COL202: Discrete Mathematical Structures. II semester, 2024-25.
Amitabha Bagchi
Tutorial Sheet 10: Generating functions and recurrences.
11 April 2025

Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM. Questions marked with a * may be a little harder and can be considered optional.

Problem 1 [LLM18, Prob. 16.6]
Suppose that

$$S(x) := \frac{x^2 + x}{(1-x)^3}.$$

Find $[x^n]S(x)$. Then find $[x^n](S(x)/(1-x))$. Using this prove the identity that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 2 [LLM18, Prob. 16.8]
Without using generating functions prove that there are $n+1$ ways of choosing n fruit if the number of apples must be even, the number of bananas must be a multiple of 5, the number of oranges can be at most 4 and the number of pears can be at most 1.

Problem 3 [LLM18, Prob. 16.13]
Suppose we are given a sequence $\{a_i\}_{i\geq 0}$ and numbers b, c such that

$$a_n = b \cdot a_{n-1} + c$$

for $n \geq 1$ and $G(x)$ is the generating function of the sequence. Find $[x^n]bxG(x)$ and $[x^n]cx/(1-x)$. Using these find a simple expression for $G(x) - xbG(x) - cx/(1-x)$. From here use the method of partial fractions to simplify $G(x)$.

Problem 4 [LLM18, Prob. 16.16]
Prove that for all $k \in \mathbb{N}$ there are polynomials $R(x)$ and $S(x)$ such that

$$[x^n]\left(\frac{R(x)}{S(x)}\right) = n^k.$$

Problem 5
In each part below the sequence $\{a_n\}_{n\geq 0}$ satisfies the given recurrence. Find the ordinary power series generating function in each case and solve to find a_n where possible.

Problem 5.1

$$a_{n+2} = 2a_{n+1} - a_n, (n \geq 0, a_0 = 0, a_1 = 1).$$

Problem 5.2

$$a_{n+1} = a_n/3 + 1, (n \geq 0, a_0 = 0).$$

Problem 6
Let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive integers. Find a recurrence for $f(n)$ and try to solve it to the extent possible using generating functions.

Problem 7 ♠
In the 2-dimensional plane we have n lines such that no two lines are parallel and no three lines intersect at one point. If R_n is the number of regions created by these n lines, find a recurrence for R_n and solve it.

Problem 8
Find a recurrence relation for the number of bit strings of length n that contain the string 01. Try and solve it if possible.

Problem 9
Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s. Try and solve it if possible.

Problem 10
Let A_n be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of A_n . Solve this recurrence relation to find a formula for d_n.

Problem 11
In how many ways can $3r$ balls be chosen from $2r$ red balls, $2r$ blue ballls and $2r$ green balls?

Problem 12
Evaluate the following sums:

Problem 12.1

$$\binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + i \cdot \binom{n}{i} + \cdots + n \cdot \binom{n}{n}$$

Problem 12.2
Given that $k \leq m$ and $k \leq n$

$$\binom{n}{0} \cdot \binom{m}{k} + \binom{n}{1} \cdot \binom{m}{k-1} + \binom{n}{2} \cdot \binom{m}{k-2} + \cdots + \binom{n}{k} \cdot \binom{m}{0},$$

Problem 12.3

$$\binom{2n}{n} + \binom{2n-1}{n-1} + \cdots + \binom{2n-i}{n-i} + \cdots + \binom{n}{0}$$

Problem 13 * (pp 25 of [Wilf94])
Let X be a random variable that takes values $0, 1, 2, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$ respectively. Clearly we must have p_i is nonegative for each i and the sum of p_is is 1. Let $P(x)$ be the ordinary power series generating function (opsgf) of $\{p_n\}_{n \geq 0}$.

Problem 13.1
Express the mean and standard deviation of X in terms of $P(x)$.

Problem 13.2
Let X_1 and X_2 be two independent random variables with the same distribution as X. Let $p_n^{(2)}$ be the probability that $X_1 + X_2 = n$. What is the opsgf of $\{p_n^{(2)}\}_{n \geq 0}$?

Problem 13.3
For $k \geq 2$, let $X_1, \ldots, X_k$ be k independent random variables with the same distribution as X. Let $p_n^{(k)}$ be the probability that $\sum_{i=1}^{k} X_i = n$. What is the opsgf of $\{p_n^{(k)}\}_{n \geq 0}$?

Problem 13.4
Use the results above to write out the mean and standard deviation of $\sum_{i=1}^{k} X_i$ where the X_i are independently chosen with the same distribution as X.

References

[LLM18] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science, June 2018, MIT Open Courseware.

[Wilf94] Herbert S. Wilf, generatingfunctionology, 1994, Academic Press.