COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 23

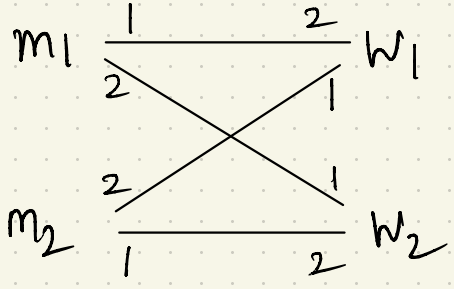
QUIZ 2 DISCUSSION & MINIMUM SPANNING TREES

MAR 01, 2023 | ROHIT VAISH

Quiz 2

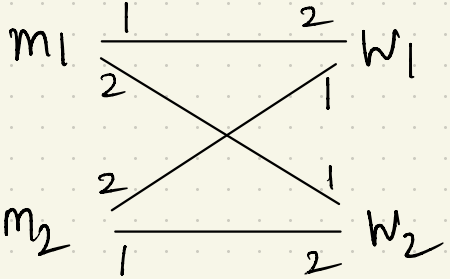
Problem 1 Construct an instance with at least $2^{n/2}$ stable matchings.

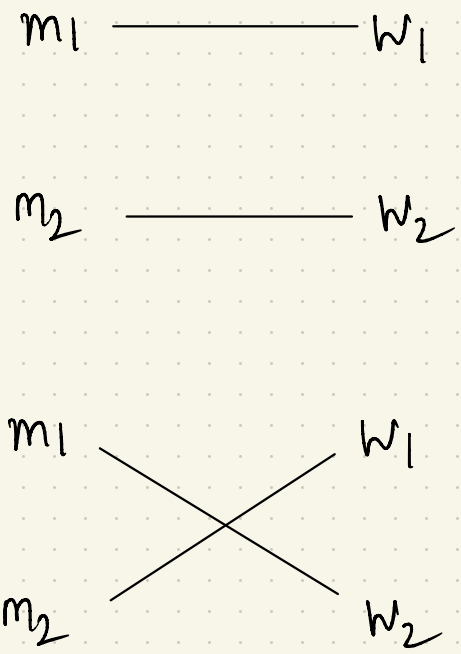
Building block



Two stable matchings

Building block





$w_1 \succ w_2 \succ$ rest $\quad m_1$ $\qquad$ w_1 $\quad m_2 \succ m_1 \succ$ rest

$w_2 \succ w_1 \succ$ rest $\quad m_2$ $\qquad$ w_2 $\quad m_1 \succ m_2 \succ$ rest

$w_1 \succ w_2 \succ$ rest

$w_2 \succ w_1 \succ$ rest

$w_3 \succ w_4 \succ$ rest

$w_4 \succ w_3 \succ$ rest

m_1

m_2

m_3

m_4

w_1 $m_2 \succ m_1 \succ$ rest

w_2 $m_1 \succ m_2 \succ$ rest

w_3 $m_4 \succ m_3 \succ$ rest

w_4 $m_3 \succ m_4 \succ$ rest

$w_1 \succ w_2 \succ$ rest

m_1

w_1

$m_2 \succ m_1 \succ$ rest

$w_2 \succ w_1 \succ$ rest

m_2

w_2

$m_1 \succ m_2 \succ$ rest

$w_3 \succ w_4 \succ$ rest

m_3

w_3

$m_4 \succ m_3 \succ$ rest

$w_4 \succ w_3 \succ$ rest

m_4

w_4

$m_3 \succ m_4 \succ$ rest

$\vdots$

$w_{n-1} \succ w_n \succ$ rest

m_{n-1}

w_{n-1}

$m_n \succ m_{n-1} \succ$ rest

$w_n \succ w_{n-1} \succ$ rest

m_n

w_n

$m_{n-1} \succ m_n \succ$ rest

$w_1 \succ w_2 \succ$ rest — m_1 — w_1 — $m_2 \succ m_1 \succ$ rest

$w_2 \succ w_1 \succ$ rest — m_2 — w_2 — $m_1 \succ m_2 \succ$ rest

$w_3 \succ w_4 \succ$ rest — m_3 — w_3 — $m_4 \succ m_3 \succ$ rest

$w_4 \succ w_3 \succ$ rest — m_4 — w_4 — $m_3 \succ m_4 \succ$ rest

⋮

$w_{n-1} \succ w_n \succ$ rest — m_{n-1} — w_{n-1} — $m_n \succ m_{n-1} \succ$ rest

$w_n \succ w_{n-1} \succ$ rest — m_n — w_n — $m_{n-1} \succ m_n \succ$ rest

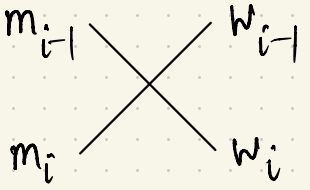
Any stable matching must have

either

m_{i-1} —— w_{i-1}

m_i —— w_i

or



for every $i \in \{2, 4, \cdots, n\}$

$w_1 \succ w_2 \succ$ rest m_1 w_1 $m_2 \succ m_1 \succ$ rest

$w_2 \succ w_1 \succ$ rest m_2 w_2 $m_1 \succ m_2 \succ$ rest

$w_3 \succ w_4 \succ$ rest m_3 w_3 $m_4 \succ m_3 \succ$ rest

$w_4 \succ w_3 \succ$ rest m_4 w_4 $m_3 \succ m_4 \succ$ rest

$\vdots$

$w_{n-1} \succ w_n \succ$ rest m_{n-1} w_{n-1} $m_n \succ m_{n-1} \succ$ rest

$w_n \succ w_{n-1} \succ$ rest m_n w_n $m_{n-1} \succ m_n \succ$ rest

Every 2×2 block has two stable matchings

$\Downarrow$

$2^{n/2}$ stable matchings in total

Problem 1

Identifying 2x2 instance with two stable matchings [3 pts]

Constructing a correct $n \times n$ instance [6 pts]

Justifying the number of stable matchings [6 pts]

Quiz 2

Problem 2 (a) Identifying men/women-optimal matchings in

$m_1 : w_2\ w_1\ w_3\ w_4$ $\qquad$ $w_1 : m_1\ m_2\ m_3\ m_4$

$m_2 : w_3\ w_2\ w_1\ w_4$ $\qquad$ $w_2 : m_2\ m_3\ m_4\ m_1$

$m_3 : w_1\ w_3\ w_2\ w_4$ $\qquad$ $w_3 : m_3\ m_1\ m_4\ m_2$

$m_4 : w_1\ w_4\ w_3\ w_2$ $\qquad$ $w_4 : m_4\ m_2\ m_3\ m_1$

(b) Modifying w_1's preference list to induce women-optimal matching (of old instance) as output of men-proposing DA algorithm

Women-optimal stable matching = Outcome of women-proposing DA

$m_1 : w_2\ w_1\ w_3\ w_4$

$m_2 : w_3\ w_2\ w_1\ w_4$

$m_3 : w_1\ w_3\ w_2\ w_4$

$m_4 : w_1\ w_4\ w_3\ w_2$

$w_1 : m_1\ m_2\ m_3\ m_4$

$w_2 : m_2\ m_3\ m_4\ m_1$

$w_3 : m_3\ m_1\ m_4\ m_2$

$w_4 : m_4\ m_2\ m_3\ m_1$

Women-optimal stable matching = Outcome of women-proposing DA

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_3 m_4

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Women-optimal stable matching = Outcome of women-proposing DA

$m_1 : w_2 \; w_1 \; w_3 \; w_4$

$m_2 : w_3 \; w_2 \; w_1 \; w_4$

$m_3 : w_1 \; w_3 \; w_2 \; w_4$

$m_4 : w_1 \; w_4 \; w_3 \; w_2$

$w_1 : m_1 \; m_2 \; m_3 \; m_4$

$w_2 : m_2 \; m_3 \; m_4 \; m_1$

$w_3 : m_3 \; m_1 \; m_4 \; m_2$

$w_4 : m_4 \; m_2 \; m_3 \; m_1$

Men-optimal stable matching = Outcome of men-proposing DA

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_3 m_4

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Men-optimal stable matching = Outcome of men-proposing DA

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_3 m_4

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Men-optimal stable matching = Outcome of men-proposing DA

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_3 m_4

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Men-optimal stable matching = Outcome of men-proposing DA

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_3 m_4

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Men-optimal stable matching = Outcome of men-proposing DA

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_3 m_4

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Let's modify the instance so that the new men-proposing DA matching is the same as the old women-optimal stable matching.

$m_1 : w_2\ w_1\ w_3\ w_4$

$m_2 : w_3\ w_2\ w_1\ w_4$

$m_3 : w_1\ w_3\ w_2\ w_4$

$m_4 : w_1\ w_4\ w_3\ w_2$

$w_1 : m_1\ m_2\ m_3\ m_4$

$w_2 : m_2\ m_3\ m_4\ m_1$

$w_3 : m_3\ m_1\ m_4\ m_2$

$w_4 : m_4\ m_2\ m_3\ m_1$

Let's modify the instance so that the new men-proposing DA matching is the same as the old women-optimal stable matching.

$m_1 : w_2 \; \boxed{w_1} \; w_3 \; w_4$

$m_2 : w_3 \; \boxed{w_2} \; w_1 \; w_4$

$m_3 : w_1 \; \boxed{w_3} \; w_2 \; w_4$

$m_4 : w_1 \; \boxed{w_4} \; w_3 \; w_2$

$w_1 : m_1 \; m_2 \; m_3 \; m_4$

$w_2 : m_2 \; m_3 \; m_4 \; m_1$

$w_3 : m_3 \; m_1 \; m_4 \; m_2$

$w_4 : m_4 \; m_2 \; m_3 \; m_1$

Let's modify the instance so that the new men-proposing DA matching is the same as the old women-optimal stable matching.

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_3 m_4

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

w_1 must change the order of the two proposers in the old instance to make the algorithm run differently. → m_3 and m_4

Let's modify the instance so that the new men-proposing DA matching is the same as the old **women-optimal stable matching**.

$m_1 : w_2 \; w_1 \; w_3 \; w_4$

$m_2 : w_3 \; w_2 \; w_1 \; w_4$

$m_3 : w_1 \; w_3 \; w_2 \; w_4$

$m_4 : w_1 \; w_4 \; w_3 \; w_2$

$w_1 : m_1 \; m_2 \; m_3 \; m_4$

$w_2 : m_2 \; m_3 \; m_4 \; m_1$

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m_3 and m_4

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$m_3 : w_1 \; w_3 \; w_2 \; w_4$

$m_4 : w_1 \; w_4 \; w_3 \; w_2$

$w_1 : m_1 \; m_2 \; m_4 \; m_3$

$w_2 : m_2 \; m_3 \; m_4 \; m_1$

$w_3 : m_3 \; m_1 \; m_4 \; m_2$

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m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

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m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

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m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

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m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

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w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Let's modify the instance so that the new men-proposing DA matching is the same as the old **women-optimal stable matching**.

m_1 :	~~w_2~~	w_1	w_3	w_4
m_2 :	~~w_3~~	w_2	w_1	w_4
m_3 :	~~w_1~~	w_3	w_2	w_4
m_4 :	w_1	w_4	w_3	w_2

w_1 :	m_1	m_2	m_4	~~m_3~~
w_2 :	m_2	m_3	m_4	~~m_1~~
w_3 :	m_3	m_1	m_4	~~m_2~~
w_4 :	m_4	m_2	m_3	m_1

Let's modify the instance so that the new men-proposing DA matching is the same as the old **women-optimal stable matching**.

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Let's modify the instance so that the new men-proposing DA matching is the same as the old **women-optimal stable matching**.

m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

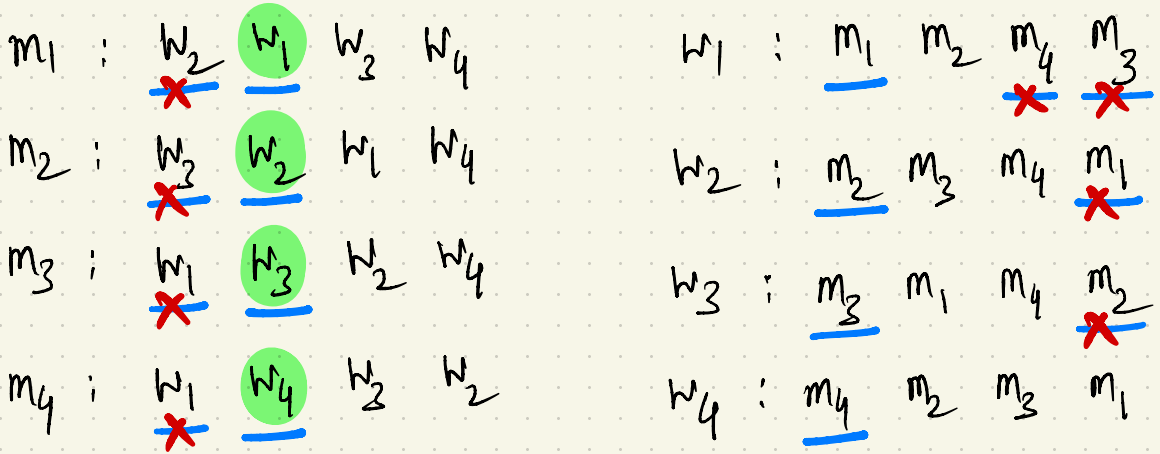
w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Let's modify the instance so that the new men-proposing DA matching is the same as the old **women-optimal stable matching**.



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m_1 : w_2 w_1 w_3 w_4

m_2 : w_3 w_2 w_1 w_4

m_3 : w_1 w_3 w_2 w_4

m_4 : w_1 w_4 w_3 w_2

w_1 : m_1 m_2 m_4 m_3

w_2 : m_2 m_3 m_4 m_1

w_3 : m_3 m_1 m_4 m_2

w_4 : m_4 m_2 m_3 m_1

Let's modify the instance so that the new men-proposing DA matching is the same as the old **women-optimal stable matching**.

m_1 :	~~w_2~~	w_1	w_3	w_4
m_2 :	~~w_3~~	w_2	w_1	w_4
m_3 :	~~w_1~~	w_3	w_2	w_4
m_4 :	~~w_1~~	w_4	w_3	w_2

w_1 :	m_1	m_2	~~m_4~~	~~m_3~~
w_2 :	m_2	m_3	m_4	~~m_1~~
w_3 :	m_3	m_1	m_4	~~m_2~~
w_4 :	m_4	m_2	m_3	m_1

* Stable marriages can be manipulated!