Recall that, $T: V \to V$ be a linear transformation.

Let B & B' be bases of V.

Then $[v]_{B'} = P[v]_B$

$$\boxed{\begin{aligned} [T(v)]_{B'} &= P[T(v)]_B \\ \Rightarrow [T]_{B'}[v]_{B'} &= P[T]_B[v]_B \end{aligned}}$$

$$\Rightarrow [T]_{B'} P[v]_B = P[T]_B[v]_B$$

$$\Rightarrow [T]_{B'} P = P[T]_B$$

$$\Rightarrow [T]_{B'} = P[T]_B P^{-1}$$

Let λ be an eigen value of $[T]_B$, $\exists\ X \neq 0$

$$[T]_B X = \lambda X \quad \Rightarrow P[T]_B X = \lambda P X$$

$$\Rightarrow P[T]_B P^{-1}(PX) = \lambda(PX)$$

$$\Rightarrow [T]_{B'}(PX) = \lambda(PX)$$

Theorem :- Let $T: V \to V$ be a linear transformation, where V is a finite dimensional vector space.
The eigen spaces corresponding to distinct eigen values are independent sub-space of V.

Proof :- Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be distinct eigen values of T. Let $W_i = \left\{ x \in V : T(x) = \lambda_i x \right\}$, $1 \le i \le m$.

We claim, $W_1, W_2, \cdots, W_m$ are independent, $m \le \dim(V)$

i.e., if $w_1 + w_2 + \cdots + w_m = 0$, $w_i \in W_i$, $1 \le i \le m$

then $w_i = 0$, $1 \le i \le m$.

For $m = 1$, there is nothing to prove.

Take $m = 2$, we show that W_1 & W_2 are independent.

Take $w_1 + w_2 \in W_1 + W_2$, $w_1 \in W_1$, $w_2 \in W_2$ such that

$$\omega_1 + \omega_2 = 0$$

Here $T(\omega_1) = d_1 \omega_1$ & $T(\omega_2) = d_2 \omega_2$

Again, $\omega_1 + \omega_2 = 0 \Rightarrow \omega_2 = -\omega_1$ ——— (1)

$$\Rightarrow T(-\omega_1) = d_2(-\omega_1)$$

$$\Rightarrow T(\omega_1) = d_2 \omega_1$$

$$\Rightarrow d_1 \omega_1 = d_2 \omega_1$$

$$\Rightarrow (d_1 - d_2)\omega_1 = 0$$

$$\Rightarrow \omega_1 = 0 \quad [\because d_1 - d_2 \neq 0$$

$\Rightarrow$ From (1), $\omega_2 = 0$

$\Rightarrow W_1$ & W_2 are independent.

Any two eigen spaces are independent.

Assume that any $m-1$ number of eigen spaces are independent.

Then we prove, $W_1, W_2, \ldots, W_m$ are independent.

Take $\omega_i \in W_i$, $1 \le i \le m$ such that

$$\omega_1 + \omega_2 + \cdots + \omega_m = 0 \longrightarrow (2)$$

$$\Rightarrow T(\omega_1 + \cdots + \omega_m) = 0$$

$$\Rightarrow T(\omega_1) + T(\omega_2) + \cdots + T(\omega_m) = 0$$

$$\Rightarrow d_1\omega_1 + d_2\omega_2 + \cdots + d_m\omega_m = 0 \longrightarrow (3)$$

Apply, $(3) - d_m \times (2)$,

$$(d_1 - d_m)\omega_1 + (d_2 - d_m)\omega_2 + \cdots + (d_{m-1} - d_m)\omega_{m-1} = 0$$

Each $(d_i - d_m)\omega_i \in W_i$, $1 \le i \le m-1$

By given assumption, $W_1, W_2, \ldots, W_{m-1}$ are independent.

$$\Rightarrow \quad (\lambda_i - \lambda_m) w_i = 0 \qquad 1 \le i \le m-1$$

$$\Rightarrow \quad w_i = 0, \qquad 1 \le i \le m-1$$

$$\Rightarrow \text{ By } (2) \quad w_m = 0$$

Thus, $W_1, \dots, W_m$ are independent.

We are done.

Diagonalizable :- A matrix $A_{n \times n}$ is diagonalizable if there exists an invertible matrix P such that $PAP^{-1} = D$ is a diagonal matrix.

A linear transformation $T: V \to V$ is called diagonalizable, if there exists a basis B of V s.t. $[T]_B$ is a diagonal matrix.

Theorem :- Let $T: V \to V$ be a linear transformation. T is diagonalizable if and only if the sum of the dimensions of the eigen spaces corresponding to the distinct eigen values equals to $\dim(V)$.

Proof :- Let $\dim(V) = n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigen values of T. Let $W_i = \{X \in V : TX = \lambda_i X\}$

Then T is diagonalizable if and only if

$$\dim(W_1) + \dim(W_2) + \cdots + \dim(W_m) = n.$$

$\boxed{\Rightarrow}$ T is diagonalizable.

There exists a basis B such that

$$[T]_B = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n),$$ some μ_i may be same as μ_j

& $\{\mu_1, \ldots, \mu_n\} = \{\lambda_1, \ldots, \lambda_m\}$

For μ_i, an eigen vector is $[0,\cdots,0,\underset{i\text{th}}{1},0,\cdots,0]$ for the matrix $[T]_B$

If λ_i is repeated r_i times, then $\dim(W_i) = r_i$

Also, $r_1 + r_2 + \cdots + r_m = n$

$$\Rightarrow \dim(W_1) + \dim(W_2) + \cdots + \dim(W_m) = \dim(V)$$

$\boxed{\Leftarrow}$ Let $\dim(W_1) + \cdots + \dim(W_m) = \dim(V)$

Let B_i be a basis of W_i, $1 \le i \le m$

Let $B = \bigcup_{i=1}^{m} B_i$

B is a basis of $W_1 + \cdots + W_m = V$ (We have proved before)

$$[T]_B = \mathrm{diag}(\mu_1, \cdots, \mu_n)$$

$W_1 \longrightarrow B_1 = \{x_1, x_2, x_3\}$

$W_2 \longrightarrow B_2 = \{x_4, x_5\}$

$$T(x_1) = d_1 x_1$$
$$T(x_2) = d_1 x_2$$
$$T(x_3) = d_1 x_3$$
$$T(x_4) = d_2 x_4$$
$$T(x_5) = d_2 x_5$$

$B = \{x_1, \dots, x_5\}$ is a basis of $W_1 + W_2 = V$

$$[T]_B = \begin{bmatrix} d_1 & & & & 0 \\ & d_1 & & & \\ & & d_1 & & \\ & & & d_2 & \\ 0 & & & & d_2 \end{bmatrix}$$

Corollary :- If $\dim(V) = n$, $T : V \to V$ has n distinct eigen value then T is diagonalizable.