COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 8

Announced on: Mar 06 (Wed)

1. Based on Problem 12.53 in [LLM17].

   Procedure *Mark* starts with a connected, unweighted, and simple graph with all edges unmarked and then marks some edges. At any point in the procedure a path that includes only marked edges is called a *fully marked* path, and an edge that has no fully marked path between its endpoints is called *eligible*.

   Procedure Mark simply keeps marking eligible edges, and terminates when there are none. Prove that Mark terminates, and that when it does, the set of marked edges forms a spanning tree of the original graph.

2. This problem describes the *cut property* of minimum spanning trees (MSTs).

   Let $G = (V, E)$ be a connected and weighted graph. For any subset $S \subseteq V$ of the vertices, let $\bar{S} \coloneqq V \setminus S$ denote the vertices not in S. Let $E(S, \bar{S})$ denote the set of all edges with one endpoint each in S and $\bar{S}$. Show that if there exists a unique edge $e \in E(S, \bar{S})$ with the smallest weight among the edges in $E(S, \bar{S})$, then e must belong to every MST of graph G.

3. This problem describes the well-known *Prim's algorithm* for constructing an MST.

   Consider a procedure that, given a connected and weighted graph $G = (V, E)$, grows a tree as follows: Start with a vertex $u \in V$, and then successively add a minimum weight edge with exactly one endpoint in the tree. The procedure stops when the tree spans the vertices in V.

   Prove that the above procedure terminates and returns an MST.

   Hint: You may use the cut property in Problem 2.

4. Given a connected and weighted graph, show that if each edge has a distinct weight, then there is a unique minimum spanning tree.

5. Prove or disprove: A graph is a tree if and only if there is a unique path between any pair of vertices.

6. Prove or disprove: Let $G = (V, E)$ be a connected graph with $|V| = n$ vertices. Then, G is a tree if and only if $|E| = n - 1$.

7. Prove or disprove: A graph is bipartite if and only if it has no cycle of odd length.

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.