Q1 b) $y_{n+1} = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds$ is the ~~Pik~~ Picard's formula. Hence

$$y_1 = 0 + \int_0^t (\sqrt{s} + 0^2)\, ds = \frac{2}{3} t^{3/2}.$$

$$y_2 = 0 + \int_0^t \sqrt{s} + \left(\frac{2}{3} s^{3/2}\right)^2 ds$$

$$= \frac{2}{3} t^{3/2} + \frac{4}{9} \times \frac{t^4}{4}$$

$$= \frac{2}{3} t^{3/2} + \frac{t^4}{9}$$

Major Q 2 Solution : $t^2y'' - 4ty' + 6y = \sin(\ln t)$.

This is a Cauchy-Euler equation.

Let $s = \ln t$. Then $\frac{dy}{dt} = \frac{ds}{dt}\frac{dy}{ds} = \frac{1}{t}\frac{dy}{ds}$

so that $t\frac{dy}{dt} = \frac{dy}{ds}$ & $\frac{d^2y}{dt^2} = -\frac{1}{t^2}\frac{dy}{ds} + \frac{1}{t}\frac{d}{ds}\left(\frac{dy}{ds}\right)\frac{ds}{dt}$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{1}{t^2}\cdot\frac{dy}{ds} + \frac{1}{t^2}\frac{d^2y}{ds^2}$$

$$\Rightarrow t^2\frac{d^2y}{dt^2} = \left(\frac{d^2y}{ds^2} - \frac{dy}{ds}\right)$$

Substituting, $\left(\frac{d^2y}{ds^2} - \frac{dy}{ds}\right) - 4\times\frac{dy}{ds} + 6y = \sin s$

$$\Rightarrow \frac{d^2y}{ds^2} - 5\frac{dy}{ds} + 6y = \sin s.$$

The char. poly of the hom. part $y'' - 5y' + 6y = 0$ is $m^2 - 5m + 6 = 0$. Hence $m = 2, 3$ Thus,

$y_h(s) = c_1 e^{2s} + c_2 e^{3s}$. Then the particular solution is given by $y_p(s) = A\sin s + B\cos s$ (using method of ~~Variat~~ undetermined coeffnts)

$y_p'(s) = A\cos s + B(-\sin s)$ & $y_p''(s) = -(A\sin s + B\cos s)$

Substiting y_p in the eqn we have

$-A\sin s - B\cos s - 5A\cos s + 5B\sin s + 6A\sin s + 6B\cos s = \sin s$

Comparing the coeffnts of $\cos s$ & $\sin s$ we have

$-A + 5B + 6A = 1$ & $-B - 5A + 6B = 0$

$A + B = 1/5$, $A - B = 0 \Rightarrow A = B = \frac{1}{10}$.

Thus $y_p = \frac{1}{10}(\cos s + \sin s)$. Thus the general solution is

$y(t) = c_1 t^2 + c_2 t^3 + \frac{1}{10}\{\cos(\ln t) + \sin(\ln t)\}$.

Q3. $y'' - 4y' + 4y = \delta(t-2) + H(t)$, $y(0) = y'(0) = 0$

Taking Lapalace Transformation on both sides, with $\mathcal{L}[y] = Y$, we get

$$(s^2 - 4s + 4)\, Y = e^{-2s} + \frac{e^{-s}}{s}$$

$$\Rightarrow \quad Y = \frac{e^{-2s}}{s^2 - 4s + 4} + \frac{e^{-s}}{s(s^2 - 4s + 4)}$$

Consider $f_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 - 4s + 4}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right]$

$$= t e^{2t}$$

so $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 - 4s + 4}\right] = H_2(t)\, f_1(t-2)$

Consider $f_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2 - 4s + 4)}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2}\right]$$

$$= \frac{1}{4} - \frac{1}{4} e^{2t} + \frac{1}{2} t e^{2t}$$

so $\mathcal{L}^{-1}\left[\frac{e^{-s}}{s(s^2 - 4s + 4)}\right] = H_1(t)\, f_2(t-1)$

so $y = \mathcal{L}^{-1}[Y] = H_2(t)\, f_1(t-2) + H_1(t)\, f_2(t-1)$

$$= \boxed{(t-2) e^{2(t-2)} H_2(t) + \left[\frac{1}{4} - \frac{1}{4} e^{2t} + \frac{1}{2} t e^{2t}\right] H_1(t)}$$

Q4. $A = \begin{pmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{pmatrix}$

characteristic polynomial $= \lambda^3 - 6\lambda^2 + 12\lambda - 8$

eigen values are 2, 2, 2

For X_1,
First eigen vector, $v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$(2I - A) v_1 = 0$$

$$X_1 = v_1 e^{2t}$$

For X_2, $(2I - A) v_2 = v_1$

$$v_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$X_2 = (t v_1 + v_2) e^{2t}$$

For X_3, $(2I - A) v_3 = v_2$

$$v_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$X_3 = \left(\frac{1}{2} t^2 v_1 + t v_2 + v_3\right) e^{2t}$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

(5)

$$\frac{1}{s(s^2+2s+5)} = \frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right]$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{s(s^2+2s+5)}\right] = \frac{1}{5}\left[1 - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)\right] = f_1(t)$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi}{2}s} - e^{-\frac{3\pi}{2}s}}{s(s^2+2s+5)}\right]$$

$$= H_{\frac{\pi}{2}}(t)\, f_1\left(t-\frac{\pi}{2}\right) - H_{\frac{3\pi}{2}}(t)\, f_1\left(t-\frac{3\pi}{2}\right)$$

$$f(2\pi) = f_1\left(\frac{3\pi}{2}\right) - f_1\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{5}\left[\cancel{1} - e^{-\frac{3\pi}{2}}\cos(3\pi) - \frac{1}{2}e^{-\frac{3\pi}{2}}\cancel{\sin(3\pi)}^{0}\right]$$

$$- \frac{1}{5}\left[\cancel{1} - e^{-\frac{\pi}{2}}\cos\pi - \frac{1}{2}e^{-\frac{\pi}{2}}\cancel{\sin\pi}^{0}\right]$$

$$\boxed{f(2\pi) = \frac{1}{5}\left(e^{-\frac{3\pi}{2}} - e^{-\pi/2}\right)}$$

Q6 Find the general Soln of

$$x' = 2x + 6y + e^t$$
$$y' = x + 3y - e^t$$

Soln. Consider the homogeneous Problem:

$$\begin{aligned} x' &= 2x + 6y \\ y' &= x + 3y \end{aligned} = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ eigenvalues are $\lambda = 0, 5$

$$(A - 0I)\vec{x} = A\vec{x} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 2x_1 + 6x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = -3x_2$$

So $\vec{u}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Similarly $(A - 5I) = \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow x_1 = 2x_2 \Rightarrow \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So general homogeneous Soln is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{5t}c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We now need to find one non-homogeneous soln.

ⓐ Using method of undetermined coefficients

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t = \begin{pmatrix} 2c_1 + 6c_2 \\ c_1 + 3c_2 \end{pmatrix} e^t + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$\Rightarrow A = 2, \quad B = -\frac{1}{2} \Rightarrow \begin{aligned} x(t) &= 2e^t \\ y(t) &= -\frac{1}{2}e^t \end{aligned}$$

So general Solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2e^t \\ -\frac{1}{2}e^t \end{pmatrix}$$

Alternative soln

(B) Using method of variation of parameters.

Fundamental Matrix

$$X = \begin{bmatrix} -3 & 2e^{5t} \\ 1 & e^{5t} \end{bmatrix} \qquad \vec{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

$$X^{-1} = \frac{1}{-5e^{5t}} \begin{bmatrix} e^{5t} & -2e^{5t} \\ -1 & -3 \end{bmatrix}$$

$$X^{-1}\vec{f} = -\frac{1}{5e^{5t}} \begin{bmatrix} e^{5t} & -2e^{5t} \\ -1 & -3 \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

$$= -\frac{1}{5e^{5t}} \begin{bmatrix} 3e^{6t} \\ 2e^t \end{bmatrix} = \begin{pmatrix} -\frac{3}{5}e^t \\ -\frac{2}{5}e^{-4t} \end{pmatrix}$$

$$\int X^{-1}\vec{f}\,dt = \begin{bmatrix} -\frac{3}{5}e^t \\ \frac{1}{10}e^{-4t} \end{bmatrix}$$

$$X\int X^{-1} f\,dt = \begin{bmatrix} -3 & 2e^{5t} \\ 1 & e^{5t} \end{bmatrix} \begin{bmatrix} -\frac{3}{5}e^t \\ \frac{1}{10}e^{-4t} \end{bmatrix} = \begin{bmatrix} 2e^t \\ -\frac{1}{2}e^t \end{bmatrix}$$

So general soln is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2e^t \\ -\frac{1}{2}e^t \end{pmatrix}$$

Major Solution Q7

The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 2 & 3 & 1 & -1 & 4 \\ 3 & 4 & -1 & 2 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{array}\right) \underset{R_4 \to R_4 - 2R_1}{\overset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 0 & -1 & -5 & -3 & -4 \\ 0 & -2 & -10 & -1 & -7 \\ 0 & -1 & -5 & 2 & -3 \end{array}\right)$$

$$\underset{\substack{R_4 \to R_4 - R_2 \\ R_2 \to (-1) \times R_2}}{\overset{R_3 \to R_3 - 2R_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 0 & 1 & 5 & 3 & 4 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 & 1 \end{array}\right) \underset{R_1 \to R_1 - 2R_2}{\overset{\substack{R_4 \to R_4 - R_3 \\ R_3 \to \frac{1}{5} \times R_3}}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & -7 & -5 & -4 \\ 0 & 1 & 5 & 3 & 4 \\ 0 & 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{R_2 \to R_2 - 3R_3}{\overset{R_1 \to R_1 + 5R_3}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & -7 & 0 & -3 \\ 0 & 1 & 5 & 0 & 4 - \frac{3}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \simeq \left(\begin{array}{cccc|c} 1 & 0 & -7 & 0 & -3 \\ 0 & 1 & 5 & 0 & \frac{17}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Where the coeffient matrix is an RRE. The unknown z is free. Assign $z = \lambda \in \mathbb{R}$. Then

$x - 7\lambda = -3$, $y + 5\lambda = \frac{17}{5}$, $w = \frac{1}{5}$. Hence

the general solution is

$x = -3 + 7\lambda$, $y = \frac{17}{5} - 5\lambda$, $z = \lambda$, $w = \frac{1}{5}$

for $\lambda \in \mathbb{R}$. This is an infinitely many solution case.

Major, Solution Q~~2~~8.

By inspection, $W_1 = \{(0, \lambda, \mu, -\lambda-\mu) : \lambda, \mu \in \mathbb{R}\}$

$W_2 = \{(\alpha, +\alpha, \beta, -2\alpha-\beta) : \alpha, \beta \in \mathbb{R}\}$

& $W_1 \cap W_2 = \{(x, y, z, \omega) \in \mathbb{R}^4 : x=0, x-y=0, y+z+\omega=0$
$x+y+z+\omega=0\}$

(by definition of intersection)

$= \{(x, y, z, \omega) \in \mathbb{R}^4 ; x = y = 0, z + \omega = 0\}$

$= \{(0, 0, \gamma, -\gamma) \in \mathbb{R}^4 : \gamma \in \mathbb{R}\}$.

Since $\gamma(0, 0, 1, -1) = (0, 0, \gamma, -\gamma)$ &

$\{(0, 0, 1, -1)\}$ is linearly independent,

~~Hi~~ $B_1 = \{(0, 0, 1, -1)\}$ is a basis of $W_1 \cap W_2$.

To find a basis of $W_1 + W_2$, we extend B_1 to a basis A of W_1 & C of W_2. Then $B_2 := A \cup C$ is a basis $W_1 + W_2$.

$\lambda = 0, \mu = 1$ gives $(0, 0, 1, -1)$ in W_1

& $\lambda = 1, \mu = 0$ gives $(0, 1, 0, -1)$ in ~~W_2~~ W_1 & so that

$(0, \lambda, \mu, -\lambda-\mu) = \lambda(0, 1, 0, -1) + \mu(0, 0, 1, -1)$

$A = \{(0, 0, 1, -1), (0, 1, 0, -1)\}$ is a basis of W_1

$\alpha = 1, \beta = 0$ gives $(1, 1, 0, -2)$ in W_2 & $(0, 0, 1, -1) \in W_2$.

& $(\alpha, \alpha, \beta, -2\alpha-\beta) = \alpha(1, 1, 0, -2) + \beta(0, 0, 1, -1)$ so that

$C = \{(0, 0, 1, -1), (1, 1, 0, -2)\}$ is a basis of W_2

Hence $B_2 = A \cup C = \{(0, 0, 1, -1), (0, 1, 0, -1), (1, 1, 0, -2)\}$ is a basis of $W_1 + W_2$. (Using dimension formula one can verify that $\dim(W_1 + W_2)$ is 3).

Major, Solution, Q9: It is given that $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T(1,1,1) = (1,-1,0)$, $T(1,-1,1) = (0,1,-1)$, $T(0,1,-1) = (-1,0,1)$.

First, we ~~obj~~ observe that $B = \{(1,1,1), (1,-1,1), (0,1,-1)\}$ is a basis of $\mathbb{R}^3$. For instance, the matrix (obtained from ~~the ba~~ B)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \underset{R_3 \to R_3 - R_2}{\overset{R_2 \to -\frac{1}{2} \times R_2}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \underset{R_1 \to R_1 - R_2 - R_3}{\overset{R_3 \to R_3 \times (-1)}{\sim}} I_3$$

has rank 3 so that the row space has dimension 3.

Therefore span $\{(1,-1,0), (0,1,-1), (-1,0,1)\}$ is the range space of T. ~~To find~~ The rank of T ~~is~~ is the rank of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \overset{R_3 \to R_3 + R_1}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\overset{R_3 \to R_3 + R_2}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1 \to R_1 + R_2}{\sim} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

whose rank is 2. Hence nullity of T is $3 - \text{rank } T = 3 - 2 = 1$ (by using rank-nullity theorem)

$\frac{1}{2}\{(1,1,1) - (1,-1,1)\} = (0,1,0)$

Hence $\frac{1}{2}\{(1,1,1) - (1,-1,1)\} - (0,1,-1) = (0,0,1)$. Now apply T

$T(0,0,1) = \frac{1}{2} T(1,1,1) - \frac{1}{2} T(1,-1,1) - T(0,1,-1) = (\frac{1}{2}, -\frac{1}{2}, 0) - (0, \frac{1}{2}, -\frac{1}{2}) - (-1,0,1)$

$= (\frac{3}{2}, -1, -\frac{1}{2})$ $\cancel{= -2(\frac{1}{2}, 1, \frac{1}{2})}$

Major Solution Q10:

Let $A = \left\{ \begin{pmatrix} 1 & t \\ t^2 & t^3 \end{pmatrix} : t \in \mathbb{R} \right\}$ & let $A_t = \begin{pmatrix} 1 & t \\ t^2 & t^3 \end{pmatrix}$.

Span A is a subspace of $M_2(\mathbb{R})$. Then

$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$,

$A_2 = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ and so on. (Here we expect that four diffnt values of t ~~with~~ g produces a set of four linearly independent set of vectors. In fact,

$A_1 + A_{-1} = 2\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ & $P = \frac{1}{2}(A_1 + A_{-1}) - A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \text{span}(A)$

Now ~~$Q = A_1 - A_0 - P =$~~ $Q = \frac{1}{2}(A_1 - A_{-1}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \text{span}(A)$

$A_2 - A_0 - 4P - 2Q = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \in \text{span} A$.

Thus $R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{6}(A_2 - A_0 - 4P - 2Q) \in \text{span}(A)$

Lastly, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = Q - R \in \text{span}(A)$

We have seen that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are in span(A). Hence A spans entire $M_2(\mathbb{R})$.