

Tutorial Sheet 9

2. Proof By contradiction

Let  $v$  be a ~~node~~ vertex in a tournament digraph  $G = (V, E)$  with maximum outdegree and suppose  $v$  is not a king.

Let  $S = \{w : (v, w) \in E\}$  be the set of out-neighbours of  $v$ . Then, outdegree of  $v = |S|$ .

Since  $v$  is not a king,  $\exists$  a vertex  $v' \notin S$ , i.e. a vertex  $v'$  which  $v$  does not beat, and no out-neighbour of  $v$  beat  $v'$ . That is  $(v, v') \notin E$  and  $\forall w \in S, (w, v') \notin E$ .

But by the definition of a tournament digraph, every vertex pair has exactly one edge between them.

$\therefore (v', v) \in E$  and  $(v', w) \in E \quad \forall w \in S$

that is,  $v$  and all out-neighbours of  $v$  are outneighbours of  $v'$ .  
 $v'$  has an outgoing edge to  $v$  and to every vertex in  $S$ .

$\therefore$  outdegree of  $v' = |S| + 1 > |S| =$  outdegree of  $v$   
which contradicts the assumption that  $v$  has the maximum outdegree in  $G$ .

Hence proved. ■