1) Consider the homogeneous system of equations $AX = 0$ with coefficients from $\mathbb{R}$, where

$$A = \begin{pmatrix} 1 & 5 & 1 & 5 & 1 \\ -1 & -5 & 1 & -1 & 0 \\ 2 & 10 & 1 & 8 & 1 \\ 1 & 5 & 2 & 7 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

and $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

a) By converting A into its row reduced echelon (RRE) matrix find all the free (independent) variables in the given system.

b) Find the rank of A.

c) Write all the solutions of the given system $AX = 0$.

Solution.

a) $$A = \begin{pmatrix} 1 & 5 & 1 & 5 & 1 \\ -1 & -5 & 1 & -1 & 0 \\ 2 & 10 & 1 & 8 & 1 \\ 1 & 5 & 2 & 7 & 0 \end{pmatrix}$$

$R_2 \longrightarrow R_2 + R_1$
$R_3 \longrightarrow R_3 - 2R_1$
$R_4 \longrightarrow R_4 - R_1$

$$\begin{pmatrix} 1 & 5 & 1 & 5 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix} \quad (1/2)$$

$R_2 \longrightarrow \frac{1}{2} R_2$

$$\begin{pmatrix} 1 & 5 & 1 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{pmatrix}$$

$R_1 \longrightarrow R_1 - R_2$
$R_3 \longrightarrow R_3 + R_2$
$R_4 \longrightarrow R_4 - R_2$

$$\begin{pmatrix} 1 & 5 & 0 & 3 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & -3/2 \end{pmatrix} \quad (1/2)$$

$R_3 \longrightarrow -2 R_3$

$$\begin{pmatrix} 1 & 5 & 0 & 3 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3/2 \end{pmatrix}$$

$R_1 \longrightarrow R_1 - 1/2\, R_3$
$R_2 \longrightarrow R_2 - 1/2\, R_3$
$R_4 \longrightarrow R_4 + 3/2\, R_3$

$$\begin{pmatrix} 1 & 5 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow (1)$$

The free variables are x_2 & x_4. $\longrightarrow$ ①

b) The rank of the matrix A is 3. $\longrightarrow$ ①

c) Let $x_2 = \lambda$ & $x_4 = \mu$. Then

$$x_1 = -5\lambda - 3\mu$$

$$x_3 = -2\mu$$

$$x_5 = 0. \longrightarrow (1)$$

Thus the set of all solutions is given by $\{(-5\lambda - 3\mu, \lambda, -2\mu, \mu, 0) : \lambda, \mu \in \mathbb{R}\}$.

Que 2 (a) (Method 1)

$$W_1 = \{ A \in M_3(\mathbb{R}) : Tr(A + 2A^t) = 0 \}.$$

Note: $Tr(A+2A^t)$

(1) $= Tr(A) + Tr(2A^t)$ $\quad (\because Tr(A+B) = Tr(A) + Tr(B))$

$= Tr(A) + 2\,Tr(A^t)$ $\quad (\because Tr(\lambda A) = \lambda\, Tr(A))$

$= Tr(A) + 2\,Tr(A)$ $\quad (\because Tr(A) = Tr(A^t))$

$= 3\,Tr(A).$

Then $Tr(A + 2A^t) = 0 \Leftrightarrow Tr(A) = 0.$

Therefore, $W_1 = \{ A \in M_3(\mathbb{R}) : Tr(A) = 0 \}.$

($\frac{1}{2}$) Since 3×3 zero matrix $O \in W_1$ as $Tr(O) = 0$, we get $W_1 \neq \phi$

For $A, B \in W_1$ and $\lambda \in \mathbb{R}$,

($1\frac{1}{2}$)
$$\begin{aligned} Tr(\lambda A + B) &= Tr(\lambda A) + Tr(B) \\ &= \lambda\, Tr(A) + Tr(B) \\ &= \lambda \cdot 0 + 0 \\ &= 0. \end{aligned}$$

$\Rightarrow \lambda A + B \in W_1$

i.e W_1 is a subspace.

Que 2 (a)   Method 2

$$W_1 = \{ A \in M_3(\mathbb{R}) : Tr(A + 2A^t) = 0 \}.$$

Note that the zero matrix $0 \in W_1$ since

$\frac{1}{2}$

$$Tr(0 + 20^t) = Tr(0) = 0.$$

i.e. $W_1 \neq \phi$

Let $A, B \in W_1$ and $\lambda \in \mathbb{R}$.

$$Tr(A + 2A^t) = 0 \quad , \quad Tr(B + 2B^t) = 0$$

Consider, $Tr\left((\lambda A + B) + 2(\lambda A + B)^t\right)$

$1\frac{1}{2}$

$$= Tr\left(\lambda A + B + 2\lambda A^t + 2B^t\right)$$
$$= Tr\left(\lambda(A + 2A^t) + (B + 2B^t)\right)$$
$$= Tr\left(\lambda(A + 2A^t)\right) + Tr\left(B + 2B^t\right)$$
$$= \lambda\, Tr\left(A + 2A^t\right) + Tr\left(B + 2B^t\right)$$
$$= \lambda \cdot 0 + 0$$
$$= 0$$

$\Rightarrow \lambda A + B \in W_1$

i.e. W_1 is a subspace.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Note that $\det(A) = 0 = \det(B)$

But $A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det(A+B) = 1$$

Thus $A, B \in W_2$ but $A+B \notin W_2$

$\therefore$ W_2 is not a vector space.

- 1 mark if correct.
- 1/2 mark if do not give examples A, B with above properties BUT say that $\det(A+B) \neq \det(A)+\det(B)$.

1st Method:

$v_1 = 1$, $v_2 = 1+x$, $v_3 = 1+x+x^3$, $v_4 = 1+x^3$

Let $c_1 \cdot 1 + c_2(1+x) + c_3(1+x+x^3) + c_4(1+x^3) = 0$

$\Leftrightarrow (c_1+c_2+c_3+c_4) + (c_2+c_3)x + (c_3+c_4)x^3 = 0$

$$\Leftrightarrow \left.\begin{aligned} c_1+c_2+c_3+c_4 &= 0 \\ c_2+c_3 &= 0 \\ c_3+c_4 &= 0 \end{aligned}\right\} (*)$$

$(*)$ is a homogeneous system of 3 equations in 4 unknowns and hence must have infinitely many solns.

In particular, the system has a nonzero solution.

Hence, $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

2nd Method:

$1+x+x^3 = (1+x) + (1+x^3) - 1$

$\Rightarrow$ $1+x+x^3$ is a linear combination of 1, $1+x$ & $1+x^3$.

$\Rightarrow$ The given set is linearly dependent.

3rd method:

Let $W = \text{span}\{1, x, x^3\}$

Then $\dim W = 3$

Since the given set contains 4 vectors in W, which is of dimension 3, it must be linearly dependent

Marking scheme:

+2 if completely correct arguments.

+1 if partially correct.