Q: Let T be the linear operator on $\mathbb{R}^4$ given by the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ wrt std basis.

a) Find all values of s and t for which T is diagonalizable.

b) Find an ordered basis B of $\mathbb{R}^4$ s.t. $[T]_B$ is diagonal matrix, whenever T is diagonalizable.

Sol^n: First, Let us find the eigen values of T which is represented by the matrix A wrt the std basis. So, we have,

$$\det(\lambda I - A) = 0 \Rightarrow \det\left(\begin{bmatrix} \lambda & 0 & 0 & 0 \\ -s & \lambda & 0 & 0 \\ 0 & -t & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \lambda^2 (\lambda-1)^2 = 0 \Rightarrow \lambda = 0, 0, 1, 1$$

Hence, the e.values of T are: $\lambda_1 = 0$ with $A.M.(\lambda_1) = A.M.(0) = 2$

and $\lambda_2 = 1$ with $A.M.(\lambda_2) = A.M.(1) = 2$

We know that T is diagonalizable if $A.M.(\lambda) = G.M.(\lambda)$ $\forall \lambda \in \Lambda(T)$

$\Lambda(T)$ → Set of all e.values.

Now, for, $\lambda_1 = 0$, let's consider the matrix $A - \lambda_1 I = A$

Then we have,

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{— (I)}.$$

For, $AM(0) = 2 = GM(0)$, we must have $\dim(\text{null}(A - \lambda_1 I)) = 2$

$\Leftrightarrow \text{nullity}(A) = 2 \Leftrightarrow \text{rank}(A) = 4-2 = 2$ — (II)

From (I), (II), we can see that $\text{nullity}(A) = 2 \Leftrightarrow s = 0$. [By RNT]

So, for T to be diagonalizable, we get $s = 0$.

Now, for $\lambda_2 = 1$, consider, $A - \lambda_2 I = A - I$

$$A - I \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

if $t = 0$, $A - I \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, if $t \neq 0$, $A - I \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, in both cases, $GM(1) = \text{nullity}(A - I) = 2$ [As, $\text{rank}(A - I) = 2$].

So, from above two observation, we get that for T to be diagonalizable, $s = 0$ and $t \in \mathbb{R}$.

b) For a basis B of $\mathbb{R}^4$ st $[T]_B$ is diagonal, we can consider $B = B_1 \cup B_2$, where B_1 and B_2 are the basis of eigenspace of $\lambda_1 = 0$ and $\lambda_2 =$ respectively.

For $\lambda = 0$, $A - 0.I \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Notation:
$E_\lambda =$ Eigenspace (λ)

$\therefore$ if $(x,y,z,w) \in E_0$, then $w = 0$ and $ty + z = 0$
$\Rightarrow z = -ty$

$\therefore (x,y,z,w) = (x,y,-ty,0)$
$= x(1,0,0,0) + y(0,1,-t,0)$

$\therefore B_1 = \{(1,0,0,0), (0,1,-t,0)\}$

Similarly, $A - I \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 8 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ if $(x,y,z,w) \in E_1$, then, $x = 0, y = 0$
$\therefore (x,y,z,w) = z(0,0,1,0) + w(0,0,0,1)$

$\therefore B_2 = \{(0,0,1,0), (0,0,0,1)\}$

$\therefore$ For $B = B_1 \cup B_2 = \{(1,0,0,0), (0,1,-t,0), (0,0,1,0), (0,0,0,1)\}$

$[T]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which is diagonal with eigenvalues in the diagonal entries.

Ques Consider the IVP

$$\frac{dy}{dx} = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1/2, & (x,y) = (0,0) \end{cases}$$

with initial condition

$$y(x_0) = 0$$

(i) For what values of x_0 does this IVP have a solution according to the existence theorem?

(ii) Additionally, for such x_0 find the largest positive α such that the solution exists in the interval $(x_0 - \alpha, x_0 + \alpha)$.

Solⁿ:- Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1/2, & (x,y) = (0,0) \end{cases}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist because

along $y = mx$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{m}{1+m^2}$

$\Rightarrow$ $f(x,y)$ is not continuous on any rectangle containing $(0,0)$

$\Rightarrow$ ~~for~~ $x_0 = 0$, the existence theorem is not applicable.

For $x_0 \neq 0$:

Let $R_b = \{ (x,y) \in \mathbb{R}^2 \mid |x - x_0| < |x_0|, |y| < b \}$

For $b > 0$, R_b doesn't contain $(0,0)$

So, by existence th^m, this IVP has a sol^n.

$$\alpha = \min \left\{ |x_0|, \frac{b}{K} \right\}$$

where $K = \sup_R f(x,y)$

$$= \frac{1}{2}$$

$\because x^2 + y^2 - 2xy \geq 0$

$\Rightarrow \dfrac{xy}{x^2+y^2} \leq \dfrac{1}{2}$

$$\alpha = \min \{ |x_0|, 2b \}$$

$= |x_0|$ by choosing $b > \dfrac{|x_0|}{2}$

Q3. Let A be the 3×3 matrix given by

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Using Cayley Hamilton theorem, show that A is invertible and find its inverse.

Answer: First, we calculate characteristic polynomial of A.

Let $p(x) = \det(A - xI)$

$$= \det \begin{pmatrix} 4-x & 1 & 2 \\ 1 & 3-x & 1 \\ 2 & 1 & 3-x \end{pmatrix}$$

$$= -(x^3 - 10x^2 + 27x - 21)$$

Note that $p(0) = +21 \neq 0$

$\therefore$ A is invertible

By Cayley Hamilton, $p(A) = 0$

$$\Rightarrow A^3 - 10A^2 + 27A - 21I = 0$$

$$\Rightarrow A(A^2 - 10A + 27I) = +21I$$

$$\Rightarrow A\left(\frac{+1}{21}(A^2 - 10A + 27I)\right) = I$$

$$\therefore A^{-1} = \frac{1}{21}(A^2 - 10A + 27I)$$

$$A^2 = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 9 & 15 \\ 9 & 11 & 8 \\ 15 & 8 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{21}\left\{\begin{bmatrix} 21 & 9 & 15 \\ 9 & 11 & 8 \\ 15 & 8 & 14 \end{bmatrix} - 10\begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} + 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{21}\begin{bmatrix} 8 & -1 & -5 \\ -1 & 8 & -2 \\ -5 & -2 & 11 \end{bmatrix}$$