MAXIMUM MARKS: 50

Instructions: Justify all your statements. Remember that you will be graded on what you write on the answer sheet, **NOT** on what you intend to write.

Question 1: [3+2 marks]

(a) Let V be a finite dimensional vector space over a field F and, let $T : V \to V$ be a linear transformation. Let $\mathrm{Im}(T) = \{T(v) : v \in V\}$. Prove that

$$\mathrm{Im}(T) = \mathrm{Im}(T^2) \quad \text{if and only if} \quad \ker(T) + \mathrm{Im}(T) = V.$$

(b) Consider $\mathbb{R}^4$ as a vector space over $\mathbb{R}$. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that the rank of T is 1 and $T^2 \neq 0$. Then calculate the nullity of T^2.

Question 2: [3+3 marks]

(a) Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -2 & -4 \end{pmatrix}$. Find A^{-1} (the inverse of A) by using Cayley-Hamilton theorem.

(b) Consider $\mathbb{R}^3, \mathbb{R}^4$ as vector spaces over $\mathbb{R}$. Let W be the subspace of $\mathbb{R}^3$ spanned by the subset $\{(1,2,1), (0,1,1), (1,3,2)\}$. Construct a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that the range of T equals W.

Queston 3: [2+4] marks]

(a) Justify whether the uniqueness theorem is applicable for the intial value problem (IVP):

$$\frac{dy}{dx} = (x^2+1)y^{2/3}; \quad y(0) = 0.$$

(b) For the ordinary differential equation (ODE)

$$(4xy^2 + 3y)dx + (3x^2y + 2x)dy = 0,$$

find real numbers p, q such that $x^p y^q$ is an integrating factor. Then, solve this ODE by making it an exact equation.

Question 4: [6 marks]

(a) Find the general solution of the following differential equation

$$y^{(5)} - y^{(4)} + 2y^{(3)} - 2y^{(2)} = 0.$$

Notation: Here $y^{(n)}$ denotes the n-th derivative of y.

(b) Find the general solution by using the method of undetermined coefficient of the following equation:

$$y^{(5)} - y^{(4)} + 2y^{(3)} - 2y^{(2)} = t + e^t.$$

Question 5: [6 marks]

(a) Find the general solution of the following homogeneous ordinary differential equation (ODE):

$$t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0.$$

(b) Use variation of parameter to find a particular solution of the following non-homogeneous ODE:

$$t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = t^4.$$

Question 6: [6 marks]
Solve the following initial value problem (IVP) using the Laplace transform:

$$y'' + 7y' + 12y = u(t-2) + \delta(t-3); \quad y(0) = 1, y'(0) = 3.$$

Here δ denotes the Dirac delta function and u denotes the Heaviside function.

Question 7: [4+2 marks]
Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and the inverse Laplace transform.

(a) Using convolution property of the Laplace transform, find $\mathcal{L}^{-1}\left(\dfrac{4}{(s^2+4s+8)^2}\right)$.

(b) Show that $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)(s) = \dfrac{1}{s}\mathcal{L}(f)(s)$.

Question 8: [6 marks]
Solve the following system of linear differential equations (**without** using Laplace transform):

$$\begin{aligned} x_1' &= 7x_1 - 3x_2 + x_3 \\ x_2' &= 8x_1 - 3x_2 + 2x_3 \\ x_3' &= -x_1 + 3x_3. \end{aligned}$$

Question 9: [3 marks]
Suppose the function given by the power series $\sum_{n=0}^{\infty} a_n x^n$ is the solution of the following initial value problem (IVP):

$$y'' + xy' + x^2y = 1 + x; \quad y(0) = 7, y'(0) = 11.$$

Find the values of a_i for $i \leq 5$.