

* If $y_1(t) \neq u(t)$, $y_1(t)$ are two solutions then

$$u'(t) = \frac{c}{[y_1(t)]^2} e^{(-\int p(x)dx)}$$

Homogeneous linear ODE with constant coefficient

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}, \quad a \neq 0.$$

* Let m_1, m_2 be solutions of $am^2 + bm + cm = 0$.

(i) If $m_1 \neq m_2$ real then the general solution is

given by $c_1 e^{m_1 t} + c_2 e^{m_2 t}$ for some $c_1, c_2 \in \mathbb{R}$.

(ii) If $m_1 = m_2 = m$ then the general solution is given

by $(c_1 + c_2 t) e^{mt}$ for some $c_1, c_2 \in \mathbb{R}$.

(iii) If $m_1 \neq m_2$ are complex say $m_1 = \alpha + i\beta \neq m_2 = \alpha - i\beta$

then the general solution is given by $e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t)$ for some $c_1, c_2 \in \mathbb{R}$.

Consider the ODE $a\theta^v y'' + b\theta^v y' + c y = 0$, $\theta > 0$ with $a \neq 0$.

Let $\theta = e^x$. $\frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx} = \frac{dy}{d\theta} \cdot e^x = \theta \frac{dy}{d\theta} = \theta y'$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{d\theta} \right) = \frac{d}{dx} \left(\theta \frac{dy}{d\theta} \right) = \frac{d}{d\theta} \left(\theta \frac{dy}{d\theta} \right) \frac{d\theta}{dx} \\ &= \left(\frac{dy}{d\theta} + \theta \frac{d^2y}{d\theta^2} \right) e^x \\ &= \theta^v \frac{d^2y}{d\theta^2} + \theta \frac{dy}{d\theta}\end{aligned}$$

$$\therefore a \left(\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta} \right) + b \frac{dy}{d\theta} + c y = 0$$

$$\Rightarrow a \frac{d^2y}{d\theta^2} + (b-a) \frac{dy}{d\theta} + c y = 0$$

Let m_1 & m_2 be two solution of $am^v + (b-a)m + c = 0$

(i) If $m_1 \neq m_2$ real then the general solution is

given by $c_1 e^{m_1 x} + c_2 e^{m_2 x}$ for some $c_1, c_2 \in \mathbb{R}$.

(ii) If $m_1 = m_2 = m$ then the general solution is given by $(c_1 + c_2 x) e^{mx}$ for some $c_1, c_2 \in \mathbb{R}$.

$$(c_1 + c_2 \ln(t)) t^m, \quad t > 0$$

(iii) If $m_1 \neq m_2$ are complex say $m_1 = \alpha + i\beta \neq m_2 = \alpha - i\beta$ Then the general solution is given by $e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$ for some $c_1, c_2 \in \mathbb{R}$.

$$t^\alpha [c_1 \cos(\beta \ln(t)) + c_2 \sin(\beta \ln(t))] \quad t > 0.$$

One can define for $t < 0$ as well by replacing t by $-t$.

* Homogeneous linear ODE (Second order).
with constant coefficient.

$$ay''(t) + by'(t) + cy(t) = 0 \quad \textcircled{1}$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$.

Let

$y = e^{mt}$ be a solution of $\textcircled{1}$

$$\Rightarrow (am^2 + bm + c)e^{mt} = 0$$

$$\Rightarrow am^2 + bm + c = 0 \quad (\because e^{mt} \neq 0)$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \textcircled{2}$$

Case 1, $m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ & $m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are

two different real solution of $\textcircled{2}$

The general solution of ① will be
 $y(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}, \quad C_1, C_2 \in \mathbb{R}.$

Case 2 :- Let $m_1 = m_2 = m = -\frac{b}{2a}$

Then e^{mt} is a solution.

Then let $u(t) e^{mt}$ be another solution

$$\begin{aligned}
 \text{then } u'(t) &= \frac{1}{[y_1(t)]^n} e^{(-\int p(t)dt)} \quad p(t) = \frac{b}{a} \\
 &= \frac{1}{e^{2mt}} e^{-\frac{b}{a} \int dt} \\
 &= e^{-x(-\frac{b}{2a}) \cdot t} e^{-\frac{b}{a} t} \\
 &= e^{\frac{b}{a} t} \cdot e^{-\frac{b}{a} t} = 1 \\
 u(t) &= t
 \end{aligned}$$

The other solution will be $u(t) e^{mt} = t e^{mt}$

The general solution of ① is

$$\begin{aligned}y(t) &= c_1 e^{mt} + c_2 t e^{mt} \\&= (c_1 + c_2 t) e^{mt}.\end{aligned}$$

Case 3 :- $m_1 = d + i\mu \neq m_2 = d - i\mu$

Then $e^{(d+i\mu)t}$, $e^{(d-i\mu)t}$ " are solutions of ①

$$e^{dt} [\cos \mu t + i \sin \mu t] \quad e^{dt} [\cos \mu t - i \sin \mu t].$$

We know that if y_1 & y_2 are two linearly independent solutions, then $\frac{1}{2}(y_1 + y_2)$ & $\frac{1}{2i}(y_1 - y_2)$ are " sol.

$$e^{dt} \cos \mu t \quad e^{dt} \sin \mu t$$

A general solution will be

$$c_1 e^{\gamma t} \cos \mu t + c_2 e^{\gamma t} \sin \mu t.$$

* Euler-Cauchy second order ODE.

$$a t^r y''(t) + b t y'(t) + c y(t) = 0$$

Let $t = e^x$

$$t > 0 \quad \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \cdot e^x = t y'(t)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d}{dt} (t y'(t)) \frac{dt}{dx}$$

$$= (t y''(t) + y'(t)) t$$

$$= t^r y''(t) + t y'(t)$$

$$\therefore t^r y''(t) = \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$$a \left(\frac{d^2y}{dx^2} - \frac{dy}{dx} \right) + b \frac{dy}{dx} + cy = 0$$

$$\Rightarrow a \frac{d^2y}{dx^2} + (b-a) \frac{dy}{dx} + cy = 0$$

(i) $m_1 \neq m_2 \in \mathbb{R}$ $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$= c_1 t^{m_1} + c_2 t^{m_2}$$

(ii) $m_1 = m_2 = m$, $y = (c_1 + c_2 x) e^{mx}$

$$= (c_1 + c_2 \ln(t)) t^m$$

(iii) $m_1, m_2 \in \mathbb{C}$, $y = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$.

$$= c_1 t^\lambda \cos(\mu \ln(t)) + c_2 t^\lambda \sin(\mu \ln(t)).$$

$t > 0$

If $t < 0$, then proceed with $-t$.

Look at homogenous ODE of second order.

Let $y'' + p(t)y' + q(t)y = \sigma(t)$, $\underline{\sigma(t) \neq 0}$. ①

If y_1 & y_2 are two solutions of ①

then $y_1 - y_2$ is a solution of

$$y'' + p(t)y' + q(t)y = 0.$$

$c_1 y_1 + c_2 y_2$

If you know the general solution of

$y'' + p(t)y' + q(t)y = 0$ and a particular solution

of $y'' + p(t)y' + q(t)y = \sigma(t)$, y_p

then the general solution of ① will be

$$c_1 y_1 + c_2 y_2 + y_p.$$

Method of Variation of Parameters :-

Let $y'' + p(x)y' + q(x)y = \tau(x)$ be an ODE.

Let y_1 and y_2 be two solution of

$$y'' + p(x)y' + q(x)y = 0.$$

Then $W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

Then $y_p = -y_1 \int \frac{y_2(x) \tau(x)}{W(y_1, y_2)(x)} dx + y_2 \int \frac{y_1(x) \tau(x)}{W(y_1, y_2)(x)} dx$

gives a Particular Solution of

$$y'' + p(x)y' + q(x)y = \tau(x).$$

Solve :- $y'' - 2y' + y = \frac{e^t}{t^2 + 1} \quad \text{--- } ①$

To find the general solution of $y'' - 2y' + y = 0$
 let $y = e^{mt}$ is a solution. $\text{--- } ②$

Then $m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1$$

Thus e^t and $t e^t$ are two solutions of $②$

$$\text{let } y_1 = e^t \quad y_2 = t e^t$$

let y_p be a particular solution of $①$

Then $W(y_1, y_2)(t) = \begin{vmatrix} e^t & t e^t \\ e^t & e^t + t e^t \end{vmatrix}$
 $= e^{2t}$

$$\begin{aligned}
 \text{Then } y_p &= -y_1 \int \frac{y_2(t) \gamma(t)}{w(y_1, y_2)(t)} dt + y_2 \int \frac{y_1(t) \gamma(t)}{w(y_1, y_2)(t)} dt \\
 &= -e^t \int \frac{te^t \cdot e^t}{(1+t^2)e^{2t}} dt + te^t \int \frac{e^t e^t}{(1+t^2)e^{2t}} dt \\
 &= -\frac{e^t}{2} \int \frac{2t dt}{1+t^2} + te^t \int \frac{dt}{1+t^2} \\
 &= -\frac{e^t}{2} \ln(1+t^2) + te^t \tan^{-1}(t).
 \end{aligned}$$

The general solution of ① is

$$c_1 e^t + c_2 te^t - \frac{e^t}{2} \ln(1+t^2) + te^t \tan^{-1}(t).$$