Marking Scheme for Question 1 (Quiz 2)

June 17, 2022

Question 1:[4 marks]

Consider the following initial value problem (IVP)

$$\frac{dy}{dx} = x(1+y); \quad y(0) = 1. \tag{1}$$

a) Using Picard's method compute the first three successive approximations of the above IVP.

b) Write the n-th approximation and justify it by induction (PMI).

Solution:

a) By Picard's method, we have

$$y_0(x) = y(x_0), \text{ and } y_n(x) = y(x_0) + \int_{x_0}^{x} f\left(s, y_{n-1}(s)\right) ds, \quad [0.5 \text{ Marks }].$$

This gives

$$y_1(x) = y(0) + \int_0^x f\left(s, y_0(s)\right) ds = 1 + \int_0^x s(1+1)ds = 1 + x^2 \quad [0.5 \text{ Marks}].$$

$$y_2(x) = y(0) + \int_0^x f\left(s, y_1(s)\right) ds = 1 + \int_0^x s(1+1+s^2)ds = 1 + x^2 + \frac{x^4}{4} \quad [0.5 \text{ Marks}].$$

$$y_3(x) = y(0) + \int_0^x f\left(s, y_2(s)\right) ds = 1 + \int_0^x s(1+1+s^2+\frac{s^4}{4})ds = 1+x^2+\frac{x^4}{4}+\frac{x^6}{4.6}. \quad \text{[0.5 Marks]}.$$

$b)$ The $n-$th Picard iteration is

$$y_n(x) = 1 + \sum_{j=1}^{n} \frac{x^{2j}}{2^{j-1}j!}. \quad \text{[1 Marks]}$$

Now we will justify this using PMI. Clearly from part a) we have $y_1(x) = 1 + x^2$. Now suppose that it is true for $n = k$ i.e.,

$$y_k(x) = 1 + \sum_{j=1}^{k} \frac{x^{2j}}{2^{j-1}j!}, \quad \text{[0.5 Marks]}$$

Now consider

$$y_{k+1} = 1 + \int_0^x s\left(1 + y_k(s)\right)ds$$

$$= 1 + \int_0^x s\left(1 + 1 + \sum_{j=1}^{k} \frac{s^{2j}}{2^{j-1}j!}\right)ds$$

$$= 1 + x^2 + \sum_{j=1}^{k} \frac{x^{2j+2}}{2^{j-1}(2j+2)j!}$$

$$= 1 + x^2 + \sum_{j=1}^{k} \frac{x^{2j+2}}{2^j(j+1)!}$$

$$= 1 + x^2 + \sum_{j=2}^{k+1} \frac{x^{2j}}{2^{j-1}j!} = 1 + \sum_{j=1}^{k+1} \frac{x^{2j}}{2^{j-1}j!}$$

Hence holds true for $n = k + 1$. [0.5 Marks]

Therefor, by PMI, we have

$$y_n(x) = 1 + \sum_{j=1}^{n} \frac{x^{2j}}{2^{j-1}j!}.$$

Marking scheme for Q2 : $\frac{dy}{dx} = x - y^2 + e^x$

$$|x| \le 1, \ |y| \le 1 \qquad y(0) = 0.$$

- The function $f(x,y) = x - y^2 + e^x$ is continuous

$$\left[\tfrac{1}{2}\right]$$

- On the given rectangle $|f(x,y)| \le 1 + e$

$$\left[\tfrac{1}{2}\right]$$

- By existence theorem, there is a soln on $(-\alpha, \alpha)$ where $\alpha = \operatorname{Min}\left\{a, \frac{b}{M}\right\}$

  where $a = 1$, $b = 1$, $M \ge 1 + e$. $\left[\tfrac{1}{2}\right]$

- So $\alpha \le \frac{1}{1+e}$ $\left(e.g. \ \alpha = \frac{1}{1+e}\right)$ $\left[\tfrac{1}{2}\right]$

MTL101 Quiz 2
Sem 2: 2021-2022

Solution. We are given a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (3x + y, -2x, -x - y + 2z)$$

(a) The matrix of T (say A) w.r.t. standard basis is

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then, the eigenvalues of T are given by $\mathrm{Det}(xI - A) = 0$ i.e.

$$= \begin{vmatrix} x - 3 & -1 & 0 \\ 2 & x & 0 \\ 1 & 1 & x - 2 \end{vmatrix} = 0$$

characteristic polynomial $f(x) = x^3 - 5x^2 + 8x - 4$ and the eigenvalues are $1, 2, 2$. [1]

(b) Observe that

$$A - I = \begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad A - 2I = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$\mathrm{Rank}(A - I) = 2 \implies \mathrm{Nullity}(A - I) = 1.$
$\mathrm{Rank}(A - 2I) = 1 \implies \mathrm{Nullity}(A - 2I) = 2.$

Now,

$$Dim(E_1) + Dim(E_2) = Nullity(A - I) + Nullity(A - 2I)) = 1 + 2 = 3.$$

As dim of Eigenspaces sum up to the dimension of $\mathbb{R}^3$, T is diagonalizable. [1]

(c) Now, the eigenspace $E_1 = \mathrm{Null}\,(A - I)$ is given by

$$E\left(1\right) = \{(a, -2a, -a) : a \in \mathbb{R}\}$$

and the eigenspace E_2 is given by

$$E\left(2\right) = \{(a, -a, b) : a, b \in \mathbb{R}\}.$$

An ordered basis B of T such that $[T]_B$ is diagonal,

$$B = \{(1, 2, -1), (1, -1, 0), (0, 0, 1)\}.$$

[2]