Lecture 4

§ Elementary Matrices.

$$A = \begin{bmatrix} 3 & 1 & -1 & 2 \\ 6 & -2 & 0 & 1 \\ 4 & 5 & 1 & -1 \end{bmatrix} \qquad R_2 \to R_2 + 2R_1$$

$$\sim \begin{bmatrix} 3 & 1 & -1 & 2 \\ 12 & 0 & -2 & 5 \\ 4 & 5 & 1 & -1 \end{bmatrix}$$

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2 \to R_2 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 & 2 \\ 6 & -2 & 0 & 1 \\ 4 & 5 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 & 2 \\ 12 & 0 & -2 & 5 \\ 4 & 5 & 1 & -1 \end{bmatrix}$$

Def: If we perform a single elementary row operation on an identity matrix I, then the result is called an elementary matrix.

Theorem: Let E be an elementary matrix obtained by performing an elementary row operation R on I (identity matrix). If we perform the same elementary row operation R on A then the resulting matrix is EA.

row Operation	Elementary matrix.
$R_3 \to R_3 - 4R_1$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$
$R_2 \to 5R_2$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$R_2 \leftrightarrows R_1$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex: $A = \begin{bmatrix} -3 & -2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$

(1) $R_1 \Leftrightarrow R_2$

(2) $R_2 \to R_2 + 3R_1$

(3) $R_2 \to -2R_2$

(1) $R_1 \Leftrightarrow R_2$ $\quad E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(2) $R_2 \to R_2 + 3R_1$ $\quad E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

(3) $R_2 \to -2R_2$ $\quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$$A_1 = E_1 A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & -2 & 3 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & -2 & 3 \end{bmatrix}$$

$$A_2 = E_2 A_1 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

$$A_3 = E_3 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 6 \end{bmatrix}$$

$$B = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} -3 & -2 & 3 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 6 \end{bmatrix} = A_3.$$

$$A_3 = E_3 E_2 E_1 A.$$

Ex: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ $\quad R_3 \to R_3 - 4R_1$

$R_3 \to R_3 + 4R_1$ on $I_{3\times3}$ $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = E'$

$$E'E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times3}.$$

$$EE' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E' is the inverse of E!

(Def of inverse: If A is an $n\times n$ matrix. Then A is invertible if $\exists$ an $n\times n$ matrix B s.t. $AB = BA = I$)

Theorem: Elementary matrices are invertible

Theorem: A is an $n\times n$ matrix then the following are equivalent.

1. A is invertible
2. $A\underline{x} = \underline{b}$ has unique solution for all $\underline{b}$
3. $A\underline{x} = 0$ has only the trivial solution
4. A can be row reduced to an identity matrix.

$\left(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1 \right)$

next lec.

$3 \Rightarrow 4$ Suppose $A\underline{x} = 0$ has only trivial solution.

Let U be the reduced echelon form of A.

Since there is only one sol to the above system $A\underline{x} = 0$,

there is only one sol to the linear system $U\underline{x} = 0$

$\Rightarrow$ Every column of U is a pivot col

$\Rightarrow$ U has n leading variable & n columns, n rows.

$\Rightarrow$ $U = I_{n\times n}$

$\Rightarrow$ A can be row reduced to the identity matrix.

$4 \Rightarrow 1$ Suppose A can be row reduced to I then $\exists$ elementary matrices $E_1, \dots, E_k$ s.t.

$E_k \cdots E_2 E_1 A = I$

$B = E_k \cdots E_2 E_1 \Rightarrow BA = I.$

§. Computing inverse of $n \times n$ matrix A.

A is an invertible $\Rightarrow$ There is B s.t $AB = I_{n\times n}$.

$$A = [\underline{a}_1 \cdots \underline{a}_n] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

vectors

$\underline{a}_1$ $\underline{a}_n$

$$B = [\underline{b}_1 \cdots \underline{b}_n] \quad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix} = [\underline{e}_1 \cdots \underline{e}_n]$$

$A\underline{b}_1$ = first ~~col~~ col of the matrix AB $= \underline{e}_1$

$A\underline{b}_2$ = Second col " $= \underline{e}_2$

$\vdots$ $\vdots$

$A\underline{b}_n$ = n^{th} col " $= \underline{e}_n$

$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\Rightarrow$ $\underline{b}_1$ is a solution to the linear system $A\underline{x} = \underline{e}_1$

$\underline{b}_2$ " $A\underline{x} = \underline{e}_2$

$\vdots$

$\underline{b}_n$ " $A\underline{x} = \underline{e}_n$

Solve $A\underline{x} = e_i$ we take the augmented system

$$[\underline{a_1} \ldots \underline{a_n} \mid \underline{e_i}] \longrightarrow \text{reduced echelon form}$$

Consider $[\underline{a_1} \ldots \underline{a_n} \mid \underline{e_1} \ldots \underline{e_n}]$

$$[A \mid I_n] \longrightarrow \text{row reduced echelon form}$$

will be

$$[I_n \mid A^{-1}].$$

Ex: $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$.

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$R_2 \to (-1) R_2$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$R_1 \to R_1 - 3R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] = [I_n \mid B]$$

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix}$

$[A|I] =$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_3 \\ R_1 \to R_1 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 6 & 1 & -1 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] \quad R_1 \to R + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -18 & -3 & 5 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] = [I \mid B].$$

$$AB = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Ex</u>: $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ -3 & -1 & 4 \end{bmatrix}$ find inverse, if it exists!

$$[A \mid I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 1 & 0 \\ -3 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 3 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \end{array}\right]$$

This matrix does not have an inverse.