Note Title 28-01-2025

Our focus here is mainly on examples and problems of whatever we have covered in class on vector space over F till LI/LD.

eg Let F be a field. let $m \in \mathbb{N}$.

Define $V = F^m = \{v = (v_1, .., v_m) : v_i \in F \ \forall\ 1 \le i \le m\}$

Define two operations

$$+ : V \times V \to V$$
$$\cdot : F \times V \to V$$

as

$$v + w = (v_1 + w_1, v_2 + w_2, --, v_m + w_m)$$

$$\alpha v = (\alpha v_1, \alpha v_2, -- \alpha v_n), \quad \alpha \in F$$

Then $F^m(F)$ is a vectors space
(can check all properties)

Additive identity is $(0, 0, --, 0) \in F^m$
and additive inverse is $(-v_1, -v_2, --, -v_m)$

with $v_i \in F \Rightarrow -v_i \in F \quad \forall\ 1 \le i \le m$.

In particular, $\mathbb{R}^m(\mathbb{R})$, $\mathbb{C}^m(\mathbb{C})$, $Q^m(Q)$, $(\mathbb{Z}_p)^m(\mathbb{Z}_p)$ are all v.s.; where $\mathbb{Z}_p$ is integer modulo p field, p is prime number.

eg Let F be a field. Consider

$$V = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in F \right\}$$

Define

$$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix},$$

where $A = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ in V.

Define $\alpha A = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ 0 & \alpha d_1 \end{bmatrix}$, $\alpha \in F$

One can check all properties to see V is a v.s. over F. Note $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is additive identity and $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in V$ then $-A = \begin{bmatrix} -a & -b \\ 0 & -d \end{bmatrix} \in V$

with $A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

eg. $V = C[0,1] = \{ f : [0,1] \to \mathbb{R} : f$ is continuous on $[0,1] \}$.

$F = \mathbb{R}$

Define $+$ on $V \times V$ and $\cdot$ on $F \times V$ as

$$(f+g)(x) = f(x) + g(x), \quad \forall\, x \in [0,1]$$

$$(\alpha f)(x) = \alpha (f(x)), \quad \forall\, x \in [0,1],\ \alpha \in F$$

Then we can see V is a v.s. over $\mathbb{R}$.
with

$$0 : [0,1] \to \mathbb{R}, \quad 0(x) = 0, \quad \forall\, x \in [0,1]$$

is additive identity and $f \in V$

$$\Rightarrow -f : [0,1] \to \mathbb{R} \text{ as } (-f)(x) = -f(x),$$

And $-f$ is continuous on $[0,1]$ as so is f.

$\therefore$ V is a v.s. over $\mathbb{R}$.

eg. $V = \{ \langle a_n \rangle : \lim_{n \to \infty} a_n \text{ exists} \}$

$=$ set of all real convergent sequences.

$F = \mathbb{R}$.

Define $+$ on $V \times V$ and $\cdot$ on $F \times V$ as

$$\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle, \quad n \in \mathbb{N}$$

(componentwise addition)

$$\alpha\langle a_n\rangle = \langle \alpha a_n\rangle, \quad \alpha \in F$$

(componentwise multiplying by α)

Then one can check all properties to see $(V,+,\cdot)$ is a v.s. over $\mathbb{R}$.

eg. Let $S \neq \Phi$ and F be a field. Define

$$F^{(S)} = \{ f : S \to F : f(x) = 0 \text{ except for a finitely many values } x \in S \}$$

Define $+$ on $F^{(S)} \times F^{(S)}$ as

$$(f+g)(x) = f(x) + g(x), \quad x \in S$$

$$(\alpha f)(x) = \alpha(f(x)), \quad x \in S, \quad \alpha \in F.$$

It is a v.s. over F as $0 \in F^{(S)}$

$0 : S \to F$ as $0(s) = 0, \ \forall s \in S$

$\Rightarrow 0$ (as function) $\in F^{(S)}$ is additive identity.

And $f : S \to F$ then $-f : S \to F$ as

$$(-f)(x) = -f(x), \quad x \in S.$$

$-f$ is additive inverse of f in F^S.

$\Rightarrow F^S$ is a v.s. over F.

eg For $n \in \mathbb{N} \cup \{0\}$, let $P_n(F)$ denotes the set of all polynomials of degree atmost n and coefficients of polynomials from F.

$$P_n(F) = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : a_i \in F, \forall \ 0 \le i \le n \}$$

Define $+$ on $P_n(F) \times P_n(F)$ and $\cdot$ on $F \times P_n(F)$ as follows

$$p(x) + q(x) = (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0)$$

termwise addition

$$\alpha p(x) = (\alpha a_n) x^n + (\alpha a_{n-1}) x^{n-1} + \cdots + \alpha a_1 x + \alpha a_0$$

$\alpha \in F$.

where $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in P_n(F)$

$q(x) = b_n x^n + \cdots + b_1 x + b_0 \in P_n(F)$

Then $(P_n(F), +, \cdot)$ is a vector space over F.

eg. Consider Q (set of all rationals) over the field $\mathbb{R}$, i.e., $Q(\mathbb{R})$

This is <u>not</u> a vector space because

$\alpha = \sqrt{2} \in F = \mathbb{R}$ and $1 \in \mathbb{Q}$ but $\alpha . 1 \notin \mathbb{Q}$.

<u>NOTE</u> : $\mathbb{Q}(\mathbb{Q})$, $\mathbb{R}(\mathbb{Q})$, $\mathbb{R}(\mathbb{R})$ are all v.s.

<u>eg.</u> Consider the set of all reals $\mathbb{R}$ over the field of complex numbers $\mathbb{C}$, i.e., $\mathbb{R}(\mathbb{C})$. This is not a v.s. $\because$ $\alpha = i \in \mathbb{C}$ and $1 \in \mathbb{R}$ but $\alpha . 1 = i \notin \mathbb{R}$.

<u>Note</u> :- $\mathbb{C}(\mathbb{C})$ is a v.s.
$\mathbb{C}(\mathbb{R})$ is a v.s.

<u>eg</u> Let $V = \{ A \in M_{n\times n}(\mathbb{R}) : A^T = -A \}$
set of all real skew symmetric matrices of order $n\times n$, $n \in \mathbb{N}$.

Define $A+B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

(entrywise- addition)

and $\alpha A = \alpha[a_{ij}] = [\alpha a_{ij}]$, $\alpha \in \mathbb{R}$

(entrywise multiplication)

Let $A, B \in V \Rightarrow A^T = -A$, $B^T = -B$

$\therefore (A+B)^T = A^T + B^T = -(A+B)$

and $(\alpha A)^T = \alpha A^T = -\alpha A$

$\therefore$ $A+B \in V$, $\alpha A \in V$.

Also zero matrix $0 \in V$ is additive identity.

let $A \in V$. Then $A^T = -A$

let $B = -A$. Then $B^T = (-A)^T = -(A)^T = A$

$= -(-A)$

$= -(B)$

$\therefore$ $B \in V$.

$\Rightarrow$ V is a VS over $\mathbb{R}$. (You can check rest of the properties).

e.g. $V = \{ A \in M_{n\times n}(\mathbb{C}) : (\bar{A})^T = A \}$

where $\bar{A} = \overline{[a_{ij}]} = [\bar{a}_{ij}]$

and "bar" means complex conjugate

let $F = \mathbb{C}$. be the field

Then $V(F)$ is not a v.s.

let $A \in V \Rightarrow (\bar{A})^T = A$

& let $\alpha = i \in \mathbb{C} = F$.

$\alpha A = iA \notin V$

$\because (\overline{\alpha A})^T = \bar{\alpha} \bar{A}^T = -iA = -\alpha A$

eg. Let $V = \{ f : [0,1] \to \mathbb{R}$ such that $f(\tfrac{1}{2}) = 1 \}$, and $F = \mathbb{R}$.

let $f(x) = 1, \ \forall \ x \in [0,1] \Rightarrow f \in V$

$$g(x) = \begin{cases} 0, & x \neq \tfrac{1}{2} \\ 1, & x = \tfrac{1}{2} \end{cases} \Rightarrow g \in V$$

$$(f+g)(x) = \begin{cases} 1 & \text{if } x \neq \tfrac{1}{2} \\ 2 & \text{if } x = \tfrac{1}{2} \end{cases}$$

$\Rightarrow f + g \notin V.$

$\Rightarrow V$ is <u>NOT</u> a v.s. over $\mathbb{R}$.

Some simple notings

(i) Let $V(F)$ be a v.s. and $x, y, z \in V$.
If $x + y = x + z \Rightarrow y = z$

Pf:-
$$\begin{aligned} y = 0 + y &= (-x + x) + y \\ &= -x + (x+y) \quad \text{(associative)} \\ &= -x + (x+z) \\ &= (-x+x) + z \quad \text{(associative)} \\ &= 0 + z \\ &= z. \end{aligned}$$

This is called cancellation law.

(ii) Additive identity is unique
Additive inverse is unique

Pf:- let $z_1, z_2 \in V$ be two additive identies in V

$$\Rightarrow \left.\begin{aligned} z_1 + x &= x + z_1 = x \\ z_2 + x &= x + z_2 = x \end{aligned}\right\} \quad \forall x \in V.$$

$$\Rightarrow \quad x + z_1 = x + z_2$$

Invoke cancellation law in (i) above.

$$\Rightarrow \quad z_1 = z_2.$$

$\therefore$ Additive identity in V is unique & is denoted by symbol 0.

Let $x \in V$ and it has two additive inverses

$$\begin{aligned} x + y_1 &= 0 \\ x + y_2 &= 0 \end{aligned} \qquad \text{for some } y_1, y_2 \in V$$

$$\Rightarrow \quad x + y_1 = x + y_2$$

Invoke cancellation law to get $y_1 = y_2$.

Note associativity is playing role in the cancellation law, and hence in the uniqueness result also.

(iii) Let $V(F)$ be a v.s. Then $0 . v = 0$, $\forall\, v \in V$, where LHS $0 \in F$ and RHS 0 is in V.

Pf :- $0.v + \overset{\text{in } V}{0} = 0.v = (\overset{\text{in } F}{0}+0).v$
$= 0.v + 0.v$ (Distributive)

$\Rightarrow 0.v + 0 = 0.v + 0.v$

$\Rightarrow 0.v = 0$ by cancellation law

(iv) Let $V(F)$ be a v.s. Then

$$\underbrace{-(\alpha v)} = \underbrace{(-\alpha).v}$$

Additive inverse of αv in V

$-\alpha$ is additive inverse of α in F
$(-\alpha).v \in V$.

Pf :- $0 = 0.v$
$= (-\alpha + \alpha).v$
$= (-\alpha).v + \alpha.v$

By uniqueness of additive inverse, we get
$-(\alpha.v) = (-\alpha).v$

(v) Let $V(F)$ be a v.s. Then $\alpha.0 = 0$, $\forall\, \alpha \in F$.

Pf :- $\alpha.0 + 0 = \alpha.0 = \underbrace{\alpha.(0+0) = \alpha.0 + \alpha.0}_{\text{distributive law}}$

Invoke cancellation law to get $\alpha.0 = 0$.

We now move to see some examples in LI|LD and linear span.

eg. Suppose $V(F)$ is a v.s. and

$$S = \{v_1, v_2, v_3, v_4\} \subset V$$

Let $v_3 = \alpha v_2 + \beta v_4$ for some $\alpha, \beta \in F$

Then $L(S) = L(S \setminus \{v_3\})$

Pf:- $S - \{v_3\} \subset S$

$$\Rightarrow L(S - \{v_3\}) \subseteq L(S)$$

Next, if $v \in L(S) = \text{Span}(S)$

$$\Rightarrow v = \hat{\alpha} v_1 + \hat{\beta} v_2 + \hat{\gamma} v_3 + \hat{\delta} v_4 \text{, for some } \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in F$$

$$= \hat{\alpha} v_1 + \hat{\beta} v_2 + \hat{\gamma}(\alpha v_2 + \beta v_4) + \hat{\delta} v_4$$

$$= \hat{\alpha} v_1 + (\hat{\beta} + \hat{\gamma}\alpha) v_2 + (\beta\hat{\gamma} + \hat{\delta}) v_4$$

$$:= \hat{\alpha} v_1 + \xi v_2 + \eta v_4 \text{, } \xi, \eta, \hat{\alpha} \in F$$

$$\in L(S - \{v_3\})$$

$\therefore L(S) \subseteq L(S - \{v_3\})$.

The above two relations give the equality result as required.

Note:- The idea in the above result is that if in a set any vector is already a linear combination of other vectors of the same set then its deletion from the set will not change the linear span of the set.

This could be interpreted that presence of any vector, which is linearly dependent on the other vectors of the same set, is not significant and its removal will not bring in any change in the linear span of the set.

Definition: Let $V(F)$ be a v.s and $S \subseteq V$. We say S is linearly dependent (LD) if $\exists$ vectors $v_1, v_2, --, v_k \in S$ and scalars $\alpha_1, \alpha_2, --, \alpha_k \in F$ such that

(i) $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$

(ii) $\alpha_1, \alpha_2, --, \alpha_k$ are **Not all** 0.

A set S which is not LD is called LI (linearly independent)

If S is finite say $S = \{v_1, v_2, \ldots, v_n\}$ in V then LD of S means

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ and not all $\alpha_1, \ldots, \alpha_n$ are zero.

So, LI of S means

$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0.$

eg. Let $V = \mathbb{R}^3$, $F = \mathbb{R}$

$$S = \left\{ \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ t \end{bmatrix} \right\}$$

For what value(s) of t, S is LD?

Soln Let S be LD

$\Rightarrow \exists\ \alpha, \beta, \gamma \in \mathbb{R}$, not all zero, such that

$$\alpha \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ -6 \\ t \end{pmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ -4 & 2 & -6 \\ 5 & -4 & t \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

This is a homogenous system of linear equations having a non-zero solution $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

$\Rightarrow$ A^{-1} does not exist.

$\Rightarrow$ $|A| = 0$

Calculate $|A|$ & see $|A| = 0$ when $t = 9$.

eg. $V = \mathbb{R}^3$, $F = \mathbb{R}$

$$S = \left\{ v_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, v^3 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$$

To check LI / LD of S, write

$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$, $\alpha, \beta, \gamma \in \mathbb{R} = F$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

$|A| = 1(12-8)$
$+ 2(-3) \neq 0$

$\Rightarrow$ A^{-1} exists

$\therefore$ $A \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$

$\Rightarrow$ S is LI.

eg. Let $V = \mathbb{R}^3$, $F = \mathbb{R}$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix} \right\}$$

Note $\begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

$\Rightarrow$ S is LD.

If we write $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix}$

So, $v_3 = 2v_1 + v_2$

$\Rightarrow L(S) = L(S - \{v_3\})$

But nothing special about v_3.

We could express

$$v_2 = v_3 - 2v_1$$

$\therefore L(S) = L(S - \{v_2\})$

eg Consider $V = P_2(\mathbb{R})$ and

$$S = \{ p(x) = 1 + 3x + 2x^2,\ q(x) = 3 + x + 2x^2,\ r(x) = 2x + x^2 \} \subset V$$

To check LI/LD of S.

Let $\alpha p + \beta q + \gamma r = 0$

$$\Rightarrow \quad (\alpha p + \beta q + \gamma r)(x) = 0 \,, \quad \forall x$$

$$\Rightarrow \quad \alpha\, p(x) + \beta\, q(x) + \gamma\, r(x) = 0 \,, \quad \forall x$$

$$\Rightarrow \quad \alpha(1 + 3x + 2x^2) + \beta(3 + x + 2x^2) + \gamma(2x + x^2)$$

$$= (\alpha + 3\beta) + (3\alpha + \beta + 2\gamma)x + (2\alpha + 2\beta + \gamma)x^2 = 0$$

$$\Rightarrow \quad \left.\begin{aligned} \alpha + 3\beta &= 0 \\ 3\alpha + \beta + 2\gamma &= 0 \\ 2\alpha + 2\beta + \gamma &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

$$|A| = 1(1-4) + 3(4-3) = 0 \Rightarrow \underline{S \text{ is LD.}}$$

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1 \,, \quad R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -8 & 2 \\ 0 & -4 & 1 \end{bmatrix}$$

$$R_2 \to -\frac{R_2}{8} \qquad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1/4 \\ 0 & -4 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + 4R_2 \qquad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$ $\begin{bmatrix} 1 & 0 & 3/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$

This is RRE form of A. and γ is free

$\alpha = -\frac{3}{4}\gamma$, $\beta = \frac{1}{4}\gamma$

Take $\gamma = 4$

$\alpha = -3$, $\beta = 1$

$\therefore$ $-3p(x) + q(x) + 4r(x) = 0$

eg. Let $V = C(\mathbb{R})$, $F = \mathbb{R}$

$V = \{ f : R \to \mathbb{R} : f \text{ is continuous on } \mathbb{R} \}$

Let $S = \{ f, g, h \}$, $f(x) = \cos x$

$g(x) = \sin x$

$h(x) = x$.

To check LI / LD of S in V.

Let $\alpha f + \beta g + \gamma h = 0$, $\alpha, \beta, \gamma \in \mathbb{R}$

$\Rightarrow (\alpha f + \beta g + \gamma h)(x) = 0$, $\forall x \in \mathbb{R}$

$\Rightarrow \alpha f(x) + \beta g(x) + \gamma h(x) = 0$, $\forall x \in \mathbb{R}$

$\Rightarrow \alpha \cos x + \beta \sin x + \gamma x = 0$, $\forall x \in \mathbb{R}$

let $x = 0 \Rightarrow \alpha = 0$

let $x = \frac{\pi}{2} \Rightarrow \beta + \gamma \frac{\pi}{2} = 0$

let $x = \frac{\pi}{4} \Rightarrow \frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}} + \frac{\gamma\pi}{4} = 0$

$$\therefore \left.\begin{array}{l} 2\beta + \gamma\pi = 0 \\ 2\sqrt{2}\,\beta + \gamma\pi = 0 \end{array}\right\} \Rightarrow \beta = 0 = \gamma, \text{ and } \alpha = 0$$

$\therefore$ S is LI

eg. $V = C(\mathbb{R})$, $F = \mathbb{R}$

$S = \{f, g, h\} \subset V$, $f(x) = e^{x}$
$g(x) = e^{2x}$
$h(x) = e^{3x}$

To check LI / LD of S

$$\alpha f + \beta g + \gamma h = 0$$

$$\Rightarrow (\alpha f + \beta g + \gamma h)(x) = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \alpha f(x) + \beta g(x) + \gamma h(x) = 0, \ \forall x \in \mathbb{R}$$

$$\Rightarrow \alpha e^{x} + \beta e^{2x} + \gamma e^{3x} = 0, \ \forall x \in \mathbb{R}$$

$x = 0 \Rightarrow \alpha + \beta + \gamma = 0$
$x = 1 \Rightarrow \alpha e + e^{2}\beta + e^{3}\gamma = 0$

$x = 2 \Rightarrow \alpha e^{2} + \beta e^{4} + \gamma e^{6} = 0$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ e & e^2 & e^3 \\ e^2 & e^4 & e^6 \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

$$|A| = 1(e^8 - e^7) + 1(e^5 - e^7) + 1(e^5 - e^4) \neq 0$$

$\Rightarrow$ A^{-1} exists.

$$\therefore \quad A \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

$\Rightarrow$ S is LI in V.

eg. Consider vector space $\mathbb{C}^2(\mathbb{R})$ means $F = \mathbb{R}$ and

$$V = \mathbb{C}^2 = \{(a + ib, c + id) : a, b, c, d \in \mathbb{R}\}$$

$$\mathbb{C}^2 \rightarrow \mathbb{C} \times \mathbb{C}$$

let $S = \{v_1 = (2+i, 1), v_2 = (1, 1-i)\} \subset V$

To check LI/LD of S.

let $\alpha v_1 + \beta v_2 = 0$, $\alpha, \beta \in F = \mathbb{R}$

$\Rightarrow \alpha(2+i, 1) + \beta(1, 1-i) = 0 \longrightarrow$ 0 in V

$\Rightarrow (2\alpha + \beta + i\alpha, \alpha + \beta - \beta i) = (0, 0)$

$\Rightarrow \quad 2\alpha + i\alpha + \beta = 0$

$\alpha + \beta - \beta i = 0$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 2+i & 1 \\ 1 & 1-i \end{bmatrix}}_{A} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

$|A| = (2+i)(1-i) - 1 = 2 - i \neq 0$

$\Rightarrow A^{-1}$ exists

$\Rightarrow A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \Rightarrow S$ is LI.

eg For what value(s) of m the vectors

$\begin{bmatrix} m+1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -m \\ -1 \end{bmatrix}, \begin{bmatrix} m \\ 1-m \\ 2 \end{bmatrix}$ are LD in $\mathbb{R}^3(\mathbb{R})$.

Let the vectors be LD in $\mathbb{R}^3(\mathbb{R})$

$\Rightarrow \exists\ \alpha, \beta, \gamma \in \mathbb{R}$, not all zero, such that

$$\alpha \begin{bmatrix} m+1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -m \\ -1 \end{bmatrix} + \gamma \begin{bmatrix} m \\ 1-m \\ 2 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} m+1 & 1 & m \\ 1 & -m & 1-m \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

has a non-zero solution $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$. $\Rightarrow |A| = 0$.

Apply elementary row transformations in A

$R_1 \leftrightarrow R_3$ $\begin{bmatrix} 1 & -1 & 2 \\ 1 & -m & 1-m \\ m+1 & 1 & m \end{bmatrix}$

$R_2 \to R_2 - R_1$ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -m+1 & -1-m \\ m+1 & 1 & m \end{bmatrix}$

$R_3 \to R_3 - (m+1) R_1$ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -m+1 & -1-m \\ 0 & m+2 & -2-m \end{bmatrix}$

$R_2 \to R_2 + R_3$ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3-2m \\ 0 & m+2 & -2-m \end{bmatrix}$

$R_3 \to R_3 - \left(\frac{m+2}{3}\right) R_2$ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3-2m \\ 0 & 0 & \frac{2}{3} m(m+2) \end{bmatrix}$

$\therefore \quad |A| = 0$

$\Leftrightarrow \quad m = 0 \quad \text{or} \quad m = -2$

eg. Consider a vector space $\mathbb{C}^2(\mathbb{C})$

means $V = \mathbb{C}^2 = \mathbb{C} \times \mathbb{C} = \{ (a+ib, c+id) : a, b, c, d \in \mathbb{R} \}$

$F = \mathbb{C}$

let $S = \{ (1+i, 1+i), (1+i, 1-i) \} \subset V$

To check LI | LD of S.

Note $\alpha, \beta \in F = \mathbb{C}$. let $\alpha = a_1 + ib_1$
$\beta = a_2 + ib_2$

and $\alpha(1+i, 1+i) + \beta(1+i, 1-i) = 0$

$\Rightarrow (a_1+ib_1)(1+i, 1+i) + (a_2+ib_2)(1+i, 1-i) = 0$

$\Rightarrow ((a_1-b_1)+i(a_1+b_1), (a_1-b_1)+i(a_1+b_1))$
$+ ((a_2-b_2)+i(a_2+b_2), (a_2+b_2)+i(-a_2+b_2)) = \underset{\text{in } \mathbb{C}^2}{0}$

$\Rightarrow ((a_1-b_1+a_2-b_2)+i(a_1+b_1+a_2+b_2),$
$(a_1-b_1+a_2+b_2)+i(a_1+b_1-a_2+b_2)) = (0,0)$

$\Rightarrow \left.\begin{array}{l} a_1-b_1+a_2-b_2 = 0 \\ a_1+b_1+a_2+b_2 = 0 \end{array}\right] \Rightarrow a_1+a_2 = 0 = b_1+b_2$

$\left.\begin{array}{l} a_1-b_1+a_2+b_2 = 0 \\ a_1+b_1-a_2+b_2 = 0 \end{array}\right] \Rightarrow a_1+b_2 = 0 = a_2-b_1$

$\Rightarrow a_1 = -a_2, \; a_2 = b_1, \; b_1 = -b_2, \; a_1 = -b_2$

$\therefore \quad a_1 = -a_2 = -b_1 = b_2 = -a_1$

$\Rightarrow \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 0$

$\Rightarrow \quad \alpha = 0, \quad \beta = 0$

$\Rightarrow$ S is LI

Suppose in above example we change $F = \mathbb{R}$.

Then $\alpha, \beta \in \mathbb{R}$

$\therefore \quad \alpha(1+i, 1+i) + \beta(1+i, 1-i) = \underset{\text{in } \mathbb{C}^2}{0}$

$\Rightarrow \quad (\alpha + \alpha i + \beta + \beta i, \quad \alpha + \alpha i + \beta - \beta i) = (0, 0)$

$\Rightarrow \quad (\alpha + \beta)(1+i) = 0$

$(\alpha + \beta) + i(\alpha - \beta) = 0 \Rightarrow \left.\begin{array}{l} \alpha + \beta = 0 \\ \alpha - \beta = 0 \end{array}\right\}$

$\Rightarrow \quad \alpha = 0 = \beta$

$\Rightarrow$ S is LI. .

Now if we express

$(1+i, 1+i) = (1,1) + i(1,1)$

$(1+i, 1-i) = (1,1) + i(1,-1)$

and consider

$S = \{(1,1), (1,1), (1,1), (1,-1)\} \subset \mathbb{R}^2 = V.$

with $F = \mathbb{C}$. or $F = \mathbb{R}$

then S is LD in $\mathbb{R}^2$

eg. Let $A \in M_{n\times n}(\mathbb{R})$ & let $v_i \in M_{n\times 1}(\mathbb{R})$
$1 \le i \le n$.

If $\{\underbrace{Av_1}_{n\times 1}, \underbrace{Av_2}_{n\times 1}, \ldots, \underbrace{Av_n}_{n\times 1}\}$ be LI in $M_{n\times 1}(\mathbb{R})$

then prove that $\{v_1, v_2, \ldots, v_n\}$ are LI in $M_{n\times 1}(\mathbb{R})$.

Pf :- Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in F = \mathbb{R}$. such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0_{n\times 1}$$

$$\Rightarrow A(\alpha_1 v_1 + \alpha_2 v_2 + \text{———} + \alpha_n v_n) = A(0) = 0_{n\times 1}$$

$$\Rightarrow \alpha_1 Av_1 + \alpha_2 Av_2 + \cdots + \alpha_n Av_n = 0$$

Since $\{Av_1, Av_2, \ldots, Av_n\}$ is LI

so $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$

$\Rightarrow \{v_1, v_2, \ldots, v_n\}$ is LI.

Note Converse is not true; means

$\{v_1, v_2, \ldots, v_n\}$ LI $\not\Rightarrow \{Av_1, Av_2, \ldots, Av_n\}$ LI.

For instance, we can take $A = 0$ (zero matrix) in $M_{n\times n}(\mathbb{R})$.

eg. Let $V = \{ f : \mathbb{R} \to \mathbb{R} \}$, $F = \mathbb{R}$

Let $S = \{ f, g \} \subset V$, $f(x) = \sin 2x$
$g(x) = \sin ax$, $a \in \mathbb{R}$

Find all a for which S is LI.

Soln Let $\alpha, \beta \in \mathbb{R}$ and $\alpha f + \beta g = 0$

$$\Rightarrow (\alpha f + \beta g)(x) = 0 \quad \forall \; x \in \mathbb{R}$$

$$\Rightarrow \alpha f(x) + \beta g(x) = 0 \quad \forall \; x \in \mathbb{R}$$

$$\Rightarrow \alpha \sin 2x + \beta \sin ax = 0 \quad \forall \; x \in \mathbb{R} \quad -(1)$$

For S to be LI, (1) must implies $\alpha = \beta = 0$

Here, note putting values of x will not yield anything simple to get value of a (as in earlier examples)

So, we need to think something else to get 'a'.

let us differentiate (1)

$$\alpha (2 \cos 2x) + \beta (a \cos ax) = 0 \quad \forall x$$

Again differentiate

$$-4\alpha \sin 2x - a^2 \beta \sin ax = 0 \quad \forall x$$

$$\Rightarrow 4\alpha \sin 2x + a^2 \beta \sin ax = 0 \quad \forall x \quad -(2)$$

Look at ① & ②

$$\left.\begin{aligned} \alpha \sin 2x + \beta \sin ax &= 0 \\ 4\alpha \sin 2x + a^2 \beta \sin ax &= 0 \end{aligned}\right\} \forall x \in \mathbb{R}$$

$$4\alpha \sin 2x + 4\beta \sin ax = 0$$
$$4\alpha \sin 2x + a^2\beta \sin ax = 0$$

$$\Rightarrow (4-a^2)\beta \sin ax = 0 \quad \forall x.$$

$$\Rightarrow 4-a^2=0 \quad \text{or} \quad \beta = 0 \quad \text{or} \quad \sin ax = 0 \ \forall x$$
$$\Rightarrow a = \pm 2 \text{ or } \beta=0 \text{ or } a=0$$

$\therefore$ $a = 0, \pm 2$ gives LD

$\because$ then $\alpha \sin 2x + \beta \sin ax = 0 \ \forall x$ with $\beta \neq 0$ is possible.

$\therefore$ For S to be LI

$$a \in \mathbb{R} - \{0, \pm 2\}.$$

eg. Let A and B be subsets of a vector space V. Show that $L(A \cap B) \subseteq L(A) \cap L(B)$

Give an example to show that

$$L(A) \cap L(B) \not\subseteq L(A \cap B).$$

Soln let $v \in L(A \cap B) = \text{span}(A \cap B)$

$\Rightarrow \exists$ finite vectors $u_1, u_2, -, u_n \in A \cap B$

and scalars $\alpha_1, \alpha_2, -- \alpha_n \in F$ such that

$v = \alpha_1 u_1 + \alpha_2 u_2 + --- + \alpha_n u_n$

Now $u_j \in A \quad \forall j, \ 1 \le j \le n \quad (\because A \cap B \subseteq A)$

$\Rightarrow \quad v \in L(A)$

and $u_j \in B, \ \forall j, \ 1 \le j \le n \quad (\because A \cap B \subseteq B)$

$\therefore \quad v \in L(B)$

$\Rightarrow \quad v \in L(A) \cap L(B)$

$\Rightarrow \quad L(A \cap B) \subseteq L(A) \cap L(B)$

Next, for other side of containment,

let $V = \mathbb{R}^3, \ F = \mathbb{R}$

$A = \{ e_1 = (1,0,0), \ e_2 = (0,1,0) \}$

$B = \{ e_1 + e_2 = (1,1,0), \ e_2 = (0,1,0) \}$

$A \cap B = \{ e_2 \}$

$\Rightarrow L(A \cap B) = \{ (0, y, 0) : y \in \mathbb{R} \}$

But $L(A) = \{(x, y, 0) : x, y \in \mathbb{R}\}$

& $L(B) = \{\gamma(e_1 + e_2) + \delta e_2 : \gamma, \delta \in \mathbb{R}\}$

$= \{(\gamma, \xi, 0), \gamma, \xi \in \mathbb{R}\} = L(A)$

$\Rightarrow L(A) \cap L(B) = L(A)$

$= \{(x, y, 0) : x, y \in \mathbb{R}\}$

One can see

$$L(A \cap B) \subsetneq L(A) \cap L(B).$$

eg. Prove that $L(L(S)) = L(S)$

means $\text{span}(\text{span}(S)) = \text{span}(S)$

Pf: Let $L(S) = A$. A is the smallest subspace of V containing S.

To show : $L(A) = A$

(i) $A \subseteq L(A)$ always by definition of $L(A)$

(ii) let $v \in L(A)$

$\Rightarrow v$ is a linear combination of a finite number of vectors in A.

$\Rightarrow v = \alpha_1 v_1 + \cdots + \alpha_k v_k$

for some $\alpha_i \in F$, $v_i \in A$, $1 \le i \le k$.

But A is a subspace of V

$\therefore$ if $v_i \in A$ and $\alpha_i \in F$ then

$\alpha_1 v_1 + \text{———} + \alpha_k v_k \in A$

$\Rightarrow v \in A$

$\Rightarrow L(A) \subseteq A$

Finally, we get $A = L(A)$

$\Rightarrow L(S) = L(L(S))$.

eg. If $\{v_1, \ldots, v_n\}$ is LI in a v.s. V and $\{w_1, w_2, \ldots, w_n\}$ is also LI in V

Then is $\{v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n\}$ LI in V?

Soln No.
Take $w_i = -v_i$, $\forall\ 1 \le i \le n$, in V

Then $v_i + w_i = 0$

and any set containing 0 vector is always LD.

eg: Let $n \in \mathbb{N}$, $n \geq 3$.

Let $\{u_1, u_2, \dots, u_n\}$ be LI in a v.s. $V(F)$.

Define $u_0 = 0$, $u_{n+1} = u_1$.

Set $v_i = u_i + u_{i+1}$, $1 \leq i \leq n$

Is $\{v_1, v_2, \dots, v_n\}$ LI in V?

Soln

$$v_1 = u_1 + u_2$$
$$v_2 = u_2 + u_3$$
$$v_3 = u_3 + u_4 \quad \dots \quad v_n = u_n + u_{n+1}$$
$$= u_n + u_1$$

To check LI / LD of $\{v_1, v_2, \dots, v_n\}$.

let $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 (u_1 + u_2) + \alpha_2 (u_2 + u_3) + \dots + \alpha_n (u_n + u_1) = 0$

$\Rightarrow (\alpha_1 + \alpha_n) u_1 + (\alpha_1 + \alpha_2) u_2 + (\alpha_2 + \alpha_3) u_3$

$+ \dots + (\alpha_{n-1} + \alpha_n) u_n = 0$

As $\{u_1, u_2, \dots, u_n\}$ is LI (given)

$\therefore \quad \alpha_1 + \alpha_n = 0$

$\alpha_1 + \alpha_2 = 0$

$\alpha_2 + \alpha_3 = 0$

$\alpha_3 + \alpha_4 = 0 \quad --- \quad \alpha_{n-1} + \alpha_n = 0$

we can see a cyclic pattern. So discuss two cases:

<u>let n be odd</u>

then $\quad \alpha_1 + \alpha_n = 0$

and $\quad \alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_4 = \alpha_5 = --- = \alpha_n$

$\therefore \quad \alpha_1 = \alpha_n = 0$

$\Rightarrow \quad \alpha_i = 0 \quad \forall \quad 1 \leq i \leq n$

$\Rightarrow \{v_1, v_2, --, v_n\}$ are LI

<u>let n be even</u>

$\alpha_1 + \alpha_n = 0$

$\alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_4 = --- = -\alpha_n = 0$

$\therefore$ we get $\alpha_1 + \alpha_n = 0$ only

we can take $\alpha_1 = 1, \quad \alpha_n = -1$

$\alpha_2 = -1, \quad \alpha_3 = 1, \quad \alpha_4 = -1, --, \quad \alpha_{n-1} = 1$

to see

$\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_n v_n = 0$

$\Rightarrow \{v_1, v_2, \ldots, v_n\}$ is LD.

Now in the same setting, define another set of vectors

$$w_i = u_{i-1} + u_i \ , \quad 1 \le i \le n$$

what can you say about LI of $\{w_1, w_2, \ldots, w_n\}$.

Soln

$$w_1 = u_0 + u_1 = u_1 \qquad (\because u_0 = 0 \text{ given})$$

$$w_2 = u_1 + u_2$$

$$w_3 = u_2 + u_3 \ \ldots, \ w_n = u_{n-1} + u_n$$

Let $\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n = 0$

for $\alpha_i \in F$, $1 \le i \le n$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 (u_1 + u_2) + \alpha_3 (u_2 + u_3) + \ldots + \alpha_{n-1}(u_{n-2} + u_{n-1}) + \alpha_n (u_{n-1} + u_n) = 0$$

$$\Rightarrow (\alpha_1 + \alpha_2) u_1 + (\alpha_2 + \alpha_3) u_2 + (\alpha_3 + \alpha_4) u_3 + \ldots + (\alpha_{n-1} + \alpha_n) u_{n-1} + \alpha_n u_n = 0$$

As $\{u_1, u_2, \ldots, u_n\}$ is LI, we get

$\alpha_1 + \alpha_2 = 0, \ \alpha_2 + \alpha_3 = 0, \ \ldots, \ \alpha_{n-1} + \alpha_n = 0, \ \alpha_n = 0$

$\Rightarrow \alpha_i = 0 \ \forall \ 1 \le i \le n \Rightarrow \{w_1, \ldots, w_n\}$ is LI.

Here, note that for any n (odd or even), $\{w_1, w_2, \ldots, w_n\}$ are LI.

e.g. Let $V = M_{2\times 2}(\mathbb{R})$

Let $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in V$

& $W = \{ A \in V \mid EAE = A^T \}$

(i) W is non-empty as $0 \in W$.

(ii) W is a subspace of V

$\because$ if $A, B \in W$ & $\alpha, \beta \in \mathbb{R} = F$

then $EAE = A^T$, $EBE = B^T$

$$\Rightarrow \alpha EAE + \beta EBE = E(\alpha A + \beta B)E$$

$$= \alpha A^T + \beta B^T = (\alpha A + \beta B)^T$$

$$\Rightarrow \alpha A + \beta B \in W$$

Also, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$.

$$EAE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$EAE = A^T$$

$$\Rightarrow \begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Rightarrow d = a \ , \quad c, b \in \mathbb{R}$$

$$\Rightarrow W = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= L(S)$$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \subset V.$$

Also, note S is LI over the field $\mathbb{R}$.

$$\because \quad \alpha\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \underbrace{0}_{\text{in } V}$$

$$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \alpha = \beta = \gamma = 0$$

$\therefore$ S is LI and $L(S) = W$.

e.g. Let $W = L(\{(1,-1,1), (1,1,0)\})$

$U = L(\{(1,1,1), (1,2,1)\})$

Show that $W \cap U \neq \{0\}$.

Note W & U are subspaces of $\mathbb{R}^3(\mathbb{R})$

So, $0 \in W \cap V$ always.

Let $v \in W \cap U$, $v = (x, y, z)$

$\Rightarrow v \in W$ & $v \in U$

$\Rightarrow (x, y, z) = \alpha(1,-1,1) + \beta(1,1,0)$

and $(x, y, z) = \gamma(1,1,1) + \delta(1,2,1)$

$\Rightarrow \alpha + \beta = \gamma + \delta = x$

$-\alpha + \beta = \gamma + 2\delta = y$

$\alpha = \gamma + \delta = z.$

$$\therefore \left.\begin{array}{l} \alpha + \beta - \gamma - \delta = 0 \\ \alpha - \beta + \gamma + 2\delta = 0 \\ \alpha - \gamma - \delta = 0 \end{array}\right\} \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

We can set $\beta = 0$ so (1) & (3) hold

$$\left.\begin{array}{l} \alpha = \gamma + \delta \\ \alpha = -\gamma - 2\delta \end{array}\right\} \Rightarrow \begin{array}{l} \gamma + \delta = -\gamma - 2\delta \\ 2\gamma + 3\delta = 0 \end{array}$$

$\therefore$ we can take $\gamma = 3$, $\delta = -2$, $\alpha = 1$, $\beta = 0$

(infact multiple choices for α, γ, δ)

$\therefore \quad (x, y, z) = (1, -1, 1) \in W$

$(1, -1, 1) = 3(1, 1, 1) - 2(1, 2, 1) \in U$

$\therefore \quad (1, -1, 1) \in W \cap U$

$\Rightarrow \quad W \cap U \neq \{0\}.$

Note: Here $\beta = 0 \quad \alpha = \gamma + \delta \quad , \quad \gamma = -\frac{3}{2}\delta$

$\Rightarrow \quad \alpha = -\frac{1}{2}\delta$

$$\therefore \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} -1/2\,\delta \\ 0 \\ -3/2\,\delta \\ \delta \end{pmatrix} = \delta \begin{pmatrix} -1/2 \\ 0 \\ -3/2 \\ 1 \end{pmatrix}, \ \delta \in \mathbb{R}$$

So, we have multiple choices for non-zero vectors in $W \cap U$.

Also, note $W \cap U$ is a subspace of V as W, U both are subspaces of V.