

**Important:** The question marked with a ♠ is to be submitted via gradescope by 11:59PM on the day that you have your tutorial.

**Problem 1**

Suppose you have  $n$  men and  $n$  women and you have to seat them around a circular table so that men and women sit alternately. How many ways can you seat them?

**Problem 2**

Suppose there are  $n$  tennis players. How many ways can we pair them up so that everyone has a partner to play a singles match?

**Problem 3**

Given a set  $A$  of size  $m$  and set  $B$  of size  $n$  count the number of

1. Relations from  $A$  to  $B$ .
2. Total functions from  $A$  to  $B$ .
3. Partial functions from  $A$  to  $B$ .
4. Surjections from  $A$  to  $B$  (assume  $m \geq n$ ). These could be partial or total functions.
5. Injections from  $A$  to  $B$ . These could be partial or total. State whatever you assume about  $m$  and  $n$ .

**Problem 4**

Given a plane with integer points of the type  $(x, y)$  where both  $x$  and  $y$  are integers, we define a *lattice path* from  $(x_1, y_1)$  to  $(x_2, y_2)$  to be a set of line segments that go from a point  $(i, j)$  to  $(i + 1, j)$  or  $(i, j + 1)$ , i.e., all steps in the path either move right or up. Count the number of lattice paths between  $(0, 0)$  and  $(m, n)$ ?

**Problem 5**

A lattice path from  $(0, 0)$  to  $(n, n)$  is called a *Catalan path* if it only visits points  $(x, y)$  such that  $y \leq x$ . Count the number of Catalan paths between  $(0, 0)$  and  $(n, n)$ . (**Hint.** Argue that every lattice path that is *not* a Catalan path must touch or cross the line  $y = x + 1$ . Find a bijection between the set of lattice paths that touch or cross the line  $y = x + 1$  and the set of lattice paths between  $(-1, 1)$  and  $(n, n)$ .)

**Problem 6 ♠**

We say that a function  $\pi$  is a *derangement of size  $n$*  if it is a bijection from  $\{1, \dots, n\}$  to itself (i.e., it is a permutation) and it has no fixed points, i.e.,  $\forall i : \pi(i) \neq i$ . Count the number of derangements of size  $n$ . The solution you submit must use Inclusion-Exclusion. Separately also try to see if you can solve the problem without the use of Inclusion-Exclusion, but you don't have to submit the other solution(s).

**Problem 7**

Prove the following identities regarding binomial coefficients by making counting arguments. Give as many different arguments as possible

**Problem 7.1**

$$\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}.$$

**Problem 7.2**

$$\binom{n}{k} \binom{n-k}{j} = \binom{n}{j} \binom{n-j}{k}.$$

**Problem 7.3**

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}.$$

**Problem 8**

Show that any finite connected graph on  $n$  vertices must have two vertices with the same degree.

**Problem 9**

Let  $A$  be any set of 20 numbers chosen from the arithmetic progression  $1, 4, 7, \dots, 100$ . Prove that there must be two distinct integers in  $A$  which sum to 104.

**Problem 10**

Consider any five points in the interior of a square of side 1. Show that there are two points which are at most  $1/\sqrt{2}$  units apart. Is this the best possible bound i.e. is there a placement of five points such that the maximum interpoint distance is exactly  $1/\sqrt{2}$ .