

Q1) Consider the initial value problem (IVP):

$$\frac{dy}{dx} = (\sin x + x^2) y^{1/5}, \quad y(0) = \beta.$$

a) If  $\beta > 0$ , then prove that the IVP has a unique solution.

b) If  $\beta = 0$ , then write down two distinct solutions of the IVP.

Solution.

The given IVP is

$$\frac{dy}{dx} = (\sin x + x^2) y^{1/5}, \quad y(0) = \beta.$$

Here  $f(x, y) = (\sin x + x^2) y^{1/5}$ . Notice that in any rectangle  $R$  containing  $(0, \beta)$ , the function  $f$  is continuous.

a) Suppose  $\beta > 0$ . choose a rectangle  $R$  with  $(0, \beta)$  as centre but does not contain  $(0, 0)$ .  $\rightarrow$  ①

Notice that the partial derivative

$$\frac{\partial f}{\partial y} = \frac{\sin x + x^2}{5 y^{4/5}} \rightarrow \textcircled{1.5}$$

is bounded in  $R$ . Thus, by the  $f$  is Lipschitz and hence by existence and uniqueness theorem, the given IVP has a unique solution  $\rightarrow$  (0.5)

b) One solution can be obtained by solving the given IVP, i.e., the solution

$$y(x) = \left( \frac{4}{5} \left( 1 + \frac{x^3}{3} - \cos x \right) \right)^{5/4}.$$

$\rightarrow$  ②

The other solution is  $y(x) = 0$ .

$\rightarrow$  ①

$$2a) \quad y' + y \tan x = \cos^2 x, \quad y\left(\frac{3\pi}{4}\right) = -1$$

$$\text{I.F.} \quad e^{\int \tan x dx} = e^{\log|\sec x|} = |\sec x| = -\sec x \text{ in a neighbourhood of } x = \frac{3\pi}{4}. \quad (\sec\left(\frac{3\pi}{4}\right) = -\sqrt{2})$$

So the sol<sup>n</sup> of the linear ODE is ①

$$y(-\sec x) = \int (-\sec x) \cos^2 x dx + C$$

$$\Rightarrow y = \cos x \sin x - C \cos x \quad \text{②}$$

$$\text{Using initial cond<sup>n</sup>: } -1 = -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - C \left(-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow C = \sqrt{2} \left(-\frac{1}{2}\right) = -\frac{1}{\sqrt{2}} \quad \text{①}$$

$$\text{The sol<sup>n</sup> is } y = \cos x \sin x + \frac{\cos x}{\sqrt{2}}$$

$$2b) \text{ The Wronskian of } y_1 = x, y_2 = \sin x \text{ is } x \cos x - \sin x \quad \text{①}$$

It vanishes at  $x=0 \in (-1,1)$  & nonzero elsewhere e.g.  $x=\frac{\pi}{4}$ .

If the LI fns  $x$  &  $\sin x$  were sol<sup>n</sup>s of  $y'' + p(x)y' + q(x)y = 0$ , then the Wronskian would have been nonzero on  $(-1,1)$  ②

Q73.

(a) Given  $y_1(x) = x$  is a solution of the following homogeneous linear ODE.

$$(1+x^2)y'' - 2xy' + 2y = 0$$

find the general solution, using the method of reduction of order.

Sol<sup>n</sup>  $\Rightarrow$  Given  $y_1(x) = x$  is one solution.

Let  $y_2(x) = v(x)y_1(x)$  be 2nd solution.

$$\text{i.e., } y_2(x) = xv$$

$$\therefore y_2' = xv' + v$$

$$y_2'' = xv'' + 2v'$$

$$\Rightarrow (1+x^2)(xv'' + 2v') - 2x(xv' + v) + 2xv = 0.$$

$$\text{Let } w = v'$$

$$\text{then } xw' + x^3w' + 2w = 0$$

$$\Rightarrow (x+x^3)w' + 2w = 0$$

$$\Rightarrow w' + \frac{2}{x(1+x^2)}w = 0$$

① marks  
for reduced order  
equation

$$\Rightarrow \frac{w'}{w} + \left[ \frac{2}{x} - \frac{2x}{1+x^2} \right] = 0$$

Integrating both sides

$$\log w + 2 \log x - \log(1+x^2) = \log c$$

$$\Rightarrow \frac{wx^2}{1+x^2} = c$$

$$\Rightarrow w = c \left( \frac{1+x^2}{x^2} \right)$$

$$\Rightarrow v' = c \left( \frac{1}{x^2} + 1 \right)$$

Integrating both sides

$$v = c \left( -\frac{1}{x} + x \right) + c' \quad \text{--- 1 marks for finding } v$$

$$\Rightarrow y_2 = vx$$

$$\Rightarrow \boxed{y_2 = c(x^2 - 1) + c'x} \quad \text{--- 1 marks for writing general solution.}$$

is the general solution

(b) Using the method of variation of parameter find a particular solution of the following non-homogeneous linear ODE:

$$(1+x^2)y'' - 2xy' + 2y = x^3 + x.$$

Sol<sup>n</sup> → From part (a), we get the fundamental solutions of the homogeneous equation as

$$y_1 = x \quad \text{and} \quad y_2 = x^2 - 1$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix} = x^2 + 1 \quad \text{① marks for Wronskian}$$

$$\text{Now, } (1+x^2)y'' - 2xy' + 2y = x^3 + x$$

$$\Rightarrow y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = x.$$

$$\Rightarrow y'' + P(x)y' + Q(x)y = F(x)$$

Then Particular solution  $y_p = y_1 u_1 + y_2 u_2$   
where

$$u_1 = - \int \frac{y_2 F(x)}{W(x)} \quad \text{and} \quad u_2 = \int \frac{y_1 F(x)}{W(x)}$$

$$\text{So, } u_1 = - \int \frac{(x^2-1)x}{(x^2+1)} dx \quad \text{--- } \left(\frac{1}{2}\right) \text{ marks for } u_1$$

$$= - \int \left(1 - \frac{2}{1+x^2}\right) x dx = -\frac{x^2}{2} + \log(1+x^2)$$

$$\text{and } u_2 = \int \frac{x^2}{x^2+1} dx \quad \text{--- } \left(\frac{1}{2}\right) \text{ marks for } u_2$$

$$= \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \tan^{-1} x$$

$$\text{So, } y_p = y_1 u_1 + y_2 u_2. \quad \text{--- } \textcircled{1} \text{ marks}$$

for particular  
solution

$$\therefore y_p = x \log(1+x^2) - \frac{x^3}{2} + x(x^2-1) - (x^2-1) \tan^{-1} x$$

$$= x \left( \log(1+x^2) - 1 \right) + \frac{x^3}{2} - (x^2-1) \tan^{-1} x$$

#### Question 4:

Solve the following system of differential equations (without using Laplace transform)

$$x_1' = 5x_1 + 2x_2 + x_3$$

$$x_2' = -8x_1 - 3x_2 - 2x_3$$

$$x_3' = 6x_1 + 4x_2 + 3x_3$$

#### Solution:

$$\text{Let } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then the above system can be rewritten as,

$$x' = Ax, \text{ where}$$

$$A = \begin{pmatrix} 5 & 2 & 1 \\ -8 & -3 & -2 \\ 6 & 4 & 3 \end{pmatrix}$$

To solve this system we have to find the eigen values and the corresponding eigenvectors of  $A$ .

The characteristic equation is given by,

$$\det(A - \lambda I) = 0$$

ie,

$$\begin{vmatrix} 5-\lambda & 2 & 1 \\ -8 & -3-\lambda & -2 \\ 6 & 4 & 3-\lambda \end{vmatrix} = 0$$

Solving, we get,

$$\lambda = 1, \quad 2-i, \quad 2+i$$

+2

• For  $\lambda = 1$ , solving

$$A v_1 = v_1$$

we get

$$v_1 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

[ $\therefore x_3$  was the free variable]

+1

• For  $\lambda = 2 - i$ , solving

$$A v_2 = (2 - i) v_2$$

we get

$$v_2 = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} \\ -1 - i \\ 1 \end{pmatrix} \quad \text{---} \quad (+1)$$

• For  $\lambda = 2 + i$ , solving

$$A v_3 = (2 + i) v_3$$

we get

$$v_3 = \begin{pmatrix} \frac{1}{2} - \frac{i}{2} \\ -1 + i \\ 1 \end{pmatrix} \quad \text{---} \quad (+1)$$

Then the general solution is

$$x(t) = C_1 e^t v_1 + C_2 e^{(2+i)t} v_2 + C_3 e^{(2-i)t} v_3.$$

$$\text{---} \quad (+1)$$

## Marking scheme of Question 5: [4+3 marks]

Let  $f^{(n)}$  denote the  $n$ -th derivative of  $f$  and  $\mathcal{L}$  denote the Laplace transform.

- (1) Suppose  $f$  is a function such that for all integers  $n \geq 0$ ,  $f^{(n)}$  is continuous function on  $\mathbb{R}$ . Moreover, assume that there exists  $M > 0, K > 0$  satisfying  $|f^{(n)}(t)| \leq Me^{Kt}$ . Using induction prove that for  $n \geq 1$ ,

$$\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - f^{(n-1)}(0).$$

- (2) Compute the Laplace transform  $\mathcal{L}(f)$  where  $f$  is given by  $f(t) = t \sin(2t)$ .

### Solution of Part (a):

[1 mark] For  $n = 1$  the statement is  $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$ , which we prove now. By definition of Laplace transform and doing integration by parts:

$$\begin{aligned}\mathcal{L}(f')(s) &= \int_0^\infty e^{-st} f'(t) dt \\ &= e^{-st} f(t) \Big|_0^\infty + \int_0^\infty s e^{-st} f(t) dt\end{aligned}$$

[1 mark] Since  $|f(t)| \leq Me^{Kt}$  is given, we get  $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$  for  $s > K$ . Then,

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0).$$

We have verified the case  $n = 1$ .

[1 mark] Now assume that the statement is true for  $n = k$ , i.e.,

$$\mathcal{L}(f^{(k)})(s) = s^k \mathcal{L}(f)(s) - s^{k-1} f(0) - s^{k-2} f^{(1)}(0) - \dots - f^{(k-1)}(0).$$

Now we claim that the statement is true for  $n = k + 1$ .

[1 mark] Consider

$$\begin{aligned}\mathcal{L}(f^{(k+1)})(s) &= \mathcal{L}((f^{(k)})')(s) \\ &= s(\mathcal{L}(f^{(k)})(s) - f^{(k)}(0)) \quad (\text{from the case } n = 1 \text{ since } |f^{(k)}(t)| \leq Me^{Kt}) \\ &= s(s^k \mathcal{L}(f)(s) - s^{k-1} f(0) - s^{k-2} f^{(1)}(0) - \dots - f^{(k-1)}(0)) - f^{(k)}(0) \\ &= s^{k+1} \mathcal{L}(f)(s) - s^k f(0) - s^{k-1} f^{(1)}(0) - \dots - f^{(k)}(0).\end{aligned}$$

Thus, the statement for  $n = k + 1$ .

Therefore by principle of mathematical induction, the given statement is true for all  $n \geq 1$ .

### Solution of Part (b):

[1 mark] Recall that  $\mathcal{L}(tf(t))(s) = -\frac{d}{ds} \mathcal{L}(f(t))(s)$ . Using this we get  $\mathcal{L}(t \sin 2t)(s) = -\frac{d}{ds} \mathcal{L}(\sin 2t)(s)$ .

[2 mark] Using definition and simple computation gives

$$\mathcal{L}(\sin 2t)(s) = \int_0^\infty e^{-st} t \sin(2t) dt = \dots = \frac{2}{s^2 + 4}.$$

Therefore,

$$\mathcal{L}(tf(t))(s) = -\frac{d}{ds} \left( \frac{2}{s^2 + 4} \right) = \frac{4s}{(s^2 + 4)^2}.$$

# MTL 101, RE-MAJOR, PROBLEM 6

Solve the following IVP using the Laplace Transform:

$$y'' + 2y = g(t) + \delta(t - \pi/2); y(0) = 1, y'(0) = 0$$

$$g(t) = \begin{cases} 0 & \text{if } t < \pi/2 \\ \sin t & \text{if } t \geq \pi/2 \end{cases}$$

*Solution.* Apply Laplace transform to get

$$\mathcal{L}(y'') + 2\mathcal{L}(y) = \mathcal{L}(g(t) + \delta(t - \pi/2))$$

$$s^2\mathcal{L}(y) - sy'(0) - y'(0) + 2\mathcal{L}(y) = \mathcal{L}(g(t) + \delta(t - \pi/2))$$

$$(s^2 + 2)\mathcal{L}(y) = s + \mathcal{L}(g(t)) + \mathcal{L}(\delta(t - \pi/2))$$

$$\mathcal{L}(y) = \frac{s}{s^2 + 2} + \frac{\mathcal{L}(g(t))}{s^2 + 2} + \frac{\mathcal{L}(\delta(t - \pi/2))}{s^2 + 2}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left( \frac{s}{s^2 + 2} + \frac{\mathcal{L}(g(t))}{s^2 + 2} + \frac{\mathcal{L}(\delta(t - \pi/2))}{s^2 + 2} \right) \\ &= \mathcal{L}^{-1} \left( \frac{s}{s^2 + 2} \right) + \mathcal{L}^{-1} \left( \frac{\mathcal{L}(g(t))}{s^2 + 2} \right) + \mathcal{L}^{-1} \left( \frac{\mathcal{L}(\delta(t - \pi/2))}{s^2 + 2} \right) \\ &= \cos \sqrt{2}t + \mathcal{L}^{-1} \left( \frac{\mathcal{L}(g(t))}{s^2 + 2} \right) + \mathcal{L}^{-1} \left( \frac{\mathcal{L}(\delta(t - \pi/2))}{s^2 + 2} \right) \\ &= \cos \sqrt{2}t + \mathcal{L}^{-1} \left( \frac{e^{-\pi s/2}}{s^2 + 2} \right) + \mathcal{L}^{-1} \left( \frac{\mathcal{L}(g(t))}{s^2 + 2} \right) \\ &= \cos \sqrt{2}t + \mathcal{L}^{-1} \left( e^{-\pi s/2} \frac{1}{\sqrt{2}} \mathcal{L}(\sin \sqrt{2}t) \right) + \mathcal{L}^{-1} \left( \mathcal{L}(g(t)) \frac{1}{\sqrt{2}} \mathcal{L}(\sin(\sqrt{2}t)) \right) \\ &= \cos \sqrt{2}t + \frac{1}{\sqrt{2}} \mathcal{L}^{-1} \left( \mathcal{L}(u(t - \pi/2) \sin \sqrt{2}(t - \pi/2)) \right) + \frac{1}{\sqrt{2}} \mathcal{L}^{-1} \left( \mathcal{L}(g(t)) \mathcal{L}(\sin(\sqrt{2}t)) \right) \\ &= \cos \sqrt{2}t + \frac{1}{\sqrt{2}} u(t - \pi/2) \sin \sqrt{2}(t - \pi/2) + \frac{1}{\sqrt{2}} g(t) * \sin(\sqrt{2}t) \end{aligned}$$

Now computing the convolution we get

$$\begin{aligned} g(t) * \sin(\sqrt{2}t) &= \int_0^t g(\tau) \sin(\sqrt{2}(t - \tau)) d\tau \\ &= \begin{cases} 0 & t < \pi/2 \\ \sqrt{2} \sin t - \sqrt{2} \cos(\sqrt{2}(t - \pi/2)) & t \geq \pi/2 \end{cases} \end{aligned}$$

□

Ques 7(a)

$$T_a(x, y) = (x + y + a^2 + 1, y)$$

$$T_a(0, 0) = (a^2 + 1, 0) \quad [1 \text{ mark}]$$

Since  $a^2 + 1 \neq 0 \quad \forall a \in \mathbb{R}$ ,  $T_a(0, 0) \neq (0, 0)$ .

As every linear transformation must map the zero vector to the zero vector,  $T_a$  is NOT a linear transformation for any  $a \in \mathbb{R}$  [1 mark]

Ques 7(b)

Characteristic polynomial of the given matrix  $A$  is

$$p(x) = \det(xI - A) = \begin{vmatrix} x-11 & -4 & -4 \\ -24 & x-5 & -7 \\ 48 & 14 & x+16 \end{vmatrix}$$

$$= (x-11)[(x-5)(x+16)+98] + 4[-24(x+16)+336] \\ - 4[-336 - 48(x-5)]$$

$$= (x-11)[x^2+11x+18] + 96x+192$$

$$= (x-11)(x+2)(x+9) + 96(x+2) \quad [1 \text{ mark}]$$

$$= (x+2)[(x-11)(x+9)+96]$$

$$= (x+2)(x^2-2x-3) = (x+2)(x+1)(x-3)$$

$\therefore$  The eigenvalues of  $A$  are  $-2, -1$  and  $3$  [1 mark]

Since  $A$  has 3 distinct eigenvalues, it is diagonalizable. [1 mark]

Then  $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ , consisting of eigenvalues on the diagonal,

is similar to the given matrix  $A$ . [1 mark]

## Marking scheme of Question 8: [3+3 marks]

Prove or disprove the following statements.

- (a) There exists  $A \in M_{3 \times 3}(\mathbb{R})$  such that the subset  $\{I, A, A^2, A^3\}$  is linearly independent in the vector space  $M_{3 \times 3}(\mathbb{R})$  over  $\mathbb{R}$ .
- (b) Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$ . If  $T, S : V \rightarrow V$  are linear transformations satisfying

$$\ker(T) + \ker(S) = V = \text{Range}(T) + \text{Range}(S),$$

then

$$\ker(T) \cap \ker(S) = \{0\} = \text{Range}(T) \cap \text{Range}(S).$$

[Notation:  $\ker(T)$  and  $\text{Range}(T)$  denote the kernel (i.e., the null space) of  $T$  and the range of  $T$ .]

**Solution of Part (a) :** The statement is NOT TRUE.

[1 mark] By Cayley-Hamilton theorem every matrix satisfies its characteristic polynomial.

[1 mark] For any  $3 \times 3$  matrix  $A$ , its characteristic polynomial has degree equal to 3. Therefore,  $A$  will satisfy a degree 3 polynomial, say  $x^3 + a_2x^2 + a_1x + a_0$ .

[1 mark] Then  $A^3 + a_2A^2 + a_1A + a_0I = 0$ , which is a non-trivial linear combination of  $I, A, A^2, A^3$  (because the coefficient of  $A^3$  is 1) which equals 0. Therefore the given subset is linearly dependent for any  $A$ .

**Solution of Part (b) :** The statement is TRUE.

Write  $n(T)$  and  $r(T)$  for the nullity of  $T$  and the rank of  $T$  respectively. Similarly for  $S$ .

[1 mark] Given  $\ker(T) + \ker(S) = V = \text{Range}(T) + \text{Range}(S)$  implies

$$n(T) + n(S) - \dim(\ker(T) \cap \ker(S)) = \dim V = r(T) + r(S) - \dim(\text{Range}(T) \cap \text{Range}(S)).$$

[1 mark] On the other hand, by rank-nullity theorem we have

$$r(T) + n(T) = \dim V = r(S) + n(S).$$

This give us

$$\begin{aligned} r(T) &= n(S) - \dim(\ker(T) \cap \ker(S)) \text{ and } n(S) = r(T) - \dim(\text{Range}(T) \cap \text{Range}(S)) \\ \Rightarrow \dim(\ker(T) \cap \ker(S)) + \dim(\text{Range}(T) \cap \text{Range}(S)) &= 0. \end{aligned}$$

[1 mark] Since dimension is always non-negative integer, we get

$$\dim(\ker(T) \cap \ker(S)) = 0 = \dim(\text{Range}(T) \cap \text{Range}(S)) = 0.$$

This implies

$$\ker(T) \cap \ker(S) = \{0\} = \text{Range}(T) \cap \text{Range}(S).$$