COL202: Discrete Mathematical Structures. I semester, 2022-23.
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Tutorial Sheet 1: Logic and Proofs.
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Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session.

Note: Please read *all* of Chapter 3 of [1] before attempting this sheet. All questions in this sheet are from that book, question numbers and page indicated in brackets refer to the pdf linked on the course page.

Problem 1 (Prob 1, pp 94)
Give truth tables for the following compound propositions

1. $(s \vee t) \wedge (\neg s \vee t) \wedge (s \vee \neg t)$
2. $(s \Rightarrow t) \wedge (t \Rightarrow u)$
3. $(s \vee t \vee u) \wedge (s \vee \neg t \vee u)$

Problem 2 (Prob 8, pp 94)
Use a truth table to show that $(s \vee t) \wedge (u \vee v)$ is equivalent to $(s \wedge u) \vee (s \wedge v) \vee (t \wedge u) \vee (t \wedge v)$.

Problem 3 (Prob 8, pp 107)
Write the following statement as a logical expression: The product of odd integers is odd. You may assume that $\texttt{odd} : \mathbb{Z} \to \{T, F\}$ is a predicate that maps odd integers to T and even integers to F.

Problem 4 (Theorem 3.2, pp 100)
Here is the statement of a theorem given in [1] written in slightly different terms.

Theorem 1
Suppose we have a domain D and two predicates $P, Q : D \to \{T, F\}$. Let $A = \{x \in D : Q(x) \text{ is } T\}$. Show that

1. $\forall x \in A : P(x)$ is logically equivalent to $\forall x \in D : Q(x) \Rightarrow P(x)$.
2. $\exists x \in A : P(x)$ is logically equivalent to $\exists x \in D : Q(x) \wedge P(x)$.

Write a proof for this. You may read the proof in the book and then write it in your own words.

Problem 5 (Prob 10, pp 107)
Rewrite the following statement without any negations. It is not the case that there exists an integer n such that $n > 0$ and for all integers $m > n$, for every polynomial equation $p(x) = 0$ of degree m there are no real numbers for solutions.

Problem 6 (Prob 11, pp 107)
Consider the following slight modification of Theorem 3.2. For each part below, either prove that it is true or give a counterexample. Let U_1 be a universal set contained in another universal set U_2, i.e., $U_1 \subseteq U_2$. Suppose that $q(x)$ is a statement such that $U_1 = \{x \in U_2 \mid q(x) \text{ is true}\}$.

1. $\forall x \in U_1 : p(x)$ is equivalent to $\forall x \in U_2 : q(x) \wedge p(x)$.
2. $\exists x \in U_1 : p(x)$ is equivalent to $\exists x \in U_2 : q(x) \Rightarrow p(x)$.

Problem 7 ♠
Given two predicates $P, Q : \mathbb{N} \to \{T, F\}$, suppose we know that $\forall n \in \mathbb{N} : (n \geq 5) \Rightarrow P(n)$ and $\forall n \in \mathbb{N} : (n \geq 6) \Rightarrow Q(n)$ are true. Prove that $\exists n \in \mathbb{N} : P(n) \wedge Q(n)$. Can we prove or disprove that $\exists n \in \mathbb{N} : P(n) \wedge \neg Q(n)$?

Problem 8 ([1], Prob 7, pp 115)
Prove using the contrapositive method that for all real numbers x, if $x^2 - 2x \neq -1$ then $x \neq 1$.

Problem 9 ([1], Prob 8, pp 115)
Prove using the proof by contradiction method that for all real numbers x, if $x^2 - 2x \neq -1$ then $x \neq 1$.

Problem 10 ([1], Prob 15, pp 115)
Given function $f, g, h : \mathbb{Z}_+ \to \mathbb{R}_+$ Recall that $f(n) = O(g(n))$ by definition if

$$\exists c \in \mathbb{R}_+ : \exists n_0 \in \mathbb{Z}_+ : \forall n \geq n_0 f(n) \leq cg(n).$$

Prove that if $f(n) = O(g(n))$ and $g(n) = O(h(n))$ then $f(n) = O(h(n))$.

References

[1] K. Bogart, S. Drysdale, C. Stein Discrete Math for Computer Science Students. 2005.