COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 1: Logic.
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Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM.

Note: Please read *all* of Chapter 3 of [BSD05] before attempting this sheet.

Problem 1 ([BSD05] Prob 1, pp 94)
Give truth tables for the following compound propositions

1. $(s \vee t) \wedge (\neg s \vee t) \wedge (s \vee \neg t)$
2. $(s \Rightarrow t) \wedge (t \Rightarrow u)$
3. $(s \vee t \vee u) \wedge (s \vee \neg t \vee u)$

Problem 2 ([BSD05] Prob 8, pp 94)
Use a truth table to show that $(s \vee t) \wedge (u \vee v)$ is equivalent to $(s \wedge u) \vee (s \wedge v) \vee (t \wedge u) \vee (t \wedge v)$.

Problem 3 ([BSD05] Prob 8, pp 107)
Write the following statement as a logical expression: The product of odd integers is odd. You may assume that $\texttt{odd} : \mathbb{Z} \to \{T, F\}$ is a predicate that maps odd integers to T and even integers to F.

Problem 4 ([BSD05] Theorem 3.2, pp 100)
Here is the statement of a theorem given in [BSD05] written in slightly different terms.

Theorem 1
Suppose we have a domain D and two predicates $P, Q : D \to \{T, F\}$. Let $A = \{x \in D : Q(x) \text{ is } T\}$. Show that

1. $\forall x \in A : P(x)$ is logically equivalent to $\forall x \in D : Q(x) \Rightarrow P(x)$.
2. $\exists x \in A : P(x)$ is logically equivalent to $\exists x \in D : Q(x) \wedge P(x)$.

Write a proof for this. You may read the proof in the book and then write it in your own words.

Problem 5 ([BSD05] Prob 10, pp 107)
Rewrite the following statement without any negations. It is not the case that there exists an integer n such that $n > 0$ and for all integers $m > n$, for every polynomial equation $p(x) = 0$ of degree m there are no real numbers for solutions.

Problem 6.1 ([LLM18] Prob 3.16(a), pp 81)
Verify by truth table that $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ is valid.

Problem 6.2 ([LLM18] Prob 3.16(b), pp 81)
Suppose P and Q are propositions. Describe a single formula (i.e. compound proposition) R using only AND, OR, NOT and copies of P and Q such that R is valid iff P and Q are equivalent.

Problem 6.3 ([LLM18] Prob 3.16(c), pp 81)
A propositional formula is satisfiable iff there is an assignment of truth values to its variables—-an *environment*—-that makes it true. Explain why P is valid iff $\neg P$ is not satisfiable.

Problem 6.4 ([LLM18] Prob 3.16(d), pp 81)
A set of propositional formulas $P_1, \ldots, P_k$ is consistent iff there is an environment in which they are all true. Write a formula S such that the set $P_1, \ldots, P_k$ is *not* consistent iff S is valid.

Problem 7 ([BSD05] Prob 11, pp 107)
Consider the following slight modification of Theorem 3.2. For each part below, either prove that it is true or give a counterexample. Let U_1 be a universal set contained in another universal set U_2, i.e., $U_1 \subseteq U_2$. Suppose that $q(x)$ is a statement such that $U_1 = \{x \in U_2 \mid q(x) \text{ is true}\}$.

1. $\forall x \in U_1 : p(x)$ is equivalent to $\forall x \in U_2 : q(x) \wedge p(x)$.
2. $\exists x \in U_1 : p(x)$ is equivalent to $\exists x \in U_2 : q(x) \Rightarrow p(x)$.

Problem 8
Recall that for two functions f, g that map the natural numbers to non-negative real numbers, we say that $f(n)$ is $O(g(n))$ if there is a positive real c and a natural number n_0 such that $f(n) \leq cg(n)$ for all $n > n_0$.

Problem 8.1 ♠
Write both "$f(n)$ is $O(g(n))$" and "$f(n)$ is not $O(g(n))$" as logical sentences with appropriate quantifiers. You may use mathematical notation like $\leq$, $\geq$ etc. where required.

Problem 8.2
Recall that we say that $f(n)$ is $\omega(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$. Write "$f(n)$ is $\omega(g(n))$ as a logical sentence and compare it with the logical sentence for "$f(n)$ is not $O(g(n))$." Are they the same thing?

Problem 9
Given two predicates $P, Q : \mathbb{N} \to \{T, F\}$, suppose we know that $\forall n \in \mathbb{N} : (n \geq 5) \Rightarrow P(n)$ and $\forall n \in \mathbb{N} : (n \geq 6) \Rightarrow Q(n)$ are true. Prove that $\exists n \in \mathbb{N} : P(n) \wedge Q(n)$. Can we prove or disprove that $\exists n \in \mathbb{N} : P(n) \wedge \neg Q(n)$?

References

[BSD05] K. Bogart, S. Drysdale, C. Stein. Discrete Math for Computer Science Students. 2005.

[LLM18] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science, June 2018, MIT Open Courseware.