

# Linear Algebra: Assignment Sheet-I

In the following,  $V$  denotes a vector space over a field  $\mathbb{F}$ .

For  $i, j \in \mathbb{N}$ , we denote  $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

1. Prove that  $0x = 0_V$  and  $(-1)x = -x$  for all  $x \in V$ .
2. Prove that for  $x \in V$  and  $\alpha \in \mathbb{F}$ , if  $\alpha x = 0$ , then either  $\alpha = 0$  or  $x = 0$ .
3. Verify (prove) the following:
  - (a)  $\mathbb{R}^n$  with coordinate-wise addition and scalar multiplication is a vector space over  $\mathbb{R}$  and over  $\mathbb{Q}$ , but not a vector space over  $\mathbb{C}$ .
  - (b)  $\mathbb{F}^n$  with coordinate-wise addition and scalar multiplication is a vector space over  $\mathbb{F}$  but not a vector space over a field  $\tilde{\mathbb{F}} \supseteq \mathbb{F}$  with  $\tilde{\mathbb{F}} \neq \mathbb{F}$ .
  - (c)  $\mathbb{R}^{m \times n}$ , the set of all real  $m \times n$  matrices is a vector space over  $\mathbb{R}$  under usual matrix multiplication and scalar multiplication.
  - (d) Let  $\Omega$  be a nonempty set. Then the set  $\mathcal{F}(\Omega, \mathbb{F})$ , the set of all  $\mathbb{F}$ -valued functions defined on  $\Omega$ , is a vector space over  $\mathbb{F}$  with respect to the pointwise addition and pointwise scalar multiplication.  
Is the set of all scalar sequences a special case of the above?.
4. Which of the following subset of  $\mathbb{C}^3$  a subspace of  $\mathbb{C}^3$ ?
  - (a)  $\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \alpha_1 \in \mathbb{R}\}$ .
  - (b)  $\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \text{either } \alpha_1 = 0 \text{ or } \alpha_2 = 0\}$ .
  - (c)  $\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \alpha_1 + \alpha_2 = 1 \in \mathbb{R}\}$ .
5. Which of the following subset of  $\mathcal{P}$  a subspace of  $\mathcal{P}$ ?
  - (a)  $\{x \in \mathcal{P} : \text{degree of } x \text{ is } 3\}$ .
  - (b)  $\{x \in \mathcal{P} : 2x(0) = x(1)\}$ .
  - (c)  $\{x \in \mathcal{P} : x(t) \geq 0 \text{ for } t \in [0, 1]\}$ .
  - (d)  $\{x \in \mathcal{P} : x(t) = x(1-t) \forall t\}$ .
6. Prove the following:
  - (a) The spaces  $\mathcal{P}_n(\mathbb{F})$  and  $\mathbb{F}^{n+1}$  are isomorphic, and find an isomorphism.

- (b) The space  $\mathbb{R}^n := \mathbb{R}^{n \times 1}$ , the space of all column  $n$ -vectors is isomorphic with  $\mathbb{R}^n$ , and find an isomorphism.
- (c) The space  $\mathbb{R}^{m \times n}$  is isomorphic with  $\mathbb{R}^{mn}$ , and find an isomorphism.
7. Prove the assertions in the following:
- $S = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 : \alpha_1 + \alpha_2 = 0\}$  is a subspace of  $\mathbb{R}^2$ .
  - $S = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 : \alpha_1 + \alpha_2 - \alpha_3 = 0\}$  is a subspace of  $\mathbb{R}^3$ .
  - For each  $k \in \{1, \dots, n\}$ ,  $S_k = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_k = 0\}$  is a subspace of  $\mathbb{R}^n$ .
  - For  $n \in \mathbb{N}$  with  $n \geq 2$  and each  $k \in \{1, \dots, n-1\}$ ,  
 $S_k = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_i = 0 \forall i > k\}$  is a subspace of  $\mathbb{R}^n$ .
  - For each  $n \in \mathbb{N}$ ,  $\mathcal{P}_n$  is a subspace of  $\mathcal{P}$ .
  - For each  $n \in \mathbb{N}$ ,  $V_n := \{x \in \mathcal{F}(\mathbb{N}, \mathbb{F}) : x(j) = 0 \forall j \geq n\}$  is a subspace of  $\mathcal{F}(\mathbb{N}, \mathbb{F})$ , and  $e_{00} := \bigcup_{n=1}^{\infty} V_n$  is a subspace of  $\mathcal{F}(\mathbb{N}, \mathbb{F})$ .  
 (Note that elements of  $W$  are sequences having only a finite number of nonzero entries.)
  - For an interval  $\Omega := [a, b] \subseteq \mathbb{R}$ ,
    - $\mathcal{R}(\Omega)$ , the set of all Riemann integrable real valued continuous functions defined on  $\Omega$  is a subspace of  $\mathcal{F}(\Omega, \mathbb{R})$ .
    - $C(\Omega)$  is a subspace of  $\mathcal{R}(\Omega)$
    - $C^1(\Omega)$ , the set of all real valued continuous functions defined on  $\Omega$  and having continuous derivative in  $\Omega$  is a subspace of  $C(\Omega)$ .
    - $S = \{x \in C(\Omega) : \int_a^b x(t) dt = 0\}$  is a subspace of  $C(\Omega)$ .
    - $S = \{x \in C(\Omega) : x(a) = 0\}$  is a subspace of  $C(\Omega)$ .
    - $S = \{x \in C(\Omega) : x(a) = 0 = x(b)\}$  is a subspace of  $C(\Omega)$ .
  - Let  $A \in \mathbb{R}^{m \times n}$ . Then
    - $\{x \in \mathbb{R}^n : Ax = 0\}$  is a subspace of  $\mathbb{R}^n$ ,
    - $\{Ax : x \in \mathbb{R}^n\}$  is a subspace of  $\mathbb{R}^m$ ,
  - $\{(\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 = 0\}$  is a subspace of  $\mathbb{R}^2$ .
  - $\{(\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 - \alpha_3 = 0\}$  is a subspace of  $\mathbb{R}^3$ .
  - For  $i \in \{1, \dots, n\}$ , let  $e_i = (\delta_{i1}, \dots, \delta_{in})$ . Let  $V = \mathbb{R}^n$ . Then  $\{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_i = 0 \text{ for } i > k\}$  is a subspace of  $\mathbb{R}^n$ .
  - If  $V_1$  and  $V_2$  are subspaces of  $V$ , then  $V_1 + V_2 = \text{span}(V_1 \cup V_2)$ .

- (m) If  $V_1$  and  $V_2$  are subspaces of  $V$  and if  $V_1 \subseteq V_2$ , then  $V_1 \cup V_2$  is a subspace of  $V$ .
- (n) If  $V_1$  and  $V_2$  are subspaces of  $V$ , then  $V_1 \cap V_2$  is a subspace of  $V$ ; but,  $V_1 \cup V_2$  need not be a subspace of  $V$ .
8. Let  $V$  be a vector space and  $S \subset V$ . Prove the following:
- $\text{span}(S)$  is a subspace of  $V$ .
  - If  $V_0$  is a subspace of  $V$  such that  $S \subset V_0$ , then  $\text{span}(S) \subset V_0$ .
  - $S = \text{span}(S)$  if and only if  $S$  is a subspace of  $V$ .
9. Prove the assertions in the following:
- If  $V = \mathbb{R}^2$ , then  $\text{span}(\{(1, -1)\}) = \{(\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 = 0\}$ .
  - If  $V = \mathbb{R}^3$ , then  $\text{span}(\{(1, -1, 0), (1, 0, 1)\}) = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 + \alpha_2 - \alpha_3 = 0\}$ .
  - For  $i \in \{1, \dots, n\}$ , let  $e_i = (\delta_{i1}, \dots, \delta_{in})$ . Let  $V = \mathbb{R}^n$ . Then
    - $\text{span}(\{e_1, \dots, e_k\}) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_i = 0 \text{ for } i > k\}$ .
    - $\text{span}(\{e_1, \dots, e_n\}) = \mathbb{R}^n$ .
  - If  $V = \mathcal{P}$ , then  $\text{span}(\{1, t, \dots, t^n\}) = \mathcal{P}_n$  and  $\text{span}(\{1, t, t^2, \dots\}) = \mathcal{P}$ .
  - For each  $i \in \mathbb{N}$ , let  $e_i = (\delta_{i1}, \delta_{i2}, \dots)$ . Then  $\text{span}(\{e_1, e_2, \dots\}) = c_{00}$ .
10. Prove that a set of vectors  $x_1, \dots, x_n$  in a vector space  $V$  are linearly dependent if and only if there exists  $k \in \{2, \dots, n\}$  such that  $x_k$  is a linear combination of  $x_1, \dots, x_{k-1}$ .
11. Prove that any three of the polynomials  $1, t, t^2, 1+t+t^2$  are linearly independent.
12. Give vectors  $x_1, x_2, x_3, x_4$  in  $\mathbb{C}^3$  such that any three of them are linearly independent.
13. Find conditions on  $\alpha$  such that the vectors
  - $(1 + \alpha, 1 - \alpha), (1 - \alpha, 1 + \alpha)$  are linearly dependent in  $\mathbb{C}^2$ ,
  - $(\alpha, 1, 0), (1, \alpha, 1), (0, 1, \alpha)$  are linearly dependent in  $\mathbb{R}^3$ .
14. Suppose  $x, y, z$  are linearly independent. Is it true that  $x+y, y+z, z+x$  are also linearly independent?
15. Prove the assertions in the following:
  - $\{e_1, \dots, e_n\}$  is a basis of  $\mathbb{R}^n$  and  $\mathbb{C}^n$ .
  - $\{1, t, \dots, t^n\}$  is a basis of  $\mathcal{P}_n$ .

- (c)  $\{1, 1+t, 1+t+t^2, \dots, 1+t+\dots+t^n\}$  is a basis of  $\mathcal{P}_n$ .
- (d)  $\{1, t, t^2, \dots\}$  is a basis of  $\mathcal{P}$ .
- (e) For each  $i \in \mathbb{N}$ , let  $e_i = (\delta_{i1}, \delta_{i2}, \dots)$ . Then  $\{e_1, e_2, \dots\}$  is a basis of  $c_{00}$ .
- (f) If  $E$  is linearly independent in a vector space, then  $E$  is a basis for  $V_0 := \text{span}(E)$ .

16. Prove:

- (a) If  $E$  is linearly independent and if  $x \in V$  with  $x \notin \text{span}(E)$ , then  $E \cup \{x\}$  is linearly independent.
- (b) Every vector space having a finite spanning set has a finite basis.
- (c) If a vector space  $V$  has a finite basis, then any two basis of  $V$  contains the same number of vectors.
17. Find bases  $E_1, E_2$  for  $\mathbb{C}^4$  such that  $E_1 \cap E_2 = \emptyset$  and  $\{(1, 0, 0, 0), (1, 1, 0, 0)\} \subseteq E_1$  and  $\{(1, 1, 1, 0), (1, 1, 1, 1)\} \subseteq E_2$ .

18. Prove the assertions in the following:

- (a)  $\mathbb{Z}^n$  and  $\mathcal{P}_n$  are finite dimensional spaces, and  $\dim(\mathbb{Z}^n) = n$ ,  $\dim(\mathcal{P}_n) = n+1$ .
- (b)  $\dim(\{\alpha_1, \dots, \alpha_n\} \in \mathbb{R}^n : \alpha_1 + \dots + \alpha_n = 0\}) = n-1$ .
- (c)  $\mathcal{P}, C[a, b], c_{00}$  are infinite dimensional spaces.
- (d) Every vector space containing an infinite linearly independent set is infinite dimensional.
- (e) If  $A \in \mathbb{R}^{m \times n}$  with  $n > m$ , then there exists  $x \in \mathbb{R}^n$  such that  $Ax = 0$ .

19. Prove:

- (a) If  $V_1$  and  $V_2$  are subspaces of a vector space  $V$  such that  $V_1 \cap V_2 = \{0\}$ , and if  $E_1$  and  $E_2$  are bases of  $V_1$  and  $V_2$ , respectively, then  $E_1 \cup E_2$  is a basis of  $V_1 + V_2$ ; and in particular,

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2).$$

- (b) If  $V_1$  and  $V_2$  are subspaces of a vector space  $V$ , then

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

- (c) Let  $V_1$  and  $V_2$  be vector spaces and let  $T$  be an isomorphism from  $V_1$  onto  $V_2$ . Let  $E \subseteq V_1$ . Then  $E$  is a basis of  $V_1$  if and only if  $\{T(u) : u \in E\}$  is a basis of  $V_2$ .

20. Suppose  $V_1$  and  $V_2$  are subspaces of a vector space  $V$ . Prove:

- (a) If  $V_1$  and  $V_2$  are finite dimensional such that  $\dim(V_1) = \dim(V_2)$  and  $V_1 \subset V_2$ , then  $V_1 = V_2$ .
- (b) If  $V = V_1 \cup V_2$ , then either  $V_1 = V$  or  $V_2 = V$ .

21. Prove that, if  $V_0$  is a subspace of a vector space  $V$ , then there exists a subspace  $V_1$  of  $V$  such that

$$V = V_0 + V_1 \quad \text{and} \quad V_0 \cap V_1 = \{0\}.$$

- 22. If  $V_1$  is the set of all *odd* polynomials (i.e.,  $x(-t) = -x(t)$  for all  $t$ ), and if  $V_2$  is the set of all *even* polynomials (i.e.,  $x(-t) = x(t)$  for all  $t$ ), prove that  $V_1$  and  $V_2$  are subspaces of  $\mathcal{P}$  such that  $V = V_1 + V_2$  and  $V_1 \cap V_2 = \{0\}$ .
- 23. Let  $V_1$  and  $V_2$  be vector spaces over the same field  $\mathbb{F}$ . For  $x := (x_1, x_2)$ ,  $y := (y_1, y_2)$  in  $V_1 \times V_2$ , and  $\alpha \in \mathbb{F}$ , define

$$x + y = (x_1 + y_1, x_2 + y_2), \quad \alpha x = (\alpha x_1, \alpha x_2).$$

Prove:

- (a)  $V_1 \times V_2$  is a vector space over  $\mathbb{F}$  with respect to the above operations with its zero as  $(0, 0)$  and  $-x := (-x_1, -x_2)$ .
- (b) If  $V_1$  and  $V_2$  are finite dimensional, then

$$\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2).$$

- (c) If  $\tilde{V}_1 := \{(x_1, x_2) \in V_1 \times V_2 : x_2 = 0\}$  and  $\tilde{V}_2 := \{(x_1, x_2) \in V_1 \times V_2 : x_1 = 0\}$ , then  $\tilde{V}_1$  and  $\tilde{V}_2$  are subspaces of  $V_1 \times V_2$  and

$$V_1 \times V_2 = \tilde{V}_1 + \tilde{V}_2, \quad \tilde{V}_1 \cap \tilde{V}_2 = \{(0, 0)\}.$$

In view of the above, the space  $V_1 \times V_2$  is called the *direct sum* of  $V_1$  and  $V_2$ .

- 24. Let  $V_1$  and  $V_2$  be subspaces of a finite dimensional vector space  $V$  such that  $V = V_1 + V_2$  and  $V_1 \cap V_2 = \{0\}$ . Prove that  $V$  is isomorphic with  $V_1 \times V_2$ .
- 25. Let  $V_0$  be a subspaces of a finite dimensional vector space  $V$ . Prove that  $V$  is isomorphic with  $(V/V_0) \times V_0$ .