

## Problems for Recitation 23

**Theorem 1.** Let  $E_1, \dots, E_n$  be events, and let  $X$  be the number of these events that occur. Then:

$$\text{Ex}(X) = \Pr\{E_1\} + \Pr\{E_2\} + \dots + \Pr\{E_n\}$$

**Theorem 2** (Markov's Inequality). Let  $X$  be a nonnegative random variable. If  $c > 0$ , then:

$$\Pr\{X \geq c\} \leq \frac{\text{Ex}(X)}{c}$$

**Theorem 3** (Chebyshev's Inequality). For all  $x > 0$  and any random variable  $R$ ,

$$\Pr\{|R - \text{Ex}(R)| \geq x\} \leq \frac{\text{Var}[R]}{x^2}$$

**Theorem 4** (Union Bound). For events  $E_1, \dots, E_n$ :

$$\Pr\{E_1 \cup \dots \cup E_n\} \leq \Pr\{E_1\} + \dots + \Pr\{E_n\}$$

**Theorem 5** (“Murphy’s Law”). If events  $E_1, \dots, E_n$  are mutually independent and  $X$  is the number of these events that occur, then:

$$\Pr\{E_1 \cup \dots \cup E_n\} \geq 1 - e^{-\text{Ex}(X)}$$

**Theorem 6** (Chernoff Bounds). Let  $E_1, \dots, E_n$  be a collection of mutually independent events, and let  $X$  be the number of these events that occur. Then:

$$\Pr\{X \geq c \text{Ex}(X)\} \leq e^{-(c \ln c - c + 1) \text{Ex}(X)} \quad \text{when } c > 1$$

# 1 Getting Dressed

Sometimes I forget a few items when I leave the house in the morning.

For example, here are probabilities that I forget various pieces of footwear:

left sock	0.2
right sock	0.1
left shoe	0.1
right shoe	0.3

- a. Let  $X$  be the number of these that I forget. What is  $\text{Ex}(X)$ ?
  - b. Upper bound the probability that I forget one or more items. Make no independence assumptions.
  - c. Use the Markov Inequality to upper bound the probability that I forget 3 or more items.
  - d. Now suppose that I forget each item of footwear independently. Use Chebyshev's Inequality to upper bound the probability that I forget two or more items.

- e. Use Theorem 5 to lower bound the probability that I forget one or more items.
- g. I'm supposed to remember many other items, of course: clothing, watch, backpack, notebook, pencil, kleenex, ID, keys, etc. Let  $X$  be the total number of items I remember. Suppose I remember items mutually independently and  $\text{Ex}(X) = 36$ . Use Chernoff's Bound to give an upper bound on the probability that I remember 48 or more items.
- h. Give an upper bound on the probability that I remember 108 or more items.

## 2 A Financial Crisis

There are a lot of foreign words here, but don't be scared! We will be trying to understand why the subprime mortgage collapse happened!

For a more complete story of how the crisis happened, refer to section 19.5.3 of the text. The following is a set of vocabulary that we will be using:

- A **loan** is money lent to a borrower. If the borrower does not pay on the loan, the loan is said to be in **default**, and collateral is seized. In the case of mortgage loans, the borrower's home is used as collateral.
- A **bond** is a collection of loans, packaged into one entity. A bond can be divided into **tranches**, in some ordering, which tell us how to assign losses from defaults. Suppose a bond contains 1000 loans, and is divided into 10 tranches of 100 bonds each. Then, all the defaults must fill up the lowest tranche before they affect others. For example, suppose 150 defaults happened. Then, the first 100 defaults would occur in tranche 1, and the next 50 defaults would happen in tranche 2.
- The lowest tranche of a bond is called the **mezzanine tranche**.
- We can make a “super bond” of tranches called a **collateralized debt obligation (CDO)** by collecting mezzanine tranches from different bonds. This super bond can then be itself separated into tranches, which are again ordered to indicate how to assign losses.

Armed with this knowledge, we can now solve problems about the crisis!

1. Suppose that 1000 loans make up a bond, and the fail rate is 5% in a year. Assuming mutual independence, give an upper bound for the probability that there are one or more failures in the second-worst tranche. What is the probability that there are failures in the best Tranche?
2. Now, do not assume that the loans are independent. Give an upper bound for the probability that there are one or more failures in the second tranche. What is an upper bound for the probability that the entire bond defaults? Show that it is a tight bound. (Hint: Use Markov's theorem).
3. Given this setup (and assuming mutual independence between the loans), what is the expected failure rate in the mezzanine tranche?
4. We take the mezzanine tranches from 100 bonds and create a CDO. What is the expected number of underlying failures to hit the CDO?
5. We divide this CDO into 10 tranches of 1000 bonds each. Assuming mutual independence, give an upper bound on the probability of one or more failures in the best tranche. The third tranche?
6. Repeat the previous question without the assumption of mutual independence.