System of ODEs

A system of first order ODEs are written as

$$y_1' = f_1(x, y_1, \dots, y_n),$$
$$y_2' = f_2(x, y_1, \dots, y_n),$$
$$\vdots$$
$$y_n' = f_n(x, y_1, \dots, y_n),$$

where x is the independent variable, f_i's are functions defined in some $(n+1)$-dimensional space.

By a solution of the above system we mean n many differentiable functions $y_1(x) = h_1(x), \dots, y_n(x) = h_n(x)$ on some interval I, satisfying the system.

By an IVP here we refer to the above system with initial conditions

$$y_1(x_0) = c_1,$$
$$\vdots$$
$$y_n(x_0) = c_n.$$

Existence and uniqueness theorem

Suppose $f_1, \ldots, f_n$ are continuous having continuous first order partial derivatives with respect to $y_1, \ldots, y_n$ in some given domain containing the point $(x_0, c_1, \ldots, c_n)$. Then the IVP has a unique solution in $(x_0-\alpha, x_0+\alpha)$ for some α.

Note that any order n ODE can be converted into a system of first order ODEs. Let

$$\boxed{y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})}$$

Consider $y_1 = y$,

$y_2 = y_1'$,

$\vdots$

$y_n = y_{n-1}'$.

Then $\boxed{y_n' = f(x, y_1, \ldots, y_{n-1})}$

Examples ① Consider $y'' + ay' + by = 0$, $a, b \in \mathbb{R}$.

Take $y_1 = y$,

$y_2 = y_1'$.

Then we have,

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1'' = -ay_1' - by_1 = -ay_2 - by_1. \end{cases}$$

We can write it as,

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

(2) Consider order n homogeneous linear ODE

$$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_0(x) y = 0.$$

Take $y_1 = y$,

$y_2 = y_1'$,

$\vdots$

$y_n = y_{n-1}'$.

Then we have,

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -a_0(x) & \cdots & & & -a_{n-1}(x) \end{pmatrix}}_{\text{Denote by } A} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then $\underline{y}' = A\underline{y}$ where $\underline{y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$

and $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$

Linear System of ODEs

By a linear system of ODEs we mean the following linear system in $y_1, \cdots, y_n$.

$$y_1' = a_{11}(x)\, y_1 + \cdots + a_{1n}(x) y_n + g_1(x)$$

$$y_2' = a_{21}(x)\, y_1 + \cdots + a_{2n}(x) y_n + g_2(x)$$

$$\vdots$$

$$y_n' = a_{n1}(x)\, y_1 + \cdots + a_{nn}(x) y_n + g_n(x).$$

This can be written as $\underline{y}' = A\underline{y} + \underline{g}$,

where $A = \begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{pmatrix}$,

$$\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \underline{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}.$$

The system is called homogeneous if $\underline{g} = 0$.

<u>Remark</u> Note that for a linear system we need the functions $a_{ij}(x)$ and g_i's to be continuous to have uniqueness and existence of solution as

$$\frac{\partial f_i}{\partial y_j} = a_{ij}(x).$$

Solution space of a homogeneous linear system of ODEs

Let $\underline{y}' = A\underline{y}$ be a homogeneous linear $n \times n$ system. Let $\underline{y}_1, \underline{y}_2$ be two solutions of it. Note that for any $c \in \mathbb{R}$, we have $c\,\underline{y}_1 + \underline{y}_2$ is also a solution

as $$A(c\underline{y}_1 + \underline{y}_2) = c\,A\underline{y}_1 + A\underline{y}_2$$
$$= c\,\underline{y}_1' + \underline{y}_2' = (c\,\underline{y}_1 + \underline{y}_2)'$$

So the solution space is a vector space.

If we assume $a_{ij}(x)$'s are continuous then by the existence and uniqueness theorem of IVP by varying the initial conditions $(c_1, \dots, c_n)$ over the set $\{(1,0,\dots,0), (0,1,\dots,0), \dots, (0,\dots,0,1)\}$ we get n-many solutions

$\underline{P}_1, \dots, \underline{P}_n$, respectively.

<u>Claim</u> $\underline{P}_1, \dots, \underline{P}_n$ is a basis of the solution space.

Let $a_1\underline{P}_1 + \dots + a_n\underline{P}_n = 0$, $a_i \in \mathbb{R}$ $\forall$ $1 \leq i \leq n$.

So, $(a_1\underline{P}_1 + \dots + a_n\underline{P}_n)(x_0) = 0$

Write $\underline{P}_1 = \begin{pmatrix} p_{11} \\ p_{12} \\ \vdots \\ p_{1n} \end{pmatrix}$, then $\underline{P}_1(x_0) = \begin{pmatrix} p_{11}(x_0) \\ \vdots \\ p_{1n}(x_0) \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So, $(a_1\underline{P}_1 + \dots + a_n \underline{P}_n)(x_0)$

$$= \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

So, $a_i = 0 \quad \forall \; 1 \leq i \leq n.$

So, $\underline{P}_1, \dots, \underline{P}_n$ are linearly independent.

We show they are spanning.

Let $\underline{y}(x)$ be a solution of the system.

Write $\underline{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$

Consider,

$$\underline{\hat{y}}(x) = y_1(x_0)\underline{P}_1(x) + \dots + y_n(x_0)\underline{P}_n(x)$$

Clearly $\underline{\hat{y}}(x)$ is a solution of the system.

Now we can note that,

$$\underline{\hat{y}}(x_0) = \begin{pmatrix} y_1(x_0) \\ \vdots \\ y_n(x_0) \end{pmatrix} = \underline{y}(x_0)$$

So by the uniqueness of solution of IVP we have $\underline{\hat{y}}(x) = \underline{y}(x)$.

<u>Remark</u> Hence the dimension of the above solution space is n.

<u>How to check linear independence of solutions?</u>

Consider the system $A\underline{y} = \underline{y}'$, A is of order $n \times n$.

Let $\underline{P}_1, \dots, \underline{P}_n$ be n many solutions of it. We want to check they are linearly independent or not.

Consider the matrix

$$\underbrace{(\underline{P}_1 \; \cdots \; \underline{P}_n)}_{B} = \begin{pmatrix} P_{11} & P_{21} & \cdots & P_{n1} \\ P_{12} & P_{22} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ P_{1n} & P_{2n} & \cdots & P_{nn} \end{pmatrix}$$

$\underline{P}_1, \dots, \underline{P}_n$ are linearly independent $\Leftrightarrow \det B \neq 0$.

In fact, det B is either identically 0 or never 0. This property of det B is similar to the property of Wronskian we noted in second and higher order ODES, so det B is referred as Wronskian in this context.

Homogeneous linear system with constant coefficients

consider $\underline{y}' = A\underline{y}$, where $A \in M_{n\times n}(\mathbb{R})$.

clearly $\underline{y} = \underline{0}$ is a solution.

In search of non-zero solution take $\underline{y} = \underline{z}e^{\lambda x}$, where

$$\underline{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, $\underline{y}' = \lambda \underline{z} e^{\lambda x}$.

Now, $A\underline{y} = A\underline{z}e^{\lambda x}$.

If $\underline{y}$ is a solution of $A\underline{y} = \underline{y}'$, then $\lambda \underline{z} e^{\lambda x} = A\underline{z}e^{\lambda x}$

So, $A\underline{z} = \lambda \underline{z}$.

so λ has to be an eigen value of A and $\underline{z}$ a corresponding eigen vector.

Remark Note that if λ is an eigen value of A and $\underline{z}$ a corresponding eigen vector, then $\underline{z}e^{\lambda x}$ is a solution of the system.

Remark If $\lambda_1, \dots, \lambda_n$ are distinct eigen values, and $\underline{z}_1, \dots, \underline{z}_n$ respectively are their corresponding eigen vectors, then we have $\underline{z}_1 e^{\lambda_1 x}, \dots, \underline{z}_n e^{\lambda_n x}$ are linearly independent solutions.

$$\det\left(\underline{z}_1 e^{\lambda_1 x} \quad \cdots \quad \underline{z}_n e^{\lambda_n x}\right)$$

$$= \underbrace{e^{(\lambda_1 + \cdots + \lambda_n)x} \det(\underbrace{\underline{z}_1 \cdots \underline{z}_n}_{\text{matrix}})}_{\neq 0 \text{ as } \underline{z}_1, \dots, \underline{z}_n \text{ are l.I.}}$$

In fact, if A is diagonalizable, i.e. if $\exists$ a basis of $\mathbb{R}^n$ consisting of eigen vectors of A, then also we can list down n-many linearly independent solutions.

Thm Consider the homogeneous linear system with constant coefficients $\underline{y}' = A\underline{y}$, $A \in M_{n\times n}(\mathbb{R})$. Suppose that A is diagonalizable. Let $\underline{z}_1, \dots, \underline{z}_n$ be linearly independent eigen vectors of A corresponding to $\lambda_1, \dots, \lambda_n$ (need not be distinct). Then the general solution of this system is $c_1 \underline{z}_1 e^{\lambda_1 x} + \cdots + c_n \underline{z}_n e^{\lambda_n x}$.

The matrix $S = (\underline{z}_1 e^{\lambda_1 x} \; \dots \; \underline{z}_n e^{\lambda_n x})$ is referred as a fundamental solution matrix. Note that

$$
\begin{aligned}
S' &= (\underline{z}_1 \lambda_1 e^{\lambda_1 x} \; \dots \; \underline{z}_n \lambda_n e^{\lambda_n x}) \\
&= (A\underline{z}_1 e^{\lambda_1 x} \; \dots \; A\underline{z}_n e^{\lambda_n x}) \\
&= A(\underline{z}_1 e^{\lambda_1 x} \; \dots \; \underline{z}_n e^{\lambda_n x}) \\
&= AS
\end{aligned}
$$

Examples ① $A\underline{y} = \underline{y}'$ where $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$.

Note that A is symmetric and hence diagonalizable.

$$\det(A - \lambda I) = \det\begin{pmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{pmatrix} = (\lambda+3)^2 - 1 = \lambda^2 + 6\lambda + 8 = (\lambda+2)(\lambda+4).$$

so eigen values of A are $-2, -4$.

$$E(A,-2) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 = x_2 \right\}$$

$$= \text{Span of } \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$E(A,-4) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4x_1 \\ -4x_2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 0 \right\}$$

$$= \text{span of } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so a general solution of the system is

$$c_1 e^{-2x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A fundamental solution matrix is

$$\begin{pmatrix} e^{-2x} & e^{-4x} \\ e^{-2x} & -e^{-4x} \end{pmatrix}.$$

② $\underline{y}' = A\underline{y}$ where $A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -4 & -\lambda \end{pmatrix}$$

$$= \lambda^2 + 4.$$

So no eigen value in $\mathbb{R}$.

But $\pm 2i$ are eigen values of A in $\mathbb{C}$.

$$E(A, 2i) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{array}{l} x_2 = 2ix_1 \\ -2x_1 = ix_2 \end{array} \right\}$$

$$= \text{span of } \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$E(A, -2i) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{array}{l} x_2 = -2ix_1 \\ 2x_1 = ix_2 \end{array} \right\}$$

$$= \text{span of } \begin{pmatrix} 1 \\ -2i \end{pmatrix}.$$

Two l.I complex solutions are

$$e^{2ix}\begin{pmatrix} 1 \\ 2i \end{pmatrix}, \quad e^{-2ix}\begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

$\cos 2x + i\sin 2x$ $\qquad$ $\cos 2x - i\sin 2x$

$$\frac{1}{2}\left\{\begin{pmatrix} \cos 2x + i \sin 2x \\ -2\sin 2x + 2i\cos 2x \end{pmatrix} + \begin{pmatrix} \cos 2x - i\sin 2x \\ -2\sin 2x - 2i\cos 2x \end{pmatrix}\right\}$$

$$= \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix}$$

$$\frac{1}{2i}\left\{\begin{pmatrix} \cos 2x + i \sin 2x \\ -2\sin 2x + 2i\cos 2x \end{pmatrix} - \begin{pmatrix} \cos 2x - i\sin 2x \\ -2\sin 2x - 2i\cos 2x \end{pmatrix}\right\}$$

$$= \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}$$

A general solution of the given system is $c_1 \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix} + c_2 \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}$.

3. $\underline{y}' = A\underline{y}$ where $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 3)^2$.

$$E(A,3) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 3x_2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 0 \right\}$$

$$= \text{span of } \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

alg. mult. of $3 = 2$

geo. mult. of $3 = 1$.

So A is not diagonalizable.

We have one solution $\underline{y}_1 = e^{3x}\begin{pmatrix}1\\-1\end{pmatrix}$.

To find another linearly independent solution we guess $\underline{y}_2 = xe^{3x}\begin{pmatrix}1\\-1\end{pmatrix}$.

$$\underline{y}_2' = e^{3x}\begin{pmatrix}1\\-1\end{pmatrix} + 3xe^{3x}\begin{pmatrix}1\\-1\end{pmatrix}.$$

To have $\underline{y}_2$ as a solution we need

$$A\underline{y}_2 = \underline{y}_2'.$$

But
$$A\underline{y}_2 = Axe^{3x}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$= xe^{3x}\underbrace{A\begin{pmatrix}1\\-1\end{pmatrix}}_{3\begin{pmatrix}1\\-1\end{pmatrix}}$$
$$= 3xe^{3x}\begin{pmatrix}1\\-1\end{pmatrix} \neq \underline{y}_2'.$$

So this choice of $\underline{y}_2$ does not work!

Take $\underline{y}_2 = xe^{3x}\begin{pmatrix}1\\-1\end{pmatrix} + e^{3x}\begin{pmatrix}c_1\\c_2\end{pmatrix}$ where

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is to be determined.

So, $\underline{y}_2' = e^{3x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3xe^{3x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3e^{3x}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$A\underline{y}_2 = xe^{3x}A\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{3x}A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Equating $A\underline{y}_2 = \underline{y}_2'$ we get

$$e^{3x}A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3e^{3x}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

i.e. $A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

i.e. $(A - 3I)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

i.e. $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So, $c_1 + c_2 = 1$

Take $c_1 = 1$, $c_2 = 0$. So we can take $\underline{y}_2 = xe^{3x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{3x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Hence a general solution is of the form

$$e^{3x}\left\{c_1\begin{pmatrix}1\\-1\end{pmatrix}+c_2 x\begin{pmatrix}1\\-1\end{pmatrix}+c_3\begin{pmatrix}1\\0\end{pmatrix}\right\}.$$

<u>Remark</u> Here we had,

$$(A-3I)\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}.$$

← eigen vector corresponding to 3.

So, $(A-3I)^2\begin{pmatrix}c_1\\c_2\end{pmatrix}=(A-3I)\begin{pmatrix}1\\-1\end{pmatrix}=0.$

So, although $\begin{pmatrix}c_1\\c_2\end{pmatrix}\notin \operatorname{Ker}(A-3I)$ it belongs to $\operatorname{Ker}(A-3I)^2$. This $\begin{pmatrix}c_1\\c_2\end{pmatrix}$ is often referred as generalized eigen vector corresponding to eigen value 3.

<u>Recall</u> For first order linear ODE $y'=ay$, $a\in\mathbb{R}$, we know a general solution is of the form $y=ce^{ax}$.

Now for a linear system of first order ODEs with constant coefficients $\underline{y}'=A\underline{y}$, we can analogously have a solution $\underline{y}=\underline{c}e^{Ax}$.

For $A \in M_{n\times n}(\mathbb{R})$,

$$e^{A} := \sum_{k\geq 0} \frac{A^k}{k!}, \quad A^0 = \text{Identity matrix}.$$

Let $\max |a_{ij}| = h$.

Then each entry of A^2 is bounded by $nh \cdot h = nh^2$.

For A^k bound is $n^{k-1} h^k = \frac{(nh)^k}{n}$

$$|a_{ij}| = \left| \sum_{k=1}^{n} a_{ik} a_{kj} \right| \leq \sum_{k=1}^{n} |a_{ik}||a_{kj}| \leq nh^2$$

Each entry of $\sum_{k=0}^{m} \frac{A^k}{k!} \leq \sum_{k=0}^{m} \frac{(nh)^k}{n} \cdot \frac{1}{k!}$

$$\leq \frac{1}{n} e^{nh}$$

so e^A exists.

Thm Consider $\underline{y}' = A\underline{y}$ with $A \in M_{n\times n}(\mathbb{R})$.

Then the general solution of this system is given by $\underline{y}(x) = \underline{c}\, e^{xA}$ where $\underline{c}$ is an arbitrary column vector in $\mathbb{R}^n$.

Further if $\underline{y}(x_0) = \underline{y}_0$, then the particular solution $\underline{y}(x) = \underline{y}_0\, e^{(x-x_0)A}$.

Remark If A is not diagonal, then finding out e^A explicitly is difficult. If one can write $A = A_1 + A_2$ where $A_1A_2 = A_2A_1$ and e^{A_1}, e^{A_2} are known then one has

$$e^A = e^{A_1 + A_2} = e^{A_1} \cdot e^{A_2}.$$

If A is diagonalizable then $\exists$ invertible matrix P such that $D = P^{-1}AP$. So, $A = PDP^{-1}$.

So, $A^n = PD^nP^{-1}$.

So, $e^A = Pe^DP^{-1}$.

For a diagonal matrix $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$, one has $e^D = \begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{pmatrix}$.

Non-homogeneous linear system of ODEs

Suppose $\underline{y}' = A\underline{y} + \underline{g}$ with $\underline{g} \neq 0$.

Let $\underline{y}_h$ be the general solution of $\underline{y}' = A\underline{y}$.

Then the general solution of the given non-homogeneous linear system is $\underline{y}_h + \underline{y}_p$ where $\underline{y}_p$ is a particular solution of it.

Method of undetermined coefficients

$A \in M_{n\times n}(\mathbb{R})$ and $g(x) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} f(x)$

where $f(x)$ is polynomial or sine, or cosine or exponential function.

Example $\underline{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \underline{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2x}$.

$$\underline{y}_h = c_1 e^{-2x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Take $\underline{y}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2x}$.

$$\underline{y}_p' = -2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2x}$$

$$A \underline{y}_p = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2x}$$

So we want,

$$-2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2x} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2x} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2x}$$

i.e. $\begin{pmatrix} -2a_1 \\ -2a_2 \end{pmatrix} = \begin{pmatrix} -3a_1 + a_2 - 6 \\ a_1 - 3a_2 + 2 \end{pmatrix}$

i.e. $\left.\begin{array}{l} a_1 - a_2 + 6 = 0 \\ a_2 - a_1 - 2 = 0 \end{array}\right\}$ Not possible.

So this choice of $\underline{y}_p$ does not work!

$\boxed{\text{Check } \underline{y}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} x e^{-2x} \text{ does not work too!}}$

Exercise! This will help to decide the right choice of $\underline{y}_p$.

So take

$$\underline{y}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} x e^{-2x} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-2x}.$$

$$\underline{y}_p' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-2x} - 2\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} x e^{-2x} - 2\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-2x}$$

$$A\underline{y}_p + \underline{g} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} x e^{-2x} + \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{-2x} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2x}.$$

We want

$A\underline{y}_p + \underline{g} = \underline{y}_p'$. comparing coefficients of xe^{-2x} we get,

$$\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -2\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

So, $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in E(A, -2)$. We can take $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $a \neq 0$.

Then $\begin{pmatrix} a \\ a \end{pmatrix} - 2\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -3b_1 + b_2 - 6 \\ b_1 - 3b_2 + 2 \end{pmatrix}.$

$$\Rightarrow \quad b_2 - b_1 = a + 6,$$
$$b_2 - b_1 = 2 - a.$$

$$\Rightarrow \quad a = -2.$$

So we need $b_2 - b_1 = 4$.

Take $b_1 = 0$, $b_2 = 4$.

So, $$\underline{y}_p = \begin{pmatrix} -2 \\ -2 \end{pmatrix} x e^{-2x} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-2x}.$$

So the general solution is

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2x} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4x} + \begin{pmatrix} -2 \\ -2 \end{pmatrix} x e^{-2x} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-2x}.$$

Variation of parameters

We discuss the same example with variation of parameters.

$$\underline{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \underline{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2x}$$

$$\underline{y}_h = c_1 e^{-2x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A fundamental solution matrix

$$S = \begin{pmatrix} e^{-2x} & e^{-4x} \\ e^{-2x} & -e^{-4x} \end{pmatrix}.$$

We know $\boxed{AS = S'}$.

Note that $\underline{y}_h = S \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Set $\underline{y}_p = S\underline{u}$.

Want $\underline{y}_p' = A\underline{y}_p + \underline{g}$.

$$\begin{cases} \underline{y}_p' = S'\underline{u} + S\underline{u}' \\ A\underline{y}_p + \underline{g} = AS\underline{u} + \underline{g} \\ \qquad\qquad\;\; = S'\underline{u} + \underline{g}. \end{cases}$$

So we need $S\underline{u}' = \underline{g}$.

So, $\underline{u}' = S^{-1}\underline{g} = -\dfrac{1}{2e^{-6x}} \begin{pmatrix} -e^{-4x} & -e^{-4x} \\ -e^{-2x} & e^{-2x} \end{pmatrix} \begin{pmatrix} -6e^{-2x} \\ 2e^{-2x} \end{pmatrix}$

$$= -\frac{1}{2e^{-6x}} \begin{pmatrix} 4e^{-6x} \\ 8e^{-4x} \end{pmatrix} = \begin{pmatrix} -2 \\ -4e^{2x} \end{pmatrix}$$

So, $\underline{u} = \begin{pmatrix} -\int_0^x 2\,dx \\ -4\int_0^x e^{2x}\,dx \end{pmatrix} = \begin{pmatrix} -2x \\ -2e^{2x} + 2 \end{pmatrix}.$

So, $\underline{y}_p = S\underline{u} = \begin{pmatrix} e^{-2x} & e^{-4x} \\ e^{-2x} & -e^{-4x} \end{pmatrix} \begin{pmatrix} -2x \\ -2e^{2x} + 2 \end{pmatrix}$

$$= \begin{pmatrix} -2xe^{-2x} - 2e^{-2x} + 2e^{-4x} \\ -2xe^{-2x} + 2e^{-2x} - 2e^{-4x} \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -2 \end{pmatrix} xe^{-2x} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} e^{-2x} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} e^{-4x}.$$

Method of diagonalization

Suppose we have $\underline{y}' = D\underline{y} + \underline{g}$, where D is diagonal.

Then this system

$$\left.\begin{aligned} y_1' &= d_1 y_1 + g_1 \\ &\vdots \\ y_n' &= d_n y_n + g_n \end{aligned}\right\}$$

has solution

$$y_1 = c_1 e^{d_1 x} + e^{d_1 x}\int_0^x e^{-d_1 t} g_1(t)\,dt,$$

$$\vdots$$

$$y_n = c_n e^{d_n x} + e^{d_n x}\int_0^x e^{-d_n t} g_n(t)\,dt.$$

Suppose we have, $\underline{y}' = A\underline{y} + \underline{g}$, where A is diagonalizable i.e. $A = PDP^{-1}$ for some invertible matrix P and D Diagonal. In fact we know P precisely. P is the matrix having columns as the L.I eigen vectors.

$$\underline{y}' = PD\underbrace{P^{-1}\underline{y}} + \underline{g}.$$

$$= PD\underline{z} + \underline{g} \qquad \text{by writing } P^{-1}\underline{y} = \underline{z}$$

$$\underline{y} = P\underline{z}$$

So, $\underline{y}' = P\underline{z}'$

So, $P\underline{z}' = PD\underline{z} + \underline{g}$.

So, $\boxed{\underline{z}' = D\underline{z} + P^{-1}\underline{g}}$

diagonal form of $\underline{y}' = A\underline{y} + \underline{g}$

$$\begin{cases} z_1' = d_1 z_1 + h_1(x) \\ \vdots \\ z_n' = d_n z_n + h_n(x) \end{cases}$$

Hence $z_i = c_i e^{d_i x} + e^{d_i x} \int_0^x e^{-d_i t} h_i(t) dx$

$\forall \; 1 \leq i \leq n$.

Then one can obtain $\underline{y}$ by finding $P\underline{z}$.

<u>Example</u>

We discuss $\underline{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \underline{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2x}$.

$$P^{-1} A P = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

So, $P^{-1} \underline{g} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -6e^{-2x} \\ 2e^{-2x} \end{pmatrix}$

$$= \begin{pmatrix} -2e^{-2x} \\ -4e^{-2x} \end{pmatrix}$$

We need to solve,

$$\underline{z}' = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \underline{z} + \begin{pmatrix} -2e^{-2x} \\ -4e^{-2x} \end{pmatrix}$$

i.e. $\left.\begin{aligned} z_1' &= -2z_1 - 2e^{-2x} \\ z_2' &= -4z_2 - 4e^{-2x} \end{aligned}\right\}$

So, $$z_1 = c_1 e^{-2x} - 2x e^{-2x}$$
$$z_2 = c_2 e^{-4x} - 2 e^{-2x}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{-2x} + c_2 e^{-4x} - 2x e^{-2x} - 2 e^{-2x} \\ c_1 e^{-2x} - c_2 e^{-4x} - 2x e^{-2x} + 2 e^{-2x} \end{pmatrix}.$$