Problem :- Determine the conditions for which the system

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits

(i) only one solution (ii) no solution (iii) many solutions.

Solution :- Let the system is $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$$

$$[A \mid B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & a & b^2 \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 2 & a-5 & b^2-5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 2 & a-5 & b^2-5 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & a-1 & b^2-2b-3 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 3 & -b+2 \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & a-1 & (b+1)(b-3) \end{bmatrix}$$

The system has a unique solution if $\text{rank}(A) = 3$

i.e., $a \neq 1$.

The system has no solutions if $\text{rank}(A) < \text{rank}(A|B)$

i.e., if $a = 1$ and $b \neq -1, 3$

The system has an infinite number of Solution if $\text{rank}(A) = \text{rank}(A|B) < 3$, i.e., either $a = 1, b = -1$

or $a = 1, b = 3$

Problem :- Show that the planes $bx - ay = n$, $cy - bz = l$, $az - cx = m$ intersect in a line if $al + bm + cn = 0$ and the direction of the line is (a, b, c), where $a, b, c \neq 0$.

Solution :- $\det A = \begin{bmatrix} b & -a & 0 & n \\ 0 & c & -b & l \\ -c & 0 & a & m \end{bmatrix}$

$$\xrightarrow[bR_3]{cR_1,\ aR_2} \begin{bmatrix} bc & -ac & 0 & cn \\ 0 & ac & -ab & al \\ -bc & 0 & ab & bm \end{bmatrix}$$

$$\xrightarrow[R_3 + R_2]{R_3 + R_1} \begin{bmatrix} bc & -ac & 0 & cn \\ 0 & ac & -ab & al \\ 0 & 0 & 0 & cn+al+bm \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} bc & 0 & -ab & cn+al \\ 0 & ac & -ab & al \\ 0 & 0 & 0 & cn+al+bm \end{bmatrix}$$

$$\xrightarrow[\frac{1}{ac} R_2]{\frac{1}{bc} R_1} \begin{bmatrix} 1 & 0 & -\frac{a}{c} & \frac{cn+al}{bc} \\ 0 & 1 & -\frac{b}{c} & \frac{l}{c} \\ 0 & 0 & 0 & cn+al+bm \end{bmatrix}$$

The planes intersect in a line if $\operatorname{rank}(A) = \operatorname{rank}(A|B) = 2$

i.e., $al + bm + cn = 0$.

Then the equations reduce to

$$x_1 - \frac{a}{c} z = \frac{cn + al}{bc}$$

$$x_2 - \frac{b}{c} z = \frac{l}{c}$$

Let $z = k$. Then $x_1 = \frac{cn+al}{bc} + \frac{ak}{c}$

$$x_2 = \frac{l}{c} + \frac{bk}{c}$$

$$\Rightarrow \left(x_1 - \frac{cn+al}{bc}\right)\frac{c}{a} = \left(x - \frac{l}{c}\right)\frac{c}{b} = z$$

$$\Rightarrow \frac{x_1 - \frac{cn+al}{bc}}{a} = \frac{x - \frac{l}{c}}{b} = \frac{z}{c}$$

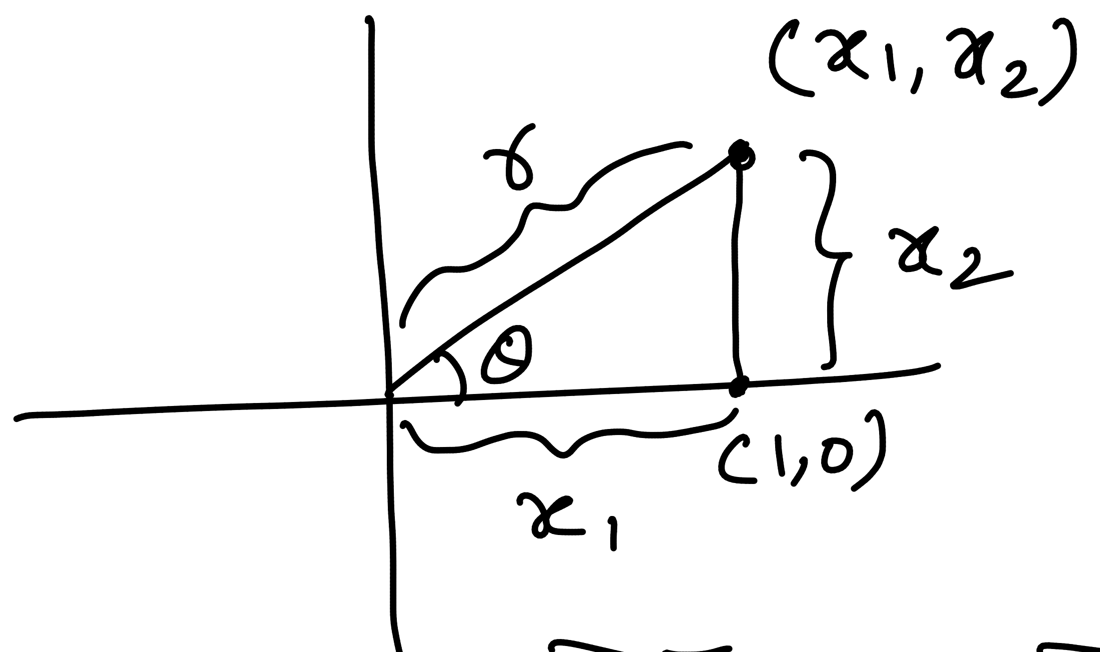
Therefore, the direction of the intersection line is (a, b, c)

Problem:- Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \text{ and } B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}.$$

Show that $\vec{y} = A\vec{x}$ is the rotation of vector $\vec{x}$ counter clockwise by angle α. Also compute $\vec{y} = A^n\vec{x}$ and $\vec{y} = AB\vec{x}$.

Solution:- Let θ be the angle between $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



Then $x_1 = r\cos\theta$

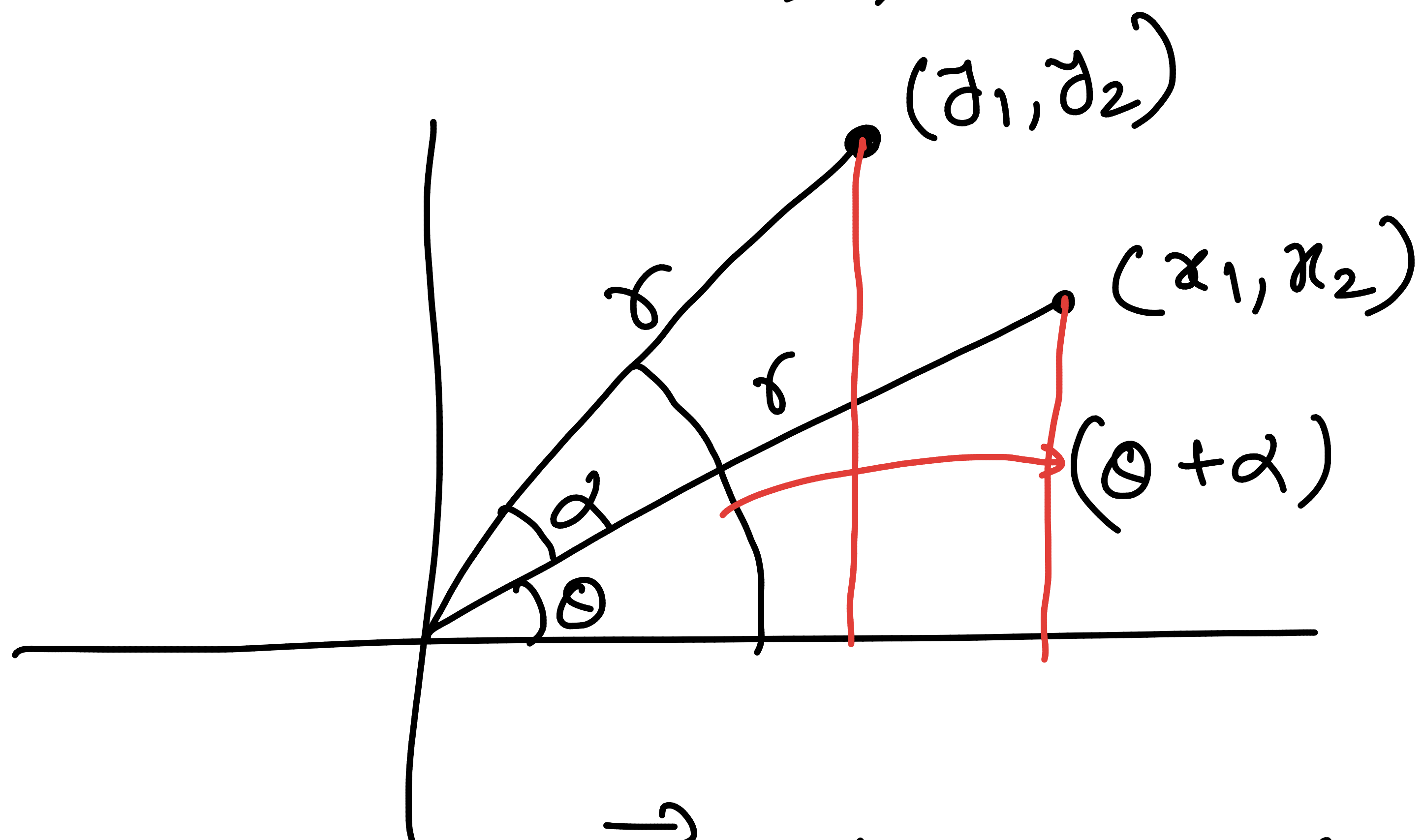
$x_2 = r\sin\theta$

where $r = \sqrt{x_1^2 + x_2^2}$

Now, $$A\vec{x} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ r(\cos\theta\sin\alpha + \sin\theta\cos\alpha) \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix} = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Therefore, $y_1 = r\cos(\theta+\alpha)$, $y_2 = r\sin(\theta+\alpha)$



Therefore, A rotates $\vec{x}$ by angle α.

Now, $$AB = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\alpha \\ \sin\beta & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

Therefore, AB rotates $\vec{x}$ by angle $\alpha+\beta$.

Note that,

$$A^2 = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

By induction, we can prove that,

$$A^n = \begin{bmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{bmatrix}$$

Therefore, A^n rotates $\vec{x}$ by angle $n\alpha$.

Problem :- Let $C[0,1]$ be the collection of all real valued continuous function on domain $[0,1]$. Show that $C[0,1]$ is a vector space over the field $\mathbb{R}$.

Solution :- For any two $f, g \in C[0,1]$

$$(f+g)(x) = f(x) + g(x)$$

$$(cf)(x) = c f(x)$$

It is easy to verify that $C[0,1]$ satisfies all properties of the vector space.

Problem :- Show that $\{x, x^2, \sin x, \cos x\}$ is linearly independent in $C[0,10]$.

Solution :- Let $c_1 x + c_2 x^2 + c_3 \sin x + c_4 \cos x = 0$

By putting $x = 0$, we get $c_4 = 0$

By putting $\pi, 2\pi$ we get, $c_1(2\pi) + c_2(2\pi)^2 = 0$

$$c_1 \pi + c_2 \pi^2 = 0$$

$$\Rightarrow c_1 = c_2 = 0$$

Then $c_3 \sin x = 0$

by putting $\frac{\pi}{2}$ we get $c_3 = 0$

Therefore, the set is linearly independent.

Problem :- Let $V, W \subseteq C[0,1]$ such that

$$V = \{ f \in C[0,1] : f(0) = 0 \}$$

$$W = \{ f \in C[0,1] : f(0) = 1 \}.$$ Are they subspace?

Solution :- Let $f, g \in V$. Then $cf + dg \in C[0,1]$ and $(cf+dg)(0) = cf(0) + dg(0) = c0 + d.0 = 0$

for every $c, d \in \mathbb{R}$ and $f, g \in V$.

Therefore, V is a subspace of $C[0,1]$

For $f, g \in W$ $\quad f(0) = 1 = g(0)$

$$(cf + dg)(0) = cf(0) + dg(0) = c + d$$

Since $c+d$ may not be 1 for arbitrary $c, d \in \mathbb{R}$, W is not a subspace of $C[0,1]$.

Problem :- Let $F^{\infty} = \{ (x_1, x_2, x_3, \cdots) : x_1, x_2, \cdots \in F \}$. Then F^{∞} is a vector space. Let $e_i = (0, \cdots, 0, 1, 0, \cdots)$ ↳ i th place

Show that $\{e_1, e_2, e_3, \cdots\}$ is linearly independent. Find $\text{span}(\{e_1, e_2, \cdots\})$.

Soln :- Choose any arbitrary finite subset $\{e_{i_1}, \cdots, e_{i_n}\}$ $\subseteq \{e_1, e_2, \cdots\}$ such that $c_1 e_{i_1} + c_2 e_{i_2} + \cdots + c_n e_{i_n} = 0$

Then $(0,\cdots,0,\underset{\hookrightarrow i_1}{c_1},0,\cdots,0,\underset{\hookrightarrow i_2}{c_2},0,\cdots,\cdots,\cdots,0,\underset{\llcorner i_n}{c_n},0,\cdots,0) = (0,0,0,0,\cdots)$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0.$

Therefore, $\{e_1, e_2, \cdots\}$ is linearly independent.

claim, $\operatorname{span}(\{e_1, e_2, \cdots\}) = \{(x_1, x_2, x_3, \cdots) : x_1, x_2, \cdots \in F$ & $x_i = 0$ after a finite stage$\}$

$= V$

Note that any finite linear combination of vectors from $\{e_1, \cdots, e_n, \cdots\}$ lies in V.

Therefore, $\operatorname{span}(\{e_1, e_2, e_3, \cdots\}) \subseteq V$.

On the other hand, let $(y_1, y_2, \cdots, y_n, 0, 0, \cdots) \in V$

then $(y_1, y_2, \cdots, y_n, 0, 0, \cdots) = y_1 e_1 + \cdots + y_n e_n \in \operatorname{span}\{e_1, \cdots e_2 \cdots$

$\Rightarrow V \subseteq \operatorname{span}(\{e_1, e_2, \cdots\})$. Therefore, $V = \operatorname{span}(\{e_1, e_2, \cdots\})$

Note that some y_i may be 0.

Problem :- Let V be a vector space over a field F. Let $\dim(V) = n$ and F has p elements. Then what is the number of elements of V.

Solution :- Since $\dim(V) = n$, let $\{v_1, v_2, \dots, v_n\}$ be a basis of V.

Therefore, each element of V can be written as uniquely $c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ where $c_1, c_2, \dots, c_n \in F$.

For each c_i, we have exactly p choices. Therefore V has $\underbrace{p \times \dots \times p}_{n \text{ times}} = p^n$ elements.

Problem :- If $\{a, b, c\}$ is a basis of V and V is a vector space over the field $\mathbb{Z}_3$ then find all the elements of V.

Solution:- The elements of the field is $\{0,1,2\}$.
Thus V has $3^3 = 27$ elements and $V =$
$\{0, a, b, c, 2a, 2b, 2c, a+b, a+c, b+c, 2a+b, 2a+c,$
$2a+2b, 2a+2c, a+2b, a+2c, b+2c, 2b+c, 2b+2c,$
$a+b+c, 2a+b+c, a+2b+c, a+b+2c, 2a+2b+c,$
$2a+b+2c, a+2b+2c, 2a+2b+2c\}$

What is the dimension of the vector space $M_{m\times n}(F)$. What is the dimension of the vector space which is the collection of all $n\times n$ symmetric (resp. anti-symmetric) matrices.

Sol^n:- $\dim M_{m\times n}(F) = mn$.

Dimension of the collection of all symmetric $n\times n$ matrices
$= \frac{n^2-n}{2} + n = \frac{n(n+1)}{2}$. For anti-symmetric dimension is $\frac{n^2-n}{2}$.