Recall that, $T: V \to V$ be a linear transformation.

Let B & B' be bases of V.

Then $[v]_{B'} = P[v]_B$

$$\boxed{[T(v)]_{B'} = P[T(v)]_B}$$

$$\Rightarrow \boxed{[T]_{B'}[v]_{B'} = P[T]_B[v]_B}$$

$$\Rightarrow [T]_{B'}P[v]_B = P[T]_B[v]_B$$

$$\Rightarrow [T]_{B'}P = P[T]_B$$

$$\Rightarrow [T]_{B'} = P[T]_B P^{-1}$$

Let λ be an eigen value of $[T]_B$, $\exists\ X \neq 0$

$$[T]_B X = \lambda X \Rightarrow P[T]_B X = \lambda PX$$

$$\Rightarrow P[T]_B P^{-1}(PX) = \lambda(PX)$$

$$\Rightarrow [T]_{B'}(PX) = \lambda(PX)$$

$T(x,y) = (x, 3x+2y)$

Question :- Is it necessary to take $T : V \to V$ to define eigen value.

definition.

In general, T should be defined from V to V.

If $T : V \to W$ and $\text{Range}(T) \subseteq V$, then also we can define, eigen value.

$$T : V \longrightarrow \text{Rang}(T) \subseteq V$$

Question :- To find eigen value of $T: V \to V$, why shall we take same basis B for $[T]_B^B$ (Not $[T]_B^{B'}$) $\dim(V) < \infty$

λ is an eigen value if $\exists\ X \neq 0 \in V$ s.t.

$$T(X) = \lambda X$$

$$[T(X)]_B = \lambda [X]_B$$

$$\Rightarrow [T]_B [X]_B = \lambda [X]_B$$

$\because X \neq 0$, $[X]_B \neq 0$

$$\Rightarrow \left([T]_B - \lambda I_{n\times n}\right) [X]_B = 0 \qquad [X]_B \in \mathbb{R}^n$$

Since $[X]_B \neq 0$, $\det\left([T]_B - \lambda I_{n\times n}\right) = 0$

$T: \mathbb{R}^2 \xrightarrow{\times} \mathbb{R}^2$

$T(x, y) = (x, y)$

1 is an eigen value.

$B = \{(1,0), (0,1)\} \longrightarrow \lambda = 1$

$B = \{(1,0), (0,1)\}$ $B' = \{(0,1), (1,0)\}$

$$[T]_B^{B'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda \to \pm 1$

Question :- The eigen value of $T: V \to V$ is same as the eigen values of the matrix $[T]_B$. But the eigen vector of T corresponds to λ are generally different than eigen value of $[T]_B$ corresponds to λ.

$T(x,y) = (x, 3x+2y)$ $\quad \therefore \quad T: \mathbb{R}^2 \to \mathbb{R}^2$

$$[T]_B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \qquad \lambda = 1, 2$$

$B = \{(1,0), (0,1)\}$

$T(x,y) = 1\,(x,y)$

$\Rightarrow (x, 3x+2y) = (x,y)$

$\Rightarrow \quad 3x+2y = y$

$\therefore \quad 3x = -y$

eigen space $\{a(1,-3) : a \in \mathbb{R}\}$ corresponds to 1.

$B' = \{(1,-3), (0,1)\}$

$$T(1,-3) = (1,-3) = 1\cdot(1,-3) + 0\cdot(0,1)$$

$$T(0,1) = (0,2) = 0\cdot(1,-3) + 2(0,1)$$

$$[T]_{B'} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

eigen value of B' are 1 & 2.

eigen ~~vector~~ space corresponds to 1 is $\{a(1,0) : a \in \mathbb{R}\}$

" " " " 2 is $\{b(0,1) : b \in \mathbb{R}\}$.

Invariant Subspace :- Let $T : V \to V$ be a linear transformation. Then $W \subseteq V$ is said to be invariant subspace if $T(W) \subseteq W$.

* $T : \mathbb{R}^2 \to \mathbb{R}^2$. Does T have an 1-dimensional invariant subspace ? may not be

* $T : \mathbb{R}^3 \to \mathbb{R}^3$. Does T have an 1-dimensional invariant subspace ? Yes.

Let $W = \{cv : c \in \mathbb{R}\}$ be a 1-dim invariant subspace

$v \in W \quad T(v) \in W$

$\Rightarrow T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$

$\Rightarrow \lambda$ is a real eigen value & v is an eigen vector of T corresponds to λ.

<u>Conversely</u>, Let λ be a real value of T

Let v be a eigen vector of T corresponds to λ.

Then $W = \{cv : c \in \mathbb{R}\}$ is an 1-dim invariant subspace of V.

If we chose $T(x,y) = (-y, x)$

Then $T : \mathbb{R}^2 \to \mathbb{R}^2$ does not have any real eigen value.

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ then characteristic Polynomial is of odd degree (3), i.e., has one real root. So it has one 1-dim invariant subspace.

<u>Question</u> :- Suppose that V is finite dimensional. Prove that any linear map on a subspace can be extended to a linear map on V.

$U \subseteq V$. Let $T : U \longrightarrow W$ be a linear map

linear operator

linear transform

Then $\exists\ S : V \longrightarrow W$ s.t.

$S|_U = T$, i.e., $S(v) = T(v)\ \forall\ v \in U$.

$$S : V \longrightarrow W$$

$$S(v) = T(v) \quad \text{if } v \in U$$

$$= 0 \quad \text{if } v \notin U$$

$v_1, v_2 \in V$

$$S(v_1 + v_2) = 0 \qquad \begin{cases} v_1 \in U \\ v_2 \notin U \end{cases}$$

$$S(v_1) = T(v_1) \quad S(v_2) = 0$$

Let $\{u_1, u_2, \cdots, u_m\}$ be a basis of U.

Then $\{u_1, u_2, \cdots, u_m\}$ is L.I. in V.

Then $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_k\}$ is a basis of V

Any element $v \in V$ can be written as

$$v = c_1 u_1 + \cdots + c_m u_m + d_1 v_1 + \cdots + d_k v_k$$

Define $S : V \to W$

$$S(v) = c_1 T(u_1) + \cdots + c_m T(u_m)$$

For any $u \in U$, $u = a_1 u_1 + \cdots + a_m u_m$

$$S(u) = T(u) = a_1 T(u_1) + \cdots + a_m T(u_m)$$

S is linear.

Question :- Suppose W is finite dimensional and $T_1, T_2 \in L(V, W)$. Prove that $Null(T_1) \subseteq Null(T_2)$ if and only if $\exists\, S \in L(W, W)$ such that $\underline{T_2 = ST_1}$.
$\dim(V) < \infty$.

$T_2 = ST_1 \Rightarrow Null(T_1) \subseteq Null(T_2)$. (Easy)

Assume, $Null(T_1) \subseteq Null(T_2)$

Take $T_1(V) \subseteq W$

Let $\underline{\{T_1(v_1), T_1(v_2), \ldots, T_1(v_n)\}}$ is a basis of $T_1(V)$.

$\Rightarrow \{v_1, v_2, \ldots, v_n\}$ is L·I (Prove it)

Then $V = \operatorname{Span}(\{v_1, \cdots, v_n\}) \oplus \operatorname{Null}(T_1)$

(Prove it)

$v \in V,$ $\quad v = c_1 v_1 + \cdots + c_n v_n + u,$

$u \in \operatorname{Null}(T_1)$

Define a map $S' : T_1(V) \longrightarrow W$

$T_1(v_i) \longmapsto T_2(v_i)$

(Linear map Prove it)

$\Rightarrow$ Extend $S : W \longrightarrow W$ from S'

Then $\boxed{T_2 = S T_1}$ (Prove it)