Problem Set 10

Problem 1. [15 points] Suppose $\Pr\{\} : \mathcal{S} \to [0, 1]$ is a *probability function* on a sample space, $\mathcal{S}$, and let B be an event such that $\Pr\{B\} > 0$. Define a function $\Pr_B\{\cdot\}$ on outcomes $w \in \mathcal{S}$ by the rule:

$$\Pr_B\{w\} = \begin{cases} \Pr\{w\} / \Pr\{B\} & \text{if } w \in B, \\ 0 & \text{if } w \notin B. \end{cases} \tag{1}$$

(a) [7 pts] Prove that $\Pr_B\{\cdot\}$ is also a probability function on $\mathcal{S}$ according to Definition 14.4.2.

(b) [8 pts] Prove that

$$\Pr_B\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

for all $A \subseteq \mathcal{S}$.

Problem 2. [20 points]

(a) [10 pts] Here are some handy rules for reasoning about probabilities that all follow directly from the Disjoint Sum Rule. Use Venn Diagrams, or another method, to prove them.

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\} \tag{Difference Rule}$$

$$\Pr\left\{\bar{A}\right\} = 1 - \Pr\{A\} \tag{Complement Rule}$$

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \tag{Inclusion-Exclusion}$$

$$\Pr\{A \cup B\} \le \Pr\{A\} + \Pr\{B\}. \tag{2-event Union Bound}$$

$$\text{If } A \subseteq B, \text{ then } \Pr\{A\} \le \Pr\{B\}. \tag{Monotonicity}$$

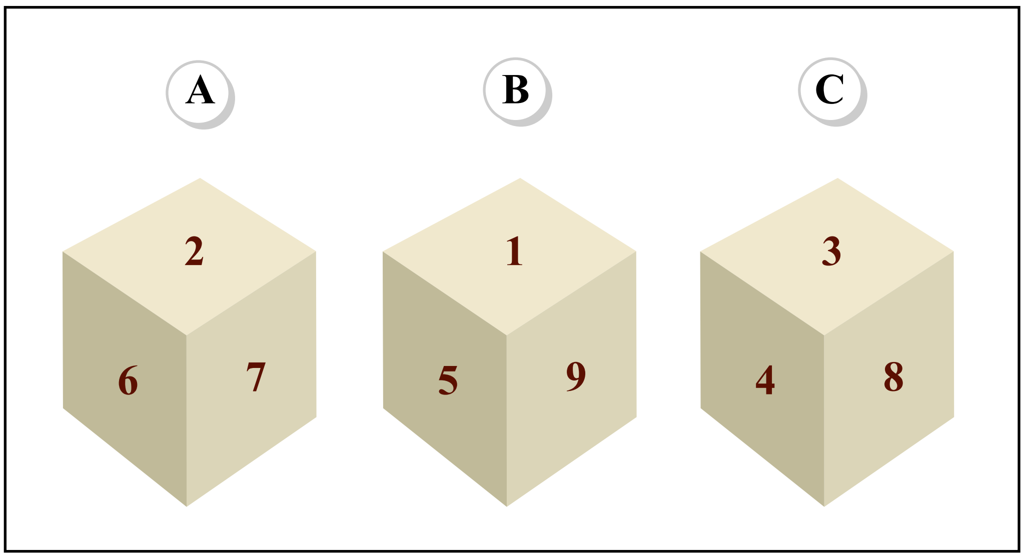
(b) [10 pts] Prove the following probabilistic identity, referred to as the **Union Bound**. You may assume the theorem that the probability of a union of *disjoint* sets is the sum of their probabilities.

Theorem. *Let $A_1, \ldots, A_n$ be a collection of events on some sample space. Then*

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} \Pr(A_i).$$

(*Hint: Induction*)

Problem 3. [15 points] Recall the strange dice from lecture:



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In the book we proved that if we roll each die once, then die A beats B more often, die B beats die C more often, and die C beats die A more often. Thus, contrary to our intuition, the "beats" relation $>$ is not transitive. That is, we have $A > B > C > A$.

We then looked at what happens if we roll each die twice, and add the result. In this problem we will show that the "beats" relation reverses in this game, that is, $A < B < C < A$, which is very counterintuitive!

(a) [5 pts] Show that rolling die C twice is more likely to win than rolling die B twice.

(b) [5 pts] Show that rolling die A twice is more likely to win that rolling die C twice.

(c) [5 pts] Show that rolling die B twice is more likely to win than rolling die A twice.

Problem 4. [14 points] Let's play a game! We repeatedly flip a fair coin. You have the sequence HHT, and I have the sequence HTT. If your sequence comes up first, then you win. If my sequence comes up first, then I win. For example, if the sequence of tosses is:

$$TTHTHT\underline{HHT}$$

then you win. What is the probability that you win? It may come as a surprise that the answer is very different from 1/2.

This problem is tricky, because the game could go on for an arbitrarily long time. Draw enough of the tree diagram to see a pattern, and then sum up the probabilities of the (infinitely many) outcomes in which you win.

It turns out that for any sequence of three flips, there is another sequence that is likely to come up before it. So there is no sequence of three flips which turns up earliest! ... and given any sequence of three flips, knowing how to pick another sequence that comes up sooner more than half the time gives you a nice chance to fool people gambling at a bar :-)

Problem 5. [15 points] We're interested in the probability that a randomly chosen poker hand (5 cards from a standard 52-card deck) contains cards from at most two suits.

(a) [7 pts] What is an appropriate sample space to use for this problem? What are the outcomes in the event, $\mathcal{E}$, we are interested in? What are the probabilities of the individual outcomes in this sample space?

(b) [8 pts] What is $\Pr(\mathcal{E})$?

Problem 6. [21 points]

I have a deck of 52 regular playing cards, 26 red, 26 black, randomly shuffled. They all lie face down in the deck so that you can't see them. I will draw a card off the top of the deck and turn it face up so that you can see it and then put it aside. I will continue to turn up cards like this but at some point while there are still cards left in the deck, you have to declare that you want the next card in the deck to be turned up. If that next card turns up black you win and otherwise you lose. Either way, the game is then over.

(a) [4 pts] Show that if you take the first card before you have seen any cards, you then have probability 1/2 of winning the game.

(b) [4 pts] Suppose you don't take the first card and it turns up red. Show that you have then have a probability of winning the game that is greater than 1/2.

(c) [4 pts] If there are r red cards left in the deck and b black cards, show that the probability of winning in you take the next card is $b/(r+b)$.

(d) [9 pts] Either,

1. come up with a strategy for this game that gives you a probability of winning strictly greater than 1/2 and prove that the strategy works, or,
2. come up with a proof that no such strategy can exist.