Linear transformation :-

Let V and W be two vector spaces over the same field F. Then a function $T: V \to W$ is said to be a linear transformation if $T(ax + by) = aT(x) + bT(y)$, $x, y \in V$, $a, b \in F$.

* dimension of V and W may be infinite.

* Today we focus only on finite dimensional vector spaces V and W.

Example :- $T: \mathbb{R}^2 \to \mathbb{R}^2$ ($\mathbb{R}^2$ is a 2-dimensional vector space over $\mathbb{R}$)

$$T(x, y) = (2x + 2y, x - y)$$

Say, $F: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 2x + 2y \\ x - y \end{bmatrix}$$

$T \equiv F$

For any 2×2 matrix A, $G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ is a linear transformation.

<u>Observe that</u>, $B = \{\underset{e_1}{\underset{\shortparallel}{(1,0)}}, \underset{e_2}{\underset{\shortparallel}{(0,1)}}\}$ is a basis of $\mathbb{R}^2$. We take this basis as an ordered basis.

$$T(1,0) = (2,1) = 2\,(1,0) + 1\,(0,1)$$
$$T(0,1) = (2,-1) = 2\,(1,0) - 1\,(0,1)$$

$$[T]_B^B \quad \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

* Let $\underset{B}{\underset{\shortparallel}{\{v_1, v_2, \ldots, v_m\}}}$ be a basis of V and $B' = \{w_1, w_2, \ldots, w_n\}$ be a basis of W where V and W are two vector spaces over a field F.

Let $T : V \to W$ be a linear transformation.

$$T(v_1) = t_{11} w_1 + t_{21} w_2 + \cdots + t_{n1} w_n$$

$$T(v_2) = t_{12} w_1 + t_{22} w_2 + \cdots + t_{n2} w_n$$

$$\vdots$$

$$T(v_m) = t_{1m} w_1 + t_{2m} w_2 + \cdots + t_{nm} w_n$$

$$T(v_j) = \sum_{i=1}^{n} t_{ij} w_i$$

Let $T = (t_{ij})_{n \times m}$

<u>Claim</u> $[T(v)]_{B'} = (t_{ij})_{n \times m} [v]_B$

Let $v = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$ $\quad [v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$

$$T(v) = T\left(\sum_{j=1}^{m} a_j v_j\right)$$

$$= \sum_{j=1}^{m} a_j \, T(v_j)$$

$$= \sum_{j=1}^{m} a_j \left(\sum_{i=1}^{n} t_{ij} w_i\right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} t_{ij} a_j\right) w_i$$

$$\left[T(v)\right]_{B'} = \begin{bmatrix} t_{11}a_1 + t_{12}a_2 + \cdots + t_{1m}a_m \\ t_{12}a_1 + t_{22}a_2 + \cdots + t_{2m}a_m \\ \cdots \\ t_{1n}a_1 + t_{2n}a_2 + \cdots + t_{nm}a_m \end{bmatrix}$$

$$= \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ \cdot & \cdot & & \cdot \\ t_{1n} & t_{2n} & \cdots & t_{nm} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = (t_{ij}) \, [v]_B$$

To find a matrix representation of $T: V \to W$, we need to fix ordered basis for V and W, say B & B'. Then find $[T]_{B}^{B'}$.

* Note that if you change the basis (even the order of the basis) then matrix will be changed.

Eigen value and eigen vector of a linear transformation

Let V be a vector space over a field F (dim(V) may be infinite)
Let $T: V \to V$ be a linear transformation. A scalar $\lambda \in F$ is an eigen value of T if there exists a non-zero vector $X \in V$ s.t. $T(X) = \lambda X$. The non-zero vector X is called an eigen vector corresponding to the eigen value λ.

Example :-

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad [T]_{SB} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Then 2, 3 are eigen values of T.

(ii) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad [T]_{SB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then T does not have any eigen value.

(iii) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\quad T(x,y) = (2x+3y, 3x+2y)$

Find the eigenvalue of T (if exist).

Soln :- If λ is an eigen value of T then there exists a non-zero vector $(x,y) \in \mathbb{R}^2$ s.t.

$$T(x,y) = \lambda(x,y)$$

$$\Rightarrow (2x+3y, 3x+2y) = (\lambda x, \lambda y) \to \begin{aligned} (2-\lambda)x + 3y &= 0 \\ 3x + (2-\lambda)y &= 0 \end{aligned}$$

The system has a non-zero solution if and only if

$$\det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \lambda = -1, 5$$

If $\lambda = -1$ then $x + y = 0 \Rightarrow y = -x$

$\Rightarrow (x, -x)$ is a eigen vector.

$(1, -1)$

If $\lambda = 5$ then $x = y = a$

$\Rightarrow (a, a)$ is a eigen vector.

$(1, 1)$

Note that,

$$T(1,0) = 2(1,0) + 3(0,1)$$
$$T(0,1) = 3(1,0) + 2(0,1)$$
$$[T]_B^B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Characteristic Polynomial :- Let V be a finite dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Then $\det(xI - T)$ is called the characteristic polynomial.

Eigen values of T are the roots of the characteristic polynomial of T.