1. If $A = (a_{ij})$ is an $n \times n$ matrix and trace $A = a_{11} + a_{22} + \ldots \ldots + a_{nn}$, prove that

(i) trace $(A + B) = $ trace $A + $ trace B.

(ii) trace $(\alpha A) = \alpha$ trace $A, \alpha \in R$.

(iii) trace $(AB) = $ trace (BA).

(iv) trace $(A^T) = $ trace A.

Conclude that $AB - BA = I$ is never true.

2. By sequence of elementary row operations reduce the following matrices to upper triangular form and hence evaluate the values their determinants.

$(i).$ $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ $\qquad$ $(ii).$ $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

3. By sequence of elementary column operations reduce the following matrices to lower triangular form and hence evaluate the values of their determinants.

$(i).$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 7 & 3 & -1 \end{bmatrix}$ $\qquad$ $(ii).$ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 4 & 1 & 2 & 9 \\ -2 & 4 & -1 & 5 \\ 2 & 4 & 1 & 6 \end{bmatrix}$

4. Diagonalize the following matrices through sequence of elementary row operations in other than unit matrix form and find values of their determinants.

$(i).$ $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$ $\qquad$ $(ii).$ $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 3 & 5 & 4 \\ 3 & -4 & -6 & 7 \\ 4 & 5 & 8 & 2 \end{bmatrix}$

5. Reduce the following matrices into row reduced echelon form and find their ranks

$(i).$ $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$ $\qquad$ $(ii).$ $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 3 & 5 & 4 \\ 3 & -4 & -6 & 7 \\ 4 & 5 & 8 & 2 \end{bmatrix}$

6. Find ranks of the following matrices

$(i).$ $\begin{bmatrix} 1 & 12 & 3 & 14 \\ 5 & 6 & 17 & 8 \\ 19 & 10 & 11 & 12 \\ 15 & 14 & 13 & 10 \end{bmatrix}$ $\qquad$ $(ii).$ $\begin{bmatrix} 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

7. If a matrix $A = (a_{ij})_{m \times n}$ has rank r, prove that there exists a nonsingular square matrices P and Q of order m and n such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

8. Find the rank of the following matrix by reducing to row reduced echelon form

$$\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$$

9. If submatrices P and Q are of ranks p and q respectively, and matrix A is of the following form.

$$A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}.$$

Show rank of A is $p + q$.

10. A and B are nonsingular matrices of the same order show that C, BC, CA and ACB have the same rank where C is also a square matrix of the same order.

11. If A and B are matrices of the same order, show that

$$\text{rank } (A + B) \leq \text{rank } A + \text{rank } B$$

12. Solve the following system of linear algebraic equations

$$(i). \quad \begin{aligned} x_1 &+ 2x_2 + 3x_3 &= 0 \\ 2x_1 &+ x_2 + 3x_3 &= 0 \\ 3x_1 &+ 2x_2 + x_3 &= 0 \end{aligned} \qquad (ii). \quad \begin{aligned} x_1 &+ x_2 + x_3 + x_4 &= 0 \\ 3x_1 &+ x_2 + 4x_3 + 2x_4 &= 0 \\ &+ 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

13. Find the values of the λ for which following system has nontrivial solution

$$\begin{aligned} 3x &+ y - \lambda z &= 0 \\ 4x &- 2y - 3z &= 0 \\ 2\lambda &+ 4y + \lambda z &= 0 \end{aligned}$$

14. Find the solutions of those of the following systems, which are consistent.

$$(i). \quad \begin{aligned} x_1 &- x_2 + x_3 &= 2 \\ 3x_1 &- x_2 + 2x_3 &= -6 \\ 3x_1 &+ x_2 + x_3 &= -18 \end{aligned} \qquad (ii). \quad \begin{aligned} x &- 3y - 3z + 13w &= -1 \\ 2x &+ 5y + 5z - 18w &= 9 \\ 3x &+ 2y + 2z - 5w &= 8 \end{aligned}$$

$$\text{(iii).} \quad \begin{array}{ccccccccc} x & + & y & + & 2z & + & w & = & 5 \\ 2x & + & 3y & + & z & - & 2w & = & 2 \\ 4x & + & 5y & + & 3z & & & = & 7 \end{array}$$

15. Let $A(x)$ be a matrix whose entries are differentiable functions of x. Let $\frac{dA}{dx}$ denote the matrix whose entries are derivatives of the corresponding entries of A. Prove that

$$\frac{d}{dx}(AB) = A\frac{dB}{dx} + \frac{dA}{dx}B.$$

Assuming that $A(x)$ is invertible for every x, show that

$$\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1}.$$

16. The system of linear algebraic equations $Ax = B$, where A and B are functions of a real variable t. Prove that

$$\frac{dx}{dt} = A^{-1}\frac{dB}{dt} - A^{-1}\frac{dA}{dt}x$$

(Assume that A^{-1} exists for each t.)

17. $\mid A - \lambda I \mid = 0$, λ being scalar, is called characteristic equation of a square matrix A. Say it is reduced to

$$b_0\lambda^n + b_1\lambda^{n-1} + \ldots\ldots + b_n = 0.$$

Cayley Hamilton theorem says that every square matrix A satisfies its characteristic equation

$$b_0A^n + b_1A^{n-1} + b_2A^{n-2} + \ldots\ldots + b_nI = 0.$$

If A is nonsingular develop a scheme of evaluating A^{-1} by use of it.

18. Find the inverse of the following matrices by use of Cayley Hamilton theorem

$$(i). \quad \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \qquad (ii). \quad \begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ -5 & 5 & -5 \end{bmatrix}$$

19. If matrices B and C are nonsingular and involved products are conformable, show that

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$$

20. Partitioning of matrix $A = (a_{ij})_{n\times n}$ and its inverse $B = (b_{ij})_{n\times n}$ is done as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where A_{11} and B_{11} are $m \times m$ matrices; A_{12} and B_{12} are $m \times k$ matrices; A_{21} and B_{21} are $k \times m$ matrices and A_{22} and B_{22} are $k \times k$ matrices. If A_{11} is nonsingular, show that
$B_{11} = A_{11}^{-1} + (A_{11}^{-1}A_{12})P^{-1}(A_{21}A_{11}^{-1}), \quad B_{21} = -P^{-1}(A_{21}A_{11}^{-1})$
$B_{12} = (A_{11}^{-1}A_{12})P^{-1}, \quad B_{22} = P^{-1}$

where $P = A_{22} - A_{21}(A_{11}^{-1}A_{12})$.

21. Find the inverse of the following matrix by the use of partitioning method

$$\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$$

22. If A and B are square matrices of same order n and A is nonsingular, prove that

$$(A + B)A^{-1}(A - B) = (A - B)A^{-1}(A + B) \text{ for all B.}$$

23. Test the definiteness of following quadratic forms after reducing to the form $X^T BX$ where B is a symmetric matrix.
(i) $6x_1^2 + 49x_2^2 + 51x_3^2 - 82x_2x_3 + 20x_1x_3 - 4x_1x_2$
(ii) $3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_2x_3 + 2x_1x_3$
(iil) $4x_1^2 + 9x_2^2 + 25x_3^2 + x_1x_2 - 2x_2x_3 + 4x_1x_3$

24. If A is a matrix, whose elements are complex numbers, can be expressed as sum of Hermitian and skew-Hermitian matrices.

25. Prove that real quadratic form $Q = \sum a_{ij}x_ix_j = X^T AX$ can be expressed in the form $Q = X^T BX$ where B is a symmetirc matrix. Show further that symmetic matrix B is unique.

26. Test the definiteness of Hermitian forms $\overline{X}^T AX$ where Hermitian matrix A is given below. Also find X^*AX

$$(i). \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & -3+i \\ -1 & -3-i & 3 \end{bmatrix} \qquad (ii). \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

27. Find eigenvalues and eigenvectors of each of the following

$$(i). \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (ii). \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix} \qquad (iii). \begin{bmatrix} i & 1+i \\ -1+i & -1 \end{bmatrix}$$

$$(iv). \begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix} \qquad (v). \begin{bmatrix} 1 & -4 & -5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \qquad (vi). \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

28. Prove that A^T has same eigenvalues as square matrix A.

29. If A is a real square matrix, show that the eigenvalues of A are real or complex conjugate in pairs. Show further that if the order of the matrix A is odd, it has atleast one real eigenvalue.

30. Prove the following.
(a) Show that 0 is an eigenvalue of an $n \times n$ matrix A $\iff$ A is singular.
(b) Let A and B be $n \times n$ matrices. Show that AB and BA have same eigenvalues.
(c) Suppose λ is an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.
(d) A and P are $n \times n$ matrices and P^{-1} exists. Show that A and $P^{-1}AP$ have same eigenvalues.
(e) The eigenvalues of A^* are conjugate of the eigenvalues of A.

31. If A and B are square matrices of same order such that A is nonsingular. Prove the following
(i) BA^{-1} and $A^{-1}B$ have same eigenvalues.
(ii) B and $A^{-1}BA$ have same eigenvalues.

32. Prove the following.
(a) The eigenvalues of hermitian matrix are real.
(b) The eigenvalues of a skew-hermitian matrix are pure imaginary or zero.
(c) The eigenvalues of a unitary matrix have absolute value 1.
(d) If A is orthogonal, $\det A = \pm 1$.
(e) The eigenvectors of real symmetric matrix corresponding to different eigenvalues of real symmetric matrix are orthogonal.

33. For the following matrices, write $\text{adj}(A - \lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \ldots\ldots + B_n\lambda^n$. Also determine the inverse of A, if it exists, by the use of Cayley Hamilton theorem.

$$(i). \begin{bmatrix} 1 & i \\ i & 2 \end{bmatrix} \qquad (ii). \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

34. Suppose $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$. Find $(A+B)^{10}$, $(AB)^{16}$ and $f(A+B)$ where $f(t) = e^t$.

35. Let A and B be matrices such that AB=BA . If $f(t) = e^t$, show that $f(A+B) = f(A)f(B)$.

36. If A is diagonalisable show that $f(A)$, polynomial in A, is also diagonalisable.

37. How the eigenvectors of similar matrices are related?

38. If p is any number, show that $A - pI$ and A have the same eigenvectors. How are the eigenvalues related?

39. If P is unitary matrix, show that
(a) A is Normal $\iff P^*AP$ is Normal.
(b) A is Hermitian $\iff P^*AP$ is Hermitian.
(c) A is skew-Hermitian $\iff P^*AP$ is skew-Hermitian.
(d) A is Unitary $\iff P^*AP$ is Unitary.

40. If A and B are Hermitian matrices, show that AB is Hermitian if and only if $AB = BA$.

41. Give examples to show that sum $A + B$ and product AB of normal matrices may not be normal.

42. Show that the following matrix is normal

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1
\end{bmatrix}
$$

Is it (a) Symmetric, (b) Skew-symmetric, (c) Orthogonal? Can all the eigenvalues of this matrix be real? What can you say about the eigenvalues of this matrix?

43. If A is a unitary matrix, show $\begin{bmatrix} 1 & O \\ 0 & A \end{bmatrix}$ is also unitary matrix.

44. If A is Hermitian, show that B^*AB is Hermitian for all B. Further, if B is nonsingular and B^*AB is Hermitian then A is Hermitian.

45. Let $A = B + iC$ where B and C are Hermitian. Show that A is normal iff $BC = CB$.

46. If A is positive definite matrix, show that A^{-1} is also Positive definite matrix. If $A = P + iQ$ is a Positive definite matrix where P and Q are real matrices, show that P is Positive definite.