MAXIMUM MARKS: 30

Instructions: Justify all your statements. Remember that you will be graded on what you write on the answer sheet, **NOT** on what you intend to write.

Question 1: [4 marks]
Consider the following system of linear equations:

$$\begin{pmatrix} 1 & 1 & a+b & 3 \\ 0 & 1 & a+b & 0 \\ 2 & 2 & 3a+2b & a-b+6 \\ 0 & 1 & a+b & a-b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 6 \\ 2 \end{pmatrix}.$$

(a) For what $a, b \in \mathbb{R}$ the system has at least one solution.
(b) Whenever the solutions exist, write all the solutions.

Question 2: [4 marks]
Recall that a subset of a vector space is called minimal spanning subset if it spans the vector space but any of its proper subset does not span the vector space. Prove that a minimal spanning subset of a vector space is a basis of the vector space.

Queston 3: [4 marks]

Let $A = \begin{pmatrix} 3 & 8 & 16 \\ 8 & 15 & 32 \\ -4 & -8 & -17 \end{pmatrix} \in M_3(\mathbb{R})$.

(a) Prove that A is similar to a diagonal matrix D.
(b) Find an invertible matrix P such that $P^{-1}AP = D$.

Question 4: [4 marks]
Let W be a subspace of a finite dimensional vector space V over a field F. Then prove that:
$\boxed{\dim W > \frac{1}{2}\dim V} \Leftrightarrow \boxed{W \cap T(W) \neq \{0\} \text{ for all one-one and onto linear transformations } T : V \to V}$.

Question 5: [4 marks]

Let $A = \begin{pmatrix} 4 & 24 & 6 \\ -1 & -7 & -2 \\ 2 & 12 & 3 \end{pmatrix} \in M_3(\mathbb{R})$. Use Cayley-Hamilton theorem to find $A^{10} + A^{20}$.

Question 6: [4 marks]
Consider the following ordinary differential equation (ODE):

$$\cos y \; dx + \cot x \; \sin y \; dy = 0.$$

Write $M = \cos y$ and $N = \cot x \; \sin y$.

(a) Using the fact that $(M_y - N_x)/N$ is a function of x only, find an integrating factor of the ODE so that it becomes exact.

(b) Solve the ODE with the following initial conditions: $y(\pi/4) = \pi/4$.

Question 7: [2+2+2 marks]
Justify whether the following statements are true or false.

(a) Let W, W_1, W_2 be subspaces of a vector space. Then,

$$W \oplus W_1 = W \oplus W_2 \Longrightarrow W_1 = W_2.$$

(b) Let y_1, y_2 be two solutions of the homogeneous linear ordinary differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x), q(x)$ are continuous functions on an open interval I. Let $W(t) = W(y_1, y_2)(t)$ denote the Wronskian. Then there exist $a, b \in I$ such that $W(a) = 0$ and $W(b) \neq 0$.

(c) Consider the initial value problem (IVP): $y' + p(x)y = q(x); y(x_0) = y_0$ where $p(x)$ and $q(x)$ are continuous functions on $\mathbb{R}$. Then, this IVP satisfies the conditions of the uniqueness theorem.

Exam Ends Here