Recall that, for any two subspaces U and W of V,

$$U + W = \{x + y : x \in U, y \in W\}$$

Theorem :- If V is finite dimensional then

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Proof :- Note that,
$$\dim(U) \leq \dim(V)$$
$$\dim(W) \leq \dim(V)$$

Since $U+W$ is a subspace of V, $\dim(U+W) \leq \dim(V)$

Also, U and W are subspaces of $U+W$.

Therefore,
$$\dim(U) \leq \dim(U+W)$$
$$\dim(W) \leq \dim(U+W)$$

Since $U \cap W$ is a subspace of U & W, $\dim(U \cap W) \leq \dim(U)$ and
$$\dim(U \cap W) \leq \dim(W)$$

* $U \cap W$ is a subspace of U and W.
* U and W both are subspaces of $U+W$.

Let $\{v_1, v_2, \dots, v_n\}$ is a basis of $U \cap W$.

Let $\{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m\}$ is a basis of U.

$\{v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_l\}$ is a basis of W.

i.e., $\dim(U \cap W) = n$, $\dim(U) = n+m$, $\dim(W) = n+l$.

We claim that $\dim(U+W) = n+m+l$ and $\{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m, w_1, w_2, \dots, w_l\}$ is a basis of $U+W$.

Let $a_1 v_1 + \dots + a_n v_n + b_1 u_1 + \dots + b_m u_m + c_1 w_1 + \dots + c_l w_l = 0$ for some $a_1, \dots, a_n, b_1, \dots, b_m, c_1, \dots, c_l \in F$.

$$\underbrace{a_1v_1+\cdots+a_nv_n+b_1u_1+\cdots+b_mu_m}_{\in U} = \underbrace{-c_1w_1-\cdots-c_\ell w_\ell}_{\in W}$$

Since the left vector is in U & right vector is in W, both side are in $U\cap W$.

$$\Rightarrow -c_1w_1-\cdots-c_\ell w_\ell \in U\cap W$$

$$\Rightarrow -c_1w_1-\cdots-c_\ell w_\ell = d_1v_1+\cdots+d_nv_n$$

$$\Rightarrow d_1v_1+\cdots+d_nv_n+c_1w_1+\cdots+c_\ell w_\ell = 0$$

Since $\{v_1,\ldots,v_n,w_1,\ldots,w_\ell\}$ is linearly independent,

$$d_1=\cdots=d_n=c_1=\cdots=c_\ell=0$$

$$\Rightarrow a_1v_1+\cdots+a_nv_n+b_1u_1+\cdots+b_mu_m=0$$

Since $\{v_1,\ldots,v_n,u_1,\ldots,u_m\}$ is linearly independent,

$$a_1=\cdots=a_n=b_1=\cdots=b_m=0$$

Therefore, $\{v_1,\ldots,v_n,u_1\ldots,u_m,w_1,\ldots,w_\ell\}$ is linearly independent.

Let $x \in U + W$

$\Rightarrow x = u + w$ where $u \in U$ & $w \in W$

Let $u = p_1 v_1 + \cdots + p_n v_n + q_1 u_1 + \cdots + q_m u_m$

$w = r_1 v_1 + \cdots + r_n v_n + s_1 w_1 + \cdots + s_l w_l$

$\Rightarrow x = (p_1 + r_1) v_1 + \cdots + (p_n + r_n) v_n + q_1 u_1 + \cdots + q_m u_m + s_1 w_1 + \cdots + s_l w_l$

$\in \text{span}(\{v_1, \cdots, v_n\ u_1, \cdots, u_m, w_1, \cdots, w_l\})$

$\Rightarrow U + W \subseteq \text{span}(\{v_1, \cdots, v_n, u_1, \cdots, u_m, w_1, \cdots, w_l\})$

Also, $v_1, \cdots, v_n, u_1, \cdots, u_m, w_1, \cdots, w_l \in U + W$

Therefore, $\text{span}(\{v_1, \cdots, v_n, u_1, \cdots, u_m, w_1, \cdots, w_l\}) \subseteq U + W$.

Thus, $U + W = \text{span}(\{v_1, \cdots, v_n, u_1, \cdots, u_m, w_1, \cdots, w_l\})$

Therefore, $\{v_1, \cdots, v_n, u_1, \cdots, u_m, w_1, \cdots, w_l\}$ is a basis of $U+W$.

$\dim(U+W) = n + m + l = \dim(U) + \dim(W) - \dim(U \cap W)$

If $U \cap W = 0$ then $\dim(U) + \dim(W) = \dim(U+W)$.

Theorem :- $U + W$ is a direct sum if and only if $U \cap W = \{0\}$

Recall that, $U+W = U \oplus W$ if every vector of $U+W$ can be expressed uniquely as $u + w$, where $u \in U$, $w \in W$.

Proof :- If $U \cap W \neq \{0\}$

Let $v \in U \cap W$, where $v \neq 0$.

Then for $u \in U$ $\quad u + v \in U$ and

for $w \in W$ $\quad w - v \in W$

Therefore $\quad u + w = (u+v) + (w-v) = u' + w'$

Therefore, $\quad u + w = u' + w'$ where $u \neq u'$ & $w \neq w'$

Thus $U + W$ is not a direct sum of U & W.

Therefore, $U + W$ is a direct sum implies $U \cap W = \{0\}$.

Now assum $U \cap W = \{0\}$

claim, $U+W$ is a direct sum of $U+W$.

If $u+w = u'+w'$ for $u, u' \in U$ & $w, w' \in W$

then $\underbrace{u-u'}_{\in U} = \underbrace{w'-w}_{\in W}$

Therefore $u-u' = w'-w \in U \cap W = \{0\}$

$\Rightarrow u = u'$ & $w = w'$

Therefore any vector of $U+W$ has unique expression, i.e., $U+W$ is a direct sum of U and W.

Problem :- If $U = \text{Span}(\{(1,2,1), (2,1,3)\})$, $W = \text{Span}(\{(1,0,0), (0,0,1)\})$

Find $\dim(U \cap W)$ and $\dim(U+W)$.

Solution :- Since each span set contains two vectors and one is not multiple of other, $\{(1,2,1),(2,1,3)\}$ and $\{(1,0,0),(0,0,1)\}$ are linearly independent.

$$\dim(U) = \dim(W) = 2.$$

Let $(x,y,z) \in U \cap W$

$$\Rightarrow (x,y,z) = a(1,2,1) + b(2,1,3)$$
$$= c(1,0,0) + d(0,0,1)$$

Therefore,
$$a + 2b = c$$
$$2a + b = 0$$
$$a + 3b = d$$

$$\Rightarrow a = -\frac{b}{2}, \quad c = \frac{3b}{2}, \quad d = \frac{5}{2}b$$

Therefore, $(x,y,z) = \frac{3b}{2}(1,0,0) + \frac{5b}{2}(0,0,1) = b\left(\frac{3}{2},0,\frac{5}{2}\right)$ where $b \in \mathbb{R}$.

Therefore $U \cap W = \text{span}\left(\left\{\left(\frac{3}{2},0,\frac{5}{2}\right)\right\}\right)$

Thus, $\dim(U \cap W) = 1$ and $\dim(U+V) = \dim(U) + \dim(W) - \dim(U \cap W) = 3$

Problem :- Let V be a vector space over $\mathbb{R}$. Let U and W be two subspaces such that $U \cup W$ is a subspace. Prove that either $U \subseteq W$ or $W \subseteq U$.

Proof :- If possible let $U \nsubseteq W$ and $W \nsubseteq U$. Let $x \in U \setminus W$ and $y \in W \setminus U$. Then $x, y \in U \cup W$

Then $x+y \in U \cup W$.

If $x+y \in U$ then, $x \in U$ implies $y = (x+y) - x \in U$

This is a contradiction.

If $x+y \in W$ then $y \in W$ implies $x = (x+y) - x \in W$.

This is a contradiction.

Therefore, either $U \subseteq W$ or $W \subseteq U$.

Problem :- Let V be a vector space over $\mathbb{R}$. Let V_1, V_2, V_3 be three subspaces such that $V_1 \cup V_2 \cup V_3$ is a subspace. Then one of them contains the other two.

Solution :- If $V_1 \subseteq V_2$ then $V_1 \cup V_2 \cup V_3 = V_2 \cup V_3$

Then by previous solution either $V_2 \subseteq V_3$ or $V_3 \subseteq V_2$

Then either $V_1, V_2 \subseteq V_3$ or $V_1, V_3 \subseteq V_2$.

Similarly, we are done if $V_2 \subseteq V_1$.

Now assume, $V_1 \setminus V_2$ & $V_2 \setminus V_1$ is non-empty.

Let $x_1 \in V_1 \setminus (V_1 \cap V_2)$ and $x_2 \in V_2 \setminus (V_1 \cap V_2)$

Then $x_1, x_2 \in V_1 \cup V_2 \cup V_3$ and hence $x_1 + x_2 \in V_1 \cup V_2 \cup V_3$

If $x_1 + x_2 \in V_1$ then $x_1 \in V_1$ implies $x_2 = (x_1 + x_2) - x_1$

$\in V_1$, i.e, $x_2 \in (V_1 \cap V_2)$. This is a contradiction.

If $x_1 + x_2 \in V_2$ then $x_2 \in V_2$ implies $x_1 = (x_1 + x_2) - x_2 \in V_2$

i.e., $x_1 \in V_1 \cap V_2$. This is a contradiction.

Therefore, $x_1 + x_2 \in V_3$.

Similarly, we can prove that $x_1 - x_2 \in V_3$.

Therefore, $x_1 = \frac{1}{2}\left[(x_1+x_2)+(x_1-x_2)\right] \in V_3$

and $x_2 = \frac{1}{2}\left[(x_1+x_2)-(x_1+x_2)\right] \in V_3$.

Thus, $$\boxed{\begin{aligned} V_1 \setminus (V_1 \cap V_2) &\subseteq V_3 \\ V_2 \setminus (V_1 \cap V_2) &\subseteq V_3 \end{aligned}}$$

Let $v \in V_1 \cap V_2$. Now, take $w \in V_1 \setminus (V_1 \cap V_2) \subseteq V_3$

If $v+w \in (V_1 \cap V_2)$ then $w = (v+w) - v \in V_1 \cap V_2$

This is a contradiction.

But $v, w \in V_1$. Therefore $v+w \in V_1$

Therefore, $v+w \in V_1 \setminus (V_1 \cap V_2) \subseteq V_3$.

Then $v = (v+w) - w \in V_3$. Therefore, $\boxed{V_1 \cap V_2 \subseteq V_3}$

Thus, $V_1 \subseteq V_3$ & $V_2 \subseteq V_3$.