MTL101:: Tutorial 1 :: Linear Algebra

(1) Suppose we have a system of three linear equations in real coefficients and in two unknowns:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$

Interpret geometrically the following statements:
(a) the system has no solutions.
(b) the system has a unique solution.
(c) the system has an infinitely many solutions.
Further, provide values to $a_i, b_i, c_i \in \mathbb{R}$ $(i = 1, 2, 3)$ so that the above statements hold.

(2) Suppose we have a system of three linear equations in real coefficients and in three unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Interpret geometrically the following statements:
(a) the system has no solutions.
(b) the system has a unique solution.
(c) the system has an infinitely many solutions.
Further, provide values to a_i, b_i, c_i, d_i $(i = 1, 2, 3)$ so that the above statements hold.

(3) Suppose we have a system of three linear equations in two unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \end{aligned}$$

Can this system have a unique solution? Interpret geometrically the following statements:
(a) the system has no solutions.
(b) the system has an infinitely many solutions.
Further, provide values to a_i, b_i, c_i, d_i $(i = 1, 2)$ so that the above statements hold.

(4) Suppose we have a system of linear equations in complex coefficients. Can we interpret the statements in question 1, 2 and 3 exactly in the same way? Solve the following system of equations (for finding x, y in $\mathbb{C}$):

$$\begin{aligned} (1-i)x + (1+i)y &= 2+3i \\ (1+i)x + (1-i)y &= 3-i \end{aligned}$$

(5) Suppose the lines L_1 and L_2 are defined by $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z-1}{3}$. Show that $L_1 \cap L_2$ is empty. Write a system of four linear equations in three unknowns in **standard form** which has no solutions using the fact that $L_1 \cap L_2$ is empty.

(6) Suppose A, B are square matrices of same size. Prove the following statements
(a) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
(b) $\det(AB) = \det(A)\det(B)$ (assume A and B are 2×2 matrices) and in particular, $\det(AB) = \det(BA)$.
(c) $\det(A+B) = \det(A) + \det(B)$ is **false**.

(7) Which of the following matrices are row reduced echelon matrix. Give a reason when the matrix is not row reduced echelon.

$$\begin{pmatrix}1&0&5\\0&2&3\\0&0&0\end{pmatrix},\ \begin{pmatrix}1&0&5\\0&1&3\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&2\\1&0&4\\0&0&0\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}.$$

(8) Find row reduce echelon matrix row equivalent to the matrices in the previous question.

(9) Show that every elementary matrix is invertible and the inverse is an elementary matrix.

(10) Compute the rank of the following matrices. Determine which are invertible.

$$\begin{pmatrix}1&1&2\\2&3&8\\-3&-1&2\end{pmatrix},\ \begin{pmatrix}1&2&-4\\-1&-1&5\\2&7&-3\end{pmatrix},\ \begin{pmatrix}1&3&-4\\1&5&-1\\3&13&-6\end{pmatrix}.$$

(11) Find inverse of the invertible matrices in the previous question by reducing the matrix to row reduced echelon form (identity matrix).

(12) Write the following matrices as the product of elementary matrices (whenever possible):

$$\begin{pmatrix}1&-3\\-2&4\end{pmatrix},\ \begin{pmatrix}1&2&3\\0&1&4\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&1&2\\2&3&8\\-3&-1&2\end{pmatrix}.$$

(13) Solve the following system of homogeneous linear equations by reducing the coefficient matrix into row reduced echelon form:
a) $2x_1+4x_2-5x_3+3x_4=0,\ 3x_1+6x_2-7x_3+4x_4=0,\ 5x_1+10x_2-11x_3+6x_4=0$
b) $x-2y-3z=0,\ 2x+y+3z=0,\ 3x-4y-2z=0,$
c) $x_1+2x_2+3x_3-2x_4+4x_5=0,\ 2x_1+4x_2+8x_3+x_4+9x_5=0,\ 3x_1+6x_2+13x_3+4x_4+14x_5=0.$

(14) Solve the following systems of equations by reducing the augmented matrix to the row reduced echelon form:
a) $x_1-x_2+2x_3=1, 2x_1+2x_3=1, x_1-3x_2+4x_3=2,$
b) $x_1+7x_2+x_3=4,\ x_1-2x_2+x_3=0,\ -4x_1+5x_2+9x_3=-9,$
c) $x_2+5x_3=-4,\ x_1+4x_2+3x_3=-2,\ 2x_1+7x_2+x_3=-1,$
d) $-2x_1-3x_2+4x_3=5,\ x_2-x_3=4,\ x_1+3x_2-x_3=2.$

(15) Consider the following system of equations:
$x+2y+z=3,\ ay+5z=10,\ 2x+7y+az=b.$
a) Find all values of a for which the following system of equations has a unique solution.
b) Find all pairs (a,b) for which the system has more than one solution.

(16) Find $a,b,c,p,q\in\mathbb{R}$ such that the following system has a solution: $\begin{pmatrix}1&2&3\\0&1&p\\0&0&q\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}.$

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