1.

$$\langle (x_1,x_2,x_3) | (y_1,y_2,y_3) \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + x_1y_2 + x_2y_1 + x_3y_1 + x_1y_3$$

$\mathcal{B} = \{e_1, e_3\}$ is a basis for W = XZ-plane.

We use the Gram-Schmidt process to convert $\mathcal{B}$ into an orthogonal basis $\mathcal{B}' = \{v_1, v_2\}$ given by

$$v_1 = e_1$$

$$v_2 = e_3 - \frac{\langle e_3 | e_1 \rangle}{\|e_1\|^2} e_1 = e_3 - e_1 \quad (\langle e_3|e_1\rangle = 1 ; \langle e_1, e_1 \rangle = 1)$$

[1 mark]

So, $\mathcal{B}' = \{e_1, e_3 - e_1\}$

The best approx. of v by a vector of W is given by

$$w_0 = \frac{\langle v|e_1\rangle}{\|e_1\|^2} e_1 + \frac{\langle v|e_3-e_1\rangle}{\|e_3-e_1\|^2}(e_3-e_1) \quad \text{[1 mark]}$$

$$= \frac{6}{1} e_1 + \frac{4}{2}(e_3 - e_1)$$

$$= 4e_1 + 2e_3 \quad \therefore \boxed{w_0 = (4,0,2)} \quad \text{[1 mark]}$$

Shortest distance, $d = \|v - w_0\|$

$$= \|(-3,2,1)\|$$

$$= \sqrt{\langle(-3,2,1)|(-3,2,1)\rangle}$$

$$= \sqrt{9+8+3-6-6-3-3}$$

[1 mark]

$$\boxed{d = \sqrt{2}}$$

2. The matrix of T w.r.t. to the standard basis is

$$A = \begin{pmatrix} 7 & 2 & 3 \\ 0 & 8 & 0 \\ 1 & -2 & 5 \end{pmatrix}$$

$\therefore$ Characteristic polynomial of T is

$$p(x) = \det(xI - A) = \begin{vmatrix} x-7 & -2 & -3 \\ 0 & x-8 & 0 \\ -1 & 2 & x-5 \end{vmatrix}$$

$$= (x-8)\left[(x-7)(x-5) - 3\right]$$
$$= (x-8)(x^2 - 12x + 32)$$
$$= (x-8)(x-8)(x-4)$$

[1 mark]

Thus the eigenvalues are $4, 8, 8$.

For $\lambda = 4$: $4I - A = \begin{pmatrix} -3 & -2 & -3 \\ 0 & -4 & 0 \\ -1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow (1, 0, -1)$ is an eigenvector with evalue 4. [1 mark]

For $\lambda = 8$: $8I - A = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ -1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \{(2,1,0), (3,0,1)\}$ is a basis for the eigenspace corresponding to eigenvalue 8. [1 mark]

If we take $\boxed{\mathcal{B} = \{(1,0,-1), (2,1,0), (3,0,1)\}}$, [1 mark]

then $[T]_{\mathcal{B}}$ is the diagonal matrix $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

3. (a) $\langle (x_1, x_2) | (x_1, x_2) \rangle = x_1^2 + 4x_1x_2 + x_2^2$
$= (x_1 + 2x_2)^2 - 3x_2^2$

So, if we choose $x_2 = 1$ & $x_1 + 2x_2 = 0$ ie. $x_1 = -2, x_2 = 1$,
we get $\langle (-2,1) | (-2,1) \rangle = -3 < 0$
which contradicts a property of inner product.

Hence, the statement is $\boxed{\text{FALSE}}$.

Another way: Use the fact that
$\langle (x_1, x_2) | (y_1, y_2) \rangle = a x_1 y_1 + b x_1 y_2 + c x_2 y_1 + d x_2 y_2$
is an inner prod. if and only if $a > 0$, $b = c$ & $ad - b^2 > 0$.

Comparing with the given eqn., we have $a = 1, b = c = 2, d = 1$.
Then $ad - b^2 = -3 < 0$
Hence, $\boxed{\text{FALSE}}$

(b) If 0 is an eigenvalue of T, then $\exists\, v \neq 0$ s.t.
$T(v) = 0v = 0$
$\Rightarrow T$ is not 1-1 $\quad (\because T(0) = 0 = T(v);\ v \neq 0)$
$\Rightarrow T$ is not invertible.

Hence, $\boxed{\text{TRUE}}$

4. The eqn. is of the form $M\,dx + N\,dy = 0$

with $M = 3xy + 2y^2 + 4y$

$N = x^2 + 2xy + 2x$

Here $M_y = 3x + 4y + 4$ & $N_x = 2x + 2y + 2$

Since $M_y \neq N_x$ the eqn. is not exact. (1 mark)

But, $\dfrac{M_y - N_x}{N} = \dfrac{x + 2y + 2}{x^2 + 2xy + 2x} = \dfrac{1}{x}$

$\therefore$ There is an integrating factor

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right) = \exp\left(\int \tfrac{1}{x}\,dx\right) = x$$ [1 mark]

Multiplying the given eqn. by x, we get

$$(3x^2y + 2xy^2 + 4xy)\,dx + (x^3 + 2x^2y + 2x^2)\,dy = 0$$

which is an exact eqn.

We find $u(x,y) = x^3y + x^2y^2 + 2x^2y + h(y)$ [1 mark]

$$u_y = x^3 + 2x^2y + 2x^2 + h'(y)$$

and $x^3 + 2x^2\cancel{y} + 2x^2 = u_y = x^3 + 2x^2y + 2x^2 + h'(y)$

$\Rightarrow h'(y) = 0$ So, we can take $h(y) = 0$.

$\therefore u(x,y) = x^3y + x^2y^2 + 2x^2y$

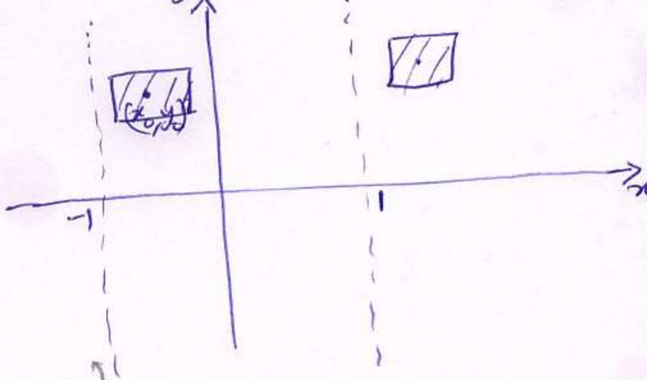
The general solution is [1 mark]

$$\boxed{x^3y + x^2y^2 + 2x^2y = c}$$

5. (a) The given ODE can be written as

$$y' = f(x,y) = \frac{4y}{x^2-1} \quad \text{for } x \neq \pm 1 .$$

$f(x,y)$ & $\frac{\partial f}{\partial y} = \frac{4}{x^2-1}$ are both continuous at all points in the xy-plane except on the lines $x=-1$ & $x=1$. [1 mark]



If $x_0 \neq -1$ & $x_0 \neq 1$, then for any $y_0 \in \mathbb{R}$, we can find a small enough closed rectangle R centered at (x_0, y_0) so that $f(x,y)$ & $\frac{\partial f}{\partial y}(x,y)$ are both cont. on R. [1 mark]

Thus the existence-uniqueness thm. guarantees a unique soln. for the IVP if $(x_0,y_0) \in \{(x,y)\in\mathbb{R}^2 : x \in \mathbb{R}\setminus\{\pm 1\},\ y\in\mathbb{R}\}$

If $x_0=-1$ or $x_0=1$, the theorem is not applicable.

(b) Solving the ODE $(x^2-1)\frac{dy}{dx} = 4y$ using the method of separation of variables, we get [mark]

$$y = c\left(\frac{x-1}{x+1}\right)^2 \ ; \ c \in \mathbb{R}.$$

are solutions. if $x^2-1 \neq 0, y \neq 0$

But the solution is defined for $x \neq -1$, and by differentiating, we can check that

$$y = \frac{c(x-1)^2}{(x+1)^2}, \quad x \neq -1$$

is a soln.

Since this satisfies $y(1)=0$ for every $c\in\mathbb{R}$, we get infinitely many solns.

$$\boxed{y = c\left(\frac{x-1}{x+1}\right)^2}, \quad x \in (-1,\infty) \qquad \text{(1 mark]}$$

to the IVP with $y(1)=0$.