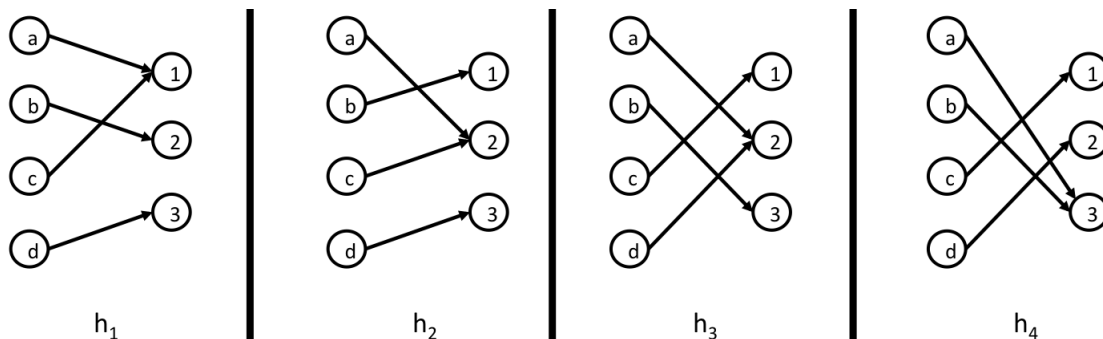


Name: \_\_\_\_\_

ID number: \_\_\_\_\_

There are 2 questions for a total of 10 points.

1. Consider the hash function family  $H = \{h_1, h_2, h_3, h_4\}$  where  $h_i$ 's are defined below and answer the questions that follow.



- (a) (1 point) State true or false:  $H$  is 2-universal.

(a) False

- (b) (2 points) Give reasons for your answer to part (a).

**Solution:** We have :

1.  $\Pr_{h \leftarrow H}[h(a) = h(b)] = 1/4$
2.  $\Pr_{h \leftarrow H}[h(a) = h(c)] = 1/2$
3.  $\Pr_{h \leftarrow H}[h(a) = h(d)] = 1/4$
4.  $\Pr_{h \leftarrow H}[h(b) = h(c)] = 0$
5.  $\Pr_{h \leftarrow H}[h(b) = h(d)] = 0$
6.  $\Pr_{h \leftarrow H}[h(c) = h(d)] = 0$

Since  $\Pr_{h \leftarrow H}[h(a) = h(c)] = 1/2 > 1/3$ ,  $H$  is not a 2-universal hash function family.

2. (7 points) Use generating functions to solve the recurrence relation  $a_k = a_{k-1} + 2a_{k-2} + 2^k$  with initial conditions  $a_0 = 4$  and  $a_1 = 12$ .

**Solution:** Let  $G(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots$ . Then we have:

$$\begin{aligned} G(x) &= a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots \\ x \cdot G(x) &= a_0x^1 + a_1x^2 + a_2x^3 + \dots \\ 2x^2 \cdot G(x) &= 2a_0x^2 + 2a_1x^3 + 2a_2x^4 + \dots \end{aligned}$$

This gives:

$$\begin{aligned} (1 - x - 2x^2) \cdot G(x) &= a_0 + (a_1 - a_0)x + \sum_{k=2}^{\infty} (a_k - a_{k-1} - 2a_{k-2})x^k \\ &= 4 + 8x + \sum_{k=2}^{\infty} 2^k x^k \quad (\text{since } a_k = a_{k-1} + 2a_{k-2} + 2^k) \\ &= 4 + 8x + 4x^2 \cdot (1 + (2x) + (2x)^2 + \dots) \\ &= 4 + 8x + \frac{4x^2}{1 - 2x} \end{aligned}$$

This implies that:

$$G(x) = \frac{4 - 12x^2}{(1+x)(1-2x)^2} = \frac{a}{1+x} + \frac{bx+c}{(1-2x)^2}.$$

We solve for constants  $a, b, c$ . We get the following equations:

$$\begin{aligned} 4a + b &= -12 \\ -4a + b + c &= 0 \\ a + c &= 4 \end{aligned}$$

which gives  $a = -\frac{8}{9}$ ,  $b = -\frac{76}{9}$ , and  $c = \frac{44}{9}$ . Collecting the coefficients of  $x^k$ , we get that:

$$\begin{aligned} a_k &= -\frac{8}{9}(-1)^k + \frac{44}{9}(k+1)2^k + (-\frac{76}{9})k2^{k-1} \\ &= -\frac{8}{9}(-1)^k + \frac{2}{3}k2^k + \frac{44}{9}2^k \end{aligned}$$