COL202: Discrete Mathematical Structures

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Tutorial Sheet 11

2.

We know that, the total number of permutations of a set of n distinct elements is $= n!$ —— ①

Now, we will count the number of permutations of set $\{1, 2, \ldots, n\}$ where atleast one element is in the correct position.
By correct position, we mean, in the permutation $(x_1, x_2, \ldots, x_n)$, x_i is at correct position $\Leftrightarrow x_i = i$

Let S_k denote the set of permutations of $\{1, 2, \ldots, n\}$ where k is fixed at k^{th} position. (i.e. k is at its correct position).
Then, we want to count $|S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n|$.

By inclusion - exclusion rule, we know:

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum_{i=1}^{n} |S_i| - \sum_{1 \le i < j \le n} |S_i \cap S_j| + \sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k|$$
$$- \cdots + (-1)^{n+1} |S_1 \cap S_2 \cap \ldots \cap S_n|$$

For any intersection of k such sets of S_i $(i \in \{1, \ldots, n\})$, we can choose the k sets in ${}^nC_k$ ways. An intersection of k sets has the permutations in which the positions of the corresponding k elements have been fixed at their correct positions. The remaining $(n-k)$ element can be arranged in any order and hence the number of such permutations is $(n-k)!$.

Hence, there are ${}^nC_k$ different possible intersections of k sets, each having cardinality $(n-k)!$.

$$\therefore |S_1 \cup S_2 \cup \dots \cup S_n| = \sum_{i=1}^{n} |S_i| - \sum_{1\le i<j\le n} |S_i \cap S_j| + \sum_{1\le i<j<k\le n} |S_i \cap S_j \cap S_k| - \dots$$

$$+ (-1)^{n+1} |S_1 \cap S_2 \cap \dots \cap S_n|$$

$$= {}^nC_1 \cdot (n-1)! - {}^nC_2 \cdot (n-2)! + {}^nC_3 \cdot (n-3)! - \dots + (-1)^{n+1}\, {}^nC_n \cdot 0!$$

$$= \frac{n!}{1!\,\cancel{(n-1)!}}\cancel{(n-1)!} - \frac{n!}{2!\,\cancel{(n-2)!}}\cancel{(n-2)!} + \frac{n!}{3!\,\cancel{(n-3)!}}\cancel{(n-3)!} - \dots + (-1)^{n+1} \frac{n!}{n!\,\cancel{0!}}\cancel{0!}$$

$$= \frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \dots + (-1)^{n+1}\frac{n!}{n!} \qquad \text{(ii)}$$

Since in a derangement, no element is at its correct position, the number of derangements is given as total number of permutations minus the number of permutations where at least one element is at its correct position. That is, from (i) and (ii):

$$\text{Total number of derangements} = n! - \left(\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \dots + (-1)^{n+1}\frac{n!}{n!}\right)$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$$