

① Find the inverse of  $A$ , where  $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}$ .

Solution :- Note that, if  $E_k E_{k-1} \cdots E_2 E_1 A = I_{n \times n}$  then

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1 I_{n \times n}$$

Thus, we form  $3 \times 6$  matrix  $(A | I_{3 \times 3})$  and perform elementary row operations to reduce  $A$  to a row-reduced echelon matrix.

$$(A | I_{3 \times 3}) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2} R_2} \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & 8 & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{pmatrix} = \left( I_{3 \times 3} \mid A^{-1} \right)$$

Therefore,  $A^{-1} = \begin{pmatrix} 8 & -\frac{1}{2} & -2 \\ -1 & \frac{1}{2} & 0 \\ -3 & 0 & 1 \end{pmatrix}$

Rank of a Matrix :- Let  $A$  be a non-zero matrix of order  $m \times n$ . The rank of  $A$  is defined to be the greatest positive integer  $r$  such that  $A$  has at least one non-zero minor of order  $r$ .

- The rank of a zero matrix is defined to be 0.
- If  $\text{rank}(A) = r$  then every minor of order  $m$  is zero for  $m > r$ .

Theorem :- If a row-reduced echelon matrix  $A$  has  $r$  non-zero rows, then  $\text{rank}(A) = r$ .

Proof :- Since  $A$  has  $r$  non-zero rows, every square sub-matrix of order  $r+1$  containing a zero row. So each minor of  $A$  of order  $r+1$  is zero. Thus  $\text{rank}(A) < r+1$ .

If we take the sub-matrix formed from  $1, 2, \dots, r$ -th rows and  $k_1, k_2, \dots, k_r$ -th columns, then it is  $I_{r \times r}$ , and has determinant 1.

Thus  $A$  has one non-zero minor of order  $r$ .

Thus  $\text{rank}(A) = r$

Theorem :- If  $A$  is row equivalent to  $B$  then  $\text{rank}(A) = \text{rank}(B)$ .

Let a system of linear equations have  $m$  number of equations (row) and  $n$  number of variables (column).

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

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$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Then the system can be expressed as  $AX = B$

where,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix  $(A|B)_{m \times (n+1)}$  is called the augmented matrix.

Note that  $\text{rank}(A|B) \geq \text{rank}(A)$  and  
 $\text{rank}(A) \leq n$ .

\* If  $\text{rank}(A|B) = \text{rank}(A) = n$  then the system of linear equations has a unique solution.

\* If  $\text{rank}(A|B) = \text{rank}(A) < n$  then the system of linear equations has an infinite number of solutions.

\* If  $\text{rank}(A|B) > \text{rank}(A)$  then the system of linear equations has no solutions.

\* If  $B=0$ , i.e., for a system of homogeneous linear equations  $(x_1=0, \dots, x_n=0)$  is always a solution.

Problem:- Solve the system of linear equation (over the field  $\mathbb{R}$ ).

$$x + 3y + 2z = 7$$

$$2x + y - z = 5$$

$$-x + 2y + 3z = 4$$

Solution:- The augmented matrix is

$$\left( \begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 2 & 1 & -1 & 5 \\ -1 & 2 & 3 & 4 \end{array} \right)$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array} \rightarrow \left( \begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 0 & -5 & -5 & -9 \\ 0 & 5 & 5 & 11 \end{array} \right) \xrightarrow{R_3 + R_2} \left( \begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 0 & -5 & -5 & -9 \\ 0 & 0 & 0 & 2 \end{array} \right)$$

$$\xrightarrow[\frac{1}{2} R_3]{-\frac{1}{5} R_2} \begin{pmatrix} 1 & 3 & 2 & 7 \\ 0 & 1 & 1 & \frac{2}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 3R_2} \begin{pmatrix} 1 & 0 & -1 & \frac{8}{5} \\ 0 & 1 & 1 & \frac{2}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 - \frac{2}{5} R_3]{R_1 - \frac{8}{5} R_3} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

$$\text{Rank}(A|B) = 3 \quad \text{and} \quad \text{Rank}(A) = 2$$

Since  $\text{Rank}(A|B) > \text{Rank}(A)$ , the system of linear equation has no solution.

Solve the system of linear equations (over the field  $\mathbb{R}$ )

$$x_1 - 2x_2 + 2x_3 - x_4 = -14$$

$$3x_1 + 2x_2 - x_3 + 2x_4 = 17$$

$$2x_1 + 3x_2 - x_3 - x_4 = 18$$

$$-2x_1 + 5x_2 - 3x_3 - 3x_4 = 26$$

Solution :- The Augmented matrix is

$$\left( \begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 3 & 2 & -1 & 2 & 17 \\ 2 & 3 & -1 & -1 & 18 \\ -2 & 5 & -3 & -3 & 26 \end{array} \right) \xrightarrow[\substack{R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 + 2R_1}]{\phantom{R_2 - 3R_1}} \left( \begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 0 & 8 & -7 & 5 & 59 \\ 0 & 7 & -5 & 1 & 46 \\ 0 & 1 & 1 & -5 & -2 \end{array} \right)$$

$$\begin{array}{l} R_2 - 8R_4 \\ R_3 - 7R_4 \end{array} \rightarrow \begin{pmatrix} 1 & -2 & 2 & -1 & -14 \\ 0 & 0 & -15 & 45 & 75 \\ 0 & 0 & -12 & 36 & 60 \\ 0 & 1 & 1 & -5 & -2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & -2 & 2 & -1 & -14 \\ 0 & 1 & 1 & -5 & -2 \\ 0 & 0 & -12 & 36 & 60 \\ 0 & 0 & -15 & 45 & 75 \end{pmatrix}$$

$$\begin{array}{l} -\frac{1}{12}R_3 \\ -\frac{1}{15}R_4 \end{array} \rightarrow \begin{pmatrix} 1 & -2 & 2 & -1 & -14 \\ 0 & 1 & 1 & -5 & -2 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & -3 & -5 \end{pmatrix} \xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & -2 & 2 & -1 & -14 \\ 0 & 1 & 1 & -5 & -2 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} R_2 - R_3 \\ R_1 - 2R_3 \end{array} \rightarrow \begin{pmatrix} 1 & -2 & 0 & 5 & -4 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since  $\text{rank}(A|B) = \text{rank}(A) = 3 < 4 = \text{number of variables}$   
 The system of linear equations has an infinite number of solutions. Let  $x_4 = c \in \mathbb{R}$  arbitrary. Then  $x_1 = -c + 2$ ,  $x_2 = 2c + 3$   
 $x_3 = 3c - 5$

Cramer's rule for  $AX = B$  (if  $\det(A) \neq 0$ ).

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$\text{Then } x, \det(A) = \begin{vmatrix} x_1 a_{11} & a_{12} & \dots & a_{1n} \\ x_1 a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ x_1 a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & a_{12} & \dots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

i-th  
column  
↓

$$\text{Thus, } x_i = \frac{\begin{vmatrix} a_{11} & a_{12} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & b_2 & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & b_n & \dots & a_{nn} \end{vmatrix}}{\det(A)}$$

$$= \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$