Laplace transform

Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a piecewise continuous function having possible discontinuities only as removable or jump and $\boxed{|f(t)| \leq M e^{kt}}$ growth condition for some constants $M, k > 0$. Then the Laplace transform of f is denoted by $\mathcal{L}(f)$.

$$\mathcal{L}(f) := F(s) := \int_0^\infty e^{-st} f(t)\,dt \quad \text{for } s > k.$$

Let $a > 0$.

$$\left|\int_0^a e^{-st} f(t)\,dt\right| \leq \int_0^a e^{-st} |f(t)|\,dt$$

$$\leq M \int_0^a e^{-(s-k)t}\,dt$$

$$\leq M \int_0^\infty e^{-(s-k)t}\,dt \;, \text{ as } e^{-(s-k)t} \text{ is a positive function}$$

$$= \frac{M}{(s-k)}$$

So, $\displaystyle\int_0^\infty e^{-st} f(t)\,dt := \lim_{a\to\infty} \int_0^a e^{-st} f(t)\,dt$ exists.

Remarks ① For two different functions f_1, f_2 we can have $\mathcal{L}(f_1) = \mathcal{L}(f_2)$. For example let, $f_1(t) = 0 \quad \forall\, t \geq 0$

and $f_2(t) = \begin{cases} 1, & t = 0 \\ 0, & t > 0 \end{cases}$

Note that,

$$\int_0^\infty e^{-st} f_1(t)\,dt = 0 = \int_0^\infty e^{-st} f_2(t)\,dt.$$

Statement only

2. If f_1, f_2 are continuous functions and $\mathcal{L}(f_1) = \mathcal{L}(f_2)$, then $f_1 = f_2$.

Notation

For a function f, we denote its Laplace transform $\mathcal{L}(f)$ by F. For f we use variable t and for F we use variable s. By the notation $\mathcal{L}^{-1}(F) = f$ we mean $\mathcal{L}(f) = F$ for a continuous function f.

Thm Let $f, g : [0, \infty) \longrightarrow \mathbb{R}$ be such that $\mathcal{L}(f), \mathcal{L}(g)$ exist. Then for any $c \in \mathbb{R}$, $\mathcal{L}(cf + g)$ exists and $\mathcal{L}(cf + g) = c\mathcal{L}(f) + \mathcal{L}(g)$.

Pf $\mathcal{L}(cf+g) = \int_0^\infty e^{-st}(cf+g)\,dt$

$$= c\int_0^\infty e^{-st} f\,dt + \int_0^\infty e^{-st} g\,dt$$

$$= c\mathcal{L}(f) + \mathcal{L}(g).$$

Examples

① Let $f(t) = 1$ for $t \geq 0$.

Then $\mathcal{L}(f) = \int_0^\infty e^{-st}\,dt = \left.\dfrac{e^{-st}}{-s}\right|_0^\infty$

$$= \frac{1}{s}$$

② Let $f(t) = e^{at}$ for $t \geq 0$.

Then $\mathcal{L}(f) = \int_0^\infty e^{-(s-a)t}\,dt$ $< \infty$ for $s > a$.

$$= \frac{1}{s-a}.$$

③ $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$

So, $\mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

for $s > |a|$.

$$= \frac{s}{s^2 - a^2}.$$

④ $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$.

$$\mathcal{L}(\sinh at) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \text{ for } s>|a|.$$

$$= \frac{a}{s^2-a^2}.$$

⑤ $\mathcal{L}(\cos at) = \int_0^\infty e^{-st}\cos at \, dt$

$$= \frac{e^{-st}}{-s}\cos at\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin at \, dt \quad \text{for } s>0$$

$$= \frac{1}{s} - \frac{a}{s}\mathcal{L}(\sin at)$$

$$\mathcal{L}(\sin at) = \int_0^\infty e^{-st}\sin at \, dt$$

$$= \frac{e^{-st}}{-s}\sin at\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at \, dt \quad \text{for } s>0$$

$$= \frac{a}{s}\mathcal{L}(\cos at)$$

So, $\mathcal{L}(\cos at) = \frac{1}{s} - \frac{a^2}{s^2}\mathcal{L}(\cos at)$.

So, $\mathcal{L}(\cos at) = \dfrac{1/s}{(1+a^2/s^2)} = \dfrac{s}{s^2+a^2}$.

So, $\mathcal{L}(\sin at) = \dfrac{a}{s^2+a^2}$.

(6) $$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

$$= \frac{e^{-st}}{-s} t^n \Big|_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt$$

$$= \frac{n}{s} \mathcal{L}(t^{n-1}).$$

$$= \frac{n!}{s^n} \mathcal{L}(1) = \frac{n!}{s^{n+1}}.$$

(7) $f : [0, \infty) \longrightarrow \mathbb{R}$ defined by $f(t) = t^a, a > 0$.

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt$$

$$= \int_0^\infty e^{-x} \frac{x^a}{s^a} \frac{dx}{s}$$

Put $st = x$
So, $s\,dt = dx$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^{(a+1)-1} dx$$

$$= \frac{\Gamma(a+1)}{s^{a+1}}.$$

(8) Suppose $g(t) = f(at)$ for $a > 0$ and $\mathcal{L}(f) = F(s)$ exists. Then

$$\mathcal{L}(g) = \int_0^\infty e^{-st} f(at) \, dt$$

$$= \frac{1}{a} \int_0^\infty e^{-\frac{sx}{a}} f(x) \, dx$$

$at = x$
$a\,dt = dx$

$$= \frac{1}{a} F\left(\frac{s}{a}\right).$$

Thm (First shifting theorem)

Let $g(t) = e^{at} f(t)$ and $\mathcal{L}(f) = F(s)$ exists for $s > k$. Then $\mathcal{L}(g)$ exists for $s-a > k$ and $\mathcal{L}(g) = F(s-a)$.

Pf
$$\mathcal{L}(g) = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$
$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$
$$= F(s-a).$$

Corollary
$$\mathcal{L}(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$
and
$$\mathcal{L}(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}.$$

Exercise Find f such that $\mathcal{L}(f) = \frac{3s-137}{s^2+2s+401}$

Solution Write,
$$\frac{3s-137}{s^2+2s+401} = \frac{3(s+1)-140}{(s+1)^2+400}$$
$$= 3\,\frac{(s+1)}{(s+1)^2+400} - 7\,\frac{20}{(s+1)^2+(20)^2}$$
$$= \mathcal{L}(3e^{-t}\cos 20t - 7e^{-t}\sin 20t).$$

Transformation of derivatives and integrals

Thm Let f be n-times differentiable function such that $\mathcal{L}(f), \dots, \mathcal{L}(f^{(n)})$ exist. Then

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

Pf

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}(f).$$

$$\mathcal{L}(f'') = -f'(0) + s\mathcal{L}(f')$$

$$= -f'(0) - sf(0) + s^2\mathcal{L}(f).$$

$$\mathcal{L}(f''') = -f''(0) + s\mathcal{L}(f'')$$

$$= -f''(0) - sf'(0) - s^2 f(0) + s^3 \mathcal{L}(f).$$

Proceeding this way we get,

$$\mathcal{L}(f^{(n)}) = -f^{(n-1)}(0) - \dots - s^{n-1} f(0) + s^n \mathcal{L}(f).$$

Applications

① $f(t) = t\sin at$.

so, $f(0) = 0$.

$f'(t) = \sin at + at\cos at$.

$f'(0) = 0$.

$f''(t) = a\cos at + a\cos at - a^2 t\sin at$

$= 2a\cos at - a^2 t\sin at$.

so, $\mathcal{L}(f'') = 2a\,\mathcal{L}(\cos at) - a^2\mathcal{L}(t\sin at)$

$\quad\quad\; \shortparallel$

$-f'(0) - sf(0) + s^2\mathcal{L}(t\sin at)$

$$\mathcal{L}(t\sin at) = \frac{2a}{s^2+a^2}\,\mathcal{L}(\cos at)$$

$$= \frac{2as}{(s^2+a^2)^2}$$

② $f(t) = \cos at$.

$f(0) = 1$. $\quad f'(t) = -a\sin at$

$f'(0) = 0$

$f''(t) = -a^2\cos at$

so, $\mathcal{L}(f'') = -a^2\mathcal{L}(\cos at)$

$\quad\quad\; \shortparallel$

$-f'(0) - sf(0) + s^2\mathcal{L}(f)$

so, $\mathcal{L}(\cos at) = \dfrac{s}{s^2+a^2}$.

③ $f(t) = \sin at$

$f(0) = 0$

$f'(t) = a\cos at \;, \quad f'(0) = a.$

$f''(t) = -a^2 \sin at$

$$\mathcal{L}(f'') = -a^2 \mathcal{L}(\sin at)$$

$$\parallel$$

$$-f'(0) - sf(0) + s^2 \mathcal{L}(f)$$

So, $\mathcal{L}(\sin at) = \dfrac{a}{s^2 + a^2}$.

Thm Let $f(t)$ be such that $\mathcal{L}(f) = F(s)$ exists. Then $\mathcal{L}\left(\int_0^t f(\tau)\,d\tau \right) = \frac{1}{s} F(s)$.

Pf Denote $g(t) := \int_0^t f(\tau)\,d\tau$

Then $g'(t) = f(t)$ except the points where f is not continuous.

So, $\mathcal{L}(g') = F(s)$.

Also, $|g(t)| = \left| \int_0^t f(\tau)\,d\tau \right|$

$$\leq \int_0^t |f(\tau)|\,d\tau \leq M \int_0^t e^{k\tau}\,d\tau$$

$$= \frac{M}{k} e^{kt}, \quad k > 0$$

Now, $\mathcal{L}(g') = -g(0) + s\mathcal{L}(g)$.

$= s\mathcal{L}(g)$ as $g(0)=0$.

So, $\mathcal{L}(g) = \dfrac{F(s)}{s}$.

<u>Application</u>

Find functions f, g such that

$\mathcal{L}(f) = \dfrac{1}{s(s^2+a^2)}$ and $\mathcal{L}(g) = \dfrac{1}{s^2(s^2+a^2)}$.

<u>Solution</u> We know,

$$\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}.$$

$$\Rightarrow \mathcal{L}\left(\int_0^t \sin a\tau \, d\tau\right) = \frac{1}{s}\,\frac{a}{s^2+a^2}$$

So, $\mathcal{L}\left(\dfrac{1}{a} - \dfrac{1}{a}\cos at\right) = \dfrac{a}{s(s^2+a^2)}$.

So, $\boxed{\mathcal{L}\left(\dfrac{1}{a^2} - \dfrac{1}{a^2}\cos at\right) = \dfrac{1}{s(s^2+a^2)}}$.

Also, $\mathcal{L}\left(\int_0^t \left(\dfrac{1}{a^2} - \dfrac{1}{a^2}\cos a\tau\right) d\tau\right) = \dfrac{1}{s^2(s^2+a^2)}$.

So, $\mathcal{L}\left(\dfrac{t}{a^2} - \dfrac{\sin at}{a^3}\right) = \dfrac{1}{s^2(s^2+a^2)}$.

Multiplication and division by t

Let $f:[0,\infty)\longrightarrow \mathbb{R}$ be such that $\mathcal{L}(f)=F(s)$ exists. We want to know $\mathcal{L}(t f(t))$ and $\mathcal{L}(f(t)/t)$ in terms of F.

Thm ① If $\mathcal{L}(f)=F(s)$ exists, then

$$\mathcal{L}(t f(t)) = -F'(s).$$

② If $\mathcal{L}(f)=F(s)$ exists and $\lim_{t\to 0}\frac{f(t)}{t}$ exists, then $\mathcal{L}(f(t)/t)=\int_s^\infty F(u)\,du$.

Pf ①

$$F(s)=\int_0^\infty e^{-st} f(t)\,dt.$$

$$\Rightarrow F'(s)=\frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right)$$

$$\Rightarrow F'(s)=\int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt$$

we could interchange the derivative and integral as we have $\int_0^\infty e^{-st} f(t)\,dt$ is absolutely convergent.

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -\mathcal{L}(t f(t))$$

So, $\boxed{\mathcal{L}(t f(t)) = -F'(s)}$.

② $$\int_s^\infty F(u)\,du = \int_s^\infty \int_0^\infty e^{-ut} f(t)\,dt\,du$$

$$= \int_0^\infty\left(\int_s^\infty e^{-ut}\,du\right) f(t)\,dt$$

So, $$\int_s^\infty F(u)\,du = \int_0^\infty \frac{e^{-ut}}{-t}\Big|_s^\infty f(t)\,dt$$

$$= \int_0^\infty e^{-st}\,\frac{f(t)}{t}\,dt$$

$$= \mathcal{L}\left(f(t)/t\right).$$

Examples

① $$\mathcal{L}(t\cos at) = -\frac{d}{ds}\left(\mathcal{L}(\cos at)\right)$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

② $$\mathcal{L}\left(\frac{\sin at}{t}\right) = \int_s^\infty F(u)\,du \quad \text{where } F(u) = \mathcal{L}(\sin at) = \frac{a}{u^2+a^2}$$

$$= \int_s^\infty \frac{a}{u^2+a^2}\,du$$

$$= \int_s^\infty \frac{d(u/a)}{1+\left(\frac{u}{a}\right)^2}$$

$$= \tan^{-1}\left(\frac{u}{a}\right)\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

$$= \cot^{-1}\frac{s}{a}.$$

Exercise Find inverse Laplace transform of $\log\left(1+\frac{a^2}{s^2}\right)$.

Solution $F(s) = \log\left(1+\frac{a^2}{s^2}\right)$

$$= \log(s^2+a^2) - \log s^2$$

To find $f(t)$ such that $\mathcal{L}(f(t)) = F(s)$.

Note that,

$$F'(s) = \frac{2s}{a^2+s^2} - \frac{2s}{s^2} = \frac{2s}{a^2+s^2} - \frac{2}{s}.$$

$$= 2\mathcal{L}(\cos at) - 2\mathcal{L}(1)$$

$$= \mathcal{L}(2\cos at - 2)$$

We know, $\mathcal{L}(t f(t)) = -F'(s)$.

So we set $t f(t) = 2 - 2\cos at$.

So, $\boxed{f(t) = \frac{2}{t}(1-\cos at)}$

Note that,

$$\mathcal{L}\left(\frac{2}{t}(1-\cos at)\right) = \int_s^\infty \mathcal{L}(2-2\cos at)\,du$$

$$= \int_s^\infty \left(\frac{2}{u} - \frac{2u}{u^2+a^2}\right) du$$

$$= \left\{2\log u - \log(u^2+a^2)\right\}\Big|_s^\infty$$

So, $\mathcal{L}\left(\frac{2}{t}(1-\cos at)\right) = \log \frac{u^2}{u^2+a^2}\Big|_s^\infty$

$$= \log\left(\frac{s^2+a^2}{s^2}\right)$$

$$= \log\left(1+\frac{a^2}{s^2}\right)$$

Laplace transform to solve IVPs

Idea (For linear ODE with constant coefficients)

consider $y''+ay'+by = r(t)$,

$y(0) = c_0$,

$y'(0) = c_1$.

Suppose Laplace transform of the function $y(t), r(t)$ exist.

$$\mathcal{L}(y'') + a\mathcal{L}(y') + b\mathcal{L}(y) = \mathcal{L}(r).$$

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + a\,s\mathcal{L}(y) - a\,y(0) + b\,\mathcal{L}(y) = \mathcal{L}(r)$$

so, $(s^2+as+b)\mathcal{L}(y) = (s+a)y(0) + y'(0) + \mathcal{L}(r)$

so, $\mathcal{L}(y) = \dfrac{(s+a)c_0 + c_1 + \mathcal{L}(r)}{s^2+as+b}$

From here one can obtain y considering the inverse Laplace transform.

<u>Examples</u> ① $y'' - y = t$,
$y(0) = 1,\ y'(0) = 1$.

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(t)$$

$$\Rightarrow s^2\mathcal{L}(y) - s y(0) - y'(0) - \mathcal{L}(y) = \frac{1}{s^2}.$$

$$\Rightarrow (s^2-1)\mathcal{L}(y) = s+1+\frac{1}{s^2}$$

$$\Rightarrow \mathcal{L}(y) = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

Taking inverse Laplace transform we get,

$$y(t) = e^t + \sinh t - t.$$

② $y'' + y = 2t,\ y(\frac{\pi}{4}) = \frac{\pi}{2},\ y'(\frac{\pi}{4}) = 2-\sqrt{2}$

Here the initial point is $\frac{\pi}{4}$, not 0.
So we set $\tilde{t} = t - \frac{\pi}{4}$.

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{\pi}{4}),$$
$$\tilde{y}(0) = \frac{\pi}{2},$$
$$\tilde{y}'(0) = 2-\sqrt{2}.$$

$$y(t) = y(\tilde{t} + \tfrac{\pi}{4}) = \tilde{y}(\tilde{t})$$

$$\mathcal{L}(\tilde{y}'') + \mathcal{L}(\tilde{y}) = 2\frac{1}{s^2} + \frac{\pi}{2}\cdot\frac{1}{s}$$

$$\Rightarrow s^2\mathcal{L}(\tilde{y}) - s\tilde{y}(0) - \tilde{y}'(0) + \mathcal{L}(\tilde{y}) = \frac{2}{s^2} + \frac{\pi}{2s}.$$

$$\Rightarrow \mathcal{L}(\tilde{y})(s^2+1) = \frac{2}{s^2} + \frac{\pi}{2s} + \frac{s\pi}{2} + 2-\sqrt{2}$$

$$= \frac{2}{s^2} + \frac{\pi}{2}\,\frac{s^2+1}{s} + 2-\sqrt{2}$$

So, $$\mathcal{L}(\tilde{y}) = \frac{2}{s^2(s^2+1)} + \frac{\pi}{2}\cdot\frac{1}{s} + \frac{2-\sqrt{2}}{s^2+1}.$$

$$= \frac{2}{s^2} - \frac{2}{s^2+1} + \frac{\pi}{2}\cdot\frac{1}{s} + \frac{2-\sqrt{2}}{s^2+1}.$$

Taking inverse Laplace transform we get,

$$\tilde{y} = 2\tilde{t} - 2\sin\tilde{t} + \frac{\pi}{2} + (2-\sqrt{2})\sin\tilde{t}$$

$$= 2\tilde{t} - \sqrt{2}\sin\tilde{t} + \frac{\pi}{2}.$$

So, $y(t) = 2t - \sqrt{2}\sin\left(t-\frac{\pi}{4}\right)$.

<u>Idea</u> (For linear ODE with variable coefficients)

consider $a(t)y'' + b(t)y' + c(t)y = r(t)$.

Suppose $a(t), b(t), c(t)$ are linear polynomial in t.

Denote $\mathcal{L}(y) = Y(s)$.

Now $\mathcal{L}(y') = s\mathcal{L}(y) - y(0) = sY(s) - y(0)$

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0).$$

$$= s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}(ty) = -Y'(s)$$

$$\mathcal{L}(ty') = -sY'(s) - Y(s)$$

$$\mathcal{L}(ty'') = -s^2 Y'(s) - 2sY(s) + y(0)$$

So substituting in

$$\mathcal{L}(a(t)y'' + b(t)y' + c(t)y) = \mathcal{L}(r),$$

we get a first order linear ODE in Y.

<u>Example</u> (Laguerre's ODE)

Let $n \geq 0$ be an integer. Consider

$$ty'' + (1-t)y' + ny = 0.$$

Taking Laplace transform we get

$$\mathcal{L}(ty'') + \mathcal{L}(y') - \mathcal{L}(ty') + n\mathcal{L}(y) = 0.$$

$$-s^2Y'(s) - 2sY(s) + y(0) + sY(s) - y(0)$$
$$+ sY'(s) + Y(s) + nY(s) = 0$$

So, $(s - s^2)Y' + (n+1-s)Y = 0$

$$\Rightarrow \frac{Y'}{Y} = \frac{n+1-s}{s(s-1)} = \frac{n}{s-1} - \frac{n+1}{s}$$

$$\Rightarrow Y = \frac{(s-1)^n}{s^{n+1}}.$$ So $y(t) = \mathcal{L}^{-1}\left(\frac{(s-1)^n}{s^{n+1}}\right)$.

If $n=0$, then $y(t)=1$.

If $n=1$, then $y = \mathcal{L}^{-1}\left(\frac{s-1}{s^2}\right) = 1-t$.

For $n \geq 0$ $\frac{(s-1)^n}{s^{n+1}}$ is a polynomial in $\frac{1}{s}$ of degree $(n+1)$. So by taking inverse Laplace transform we get y is a polynomial in t of degree n. The polynomials $\mathcal{L}^{-1}\left(\frac{(s-1)^n}{s^{n+1}}\right) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})$ for $n \geq 0$ are called Laguerre's polynomial.

We know, $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$.

So, $\mathcal{L}(e^{-t}t^n) = \frac{n!}{(s+1)^{n+1}}$

$$\text{So, } \mathcal{L}\left(\frac{d^n}{dt^n}(t^n e^{-t})\right) = s^n \frac{n!}{(s+1)^{n+1}} - s^{n-1} e^{-t}t^n\Big|_{t=0} - \dots - \frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\Big|_{t=0}$$

$$= \frac{s^n\, n!}{(s+1)^{n+1}}$$

So, $\mathcal{L}\left(e^t \frac{d^n}{dt^n}(t^n e^{-t})\right) = \frac{n!\,(s-1)^n}{s^{n+1}}$.

So, $\mathcal{L}^{-1}\left(\frac{(s-1)^n}{s^{n+1}}\right) = \frac{1}{n!} e^t \frac{d^n}{dt^n}(t^n e^{-t})$.

More tools from Laplace transform

Let $a \geq 0$. Define $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$

Note that $\mathcal{L}(u(t-a)) = \int_a^\infty e^{-st}\,dt$

$$= \frac{e^{-sa}}{s}.$$

Thm (Second shifting theorem)
Let $f(t)$ for $t \geq 0$ be such that $F(s) = \mathcal{L}(f(t))$ exists. Then $\mathcal{L}(\tilde{f}(t)) = e^{-as} F(s)$

where $\tilde{f}(t) = f(t-a)\,u(t-a) = \begin{cases} 0, & \text{if } t < a \\ f(t-a), & \text{if } t \geq a. \end{cases}$

Pf
$$\mathcal{L}(\tilde{f}(t)) = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_0^\infty e^{-s(a+\tau)} f(\tau)\,d\tau = e^{-as} F(s).$$

Corollary If $\mathcal{L}(f(t)) = F(s)$, then

$$\mathcal{L}(f(t)u(t-a)) = e^{-as}\mathcal{L}(f(t+a)).$$

Pf $g(t) = f(t+a)$.

So, $f(t) = g(t-a)$

So, $\mathcal{L}(g(t-a)u(t-a)) = e^{-as}\mathcal{L}(g(t))$

$$= e^{-as}\mathcal{L}(f(t+a)).$$

Examples

① $\mathcal{L}(u(t-3)t^2) = e^{-3s}\mathcal{L}((t+3)^2)$

$$= e^{-3s}\mathcal{L}(t^2+6t+9)$$

$$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$

② Let $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & 1 \leq t < 2 \\ 2, & t \geq 2 \end{cases}$

Note that,

$$f(t) = 1(u(t) - u(t-1)) + t(u(t-1) - u(t-2)) + 2u(t-2)$$

$$= u(t) + (t-1)u(t-1) + (2-t)u(t-2).$$

So, $\mathcal{L}(f(t)) = \mathcal{L}(u(t)) + \mathcal{L}((t-1)u(t-1))$
$- \mathcal{L}((t-2)u(t-2))$.

$$= \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

③ Solve $y'(t) + 2y(t) = 6u(t-4)$, $y(0) = 1$.

$$\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) = 6\mathcal{L}(u(t-4))$$

$$\Rightarrow sY(s) - y(0) + 2Y(s) = 6\frac{e^{-4s}}{s}$$

$$\Rightarrow (s+2)Y(s) = 1 + 6\frac{e^{-4s}}{s}.$$

$$\Rightarrow Y(s) = \frac{1}{s+2} + 3e^{-4s}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

Taking inverse Laplace transform we get

$$y(t) = e^{-2t} + 3u(t-4) + 3u(t-4)e^{-2(t-4)}.$$

④ Find $f(t)$ such that

$$\mathcal{L}(f(t)) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}.$$

Recall that, $\mathcal{L}(\sin \pi t) = \dfrac{\pi}{s^2+\pi^2}$.

So, $\mathcal{L}(u(t-1)\sin\pi(t-1)) = \dfrac{\pi e^{-s}}{s^2+\pi^2}$,

$$\mathcal{L}\left(u(t-2)\sin\pi(t-2)\right) = \frac{\pi e^{-2s}}{s^2+\pi^2}.$$

Now, $\mathcal{L}(t) = \frac{1}{s^2}$, $\mathcal{L}(t e^{-2t}) = \frac{1}{(s+2)^2}$.

So, $\mathcal{L}\left(u(t-3)(t-3)e^{-2(t-3)}\right)$

$$= \frac{e^{-3s}}{(s+2)^2}.$$

So we can take

$$f(t) = \frac{1}{\pi}u(t-1)\sin\pi(t-1) + \frac{1}{\pi}u(t-2)\sin\pi(t-2) + u(t-3)(t-3)e^{-2(t-3)}.$$

Exercises

1. Let $f(t) = \begin{cases} 2 & \text{if } 0 \leq t < 1, \\ \frac{t^2}{2} & \text{if } 1 \leq t < \frac{\pi}{2}, \\ \cos t & \text{if } t \geq \frac{\pi}{2}. \end{cases}$

   Find $\mathcal{L}(f(t))$.

2. Solve $y'' + 3y' + 2y = u(t-1) - u(t-2)$,
   $y(0) = 0$,
   $y'(0) = 0$.

Impulse function

Let $a>0$ be a real number. For $\epsilon>0$ (small) we define $f_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & , a-\epsilon<t<a+\epsilon \\ 0 & , \text{otherwise} \end{cases}$

The impulse of this function is defined as the integral $\int_0^\infty f_\epsilon(t)\,dt$.

Note that $\int_0^\infty f_\epsilon(t)\,dt = 1$.

Define $\boxed{\delta(t-a) := \lim_{\epsilon\to 0} f_\epsilon(t)}$ Dirac delta function

If $t\neq a$ then $\exists$ $\epsilon_0>0$ such that $t\notin(a-\epsilon_0, a+\epsilon_0)$. So $f_{\epsilon_0}(t)=0$.

Note that $\forall\ 0<\epsilon\leq\epsilon_0$ $f_\epsilon(t)=0$.

So, $\delta(t-a)=0$.

If $t=a$ then the limit is ∞.

Note $\delta(t-a)$ is not actually a function.

$\lim_{\epsilon\to 0}\int_0^\infty f_\epsilon(t) = 1$. We write it as $\int_0^\infty \delta(t-a) = 1$.

Can we define Laplace transform of $\delta(t-a)$?

we find out $\mathcal{L}(f_\epsilon(t))$.

$$\mathcal{L}(f_\epsilon(t)) = \int_0^\infty e^{-st} f_\epsilon(t)\,dt = \int_{a-\epsilon}^{a+\epsilon} e^{-st} \frac{1}{2\epsilon}\,dt$$

$$= \frac{1}{2\epsilon} \left. \frac{e^{-st}}{-s} \right|_{a-\epsilon}^{a+\epsilon}$$

$$= \frac{e^{-as}}{2\epsilon s} \left(e^{\epsilon s} - e^{-\epsilon s}\right).$$

As we defined $\delta(t-a)$ as $\lim_{\epsilon\to 0} f_\epsilon(t)$,

we define $\mathcal{L}(\delta(t-a))$ as $\lim_{\epsilon\to 0} \mathcal{L}(f_\epsilon(t))$.

$$\lim_{\epsilon\to 0} \mathcal{L}(f_\epsilon(t))$$

$$= \lim_{\epsilon\to 0} \frac{e^{-as}(e^{\epsilon s} - e^{-\epsilon s})}{2\epsilon s}$$

$$= \frac{e^{-as}}{2s} \lim_{\epsilon\to 0} \frac{e^{\epsilon s} - e^{-\epsilon s}}{\epsilon}$$

$$= \frac{e^{-as}}{2s} \lim_{\epsilon\to 0} \frac{s e^{\epsilon s} + s e^{-\epsilon s}}{1}$$

$$= e^{-as}.$$

Hence $\mathcal{L}(\delta(t-a)) = e^{-as}$.

So, $\mathcal{L}(\delta(t)) = 1$, by taking $a = 0$.

Observation

Let f be a continuous function on $[0,\infty)$.

we define

$$\int_0^\infty \delta(t-a)\, f(t)\,dt := \lim_{\epsilon\to 0}\int_0^\infty f_\epsilon(t)\, f(t)\,dt$$

$$= \lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon} \frac{1}{2\epsilon} f(t)\,dt$$

Define $\quad g(t) := \int_0^t f(\tau)\,d\tau.$

So, $\quad g'(t) = f(t)$, (by the fundamental theorem of calculus)

So, $$\int_{a-\epsilon}^{a+\epsilon} \frac{1}{2\epsilon} f(t)\,dt = \int_{a-\epsilon}^{a+\epsilon} \frac{g'(t)}{2\epsilon}\,dt$$

$$= \frac{g(a+\epsilon)-g(a-\epsilon)}{2\epsilon}$$

$$= g'(\tilde{a}) \text{ for some } \tilde{a}\in(a-\epsilon, a+\epsilon)$$

$$= f(\tilde{a}).$$

Therefore,

$$\int_0^\infty \delta(t-a)\, f(t)\,dt = \lim_{\epsilon\to 0} f(\tilde{a})$$

$$= f(a) \quad \text{as } \tilde{a}\in(a-\epsilon, a+\epsilon).$$

Examples

(1) $y'' + 2y' + 5y = \delta(t-1),\ y(0) = 0,\ y'(0) = 0.$

Denote $\mathcal{L}(y(t)) = Y(s)$

So, $s^2 Y + 2sY + 5Y = e^{-s}.$

So, $Y = \dfrac{e^{-s}}{s^2+2Y+5} = \dfrac{e^{-s}}{(s+1)^2+4}.$

Recall, $\mathcal{L}(\sin 2t) = \dfrac{2}{s^2+4}.$

So, $\mathcal{L}(e^{-t}\sin 2t) = \dfrac{2}{(s+1)^2+4}.$

So, $\mathcal{L}\left(\dfrac{e^{-t}\sin 2t}{2}\right) = \dfrac{1}{(s+1)^2+4}.$

So, $\boxed{y(t) = \frac{1}{2} u(t-1)\, e^{-(t-1)} \sin 2(t-1)}.$

(2) $y'' + 3y' + 2y = \delta(t-1),\ y(0) = 0,\ y'(0) = 0.$

$s^2 Y + 3sY + 2Y = e^{-s}$

So, $Y = \dfrac{e^{-s}}{s^2+3s+2} = e^{-s}\left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right)$

Recall, $\mathcal{L}(e^{-t}) = \dfrac{1}{s+1},\ \mathcal{L}(e^{-2t}) = \dfrac{1}{s+2}.$

So, $\mathcal{L}(e^{-t} - e^{-2t}) = \frac{1}{s+1} - \frac{1}{s+2}$.

So, $y(t) = u(t-1)\left(e^{-(t-1)} - e^{-2(t-1)}\right)$

$$= \begin{cases} 0, & 0 \leq t < 1 \\ e^{-(t-1)} - e^{-2(t-1)}, & t \geq 1 \end{cases}$$

Partial fraction decomposition

Any polynomial over $\mathbb{R}$ can be written as a product of polynomials over $\mathbb{R}$ of degree ≤ 2. So if we encounter $\frac{1}{P(s)}$, P is over $\mathbb{R}$, then we can do the following:

denominator	partial fraction
$(s-a)^k$	$\frac{A_1}{(s-a)} + \cdots + \frac{A_k}{(s-a)^k}$
$(s^2+as+b)^k$ with $a^2-4b<0$	$\frac{A_1 s + B_1}{s^2+as+b} + \cdots + \frac{A_k s + B_k}{(s^2+as+b)^k}$

Here $k \in \mathbb{N}$, $a, b, A_i, B_i \in \mathbb{R}$ $\forall\, 1 \leq i \leq k$.

Example $y'' + 2y' + 2y = (u(t) - u(t-\pi))10\sin 2t$,

$y(0) = 1$, $y'(0) = -5$.

$$(s^2 Y - s + 5) + 2(sY - 1) + 2Y = 20\,\frac{1 - e^{-\pi s}}{s^2 + 4}.$$

So, $$Y = \frac{20}{(s^2+2s+2)(s^2+4)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2}.$$

Now,

$$\frac{s-3}{s^2+2s+2} = \frac{s+1-4}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{4}{(s+1)^2+1}.$$

$$\mathcal{L}^{-1}\left(\frac{s-3}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1} - \frac{4}{(s+1)^2+1}\right)$$

$$= e^{-t}\cos t - 4e^{-t}\sin t$$

Now,

$$\frac{20}{(s^2+2s+2)(s^2+4)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+2s+2}.$$

Solving this we get $A = B = -2$, $C = 2$, $D = 6$.

$$\mathcal{L}^{-1}\left(-\frac{2(s+1)}{s^2+4}\right) = -2\cos 2t - \sin 2t$$

$$\mathcal{L}^{-1}\left(\frac{2s+6}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{2(s+1)+4}{(s+1)^2+1}\right) = 2e^{-t}\cos t + 4e^{-t}\sin t$$

So, $\mathcal{L}^{-1}\left(\dfrac{20}{(s^2+2s+2)(s^2+4)}\right) = \boxed{-2\cos 2t - \sin 2t + 2e^{-t}(\cos t + 2\sin t)}$

$g(s) =$

$$\mathcal{L}^{-1}\left(\frac{20e^{-\pi s}}{(s^2+2s+2)(s^2+4)}\right) = u(t-\pi)\, g(t-\pi)$$

Laplace transform of convolution of functions

Note $\mathcal{L}(1) = \dfrac{1}{s}$, $\mathcal{L}(e^t) = \dfrac{1}{s-1}$

So, $\mathcal{L}(1 \cdot e^t) \neq \mathcal{L}(1)\mathcal{L}(e^t)$.

Convolution Let $f(t), g(t)$ for $t \geq 0$ be two integrable functions. Then the convolution of f, g, denoted by $f * g$ is defined by

$$f * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau .$$

Properties

1. $f * g = g * f$
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0$. 0 is the identically 0 function.

Remark In general $f * 1 \neq f$.
For example $t * 1 = t^2/2 \neq t$.

Thm

Let $f(t), g(t)$ for $t \geq 0$ be such that $F(s) = \mathcal{L}(f)$, $G(s) = \mathcal{L}(g)$ exist. Then
$$F(s)G(s) = \mathcal{L}(f * g).$$

Remark To know, $\mathcal{L}^{-1}\left(\frac{1}{(s^2+as+b)^2}\right)$

we need to then know

$f * f$ where $\mathcal{L}(f) = \frac{1}{s^2+as+b}$.

Examples (1) Find f such that
$$\mathcal{L}(f) = \frac{1}{(s-a)s}.$$

We know, $\mathcal{L}(1) = \frac{1}{s}$.

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}.$$

So $f = e^{at} * 1$
$$= \int_0^t e^{a\tau}\, d\tau = \frac{1}{a}(e^{at}-1).$$

(2) Find f such that $\mathcal{L}(f) = \frac{1}{(s^2+a^2)^2}$.

We know, $\mathcal{L}\left(\frac{1}{a}\sin at\right) = \frac{1}{s^2+a^2}$.

So, $f(t) = \frac{1}{a}\sin at * \frac{1}{a}\sin at$

So $f(t) = \frac{1}{a^2}\int_0^t \sin a\tau \sin a(t-\tau)\,d\tau.$

$$= \frac{1}{2a^2}\int_0^t \left(\cos a(2\tau - t) - \cos at\right)d\tau$$

$$= \frac{1}{2a^2}\left(\frac{\sin a(2\tau-t)}{2a} - \tau\cos at\right)\Bigg|_{\tau=0}^{t}$$

$$= \frac{1}{2a^2}\left(\frac{\sin at}{a} - t\cos at\right).$$

(3) $y'' + a^2 y = A\sin at,\ y(0) = y'(0) = 0.$

$$s^2Y + a^2Y = A\frac{a}{s^2+a^2}$$

So, $Y = \frac{Aa}{(s^2+a^2)^2}.$

So, $y(t) = \frac{A}{2a^2}\left(\sin at - at\cos at\right).$

Integral equations

Equations where unknown functions $y(t)$ appears inside integral.

Examples (1) $y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t.$

Taking Laplace transform we get,

$$Y(s) - Y(s)\frac{1}{s^2+1} = \frac{1}{s^2}.$$

$$\Rightarrow Y(s) = \frac{s^2+1}{s^4}.$$

So, $Y(s) = \frac{1}{s^2} + \frac{1}{s^4}$.

So, $y(t) = t + \frac{t^3}{3!}$.

(2) $y(t) - \int_0^t (1+\tau)\, y(t-\tau)\, d\tau = 1 - \sinh t$.

Taking Laplace transform we get,

$$Y(s) - Y(s)\left(\frac{1}{s} + \frac{1}{s^2}\right) = \frac{1}{s} - \frac{1}{s^2-1}.$$

So, $Y(s) = \frac{s}{s^2-1}$.

So, $y(t) = \cosh t$.

To solve system of ODEs with constant coefficients using Laplace transform

Consider
$$\left.\begin{aligned} y_1' &= a_{11} y_1 + a_{12} y_2 + g_1(t), \\ y_2' &= a_{21} y_1 + a_{22} y_2 + g_2(t). \end{aligned}\right\}$$

Taking Laplace transform we get,

$$sY_1 - y_1(0) = a_{11} Y_1 + a_{12} Y_2 + G_1(s),$$
$$sY_2 - y_2(0) = a_{21} Y_1 + a_{22} Y_2 + G_2(s).$$

$$\Rightarrow \begin{cases} (a_{11} - s) Y_1 + a_{12} Y_2 = -y_1(0) - G_1(s), \\ a_{21} Y_1 + (a_{22} - s) Y_2 = -y_2(0) - G_2(s). \end{cases}$$

$$(A - sI)\underline{Y} = -\underline{y}(0) - \underline{G}$$

$$\Rightarrow \underline{Y} = (sI - A)^{-1}(\underline{y}(0) + \underline{G})$$ $s \neq$ eigen value of A.

Then taking Laplace inverse we get $\underline{y}(t)$.

Example $\quad \underline{y}' = \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \underline{y} + \begin{pmatrix} e^{2t} \\ 3 \end{pmatrix}, \quad \underline{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

$$\text{So, } \underline{Y} \begin{pmatrix} s-1 & 0 \\ 1 & s-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{s-2} \\ \frac{3}{s} \end{pmatrix}.$$

$$\underline{Y} = \begin{pmatrix} s-1 & 0 \\ 1 & s-3 \end{pmatrix}^{-1} \begin{pmatrix} \frac{s-1}{s-2} \\ \frac{3}{s} \end{pmatrix}$$

$$= \frac{1}{(s-1)(s-3)} \begin{pmatrix} s-3 & 0 \\ -1 & s-1 \end{pmatrix} \begin{pmatrix} \frac{s-1}{s-2} \\ \frac{3}{s} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{s-2} \\ \frac{1}{s-2} - \frac{1}{s} \end{pmatrix}$$

$$\text{So, } \underline{y}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} - 1 \end{pmatrix}.$$