Name: ______________________ Group No: ______ Entry No: ______________

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI
MAJOR TEST 2018-2019 SECOND SEMESTER
MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)

Time: 2 hours | Max. Marks: 40

Write your Name and Group number and Entry number at the places specified above. Attempt all questions. All notations are standard. All parts of a question must be answered at one place. Exhibit clearly all the steps. Use of any electronic gadget including calculator is NOT allowed. Attach question paper with the answer book. Do not do any rough work on the question paper. No query will be entertained.

1a. Suppose that $\{x_1, \ldots\ldots\ldots, x_n\}$ is a linearly independent set of vectors in a vector space. Let $x = \sum_{i=1}^{n} \beta_i x_i$, $\beta_i \neq 0\ \forall i, 1 \leq i \leq n$. Find the condition under which $\{x_1 - x, \ldots\ldots\ldots, x_n - x\}$ is linearly dependent. (3)

1b. Let $V = \{p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad : \quad a_0, a_1, a_2, a_3 \in R\}$ be a vector space. Let $T : V(R) \longrightarrow V(R)$ be a linear transformation defined by

$$T(p(x)) = a_0 + a_1(1+x) + a_2(1+x)^2 + a_3(1+x)^3.$$

What is the matrix of T with respect to ordered basis $B = \{1, (1+x), (1+x^2), (1+x^3)\}$? (4)

2a. Let a linear transformation $T : R^{4\times 1} \to R^{4\times 1}$ be defined by $TX = AX$ where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

Find a basis for the null space of T. (3)

2b. Find the values of λ and k for which the following system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ x & + & 2y & + & 3z & = & 10 \\ x & + & 2y & + & \lambda z & = & k \end{array}$$

has (i). no solution (ii). unique solution (iii). infinite number of solutions (4)

3. Find general solution of the system of differential equations $Y' = AY + b(x)$ where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 2e^{-x} \\ 3x \end{bmatrix}$$

by determining the eigenvalues and eigenvectors of A. (5)

P.T.O.

4a. Use Frobenius method of extended power series to find complete solution of $xy'' + y' + y = 0$. (5)

4b. Show that between any two positive zeros of $J_{n+1}(x)$ there is precisely one zero of $J_n(x)$. (3)

5a. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$((x^2+1)y')' + \lambda(x^2+1)^{-1}y = 0 \quad y(0) = y(1) = 0,$$

where $\lambda > 0$. (4)

5b. Examine whether the set of functions $\{\cos\frac{n\pi x}{L}\}$, $L > 0$, $n = 0, 1, 2, \ldots\ldots$ is orthogonal or not with respect to the weight function $r(x) = 1$ on the interval $[-L, L]$. If so, find the orthonormal set. (3)

6a. Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies $|\ f(t)\ | \leq Me^{kt}$ for all $t \geq 0$ and for some constants k and M. Then prove or disprove that the Laplace transform of $f(t)$ exists for all $s > k$ where s is the Laplace transform parameter. (3)

6b. Find Laplace transform of the function $\frac{e^{-t}}{\sqrt{t}}$. Hence Show that $L^{-1}(\frac{1}{s\sqrt{s+1}}) = erf(\sqrt{t})$ where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. (3)