COL202 : Discrete Mathematical Structures

Name: T Pratham
Entry no.: 2020EE10559
Group : 2

Tutorial Sheet - 7

2. Theorem: Under the men-proposing deferred acceptance algorithm, there is always atleast one woman who receives exactly one proposal during the execution of the algorithm.

Lemma 1: At ~~woman~~ the end of any round of algorithm, a woman is not yet proposed by a man if and only if she does not have a tentative match at the end of that round.

Proof: ($\Rightarrow$) By contradiction.

Suppose at the end of round r, a woman w has no tentative match.

Let us assume, in some round $r' < r$, she was proposed by some man (possibly many men). Then at the end of r', she has a tentative match (as per algorithm). Now, according to the algorithm, in all subsequent rounds, only two possibilities arise:-

a) w is not proposed by any other man. In which case, the tentative match remains intact.

b) In some future round, w is proposed by another man (possibly multiple men). In this case, w will either remain tentatively matched with her old partner or break that match and form a new tentative match with the man who proposed now, according to her preference.

In any case, w remains tentatively matched with a man in all future rounds. In particular, at the end of round r, w has a tentative match. Which is a contradiction.

Hence w was never proposed.

($\Leftarrow$)

Given W was never proposed till the end of round r.

According to the algorithm, a tentative match is only formed after a woman has been proposed by one (or more) man in a round.

~~Hence~~ Hence, w has no tentative match till the end of round r.

Lemma 2: In r^{th} round, if k men are proposing, then before the start of r^{th} round, there are k women who were not proposed yet.

Proof: According to the algorithm, in any round, only those men propose who do not have a tentative match at the start of that round.

Then, if in round r, exactly k men are proposing, there are exactly $(n-k)$ men who are not proposing.

$\Rightarrow$ $(n-k)$ men have a tentative match.

$\Rightarrow$ $(n-k)$ women have a tentative match.

$\Rightarrow$ k women do not have a tentative match at the start of round r.

$\Rightarrow$ k women are not yet proposed before the start of round r (Using Lemma 1)

Hence proved.

~~Proof of theorem~~ :

Lemma 3 : Before r^{th} round, if k women are not proposed yet, then k men will propose in r^{th} round.

Proof: From lemma 1, the k women who are not proposed yet do not have a tentative match.

⇒ $(n-k)$ women who are proposed have a tentative match.

⇒ $(n-k)$ men have a tentative match before round r

⇒ k men do not have a match before round r.

⇒ As per the algorithm, only men who do not have a match propose in a round. So only these k men will propose in r^{th} round.

Hence proved.

Proof of theorem :

Consider the last round of DA algorithm (we know that DA algorithm terminates after finite number of rounds, hence ~~last~~ there is a last round).

Let k women be unproposed till last round.

$k > 0$ since if $k=0$ then algorithm has already terminated and this is not the last round (using lemma 3).

Then the k men will propose each of these k women and these k women receive exactly one proposal, which is in this last round.

If any woman receives more than one proposal, then all but one proposal will be rejected and this will not be the last round. Which is a contradiction

Hence, there is atleast one woman $(k \geq 1)$ who receive exactly one proposal during the execution of DA algorithm ▨