MTL101-Minor-Question-1

Solution 1. For $\lambda, \mu \in \mathbb{R}$, consider the augmented matrix (A, B)

$$(A,B) = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & \lambda & 5 \\ 1 & 1 & 2 & 4 & \mu \\ 2 & 2 & 3 & 6 & \mu+3 \end{bmatrix}$$

Step 1 We apply row operations to form its row reduced echleon form.

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & \lambda & 5 \\ 1 & 1 & 2 & 4 & \mu \\ 2 & 2 & 3 & 6 & \mu+3 \end{pmatrix} \xrightarrow{(R_3 \to R_3 - R_1), (R_2 \to R_2 - R_1), (R_4 \to R_4 - 2R_1)}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 1 & 2 & \mu-3 \\ 0 & 0 & 1 & 2 & \mu-3 \end{pmatrix} \xrightarrow{(R_4 \to R_4 - R_3)} \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 1 & 2 & \mu-3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(R_3 \to R_3 - R_2)}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 4-\lambda & \mu-5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[1]

Step 2 System is consistent for: 1) $\lambda = 4$ and $\mu = 5$ [0.5]

2) $\lambda \neq 4$ and $\mu \in \mathbb{R}$ [0.5]

Step 3

1. When $\lambda = 4$ and $\mu = 5$, we have

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying row operations again to obtain the solution set;

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(R_1 \to R_1 - R_2)} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have x_2, x_4 as free variables and $x_1 + x_2 = 1, x_3 + 2x_4 = 2$. Hence, solution set is given as:
$\{(1-a, a, 2-2b, b) | a, b \in \mathbb{R}\}$ [1]

Step 4

2. When $\lambda \neq 4$ and $\mu \in \mathbb{R}$ we have the augmented matrix as;

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 1 & ((\mu-5)/(4-\lambda)) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying row operations again to obtain the solution set;

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 1 & ((\mu-5)/(4-\lambda)) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(R_1 \to R_1 - R_2)} \begin{pmatrix} 1 & 1 & 0 & 4-\lambda & 1 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 1 & ((\mu-5)/(4-\lambda)) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case, x_2 is the free variable, $x_4 = (\mu-5)/(4-\lambda)$, $x_3 = 2 - (\lambda-2)((\mu-5)/(4-\lambda))$ and $x_1 + x_2 + (4-\lambda)x_4 = 1$ i.e $x_1 = 6 - x_2 - \mu$. Hence, the solution set is given as :

$\{(6-a-\mu, a, 2-(\lambda-2)((\mu-5)/(4-\lambda)), (\mu-5)/(4-\lambda)) | a \in \mathbb{R}\}$ [1]

MTL 101, MINOR 1, PROBLEM 2

2. [2+2] Let $M_2(\mathbb{C})$ be the set of all 2×2 matrices with entries in $\mathbb{C}$. Observer that $M_2(\mathbb{C})$ is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$. Let $\overline{d}$ denote the complex conjugate of $d \in \mathbb{C}$. Let

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) \ : \ a + \overline{d} = 0 \right\} \subseteq M_2(\mathbb{C})$$

Consider $V_1 = M_2(\mathbb{C})$ over $\mathbb{R}$ and $V_2 = M_2(\mathbb{C})$ over $\mathbb{C}$.

(a) Is W a subspace of V_1? Justify your answer.

Let $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, v = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in W$. Then $a + \overline{d} = 0 = e + \overline{h}$. For $c_1, c_2 \in \mathbb{R}$, we get

$$c_1 u + c_2 v = c_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} + c_2 \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} c_1 a + c_2 e & c_1 b + c_2 f \\ c_1 c + c_2 g & c_1 d + c_2 h \end{bmatrix}$$

Notice that $c_1 a + c_2 e + \overline{c_1 d + c_2 h} = c_1 a + c_2 e + c_1 \overline{d} + c_2 \overline{h} = c_1(a + \overline{d}) + c_2(e + \overline{h}) = 0$ (as $c_1, c_2 \in \mathbb{R}$). Thus $c_1 u + c_2 v \in W$ and hence W is a subspace of V_1.

(b) Is W a subspace of V_2? Justify your answer.

For $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & d_1 + id_2 \end{bmatrix} \in W$, since $a + \overline{d} = 0$, we have $a_1 + ia_2 + \overline{d_1 + id_2} = a_1 + ia_2 + d_1 - id_2 = (a_1 + d_1) + i(a_2 - d_2) = 0$. Thus $d_1 = -a_1, d_2 = a_2$. Thus

$$W = \left\{ \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & -a_1 + ia_2 \end{bmatrix} \right\} \tag{0.1}$$

For $u \in W$, notice that

$$iu = i \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & -a_1 + ia_2 \end{bmatrix} = \begin{bmatrix} ia_1 - a_2 & ib_1 - b_2 \\ ic_1 - c_2 & -ia_1 - a_2 \end{bmatrix}$$

But $ia_1 - a_2 + \overline{-ia_1 - a_2} = (-a_2 - a_2) + i(a_1 + a_1) \neq 0$ (if either of a_1, a_2 is nonzero). Thus $iu \notin W$ if either of a_1, a_2 is nonzero. Thus W is NOT subspace of V_2.

3) Let $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the linear transformation defined by

$$T(x, y, z, w) = (y + 2z + 3w,\ x - y,\ x + 2z + 3w,\ 3x - y + 4z + 6w).$$

a) Find a basis of the range of T.

b) Find the nullity of T.

Solution.

a) Let $B = \{e_1, e_2, e_3, e_4\}$ denote the standard basis for $\mathbb{R}^4$. Then

$$\begin{aligned} T(e_1) &= (0, 1, 1, 3) \\ T(e_2) &= (1, -1, 0, -1) \\ T(e_3) &= (2, 0, 2, 4) \\ T(e_4) &= (3, 0, 3\ 6) \end{aligned} \qquad (1)$$

Now, $\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & -1 & 0 & -1 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 3 & 6 \end{pmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 3 & 6 \end{pmatrix}$$

$$\begin{array}{l} R_3 \longrightarrow R_3 - 2R_1 \\ R_4 \longrightarrow R_4 - 3R_1 \end{array} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \\ 0 & 3 & 3 & 9 \end{pmatrix}$$

$$\begin{array}{l} R_1 \longrightarrow R_1 + R_2 \\ R_3 \longrightarrow R_3 - 2R_2 \\ R_4 \longrightarrow R_4 - 3R_2 \end{array} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (1\tfrac{1}{2})$$

A basis of the range of T is $\{(1,0,1,2), (0,1,1,3)\}$. $(1/2)$.

b) By a) $\text{rank}(T) = 2$. Now, using rank-nullity theorem,

$$\begin{aligned} \text{nullity}(T) &= \dim(\mathbb{R}^4) - \text{rank}(T) \\ &= 4 - 2 = 2. \qquad (1). \end{aligned}$$

MARKING SCHEME FOR QUESTION 4

Question 4(a):[2 marks]
Justify if there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the following:

$$T(1,1,0) = (3,5), \quad T(1,0,1) = (5,3), \quad T(2,1,1) = (4,4)$$

If it exists, define one such T.
Solution:
Suppose there exists a linear transformation T such that

$$T(1,1,0) = (3,5), \quad T(1,0,1) = (5,3), \quad T(2,1,1) = (4,4).$$

Since

$$(2,1,1) = (1,1,0) + (1,0,1), \qquad \text{[0.5 marks]}$$

then using the linearity of T, we have

$$T(2,1,1) = T(1,1,0)+T(1,0,1) = (3,5)+(5,3) = (8,8) \neq T(2,1,1). \quad \text{[1 marks]}$$

Therefore T doesn't satisfy linearity for $u = (1,1,0)$ and $v = (1,0,1)$. This contradicts our assumption. Hence there doesn't exists any such linear transformation. [0.5 marks]
Question 4(b):[2 marks]

Justify if there exists a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the following:

$$S(1,0,0) = (1,2), S(0,1,0) = (7,5), S(1,1,0) = (8,7).$$

If it exists, define one such S.

Solution:

Method I:

Since $\{(1,0,0),(0,1,0)\}$ is a linearly independent set, therefore it can be extended to a basis of $\mathbb{R}^3$. Let $\{(1,0,0),(0,1,0),v\}$ be the basis of $\mathbb{R}^3$ (one such choice of v is $(0,0,1)$). [1 marks]

Now, corresponding to each $\gamma \in \mathbb{R}^2$, we can define a linear transformation by fixing the image of v equal to γ, therefore we can construct infinitely many such linear transformations. For simplicity, we choose $\gamma = (0,0)$ and set $S(0,0,1) = (0,0)$. [0.5 marks]

For $(x,y,z) \in \mathbb{R}^3$, we have $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$. Now we can define

$$\begin{aligned} S(x,y,z) &= S(1,0,0) + S(0,1,0) + S(0,0,1) \\ &= x(1,2) + y(7,5) + z(0,0) \\ &= (x+7y, 2x+5y) \quad \text{[0.5 marks]} \end{aligned}$$

Method II:

Since

$$(1,1,0) = (1,0,0) + (0,1,0),$$

observe that

$$S(1,0,0) + S(0,1,0) = (1,2) + (7,5) = (8,7) = S(1,1,0).$$

This gives the guarantee for existence of such a linear transformation. [1 marks]

Now, a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$ must be of the form

$$S(x, y, z) = (ax + by + cz, dx + ey + fz), \quad a, b, c, d, e, f \in \mathbb{R}. \tag{1}$$

Now using the given conditions, we get

$$\begin{aligned} S(1,0,0) &= (a,d) = (1,2) \\ S(0,1,0) &= (b,e) = (7,5) \\ S(1,1,0) &= (a+b, d+e) = (8,7) \end{aligned} \tag{2}$$

this gives $a = 1, b = 7, d = 2, e = 5$ and c, f can be any real number. Therefore, for different choice of c, d we can define a linear transformation satisfying our given conditions. For example take $c = d = 0$, then

$$S(x, y, z) = (x + 7y, 2x + 5y), \quad \text{[1 marks]}.$$

Q5 a) • Describing $W_1 \cap W_2$ as solution space [$\frac{1}{2}$]
- Finding RRE form or a reduced form to find rank of the coefficient matrix [$\frac{1}{2}$]
- Finding rank of this matrix [$\frac{1}{2}$]
- Finding $\dim W_1 \cap W_2 = 1$ [$\frac{1}{2}$]
- Giving a reason for [$\frac{1}{2}$]

Q 5 c) • Using dimension formula with arguments [$\frac{1}{2}+\frac{1}{2}$]
- Finding $\dim W_1$ & $\dim W_2$
- Giving reasons why $W_1 + W_2 \neq V$ [$\frac{1}{2}$]

Writing a basis B of $W_1 \cap W_2$ [$\frac{1}{2}$]

5 c) Writing a basis of W_1 containing B or any basis of W_1 [$\frac{1}{2}$]
- Writing a basis of W_1 containing B or any basis of W_2 [$\frac{1}{2}$]
- Writing a basis of W_2 containing B [$\frac{1}{2}$]
- Finding a basis of $W_1 + W_2$ containing B
- Finding a basis of W_1+W_2 using basis of W_1 & W_2 which donot intersect [1]

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5a) $W_1 \cap W_2 = \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 : \begin{array}{r} a_0 + a_1 + a_2 + a_3 = 0 \\ a_1 + 2a_2 + 3a_3 = 0 \\ a_0 + 2a_1 + 3a_2 + 4a_3 = 0 \\ a_2 + 3a_3 = 0 \end{array} \right\}$

The coeff. matrix is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is RRE. Its rank is 3 so that $\dim W_1 \cap W_2$ is $4-3=1$

5b) The coeff. matrix for W_1 is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ which is RRE & its rank is 2 so that $\dim W_1 = 4-2=2$

The coeff. matrix of W_2 is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ which is RRE & its rank is 2 so that $\dim W_2 = 4-2=2$.

By dimension formula $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$
$= 2+2-1=3 \neq 4 = \dim V$ (since $\{1, x, x^2, x^3\}$ is a basis of V.

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5 c) By using 5 a) $\{1-3x+3x^2-1\}$ is a basis of $W_1 \cap W_2$.
Using RRE form of the matrix defining W_1, $(2, -3, 0, 1)$
i.e. $2-3x+x^3 \in W_1$ which is not a multiple of $1-3x+3x^2-1$
so that $\{1-3x+3x^2-1, 2-3x+x^3\}$ is a basis of W_1.
Use the same argument for W_2 to see
$\{1-3x+3x^2-1, 2-x\}$ is a basis of W_2
So, $\{1-3x+3x^2-1, 2-3x+x^3, 2-x\}$ is a basis of W_1+W_2

Alternatively, pick any bases B_1 & B_2 of $W_1 \cap W_2$.
If $B_1 \cap B_2 = \phi$, $B_1 \cup B_2$ is not LI (although $B_1 \cup B_2$ spans W_1+W_2). Find a basis of $W_1 \cup W_2$ from $B_1 \cup B_2$
You form matrix whose rows are coordinate vectors of vectors in $B_1 \cup B_2$
& find its RRE form. The non zero rows gives a basis of W_1+W_2.

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Ques 6 $\quad S = \{(1,2,3,4), (0,1,2,3)\} \subseteq \mathbb{R}^4$

Extend S to a basis of $\mathbb{R}^4$

Solution: Since $\dim(\mathbb{R}^4) = 4$, we need to add two more vectors to S in order to get a basis of $\mathbb{R}^4$.

It is enough to find $v_3, v_4 \in \mathbb{R}^4$ such that $\{(1,2,3,4), (0,1,2,3), v_3, v_4\}$ is linearly indep.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix} \overset{R_1 \to R_1 - 2R_2}{\sim} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

If we take $v_3 = (0,0,1,0)$ and $v_4 = (0,0,0,1)$, then the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is of rank 4 which implies the rows are linearly indep.

Hence, $\mathcal{B} = S \cup \{(0,0,1,0), (0,0,0,1)\}$ is a basis of $\mathbb{R}^4$.

Marking scheme:

+3 : If a correct basis is given and justified.

+2 : If there is some minor mistake or slight justification is missing.

+1 : For some major mistakes.

0 : If completely wrong.

7.a

$$T(x,y,z) = (x+2y+3z,\ y+2z,\ -x-z)$$

$$B = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

$\frac{1}{2}$ Mark
$$\left\{\begin{aligned} T(1,0,0) &= (1,0,-1) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1) \\ T(0,1,0) &= (2,1,0) = 2(1,0,0) + 1(0,1,0) + 0(0,1,0) \\ T(0,0,1) &= (3,2,-1) = 3(1,0,0) + 2(0,1,0) - 1(0,1,0) \end{aligned}\right.$$

$\frac{1}{2}$ mark
$$[T]_B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix}^T$$

7.b

$$B' = \{(1,0,0),\ (1,1,0),\ (1,1,1)\}$$

1 mark
$$\left\{\begin{aligned} T(1,0,0) &= (1,0,-1) = 1(1,0,0) + 1(1,1,0) - 1(1,1,1) \\ T(1,1,0) &= (3,1,-1) = 2(1,0,0) + 2(1,1,0) - 1(1,1,1) \\ T(1,1,1) &= (6,3,-2) = 3(1,0,0) + 5(1,1,0) - 2(1,1,1) \end{aligned}\right.$$

1 mark
$$[T]_{B'} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -1 \\ 3 & 5 & -2 \end{pmatrix}^T$$

7.c

$$\left.\begin{aligned} (1,0,0) &= 1(1,0,0) + 0(0,1,0) + 0(0,0,1) \\ (1,1,0) &= 1(1,0,0) + 1(0,1,0) + 0(0,0,1) \\ (1,1,1) &= 1(1,0,0) + 1(0,1,0) + 1(0,0,1) \end{aligned}\right\}$$ 1 mark

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ Satisfying}$$ 1 mark

$$P[T]_{B'} = [T]_B P$$

N.B If anyone construct other P Satisfying the relation $P[T]_{B'} = [T]_B P$, we are giving 2 marks to them.