Independent Subspaces :- Let V be a vector space over F. Let $W_1, W_2, \ldots, W_m$ be subspaces of V. Then $W_1, W_2, \ldots, W_m$ are called independent if for any elements $\vartheta_1 + \vartheta_2 + \cdots + \vartheta_m = 0 \in W_1 + \cdots + W_m$, $\vartheta_i \in W_i$, $1 \le i \le m$ implies $\vartheta_i = 0$ for each i.

Theorem :- $W_1, W_2, \ldots, W_m$ are independent if and only if $\dim(W_1 + \cdots + W_m) = \dim(W_1) + \cdots + \dim(W_m)$

Proof :-

Let B_i be a basis of W_i, $1 \le i \le m$

$$B_i = \{ \vartheta_1^i, \vartheta_2^i, \ldots, \vartheta_{k_i}^i \}, \text{ i.e., } \dim(W_i) = \underset{|B_i|}{\overset{\shortparallel}{k_i}}$$

Take $B = \bigcup_{i=1}^{m} B_i = \{v_1^1, \dots, v_{k_1}^1, v_1^2, \dots, v_{k_2}^2, \dots, v_1^m, \dots, v_{k_m}^m\}$

$|B| \leq \sum_{i=1}^{m} |B_i|$

Some of them may be repeated according to current information.

$$\dim(W_1 + \dots + W_m) = \dim(W_1) + \dim(W_2 + \dots + W_m) - \dim(W_1 \cap (W_2 + \dots + W_m))$$
$$\leq \dim(W_1) + \dim(W_2 + \dots + W_m)$$
$$\dots$$
$$\leq \dim(W_1) + \dim(W_2) + \dots + \dim(W_m)$$
$$= \sum_{i=1}^{m} |B_i|$$

For any $v \in W_1 + \dots + W_m$, $v = v_1 + v_2 + \dots + v_m$

each $v_i \in W_i$, i.e., $v_i = c_1^i v_1^i + \dots + c_{k_i}^i v_{k_i}^i$, $1 \leq i \leq m$

$\Rightarrow B$ spans $W_1 + \dots + W_m$

$$\therefore \quad \dim(W_1 + W_2 + \cdots + W_m) \le |B| \le \sum_{i=1}^{m} |B_i|$$

$\boxed{\Rightarrow}$ Let $W_1, W_2, \cdots, W_m$ are independent.

Then claim, $B_i \cap B_j = \phi$ for $i \neq j$

Because, if $v \in B_i \cap B_j$ then $v \neq 0$

$$0 + 0 + \cdots + \underbrace{v}_{i\text{th}} + 0 \cdots + \underbrace{(-v)}_{j\text{th}} + \cdots + 0 = 0$$

This contradicts that $W_1, \cdots, W_m$ are independent.

$$\dim(W_1 + \cdots + W_m) \le |B| = \sum_{i=1}^{m} |B_i|$$

Let $\underbrace{c_1^1 v_1^1 + \cdots + c_{k_1}^1 v_{k_1}^1}_{\in W_1,\ = 0} + \cdots + \underbrace{c_1^m v_1^m + \cdots + c_{k_m}^m v_{k_m}^m}_{\in W_m,\ = 0} = 0$

$\Rightarrow c_1^i v_1^i + \cdots + c_{k_i}^i v_{k_i}^i = 0, \quad 1 \le i \le m$

$\Rightarrow c_1^i = c_2^i = \cdots = c_{k_i}^i = 0$, $1 \le i \le m$ as $\{v_1^i, v_2^i, \ldots, v_{k_i}^i\}$ is a basis of W_i.

$\Rightarrow B$ is linearly independent.

Since $\dim(W_1 + \cdots + W_m) \le |B|$, B is a basis of $W_1 + \cdots + W_m$.

$$\therefore \quad \dim(W_1 + \cdots + W_m) = |B| = \sum_{i=1}^{m} |B_i| = \sum_{i=1}^{m} \dim(W_i)$$

$\boxed{\Leftarrow}$ Given

$$\dim(W_1 + \cdots + W_m) = \sum_{i=1}^{m} \dim(W_i) = \sum_{i=1}^{m} |B_i|$$

Since,

$$\dim(W_1 + W_2 + \cdots + W_m) \le |B| \le \sum_{i=1}^{m} |B_i|$$

$\dim(W_1 + \cdots + W_m) = |B|$

Since B spans $W_1 + \cdots + W_m$, B is a basis of $W_1 + \cdots + W_m$

Let $w_1 + \cdots + w_m = 0$, where $w_i \in W_i$

Let $w_i = c_1^i v_1^i + \cdots + c_{k_i}^i v_{k_i}^i$

Then

$$c_1^1 v_1^1 + \cdots + c_{k_1}^1 v_{k_1}^1 + \cdots + c_1^m v_1^m + \cdots + c_{k_m}^m v_{k_m}^m = 0$$

Since B is a basis, $c_i^1 = \cdots = c_{k_i}^i = 0$, $1 \le i \le m$

$\Rightarrow w_i = 0$

Then $W_1, \cdots, W_m$ are independent.

Theorem :- Let V be a vector space of dimension n. Let $\operatorname{span}(S) = V$ and $|S| = n$, then S is a basis of V.

Proof :- If possible S is linearly dependent, where $S = \{v_1, \ldots, v_n\}$.

Then $\exists\ c_1, \ldots, c_n$ not all zero such that

$$c_1 v_1 + \cdots + c_n v_n = 0$$

W.L.O.G, $c_n \neq 0$, $v_n = -c_n^{-1}(c_1 v_1 + \cdots + c_{n-1} v_{n-1})$

$$\in \operatorname{span}\{v_1, \ldots, v_{n-1}\}$$

$$\Rightarrow v_1, \ldots, v_{n-1} \in \operatorname{span}\{v_1, \ldots, v_{n-1}\}$$

$$\Rightarrow \underbrace{\operatorname{span}\{v_1, \ldots, v_n\}}_{= V} \subseteq \operatorname{span}\{v_1, \ldots, v_{n-1}\}$$

$\Rightarrow \dim(V) \leq n-1$, this is a contradiction

Theorem:- Let V be a vector space, let $n = \dim(V)$.

(i) If $\text{span}(S) = V$ then $n \leq |S|$

(ii) If $\text{span}(S) = V$ and $n = |S|$ then S is a basis of V.

(iii) If $\text{span}(S) = V$ and S is linearly independent then S is a basis of V, and hence $|S| = n$.

(iv) $|S| = n$ and S is linearly independent, then S is a basis of V.

(v) If $|S| > n$, then S is linearly dependent.

(vi) If S is linearly independent, then $|S| \leq n$ and there is a basis B of V such that $S \subseteq B$