COL 202: Discrete Mathematical Structures

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Tutorial Sheet - 6

2. a) <u>Theorem</u>: Every graph with width at most w is $(w+1)$ - colorable.

<u>Proof</u>: By induction, on number of vertices n in the graph.

$P(n)$: A graph $G = (V, E)$ having n vertices (i.e. $|V| = n$) with width atmost w is $(w+1)$ - colorable.

Base case: $n = 1$, i.e. G has only 1 vertex, so its width $w = 0$
Hence, it can be colored in $w+1 = 1$ color. $\therefore P(1)$ is true.

Induction hypothesis: Assume, for some $n \in \mathbb{N}$, $P(n)$ is true.

Induction step: We need to show $P(n+1)$ is true.

Consider graph $G = (V, E)$, with $|V| = n+1$ and width at most w.
That is, the vertices can be arranged in a sequence $v_1, v_2, \ldots, v_n, v_{n+1}$ such that $\forall\ i \in \{1, \ldots, n+1\}$, v_i is connected to atmost w vertices preceding it.

Remove v_{n+1} and all edges incident to it from G to obtain a new graph G' with n-vertices and width atmost w. Since removing v_{n+1} and retaining the same sequence does not affect the number of edges from a vertex v_i $(i \in \{1, 2, \ldots, n\})$ to a preceeding vertex.

Hence, from induction hypothesis, G' is $(w+1)$ - colorable.

Now, replacing v_{n+1} with all its incident edges. Since G has width atmost w and there are no vertices succeeding v_{n+1}, v_{n+1} has at most w neighbours which preceed v_{n+1} in the sequence. So, we can colour v_{n+1} differently from all its neighbours and this will require only $(w+1)$ colors.

$\Rightarrow P(n+1)$ is true

Hence Proved.

b) Theorem: The average degree of a graph of width w is atmost $2w$.

Proof: By induction, on number of vertices n in the graph.

$P(n)$: A graph $G = (V, E)$ with $|V| = n$ and width w has an average degree of atmost $2w$.

Base case: $n = 1$, width of the graph, $w = 0$ and ~~its~~ its average degree is $\frac{0}{1} = 0 = 2w$.

$\therefore$ $P(1)$ is true.

Induction hypothesis: Assume for some $n \in \mathbb{N}$, $P(n)$ is true.

i.e. $$\frac{\sum_{v \in V} \deg(v)}{|V|} \leq 2w$$

Induction step: We need to show $P(n+1)$ is true.

Consider graph $G = (V, E)$ with $|V| = n+1$ and width w. The vertices can be arranged in a sequence $v_1, v_2, \ldots, v_n, v_{n+1}$ such that $\forall i \in \{1, \ldots, n+1\}$, v_i is connected to atmost w vertices preceding it.

Remove v_{n+1} and all edges incident to it to obtain a new graph $G' = (V', E')$ with $|V'| = n$ and width w (reason for width $= w$ explained in part a))

From induction hypothesis, $\sum_{v \in V'} \deg(v) \leq 2wn$

Now, replacing v_{n+1}, with all its incident edges. Since width of G is w, atmost w edges are incident to v_{n+1} and only the degree of atmost w vertices preceding v_{n+1} increases by 1 on replacing v_{n+1}.

Hence, average degree $= \frac{\sum_{v \in V} \deg(v)}{n+1} \leq \frac{\sum_{v \in V'} \deg(v) + 2w}{n+1}$ [sum of degrees of 1^{st} n vertices when v_{n+1} removed + $\deg(v_{n+1})$ + at most w edges to v_{n+1}]

$$\leq \frac{2wn + 2w}{n+1} = \frac{2w\cancel{(n+1)}}{\cancel{(n+1)}} = 2w$$

$\therefore$ $P(n+1)$ is true. Hence proved.