COL202: Discrete Mathematical Structures. I semester, 2020-21.
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Tutorial Sheet 9: Recurrences and generating functions.
17 December 2020

Important: The question marked with a ♠ is to be submitted via gradescope by 11:59PM on the day that you have your tutorial.

Problem 1
In the 2-dimensional plane we have n lines such that no two lines are parallel and no three lines intersect at one point. If R_n is the number of regions created by these n lines, find a recurrence for R_n and solve it.

Problem 2
Find a recurrence relation for the number of bit strings of length n that contain the string 01. Try and solve it if possible.

Problem 3
Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s. Try and solve it if possible.

Problem 4
Let A_n be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of A_n . Solve this recurrence relation to find a formula for d_n.

Problem 5
In how many ways can $3r$ balls be chosen from $2r$ red balls, $2r$ blue ballls and $2r$ green balls?

Problem 6
Evaluate the following sums:

Problem 6.1

$$\binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + i \cdot \binom{n}{i} + \cdots + n \cdot \binom{n}{n}$$

Problem 6.2
Given that $k \leq m$ and $k \leq n$

$$\binom{n}{0} \cdot \binom{m}{k} + \binom{n}{1} \cdot \binom{m}{k-1} + \binom{n}{2} \cdot \binom{m}{k-2} + \cdots + \binom{n}{k} \cdot \binom{m}{0},$$

Problem 6.3

$$\binom{2n}{n} + \binom{2n-1}{n-1} + \cdots + \binom{2n-i}{n-i} + \cdots + \binom{n}{0}$$

Problem 7
Prove that

$$\binom{r}{0}^2 + \binom{r}{1}^2 + \binom{r}{2}^2 + \cdots + \binom{r}{i}^2 + \cdots + \binom{r}{r}^2 = \binom{2r}{r}$$

And also that the generating funtion for $a_r = \binom{2r}{r}$ is

$$A(z) = (1 - 4z)^{-1/2}.$$

Problem 8
Let $f(n,k,h)$ be the number of ordered representations of n as a sum of exactly k integers each of which is $\geq h$. Find the generating function $\sum_n f(n,k,h)x^n$. By ordered representation we mean that that if $n = 10$, $k = 3$ and $h = 2$ then we will consider $5 + 3 + 2$ and $2 + 3 + 5$ as two *different* representations.

Problem 9
In each part below the sequence $\{a_n\}_{n\geq 0}$ satisfies the given recurrence. Find the ordinary power series generating function in each case and solve to find a_n where possible.

Problem 9.1

$$a_{n+1} = 3a_n + 2, (n \geq 0, a_0 = 0).$$

Problem 9.2

$$a_{n+1} = \alpha a_n + \beta, (n \geq 0, a_0 = 0).$$

Problem 9.3

$$a_{n+2} = 2a_{n+1} - a_n, (n \geq 0, a_0 = 0, a_1 = 1).$$

Problem 9.4

$$a_{n+1} = a_n/3 + 1, (n \geq 0, a_0 = 0).$$

Problem 10
Let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive integers. Find a recurrence for $f(n)$ and try to solve it to the extent possible using generating functions.

Problem 11
In the Double Tower of Hanoi problem there are $2n$ disks of n different sizes, 2 of each size. As before we are to move all the disks from tower 1 to tower 3 using tower 2 for help, without placing a disk of (strictly) larger radius on top of a disk of (strictly) smaller radius. How many moves will it take to transfer the disks if disks of the same radius are indistinguishable from each other.

Problem 12
Solve the recurrence

$$g_n = g_{n-1} + 2g_{n-2} + \cdots + ng_0, \text{ for } n > 0,$$

with $g_0 = 1$. Try and solve it in multiple ways.

Problem 13
In the following assume that $A(x)$, $B(x)$ and $C(x)$ are the ordinary power series generating functions of the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ respectively. With this notation attempt the following problems:

Problem 13.1
If $c_n = \sum_{j+2k\leq n} a_j b_k$, express $C(x)$ in terms of $A(x)$ and $B(x)$.

Problem 13.2
If

$$nb_n = \sum_{k=0}^{n} 2^k \frac{a_k}{(n-k)!},$$

express $A(x)$ in terms of $B(x)$.

Problem 14 ♠

Solve the recurrence $g_0 = 0, g_1 = 1$ and

$$g_n = -2ng_{n-1} + \sum_{k=0}^{n} \binom{n}{k} g_k g_{n-k}, \text{ for } n > 1,$$

using an exponential generating function.