Lemma:- Suppose $A, B \in M_{n\times m}(F)$. If $AX = BX$ for every $X \in M_{m\times 1}(F)$ then $A = B$.

Solution:- Let $X = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Then $AX = BX$ implies

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \Rightarrow \begin{array}{l} a_{11} = b_{11} \\ a_{21} = b_{21} \\ \vdots \\ a_{n1} = b_{n1} \end{array}$$

Similarly,

By taking $X = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow$ i th place

we have

$$\begin{aligned} a_{1i} &= b_{1i} \\ a_{2i} &= b_{2i} \\ &\vdots \\ a_{ni} &= b_{ni} \quad \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

Therefore, $A = B$.

Lemma:- The map $T : V \to F^n$ defined by $T(v) = [v]_B$ is an isomorphism. Let $B = \{b_1, \cdots, b_n\}$.

Proof:- For each vector the coefficients with respect to the basis B is unique. Therefore, T is well defined.

If $[v]_B = [w]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow v = c_1 b_1 + \cdots + c_n b_n = w$

$\Rightarrow T(v) = T(w) \Rightarrow v = w$. Therefore, T is injective.

For any $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in F^n$. Take $v = c_1 b_1 + \cdots + c_n v_n \Rightarrow T(v) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Therefore, T is surjective.

Let $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ & $[w]_B = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$. Then $v = c_1 b_1 + \cdots + c_n b_n$
$w = d_1 b_1 + \cdots + d_n b_n$

$$[pv + qw]_B = [(pc_1 + qd_1) b_1 + \cdots + (pc_n + qd_n) b_n]_B$$

$$= \begin{bmatrix} pc_1 + qd_1 \\ \vdots \\ pc_n + qd_n \end{bmatrix} = p \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + q \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \Rightarrow T(pv + qw) = pT(v) + qT(w).$$

Let $T: V \to W$ be a linear transformation of finite dimensional vector spaces and let A, B be bases of V and C, D be bases of W.

$$a_1 = P_{11} b_1 + P_{21} b_2 + \cdots + P_{m1} b_m$$
$$\vdots$$
$$a_m = P_{1m} b_1 + P_{2m} b_2 + \cdots + P_{mm} b_m$$

$\dim(V) = m$

$\dim(W) = n$

Let $[v]_B = P [v]_A$ in V

and $[T(v)]_D = Q [T(v)]_C$ in W

where P and Q are invertible matrices in $M_{m \times m}(F)$ and $M_{n \times n}(F)$ respectively.

Recall, $[T(v)]_D = [T]_B^D [v]_B$

$$[T(v)]_C = [T]_A^C [v]_A$$

$$\Rightarrow [T]_B^D [v]_B = Q [T]_A^C [v]_A$$

$$\Rightarrow [T]_B^D P [v]_A = Q [T]_A^C [v]_A$$

For any $X \in F^n$, we have $v \in V$ s.t. $[v]_A = X$

Therefore, $[T]_B^D PX = Q[T]_A^C X$ for every $X \in F^n$

$$\Rightarrow [T]_B^D P = Q[T]_A^C$$

$$\Rightarrow [T]_B^D = Q[T]_A^C P^{-1} \Big/ Q^{-1}[T]_B^D P = [T]_A^C$$

In particular, when $V = W$, $A = C$, $B = D$ & $P = Q$

we have, $$[T]_B = P[T]_A P^{-1}$$

$$\text{or} \quad P^{-1}[T]_B P = [T]_A$$

Problem :- Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ $T(x, y, z) = (2x + z, y + 3z)$ be the linear transformation. Let $A = \{(1,1,0), (1,0,1), (1,1,1)\}$ and $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ be bases of $\mathbb{R}^3$. Let $C = \{(2,3), (3,2)\}$

and $D = \{(1,0),(0,1)\}$ be basis of $\mathbb{R}^2$. Then find P, Q, $[T]_A^C$, $[T]_B^D$ and verify $[T]_B^D P = Q [T]_A^C$.

Solution :-

$$(1,1,0) = 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$$
$$(1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$
$$(1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

$$(2,3) = 2(1,0) + 3(0,1)$$
$$(3,2) = 3(1,0) + 2(0,1)$$

$$\Rightarrow \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Now, $T(1,0,0) = (2,0) = 2(1,0) + 0(0,1)$

$T(0,1,0) = (0,1) = 0(1,0) + 1(0,1)$

$T(0,0,1) = (1,3) = 1(1,0) + 3(0,1)$

Therefore, $[T]_B^D = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

$T(1,1,0) = (2,1) = -\frac{1}{5}(2,3) + \frac{4}{5}(3,2)$

$T(1,0,1) = (3,3) = \frac{3}{5}(2,3) + \frac{3}{5}(3,2)$

$T(1,1,1) = (3,4) = \frac{6}{5}(2,3) + \frac{1}{5}(3,2)$

Therefore, $[T]_A^C = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

$$[T]_B^D\, P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$Q[T]_A^C = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & \frac{6}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

<u>Lemma</u>:- Let $S, T : V \to W$ be two linear transformation. Let A and B be two bases of V and W respectively. Then

(a) $[S+T]_A^B = [S]_A^B + [T]_A^B$

(b) $[\lambda T]_A^B = \lambda [T]_A^B$

(c) If $R : W \longrightarrow U$ is a linear transformation with C is a basis of U then
$$[R \circ T]_A^C = [R]_B^C\, [T]_A^B .$$

Let $A = \{a_1, a_2, \cdots, a_n\}$, $B = \{b_1, b_2, \cdots, b_p\}$ & $C = \{c_1, c_2, \cdots, c_m\}$ be are ordered bases of V, W and U respectively.

Let $$[T]_A^B = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots \\ t_{p1} & t_{p2} & \cdots & t_{pn} \end{bmatrix}_{p\times n} \qquad [R]_B^C = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & r_{22} & \cdots & r_{2p} \\ \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mp} \end{bmatrix}_{m\times p}$$

Let $$[R\circ T]_A^C = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{bmatrix}_{m\times n}$$

Then, $$T(a_j) = t_{1j}b_1 + t_{2j}b_2 + \cdots + t_{pj}b_p, \quad 1 \le j \le n$$

$$R(b_j) = r_{1j}c_1 + r_{2j}c_2 + \cdots + r_{mj}c_m, \quad 1 \le j \le p$$

$$(R\circ T)(a_j) = s_{1j}c_1 + s_{2j}c_2 + \cdots + s_{mj}c_m, \quad 1 \le j \le n$$

$$T(a_j) = t_{1j} b_1 + t_{2j} b_2 + \cdots + t_{pj} b_p, \quad 1 \le j \le n$$

$$R(b_j) = r_{1j} c_1 + r_{2j} c_2 + \cdots + r_{mj} c_m, \quad 1 \le j \le p$$

implies, $(R \circ T)(a_j) = t_{1j}(r_{11} c_1 + r_{21} c_2 + \cdots + r_{m1} c_m) + t_{2j}(r_{12} c_1 + r_{22} c_2 + \cdots + r_{m2} c_m) + \cdots + t_{pj}(r_{1p} c_1 + r_{2p} c_2 + \cdots + r_{mp} c_m)$

$$= \left(\sum_{k=1}^{p} r_{1k} t_{kj}\right) c_1 + \left(\sum_{k=1}^{p} r_{2k} t_{kj}\right) c_2 + \cdots + \left(\sum_{k=1}^{p} r_{mk} t_{kj}\right) c_m$$

$$\Rightarrow [R \circ T]_A^C = \begin{bmatrix} \sum_{k=1}^{p} r_{1k} t_{k1} & \sum_{k=1}^{p} r_{1k} t_{k2} & \cdots & \sum_{k=1}^{p} r_{1k} t_{kn} \\ \sum_{k=1}^{p} r_{2k} t_{k1} & \sum_{k=1}^{p} r_{2k} t_{k2} & \cdots & \sum_{k=1}^{p} r_{2k} t_{kn} \\ \vdots & & & \\ \sum_{k=1}^{p} r_{mk} t_{k1} & \sum_{k=1}^{p} r_{mk} t_{k2} & \cdots & \sum_{k=1}^{p} r_{mk} t_{kn} \end{bmatrix}_{m \times n}$$

$$= [R]_B^C \, [T]_A^B$$