

Theorem :- A necessary and sufficient condition for a non-empty subset W of a vector space over a field F , to be a subspace is that ' $u, v \in W, a \in F \Rightarrow au + bv \in W$ '.

Proof :- If W is a subspace and $u, v \in W \& a \in F$. Then $au \in W$ and hence $au + bv \in W$.

Conversely, suppose $u, v \in W, a \in F \Rightarrow au + bv \in W$.

- * Since W is non-empty $\exists w \in W$. Then take $a = -1, u = v = w$. Then $-w + w = 0 \in W$
- * Take $v = 0$. Then $au \in W$ for every $a \in F, u \in W$.
- * Take $a = 1$. Then $u + v \in W$ for every $u, v \in W$.

Thus, W is a subspace of the vector space.

Theorem :- W is a subspace of V if and only if

$$u, v \in W \& a, b \in F \Rightarrow au + bv \in W. \text{ (Same Proof)}$$

Problem :- Let $A \in M_{m \times n}(F)$. Let $W = \{X \in F^n \text{ such that } AX = 0\}$. Then prove that W is a subspace of F^n .

Note that, here a vector in F^n is written as column matrix instead of an n -tuple.

Solution :- For $O_{n \times 1} \in F^n$, $AO_{n \times 1} = O_{m \times 1}$. Thus $O_{n \times 1} \in W$.

If $x_1, x_2 \in W$ then $AX_1 = 0 = AX_2$

Then $A(x_1 + x_2) = AX_1 + AX_2 = 0 + 0 = 0$

Thus, $x_1 + x_2 \in W$

If $c \in F$ & $x \in W$ then $A(cx) = c(Ax) = c0 = 0$

Thus, $cx \in W$.

Therefore, W is a subspace.

Alternatively, let $x_1, x_2 \in W$ & $c \in F$. Then $A(cx_1 + x_2) = c(Ax_1) + Ax_2 = 0 \Rightarrow cx_1 + x_2 \in W$.

Linear Combination :- A linear combination of a list v_1, v_2, \dots, v_m of vectors in V is a vector of the form $a_1 v_1 + a_2 v_2 + \dots + a_m v_m$, where $a_1, a_2, \dots, a_m \in F$.

* It is defined ONLY for finite collection/list.

Linearly independent :- A list v_1, v_2, \dots, v_n of vectors in V is called linearly independent if the only choice of $a_1, a_2, \dots, a_m \in F$ that makes $a_1 v_1 + a_2 v_2 + \dots + a_m v_m$

equal 0 is $a_1 = a_2 = \dots = a_m = 0$. [Here we defined for a finite list. But it can be generalised for an infinite list as well by a suitable modification. We shall see that later.]

* The empty set ϕ is declared to be linearly independent.

Linearly dependent :- A list v_1, v_2, \dots, v_n of vectors in V is called linearly dependent if the list is not linearly independent, i.e., there exist $a_1, a_2, \dots, a_m \in F$, not all 0 , such that $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$

[Here we defined for a finite list. But it can be generalised for an infinite list as well by a suitable modification. We shall see that later.]

Example :- (i) The list $(2, 3, 1), (1, -1, 2), (7, 3, 8)$ of vectors in \mathbb{R}^3 is linearly dependent of $2(2, 3, 1) + 3(1, -1, 2) + (-1)(7, 3, 8) = (0, 0, 0)$.

(ii) The list $(1, 0, 0), (0, 1, 0), (1, 1, 1)$ of vectors in \mathbb{R}^3 is linearly independent of $c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(1, 1, 1) = (0, 0, 0)$ implies $c_1 + c_3 = 0$
 $c_2 + c_3 = 0$
 $c_3 = 0$

Thus $c_1 = 0, c_2 = 0, c_3 = 0$.

Problem :- Find the value of a such that the list $(2, 3, 1), (1, -1, 2), (7, 3, a)$ is linearly dependent.

Solution :- Let $c_1(2, 3, 1) + c_2(1, -1, 2) + c_3(7, 3, a) = (0, 0, 0)$

This implies,

$$2C_1 + C_2 + 7C_3 = 0$$

$$3C_1 - C_2 + 3C_3 = 0$$

$$C_1 + 2C_2 + aC_3 = 0$$

$$\begin{bmatrix} 2 & 1 & 7 \\ 3 & -1 & 3 \\ 1 & 2 & a \end{bmatrix}$$

$$\xrightarrow{\begin{array}{l} R_1 - 2R_3 \\ R_2 - 3R_3 \end{array}} \begin{bmatrix} 0 & -3 & 7-2a \\ 0 & -7 & 3-3a \\ 1 & 2 & a \end{bmatrix}$$

$$\xrightarrow{R_1 - \frac{3}{7}R_2} \begin{bmatrix} 0 & 0 & (7-2a) - \frac{9(1-a)}{7} \\ 0 & -7 & 3-3a \\ 1 & 2 & a \end{bmatrix}$$

$$\xrightarrow{\begin{array}{l} R_1 \leftrightarrow R_3 \\ -\frac{1}{7}R_2 \end{array}} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{3}{7}(a-1) \\ 0 & 0 & \frac{40-5a}{7} \end{bmatrix}$$

The system has unique solution, i.e., zero solution if and only if $\frac{40-5a}{7} \neq 0$, i.e., $a \neq 8$.

Thus the system is linearly dependent if and only if $a = 8$.

Some more examples :-

1. A list V of one vector $v \in V$ is linearly independent if and only if $v \neq 0$
2. A list of two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.
3. The list $1, x, x^2, x^3, \dots, x^m$ is linearly independent in $F[x]$.

Span(finite) :- The set of all linear combinations of a list of vectors v_1, v_2, \dots, v_m in V is called the span of v_1, v_2, \dots, v_m , denoted by $\text{Span}(v_1, \dots, v_m)$.

In other words,

$$\text{Span}(v_1, v_2, \dots, v_m) = \left\{ a_1 v_1 + \dots + a_m v_m : a_1, a_2, \dots, a_m \in F \right\}$$

Note that the span of the empty set \emptyset is $\{0\}$.

Theorem :- The span of a finite list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Proof :- Suppose v_1, v_2, \dots, v_m is a list of vectors in V .
Let $u, v \in \text{Span}(v_1, v_2, \dots, v_m)$.

$$\text{Then } u = a_1 v_1 + \dots + a_m v_m \text{ & } v = b_1 v_1 + \dots + b_m v_m$$

Then for any $c \in F$,

$$cu + v = (ca_1 + b_1)v_1 + \dots + (ca_m + b_m)v_m \\ \in \text{Span}(v_1, \dots, v_m)$$

Thus $\text{Span}(v_1, \dots, v_m)$ is a subspace of V .

Now $v_i = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_{i-1} + 1 \cdot v_i + 0 \cdot v_{i+1} + \dots + 0 \cdot v_m$
 $\in \text{Span}(v_1, \dots, v_m)$, for each v_i .

Let W be a subspace containing v_1, v_2, \dots, v_m .

Let $x \in \text{Span}(v_1, \dots, v_m)$

$$\Rightarrow x = c_1 v_1 + c_2 v_2 + \dots + c_m v_m, \text{ for some } c_i \in F$$

Since $v_i \in W \Rightarrow c_i v_i \in W$

$$\Rightarrow c_1 v_1 + c_2 v_2 + \dots + c_m v_m \in W$$

Thus, $\text{Span}(v_1, \dots, v_m) \subseteq W$.