Sample Solutions: MTL(101) Minor Exam

1. Prove that any maximal linearly independent set of vectors in a vector space, V (not necessarily finite dimensional), is a basis of V.

   Proof. Prof Ritumoni's lecture notes 7-8. $\square$

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T(x, y, z, w) = (x + y, z, w, w)$$

   Compute the rank and nullity of T^2.

   Proof. Let $\mathcal{B}_{\mathbb{R}^4} = \{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$. So, $\mathrm{Im}(T)$ is the subspace of $\mathbb{R}^4$ spanned by $T(e_1), T(e_2), T(e_3), T(e_4)$.

   Note that

$$T(e_1) = e_1;\ T(e_2) = e_1;\ T(e_3) = e_2;\ T(e_4) = e_3 + e_4.$$

   Thus we have

$$T^2(e_1) = e_1;\ T^2(e_2) = e_1;\ T^2(e_3) = e_1;\ T^2(e_4) = e_2 + e_3 + e_4.$$

   So, the rank of T^2 is 2. Therefore by using rank nullity we conclude that nullity(T^2) is 2. $\square$

3. Let B and B' be the following standard ordered bases of $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$, respectively:

$$B = \{1, x, x^2\},$$

   and

$$B' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

   Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, be the linear transformation given by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}.$$

   Compute the matrix of linear transformation $[T]_B^{B'}$.

Solution: First we see that

$$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

So the first column of $[T]_B^{B'}$ is the coordinate vector $[T(1)]_{B'} = (0, 2, 0, 0)$.
Next,

$$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

So the second column of $[T]_B^{B'}$ is the coordinate vector $[T(x)]_{B'} = (1, 2, 0, 0)$.
Finally,

$$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}.$$

So the third column of $[T]_B^{B'}$ is the coordinate vector $[T(x^2)]_{B'} = (0, 2, 0, 2)$.
So in total we get

$$[T]_{\mathbf{B}}^{B'} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Find the eigenvalues of the following 3×3 matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Also find eigenvectors corresponding to eigenvalues that are integers.

Solution The eigenvalues and eigenvectors are found by solving the eigenvalue problem

$$AX = \lambda X \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now

$(A - \lambda I)X = 0$ and non-trivial solutions for X will exist if $\det(A - \lambda I) = 0$

We start with:

$$\det\left(\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

which simplifies to:

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we get:

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix} = 0.$$

This simplifies to:

$$(2-\lambda)\{(2-\lambda)^2 - 1\} - (2-\lambda) = 0.$$

Factoring out $(2-\lambda)$:

$$(2-\lambda)\{4 - 4\lambda + \lambda^2 - 1\} = 0.$$

which simplifies to:

$$(2-\lambda)(\lambda^2 - 4\lambda + 2) = 0.$$

Solving for λ:

$$\lambda = 2 \quad \text{or} \quad \lambda = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}.$$

Thus, the three eigenvalues are $2, 2+\sqrt{2}, 2-\sqrt{2}$. The eigen value that is integer is $\lambda = 2$. Here $AX = 2X$ implies

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

i.e.

$$\begin{aligned} 2x - y &= 2x \\ -x + 2y - z &= 2y \\ -y + 2z &= 2z \end{aligned}$$

After simplifying the equations become:

$$\begin{aligned} -y &= 0 \quad \text{(a)} \\ -x - z &= 0 \quad \text{(b)} \\ -y &= 0 \quad \text{(c)} \end{aligned}$$

(a), (c) imply $y = 0$; (b) implies $x = -z$.

$$\therefore \text{ eigenvector has the form } \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} \text{ for any } x \neq 0.$$

That is, eigenvectors corresponding to $\lambda = 2$ are all proportional to

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

that is

$$span \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

5. Let V and W be two finite dimensional vector spaces over the field $\mathbb{F}$ and $T : V \to W$ be a linear transformation.

   i) Prove that, if $\mathrm{Ker}(T) = \{0\}$, then T sends linearly independent set to linearly independent set, i.e., if $\{v_1, \ldots, v_k\}$ is linearly independent in V, then $\{T(v_1), \ldots, T(v_k)\}$ is linearly independent in W.

   ii) Is the above result true if we drop the condition $\mathrm{Ker}(T) = \{0\}$? Justify.

Solution

(i) Let $\{v_1, v_2, \dots, v_k\}$ be a linearly independent set in V. We need to show that $\{T(v_1), T(v_2), \dots, T(v_k)\}$ is linearly independent in W.

Suppose there exist scalars $a_1, a_2, \dots, a_k \in \mathbb{F}$ such that

$$a_1 T(v_1) + a_2 T(v_2) + \cdots + a_k T(v_k) = 0.$$

Since T is linear, we can rewrite this as:

$$T(a_1 v_1 + a_2 v_2 + \cdots + a_k v_k) = 0.$$

Given that $\ker(T) = \{0\}$, it follows that:

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0.$$

But since $\{v_1, v_2, \dots, v_k\}$ is linearly independent, the only solution is:

$$a_1 = a_2 = \cdots = a_k = 0.$$

Thus, $\{T(v_1), T(v_2), \dots, T(v_k)\}$ is also linearly independent in W. □

(ii) No, the result does not necessarily hold if $\ker(T) \neq \{0\}$, meaning T may map a linearly independent set to a linearly dependent set.

Counterexample: Consider $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$ with the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T(x, y, z) = (x, y).$$

The set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly independent in V. Applying T, we obtain:

$$T(1, 0, 0) = (1, 0), \quad T(0, 1, 0) = (0, 1), \quad T(0, 0, 1) = (0, 0).$$

The set $\{(1, 0), (0, 1), (0, 0)\}$ in W is clearly linearly dependent because $(0, 0)$ is a zero vector.

This example demonstrates that T does not always preserve linear independence if $\ker(T) \neq \{0\}$. □

6. Let W_1 and W_2 be subspaces of $\mathbb{R}^5$ given by

$$W_1 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 + 2x_3 = 0, \quad 2x_4 + x_5 = 0\}$$

and

$$W_2 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_2 + 2x_3 = 0, \quad x_1 + 2x_4 + x_5 = 0\}.$$

Find $\dim(W_1 \cap W_2)$. Is $W_1 + W_2 = \mathbb{R}^5$? Justify.

Solution:

Let W_1 and W_2 be subspaces of $\mathbb{R}^5$ defined by:

$$W_1 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 + 2x_3 = 0, \quad 2x_4 + x_5 = 0\}$$

and

$$W_2 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_2 + 2x_3 = 0, \quad x_1 + 2x_4 + x_5 = 0\}.$$

To find $W_1 \cap W_2$, we solve the system consisting of all four equations:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0, \\ 2x_4 + x_5 &= 0, \\ x_2 + 2x_3 &= 0, \\ x_1 + 2x_4 + x_5 &= 0. \end{aligned}$$

The corresponding RREF is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & |0 \\ 0 & 1 & 2 & 0 & 0 & |0 \\ 0 & 0 & 0 & 1 & 1/2 & |0 \\ 0 & 0 & 0 & 0 & 0 & |0 \end{bmatrix}$$

Solving this we get the solution as the general solution is:

$$(x_1, x_2, x_3, x_4, x_5) = (0, -2x_3, x_3, \frac{-x_5}{2}, x_5).$$

The free variables in the system are x_3 and x_5. This solution space is spanned by the vectors:

$$(0, -2, 1, 0, 0) \quad \text{and} \quad (0, 0, 0, -1/2, 1).$$

Thus, the dimension of $W_1 \cap W_2$ is:

$$\dim(W_1 \cap W_2) = 2.$$

Using the dimension formula:

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Compute dimension of W_1 and W_2 . Since we have a total of 5 variables and there are 2 independent equations, the number of free variables is:

$$\text{Number of variables} - \text{Number of independent equations} = 5 - 2 = 3.$$

Thus, the dimension of W_1 and W_2 is:

$$\dim(W_1) = 3, \quad \dim(W_2) = 3.$$

Thus, we compute:

$$\dim(W_1 + W_2) = 3 + 3 - 2 = 4.$$

Since $\dim(W_1 + W_2) = 4 \neq 5$, it follows that:

$$W_1 + W_2 \neq \mathbb{R}^5.$$

□

7. Prove (if true) or disprove (if false) the following statements.

   i) If W_1, W_2 and W are subspaces of a vector space V such that

   $$W_1 \oplus W = W_2 \oplus W,$$

   then $W_1 = W_2$.

   ii) The span of the set $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 4\}$ over $\mathbb{R}$ is $\mathbb{R}^3$.

   iii) If Y_1 and Y_2 are solutions of the system of linear equations

   $$AX = B, \text{ where } B \neq \mathbf{0},$$

   then for $a_1 \neq 0 \neq a_2$, $a_1Y_1 + a_2Y_2$ is not a solution of $AX = B$.

iv) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\begin{aligned} T(1,1,1,1) &= (1,0,0,0), \\ T(1,0,1,0) &= (0,1,0,0), \\ T(0,-1,0,-1) &= (0,0,1,0), \\ T(0,0,0,1) &= (0,0,0,0). \end{aligned}$$

i) False. For example, $V = \mathbf{R}^2$, $W_1 = \mathbf{R} \times \{0\}$, $W_2 = \{0\} \times \mathbf{R}$ and $W = \{(x,y) \in \mathbf{R}^2 : x = y\}$. Then $W_1 \cap W = \{0\} = W_2 \cap W$. Also observe $(x,y) = (x-y,0) + (y,y) = (0,y-x) + (x,x)$.

ii) True. Observe $(2,0,0), (0,2,0), (0,2,1) \in S$ so the the span of S contains e_1, e_2 and $e_3 = (0,2,1) - (0,2,0)$ which span $\mathbf{R}^3$.

iii) False. $A(a_1Y_1 + a_2Y_2) = a_1AY_1 + a_2AY_2 = a_1B + a_2B = (a_1 + a_2)B = B$ if $a_1 + a_2 = 1$.

iv) False. Observe $(1,1,1,1) = (1,0,1,0) - (0,-1,0,-1)$. If T is a linear transformation $(1,0,0,0) = T(1,1,1,1) = T(1,0,1,0) - T(0,-1,0,-1) = (0,1,0,0) - (0,0,1,0) = (0,1,-1,0)$. A contradiction.