Note Title 2/28/2023

MTL 101 Lec - 1

Course Webpage : https://sites.google.com/view/kporwal/mtl101

Course Coordinator : Prof. Kamana Porwal

Instructor (G 21-25) : Prof. Amit Priyadarshi
priyadarshi@maths.iitd.ac.in
Office : MZ 162

Reference Books :

1. S. Axler, Linear Algebra done right
2. Gilbert Strang, Intro. to Linear Algebra
3. E. Kreyszig, Advanced Engineering Mathematics.

System of Linear Equations

A system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$ is of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

where $a_{ij} \in \mathbb{F}$ (a field) for $1 \le i \le m$, $1 \le j \le n$,

$b_1, b_2, \dots, b_m \in \mathbb{F}$

$x_1, x_2, \dots, x_n$ are unknowns.

The above system can be written in matrix form as:

$$AX = B,$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in M_{m\times n}(\mathbb{F})$

$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in M_{m\times 1}(\mathbb{F})$

and $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the column of unknowns.

Field:

A field $\mathbb{F}$ is a nonempty set with two binary operations on $\mathbb{F}$ called addition $(+)$ and multiplication $(\cdot)$, ie.,

$$+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \quad \text{and} \quad \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$

satisfying the following properties:

(i) $+$ is commutative, ie., $a+b = b+a \quad \forall a, b \in \mathbb{F}$

(ii) $+$ is associative, ie., $a+(b+c) = (a+b)+c \quad \forall a, b, c \in \mathbb{F}$.

(iii) $\exists$ an additive identity $0 \in \mathbb{F}$ s.t. $a+0 = a \quad \forall a \in \mathbb{F}$.

(iv) $\forall a \in \mathbb{F}$, $\exists -a \in \mathbb{F}$ s.t. $a + (-a) = 0$

(existence of additive inverse)

(v) $\cdot$ is commutative, i.e., $a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{F}$

(vi) $\cdot$ is associative, i.e., $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in \mathbb{F}$.

(vii) $\exists$ a multiplicative identity $1 \in \mathbb{F}$ $(1 \neq 0)$

i.e. $a \cdot 1 = a \quad \forall a \in \mathbb{F}$

(viii) Every nonzero element in $\mathbb{F}$ has a multiplicative inverse

i.e. $\forall a \neq 0 \in \mathbb{F}$, $\exists a^{-1} \in \mathbb{F}$ s.t. $a \cdot a^{-1} = 1$.

(ix) Distributive law : $a \cdot (b + c) = a \cdot b + a \cdot c \quad \forall a, b, c \in \mathbb{F}$

<u>Examples</u>:

- $\mathbb{Q}$ is a field with the usual $+$ & $\cdot$.
- $\mathbb{R}$ " " " " " " " "
- $\mathbb{C}$ " " " " " " " "

- $\mathbb{N}$ with usual addition & multiplication is not a field.
- $\mathbb{Z}$ " " " " " " "

Note Title 3/1/2023

Some examples of finite fields:

① $\mathbb{F} = \{0, 1\} \longleftrightarrow \{\text{even}, \text{odd}\} \longleftrightarrow \mathbb{Z}_2$

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

② $\mathbb{F} = \{0, 1, 2\}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

In this course we'll always take $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$.

Consider a system of linear equations

$$AX = B \quad , \text{ where } \quad A \in M_{m\times n}(\mathbb{F})$$

$$B \in M_{m\times 1}(\mathbb{F})$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

coefficient matrix

A solution to the above system is an element $Y \in M_{n\times 1}(\mathbb{F})$ s.t. $AY = B$.

The collection of all the solutions is called the solution set.

The system is called homogeneous if $B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

& non-homogeneous if $B \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

For the system $AX = B$, the matrix

$\underset{m \times (n+1)}{\left(A \mid B \right)}$ is called the augmented matrix.

If the field $\mathbb{F}$ is an <u>infinite</u> field, we have 3 possibilities:

1. The system $AX = B$ has no solution
2. " " " has a unique soln.
3. " " " " infinitely many solns.

Write examples for each of the 3 cases.

Note that for a homogeneous system $X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is always a solution. So, we either have a unique soln. or infinitely many solns. (for an infinite field).

Note Title 3/3/2023

Lec 3

Elementary row operations

An elementary row operation is a map from $M_{m\times n}(\mathbb{F})$ into $M_{m\times n}(\mathbb{F})$ which is any one of the following three types:

(i) Interchanging the ith & jth rows of the matrix, denoted by $R_i \longleftrightarrow R_j$.

(ii) Multiplying the ith row by a nonzero $\lambda \in \mathbb{F}$; $R_i \to \lambda R_i$

(iii) For $i \neq j$, replacing the ith row by the sum of the ith row and a multiple μ times the jth row; $R_i \to R_i + \mu R_j$

Prop: Every elem. row operation is invertible and the inverse of an elem. row operation is again an elem. row operation.

Pf: If $\rho : R_i \leftrightarrow R_j$, then $\rho^{-1} : R_i \leftrightarrow R_j$

If $\rho : R_i \to \lambda R_i$ for $\lambda \neq 0$, then $\rho^{-1} : R_i \to \frac{1}{\lambda} R_i$.

If $\rho : R_i \to R_i + \mu R_j$, then $\rho^{-1} : R_i \to R_i - \mu R_j$.

Prop: Let ρ be an elem. row oper. and $A \in M_{m \times n}(\mathbb{F})$. Then $\rho(A) = \rho(I_m) A$, where I_m is the $m \times m$ identity matrix.

Pf: Verify this for each of the 3 elem. row operations.

Defn: A square matrix $E \in M_{n\times n}(\mathbb{F})$ is called an elementary matrix if $E = \rho(I)$ for some elem. row oper. ρ.

e.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, , $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $(\lambda \neq 0)$

$\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, $\mu \in \mathbb{F}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + \mu R_1} \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

<u>Defn</u>: A row operation is a map from $M_{m \times n}(\mathbb{F})$ to itself which is a composition of finitely many elem. row opers.

$\rho = \rho_k \circ \cdots \circ \rho_2 \circ \rho_1$, where ρ_i is an elem. row operation for each i.

<u>Prop</u>: If $\rho = \rho_k \circ \cdots \circ \rho_2 \circ \rho_1$ is a row oper., then

$$\rho(A) = \rho(I)A = \rho_k(I) \cdots \rho_2(I)\rho_1(I)A$$

<u>Pf</u>: $\rho(A) = \rho_k\left(\rho_{k-1} \circ \cdots \circ \rho_2 \circ \rho_1(A)\right)$

$= \rho_k(I)\left(\rho_{k-1} \circ \cdots \circ \rho_2 \circ \rho_1(A)\right)$ (by the earlier prop.)

$$\vdots$$
$$= \rho_k(I)\,\rho_{k-1}(I)\cdots\rho_1(A)$$
$$= \rho_k(I)\,\rho_{k-1}(I)\cdots\rho_1(I)\,A$$

In particular, taking $A = I$, $\rho(I) = \rho_k(I)\cdots\rho_1(I)$

$\therefore \quad \rho(A) = \rho(I)A = \rho_k(I)\cdots\rho_1(I)\,A$.

Row equivalence

Defn : A matrix A is said to be <u>row equivalent</u> to a matrix B if $B = \rho(A)$, where ρ is a row operation.

i.e. $B = \rho_k \circ \cdots \circ \rho_2 \circ \rho_1 (A)$, ρ_i : elem. row oper.

Notation $A \overset{R}{\sim} B$ or $A \sim B$

Prop: Row equivalence is an equivalence relation.

Pf: (i) $A \overset{R}{\sim} A$ $\qquad A \xrightarrow{1 \cdot R_1} A$

(ii) $A \sim B \Rightarrow B \sim A$.

(iii) $A \sim B$ & $B \sim C \Rightarrow A \sim C$.

Note Title 3/14/2023

Lec - 4

Row reduced echelon (RRE) matrix

A matrix is called an RRE matrix if it satisfies the following four properties:

(i) Every zero row is below every nonzero row.

(ii) The leading entry (i.e. the first nonzero entry) in every nonzero row must be 1.

(iii) A column which contains a leading nonzero entry 1 of a row must have all other entries equal to zero.

(iv) Suppose the leading entries in the nonzero rows occur in k_i th column for ith row.

Then $k_1 < k_2 < \cdots < k_r$, where r is the no. of nonzero rows.

Examples: $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is not RRE because (i) is not satisfied.

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not RRE because the leading entry in 2nd row is 2.

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is RRE matrix

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is RRE Here $k_1 = 1$, $k_2 = 2$

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$ is not RRE because $k_1 = 2 > k_2 = 1$.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \to$$ identity matrix is RRE.

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ is RRE

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ is RRE.

Theorem: Every matrix A is row equivalent to a unique RRE matrix R.

Defn: This RRE matrix R is called the RRE form of A.

Example: Find the RRE form of $A = \begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix}$

$$A \overset{R_3 \leftrightarrow R_4}{\sim} \begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\sim]{R_1 \to \frac{1}{3}R_1} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\sim]{R_3 \to R_3 - 4R_1} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 2 & -4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\sim]{R_2 \to \frac{R_2}{4}} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 2 & -4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\sim]{R_3 \to R_3 - 2R_2} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & -11/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\sim]{R_3 \to \frac{-6}{11}R_3} \begin{pmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2 \to R_2 - \frac{1}{4}R_3]{R_1 \to R_1 - \frac{1}{3}R_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

✓ RRE.

Applications of RRE form

(1) Knowing whether a given square matrix is invertible or not, and if it is invertible finding its inverse.

Let A be an $n \times n$ matrix and let R be the RRE form of A.

Then $R = \rho_k \circ \rho_{k-1} \circ \cdots \circ \rho_1 (A)$

Note that A is invertible iff $\det(A) \neq 0$

iff $R = I$.

Note Title 3/15/2023

Lec - 5

- Note that an $n \times n$ matrix A is invertible if and only if the RRE form of A is I, the identity matrix.

- If the matrix A is invertible, i.e., the RRE form of A is I, how can we find A^{-1}?

Let the RRE form of A be I.

Then $I = \rho(A)$, where ρ is a row operation, i.e. $\rho = \rho_k \circ \cdots \circ \rho_2 \circ \rho_1$ for finitely many elem. row opers. ρ_i's.

But, we know that $\rho(A) = \rho(I) A$.

$$\therefore \quad \rho(I)A = I$$

$$\Rightarrow \quad A^{-1} = \rho(I)$$

(why is inverse of A unique?
Suppose $AB = I = BA$ and $AC = I = CA$.

$$B = BI = B(AC) = (BA)C = IC = C$$)

So, in order to compute A^{-1} we need to apply the same elem. row operations to I that we applied to A to get I.

This gives a method to compute A^{-1} as follows:

- Start with the augmented matrix $(A \mid I)$

- Apply elem. row operations to this matrix $(A|I)$ with the aim of obtaining the RRE form of A.
- If $R \neq I$, then A is not invertible.

  If $R = I$, then we get A^{-1} as the right matrix.

Example: Find the inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right)$$

$R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right)$$

$R_3 \to \frac{1}{2} R_3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

$R_1 \to R_1 - R_3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right)$$

$$\underbrace{}_{I} \qquad \underbrace{}_{A^{-1}}$$

Solving system of linear equations using RRE form -

Consider the system $AX = B$, where

$A \in M_{m\times n}(\mathbb{F})$, $B = M_{m\times 1}(\mathbb{F})$,

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ ← column of unknowns.

Lemma: If $(A|B)$ is row equivalent to $(A'|B')$, then the systems $AX = B$ and $A'X = B'$ have the same solutions.

Pf: We have $(A'|B') = \rho(A|B) = \big(\rho(A)|\rho(B)\big)$

$\Rightarrow \quad A' = \rho(A)$ and $B' = \rho(B)$

If Y is a soln. of $AX = B$, then

$AY = B$

$\Rightarrow \rho(AY) = \rho(B)$

But $\rho(AY) = \rho(I)AY = \rho(A)Y \quad (\because \rho(A) = \rho(I)A)$

$\therefore \quad \rho(A)Y = \rho(B) \Rightarrow A'Y = B'$

$\Rightarrow$ Y is a soln. of $A'X = B'$

Similarly if $A'Y = B'$, then $AY = B$.

If R is the RRE form of A, then
$R = \rho(A)$ for some row oper. ρ.

$\therefore$ The systems $AX = B$ and $RX = B'$ have the same solns., where $B' = \rho(B)$.

Now, we will see how to write the solutions of $RX = B'$, where R is an RRE matrix.

Example:

$$(R \mid B') = \left(\begin{array}{cccc|c} 0 & 1 & 2 & 0 & b_1 \\ 0 & 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 0 & b_3 \end{array}\right)$$

Case I : If $b_3 \neq 0$, then the system has no solution.

Case II : If $b_3 = 0$,

We have $k_1 = 2$, $k_2 = 4$

Then the system reduces to $0x_1 + x_2 + 2x_3 + 0x_4 = b_1$

$$0x_1 + 0x_2 + 0x_3 + x_4 = b_2$$

To solve this we can assign arbitrary values of x_1 and x_3 and calculate x_2 & x_4 in terms of x_1 & x_3.

Put $x_1 = \alpha$, $x_3 = \beta$. Then $x_2 = b_1 - 2\beta$

$x_4 = b_2$

$\therefore$ The solution set is $\{(\alpha, b_1 - 2\beta, \beta, b_2) : \alpha, \beta \in \mathbb{F}\}$

Lec - 6

Rank of a matrix

Defn: The rank of a matrix is the number of nonzero rows in its RRE form.

Note that if A and B row equivalent to each other, then $\text{rank}(A) = \text{rank}(B)$.

- For a square matrix A of size $n \times n$, A is invertible if and only if $\text{rank}(A) = n$.

Theorem: A system of linear equation $AX=B$ is consistent (i.e. has at least one solution) if and only if $\operatorname{rank}(A|B) = \operatorname{rank}(A)$.

The system has a unique soln. iff $\operatorname{rank}(A|B) = \operatorname{rank}(A) = \#$ of unknowns

Conclusion:

(I) No soln $\iff$ $\operatorname{rank}(A|B) > \operatorname{rank}(A)$

(II) Unique soln. $\iff$ $\operatorname{rank}(A|B) = \operatorname{rank}(A) = \#$ of unknowns

(III) Infinitely many solns (assuming $\mathbb{F}$ is infinite) $\iff$ $\operatorname{rank}(A|B) = \operatorname{rank}(A) < \#$ of unknowns.

Proof: Suppose $\mathrm{rank}(A|B) = \mathrm{rank}(A) = r$.

Let $(R|B')$ be the RRE form of $(A|B)$.

Let $k_1, k_2, \dots, k_r$ be the column nos. corresponding the leading entries in the 1st, 2nd, ..., rth rows of $(R|B')$.

Let us call $x_{k_1}, x_{k_2}, \dots, x_{k_r}$ as the dependent unknowns and the remaining $(n-r)$ as the independent (or free) unknowns.

Then we can write all solutions by assigning arbitrary values to the $(n-r)$ free unknowns and calculating $x_{k_1}, x_{k_2}, \dots, x_{k_r}$ in terms of these $(n-r)$ unknowns.

If $n = r$, we get a unique soln.

Example: Solve:

$$x_1 + x_2 + x_3 = 3$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_2 + 2x_3 = 3$$

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Here, $\text{rank}(A|B) = \text{rank}(A) = 2 < 3 = \#$ of unknowns

$\therefore$ The system has infinitely many solns.

To write all solns :

$$k_1 = 1 \ , \ k_2 = 2$$

$\therefore$ x_1 & x_2 are dependent & x_3 is indep. unknown

$$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$$

$$x_2 + 2x_3 = 3 \Rightarrow x_2 = 3 - 2x_3$$

$\therefore$ Solution set $= \{ (\lambda, 3-2\lambda, \lambda) : \lambda \in \mathbb{F} \}$.

Note Title 3/21/2023

Lec 7

Vector Spaces

Let V be a nonempty set and $\mathbb{F}$ be a field. Let $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ be two maps. Then $(V, +, \cdot)$ is said to be a vector space over the field $\mathbb{F}$ is the following are satisfied.

(1) $u + v = v + u \quad \forall u, v \in V$ (Commutativity)

(2) $u + (v + w) = (u + v) + w \quad \forall u, v, w \in V$ (Associativity)

(3) $\exists\ 0 \in V$ s.t. $u + 0 = 0 + u = u \quad \forall u \in V$ (Existence of additive identity)

(4) For any $v \in V$, $\exists -v \in V$ s.t. $v + (-v) = 0$ (Existence of additive inverses)

(5) $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v \quad \forall\, v \in V \;\&\; \alpha, \beta \in \mathbb{F}$

(6) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v \quad \forall\, \alpha, \beta \in \mathbb{F},\ v \in V$

(7) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \quad \forall\, \alpha \in \mathbb{F},\ u, v \in V$

(8) $1 \cdot v = v \quad \forall\, v \in V$, where 1 is the multiplicative identity of the field $\mathbb{F}$.

Elements of V are called vectors & elts. of $\mathbb{F}$ are called scalars. We call $+$ the vector addition and $\cdot$ as the scalar multiplication.

Remark: Additive identity in a vector space is unique.

Pf: Suppose 0 & $0'$ are both additive identities.

Then $0' = 0 + 0' = 0$

. We will call 0, the additive identity of V, as the zero vector.

Similarly, we can prove that additive inverse of any $v \in V$ is unique.

Exercise: (i) $\underset{\substack{\uparrow \\ \in \mathbb{F}}}{\overset{\text{the zero scalar}}{0}} \cdot v = \underset{\substack{\uparrow \\ \in V}}{\overset{\text{the zero vector}}{0}} \quad \forall\, v \in V.$

(ii) $\alpha \cdot \vec{0} = \vec{0}$

(iii) $(-1) \cdot v = -v \quad \forall\, v \in V$

Pf: $$(-1).v + v = (-1).v + 1.v$$
$$= ((-1)+1).v$$
$$= 0.v = 0 \quad (\text{by Ex(1)})$$

<u>Pf</u> of (1): To prove $0 \cdot \vec{v} = \vec{0}$

$$0 \cdot \vec{v} = (0+0) \cdot \vec{v}$$
$$= 0 \cdot \vec{v} + 0 \cdot \vec{v}$$

Adding the additive inverse, $-(0 \cdot \vec{v})$, of $0 \cdot \vec{v}$, we get

$$\underbrace{0 \cdot \vec{v} + \left(-(0 \cdot \vec{v})\right)}_{\vec{0}} = \underbrace{\left(0 \cdot \vec{v} + 0 \cdot \vec{v}\right) + \left(-(0 \cdot \vec{v})\right)}_{\underbrace{(0 \cdot \vec{v}) + \left(0 \cdot \vec{v} + \underbrace{\left(-(0 \cdot \vec{v})\right)\Big)}_{\vec{0}}}_{0 \cdot \vec{v}}}$$

Examples of vector spaces:

① Let $V = \{\vec{0}\}$ and $\mathbb{F}$ be any field.

Then V is a vector space over $\mathbb{F}$ with the operations:

$$\vec{0} + \vec{0} = \vec{0} \quad ; \quad \alpha \cdot \vec{0} = \vec{0} \quad \text{for } \alpha \in \mathbb{F}$$

$V = \{0\}$ is called the zero vector space.

② $V = \mathbb{F}$ is a vector space over $\mathbb{F}$ with the operations borrowed from the $+$ & $\cdot$ on $\mathbb{F}$.

In particular, $\mathbb{R}$ is a vector space over $\mathbb{R}$

$\mathbb{C}$ is a vector space over $\mathbb{C}$.

(3) $\mathbb{C}$ is a vector space over $\mathbb{R}$ with the operations $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined as the usual addition of complex nos.

(4) Let $\mathbb{F}$ be any field.

Let $V = \mathbb{F}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{F},\ 1 \le i \le n\}$.

The addition on V is defined as

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$$

$$\alpha \cdot (x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

Then V is a vector space over $\mathbb{F}$.

The zero vector in $\mathbb{F}^n$ is $0 = (0, 0, \dots, 0)$

Note Title 3/22/2023

Lec - 8

Some more examples of vector spaces

① $M_{m\times n}(\mathbb{F})$ = set of all $m\times n$ matrices with entries from field $\mathbb{F}$.

Then $M_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$ with the usual matrix addition and scalar multiplication.

② Let $\mathbb{F}[x]$ denote the set of all polynomials with coefficients from the field $\mathbb{F}$.

i.e. $\mathbb{F}[x] = \{ a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{F},\ n \in \mathbb{N} \cup \{0\} \}$.

$\mathbb{F}[x]$ is a vector space over $\mathbb{F}$.

③ Let $V = \{ f : \mathbb{R} \to \mathbb{R} \}$.

We define $+$ as $(f+g)(x) = f(x) + g(x)$

& scalar mult. as $(\alpha \cdot f)(x) = \alpha \cdot f(x)$

Then V is a vector space over $\mathbb{R}$.

(In ②, if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$q(x) = b_0 + b_1 x + \cdots + b_m x^m$

Suppose $m \geq n$. Then

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n + b_{n+1} x^{n+1} + \cdots + b_m x^m .$$)

In ~~example~~ ③ if we change the domain from $\mathbb{R}$ to any nonempty set X, i.e.

$$V = \{ f : X \to \mathbb{R} \}$$

Then V is a vector space over $\mathbb{R}$.

More generally, $V = \{ f : X \to \mathbb{F} \}$ is a vector space over $\mathbb{F}$.

④ $V = C(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \}$.

can be replaced by any nonempty subset of $\mathbb{R}$.

Then V is a vector space over $\mathbb{R}$.

(5) Let $V = \text{Maps}(\mathbb{N}, \mathbb{F})$ = the set of all sequences where elements are from $\mathbb{F}$.

$$= \{ (x_1, x_2, x_3, \dots, x_n, \dots) : x_n \in \mathbb{F} \ \forall n \in \mathbb{N} \}.$$

$\text{Maps}(\mathbb{N}, \mathbb{F})$ is a vector space over $\mathbb{F}$, where $+$ & $\cdot$ are defined as usual.

(6) Let $\mathbb{F}^\infty := \{ (x_1, x_2, \dots) \in \text{Maps}(\mathbb{N}, \mathbb{F}) \mid x_n = 0 \text{ for all but finitely many terms} \}$

Then $\mathbb{F}^\infty$ is a vector space over $\mathbb{F}$.

Note ~~that~~ the zero sequence $(0, 0, 0, \dots)$

also belongs to $\mathbb{F}^\infty$.

Subspace of a vector space

Let V be a vector space over $\mathbb{F}$. A subset W of V is said to be a subspace of V if W is a vector space over $\mathbb{F}$ with the vector addition and scalar multiplication borrowed from those on V.

e.g. $\{0\}$ & V are subspaces of V.

Note Title 3/24/2023

Lec - 9

Thm (Subspace criterion): A nonempty subset W of a vector space V is a subspace of V if and only if

$$u, v \in W \ \& \ a \in \mathbb{F} \Rightarrow au + v \in W$$

$$\left(\text{or} \quad u, v \in W \ \& \ a, b \in \mathbb{F} \Rightarrow au + bv \in W \right)$$

Pf: ($\Rightarrow$) Obvious

($\Leftarrow$) To show that W is closed under $+$:

If $u, v \in W$, then $u + v = 1 \cdot u + v \in W$

$(\text{or} \quad u + v = 1 \cdot u + 1 \cdot v \in W)$

To show: W is closed under scalar multiplication:

First, we will show that $0 \in W$

Since $W \neq \emptyset$, $\exists\, u \in W$.

Then $0 = (-1)u + u \in W$

Now if $v \in W$ & $a \in \mathbb{F}$, then

$$av = av + 0 \in W.$$

<u>Example</u>: Let $A \in M_{m\times n}(\mathbb{F})$.

Then $W = \{X \in M_{n\times 1}(\mathbb{F}) : AX = 0\}$

is a subspace of $M_{n\times 1}(\mathbb{F})$.

(Generally we will write $\mathbb{F}^n$ instead of $M_{n\times 1}(\mathbb{F})$)

Pf: $0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in W$ because $A0 = 0$.

Let $X, Y \in W$ and $a \in \mathbb{F}$

Then $A(aX+Y) = aAX + AY$

$= a \cdot 0 + 0 = 0 + 0 = 0$

$\therefore aX+Y \in W$.

$\therefore$ W is a subspace.

- Let $V = M_n(\mathbb{F}) = M_{n\times n}(\mathbb{F})$

and $W = \{A \in V : A^t = A\}$ i.e. W consists of all $n\times n$ symmetric matrices

Is W a subspace of V?

Clearly, $0 \in W$

Also if $A, B \in W$ & $\alpha \in \mathbb{F}$

then $(\alpha A + B)^t = \alpha A^t + B^t$

$= \alpha A + B$

$\Rightarrow \alpha A + B \in W$.

Ques: If W_1 & W_2 are subspaces of V, is it necessary that $W_1 \cup W_2$ a subspace?

Ans: No.

Take $V = \mathbb{R}^2$

$W_1 = \{(x,0) : x \in \mathbb{R}\} \leftarrow$ x-axis

$W_2 = \{(0,y) : y \in \mathbb{R}\} \leftarrow$ y-axis.

But, $W_1 \cup W_2$ is <u>not</u> a subspace of $\mathbb{R}^2$

(take $u = (1,0)$ & $v = (0,1)$

then $u \in W_1 \cup W_2$ and $v \in W_1 \cup W_2$

but, $u + v = (1,1) \notin W_1 \cup W_2$)

Theorem: The intersection of any collection of subspaces is also a subspace.

Pf: Let $\{W_i : i \in \Lambda\}$ is a collection of subspaces.

To show: $\bigcap_{i \in \Lambda} W_i$ is also a subspace.

- $0 \in W_i \ \forall i \in \Lambda \Rightarrow 0 \in \bigcap_{i \in \Lambda} W_i$
- Let $u, v \in \bigcap_{i \in \Lambda} W_i$ & $a \in \mathbb{F}$

$\Rightarrow u, v \in W_i \ \forall i \Rightarrow au + v \in W_i \ \forall i$

($\because W_i$ is a subsp.)

$\Rightarrow \quad au + v \in \bigcap_{i \in \Lambda} W_i$.

Defn: For any subset S of a vector space V, the "subspace generated by S" is the intersection of all subspaces containing S.
We will denote this by $\langle S \rangle$

Note that $\langle S \rangle$ is the smallest subspace containing S. If $S = \phi$, then $\langle S \rangle = \{0\}$.

Let $\operatorname{span}(S)$ = collection of all possible linear combinations of vectors from S

$$= \left\{ a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \;\middle|\; \begin{array}{l} a_i \in \mathbb{F} \\ v_i \in S \end{array}, \; 1 \le i \le n, \right.$$

$n \in \mathbb{N}$

Theorem: For any nonempty set S, $\text{span}(S) = \langle S \rangle$.

Note Title 3/28/2023

Lec - 10

Recall:

- For a subset S of a vector space V, the subspace generated by S, $\langle S \rangle$ = the intersection of all the subspaces which contains S.

- The span of S, $\text{span}(S) = \{$all linear combinations of elements from $S\}$

$\therefore$ $v \in \text{span}(S) \iff \exists\, v_1, v_2, \ldots, v_n \in S$ and scalar $a_1, a_2, \ldots, a_n \in \mathbb{F}$ s.t. $v = \sum_{i=1}^{n} a_i v_i$

Convention: For $S = \phi$, $\text{span}(S) = \{0\}$.

- Both $\langle S \rangle$ and $\mathrm{span}(S)$ are subspaces of V. In fact,

<u>Theorem</u>: For any subset S, $\mathrm{span}(S) = \langle S \rangle$.

<u>Pf</u>: first, since $\mathrm{span}(S)$ is a subspace of V and
$S \subseteq \mathrm{span}(S)$ (If $v \in S$, then $v = 1 \cdot v \in \mathrm{span}(S)$)
Since $\langle S \rangle$ is the intersection of all subspaces
containing, $\langle S \rangle \subseteq \mathrm{span}(S)$.
Now, to show that $\mathrm{span}(S) \subseteq \langle S \rangle$ it is enough
to show that if W is any subspace containing S,
then $W \supseteq \mathrm{span}(S)$.
Let $S \subseteq W$ for some subspace W.

If $v \in \text{span}(S)$, then $v = a_1 v_1 + \cdots + a_n v_n$
for some $v_1, \cdots, v_n \in S$
& $a_1, \cdots, a_n \in \mathbb{F}$

Since $S \subseteq W$, $v_1, v_2, \cdots, v_n \in W$
But W is a subspace, so $a_1 v_1 + \cdots + a_n v_n \in W$.

$\therefore \quad \text{span}(S) \subseteq W$.

Linear dependence and independence

Defn: A subset S of a vector space V is said to be linearly dependent if the zero vector of V can be written as a non-trivial linear combination of some vectors from S, i.e.,

($\exists$ distinct vectors $v_1, v_2, \dots, v_n \in S$ and non-zero scalars $a_1, a_2, \dots, a_n \in \mathbb{F}$ s.t.

$$a_1 v_1 + \dots + a_n v_n = 0$$)

$\therefore$ A finite set $S = \{v_1, v_2, \dots, v_m\}$ is linearly dependent if $\exists\, a_1, a_2, \dots, a_m \in \mathbb{F}$ s.t. at least one $a_i \neq 0$ and $a_1 v_1 + a_2 v_2 + \dots + a_m v_m = 0$

Defn: A subset S of a vector space is said to be linearly independent if S is not linearly dependent.

$\therefore$ A nonempty set S is linearly independent if and only if $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ for some $a_i \in \mathbb{F}$ and $v_1, \dots, v_n$ distinct vectors in S

$\Rightarrow a_i = 0 \;\forall i \in \{1, 2, \dots, n\}$

To show $S = \{v_1, v_2, \ldots, v_n\}$ is linearly independent, we have to show that $a_1 v_1 + \cdots + a_n v_n = 0$

$\Rightarrow a_i = 0 \quad \forall i$

Example: 1) Let $V = \mathbb{R}^3$ and $S = \{(1,0,0), (0,1,0), (0,0,1)\}$.

Is S linearly independent?

Let $a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (0,0,0)$

$\Rightarrow (a_1, a_2, a_3) = (0,0,0)$

$\Rightarrow a_1 = 0, \; a_2 = 0, \; a_3 = 0$

$\therefore$ S is linearly indep.

(2) $V = \mathbb{R}^3(\mathbb{R})$, $S = \{(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)\}$

How to check if S is linearly dependent or not?

Suppose $x_1(a_1, a_2, a_3) + x_2(b_1, b_2, b_3) + x_3(c_1, c_2, c_3) = (0,0,0)$,

where $x_1, x_2, x_3 \in \mathbb{R}$.

$$\Rightarrow \begin{cases} a_1 x_1 + b_1 x_2 + c_1 x_3 = 0 \\ a_2 x_1 + b_2 x_2 + c_2 x_3 = 0 \\ a_3 x_1 + b_3 x_2 + c_3 x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is a system of homog. linear eqns in 3 unknowns x_1, x_2, x_3.

If it has a unique soln. $(0,0,0)$ then the set S is linearly indep. and if it has more than one solns. then S is linearly dependent.

Remarks: (1) If S contains the zero vector, then S must be linearly dependent.

In particular, $\{0\}$ is linearly dependent.

(2) A superset of a linearly dependent subset must be linearly dependent.

(3) Any subset of a linearly independent set must be linearly independent. (L.I.)

Ques: If the empty set ϕ L.I. or L.D.?

Ans: ϕ is L.I. (because it is not L.D.)

Ex: let $V = \mathbb{F}[x]$ = the space of all polynomials.

and let $S = \{1, x, x^2, \dots, x^n, \dots\}$.

Prove that S is L.I.

Note Title 3/31/2023

Lec - 11

Basis & Dimension of a vector space

Defn: Let V be a vector space. A subset $\mathcal{B}$ of V is said to be a basis of V if

(i) $\mathcal{B}$ is L.I., and

(ii) $\text{span}(\mathcal{B}) = V$.

Examples: (1) $V = \mathbb{F}^n$ (over $\mathbb{F}$).

$$\mathcal{B} = \{\underbrace{(1,0,0,\dots,0)}_{e_1}, \underbrace{(0,1,0,0,\dots,0)}_{e_2}, \dots, \underbrace{(0,0,\dots,0,1)}_{e_n}\}$$

If $a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = 0$, then

$$(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$$

$$\Rightarrow \quad a_1 = a_2 = \cdots = a_n = 0$$

$\therefore$ $\mathcal{B}$ is L.I.

Also, if $x = (x_1, x_2, \ldots, x_n) \in V$, then

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in \operatorname{span}(\mathcal{B}).$$

$\therefore$ $\operatorname{span}(\mathcal{B}) = V$.

Hence, $\mathcal{B}$ is a basis of $\mathbb{F}^n$.

Ex: Show that $\{e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + e_2 + \cdots + e_n\}$ is also a basis of $\mathbb{F}^n$.

(2) $V = \mathbb{F}[x]$ = the vector space of all polynomials over $\mathbb{F}$.

$\mathcal{B} = \{1, x, x^2, x^3, \ldots\}$ ← infinite set.

If $v \in V$, then $v = a_0 + a_1 x + \cdots + a_n x^n$

$\in \text{span}\{1, x, x^2, \ldots\}$

$\therefore \text{span}(\mathcal{B}) = V$.

Let $b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m = 0$

$\Rightarrow b_i = 0 \quad \forall i \in \{0, 1, 2, \ldots, m\}$ (by the defn. of the zero polynomial)

$\Rightarrow \mathcal{B}$ is L.I.

(3) Let $V = \text{Maps}(\mathbb{N}, \mathbb{R})$ = v. sp. of all real sequences.

and $S = \{e_1, e_2, \cdots, e_n, \cdots\}$,

where $e_n = (0, \cdots, 0, \underset{\substack{\uparrow \\ n\text{th place}}}{1}, 0, 0, \cdots)$.

Ques: Is $\text{span}(S) = V$?

Ans: No, because $(1, 1, 1, 1, \cdots) \notin \text{Span}(S)$.

In fact, $\text{span}(S) = \mathbb{F}^\infty$.

$x \in \mathbb{F}^\infty \Rightarrow x = (x_1, x_2, \cdots, x_k, 0, 0, \cdots)$

$= x_1 e_1 + x_2 e_2 + \cdots + x_k e_k$.

$\in \text{Span}(S)$.

$\therefore \mathbb{F}^\infty \subseteq \text{span}(S)$. Also, $\text{span}(S) \subseteq \mathbb{F}^\infty$.

Theorem: A basis of a vector space is a maximal L.I. subset of V.

Pf: Let $\mathcal{B}$ be a basis of V.

To show: $\mathcal{B}$ is a maximal L.I. set.

Let $v \notin \mathcal{B}$. To show: $\mathcal{B} \cup \{v\}$ is L.D.

Since $\mathcal{B}$ is a basis of V, $\text{span}(\mathcal{B}) = V$.

$\therefore \exists\, v_1, v_2, \ldots, v_n \in \mathcal{B}$ & $a_1, a_2, \ldots, a_n \in \mathbb{F}$

s.t. $v = a_1 v_1 + \cdots + a_n v_n$

$\Rightarrow a_1 v_1 + \cdots + a_n v_n + (-1) v = 0$

$\nRightarrow$ $\mathcal{B} \cup \{v\}$ is L.D.

i.e. $\mathcal{B}$ is a maximal L.I. subset.

Theorem: Every maximal L.I. subset of V is a basis of V.

Pf: Let $\mathcal{B}$ be a maximal L.I. subset.

To show: $\mathcal{B}$ is a basis of V, it is enough to show that $\text{span}(\mathcal{B}) = V$.

Suppose $\text{span}(\mathcal{B}) \neq V$.

Then $\exists\, v \in V$ s.t. $v \notin \text{span}(\mathcal{B})$. ($\Rightarrow v \notin \mathcal{B}$)

Claim: $\mathcal{B} \cup \{v\}$ is L.I.

(this will contradict that $\mathcal{B}$ is a maximal L.I. set)

Let $a_0 v + a_1 v_1 + \cdots + a_n v_n = 0$, $a_i \in \mathbb{F}$, $v_1, \ldots, v_n \in \mathcal{B}$.

Suppose $a_0 \neq 0$, ~~then~~

$$v = -a_0^{-1}\left(a_1 v_1 + \cdots + a_n v_n\right)$$

$$\in \operatorname{span}(\mathcal{B}) \quad \text{a contradiction}$$

$\therefore \quad a_0 = 0 \implies a_1 v_1 + \cdots + a_n v_n = 0$

$\implies a_1 = \cdots = a_n = 0 \quad (\because \{v_1, \ldots, v_n\} \text{ is L.I.})$

Note Title 4/1/2023

Lec-12

Proposition: Suppose $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Then any set containing more than n elements must be linearly dependent.

Pf: Let $S = \{u_1, u_2, \ldots, u_m\}$, where $m > n$.

Since $\text{span}(\mathcal{B}) = V$,

$$u_j = a_{1j} v_1 + a_{2j} v_2 + \cdots + a_{nj} v_n,$$

for $j = 1, 2, \ldots, m$

Now, $c_1 u_1 + c_2 u_2 + \cdots + c_m u_m$

$$= \sum_{j=1}^{m} c_j u_j = \sum_{j=1}^{m} c_j \sum_{i=1}^{n} a_{ij} v_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} c_j \right) v_i \quad \text{——} (*)$$

Consider the homog. system $AX = 0$,

where $A = (a_{ij})_{n \times m}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

Then, $\operatorname{rank}(A) \leq n < m = \#$ of unknowns.

$\therefore$ The system has a nonzero soln $(c_1, c_2, \dots, c_m)$

Then $(*) \Rightarrow c_1 u_1 + \dots + c_m u_m = 0$

$\therefore \{u_1, u_2, \cdots, u_m\}$ is L.D.

<u>Corollary</u>: Suppose V is a vector space with a basis $\mathcal{B}$ containing exactly n elements. Then every basis of V has exactly n elements.

<u>Defn</u> (Dimension of a vector space)

The dimension of a vector space V is the number of elements in any basis of V (if a basis has finitely many elements). If the no. of elements in a basis is infinite, then we say V is infinite dimensional v. sp. or $\dim(V) = \infty$.

Fact: • Every vector space has a basis.

The proof of this fact uses the Zorn's lemma (or the axiom of choice) but we will not discuss it.

• Every L.I. subset of a vector space can be extended to a basis.

Examples: ① $\dim(\mathbb{F}^n(\mathbb{F})) = n$ because $\{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{F}^n$.

② $\dim(\mathbb{C}(\mathbb{R})) = 2$, $\mathcal{B} = \{1, i\}$ is a basis.

(3) $\dim(M_{m\times n}(\mathbb{F})) = mn$

(4) $\dim(\mathbb{F}[x]) = \infty$, $\mathcal{B} = \{1, x, x^2, \dots\}$

(5) Let $P_n[x] = \{a_0 + a_1 x + \cdots + a_n x^n : a_0, \dots, a_n \in \mathbb{F}\}$

= set of all polynomials with degree $\leq n$.

$\dim(P_n[x]) = n+1$

because $\{1, x, x^2, \dots, x^n\}$ is a basis

Note Title 4/5/2023

Lec-13

Coordinates w.r.t. an ordered basis

Defn: Let V be a finite dimensional space. A basis $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ is called an ordered basis if the order of the vectors in $\mathcal{B}$ is fixed.

Since $\mathcal{B}$ is a basis of V, given any $v \in V$, we can write $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$; $\alpha_i \in \mathbb{F}$

Also, $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is unique.

The vector $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in M_{n\times 1}(\mathbb{F})$ is called the coordinate vector of v w.r.t the ordered basis $\mathcal{B}$.

Notation : $[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$.

Change of basis

Suppose $\mathcal{B}_1 = \{v_1, v_2, \dots, v_n\}$ and $\mathcal{B}_2 = \{w_1, w_2, \dots, w_n\}$ be two ordered bases of V.

Q: How is $[v]_{\mathcal{B}_2}$ related to $[v]_{\mathcal{B}_1}$?

Suppose $[v]_{\mathcal{B}_1} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ & $[v]_{\mathcal{B}_2} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$

i.e. $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

& $v = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_n w_n$

Let $\quad v_j = \sum_{i=1}^{n} p_{ij} w_i \quad , \; j \in \{1, 2, \cdots, n\}$

Then $\begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix} = [v_j]_{\mathcal{B}_2}$.

Now, $\quad v = \sum_{j=1}^{n} \alpha_j v_j = \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{n} p_{ij} w_i \right)$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij} \alpha_j \right) w_i$$

$\Rightarrow \quad \beta_i = \sum_{j=1}^{n} p_{ij} \alpha_j \quad$ for $\; i \in \{1, 2, \cdots, n\}$

$\therefore \quad [v]_{\mathcal{B}_2} = P \, [v]_{\mathcal{B}_1} \quad ; \;$ where $P = (p_{ij})_{n \times n}$

This matrix is called the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. The columns of matrix P are the coordinate vectors of elements of $\mathcal{B}_1$ w.r.t. $\mathcal{B}_2$.

Let P be the change of bases matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$
and Q " " " " " " " $\mathcal{B}_2$ to $\mathcal{B}_1$

Then $$[v]_{\mathcal{B}_2} = P\,[v]_{\mathcal{B}_1}$$

and $$[v]_{\mathcal{B}_1} = Q\,[v]_{\mathcal{B}_2}$$

$$\therefore \quad [v]_{\mathcal{B}_2} = PQ\,[v]_{\mathcal{B}_2} \qquad \forall\, v \in V. \qquad (1)$$

<u>Claim</u>: $PQ = I_n \leftarrow n \times n$ identity matrix.

<u>Pf</u>: Taking $v = w_j$ for $1 \le j \le n$

Then $[v]_{\mathcal{B}_2} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th}$ place.

$\therefore$ (1) $\Rightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (PQ) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = j^{th}$ column of PQ.

$\Rightarrow PQ = I_n$

$\therefore$ P is invertible and $P^{-1} = Q$.

<u>Row space and column space of a matrix</u>.

Let $A \in M_{m \times n}(\mathbb{F})$

Then each row of A can be ~~thought of as~~ thought of as an element of $\mathbb{F}^n$ and each column of A can be thought of as an element of $\mathbb{F}^m$.

Let $R_1, R_2, \dots, R_m$ be the rows of A and $C_1, C_2, \dots, C_n$ be the columns of A.

Defn: The row space of A = $\text{span}\{R_1, R_2, \dots, R_m\}$
The column space of A = $\text{Span}\{C_1, C_2, \dots C_n\}$

Then $\overbrace{\text{row}(A)}^{=\text{ row space }(A)}$ is a subspace of $\mathbb{F}^n$
$\text{col}(A)$ is a subspace of $\mathbb{F}^m$

Defn: The row rank of A is the dimension of $\text{row}(A)$
The column rank " " " " " " $\text{col}(A)$

i.e. row rank (A) = dim (row (A))

col rank (A) = dim (col (A))

Note that ① row rank (A), col rank $(A) \leq \min\{m, n\}$.

② If A is row equivalent of B,

then row (A) = row (B)

(Each row of B is a linear combination of rows of A. Therefore, row $(B) \subseteq$ row (A)

Similarly, row $(A) \subseteq$ row (B))

③ The nonzero rows of an RRE matrix are L.I.

Hence, ④ row rank (A) = rank (A)

and $\{R_1, R_2, \cdots, R_r\}$ is a basis of $\mathrm{row}(A)$,
where $R_1, R_2, \cdots, R_r$ are the nonzero rows of the RRE form of A.

Note Title 4/11/2023

Lec - 14

Find a basis for the solution space of $AX = 0$:

Example : Take $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \end{pmatrix} \underset{\sim}{\xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1}}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

$$\underset{\sim}{\xrightarrow{R_3 \to R_3 - R_2}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \text{RRE form of } A.$$

Here, $\operatorname{rank}(A) = r = 2$; $n = 4$

$\dim(\text{solution space}) = n - r = 2$.

Also, $k_1 = 1$, $k_2 = 3$.

$\therefore$ x_1 & x_3 are dependent and x_2 & x_4 are indep. unknowns.

Let $x_2 = \alpha$, $x_4 = \beta$.

Then $x_1 + 2\alpha + \beta = 0 \Rightarrow x_1 = -2\alpha - \beta$

$x_3 + 3x_4 = 0 \Rightarrow x_3 = -3\beta$

$\therefore$ Solution set $= \{ (-2\alpha-\beta, \alpha, -3\beta, \beta) : \alpha, \beta \in \mathbb{R} \}$

A basis for the solution is obtained by putting $(\alpha, \beta) = (1, 0)$ & $(\alpha, \beta) = (0, 1)$.

$\mathcal{B} = \{ (-2, 1, 0, 0), (-1, 0, -3, 1) \}$

Sum of subspaces

Let W_1 & W_2 be subspaces of a vector space V.
The sum $W_1 + W_2$ is defined by

$$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1 , w_2 \in W_2 \}.$$

Prop: $W_1 + W_2$ is a subspace of V.

Pf: Let u & $v \in W_1 + W_2$ and $\alpha, \beta \in \mathbb{F}$.

To show: $\alpha u + \beta v \in W_1 + W_2$

Since $u \in W_1 + W_2$, $u = w_1 + w_2$ for some $w_1 \in W_1, w_2 \in W_2$

Since $v \in W_1 + W_2$, $v = w_1' + w_2'$ for some $w_1' \in W_1, w_2' \in W_2$.

Then $\quad \alpha u + \beta v = \alpha(w_1 + w_2) + \beta(w_1' + w_2')$

$$= \underbrace{(\alpha w_1 + \beta w_1')}_{\in W_1} + \underbrace{(\alpha w_2 + \beta w_2')}_{\in W_2}$$

$\therefore \quad \alpha u + \beta v \in W_1 + W_2$.

Example: $\quad V = \mathbb{R}^3, \quad W_1 = \{(x, 0, 0) : x \in \mathbb{R}\} \to$ x-axis

$W_2 = \{(0, y, 0) : y \in \mathbb{R}\} \to$ y-axis

Then $\quad W_1 + W_2 = \{(x, y, 0) : x \in \mathbb{R}, y \in \mathbb{R}\}$
$\downarrow$
xy-plane

Q: What is $\dim(W_1 + W_2)$?

Ex: Show that $W_1 + W_2 = \text{span}(W_1 \cup W_2)$.

Theorem: Let W_1 & W_2 be finite dimensional subspaces of a vector space V. Then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Pf: Let $\dim(W_1 \cap W_2) = k$, $\dim(W_1) = k+n$, $\dim(W_2) = k+m$, where $k, n, m \in \mathbb{N} \cup \{0\}$.

Let $\mathcal{B} = \{v_1, v_2, \dots, v_k\}$ be a basis of $W_1 \cap W_2$.

(If $k = 0$, then $\mathcal{B} = \phi$)

Since $\mathcal{B}$ is a L.I. subset of W_1, we can extend $\mathcal{B}$ to basis $\mathcal{B}_1$ of W_1 as follows:

$$\mathcal{B}_1 = \{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_n\}.$$

Similarly, a basis for W_2 is

$$\mathcal{B}_2 = \{v_1, v_2, \dots, v_k, w_1, w_2, \dots, w_m\}$$

<u>Claim</u>: $S = \{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_m\}$ is a basis for $(W_1 + W_2)$.

- First we'll show that $\text{span}(S) = W_1 + W_2$.

$$S \subseteq W_1 \cup W_2 \subseteq W_1 + W_2$$

$$\Rightarrow \text{span}(S) \subseteq W_1 + W_2.$$

Conversely, let $v \in W_1 + W_2$. Then

$$v = w_1 + w_2 \; ; \quad w_1 \in W_1 , \; w_2 \in W_2 .$$

$$= (a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_n u_n)$$

$$+ (c_1 v_1 + \cdots + c_k v_k + d_1 w_1 + \cdots + d_m w_m)$$

$$= (a_1 + c_1) v_1 + \cdots + (a_k + c_k) v_k + b_1 u_1 + \cdots + b_n u_n$$

$$+ d_1 w_1 + \cdots + d_m w_m$$

$$\in \operatorname{span}(S) .$$

$$\Rightarrow \quad W_1 + W_2 \subseteq \operatorname{span}(S)$$

- Next we'll show that S is L.I.

Let $\alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 u_1 + \cdots + \beta_n u_n + \gamma_1 w_1 + \cdots + \gamma_m w_m = 0$

$$\Rightarrow \quad \underbrace{\alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 u_1 + \cdots + \beta_n u_n}_{\in W_1} = \underbrace{-(\gamma_1 w_1 + \cdots + \gamma_m w_m)}_{\in W_2} \quad \text{---} (*)$$

$$\therefore \quad -(\gamma_1 w_1 + \cdots + \gamma_m w_m) \in W_1 \cap W_2 .$$

$$\Rightarrow \quad -(\gamma_1 w_1 + \cdots + \gamma_m w_m) = a_1 v_1 + \cdots + a_k v_k$$

$$\Rightarrow \quad a_1 v_1 + \cdots + a_k v_k + \gamma_1 w_1 + \cdots + \gamma_m w_m = 0$$

$$\Rightarrow \quad a_i = 0 = \gamma_j \text{ for } 1 \le i \le k,\ 1 \le j \le m \ (\because \mathcal{B}_2 \text{ is L.I.})$$

In particular, $\gamma_1 = \gamma_2 = \cdots = \gamma_m = 0$

Then $(*) \Rightarrow \quad \alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 u_1 + \cdots + \beta_n u_n = 0$

$$\Rightarrow \alpha_1 = \cdots = \alpha_k = 0, \quad \beta_1 = \cdots = \beta_n = 0 \quad (\because \mathcal{B}_1 \text{ is L.I.})$$

Hence, S is L.I.

$$\dim(W_1+W_2+W_3) \neq \dim W_1 + \dim W_2 + \dim W_3 - \dim(W_1\cap W_2) - \dim(W_2\cap W_3) - \dim(W_3\cap W_1) + \dim(W_1\cap W_2\cap W_3)$$

Example: $V = \mathbb{R}^2$, $W_1 = x$-axis

$W_2 = y$-axis

$W_3 = \{y = x\}$

Note Title 4/12/2023

Lec - 15

Direct sum of subspaces: Let W_1 & W_2 be subspaces of V.

A sum $W_1 + W_2$ is called a direct sum if $W_1 \cap W_2 = \{0\}$.
In this case, we denote it by $W_1 \oplus W_2$.

By, the dimension formula for $W_1 + W_2$, we get

$$\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2 .$$

Thm: Suppose $V = W_1 \oplus W_2$. Then every vector $v \in V$ can be written uniquely as $v = w_1 + w_2$, where $w_1 \in W_1, w_2 \in W_2$.

Pf: Since $V = W_1 + W_2$, given $v \in V$, $\exists\, w_1 \in W_1, w_2 \in W_2$ s.t. $v = w_1 + w_2$.

Suppose $v = w_1 + w_2 = w_1' + w_2'$, where $w_1, w_1' \in W_1$
$w_2, w_2' \in W_2$.

$\Rightarrow \quad w_1 - w_1' = w_2' - w_2$

Here, L.H.S. $\in W_1$ and R.H.S. $\in W_2$.

$\therefore \quad w_1 - w_1' = w_2' - w_2 \in W_1 \cap W_2 = \{0\}$

$\Rightarrow \quad w_1 - w_1' = w_2' - w_2 = 0$

$\Rightarrow \quad w_1' = w_1$ and $w_2' = w_2$.

Example: $\mathbb{R}^2 = \{(x,0) : x \in \mathbb{R}\} \oplus \{(0,y) : y \in \mathbb{R}\}$

Remark: The converse of the previous theorem is also true.

Pf: Let $w \in W_1 \cap W_2$.

To show: $w = 0$.

We have $w = w + 0 = 0 + w$

By the uniqueness, $w = 0$

<u>Direct sum of $W_1, W_2, \cdots, W_k$</u>:

$W_1 + W_2 + \cdots + W_k$ is the direct sum if

$$\dim (W_1 + W_2 + \cdots + W_k) = \dim W_1 + \dim W_2 + \cdots + \dim W_k .$$

Notation: $W_1 \oplus W_2 \oplus \cdots \oplus W_k$

<u>Thm</u>: $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ if and only if every $v \in V$ can be written uniquely as $v = w_1 + w_2 + \cdots + w_k$, with $w_i \in W_i$ for $1 \le i \le k$.

<u>Pf</u>:

$$\dim (W_1 + W_2 + \cdots + W_k) = \dim W_1 + \dim (W_2 + \cdots + W_k) - \dim \big(W_1 \cap (W_2 + \cdots + W_k)\big) \quad -(*)$$

($\Leftarrow$) Claim: $W_1 \cap (W_2 + \cdots + W_k) = \{0\}$.

Let $w \in W_1 \cap (W_2 + \cdots + W_k)$. Since $w \in W_2 + \cdots + W_k$, $w = w_2 + \cdots + w_k$.

Then $w = \underbrace{w}_{\in W_1} + \underbrace{0}_{\in W_2} + 0 + \cdots + \underbrace{0}_{\in W_k} = \underbrace{0}_{\in W_1} + w_2 + \cdots + w_k$

By uniqueness, $w = 0$, $w_2 = 0, \ldots, w_k = 0$

$\therefore W_1 \cap (W_2 + \cdots + W_k) = \{0\}$.

$\therefore$ By $(*)$, $\dim (W_1 + \cdots + W_k) = \dim W_1 + \dim (W_2 + \cdots + W_k)$

By induction, we get

$$\dim (W_1 + \cdots + W_k) = \dim W_1 + \dim W_2 + \cdots + \dim W_k$$

$\therefore V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

($\Rightarrow$) Assume $V = W_1 \oplus \cdots \oplus W_k$. ie.

$$\dim V = \dim W_1 + \dim W_2 + \cdots + \dim W_k.$$

Suppose $v = w_1 + w_2 + \cdots + w_k = w_1' + w_2' + \cdots + w_k'$.

$$\Rightarrow \underbrace{w_1 - w_1'}_{\in W_1} = \underbrace{(w_2' - w_2) + \cdots + (w_k' - w_k)}_{\in W_2 + \cdots + W_k} \quad \text{---} (**)$$

Now, $\dim W_1 + \dim W_2 + \cdots + \dim W_k$

$$= \dim (W_1 + W_2 + \cdots + W_k)$$

$$= \dim W_1 + \underbrace{\dim (W_2 + \cdots + W_k)}_{\dim W_2 + \cdots + \dim W_k} - \dim \big(W_1 \cap (W_2 + \cdots + W_k)\big)$$

$$\Rightarrow \dim \big(W_1 \cap (W_2 + \cdots + W_k)\big) = 0$$

$$\Rightarrow W_1 \cap (W_2 + \cdots + W_k) = \{0\}$$

By $(**)$, $w_1 - w_1' = 0 \Rightarrow w_1' = w_1$.

Similarly, $W_2 \cap (W_1 + W_3 + \cdots + W_k) = \{0\}$

$\therefore \quad W_2' = W_2$

and so on.

Thus, $W_i' = W_i \quad \forall\, i = 1, 2, \ldots, k.$

Lec - 16

Linear Transformations

Defn : Let V and W be vector spaces over the same field $\mathbb{F}$.
A function $T : V \to W$ is called a linear transformation if

$$T(au+bv) = aT(u) + bT(v) \quad \forall\ u, v \in V$$
$$\& \ \forall\ a, b \in \mathbb{F}.$$

Equivalently, (i) $T(u+v) = T(u) + T(v) \quad \forall\ u, v \in V$

(ii) $T(av) = aT(v) \quad \forall\ v \in V,\ a \in \mathbb{F}$.

Remark : If $T : V \to W$ is a linear transformation, then

$$T(0_V) = 0_W \leftarrow \text{zero vector in } W.$$

$\uparrow$ zero vector in V

Pf: $T(0) = T(0+0) = T(0) + T(0)$

Adding $-T(0)$ both sides, we get

$$T(0) = 0.$$

Example: ① $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+y, x-y+1)$

$$T(0,0) = (0,1) \neq (0,0)$$

$\therefore$ T is not a linear transf.

② $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x^2, y)$

$$T(2(1,0)) = T(2,0) = (4,0)$$

$$2T(1,0) = 2(1,0) = (2,0)$$

Since, $T(2(1,0)) \neq 2T(1,0)$, T is not a linear transf.

(3) $T : \mathbb{R} \to \mathbb{R}$, $T(x) = mx + c$

T is a linear transf. iff $c = 0$.

(4) Show that every linear transf. $T : \mathbb{R} \to \mathbb{R}$ is of the form $T(x) = mx$ for some $m \in \mathbb{R}$.

Pf: $T(x) = T(x \cdot 1) = x\,T(1) = mx$,

where $m = T(1)$.

Exercise: Show that every lin. transf. $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by $T(X) = AX$, where $A \in M_{m\times n}(\mathbb{R})$.

i.e. $T(x_1, x_2, \cdots, x_n) = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \cdots, a_{m1}x_1 + \cdots + a_{mn}x_n)$

5. $T : V \to W$, $T(v) = 0 \;\; \forall v \in V$ (Zero transf.)

T is a linear transf.

6. $T : V \to V$, $T(v) = v \;\; \forall v \in V$ (Identity transf.)

T is a linear transf.

7. Let $V = \mathbb{R}[x]$ = v. sp. of all real polynomials.

Define $T : V \to V$ by $T(p(x)) = p(x-1)$

Let $u = p(x)$, $v = q(x) \in V$

and $a, b \in \mathbb{R}$.

Then $T(au + bv) = T(a p(x) + b q(x))$

$= a p(x-1) + b q(x-1)$

$= a T(u) + b T(v)$

Null space (or kernel) and range space of a linear transf.

Defn: For a linear transf. $T : V \to W$, the null space (or kernel) of T, $\ker(T) = \{ v \in V : T(v) = 0 \}$.

range space of T, $\text{range}(T) = \{ T(v) : v \in V \}$.

Prop: $\ker(T)$ is a subspace of V and $\text{range}(T)$ is a subspace of W.

Pf: Exercise.

Prop: T is injective (or 1-1) iff $\ker(T) = \{0\}$.

Pf: ($\Rightarrow$) Let $v \in \ker(T)$. Then $T(v) = 0 = T(0)$

$\Rightarrow v = 0$ ($\because T$ is 1-1)

$\therefore \ker(T) = \{0\}$.

($\Leftarrow$) Suppose $T(u) = T(v)$

$\Rightarrow T(u-v) = T(u) - T(v) = 0$

$\Rightarrow u - v \in \ker(T) = \{0\}$

$\Rightarrow u - v = 0 \Rightarrow u = v$

$\therefore T$ is 1-1.

Defn: $\text{nullity}(T) = \dim(\ker(T))$

$\text{rank}(T) = \dim(\text{range}(T))$

Theorem (Rank-nullity theorem): Let V be a finite dimensional vector space and $T: V \to W$ be a linear transf.

Then $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

Proof: Let $\mathcal{B}_1 = \{v_1, v_2, \dots, v_k\}$ be a basis of $\ker(T)$.

(If $\ker(T) = \{0\}$, we take $\mathcal{B}_1 = \phi$)

Then $\mathcal{B}_1$ is a L.I. subset of V.

Extend $\mathcal{B}_1$ to a basis $\mathcal{B} = \{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_r\}$ of V.

Then nullity$(T) = k$, $\dim(V) = k + r$

<u>Claim</u>: $\{T(u_1), T(u_2), \dots, T(u_r)\}$ is a basis of range(T).

Let $w \in$ range(T). Then $w = T(v)$ for some $v \in V$.

Since $\{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_r\}$ is a basis of V,

$$v = a_1 v_1 + a_2 v_2 + \dots + a_k v_k + b_1 u_1 + \dots + b_r u_r,$$

$a_i, b_j \in \mathbb{F}$.

$$\Rightarrow \quad w = T(v) = a_1 \overset{0}{\cancel{T(v_1)}} + \dots + a_k \overset{0}{\cancel{T(v_k)}} + b_1 T(u_1) + \dots + b_r T(u_r)$$

$$\in \operatorname{span}\{T(u_1), \dots, T(u_r)\}.$$

$\therefore \quad \operatorname{span}\{T(u_1), \dots, T(u_r)\} = $ range(T).

Let $\quad c_1 T(u_1) + \cdots + c_r T(u_r) = 0$

$\Rightarrow \quad T(c_1 u_1 + \cdots + c_r u_r) = 0$

$\Rightarrow \quad c_1 u_1 + \cdots + c_r u_r \in \ker(T)$

$\Rightarrow \quad c_1 u_1 + \cdots + c_r u_r = a_1 v_1 + \cdots + a_k v_k$

$\Rightarrow \quad a_1 v_1 + \cdots + a_k v_k - c_1 u_1 - \cdots - c_r u_r = 0$

$\Rightarrow \quad a_1 = 0 = \cdots = a_k = c_1 = \cdots = c_r$ ($\because B$ is L.I.)

$\Rightarrow \quad c_1 = \cdots = c_r = 0$

$\therefore \quad \{T(u_1), \ldots, T(u_r)\}$ is L-I

Hence, $\{T(u_1), \ldots, T(u_r)\}$ is a basis of range(T)

$\Rightarrow \quad \text{rank}(T) = r$

$\therefore \quad \text{rank}(T) + \text{nullity}(T) = r + k = \dim(V)$

Note Title 4/26/2023

Lec - 17

Application of rank-nullity theorem

Theorem: For any $A \in M_{m \times n}(\mathbb{F})$, row rank$(A)$ = col. rank(A).

Pf: Define $T : \mathbb{F}^n \to \mathbb{F}^m$ by $T(X) = AX$.

Then T is a linear transf.

Also, $\ker(T) = \{X \in \mathbb{F}^n : T(X) = 0\}$

$= \{X \in \mathbb{F}^n : AX = 0\}$

= solution space of the system $AX = 0$

$\therefore$ nullity(T) = dim $\ker(T)$ = n − rank(A)

= n − row rank(A).

Now, $\text{range}(T) = \{ T(x) : x \in \mathbb{F}^n \}$

$$= \text{span}\{ T(e_1), T(e_2), \dots, T(e_n) \}$$

$$= \text{span}\{ \text{columns of } A \}.$$

$$= \text{column space}(A)$$

$\therefore \quad \text{rank}(T) = \text{col. rank}(A)$

By the rank-nullity thm.,

$$\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{F}^n = n$$

$$\Rightarrow \quad \text{col. rank}(T) + n - \text{row rank}(T) = n$$

$$\Rightarrow \quad \text{col. rank}(T) = \text{row rank}(T)$$

$$T(e_1) = Ae_1 = A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \text{1st column of } A, \ \dots$$

Thm: Suppose V is a finite dimensional vector space and $T: V \to V$ is a linear transformation. Then T is injective iff T is surjective.

Pf: Just use rank-nullity thm.

Remark: The above result is not true if $\dim V = \infty$.

e.g. Let $V = \mathrm{Maps}(\mathbb{N}, \mathbb{F})$ = the space of all sequences.

Define $T: V \to V$ by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

Then T is a surjective linear transformation but it is not injective.

$$\left(T(1,0,0,0,\ldots) = (0,0,0,\ldots) = T(0,0,0,\ldots) \right)$$

Define $S : V \to V$ by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$$

Then S is injective but not surjective.

Matrix representation of a linear transformation

Let V and W be finite dimensional vector spaces over $\mathbb{F}$.

Let $T : V \to W$ be a linear transf.

Suppose $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ & $\mathcal{B}' = \{w_1, w_2, \ldots, w_m\}$ be two ordered bases of V & W, respectively.

Then $T(v_1) = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$

$T(v_2) = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m$

$\vdots$

$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_n$

$$\therefore \quad [T(v_1)]_{\mathcal{B}'} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ \cdots, \ [T(v_n)]_{\mathcal{B}'} = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

So, we get a matrix $A = (a_{ij})_{m \times n} = [T]_{\mathcal{B}}^{\mathcal{B}'}$

$[T]_{\mathcal{B}}^{\mathcal{B}'}$ is called the matrix of T w.r.t. the bases $\mathcal{B}$ & $\mathcal{B}'$.

<u>Proposition</u>: $[T(v)]_{\mathcal{B}'} = [T]_{\mathcal{B}}^{\mathcal{B}'} [v]_{\mathcal{B}} \quad \forall v \in V$.

<u>Pf</u>: Let $[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

i.e. $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$,

where $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$

$$\therefore \quad T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n).$$

$$= \sum_{i=1}^{n} \alpha_i T(v_i)$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} a_{ji} w_j$$

$$= \sum_{j=1}^{m} \Big(\underbrace{\sum_{i=1}^{n} a_{ji} \alpha_i}_{j\text{th component of } A[v]_{\mathcal{B}}} \Big) w_j$$

$$\therefore \quad [T(v)]_{\mathcal{B}'} = [T]_{\mathcal{B}}^{\mathcal{B}'} [v]_{\mathcal{B}}.$$

Suppose $\mathcal{B}_1$ & $\mathcal{B}_2$ be two ordered bases of V

and $\mathcal{B}_1'$ & $\mathcal{B}_2'$ " " " " " W.

Then $\quad [T(v)]_{\mathcal{B}_1'} = [T]_{\mathcal{B}_1}^{\mathcal{B}_1'} [v]_{\mathcal{B}_1} \quad$ —— ①

and $\quad [T(v)]_{\mathcal{B}_2'} = [T]_{\mathcal{B}_2}^{\mathcal{B}_2'} [v]_{\mathcal{B}_2} \quad$ —— ②

Let P be the change of bases matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$

and Q " " " " " " " $\mathcal{B}_1'$ to $\mathcal{B}_2'$.

$P \to n \times n$, $\quad Q \to m \times m$

$[T]_{\mathcal{B}_1}^{\mathcal{B}_1'}$, $[T]_{\mathcal{B}_2}^{\mathcal{B}_2'} \to m \times n$

$$[v]_{\mathcal{B}_2} = P\,[v]_{\mathcal{B}_1} \quad \forall\, v \in V$$

$$[w]_{\mathcal{B}_2'} = Q\,[w]_{\mathcal{B}_1'} \quad \forall\, w \in W.$$

$$\therefore \quad [T(v)]_{\mathcal{B}_2'} = Q\,[T(v)]_{\mathcal{B}_1'}$$

$$1 \;\&\; 2 \Rightarrow \quad [T]_{\mathcal{B}_2}^{\mathcal{B}_2'}\,[v]_{\mathcal{B}_2} = Q\,[T]_{\mathcal{B}_1}^{\mathcal{B}_1'}\,[v]_{\mathcal{B}_1}$$

$$\Rightarrow \quad [T]_{\mathcal{B}_2}^{\mathcal{B}_2'}\,P\,[v]_{\mathcal{B}_1} = Q\,[T]_{\mathcal{B}_1}^{\mathcal{B}_1'}\,[v]_{\mathcal{B}_1}$$

The above is true for all $v \in V$.

Therefore, $[T]_{\mathcal{B}_2}^{\mathcal{B}_2'}\,P = Q\,[T]_{\mathcal{B}_1}^{\mathcal{B}_1'}$ (why?)

$$\Rightarrow \quad \boxed{[T]_{B_2'}^{B_2'} = Q\,[T]_{B_1'}^{B_1'}\,P^{-1}}$$

Note Title 4/28/2023

Lec - 18

Eigenvalues and eigenvectors

Let V be a vector space over a field $\mathbb{F}$ and let $T : V \to V$ be a linear operator.

A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of T if $\exists\ v \in V \setminus \{0\}$ s.t. $T(v) = \lambda v$.

Also, any nonzero v satisfying $T(v) = \lambda v$ will be called an eigenvector of T corresp. to the eigenvalue of λ.

Note that if v is an eigenvector corresp. to eigenvalue λ, then αv is also an eigenvector corresp. to same eigenvalue λ for any $\alpha \in \mathbb{F}, \alpha \neq 0$.

Eigenspace (λ), $E_\lambda = \{ v \in V : T(v) = \lambda v \}$

$= $ set of all eigenvectors $\cup \{0\}$.

Clearly, eigenspace (λ) is a subspace of V $\forall \lambda \in \mathbb{F}$.

Note that: $E_\lambda = \ker(\lambda I - T)$

- λ is an eigenvalue of T iff $\ker(\lambda I - T) \neq \{0\}$

iff nullity$(\lambda I - T) \neq 0$.

Examples: ① $T : V \to V$, $V \neq \{0\}$, $T(v) = 0 \ \forall v$ (Zero operator)

$\lambda = 0$ is the only eigenvalue of T.

$E_\lambda = V$.

② (Identity operator) $T(v) = v \quad \forall v \in V \setminus \{0\}$.

$\lambda = 1$ is the only eigenvalue of T

$E_\lambda = V$.

③ $T(x, y) = (y, x)$, $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$T(1,1) = (1,1) = 1 \cdot (1,1) \Rightarrow 1$ is an eigenvalue of T.

$T(1,-1) = (-1,1) = -1 \cdot (1,-1) \Rightarrow -1$ is an eigenvalue

Now, suppose $\lambda \in \mathbb{R}$ is an eigenvalue of T.

Then $\exists\ (x, y) \neq (0,0)$ s.t.

$$T(x,y) = \lambda(x,y)$$

$$\Rightarrow \quad (y, x) = (\lambda x, \lambda y)$$

$$\Rightarrow \quad \lambda x = y \text{ and } \lambda y = x$$

Putting $y = \lambda x$ in $x = \lambda y$, we get

$x = \lambda^2 x \Rightarrow$ either $\lambda^2 = 1$ or $x = 0$

But if $x = 0$, then $y = \lambda x = 0$

$\Rightarrow (x, y) = (0,0)$ $\times$

$\therefore \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

(4) $T : \mathrm{Maps}(\mathbb{N}, \mathbb{F}) \to \mathrm{Maps}(\mathbb{N}, \mathbb{F})$

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

For any $\lambda \in \mathbb{F}$,

$$T(1, \lambda, \lambda^2, \lambda^3, \dots) = (\lambda, \lambda^2, \lambda^3, \dots)$$
$$= \lambda(1, \lambda, \lambda^2, \dots)$$

$\Rightarrow$ λ is an eigenvalue of T.

(5) $T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$

$$T(x, y) = (-y, x) \longleftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

T has no eigenvalues

(6) $T : \mathbb{C}^2(\mathbb{C}) \to \mathbb{C}^2(\mathbb{C})$

$$T(x, y) = (-y, x)$$

$\lambda = \pm i$ are the eigenvalues.

Now, suppose V is a finite dimensional vector space over $\mathbb{F}$, say $\dim V = n$.

Let $T : V \to V$ be a linear operator.

Let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be an ordered basis of V.

Then $[T]_{\mathcal{B}}^{\mathcal{B}} \in M_{n \times n}(\mathbb{F})$

Notation: $[T]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}}$.

Suppose $\mathcal{B}'$ is another ordered basis of V.

Then $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P$, for some invertible matrix P.

<u>Defn</u>: Two matrices A and $B \in M_{n\times n}(\mathbb{F})$ are said to be similar if $B = P^{-1}AP$ for some invertible matrix P.

<u>Properties</u>: • Similar matrices have the same trace and determinant.

$$tr(P^{-1}(AP)) = tr((AP)P^{-1}) = tr\,A.$$

$$det(P^{-1}(AP)) = det((AP)P^{-1}) = det\,A.$$

- $[T]_{\mathcal{B}'}$ and $[T]_{\mathcal{B}}$ are similar matrices.
- $[\lambda I - T]_{\mathcal{B}'}$ and $[\lambda I - T]_{\mathcal{B}}$ are similar matrices

Note Title 4/29/2023

Lec - 19

Characteristic polynomial of T :

Let $T : V \to V$ be a linear operator, where V is a finite dimensional vector space.

Then the characteristic polynomial of T is defined as

$$p(x) = \det(xI - T) \quad \leftarrow \text{this is well-defined.}$$

$$= \det(xI - [T]_{\mathcal{B}}), \text{ where } \mathcal{B} \text{ is any ordered basis of } V.$$

Note that if $\dim V = n$, then $p(x)$ is a monic polynomial of degree n. (Monic means coeff of $x^n = 1$)

__Theorem__: For any linear operator T on a __finite dimensional__ vector space V, $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if λ is a root of the char. poly. of T

(i.e. $p(\lambda) = 0$)

__Pf__: $(\Rightarrow)$ λ is an eigenvalue of T

$\Rightarrow \exists\, v \in V,\ v \neq 0$ s.t. $T(v) = \lambda v$.

$\Rightarrow (\lambda I - T)(v) = 0$

$\Rightarrow [(\lambda I - T)(v)]_{\mathcal{B}} = 0$ for any ordered basis $\mathcal{B}$ of V.

$$\Rightarrow \underbrace{[(\lambda I - T)]_{\mathcal{B}}}_{\in M_n(\mathbb{F})} \underbrace{[v]_{\mathcal{B}}}_{\in \mathbb{F}^n} = 0$$

$$\Rightarrow \quad \underbrace{\det\left[(\lambda I - T)\right]_{\mathcal{B}}}_{\det(\lambda I - T)} = 0 \qquad (\because [v]_{\mathcal{B}} \neq 0)$$

$$\Rightarrow \quad p(\lambda) = 0.$$

$(\Leftarrow)$ Suppose $\det(\lambda I - T) = 0$ & $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ is any ordered basis of V.

Then $\det[\lambda I - T]_{\mathcal{B}} = 0$

$\Rightarrow \exists\ X \in \mathbb{F}^n,\ X \neq 0$ s.t. $[\lambda I - T]_{\mathcal{B}} X = 0$

Suppose $X = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

Then if we take $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \neq 0$, then

$$\left[(\lambda I - T)(v)\right]_{\mathcal{B}} = [\lambda I - T]_{\mathcal{B}}[v]_{\mathcal{B}} = [\lambda I - T]_{\mathcal{B}} X = 0$$

$\Rightarrow \quad (\lambda I - T)(v) = 0 \quad \Rightarrow \quad T(v) = \lambda v$

$\Rightarrow \quad \lambda$ is an eigenvalue of T.

Example: $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x+y,\ y+z,\ z+x)$

find the eigenvalues of T and for each eigenvalue find a basis for the eigenspace.

Take basis $\mathcal{B} = \{ (1,0,0), (0,1,0), (0,0,1) \}$ of $\mathbb{R}^3$.

Then $A = [T]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$\therefore$ Char. poly. of T, $p(x) = \det(xI - A) = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & -1 \\ -1 & 0 & x-1 \end{vmatrix}$

$$= (x-1)\ (x-1)^2 - 1$$

$$= (x-2)(x^2 - x + 1)$$

Since $\mathbb{F} = \mathbb{R}$, the only roots of $p(x)$ is $x = 2$.

$\therefore\ \lambda = 2$ is the only eigenvalue of T.

Eigenspace $= \ker(2I - T)$

$$2I - A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = x_3$
$x_2 = x_3$

$\therefore\ \ker(2I - A) = \operatorname{span}\{(1, 1, 1)\}$

$\therefore$ $\{(1,1,1)\}$ is a basis of the eigenspace.

For $A \in M_n(\mathbb{F})$, $\lambda \in \mathbb{F}$ is an eigenvalue of A if $\exists$ $X \in \mathbb{F}^n$, $X \neq 0$ s.t. $AX = \lambda X$.

Char. poly. of A, $p(x) = \det(x\cancel{I} - A)$

The roots of $p(x)$ are precisely the eigenvalues of A.

Char. poly of A (or T), $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$.

Putting $x = 0$, we get $a_0 = p(0) = \det(-A) = (-1)^n \det A$.

$\therefore$ A is invertible iff $a_0 \neq 0$

Suppose we take $\mathbb{F} = \mathbb{C}$, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of $p(x)$, with multiplicity.

i.e. $p(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$

$\Rightarrow \quad x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$

Comparing the coeffs. of x^{n-1} both sides, we get

$$a_{n-1} = -(\lambda_1 + \lambda_2 + \dots + \lambda_n)$$

$\therefore$ Sum of the eigenvalues (with multiplicity) $= -a_{n-1}$

In fact, $\boxed{\text{Sum of eigenvalues} = \operatorname{trace}(A)}$

Also, product of eigenvalues $= \lambda_1 \lambda_2 \cdots \lambda_n$

Putting $x=0$ in $p(x)$, $p(0) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{(-1)^n \det A}$

$\therefore \boxed{\text{Prod. of eigenvalues} = \det A}$

Note Title 5/2/2023

Lec-20

Cayley-Hamilton Theorem :

Suppose $A \in M_n(\mathbb{F})$ (or $T: V \to V$ is a linear operator, where $\dim V < \infty$) and let $p(x)$ be the characteristic polynomial of A (or T). Then $p(A) = 0$ ← zero matrix

(or $p(T) = 0$ ← zero operator)

Note that if $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$$ ← identity matrix

and $$p(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I$$ ← the identity operator on V.

$T^2 = T \circ T$, $T^3 = T \circ T \circ T$, ...

Incorrect proof:

$$p(x) = \det(xI - A) \Rightarrow \underbrace{p(A) = \det(AI - A)}_{\times} = \det(0) = 0$$

Exercise: Prove the Cayley-Hamilton thm for 2×2 matrices.

(Hint: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $p(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix}$

$$= x^2 - (a+d)x + (ad-bc)$$

Then calculate $p(A) = A^2 - (a+d)A + (ad-bc)I$)

The proof for general $n\times n$ matrix is not easy & we'll skip it.

Some applications of Cayley-Hamilton thm.

① **Finding inverse of a matrix :**

$A \in M_n(\mathbb{F})$, $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

We know that A is invertible iff $a_0 \neq 0$.

Suppose $a_0 \neq 0$. By the Cayley-Hamilton thm,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Multiplying by A^{-1}, we get

$$\boxed{A^{-1} = -\frac{1}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I\right)}$$

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\phi(x) = \begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & -1 \\ -1 & -1 & x \end{vmatrix} = (x-1)\left[x(x-1)-1\right]$$

$$= (x-1)\left(x^2-x-1\right) = x^3-2x^2+1$$

$$\therefore \quad \phi(A) = A^3-2A^2+I = 0$$

$$\Rightarrow \quad A^{-1} = 2A - A^2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

② Finding powers of a square matrix:

Suppose we want to find A^{100}.

Let $p(x)$ be the char. poly. of A.

Then by the division algorithm,

$$x^{100} = p(x)\, g(x) + r(x), \text{ where either } r(x) = 0$$
$$\text{or degree } r(x) < n.$$

If we can find $r(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$, then

$$A^{100} = \underbrace{p(A)}_{\substack{\parallel \\ 0}} g(A) + r(A) = r(A).$$

Find A^{100} for the previous example.

$$x^{100} = \underbrace{(x-1)(x^2-x-1)}_{\text{roots are } 1,\ \frac{1\pm\sqrt{5}}{2}} + (b_0 + b_1 x + b_2 x^2)$$

Putting $x=1$, we get $b_0 + b_1 + b_2 = 1$

Putting $x = \frac{1+\sqrt{5}}{2}$, -- -- -- --

$x = \frac{1-\sqrt{5}}{2}$, -- -- -- --

Solve for b_0, b_1, b_2.

$$A^{100} = b_0 I + b_1 A + b_2 A^2 = \text{---}$$

Example. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Find A^{50}.

$$p(x) = \det\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} = x^2+1$$

$$x^{50} = (x^2+1)\,g(x) + (b_0 + b_1 x)$$

Put $x = i$: $\quad b_0 + b_1 i = i^{50} = -1$

Put $x = -i$: $\quad b_0 - b_1 i = (-i)^{50} = -1$

$$\Rightarrow \quad b_0 = -1, \quad b_1 = 0$$

$$\therefore \quad x^{50} = (x^2+1)\,g(x) - 1$$

$$\therefore \quad A^{50} = \underbrace{(A^2+I)}_{\substack{\parallel \\ 0 \ (\text{by C-H})}}\,g(A) - I = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Theorem: Let v_1 and v_2 be eigenvectors a linear operator T with eigenvalues λ_1 and λ_2, where $\lambda_1 \neq \lambda_2$. Then $\{v_1, v_2\}$ is L.I.

Pf: Suppose $c_1 v_1 + c_2 v_2 = 0 \quad \text{—— (i)}$

$$\Rightarrow T(c_1 v_1 + c_2 v_2) = T(0) = 0$$

$$\Rightarrow c_1 T(v_1) + c_2 T(v_2) = 0$$

$$\Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \quad \text{—— (ii)}$$

$\lambda_2 \times (i) - (ii) \Rightarrow c_1(\lambda_2 - \lambda_1) v_1 = 0$

$$\Rightarrow c_1 = 0 \quad (\because v_1 \neq 0 \ \& \ \lambda_2 - \lambda_1 \neq 0)$$

Putting $c_1 = 0$ in (i), $c_2 v_2 = 0 \Rightarrow c_2 = 0$ ($\because v_2 \neq 0$)

$\therefore \{v_1, v_2\}$ is L.I.

Thm: If $v_1, v_2, \ldots, v_k$ are eigenvectors of T with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is L.I.

Pf: Prove using induction and the above thm.

Diagonalizability:

Defn: Let $T: V \to V$ be a linear operator, where $\dim V < \infty$. Then T is said to be diagonalizable if there exists a basis for V

consisting of eigenvectors of T.

i.e. $\exists\ \mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ a basis of V

s.t. each v_i is an eigenvector of T.

(Note that the eigenvalues for these v_i may not be distinct.)

e.g. $T: V \to V$, $T(v) = v \quad \forall v$.

If we any basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$,

then $T(v_i) = v_i \Rightarrow v_i$ is an eigenvector of T with eigenvalue 1.

$\therefore$ The identity operator is diagonalizable.

Similarly, the zero operator is diagonalizable.

Remark: If $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ is a basis of V and $T(v_i) = \lambda_i v_i$ for $i = 1, 2, \dots, n$, then

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} \leftarrow \text{a diagonal matrix.}$$

Note Title 5/3/2023

Lec-21

- A matrix $A \in M_n(\mathbb{F})$ is diagonalizable if $\exists$ a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

Equivalently, A is diagonalizable if A is similar to a diagonal matrix, i.e., $\exists$ an invertible matrix P s.t. $P^{-1}AP = D$, a diagonal matrix.

Note that similar matrices have the same characteristic polynomial and hence the same set of eigenvalues with multiplicity.

if $B = P^{-1}AP$, then
$$\begin{aligned} p_B(x) &= \det(xI - B) \\ &= \det(xI - P^{-1}AP) \\ &= \det(P^{-1}(xI - A)P) \end{aligned}$$

$= \det(xI - A) = p_A(x)$

$\therefore$ If A is diagonalizable, then $\exists$ an invertible matrix P such that $P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A

Q. What is P?

For any $1 \le j \le n$, $P^{-1}AP(e_j) = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} e_j = \lambda_j e_j$; $e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}}$ place

$\Rightarrow \quad A(Pe_j) = P(\lambda_j e_j) = \lambda_j Pe_j$

$\Rightarrow \quad Pe_j$ is an eigenvector of A corresp. to eigenvalue λ_j

$\therefore$ The columns of P are n linearly indep. eigenvectors of A corresp. to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Example: Is $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ diagonalizable?

Char poly, $p(x) = \det(xI - A) = \det\begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix} = x^2$

$\therefore$ Eigenvalues of A: 0 with multiplicity 2

Eigenspace corresp. to $\lambda = 0$: $\ker\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$\therefore$ dim. of eigenspace $= 1 < 2 =$ dim. of $\mathbb{C}^2$

$\therefore$ A is not diagonalizable.

Theorem: Let $\dim V = n$ and $T: V \to V$ be a linear operator.

Then T is diagonalizable if and only if the sum of the dimensions of eigenspaces equals n.

if and only $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$,

where $W_1, W_2, \ldots, W_k$ are eigenspaces corresp. to <u>distinct</u> eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Is A diagonalizable?

$$p(x) = \det \begin{pmatrix} x-1 & 0 & 0 \\ 0 & x-2 & -1 \\ 0 & 0 & x-2 \end{pmatrix} = (x-1)(x-2)^2$$

$\therefore$ Eigenvalues are $\lambda_1 = 1$ with algebraic multiplicity 1

$\lambda_2 = 2$ " " " 2

For $\lambda_1 = 1$: $W_1 = \ker \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \leftarrow$ rank $= 2$

$\dim W_1 = 3 - 2 = 1 \leftarrow$ geom. mult. of λ_1

For $\lambda_2 = 2$: $W_2 = \ker \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow$ rank $= 2$

$\therefore \dim W_2 = 3 - 2 = 1 \leftarrow$ geom. mult. of λ_2.

$\therefore \dim W_1 + \dim W_2 = 1 + 1 = 2 \neq 3$

$\therefore$ A is not diagonalizable

Example:
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$p(x) = (x-1)(x-2)^2$$

$\dim W_1 = 1$

$\dim W_2 = 2 \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \text{rank} = 1$

$\dim W_1 + \dim W_2 = 1 + 2 = 3$

$\therefore$ A is diagonalizable.

Note Title 5/9/2023

Lec - 22

Ordinary Differential Equations (ODEs)

Reference book : Advanced Engineering Mathematics by E. Kreyszig

First order ODEs : A first order ODE is an equation involving an independent variable x, a dependent variable $y(x)$, and the first derivative $y'(x)$ on some open interval $I \subseteq \mathbb{R}$.

Implicit form : $F(x, y, y') = 0$

Explicit form : $y' = f(x, y)$

- A solution to a first order ODE is some function $y = h(x)$ which is differentiable on an interval I and satisfies the given ODE, i.e. $F(x, h(x), h'(x)) = 0 \quad \forall x \in I$

or $h'(x) = f(x, h(x)) \quad \forall x \in I$.

Initial value problem (IVP):

A first IVP is of the form

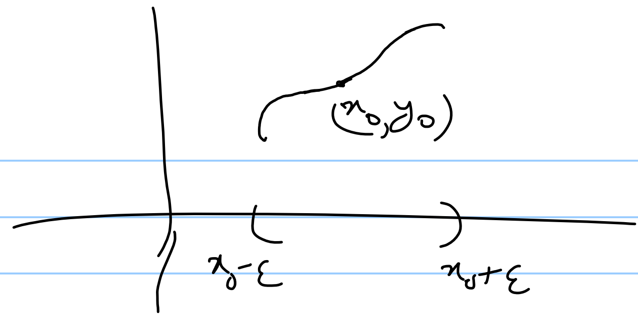
$$y' = f(x, y) \quad ; \quad y(x_0) = y_0 \quad \text{———} \quad (*)$$

Note that the ODE should be defined in some open interval containing x_0.

A soln. to the IVP $(*)$ is a function $y = h(x)$, which is differentiable on some interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$ s.t.

$$h'(x) = f(x, h(x)) \quad \forall x \in I$$

and $h(x_0) = y_0$.



Q: Does every 1st order ODE have a solution?

Ans: No.

$$\left(\frac{dy}{dx}\right)^2 + 1 = 0$$ has no solutions.

The ODE $\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ has $y \equiv 0$ as the only soln.

But, the IVP $\left(\frac{dy}{dx}\right)^2 + y^2 = 0$; $y(0) = y_0$

has no soln. if $y_0 \neq 0$.

Q: Can a 1st order IVP have more than one solutions?

Consider $xy' = y-1$; $y(0)=1$

check that $y = 1 + cx$ is a soln. for any $c \in \mathbb{R}$.

So, we can have an IVP having infinitely many solns.

Theorem 1 (Existence thm. for 1st order IVP):

Consider the IVP : $\dfrac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$.

Suppose $f(x,y)$ is continuous on some closed rectangle

$R = \{ (x,y) \in \mathbb{R}^2 : |x-x_0| \le a , |y-y_0| \le b \}$

$\hookrightarrow$ contains the point (x_0, y_0), $(a>0, b>0)$

Then the IVP has at least one solution $y = h(x)$ defined on some open interval $(x_0 - \alpha, x_0 + \alpha)$

Furthermore, if $|f(x,y)| \le K \quad \forall (x,y) \in R$ for some $K > 0$, then the solution must exist in $(x_0 - \alpha, x_0 + \alpha)$, where $\alpha = \min\{a, \frac{b}{K}\}$

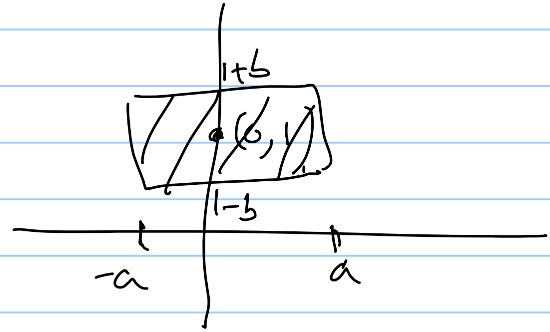
We'll not prove this theorem.

Remarks: ① The conditions in the above thm are sufficient to guarantee the existence of solutions but are not necessary conditions

i.e. the IVP can have solutions even if the function $f(x,y)$ is discontinuous on every rectangle containing (x_0, y_0).

Example: $\quad x y' = y - 1 \quad , \quad y(0) = 1$.

Here $\quad f(x,y) = \dfrac{y-1}{x} \quad$ for $x \neq 0$



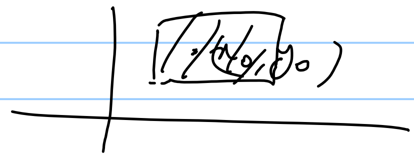
$\lim\limits_{x \to 0} f(x,y)$ does not exist.

$\therefore$ $f(x,y)$ is not continuous on any rectangle R containing $(0,1)$.

$\therefore$ The existence thm fails.

But, we know that the IVP has infinitely many solns. $\quad y = 1 + cx$.

Consider $\quad xy' = y - 1 \quad ; \quad y(x_0) = y_0 \quad$ with $x_0 \neq 0$



$f(x,y) = \dfrac{y-1}{x}$ is cont. on

the rectangle $R = \{ (x,y) : |x - x_0| \le \frac{|x_0|}{2}, |y - y_0| \le b \}$

$\therefore$ Thm. 1 can be used to say the IVP has at least one soln.

<u>Remark</u>: $\alpha = \min\{a, \frac{b}{K}\}$ need not be the best possible for a given IVP.

e.g. $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$

Check that $y = \tan x$ is a soln.

This soln. is defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

What α do we get using Theorem 1?

$f(x,y) = 1 + y^2 \rightarrow$ continuous on $R = \{(x,y) : |x| \le a, |y| \le b\}$ for any $a, b > 0$.

Also, $|f(x, y)| = 1 + y^2 \leq 1 + b^2$ on R.

So, we can take $K = 1 + b^2$

$\therefore \quad \dfrac{b}{K} = \dfrac{b}{1+b^2} \leq \dfrac{1}{2}$

$\therefore \quad \min\left\{a, \dfrac{b}{K}\right\} \leq \dfrac{1}{2}$.

Note Title 5/10/2023

Lec - 23

Existence-uniqueness thm. for 1st order IVP

Defn: A function $f(x,y)$ is said to be Lipschitz in the second variable on a region $R \subseteq \mathbb{R}^2$ if there exists a constant $L>0$ s.t.

$$|f(x,y_1) - f(x,y_2)| \le L\,|y_1 - y_2| \quad \forall (x,y_1), (x,y_2) \in R.$$

Remark: If $\frac{\partial f}{\partial y}$ exists and is bounded on R, then $f(x,y)$ satisfies the above Lipschitz condition on R.

Pf: By the mean value thm,

$$f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(x,\tilde{y})\,(y_1 - y_2) \quad \text{for some } \tilde{y} \text{ between } y_1 \text{ \& } y_2.$$

$\therefore \quad |f(x,y_1) - f(x,y_2)| = \left|\frac{\partial f}{\partial y}(x,\tilde{y})\right| \, |y_1 - y_2|$

$\leq L\,|y_1 - y_2|$

- However, $f(x,y)$ might be Lipschitz in the 2nd variable even if $\frac{\partial f}{\partial y}$ does not exist at all points in R.

e.g. $\quad f(x,y) = x + |\sin y| \quad$ on $\mathbb{R}^2$

$\frac{\partial f}{\partial y}$ does not exist if $y = n\pi$, $n \in \mathbb{Z}$.

But, $\quad |f(x,y_1) - f(x,y_2)| = \big|\,|\sin y_1| - |\sin y_2|\,\big|$

$\leq |\sin y_1 - \sin y_2|$

(by MVT) $\leq |y_1 - y_2|$

i.e. $f(x,y)$ is Lipschitz in the y-variable with $L=1$.

<u>Theorem 2</u> (Uniqueness thm)

Suppose $f(x,y)$ be continuous on a closed rectangle

$R = \{(x,y) \in \mathbb{R}^2 : |x-x_0| \le a, \; |y-y_0| \le b\}$ for some $a>0, b>0$.

Also, suppose $f(x,y)$ is Lipschitz in the y-variable.

Then the IVP: $y' = f(x,y)$; $y(x_0) = y_0$

has a unique solution defined on some interval $(x_0-\alpha, x_0+\alpha)$.

(Note that $\frac{\partial f}{\partial y}$ being continuous on R is enough to ensure the Lipschitz condition)

Remark: If the conditions in the previous ~~theorem~~ fail, we cannot conclude that the uniqueness fails.

Example: Consider $\dfrac{dy}{dx} = 2\sqrt{|y|}$; $y(x_0) = y_0$.

Discuss the existence & uniqueness of soln for different values of (x_0, y_0).

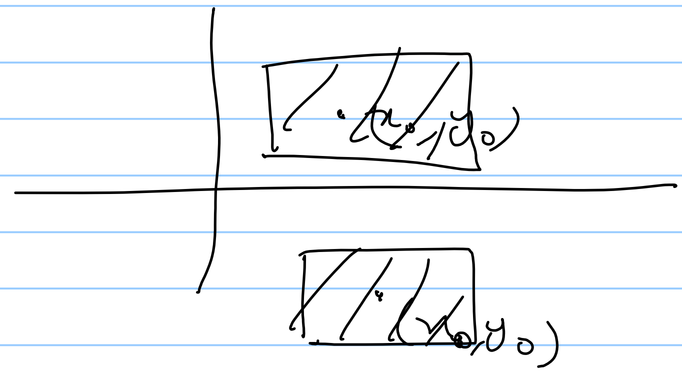
Here, $f(x,y) = 2\sqrt{|y|}$ is continuous on $\mathbb{R}^2$ and hence it is cont. on any rectangle containing (x_0, y_0).

$\therefore$ By the existence thm., the IVP has at least one soln. for any (x_0, y_0).

Note that $f(x,y)$ satisfies the Lipschitz condition on any closed & bounded region which does not intersect with the x-axis $(y=0)$.

$\left[\because \dfrac{\partial f}{\partial y}\right.$ is continuous on those regions$\left.\right)$.

$\therefore$ For any (x_0, y_0) with $y_0 \neq 0$, Theorem 2 guarantees uniqueness of soln.



Ex: Try to find the unique soln. for any $y_0 \neq 0$.

Take $(x_0, y_0) = (0,0)$.

IVP: $\dfrac{dy}{dx} = 2\sqrt{|y|}$, $y(0) = 0$.

Q. Can you find some solutions to the above IVP?

Clearly, $y \equiv 0$ is a solution.

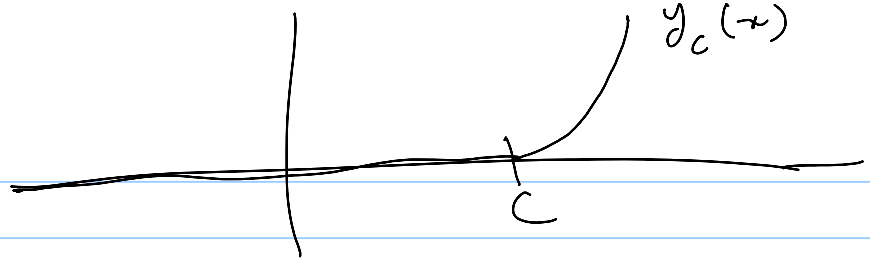
Also, $y = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ is a solution.

$$\frac{dy}{dx} = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases} = 2\sqrt{|y|}.$$

Also, $y = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is also a soln

$y = \begin{cases} 0, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ " " "

$$y_c(x) = \begin{cases} (x-c)^2, & x \geq c \\ 0, & x < c \end{cases} \quad \text{for } c \geq 0$$



Ex: (Q 24 from Tut 4)

IVP : $y \frac{dy}{dx} = x$; $y(0) = \beta$

Find the values of β for which the IVP has

(a) a unique soln.

b) no soln.

(c) more than one soln.

Note Title 5/12/2023

Lec-24

Diagonalizability

Defn: Subspaces $W_1, W_2, \dots, W_k$ of a vector space V are said to be independent if $w_1 + w_2 + \dots + w_k = 0$ for $w_i \in W_i$ implies $w_i = 0 \ \forall i$.

Thm: $W_1, W_2, \dots, W_k$ are independent subspaces

iff $W_1 + W_2 + \dots + W_k = W_1 \oplus W_2 \oplus \dots \oplus W_k$

iff $\dim (W_1 + W_2 + \dots + W_k) = \dim W_1 + \dim W_2 + \dots + \dim W_k$.

Theorem: Suppose $T: V \to V$ is a linear operator on a finite dimensional vector space V. Then the eigenspaces corresp. to distinct eigenvalues are indep. subspaces.

Pf: Let $W_1, W_2, \ldots, W_k$ be eigenspaces corresp. to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Let $\omega_1 + \omega_2 + \cdots + \omega_k = 0$ — (i), where $\omega_i \in W_i$ for $i = 1, 2, \ldots, k$.

$\Rightarrow T(\omega_1 + \omega_2 + \cdots + \omega_k) = T(0) = 0$

$\Rightarrow T\omega_1 + T\omega_2 + \cdots + T\omega_k = 0$

$\Rightarrow \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_k\omega_k = 0$ — (ii)

$\lambda_k \times$ (i) − (ii) $\Rightarrow (\lambda_k - \lambda_1)\omega_1 + \cdots + (\lambda_k - \lambda_{k-1})\omega_{k-1} = 0$

By induction hypothesis, $(\lambda_k - \lambda_i) w_i = 0 \quad \forall i = 1, 2, \dots, k-1$

$\Rightarrow \quad w_i = 0 \quad \left(\because \lambda_k \neq \lambda_i \text{ for } i = 1, 2, \dots, k-1 \right)$

$\forall i = 1, 2, \dots, k-1$

Putting $w_1 = w_2 = \dots = w_{k-1} = 0$ in (i), we get

$w_k = 0$

$\therefore$ We are done by induction.

Theorem: $T : V \to V$ is diagonalizable if and only if $\dim V$ equals the sum of dimensions of each eigenspaces.

Pf: ($\Rightarrow$) T diagonalizable implies $\exists$ a basis $\beta = \{ v_1, v_2, \dots, v_n \}$ consisting of eigenvectors.

If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues, the dim. of eigenspace corresp. to λ_i is equal to the no. of eigenvectors in $\mathcal{B}$ corresp. to the eigenvalue λ_i.

$\therefore$ $\sum_{i=1}^{k} \dim W_i = \dim V$

($\Leftarrow$) Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues and let $W_1, W_2, \dots, W_k$ be the corresponding eigenspaces.

Given: $\dim W_1 + \dim W_2 + \dots + \dim W_k = \dim V$

To prove: T is diagonalizable.

Let $\mathcal{B}_i$ be a basis of eigenspace W_i for $i = 1, 2, \dots, k$.

Take $\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i$

Then $|\mathcal{B}| = \sum_{i=1}^{k} |\mathcal{B}_i| = \sum_{i=1}^{k} \dim W_i = \dim V$.

Since $W_1, W_2, \dots, W_k$ are indep. subspaces,

$\mathcal{B}$ is L.I.

$\therefore$ We got a basis $\mathcal{B}$ of V consisting of eigenvectors.

$\Rightarrow$ T is diagonalizable.

---

e.g. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$p(x) = (x-1)^3$ $\quad \lambda = 1$ is repeated 3 times.

$I - A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow$ rank $= 2 \Rightarrow$ nullity $= 1$

$\therefore$ dim of eigenspace $= 1$.

Note Title 5/13/2023

Lec - 25

Second & higher order linear ODEs

We'll consider nth order linear ODEs of the form

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + p_{n-2}(t)\, y^{(n-2)} + \cdots + p_1(t)\, y' + p_0(t)\, y = g(t),$$

where $y^{(k)}$ denotes the kth derivative of $y(t)$.

Special cases:

① First order linear ODE : $y' + p(t)\, y = g(t)$

You already know how to find the general soln of this.

② Second order linear ODE :

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

Theorem (Existence-uniqueness thm): Consider the linear ODE :

$$y^{(n)} + p_{(n-1)}(t)\,y^{(n-1)} + \dots + p_1(t)\,y' + p_0(t)\,y = g(t).$$

Assume that $p_0(t), p_1(t), \dots, p_{n-1}(t), g(t)$ are continuous on an open interval I. If $t_0 \in I$, then the IVP given by the ODE and initial conditions

$$y(t_0) = y_0, \; y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1}$$

has a unique soln. which is defined on the interval I.

Pf: Omitted.

Homogeneous : if $g(t) = 0$

Non-homogeneous : if $g(t) \neq 0$.

Homogeneous linear ODEs

Theorem: Let $p_0(t), p_1(t), \dots, p_{n-1}(t)$ be continuous on interval I. Then the set of all solutions of the homog. linear ODE

$$y^{(n)} + p_{(n-1)}(t)\, y^{(n-1)} + \dots + p_1(t)\, y' + p_0(t)\, y = 0$$

forms a vector space over $\mathbb{R}$.

Pf: Just check that if $y_1(t)$ and $y_2(t)$ are solutions, then $c_1 y_1(t) + c_2 y_2(t)$ is also a soln. for any $c_1, c_2 \in \mathbb{R}$.

Theorem: Consider the assumptions as in the previous thm. Then the solution space is of dimension n.

Proof: **Step 1**: We'll show that $\exists$ n L.I. solns.

For $k \in \{0,1,2,\cdots,n-1\}$, consider the IVP: $y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0$;

$y^{(k)}(t_0) = 1$, $y^{(j)}(t_0) = 0$ $\forall\, j \neq k,\ j \in \{0,1,\cdots,n-1\}$

(For $n=2$: 2 IVPs

$y'' + p(t)\, y' + q(t)\, y = 0$; $y(t_0)=1$, $y'(t_0)=0$

$y'' + p(t)\, y' + q(t)\, y = 0$; $y(t_0)=0$, $y'(t_0)=1$.

By the existence-uniqueness thm., $\exists$ a soln. $y_k(t)$ for each $k = 0,1,2,\cdots,n-1$.

Claim : $\{ y_0(t), y_1(t), \ldots, y_{n-1}(t) \}$ is L.I.

Suppose $c_0 y_0(t) + c_1 y_1(t) + \cdots + c_{n-1} y_{n-1}(t) = 0$ — $(*)$

Putting $t = t_0$, we get $c_0 = 0$

Diff. $(*)$ & putting $t = t_0$, ~~we~~ get $c_1 = 0$.

Like this, we get $c_k = 0 \quad \forall k = 0, 1, \ldots, n-1$.

Step 2 : $\{ y_0(t), y_1(t), \ldots, y_{n-1}(t) \}$ spans the solution space.

Let $\tilde{y}(t)$ be any solution.

Claim : $\tilde{y}(t) = \underbrace{\tilde{y}(t_0)\, y_0(t) + \tilde{y}'(t_0)\, y_1(t) + \cdots + \tilde{y}^{(n-1)}(t_0)\, y_{n-1}(t)}_{\substack{\| \\ y(t)}}$

Then $Y(t)$ is a solution to the IVP :

$$y^{(n)} + p_{(n-1)}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = 0 \; ;$$

$$y(t_0) = \tilde{y}(t_0) \;,\; y'(t_0) = \tilde{y}'(t_0) \;, \cdots ,\; y^{(n-1)}(t_0) = \tilde{y}^{(n-1)}(t_0) .$$

But, clearly $\tilde{y}(t)$ is also a soln. to the same IVP.

By the uniqueness thm., $\tilde{y}(t) = Y(t) \quad \forall\, t \in I$.

Hence, we are done.

Examples : (1) $y'' - y = 0$

Can you give two L.I. solns ?

Clearly, $y_1(t) = e^{t}$ is a soln.

Also, $y_2(t) = e^{-t}$ is a soln.

Check that $\{e^t, e^{-t}\}$ is L.I.

By the above thm., $\{e^t, e^{-t}\}$ must be a basis of the soln. space

$\therefore$ The general soln is given by $\boxed{y(t) = c_1 e^t + c_2 e^{-t}}$.

(2) Solve : $y'' + y = 0$

$y_1(t) = \cos t$, $y_2(t) = \sin t$ are two L.I. solns.

$$\therefore \boxed{y(t) = c_1 \cos t + c_2 \sin t}$$

Note Title 5/16/2023

Lec-26 :

Wronskian & linear independence

Let $f_1(t), f_2(t), \ldots, f_n(t)$, $t \in I$ be linearly dependent.

Then $\exists\ (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = 0 \quad \forall\, t \in I.$$

Suppose $f_1(t), f_2(t), \ldots, f_n(t)$ are $(n-1)$-times differentiable on I.

Then

$$\left.\begin{aligned} c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) &= 0 \\ c_1 f_1'(t) + c_2 f_2'(t) + \cdots + c_n f_n'(t) &= 0 \\ &\vdots \\ c_1 f_1^{(n-1)}(t) + c_2 f_2^{(n-1)}(t) + \cdots + c_n f_n^{(n-1)}(t) &= 0 \end{aligned}\right\} \forall\, t \in I \quad (*)$$

i.e.
$$\begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & & & \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \forall t \in I.$$

$$\Rightarrow \quad \det \begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ \vdots & & & \\ f_1^{(n-1)}(t) & & - & f_n^{(n-1)}(t) \end{pmatrix} = 0 \quad \forall t \in I$$

Defn: For $(n-1)$-times differentiable functions, $f_1(t), \cdots, f_n(t)$ on I, the Wronskian of $f_1, f_2, \cdots, f_n$ is defined as
$$W(f_1, f_2, \cdots, f_n)(t) = \det \begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ \vdots & & & \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{pmatrix}$$

By the above, we get

Theorem: If $f_1(t), \ldots, f_n(t)$ are linearly dependent on I, where $f_1, f_2, \ldots, f_n$ are $(n-1)$-times differentiable on I, then $W(f_1, f_2, \ldots, f_n)(t) = 0 \quad \forall t \in I$.

Hence, if $W(f_1, f_2, \ldots, f_n)(t_0) \neq 0$ for some $t_0 \in I$, then $f_1(t), \ldots, f_n(t)$ must be L.I. on I.

Example: Show that $\{e^x, \sin x, \cos x\}$ is L.I.

For $n=2$:
$$W(f_1, f_2)(t) = \det\begin{pmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{pmatrix} = f_1(t)f_2'(t) - f_1'(t)f_2(t)$$

<u>Remarks</u> ① The converse of the above thm. is not true, i.e., $W(f_1, f_2, \dots, f_n)(t) = 0 \;\; \forall t \in I$ does not imply that $f_1(t), \dots, f_n(t)$ are L.D. on I.

e.g. $f_1(t) = t^2$, $f_2(t) = t|t|$ on $(-1, 1)$.

$$W(f_1, f_2)(t) = f_1(t) f_2'(t) - f_1'(t) f_2(t)$$
$$= t^2 (2|t|) - 2t\,(t|t|) = 0 \;\; \forall t$$

Let $c_1 f_1(t) + c_2 f_2(t) = 0 \;\; \forall t \in (-1, 1)$

i.e. $c_1 t^2 + c_2 t|t| = 0 \;\; \forall t \in (-1, 1)$

Putting $t = \frac{1}{2}$, $\frac{1}{4} c_1 + \frac{1}{4} c_2 = 0$

Putting $t = -\frac{1}{2}$, $\frac{1}{4} c_1 - \frac{1}{4} c_2 = 0$

$\Rightarrow c_1 = c_2 = 0$.

$\therefore$ $f_1(t)$, $f_2(t)$ are L.I. on $(-1,1)$.

<u>Ex</u>: Suppose $f_1(t)$, $f_2(t)$ are both differentiable on an open interval I. Assume that $f_2(t) \neq 0$ $\forall t \in I$.

Then $W(f_1, f_2)(t) = 0 \Rightarrow f_1(t), f_2(t)$ L.D. on I.

<u>Pf</u>: Let $g(t) = \dfrac{f_1(t)}{f_2(t)}$, $t \in I$ $\leftarrow$ defined since $f_2(t) \neq 0$ $\forall t$.

Then
$$g'(t) = \frac{f_1'(t) f_2(t) - f_1(t) f_2'(t)}{(f_2(t))^2}$$
$$= -\frac{W(f_1, f_2)(t)}{(f_2(t))^2} = 0 \quad \forall t \in I$$

$\Rightarrow$ $g(t) = c$ for some $c \in \mathbb{R}$.

$\Rightarrow \quad f_1(t) = c\, f_2(t) \quad \forall t \in I$

$\Rightarrow \quad f_1(t), f_2(t)$ are L.D. on I.

<u>Method of reduction of order :</u>

Suppose $y_1(t)$ is a solution of 2nd order linear homog. ODE

$$y'' + p(t)\, y' + q(t)\, y = 0,$$

where $p(t)$ & $q(t)$ are continuous on interval I.

Let $y_2(t) = v(t)\, y_1(t)$ be a soln. of the ODE.

Then
$$y_2' = v'(t)\, y_1 + v(t)\, y_1'$$
$$y_2'' = v''(t)\, y_1 + 2 v'(t)\, y_1' + v(t)\, y_1''$$

$$\therefore \quad \left[v''(t)y_1 + 2v'(t)y_1' + v(t)y_1''\right] + p(t)\left[v'(t)y_1 + v(t)y_1'\right] + q(t)v(t)y_1 = 0$$

$$\Rightarrow \quad v''(t)y_1 + v'(t)\left(2y_1' + p(t)y_1\right) + v(t)\underbrace{\left(y_1'' + p(t)y_1' + q(t)y_1\right)}_{=0} = 0$$

This gives a 1st linear ODE in v', ~~whose~~ soln. is

$$v'(t) = c \exp \int \left(-\frac{2y_1'}{y_1} - p(t)\right)dt$$

$$= c \cdot \frac{1}{y_1^2} \exp\left(-\int p(t)\,dt\right)$$

Integrating, we can find $v(t)$.

Example: Consider $y'' - 2y' + y = 0$

Note that $y(t) = e^t$ is a soln.

Assume $y_2(t) = v(t)e^t$ is a soln.

$y_2' = \ldots$

$y_2'' = \ldots$

We'll get

$v''(t) = 0 \Rightarrow v'(t) = c_1$

$\Rightarrow v(t) = c_1 t + c_2$

In particular, we can take $v(t) = t$

$\therefore$ $y(t) = te^t$ is also a soln.

$\therefore$ $\boxed{y = c_1 e^t + c_2 te^t}$.

Note Title 5/17/2023

Lec - 27

Theorem (Abel's theorem): If $y_1(t)$ and $y_2(t)$ are solutions of the ODE: $y'' + p(t)y' + q(t)y = 0$, where $p(t)$ and $q(t)$ are continuous on an interval I, then

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)\,dt\right) \quad \text{for } t \in I$$

for some constant $c \in \mathbb{R}$.

Pf: $W(t) = W(y_1, y_2)(t) = y_1(t)\, y_2'(t) - y_1'(t)\, y_2(t)$.

$$\Rightarrow \quad W'(t) = y_1(t)\, y_2''(t) + \cancel{y_1'(t)\, y_2'(t)} - \cancel{y_1'(t)\, y_2'(t)} - y_1''(t)\, y_2(t)$$

$$= y_1(t)\left[-p(t)\, y_2'(t) - \cancel{q(t)\, y_2(t)}\right] - y_2(t)\left[-p(t)\, y_1'(t) - \cancel{q(t)\, y_1(t)}\right]$$

$$= -p(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]$$

$$\Rightarrow \quad W'(t) = -p(t)\,W(t)$$

$$\Rightarrow \quad \boxed{W(t) = c\exp\left(-\int p(t)\,dt\right)} \text{ for some } c \in \mathbb{R}.$$

<u>Corollary</u>: If $y_1(t)$, $y_2(t)$ are solns. of $y'' + p(t)y' + q(t)y = 0$, $p(t)$, $q(t)$ are cont. on I.

then $W(y_1, y_2)(t)$ is either identically zero on I

or is never zero on I.

<u>Remark</u>: In general the above is not true, e.g.,

$$y_1(t) = t \quad , \quad y_2(t) = t^2$$

$$W(y_1, y_2)(t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$$

$W(t) = 0$ at $t=0$ & $W(t) \neq 0$ if $t \neq 0$.

Theorem: Let $y_1(t)$ & $y_2(t)$ be two solns. of $y'' + p(t)y' + q(t)y = 0$ with $p(t)$ & $q(t)$ continuous on an open interval I, and — $(*)$ let $t_0 \in I$. Then $W(y_1, y_2)(t_0) = 0 \Rightarrow y_1(t)$ & $y_2(t)$ are L.D. on I.

Pf: Since $W(y_1, y_2)(t_0) = 0$, $\exists\ (c_1, c_2) \neq (0,0)$ s.t.

$$c_1 y_1(t_0) + c_2 y_2(t_0) = 0$$

$$\& \quad c_1 y_1'(t_0) + c_2 y_2'(t_0) = 0$$

Claim: $c_1 y_1(t) + c_2 y_2(t) = 0 \quad \forall\ t \in I$.

Let $Y(t) = c_1 y_1(t) + c_2 y_2(t)$, $t \in I$.

Since $y_1(t)$ & $y_2(t)$ are solns. of $(*)$, $Y(t)$ is also a soln. of $(*)$.

Also, $Y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = 0$

$Y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = 0$

$\therefore$ $Y(t)$ is a soln. of $(*)$ with initial conds. $y(t_0)=0, y'(t_0)=0$.

By the uniqueness thm, $Y(t) = 0 \quad \forall t \in I$

ie. $c_1 y_1(t) + c_2 y_2(t) = 0 \quad \forall t \in I$.

2nd order homogeneous linear ODEs with constant coefficients:

Consider ODE of the form $ay'' + by' + cy = 0$, —— $(*)$

where $a, b, c \in \mathbb{R}$, $a \neq 0$.

In order to find the general soln. we need to find two L.I. solns.

Let $\quad y(t) = e^{mt}$

Then $\quad y'(t) = m e^{mt} \quad ; \quad y''(t) = m^2 e^{mt}$.

$\therefore \quad y = e^{mt}$ is a soln. of $(*)$

$\iff \quad a m^2 e^{mt} + b m e^{mt} + c e^{mt} = 0$

$\iff \quad \boxed{a m^2 + b m + c = 0} \quad (\because e^{mt} \neq 0 \ \forall t)$

$\hookrightarrow$ characteristic eqn.

<u>Case I</u>: $\quad a m^2 + b m + c = 0$ has two real & distinct roots m_1 & m_2
$(b^2 - 4ac > 0)$.

Then $\quad y_1 = e^{m_1 t}$, $\quad y_2 = e^{m_2 t}$ are solns. of $(*)$

Also, $\quad y_1(t)$ & $y_2(t)$ are L.I. because $m_1 \neq m_2$.

$\therefore$ The general soln. is $\boxed{y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}}$

Case II: $b^2-4ac=0$ i.e. $am^2+bm+c=0$ has equal roots

$m_1=m_2=\frac{-b}{2a}=m$

$y_1(t)=e^{mt}$ is a soln., where $m=\frac{-b}{2a}$

We use the reduction of order method to find a 2nd soln. as follows.

$$y_2(t)=v(t)\,e^{mt}$$

$$y_2'(t)=e^{mt}v'(t)+me^{mt}v(t)$$

$$y_2''(t)=e^{mt}v''(t)+2me^{mt}v'(t)+m^2e^{mt}v(t)$$

$$\therefore e^{mt}\Big[a\big(v''(t)+2mv'(t)+m^2v(t)\big)+b\big(v'(t)+mv(t)\big)+cv(t)\Big]=0$$

$$\Rightarrow \quad a\,v''(t) + \underbrace{(2am+b)}_{\substack{\shortparallel \\ 0}} v'(t) + \underbrace{(am^2+bm+c)}_{\substack{\shortparallel \\ 0}} v(t) = 0$$

$\Rightarrow \quad v''(t) = 0$. So, let's take $v(t) = t$

$\therefore \quad \boxed{y_2(t) = t e^{mt}}$ is also a soln.

$\therefore$ The general soln. is $\boxed{y(t) = (c_1 + c_2 t) e^{mt}}$

Case III : $b^2 - 4ac < 0$. Then we have two non-real roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Then $e^{m_1 t}$ and $e^{m_2 t}$ are complex-valued solns. of $(*)$

To get real-valued solns., we take

$$y_1(t) = \frac{e^{m_1 t} + e^{m_2 t}}{2} = Re(e^{m_1 t}) = e^{\alpha t}\cos\beta t$$

$$\& \quad y_2(t) = \frac{e^{m_1 t} - e^{m_2 t}}{2i} = Im(e^{m_1 t}) = e^{\alpha t}\sin\beta t.$$

Since $\beta \neq 0$, $y_1(t)$ & $y_2(t)$ are L.I.

$\therefore$ The general soln. is

$$\boxed{y(t) = (c_1\cos\beta t + c_2\sin\beta t)\,e^{\alpha t}}$$

Note Title 5/18/2023

Lec - 28

Euler - Cauchy Equations

Consider the ODE : $at^2y'' + bty' + cy = 0$; where $a, b, c \in \mathbb{R}$, $a \neq 0$.

$(t > 0)$

$$\boxed{t^m = e^{m\ln t} \text{ for } t > 0}$$

Assume $y = t^m$

Then $ty' = mt^m$

$t^2y'' = m(m-1)t^m$

$\therefore \quad (am(m-1) + bm + c)\,t^m = 0$

$\therefore$ If m is a root of $am(m-1) + bm + c = 0$, then t^m is a soln.

char. eqn. for the Euler-Cauchy eqn is

$$\boxed{am(m-1)+bm+c=0}$$

or $$\boxed{am^2+(b-a)m+c=0}$$

Case I: Two real & distinct roots, say m_1 & m_2.

Then $\boxed{y(t) = c_1 t^{m_1} + c_2 t^{m_2}}$ is the general soln.

Case II: $m_1 = m_2 = \dfrac{-(b-a)}{2a} = m$ (say)

$y_1(t) = t^m$

$$y_2(t) = t^m v(t)$$

$$y_2' = t^m v' + m t^{m-1} v$$

$$y_2'' = t^m v'' + 2m t^{m-1} v' + m(m-1) t^{m-2} v$$

$$0 = a t^2 y_2'' + b t y_2' + c y_2$$

$$= a t^{m+2} v'' + 2am t^{m+1} v' + am(m-1) t^m v$$

$$+ b t^{m+1} v' + bm t^m v + c t^m v$$

$$= a t^{m+2} v'' + (2am+b) t^{m+1} v' + \cancelto{0}{(am(m-1)+bm+c)} t^m v$$

$$= a t^{m+2} v'' + a t^{m+1} v' \qquad \left(m = \frac{-b-a}{2a} \quad 2am+b = a \right)$$

$$\Rightarrow \quad t v'' + v' = 0$$

$$\Rightarrow \quad v'' = -\frac{1}{t} v' \quad \Rightarrow \quad v' = \frac{1}{t} \quad \Rightarrow \quad v = \ln t$$

$$\therefore \quad y_2(t) = t^m \ln t$$

$\therefore$ The general soln. is $\boxed{y = (c_1 + c_2 \ln t)\, t^m}$

Case III : $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, where $\beta \neq 0$.

$$t^{m_1} = t^{\alpha + i\beta} = t^{\alpha} \cdot t^{i\beta} = t^{\alpha} e^{i\beta \ln t}$$

$$= t^{\alpha}\left[\cos(\beta \ln t) + i \sin(\beta \ln t)\right]$$

$$y_1 = t^{\alpha} \cos(\beta \ln t)$$

$$y_2 = t^{\alpha} \sin(\beta \ln t)$$

$$\boxed{y = \left[c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)\right] t^{\alpha}} \quad , \ t > 0$$

For $t < 0$, use the substitution $s = -t$

$$\frac{dy}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt} = -\frac{dy}{ds}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{dy}{ds}\right) = \frac{d}{ds}\left(-\frac{dy}{ds}\right) \cdot \frac{ds}{dt} = \frac{d^2y}{ds^2}$$

$\therefore \quad at^2 \frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0 \quad , \ t < 0$

becomes $\quad as^2 \frac{d^2y}{ds^2} + bs\frac{dy}{ds} + cy = 0 \quad , \ s = -t > 0 .$

$\therefore$ We should replace t by $|t|$ in the above formulas to get the general soln. for Euler-Cauchy eqn.

Ex: Solve: $4t^2y'' + 4ty' - y = 0$, $y(-1) = 0$, $y'(-1) = 1$.

Char. eqn. $4m(m-1) + 4m - 1 = 0$

$\Rightarrow 4m^2 - 1 = 0 \Rightarrow m = \pm\frac{1}{2}$

$\therefore$ $y(t) = c_1|t|^{1/2} + c_2|t|^{-1/2} = c_1(-t)^{1/2} + c_2(-t)^{-1/2}$, $t<0$

$0 = y(-1) = c_1 + c_2$

$y'(t) = c_1\left(\frac{1}{2}\right)(-t)^{-1/2}\cdot(-1) + c_2\left(\frac{-1}{2}\right)(-t)^{-3/2}(-1)$

$= -\frac{c_1}{2}(-t)^{-1/2} + \frac{c_2}{2}(-t)^{-3/2}$

$$1 = y'(-1) = -\frac{c_1}{2} + \frac{c_2}{2}$$

$$\therefore \left.\begin{array}{l} c_1 + c_2 = 0 \\ c_2 - c_1 = 2 \end{array}\right\} \Rightarrow c_1 = -1,\ c_2 = 1$$

$$\therefore \boxed{y(t) = -|t|^{1/2} + |t|^{-1/2}}$$

$$c_1 t^2 + c_2 t^3$$

$$(m-2)(m-3) = 0$$

$$m^2 - 5m + 6 = 0$$

$$m(m-1) - 4m + 6 = 0$$

$$t^2 y'' - 4t y' + 6y = 0, \quad y(0) = 0,\ y'(0) = 0$$

$$y = c_1 t^2 + c_2 t^3 \Rightarrow y(0) = 0$$

$$y' = 2c_1 t + 3c_2 t^2 \Rightarrow y'(0) = 0$$

$$y = c_1 t^2 + c_2 t^3 , \quad t \in \mathbb{R}$$ is a soln. for the IVP for any $c_1, c_2 \in \mathbb{R}$.

Remark: The Euler-Cauchy can be converted to a constant coeff. eqn. as follows:

$$a t^2 \frac{d^2y}{dt^2} + b t \frac{dy}{dt} + c y = 0$$

Put $t = e^s$, i.e. $s = \ln t$

$$\frac{dy}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt} = \frac{1}{t} \frac{dy}{ds} \Rightarrow t \frac{dy}{dt} = \frac{dy}{ds}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{t} \frac{dy}{ds} \right) = -\frac{1}{t^2} \frac{dy}{ds} + \frac{1}{t^2} \cdot \frac{d^2y}{ds^2}$$

$$\Rightarrow \quad t^2 \frac{d^2y}{dt} = \frac{d^2y}{ds^2} - \frac{dy}{ds}$$

$$\therefore \quad at^2 \frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0 \quad \text{becomes}$$

$$a\left(\frac{d^2y}{ds^2} - \frac{dy}{ds}\right) + b\frac{dy}{ds} + cy = 0$$

i.e. $\boxed{a\frac{d^2y}{ds^2} + (b-a)\frac{dy}{ds} + cy = 0}$

$$am^2 + (b-a)m + c = 0$$

$$m_1, m_2 \longrightarrow e^{m_1 s}, \quad e^{m_2 s}$$

$$e^{m_1 s} = (e^s)^{m_1} = t^{m_1}$$

Note Title 5/19/2023

Lec - 29

Nonhomogeneous linear ODEs :

We consider $y'' + p(t)y' + q(t)y = r(t)$, $t \in I$,

where $p(t), q(t), r(t)$ are continuous fns. on I.

Theorem: The difference of any two solns. of the nonhomog. linear ODE is a soln. of the corresponding homogeneous ODE
$y'' + p(t)y' + q(t)y = 0$.

Pf: If $y_1'' + p(t)y_1' + q(t)y_1 = r(t)$

& $y_2'' + p(t)y_2' + q(t)y_2 = r(t)$, then

$$(y_1 - y_2)'' + p(t)(y_1 - y_2)' + q(t)(y_1 - y_2) = 0$$

Theorem: The general soln. of the 2nd order nonhomog. linear ODE is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),$$

where $y_1(t)$ & $y_2(t)$ are a pair of L.I. solns. of the homog. ODE and $y_p(t)$ is a particular soln. of the nonhomog. ODE.

Pf: If $y(t)$ is any soln. of $y'' + p(t) y' + q(t) y = r(t)$, then by the prev. thm, $y(t) - y_p(t)$ is a soln. of

$$y'' + p(t) y' + q(t) y = 0$$

$$\therefore \quad y(t) - y_p(t) = c_1 y_1(t) + c_2 y_2(t) \text{ for some } c_1, c_2 \in \mathbb{R}.$$

($\because$ $\{y_1(t), y_2(t)\}$ is a basis for the soln. space of homog.)

$\Rightarrow \quad y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

Q: How to find a particular soln. $y_p(t)$?

Variation of parameters method :

Suppose $y_1(t)$ and $y_2(t)$ are two L.I. solns. of the homog. linear ODE $\quad y'' + p(t)\, y' + q(t)\, y = 0$.

Consider the nonhomog. ODE $\quad y'' + p(t)\, y' + q(t)\, y = r(t) \quad — (*)$

Assume $\quad y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$

Then $\quad y_p'(t) = u_1'(t)\, y_1(t) + u_1(t)\, y_1'(t) + u_2'(t)\, y_2(t) + u_2(t)\, y_2'(t)$

& $\quad y_p'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''$

Now, y_p is a soln. of (*) if

$$u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''$$

$$+ p(t)\left[u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'\right] + q(t)\left[u_1 y_1 + u_2 y_2\right] = r(t)$$

$$\Leftrightarrow \quad y_1 u_1'' + y_2 u_2'' + 2y_1' u_1' + p(t) y_1 u_1' + 2 y_2' u_2' + p(t) y_2 u_2'$$

$$+ u_1 \Big(\underbrace{y_1'' + p(t) y_1' + q(t) y_1}_{=0} \Big) + u_2 \Big(\underbrace{y_2'' + p(t) y_2' + q(t) y_2}_{=0} \Big) = r(t)$$

$$\Leftrightarrow \quad y_1 u_1'' + y_2 u_2'' + \qquad = r(t)$$

We impose one more condition

$$\boxed{y_1 u_1' + y_2 u_2' = 0} \qquad \text{—— (i)}$$

Then $\quad y_p' = u_1 y_1' + u_2 y_2'$

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

$\therefore \quad y_p(t)$ is a soln. if

$$\left(u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'\right) + p(t)\left(u_1 y_1' + u_2 y_2'\right) + q(t)\left(u_1 y_1 + u_2 y_2\right) = r(t)$$

$$\Leftrightarrow \left[\underbrace{y_1'' + p(t) y_1' + q(t) y_1}_{0}\right] u_1 + \left[\overbrace{y_2'' + p(t) y_2' + q(t) y_2}^{0}\right] u_2 + y_1' u_1' + y_2' u_2' = r(t)$$

$$\Leftrightarrow \quad \boxed{y_1' u_1' + y_2' u_2' = r(t)} \quad \text{—— (i)}$$

So, if we can find $u_1(t)$ & $u_2(t)$ s.t. (i) & (ii) are satisfied simultaneously, then $y_p = y_1 u_1 + y_2 u_2$ is a soln. of $(*)$.

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = r(t) \end{cases}$$

$$\Leftrightarrow \quad \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r(t) \end{pmatrix} \quad \text{——} \quad (**)$$

Since $\det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = W(y_1, y_2)(t) \neq 0 \quad \forall t \in I$,

$(**)$ has a unique soln. (u_1', u_2') given by

$$u_1' = \frac{-r(t)}{W(y_1,y_2)(t)} y_2(t)$$

$$u_2' = \frac{r(t)}{W(y_1,y_2)(t)} y_1(t)$$

Integrating this, we can find $u_1(t)$, $u_2(t)$.

$$y_p(t) = y_1(t) \int \frac{-y_2(t)\, r(t)}{W(y_1,y_2)(t)}\, dt + y_2(t) \int \frac{y_1(t)\, r(t)}{W(y_1,y_2)(t)}\, dt$$

Example: Solve : $y'' + y = \tan t$

$y_1(t) = \cos t$, $y_2(t) = \sin t$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$u_1'(t) = \frac{-y_2(t)\, r(t)}{W(y_1, y_2)(t)} = -\sin t \tan t$$

$$= -\frac{\sin^2 t}{\cos t} = \frac{\cos^2 t - 1}{\cos t}$$

$$= \cos t - \sec t$$

$$\Rightarrow \quad u_1(t) = \sin t - \ln|\sec t + \tan t|$$

$$u_2'(t) = \frac{y_1(t)\, r(t)}{W(y_1, y_2)(t)} = \cos t \tan t = \sin t$$

$$u_2(t) = -\cos t$$

$$y_p(t) = y_1(t)\, u_1(t) + y_2(t)\, u_2(t)$$

$$= \cos t\left[\cancel{\sin t} - \ln|\sec t + \tan t|\right] - \cancel{\cos t \sin t}$$

$$= -\cos t \ln|\sec t + \tan t|$$

i.e. $\boxed{y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln|\sec t + \tan t|}$

Lec-30

Note Title 5/20/2023

Variation of parameters method :

$$y'' + p(t)\,y' + q(t)\,y = r(t) \quad \text{——} (*)$$

Let y_1 & y_2 be two L.I. solns. of $y'' + p(t)y' + q(t)y = 0$.

Assume $y_p = u_1 y_1 + u_2 y_2$

Then $y_p' = (u_1 y_1' + u_2 y_2') + (u_1' y_1 + u_2' y_2)$

We put one condition : $y_1 u_1' + y_2 u_2' = 0$ —— (i)

$\therefore \quad y_p' = y_1' u_1 + y_2' u_2$

$\Rightarrow \quad y_p'' = y_1'' u_1 + y_2'' u_2 + y_1' u_1' + y_2' u_2'$

y_p is a soln. of $(*)$ if (i) is satisfied and

$$y_1'' u_1 + y_2'' u_2 + y_1' u_1' + y_2' u_2' + p(t)\left[y_1' u_1 + y_2' u_2\right] + q(t)\left[y_1 u_1 + y_2 u_2\right] = r(t)$$

$$\Rightarrow \quad y_1' u_1' + y_2' u_2' = r(t) \quad \text{—— (ii)}$$

$$\therefore \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

Since $W(y_1, y_2)(t) \neq 0 \quad \forall t \in I$, we get (by the Cramer's rule)

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ r(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{W_1(t)}{W(t)} r(t)$$

and $u_2'(t) = \dfrac{W_2(t)}{W(t)}\, r(t)$,

where $W_1(t) = \begin{vmatrix} 0 & y_2(t) \\ 1 & y_2'(t) \end{vmatrix}$ & $W_2 = \begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & 1 \end{vmatrix}$

Remark: For nth order, $y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y' + p_n(t)\, y = r(t)$.

- $y_1, y_2, \cdots, y_n$ n L.I. solns of the homog.
- $y_p = y_1 u_1 + y_2 u_2 + \cdots + y_n u_n$,

where
$$\begin{aligned}
y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0 \\
y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0 \\
&\vdots \\
y_1^{(n-2)} u_1' + \cdots + y_n^{(n-2)} u_n' &= 0 \\
y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} u_n' &= r(t)
\end{aligned}$$

i.e.
$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r(t) \end{pmatrix}$$

$$\therefore \quad u_i'(t) = \frac{W_i(t)}{W(t)} r(t), \quad i = 1, 2, \cdots, n,$$

where
$$\begin{array}{l} W_i(t) = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y_1' & & 0 & & y_n' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & & 1 & & y_n^{(n-1)} \end{vmatrix} \\ \qquad\qquad\qquad \uparrow \text{ ith column} \end{array}$$

Remark: To find a particular soln. by variation of parameters method for eqn. of the form

$$at^2y'' + bty' + cy = g(t), \quad a \neq 0$$

$$\Leftrightarrow \quad y'' + \frac{b}{at}y' + \frac{c}{at^2}y = \frac{g(t)}{at^2} = r(t)$$

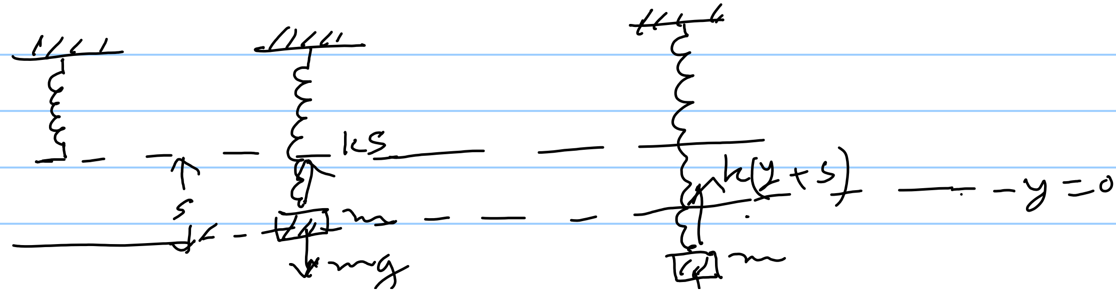
i.e. In the formula $u_i'(t) = \frac{W_i(t)}{W(t)} r(t)$, $i = 1, 2$,

we have to use $\boxed{r(t) = \frac{g(t)}{at^2}}$ & not $g(t)$

Application of 2nd order ODE with constant coefficients

- Mass-spring system

Undamped oscillation



Equilibrium : $mg = ks$

$\downarrow mg$

$F = mg - k(y+s) = -ky$

i.e. $m\dfrac{d^2y}{dt^2} + ky = 0$

i.e. $y'' + \omega^2 y = 0$; where $\omega = \sqrt{\dfrac{k}{m}}$ $\leftarrow$ angular frequency.

Soln: $\boxed{y = A\cos\omega t + B\sin\omega t}$

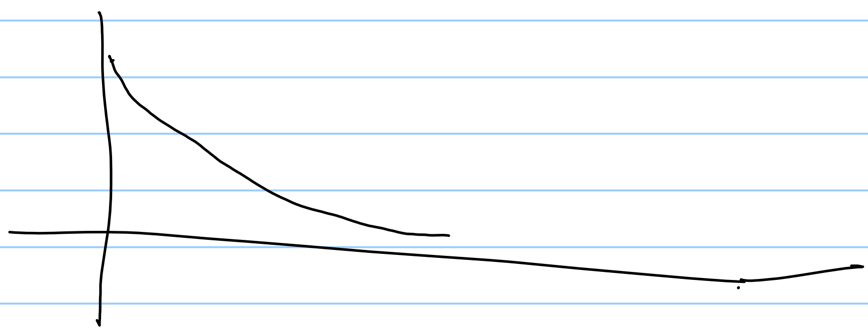
<u>Damped oscillation</u>: $my'' + by' + ky = 0$, $m>0$, $b>0$, $k>0$

Char eqn : $mx^2 + bx + k = 0$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

Case I : (Overdamping) $b^2 > 4mk$, $x = -\alpha, -\beta$

$y = c_1 e^{-\alpha t} + c_2 e^{-\beta t}$; where $\alpha, \beta > 0$



Case II : Critically damped case

$b^2 - 4mk = 0$

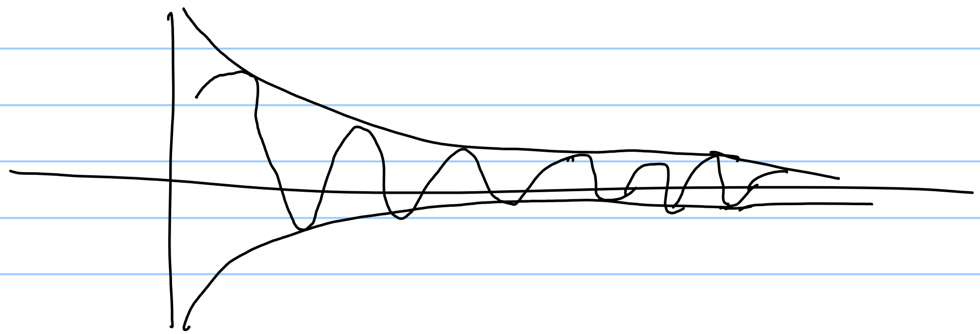
$$y = (c_1 + c_2 t) e^{-\frac{b}{2m} t}$$

Case III : (Underdamped)

$b^2 - 4mk < 0$

$r_{1,2} = -\lambda \pm i\mu \quad , \quad \lambda > 0$

$$y = (c_1 \cos \mu t + c_2 \sin \mu t) e^{-\lambda t}$$



Forced vibration :

$$my'' + by' + ky = F_0(t)$$

$$y'' + \omega^2 y = a\cos(\omega_0 t)$$

$$y_h(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

$$y_p(t) = \begin{cases} A\cos \omega t + B\sin \omega t & \text{if } \omega_0 \neq \omega \\ At\cos \omega t + Bt\sin \omega t & \text{if } \omega_0 = \omega \end{cases}$$

Note Title 5/23/2023

Lec - 31

Laplace Transform :

Defn : For a function $f(t)$ defined on $[0,\infty)$, we define the Laplace transform of f as

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$, whenever the improper integral converges.

$$= \lim_{b \to +\infty} \int_0^b e^{-st} f(t)\, dt .$$

Examples: ① $f(t) = 1 \quad \forall t$

$$F(s) = \int_0^\infty e^{-st}\, dt = \frac{1}{s} \quad \text{if } s > 0 .$$

(2) $f(t) = e^{at}$ for ~~some~~ $a \in \mathbb{R}$.

$$F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a} \quad \text{if } s>a$$

$$\boxed{\mathcal{L}(e^{at})(s) = \frac{1}{s-a}, \quad s>a}$$

(3) $f(t) = t^n$, $n \in \mathbb{N}$.

$$F(s) = \int_0^\infty t^n e^{-st}\,dt = \underbrace{t^n \frac{e^{-st}}{-s}\Big|_0^\infty}_{\substack{\parallel \\ 0 \text{ if } s>0}} - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\,dt$$

$$= \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\,dt$$

$\therefore \quad \mathcal{L}(t^n)(s) = \frac{n}{s}\mathcal{L}(t^{n-1})(s) \quad \text{if } s>0$

$\therefore \quad \mathcal{L}(t) = \frac{1}{s}\mathcal{L}(1)(s) = \frac{1}{s^2}, \quad s>0$

In general, $\boxed{\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}, \quad s>0}$

<u>Prop</u>: (Linearity of Laplace transform):

$$\mathcal{L}(af(t)+bg(t))(s) = a\mathcal{L}(f(t))(s) + b\mathcal{L}(g(t))(s).$$

<u>Pf</u>: Easy to verify.

Using the linearity, we can find the Laplace transform of any polynomial.

(4) $\cosh(at) = \dfrac{e^{at} + e^{-at}}{2}$

$\sinh(at) = \dfrac{e^{at} - e^{-at}}{2}$

$$\mathcal{L}(\cosh(at))(s) = \frac{1}{2}\mathcal{L}(e^{at})(s) + \frac{1}{2}\mathcal{L}(e^{-at})(s).$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right], \text{ if } s > a \;\&\; s > -a$$

$$\therefore \boxed{\mathcal{L}(\cosh(at))(s) = \frac{s}{s^2 - a^2}, \quad s > |a|}$$

$$\boxed{\mathcal{L}(\sinh(at))(s) = \frac{a}{s^2 - a^2}, \quad s > |a|}$$

5

$$f(t) = \cos(\omega t)$$

$$F(s) = \int_0^\infty e^{-st}\cos(\omega t)\,dt$$

$$= \frac{e^{-st}}{-s}\cos\omega t\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{s}\,\omega\sin(\omega t)\,dt$$

$$= \frac{1}{s} - \frac{\omega}{s}\left[\underbrace{\frac{e^{-st}}{-s}\sin\omega t\Big|_0^\infty}_{=0} - \int_0^\infty \frac{e^{-st}}{-s}\,\omega\cos\omega t\,dt\right]$$

$$= \frac{1}{s} - \frac{\omega^2}{s^2}F(s)$$

$$\left(1 + \frac{\omega^2}{s^2}\right)F(s) = \frac{1}{s} \Rightarrow \boxed{F(s) = \frac{s}{s^2+\omega^2}, \quad s > 0}$$

(6) $\boxed{\mathcal{L}(\sin(\omega t))(s) = \dfrac{\omega}{s^2+\omega^2}, \quad s>0}$

Existence of Laplace transform

Thm: Suppose $f(t)$, $t \geq 0$ is a piecewise continuous function and suppose $\exists\, M>0$, $k>0$

$$|f(t)| \leq M e^{kt} \quad \forall t \geq 0$$

Then $\mathcal{L}(f)(s)$ exists for $s>k$.

Proof:
$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\left| \int_0^\infty e^{-st} f(t)\,dt \right| \le \int_0^\infty e^{-st} |f(t)|\,dt$$

$$\le \int_0^\infty e^{-st} M e^{kt}\,dt = M \int_0^\infty e^{-(s-k)t}\,dt$$

$$= \frac{M}{s-k} \quad \text{if } s>k$$

$$< \infty$$

$\therefore$ $F(s)$ is defined $\forall\, s > k$.

Remark: ① For $f(t) = e^{t^2}$ the above condition is not satisfied.

② The condition in the above theorem is only a sufficient cond. for the existence of Laplace transform but it is not a necessary condition.

e.g. $f(t) = \frac{1}{\sqrt{t}}$ does not satisfy the condition.

Try to show that $\mathcal{L}(f)(s)$ exists.

Lec 32

Note Title 5/24/2023

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = ?$$

Let $f(t) = t^a$, $t \geq 0$, where $a \in \mathbb{R}$.

Then $F(s) = \int_0^\infty t^a e^{-st}\, dt$

Put $st = u$

$\Rightarrow dt = \frac{du}{s}$

$$= \int_0^\infty \left(\frac{u}{s}\right)^a e^{-u} \frac{du}{s}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty u^a e^{-u}\, du$$

$$= \frac{1}{s^{a+1}} \Gamma(a+1) \quad \text{if } a > -1$$

$\therefore \quad \mathcal{L}(t^a)$ exists if and only if $a > -1$.

$$\boxed{\mathcal{L}(t^a)(s) = \frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0}$$

Putting $a = n \in \mathbb{N}$, we get $\mathcal{L}(t^n)(s) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$, which we already derived.

Putting $a = -1/2$,

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right)(s) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \sqrt{\frac{\pi}{s}} \qquad \left(\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right)$$

Similarly,

$$\mathcal{L}(\sqrt{t})(s) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s\sqrt{s}}.$$

Some more properties:

s-shifting: Let $F(s) = \mathcal{L}(f(t))(s)$.

Then $F(s-a) = ?$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \quad F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$$

$$= \mathcal{L}\left(e^{at} f(t)\right)(s).$$

$$\Rightarrow \quad \boxed{\mathcal{L}\left(e^{at} f(t)\right)(s) = F(s-a)}$$

$$\Rightarrow \quad \boxed{\mathcal{L}^{-1}\left(F(s-a)\right) = e^{at} f(t), \text{ where } F(s) = \mathcal{L}(f(t))(s)}$$

Examples : (1) $\mathcal{L}\left(e^{2t}t^2\right) = ?$

$f(t) = t^2 \Rightarrow F(s) = \frac{2}{s^3}$

$\therefore \mathcal{L}\left(e^{2t}t^2\right) = F(s-2) = \frac{2}{(s-2)^3}$.

(2) $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$

Take $F(s) = \frac{1}{s^2+2^2}$. Then $f(t) = \mathcal{L}^{-1}(F(s)) = \frac{\sin(2t)}{2}$

$\therefore \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+2^2}\right) = \mathcal{L}^{-1}\left(F(s+1)\right) = e^{-t}f(t)$

$= \frac{1}{2}e^{-t}\sin(2t)$.

Derivatives of Laplace transform:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$F'(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt = \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$= \mathcal{L}\left(-t f(t)\right)(s)$$

$$\Rightarrow \boxed{\mathcal{L}\left(t f(t)\right)(s) = -F'(s)}$$

$$\Rightarrow \boxed{\mathcal{L}\left(t^n f(t)\right)(s) = (-1)^n F^{(n)}(s)} \quad \text{for } n \in \mathbb{N}.$$

e.g. $\mathcal{L}(t\cos\omega t) = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+\omega^2}\right)$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

- <u>Laplace transform of derivatives</u>

$$\mathcal{L}(f'(t))(s) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + \int_0^\infty s e^{-st} f(t)\,dt$$

$$= -f(0) + s F(s)$$

$$\therefore \quad \boxed{\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)}$$

$$\mathcal{L}(f'')(s) = s\mathcal{L}(f')(s) - f'(0)$$

$$= s\left[s\mathcal{L}(f)(s) - f(0)\right] - f'(0)$$

$$\boxed{\mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)}$$

$$\boxed{\mathcal{L}(f^{(n)})(s) = s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

- <u>Laplace transform of integrals</u>:

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = ?$$

Let $g(t) = \int_0^t f(\tau)\,d\tau$

Then $g'(t) = f(t)$ and $g(0) = 0$

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}(g')(s) = s\,\mathcal{L}(g)(s) - \cancel{g(0)}^{\,0}$$

$$\Rightarrow \quad \mathcal{L}(g)(s) = \frac{F(s)}{s}$$

$$\Rightarrow \quad \boxed{\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}}$$

e.g. $$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+\omega^2)}\right) = ?$$

$$F(s) = \frac{1}{s^2+\omega^2} \Rightarrow f(t) = \frac{1}{\omega}\sin(\omega t)$$

$$\therefore \quad \mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)\,d\tau = \int_0^t \frac{1}{\omega}\sin(\omega\tau)\,d\tau$$

$$= \frac{1}{\omega^2}\left[1 - \cos(\omega t)\right].$$

Using Laplace transform to solve 2nd (or higher) order IVPs.

e.g. Solve : $y'' - y = t$, $y(0) = 1$, $y'(0) = 1$.

Taking $\mathcal{L}$:

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(t)$$

$$\Rightarrow \quad s^2 \mathcal{L}(y) - s\, y(0) - y'(0) - \mathcal{L}(y) = \frac{1}{s^2}$$

$$\Rightarrow \quad s^2 Y(s) - s - 1 - Y(s) = \frac{1}{s^2}$$

$$\Rightarrow \quad (s^2 - 1) Y(s) = \frac{1}{s^2} + s + 1$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2(s^2-1)} + \frac{s+1}{s^2-1}$$

$$= \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) + \frac{1}{s-1}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= \sinh(t) - t + e^{t}$$

$$= \frac{e^{t} - e^{-t}}{2} - t + e^{t} = \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} - t$$

Lec 33

Shifted data problem :

If for an IVP, the initial conditions are given at $t = t_0$ instead of $t = 0$, we use the change of variable

$$\tau = t - t_0 \quad \text{i.e.} \quad t = \tau + t_0$$

Put $\tilde{y}(\tau) = y(t) = y(\tau + t_0)$

Then $\tilde{y}'(\tau) = y'(t)$ and $\tilde{y}''(\tau) = y''(t)$

e.g. Solve $y'' + y = 2t$, $y(\frac{\pi}{4}) = \frac{\pi}{2}$, $y'(\frac{\pi}{4}) = 2 - \sqrt{2}$

By the above change of variable : $t = \tau + \frac{\pi}{4}$,

$$\tilde{y}'' + \tilde{y} = 2\left(\tau + \frac{\pi}{4}\right), \quad \tilde{y}(0) = \frac{\pi}{2}, \quad \tilde{y}'(0) = 2 - \sqrt{2}$$

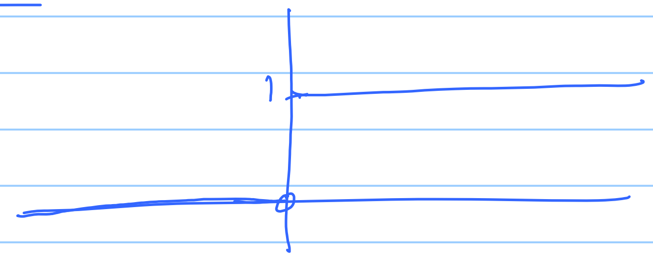
Taking the Laplace transform, we get

$$s^2 \tilde{Y}(s) - \frac{\pi}{2} s - (2 - \sqrt{2}) + \tilde{Y}(s) = 2\left(\frac{1}{s^2} + \frac{\pi}{4s}\right)$$

Solve for $\tilde{Y}(s)$ and then take $\mathcal{L}^{-1}$ to get $\tilde{y}(\tau)$. Then $y(t) = \tilde{y}\left(t - \frac{\pi}{4}\right)$.

Heaviside function (unit step function)

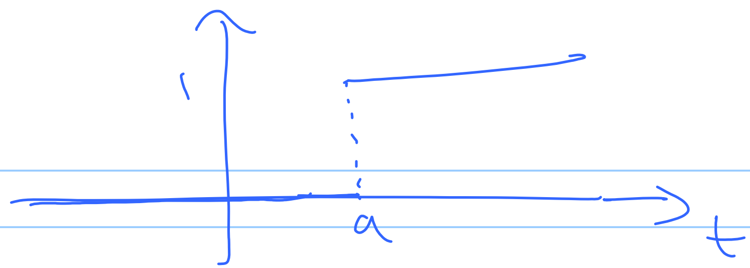
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Let $a > 0$.

$$u_a(t) = \begin{cases} 0 & , t < a \\ 1 & , t \geq a \end{cases}$$

$$= u(t-a)$$



$$\mathcal{L}(u_a(t)) = ?$$

$$\mathcal{L}(u_a(t)) = \int_0^\infty e^{-st} u_a(t)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \frac{e^{-as}}{s} \quad \text{if } s > 0$$

$$\boxed{\mathcal{L}(u_a(t))(s) = \frac{e^{-as}}{s} \quad , \quad s > 0}$$

Let $\tilde{f}(t) = f(t-a)\,u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t \geq a \end{cases}$

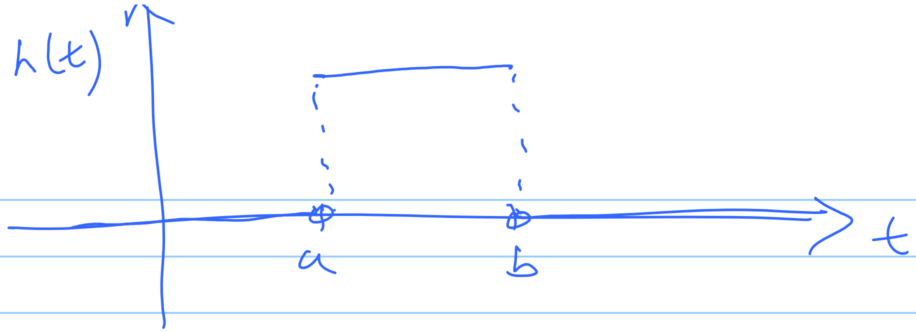
Then $\mathcal{L}(\tilde{f})(s) = \displaystyle\int_a^\infty e^{-st} f(t-a)\,dt$

$$= e^{-as} \int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$ Putting $t-a=\tau$

$$= e^{-as}\mathcal{L}(f)(s)$$

$\therefore$ $\boxed{\mathcal{L}\left(f(t-a)\,u(t-a)\right)(s) = e^{-as}\mathcal{L}(f)(s)}$

i.e. $\boxed{\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)\,u(t-a)}$



$$h(t) = \begin{cases} 1 & \text{if } a \le t < b \\ 0 & \text{otherwise} \end{cases}$$

Note that $h(t) = u(t-a) - u(t-b)$

e.g. Let $f(t) = \begin{cases} 2 & , \ t < 1 \\ t^2/2 & , \ 1 < t < \frac{\pi}{2} \\ \cos t & , \ t > \frac{\pi}{2} \end{cases}$

Find $\mathcal{L}(f)$.

Soln:

$$f(t) = 2\left[1 - u(t-1)\right] + \frac{t^2}{2}\left[u(t-1) - u\left(t-\frac{\pi}{2}\right)\right] + \cos t \; u\left(t-\frac{\pi}{2}\right).$$

$$\therefore \quad \mathcal{L}(f)(s) = 2\left(\mathcal{L}(1) - \mathcal{L}(u(t-1))\right] + \mathcal{L}\left(\frac{t^2}{2}u(t-1)\right) - \mathcal{L}\left(\frac{t^2}{2}u\left(t-\frac{\pi}{2}\right)\right] + \mathcal{L}\left(\cos t \; u\left(t-\frac{\pi}{2}\right)\right)$$

$$\mathcal{L}\left(t^2 u(t-1)\right) = \mathcal{L}\left[(t-1)^2 u(t-1) + 2(t-1)u(t-1) + u(t-1)\right]$$

$$= \frac{2}{s^3}e^{-s} + \frac{2}{s^2}e^{-s} + \frac{e^{-s}}{s}.$$

(Another way: $\mathcal{L}\left(t^2 u(t-1)\right) = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{e^{-s}}{s}\right)$

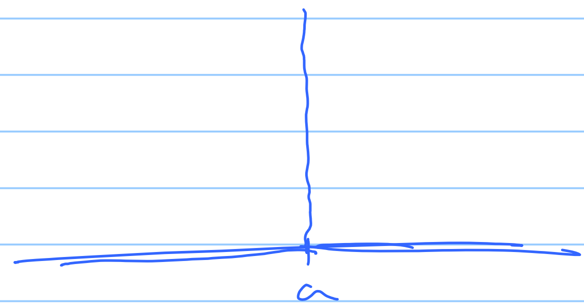
$$= \frac{d}{ds}\left(-\frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right) = \frac{2}{s^3}e^{-s} + \frac{2}{s^2}e^{-s} + \frac{e^{-s}}{s}.\ )$$

$$\mathcal{L}\left(\cos t\, u\left(t-\frac{\pi}{2}\right)\right) = \mathcal{L}\left(-\sin\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right)\right)$$

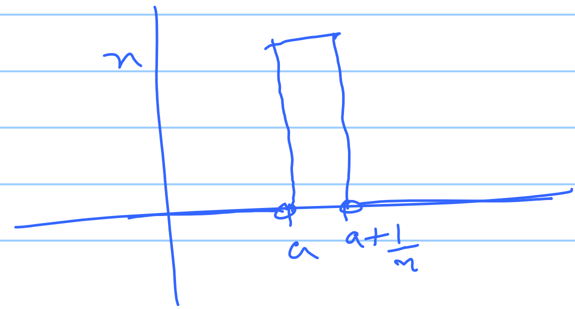
$$= -e^{-\frac{\pi}{2}s}\cdot\frac{1}{s^2+1}$$

Dirac delta "function"

$$\delta(t-a) = \begin{cases} 0 & \text{if } t \neq a \\ \infty & \text{if } t = a \end{cases}$$



Let $$f_n(t-a) = \begin{cases} n & \text{if } a \le t \le a+\frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$



Then $$\lim_{n\to\infty} f_n(t-a) = \delta(t-a)$$

Now, $f_n(t-a) = n\left[u(t-a) - u\left(t-a-\frac{1}{n}\right)\right]$

$$\mathcal{L}\left(f_n(t-a)\right) = n\left[\frac{e^{-as}}{s} - \frac{e^{-\left(a+\frac{1}{n}\right)s}}{s}\right]$$

$$= e^{-as}\left(\frac{1-e^{-s/n}}{s/n}\right)$$

$$\therefore \lim_{n\to\infty} \mathcal{L}\left(f_n(t-a)\right) = e^{-as} \lim_{n\to\infty}\left(\frac{1-e^{-s/n}}{s/n}\right) = e^{-as}.$$

$$\therefore \boxed{\mathcal{L}\left(\delta(t-a)\right) = e^{-as}}$$

Properties of Dirac delta fn :

① $\int_{0}^{\infty} \delta(t-a)\, dt = 1$

② $\int_{0}^{\infty} g(t)\, \delta(t-a)\, dt = g(a)$.

Lec 34

Note Title 5/31/2023

Example of IVP involving Dirac delta:

Solve: $y'' + 3y' + 2y = \delta(t-1)$; $y(0) = 0$, $y'(0) = 0$.

Taking the Laplace transform, we get

$$s^2 Y(s) + 3s Y(s) + 2 Y(s) = \mathcal{L}(\delta(t-1)) = e^{-s}$$

$$\Rightarrow \quad Y(s) = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = \frac{e^{-s}}{s+1} - \frac{e^{-s}}{s+2}.$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left(e^{-s} \cdot \frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(e^{-s} \cdot \frac{1}{s+2}\right)$$

$$= e^{-(t-1)} u(t-1) - e^{-2(t-1)} u(t-1)$$

$$= \begin{cases} 0 & \text{if } t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } t > 1 \end{cases}$$

Convolution:

$$\mathcal{L}(f(t)\,g(t)) \neq \mathcal{L}(f(t))\,\mathcal{L}(g(t))$$

e.g. Take $f(t) = g(t) = 1$. Then L.H.S. $= \frac{1}{s}$

R.H.S. $= \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$.

Define: The convolution of $f(t)$ and $g(t)$, $t > 0$ is defined as

$$(f * g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau \quad , \quad t > 0 .$$

Properties:

- $f * g = g * f$.
- $(f * g) * h = f * (g * h)$.

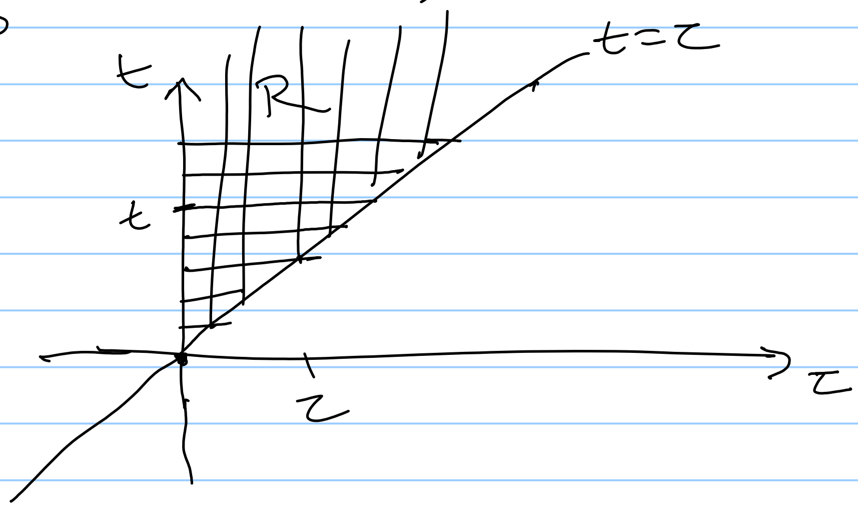
Prop: $\mathcal{L}(f*g) = \mathcal{L}(f)\,\mathcal{L}(g)$.

Pf:
$$\mathcal{L}(f*g)(s) = \int_0^\infty e^{-st}(f*g)(t)\,dt$$
$$= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)dt$$

The region of the double integral is

$R = \{(\tau, t) : 0 \le \tau \le t,\ 0 \le t < \infty\}$

$= \{(\tau, t) : \tau \le t < \infty,\ 0 \le \tau < \infty\}$



Changing the order of integrals, we get

$$\mathcal{L}(f*g)(s) = \int_0^\infty \left(\int_\tau^\infty f(\tau)\, g(t-\tau)\, e^{-st}\, dt \right) d\tau$$

$$= \int_0^\infty f(\tau) \underbrace{\left(\int_\tau^\infty e^{-st}\, g(t-\tau)\, dt \right)}_{\text{Put } t-\tau = u} d\tau$$

$$= \int_0^\infty e^{-s(u+\tau)}\, g(u)\, du = e^{-s\tau} \mathcal{L}(g)(s)$$

$$\therefore \quad \mathcal{L}(f*g)(s) = \int_0^\infty f(\tau)\, e^{-s\tau}\, G(s)\, d\tau = F(s)G(s).$$

$\therefore$ $\boxed{\mathcal{L}^{-1}\left(F(s)G(s)\right)(t) = (f*g)(t)}$

e.g.: Find $\mathcal{L}^{-1}\left(\frac{1}{(s^2+w^2)^2}\right)$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+w^2}\right) = \frac{1}{w}\sin(wt)$$

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{(s^2+w^2)^2}\right) = \frac{1}{w^2}\sin(wt) * \sin(wt)$$

$$= \frac{1}{w^2}\int_0^t \sin(w\tau)\sin(wt-w\tau)\,d\tau$$

$$= \frac{1}{2w^2}\int_0^t \left[\cos(2w\tau-wt) - \cos(wt)\right]d\tau$$

$$= \frac{1}{2\omega^2}\left[\frac{\sin(\omega t)}{\omega} - t\cos(\omega t)\right]$$

$$= \frac{1}{2\omega^2}\left[\sin(\omega t) - \omega t\cos(\omega t)\right]$$

Example: Solve: $y'' + 3y' + 2y = r(t) = \begin{cases} 1 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

$y(0) = 0, \; y'(0) = 0.$

Taking Laplace transform,

$$s^2 Y(s) + 3sY(s) + 2Y(s) = R(s)$$

$$\Rightarrow \quad Y(s) = \left(\frac{1}{s^2+3s+2}\right) R(s) = \left(\frac{1}{s+1} - \frac{1}{s+2}\right) R(s)$$

$$\Rightarrow \quad y(t) = r(t) * g(t) \quad ; \text{ where } g(t) = e^{-t} - e^{-2t}$$

$$= \int_0^t r(\tau)\, g(t-\tau)\, d\tau$$

$$= \begin{cases} 0 & \text{if } t<1 \\ \int_1^t g(t-\tau)\, d\tau & \text{if } 1<t<2 \\ \int_1^2 g(t-\tau)\, d\tau & \text{if } t>2 \end{cases}$$

$$= \ldots$$

solving integral equations if it involves convolution:

Example: Solve: $y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$

$$\Rightarrow \quad y(t) - y(t) * \sin t = t$$

$$\Rightarrow \quad Y(s) - Y(s) \cdot \frac{1}{s^2+1} = \frac{1}{s^2}$$

$$\Rightarrow \quad \frac{s^2}{s^2+1} Y(s) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\Rightarrow \quad y(t) = t + \frac{t^3}{3!} = t + \frac{t^3}{6}.$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = ?$$

Let $g(t) = \dfrac{f(t)}{t}$

$\Rightarrow \quad t\,g(t) = f(t)$

$\Rightarrow \quad \mathcal{L}(t\,g(t)) = \mathcal{L}(f(t))$

$\Rightarrow \quad -G'(s) = F(s)$

$\Rightarrow \quad G'(s) = -F(s)$

$$\Rightarrow \quad \boxed{G(s) = \int_s^{\infty} F(s)\,ds}$$

Example : ① If $F(s) = \ln\left(1 + \frac{\omega^2}{s^2}\right)$, $\mathcal{L}^{-1}(F(s)) = ?$

$$F(s) = \ln(\omega^2 + s^2) - 2\ln s$$

$$\Rightarrow \quad F'(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}$$

$$\Rightarrow \quad -t f(t) = 2\cos(\omega t) - 2$$

$$\Rightarrow \quad \boxed{f(t) = \frac{2(1 - \cos\omega t)}{t}}$$

(2) $\mathcal{L}^{-1}\left(\tan^{-1}\left(\frac{2}{s}\right)\right) = ?$

$F(s) = \tan^{-1}\left(\frac{2}{s}\right)$

$F'(s) = \frac{1}{1+\frac{4}{s^2}} \cdot \left(\frac{-2}{s^2}\right) = \frac{-2}{s^2+4}$

$\Rightarrow \quad -t\, f(t) = -\sin(2t) \quad \Rightarrow \boxed{f(t) = \frac{\sin(2t)}{t}}$

Second order linear ODEs involving ty' & ty'' terms:

$\mathcal{L}(t y') = -\frac{d}{ds}\mathcal{L}(y') = -\frac{d}{ds}\left(sY(s) - y(0)\right) = -sY'(s) - Y(s)$

$$\mathcal{L}(t y'') = -\frac{d}{ds}\left[s^2 Y(s) - s y(0) - y'(0) \right]$$

$$= -2s Y(s) - s^2 Y'(s) + y(0)$$

Substituting, we get a first order linear ODE in $Y(s)$.

e.g. (Laguerre's eqn)

$$t y'' + (1-t) y' + n y = 0 \quad , \quad n \in \mathbb{N} \cup \{0\}.$$

Try to solve this using above.

Note Title 6/2/2023

Lec 35

Solving system of ODEs using Laplace transform:

Consider the system

$$y_1' = a_{11} y_1 + a_{12} y_2 + g_1(t) \qquad ; \quad a_{ij} \in \mathbb{R}.$$

$$y_2' = a_{21} y_1 + a_{22} y_2 + g_2(t)$$

Taking the Laplace transform,

$$\begin{cases} s Y_1(s) - y_1(0) = a_{11} Y_1(s) + a_{12} Y_2(s) + G_1(s) \\ s Y_2(s) - y_2(0) = a_{21} Y_1(s) + a_{22} Y_2(s) + G_2(s) \end{cases}$$

(i) $\begin{cases} (a_{11}-s)Y_1(s) + a_{12}Y_2(s) = -y_1(0) - G_1(s) \\ a_{21}Y_1(s) + (a_{22}-s)Y_2(s) = -y_2(0) - G_2(s) \end{cases}$

(ii) $$\boxed{(A - sI)\vec{Y}(s) = -\vec{y}(0) - \vec{G}(s)}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\vec{Y}(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}$,

$\vec{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$, $\vec{G}(s) = \begin{pmatrix} G_1(s) \\ G_2(s) \end{pmatrix}$

Example: Solve: $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = y_2 \end{cases}$

Here $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Taking Laplace transf.,

$$\left(A - sI\right) \vec{y}(s) = -\vec{y}(0) = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1-s & 1 \\ 0 & 1-s \end{pmatrix} \vec{y}(s) = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \vec{y}(s) = \frac{-1}{(1-s)^2} \begin{pmatrix} 1-s & -1 \\ 0 & 1-s \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow Y_1(s) = \frac{c_1(s-1) + c_2}{(1-s)^2} = \frac{c_1}{s-1} + \frac{c_2}{(s-1)^2}$$

$$\& \quad Y_2(s) = \frac{c_2}{s-1}$$

$$\Rightarrow \boxed{\begin{aligned} y_1(t) &= c_1 e^t + c_2 t e^t \\ y_2(t) &= c_2 e^t \end{aligned}}$$

Example: Solve the system:

$$y_1'' + y_2 = -5\cos 2t \qquad y_1(0)=1,\ y_1'(0)=1$$

$$y_2'' + y_1 = 5\cos 2t \qquad y_2(0)=-1,\ y_2'(0)=1$$

Taking Laplace,

$$s^2 Y_1 - s - 1 + Y_2 = \frac{-5s}{s^2+4}$$

$$s^2 Y_2 + s - 1 + Y_1 = \frac{5s}{s^2+4}$$

1) $$\begin{cases} s^2 Y_1 + Y_2 = s+1 - \dfrac{5s}{s^2+4} \\ Y_1 + s^2 Y_2 = 1-s + \dfrac{5s}{s^2+4} \end{cases}$$

Find $Y_1(s)$, $Y_2(s)$ and then take the inverse Laplace to get $y_1(t)$, $y_2(t)$.

$$Y(s) = \frac{(s-1)^n}{s^{n+1}}, \quad \mathcal{L}^{-1}(Y(s)) = ?$$

$$\mathcal{L}\left(\frac{d^n}{dt^n}\left(t^n e^{-t}\right)\right) = ?$$

Let $f(t) = t^n e^{-t}$. Then $F(s) = \dfrac{n!}{(s+1)^{n+1}}$

$$\left(\because \mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \right)$$

$$\because \quad \mathcal{L}\left(\frac{d^n}{dt^n} f(t) \right) = s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

$$f(t) = t^n e^{-t} \quad \text{a} \quad f(0) = 0$$

$$f^{(k)}(0) = 0 \quad \forall \; k = 0, 1, \cdots, n-1.$$

$$\therefore \quad \mathcal{L}\left(\frac{d^n}{dt^n} \left(t^n e^{-t} \right) \right) = n! \frac{s^n}{(s+1)^{n+1}}$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left(\frac{(s-1)^n}{s^{n+1}} \right) = \frac{1}{n!} e^t \frac{d^n}{dt^n} \left(t^n e^{-t} \right)$$

$$F(s) = \frac{s}{s^4+4}, \quad \mathcal{L}^{-1}\left(F(s)\right) = ?$$

$$s^4+4 = \left(s^4+4s^2+4\right) - 4s^2$$

$$= \left(s^2+2\right)^2 - (2s)^2 = \left(s^2+2+2s\right)\left(s^2+2-2s\right)$$

$$= \left[(s+1)^2+1\right]\left[(s-1)^2+1\right]$$

$$F(s) = \frac{s}{(s+1)^2+1} \cdot \frac{1}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t = e^{-t}(\cos t - \sin t)$$

$$= g(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = e^t \sin t = h(t)$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = g * h\,(t) = \int_0^t g(\tau)\, h(t-\tau)\, d\tau\,.$$

$$= \ldots$$

Note Title 6/5/2023

Lec 36

$$g(t) = \frac{f(t)}{t} \Rightarrow f(t) = t\,g(t)$$

$$\Rightarrow F(s) = -G'(s)$$

$$\Rightarrow G'(s) = -F(s)$$

$$\Rightarrow G(s) = \int_s^\infty F(u)\,du + C$$

Taking lim. as $s \to \infty$, $\lim_{s\to\infty} G(s) = 0$

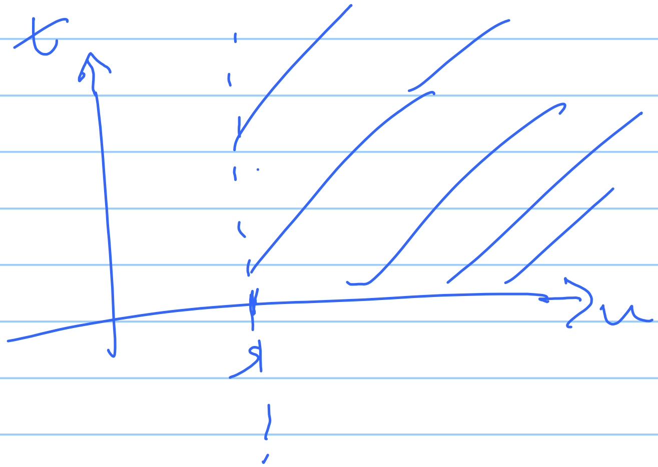
& $\lim_{s\to\infty} \int_s^\infty F(u)\,du = 0$

$\therefore C = 0$

$$\therefore G(s) = \int_s^\infty F(u)\,du$$

Another way:

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_0^\infty e^{-st}\,\frac{f(t)}{t}\,dt \qquad \int_s^\infty e^{-ut}\,du = \frac{e^{-st}}{t}$$



$$= \int_0^\infty \left(\int_s^\infty e^{-ut}\,du \right) f(t)\,dt$$

$$= \int_s^\infty \left(\int_0^\infty e^{-ut} f(t) \right) du$$

$$= \int_s^\infty F(u)\,du$$

$$4y'' - y = e^x$$

$$D = \frac{d}{dx}$$

$$(4D^2 - I)\, y = e^x$$

$$(2D+I)\cdot(2D-I)\, y = e^x$$

Let $z = (2D-I)(y) = 2y' - y$ — (1)

$$(2D+I)\, z = e^x$$

i.e. $2\frac{dz}{dx} + z = e^x$

1st order linear ODE in z,

$$z = f_{c_1}(x)$$

① $\Rightarrow$ $2y' - y = f_{c_1}(x) \longrightarrow$ 1st order linear ODE in y.