Existence Theorem for Laplace Transform :-

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies $|f(t)| \leq M e^{kt}$ for some constants M and K, then the Laplace transformation $L(f)$ exists for all $s > K$.

Proof :-

$$L(f) = \int_0^\infty e^{-st} f(t)\, dt$$

$$|L(f)| = \left| \int_0^\infty e^{-st} f(t)\, dt \right|$$

$$\leq \int_0^\infty |e^{-st}|\, |f(t)|\, dt$$

$$\leq \int_0^\infty M \cdot e^{kt} \cdot e^{-st}\, dt$$

$$= M \int_0^\infty e^{-(s-k)t}\, dt = M \lim_{b \to \infty} \left[\frac{e^{-(s-k)t}}{-(s-k)} \right]_0^b$$

$$\Rightarrow |L(f)| \leq \left[M \lim_{b\to\infty} \frac{1}{(k-s)\,e^{(s-k)b}} - M\frac{1}{k-s} \right]$$

$$= \frac{M}{s-k}$$

Therefore, $L(f)$ exists.

Uniqueness :— If the laplace transformation of a given function exists, then it is uniquely determind. Conversely, it can be shown that if two functions have same transformation, then they are equal for all values except possibly some isolated points. In particular, if two continuous functions have same transformation, they are completely identical.

Dirac's Delta Function :-

Let

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

Let $f_K(t-a) = \begin{cases} \frac{1}{K}, & a \leq t \leq a+K \\ 0, & \text{otherwise} \end{cases}$

Then $\lim\limits_{K \to 0+} f_K(t-a) = \delta(t-a)$.

Note that, $\int_0^\infty f_K(t-a)\, dt = \int_a^{a+K} \frac{1}{K}\, dt = 1$

Therefore, by taking $K \to 0+$ we have

$$\int_0^\infty \delta(t-a) = 1.$$

$\delta(t-a)$ is called the Dirac delta function.

Note that, $f_k(t-a) = \frac{1}{k}\left[U(t-a) - U(t-(a+k))\right]$

$$\Rightarrow L\left(f_k(t-a)\right) = \frac{1}{k}\left[L\left(U(t-a)\right) - L\left(U(t-(a+k))\right)\right]$$

$$= \frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right]$$

$$= e^{-as}\,\frac{1-e^{-ks}}{ks}$$

Taking $k \to 0^+$ we have,

$$L\left(\delta(t-a)\right) = e^{-as} \lim_{k\to\infty} \frac{1-e^{-ks}}{ks}$$

$$= e^{-as}$$

Formula

① $L(1) = \frac{1}{s}$, $s > 0$

② $L(t^n) = \frac{\lfloor n}{s^{n+1}}$, $s > 0$, $n \in \mathbb{N}$

③ $L(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}$, $s > 0$, $a \in (0, \infty)$

④ $L(e^{at}) = \frac{1}{s-a}$

⑤ $L(\cos at) = \frac{s}{s^2 + a^2}$

⑥ $L(\sin at) = \frac{a}{s^2 + a^2}$

⑦ $L(\cosh t) = \dfrac{s}{s^2 - a^2}$

⑧ $L(\sinh t) = \dfrac{a}{s^2 - a^2}$

⑨ $L\left(e^{at} f(t)\right) = F(s-a)$, where $L(f(t)) = F(s)$

⑩ $L\left(\int_0^t f(\tau)\, d\tau\right) = \dfrac{1}{s} F(s)$ where $L(f(t)) = F(s)$

⑪ $L(f * g) = L\left(\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right) = F(s)\, G(s)$

where $L(f(t)) = F(s)$ and $L(g(t)) = G(s)$

⑫ $L(u(t-a)) = \dfrac{e^{-as}}{s}$, where $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

⑬ $L\left(u(t-a)\, f(t-a)\right) = e^{-as} F(s)$ where $L(f(t)) = F(s)$

(14) $L(t f(t)) = -F'(s)$ where, $L(f(s)) = F(s)$

(15) $L(t^n f(t)) = (-1)^n F^n(s)$ where $L(f(s)) = F(s)$.

(16) $L(f'(t)) = s L(f(t)) - f(0)$

$L(f''(t)) = s^2 L(f(t)) - s f(0) - f'(0)$

(17) $L(\delta(t-a)) = e^{-as}$

Solve $y'' + 3y' + 2y = u(t-1) - u(t-2)$, $y(0) = 0$
$y'(0) = 0$.

$$u(t-1) - u(t-2) = \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 0, & 2 < t < \infty \end{cases}$$

Solution :- using laplace transformation on both sides we have,

$$s^2 L(y) - s y(0) - y'(0) + 3s L(y) - 3 y(0) + 2 L(y) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$\Rightarrow s^2 L(y) + 3s L(y) + 2 L(y) = \frac{1}{s}(e^{-s} - e^{-2s})$$

$$\Rightarrow (s^2 + 3s + 2) L(y) = \frac{1}{s}(e^{-s} - e^{-2s})$$

$$\Rightarrow (s+1)(s+2) L(y) = \frac{1}{s}(e^{-s} - e^{-2s})$$

$$\Rightarrow L(y) = \frac{e^{-s}}{s(s+1)(s+2)} - \frac{e^{-}}{s(s+1)(s+2)}$$

We know that $L\left(U(t-a)\, f(t-a)\right) = e^{-as} F(s)$

i.e., $L^{-1}\left(e^{-as} F(s)\right) = U(t-a)\, f(t-a)$.

Let $F(s) = \frac{1}{s(s+1)(s+2)}$

$$= \frac{1}{s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$= \left[\frac{1}{s} - \frac{1}{s+1}\right] - \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

$$= \frac{1}{2}\cdot\frac{1}{s+2} + \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow f(t) = L^{-1}(F(s)) = \frac{1}{2} L^{-1}\left(\frac{1}{s+2}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$f(t) = \frac{1}{2}e^{-2t} + \frac{1}{2} - e^{-t}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Therefore,
$$y(t) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s+1)(s+2)}\right) - \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s+1)(s+2)}\right)$$

$$= U(t-1)\,f(t-1) - U(t-2)\,f(t-2)$$

$$\therefore \quad y(t) = \begin{cases} 0, & 0 < t < 1 \\ \frac{1}{2} - e^{1-t} + \frac{1}{2}e^{2-2t}, & 1 < t < 2 \\ -e^{1-t} + \frac{1}{2}e^{2-2t} + e^{2-t} - \frac{1}{2}e^{4-2t}, & t > 2. \end{cases}$$