Lecture 1.

Matrices: is a rectangular array of numbers or functions.

numbers: Real numbers $\mathbb{R}$ Complex numbers $\mathbb{C}$.

$$A = (a_{\overset{\text{row}}{i}\underset{\text{column}}{j}}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad a_{ij} \in \mathbb{R} \text{ or } \mathbb{C}$$

Size of matrix A = (# of rows) $\times$ (# of columns)
$$m \times n$$

Matrices with single $\overset{(m=1)}{\text{row}}$ or a single $\overset{(n=1)}{\text{col}}$ is call a vector.

matrices with single rows are called row vectors
cols column vectors

<u>Ex</u>: $\begin{bmatrix} 1 & 0 \\ \pi & e \end{bmatrix}$ 2×2 matrix.

$\begin{bmatrix} 1 \\ \pi \end{bmatrix}$ is a column vector $\begin{bmatrix} 1 & 0 \end{bmatrix}$ is a row vector

- If $m = n$ then A is a square matrix
- The diagonal entries $a_{11}, a_{22}, \dots a_{nn}$ are called the main diagonal of A.

Ex:

$$\begin{array}{cc} \cancel{x} & f(x) = e^x \end{array}$$

$$\begin{bmatrix} 1 & e \\ 2 & e^2 \\ 3 & e^3 \\ 4 & e^4 \end{bmatrix}$$

- Two matrices are equal

$A = (a_{ij})$ $\qquad B = (b_{ij})$

if and only if
1. they have same size
2. corresponding entries are equal i.e., $a_{ij} = b_{ij}$ for all i, j.

- The sum of two matrices $A = (a_{ij})$ $B = (b_{ij})$ of the same size

$A + B = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$

↳ entries obtained by adding the corresponding entries of A & B.

- Matrices of different sizes cannot be added.

Ex: $A = \begin{bmatrix} -4 & 6 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 6 & 2 \end{bmatrix}$ $\quad A + B = \begin{bmatrix} -4+1 & 6+2 & 2+3 \\ 1-4 & 2+6 & 2+3 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 8 & 5 \\ -3 & 8 & 5 \end{bmatrix}$$

- scalar multiplication (Multiplication by a number)

The product of $m \times n$ matrix $A = (a_{ij})$ and a scalar $c \in \mathbb{R}$ (or $\mathbb{C}$) written as cA is a matrix (ca_{ij}), obtained by multiplying each entry of A by the scalar c.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ e & \pi \end{bmatrix}$ $c = \sqrt{2}$ $\sqrt{2}A = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 3\sqrt{2} & 4\sqrt{2} \\ e\sqrt{2} & \pi\sqrt{2} \end{bmatrix}$

- negative of A is defined as $-A = (-1) \cdot A$.

$A = (a_{ij})$ then $-A = (-a_{ij})$.

- $A + B$, $A - B$.

Rules for matrix addition & scalar multiplication.

1. $A + B = B + A$
2. $(A+B)+C = A+(B+C)$
3. $A + O = A$
4. $A + (-A) = O$

O is the zero matrix having the same size as that of A.

If A is 3×2 matrix then $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. $A + O = A$.

If $m=1$ or $n=1$ then the 0 is called a zero row or zero col.

(5) $c(A+B) = cA + cB$

(6) $(c+d)A = cA + dA$

(7) $c(dA) = (cd)A$.

(8) $I \cdot A = A$

I is the identity matrix

§ Matrix multiplication.

$A = (a_{ij})$ $\quad$ $B = (b_{ij})$

Size $\quad m \times n$ $\quad$ $r \times p$.

The product $c = AB$ is defined if and only if $r = n$ and (1) AB is an $m \times p$ matrix.

(2) $c = AB = (c_{ij})$

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$i = 1 \ldots m$

$j = 1 \ldots p$.

4×3 $\quad$ 3×2 $\quad$ 4×2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}$$

$$c_{11} = [a_{11}\ a_{12}\ a_{13}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

Ex:

$$\begin{bmatrix} 3 & 5 & 1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 3 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Rules for matrix multiplication

1. $(cA)B = c(AB) = A(cB)$
2. $A(BC) = (AB)C$
3. $(A+B)C = AC + BC$
4. $C(A+B) = CA + CB$.

- Identity: $\mathbb{1}A = A$.

$$\mathbb{1}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- It is possible that $AB = BA$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 9 & -5 \\ 1 & 1 \end{bmatrix} \neq BA = \begin{bmatrix} 6 & -10 \\ -1 & 4 \end{bmatrix}$$

- $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

$0 \cdot 2 = 0$
$1 \cdot 2 \neq 0$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- $AC = BC \not\Rightarrow A = B$ or $C = 0$

$\cancel{2} \cdot 3 = \cancel{2} \cdot 4$
$3 = 4.$

$$A = \begin{bmatrix} -3 & 3 \\ 11 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 \\ 8 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

$$AC = \begin{bmatrix} 3 & 9 \\ 5 & 15 \end{bmatrix} = BC = \begin{bmatrix} 3 & 9 \\ 5 & 15 \end{bmatrix}$$

- Transposition

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

The transpose of a $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix A^T that has the first row of A as its first column, second row of A is its second colun ....

$$A^T = (a_{ji}) = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}$$

Transposition laws ① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

(3) $(cA)^T = cA^T$

(4) $\boxed{(AB)^T = B^T A^T}$

- Skew-Symmetric matrices: are square matrices s.t. transpose equals the negative of A

$$A^T = -A.$$

- Symmetric matrices are those which equals its transpose

$$A = A^T.$$

- Diagonal matrices: Matrices which can have non zero entries only on the main diagonal

- Triangular matrices:
  - Upper Triangular matrices: Are square matrices that can have nonzero entries only on and above the main diagonal

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

  - lower Triangular matrices: Are square matrices that can have nonzero entries only on & below the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$$

- Power of Matrices : A is a square matrix

$$A^2 = A \cdot A$$
$$A^3 = A^2 \cdot A$$
$$\vdots$$
$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

Partitioned matrices.

$$A = \left[\begin{array}{ccc|cc} 2 & 0 & -3 & 1 & 7 \\ -1 & 4 & 2 & 0 & 4 \\ 6 & -1 & 1 & 8 & -3 \\ \hline 0 & 2 & 7 & -3 & 3 \\ 2 & 0 & -6 & 9 & 0 \\ \hline 1 & -1 & 8 & 5 & -1 \\ 4 & 6 & 9 & 7 & 8 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

Another matrix B of the same size as that A.

$$A+B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}+B_{11} & A_{12}\ B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \\ A_{31}+B_{31} & A_{32}+B_{22} \end{bmatrix}$$

$$rA = \begin{bmatrix} rA_{11} & rA_{12} \\ rA_{21} & rA_{22} \\ rA_{31} & rA_{32} \end{bmatrix}$$

Ex: $A = \left[\begin{array}{cc|ccc} 3 & -1 & 2 & 4 & 0 \\ 0 & 2 & 1 & -3 & 1 \\ 2 & 3 & -4 & 0 & -4 \\ \hline 1 & 6 & 0 & 2 & -2 \end{array}\right]$ $\quad B = \left[\begin{array}{c|c|c} 3 & 1 & 2 \\ 4 & 0 & 3 \\ \hline -1 & 7 & 0 \\ 2 & 4 & 1 \\ 0 & -1 & -1 \end{array}\right]$

$4 \times 5 \qquad\qquad 5 \times 3$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} 11 & 33 & 7 \\ 1 & -4 & 4 \\ 22 & 30 & 9 \\ \hline 31 & 11 & 24 \end{array}\right]$$

- Inverse of matrix: If A is a square matrix, the inverse is a matrix B s.t $A \cdot B = \underline{I}$ and $B \cdot A = I$