Name: _____

ID number: _____

There are 1 questions for a total of 10 points.

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1. (10 points) Consider the following randomized algorithm. The inputs are a positive integer n and an integer array A containing n distinct integers.

```
FindMax(A, n)
  - Randomly permute the array A
  - Max ← A[1]
  - For i = 2 to n
      - if (A[i] > Max) Max ← A[i]
  - return(Max)
```

What is the expected number of times the value of the variable Max changes within the for loop? Express your answer as a function of n using Θ notation. Show details of your calculations.

Solution: Let X_i be an indicator random variable that is 1 if the value of Max gets modified in iteration i and 0 otherwise. The value of Max changes in the i^{th} iteration iff $A[i]$ is the maximum element in the subarray $A[1], ..., A[i]$. This happens with probability $\frac{1}{i}$. So, we have for all $i = 2, 3, ..., n$

$$\mathbf{Pr}[X_i = 1] = \frac{1}{i} \qquad \text{and} \qquad \mathbf{E}[X_i] = 1 \cdot \mathbf{Pr}[X_i = 1] = \frac{1}{i}$$

The number of times the variable Max changes is given by $\sum_{i=2}^{n} X_i$. The expectation of this quantity is given by:

$$\begin{aligned} \mathbf{E}[\sum_{i=2}^{n} X_i] &= \sum_{i=2}^{n} \mathbf{E}[X_i] \quad \text{(by linearity of expectation)} \\ &= \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \\ &= \Theta(\log n) \quad \text{(using discussion in class)} \end{aligned}$$