Note that, in Lecture 1, for the existence theorem we have taken closed rectangle R.
If you wish to take an open rectangle then you must include that f is bounded.

Existence Theorem / Picard's existence theorem :- Let the right side $f(x,y)$ of the ODE in the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

be continuous at all the points (x,y) in some rectangle

$$R: |x-x_0| < a, \quad |y-y_0| < b$$

and bounded in R, i.e., $\exists$ $K \in \mathbb{R}$ such that

$$|f(x,y)| \leq K \quad \text{for all } (x,y) \in R.$$

Then the IVP has at least one solution. This solution exists at least for all x in the sub interval $(x_0-\alpha, x_0+\alpha)$ of (x_0-a, x_0+a) where $\alpha = \min\{a, b/K\}$

Uniqueness Theorem | Picard's uniqueness theorem :-

Let the IVP be $y' = f(x,y)$, $y(x_0) = y_0$. Let f and its partial derivative $f_y = \frac{\partial f}{\partial y}$ be continuous for all (x,y) in R $[R := \{(x,y) : |x-x_0| < a,\ |y-y_0| < b]$ and bounded, say

$$|f(x,y)| \le K \quad \text{and} \quad |f_y(x,y)| \le M \quad \text{for all } (x,y) \in R$$

Then the IVP has exactly one solution. This solution exists, at least, for all x in the sub interval $(x_0-\alpha, x_0+\alpha)$ where $\alpha = \min\{a, b/K\}$.

If condition fails then we can not conclude anything.

Question:- Solve the following IVP

$$(x-1)y' = 2y, \quad y(1) = 1.$$

Explain the results in view of the theory of existence and uniqueness of IVP

Solution :- The equation can be written as

$$\frac{dy}{y} = 2\,\frac{dx}{x-1}$$

$$\Rightarrow \int \frac{dy}{y} = 2\int \frac{dx}{x-1}$$

$$\Rightarrow \log y = 2\log(x-1) + \log c$$

$$\Rightarrow \log y = \log c\,(x-1)^2$$

$$\Rightarrow y = c\,(x-1)^2$$

If we put $y(1) = 1$ then we get $1 = c \cdot 0$, which is not possible.

Thus the IVP has no solution.

☒ Note that here $x_0 = 1$ & $y_0 = 1$ and $f(x,y) = \frac{2y}{x-1}$

Therefore, $f(x,y)$ is not even defined on the line $x=1$. Therefore, the existence theorem can not be applied here.

Question:- Find all the initial conditions such that corresponding IVP with ODE

$$(x^2-4x)\,y' = (2x-4)\,y$$

has no solution, unique solution and more than one solutions.

Solution:- The ODE $(x^2-4x)\frac{dy}{dx} = (2x-4)$, $y(x_0)=y_0$

Here $f(x,y) = \dfrac{(2x-4)y}{x^2-4x}$ $\quad \dfrac{\partial f}{\partial y} = \dfrac{2x-4}{x^2-4x}$

The existence and uniqueness theorem guarantees the existence of unique theorem in a neighborhood of (x_0, y_0) where f and $\frac{\partial f}{\partial y}$ are continuous and bounded.

If $x_0^2 - 4x_0 \neq 0$ in a domain then f and f_y both are continuous and bounded in a domain containing (x_0, y_0).

Hence the IVP has a unique solution if
$$x_0 \neq 0, 4.$$

If $x_0 = 0$ or $x_0 = 4$ then nothing can be said by the existence and uniqueness theorem as f & f_y are not continuous (not even defined) in any rectangle

Contaiining (x_0, y_0).

But, $$\frac{dy}{dx} = \frac{2x-4}{x^2-4x}\, y$$

$$\Rightarrow \frac{dy}{y} = \frac{(2x-4)\,dx}{x^2-4x}$$

$$\Rightarrow \frac{dy}{y} = \frac{d(x^2-4x)}{x^2-4x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{d(x^2-4x)}{x^2-4x}$$

$$\Rightarrow \log y = \log(x^2-4x) + \log c$$

$$\Rightarrow y = c(x^2-4x)$$

If $x_0 = 0$ or $x_0 = 4$ then $y_0 = 0$

Therefore,

If '$x_0 = 0$ & $y_0 \neq 0$' or '$x_0 = 4$ & $y_0 \neq 0$'

then the IVP has no solution.

If '$x_0 = 0$ & $y_0 = 0$' or '$x_0 = 4$ & $y_0 = 0$' then

$y = c\,x(x-4)$ is a solution for any $c \in \mathbb{R}$.

Therefore the IVP has an infinite number of solution.

Lipschitz condition :-

A function $f(x, y)$ is said to satisfy the Lipchitz condition on a region R if there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2|$$

$$\forall\ (x, y_1), (x, y_2) \in R.$$

Note that, If $\frac{\partial f}{\partial y}$ is bounded on R then $f(x,y)$ satisfy the lipschtiz condition on R.

Proof :- By the mean value theorem,

$$f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(x,\tilde{y})\ (y_1 - y_2)$$

for some $\tilde{y}$ between y_1 & y_2.

Let $\left|\frac{\partial f}{\partial y}\right| \leq M$ on R. Then

$$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial f}{\partial y}(x,\tilde{y})\right| \, |y_1 - y_2|$$

$$\leq M\,|y_1 - y_2|$$

Note that, $f(x,y)$ might be lipschitz in y-variable even if $\frac{\partial f}{\partial y}$ does not exist at all points in R.

example, $f(x,y) = |\sin y|$ on $\mathbb{R}^2$.

$\frac{\partial f}{\partial y}$ does not exist if $y = m\pi$.

But
$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= \big| |\sin y_1| - |\sin y_2| \big| \\ &\leq |\sin y_1 - \sin y_2| \\ &= |\cos \tilde{y}| \, |y_1 - y_2| \\ &\leq |y_1 - y_2| \end{aligned}$$