLM18, Prob. 4.40]
Let $\mathcal{R} \subset A \times B$ be a binary relation. Prove that if $\mathcal{R}$ is a function then for every $X \subseteq A$, $|X| \geq |\mathcal{R}(X)|$.

Problem 4 [LLM18, Prob. 4.30]
Let L_n be the length-n sequences of digits 0 to 9 whose sum of digits is less than $9n/2$, and let G_n be the length-n sequences whose sum of digits is greater than $9n/2$. For example, $2134 \in L_4$ and $9786 \in G_4$. Describe a simple bijection between L_n and G_n. Prove carefully that your mapping is a bijection by verifying that it is a function, total, injective, and surjective.

[1]Since [LLM18] has different notation for composition, the statement in the book is different from the one below.

Problem 5
Prove Lemma 8.1.3(3) and 8.1.3(4) of [LLM18], i.e., show that for any sets A, B, C

- If A bij B and B bij C then A bij C.
- A bij B iff B bij A.

Problem 6
Complete the proof of the Schroeder-Bernstein theorem discussed in class. See Problem 8.14 of [LLM18] for a detailed proof outline that you can fill out.

Problem 7
In class we discussed Cantor's predicament when a fleet of buses indexed by $\mathbb{N}$ appeared at his hotel, each with seats indexed by $\mathbb{N}$. We suggested that Cantor can accommodate the guests as follows

- Currently resident guest in room i can shift to room 2^i.
- The j passenger of bus k can move into room p_{k+1}^j where p_k denotes the kth prime number, i.e., $p_1 = 2$.

Problem 7.1
Extend the solution discussed in class to the case where the fleet of buses is indexed by $\mathbb{N}^2$.

Problem 7.2
Now let us consider a situation where the government declares that only rooms whose number is a multiple of 6 can be used. What should Cantor do to accomodate the guests now? In this case assume the buses are indexed by $\mathbb{N}$.

Problem 7.3
Use the solution of Problem 7.2 to give a simple way of accomodating a fleet of buses indexed by $\mathbb{N}^k$ for any $k \geq 1$.

Problem 8 [LLM18, Prob. 8.6]
Suppose we have an infinite sequence $\{f_i\}_{i \geq 1}$ of functions from $\mathbb{N}$ to $\mathbb{R}_+$ (positive reals). A function $h : \mathbb{N} \to \mathbb{R}_+$ is said to *majorize* the sequence if for each $k \in \mathbb{N}$ there is some $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 : f_k(n) \leq h(n)$.

Problem 8.1
Before going to the main question do the following

1. Show that for any finite $A \subset \mathbb{R}_+$, $\sup A \in \mathbb{R}_+$.
2. Show that there exist infinite sets of the form $A \subset \mathbb{R}_+$ such that $\sup A \notin \mathbb{R}_+$ (i.e. the supremum is ∞).

Problem 8.2
Give an explicit construction for h using Problem 8.1 as a hint. Is there some way of doing it without using this hint?

Problem 8.3
Also show that there is an h such that $f_k(n)$ is $o(h(n))$ for every $k \in \mathbb{N}$.

Problem 9 ♠
A set S is called *countable* if it can be put into bijection with the set of natural numbers $\mathbb{N}$. Prove that set $\mathcal{L}$ of all finite length propositional sentences with two atomic propositions P and Q and using the logical connectives $\vee, \wedge, \Rightarrow, \neg$ is countable by constructing a bijection between $\mathcal{L}$ and $\mathbb{N}$.

References

[Sak02] S. Arun-Kumar, Lecture notes for *Introduction to Logic for Computer Science.*, IIT Delhi, 2002. http://www.cse.iitd.ernet.in/∼sak/courses/ilcs/logic.pdf

[LLM18] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science, June 2018, MIT Open Courseware.