COL202 Major

Anish Banerjee

TOTAL POINTS

31.5 / 40

QUESTION 1

1 Problem 1 **5 / 5**

✓ **+ 1 pts** **Calculate the number of ways in which $$k$$ fixed points can be chosen out of $$n$$ points.**

✓ **+ 1 pts** **State that the left out $$(n-k)$$ elements must not be fixed.**

✓ **+ 2 pts** **Calculate the number of derangements of those $$(n-k)$$ elements.**

✓ **+ 1 pts** **Combine the two findings to reach the conclusion**

- **1 pts** Proof writing guidelines not followed.

+ **0 pts** Incorrect / Not attempted

QUESTION 2

Problem 2 5 pts

2.1 Problem 2.1 **2 / 2**

✓ **+ 2 pts** **All Correct**

+ **0 pts** Incorrect/Unattempted

+ **1 pts** Series form of Exponential generating function

+ **1 pts** Closed form of Exponential generating function

2.2 Problem 2.2 **1 / 3**

+ **0.25 pts** EGF for oddness

+ **0.75 pts** EGF for one partition even + one partition odd

+ **0.25 pts** Short explanation for the above EGF

+ **0.75 pts** EGF for both partitions even

+ **0.25 pts** Short explanation for the above EGF

+ **0.25 pts** Final EGF

+ **0.5 pts** Explicit formula for $$p_n$$

✓ **+ 1 pts** **Overcounting/undercounting the partitions**

+ **0 pts** Incorrect/Unattempted

QUESTION 3

Problem 3 8 pts

3.1 Problem 3.1 **3 / 3**

Prove that every pair of this poset has a meet and a join, thereby concluding that it is a lattice.

✓ **+ 0.75 pts** **Give expressions for the meet and join of any two arbitrary elements of the poset.**

✓ **+ 0.75 pts** **Prove that the stated meet and join are actually the meet and join of the two elements.**

Prove that every subset of this lattice has a meet and a join, thereby concluding that the lattice is a complete lattice.

✓ **+ 0.75 pts** **Give expressions for the meet and join of any arbitrary subset of the lattice.**

✓ **+ 0.75 pts** **Prove that the stated meet and join are actually the meet and join of the subset.**

- **1 pts** Proof writing guidelines not followed.

+ **0 pts** Not attempted / Incorrect.

3.2 Problem 3.2 **4.5 / 5**

+ **0.5 pts** Mention the method of proof

✓ **+ 1 pts** **The minimum value that x can take is (1,1,...,1) and the maximum value that x can take is (n,n,...,n)**

✓ **+ 1 pts** **The least change in the value of x can be in one coordinate value**

✓ **+ 1.5 pts** **As f(x) is monotonic, the value of f(x) differs from the previous f(x) at one position and is 1 more than the value at that position in x**

✓ **+ 1 pts** **Conclusion that the loop can run for a maximum of n*k times**

+ **0 pts** Incorrect

QUESTION 4

Problem 4 11 pts

4.1 Problem 4.1 **3 / 3**

✓ **- 0 pts** Correct

- 1 pts Did not argue when 3SAT is unsatisfiable.

- 2 pts Showed an un-satisfiable 3SAT but did not argue about its construction.

- 1 pts Did not show an example for un-satisfiable 3SAT

- 2 pts Did not argue about the construction of the equation and did not show an example for an un-satisfiable 3SAT..

- 3 pts Incorrectly argued about construction of 3SAT

4.2 Problem 4.2 **0 / 6**

+ 1.5 pts Observing that probability of a clause to be true is 7/8

+ 2 pts Showing expectation of these will be 7m/8

+ 1.5 pts Arguing > 0 probability for R.V. to be greater than it's expectation

+ 1 pts Argument about existence of Ceiling

- 1 pts Not following proof guidelines

✓ **+ 0 pts** **Incorrect/ Not attempted**

4.3 Problem 4.3 **2 / 2**

✓ **+ 1 pts** **If less than 8 clauses, then all the clauses will be true (using Problem 4.2)**

✓ **+ 0.5 pts** **Give examples for m = 7, 6 etc**

✓ **+ 0.5 pts** **Conclusion**

+ 0 pts incorrect

QUESTION 5

5 Problem 5 **5 / 5**

+ 5 pts Correct

+ 2 pts p1: Formally write $$f$$ and prove surjection or injection from Σ* to $$\mathcal{N}$$

+ 2 pts p2: Formally write $$g$$ and prove surjection or injection from $$\mathcal{N}$$ to Σ*

+ 1 pts p3: Use Schroder Bernstein theorem to prove the cardinality of both sets

✓ **+ 2 pts** **P1: Formally write the bijection$$f$$ from Σ* to $$\mathcal{N}$$**

✓ **+ 1.5 pts** **P2: Argue that the above function is one-one**

✓ **+ 1.5 pts** **P3: Argue that the above function is onto**

+ 1.5 pts P2: Write the inverse $$g$$ of $$f$$

+ 1.5 pts P3: Argue that the above function $$g$$ is indeed inverse of $$f$$

+ 0 pts No solution / incomplete solution

- 0.5 pts P4: For not following the guidelines of proof

QUESTION 6

6 Problem 6 **6 / 6**

✓ **- 0 pts** Correct

- 6 pts Not attempted or nothing substantial written.

- 5 pts Wrong or missing idea or proof details

- 5 pts Induction on trees cannot be done by adding a node/edge to create a larger tree. This requires a proof that all possible trees of this size can be generated this way. That proof will further bring you back to working with the tree that results from removal, so the argument is circular. This point has been made in class multiple times.

- 1 pts Induction variable not clearly and/or separately specified

- 1 pts Missed discussing the case where the walk begins at the removed vertex.

- 1 pts Missed discussing the case where the walk doesn't begin at the removed vertex.

- 4 pts Right direction but incorrect/incomplete arguments.

- 1 pts Wrong way of writing the induction hypothesis or missing induction hypothesis

- 0.5 pts Missed $$\forall v \in V$$ in the statement.

COL202: Discrete Mathematical Structures. I semester, 2022-23.
Major exam.
, Maximum Marks: .

Name (In CAPITAL letters as on Gradescope)	I	Ent. No.
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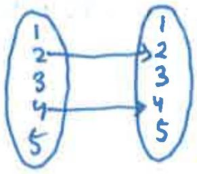
Important: If you write outside the box we may not grade it.

Problem 1 (5 marks) Given a permutation π of the set $[n]$, we say that i is a fixed point of π if $\pi(i) = i$. Count the number of permutations of $[n]$ with exactly k fixed points where $0 \leq k \leq n$. Please argue your answer. An answer without an argument will get 0.



No. of fixed points $= K$

No. of ways of choosing these fixed points $= \binom{n}{k}$

The remaining $n-k$ elements must map in such a way so that no element maps to itself. This is the number of dearrangements on a set of size $n-k$.

Thus Answer $= {}^{n}C_{k} D_{n-k}$ where D_n is the no. of dearrangements on a set of size n

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} \dots \frac{(-1)^n}{n!}\right)$$

Problem 2.1 (2 marks) We say a sequence $\{a_n\}_{n \geq 0}$ captures a property if $a_i = 1$ iff i has that property, e.g.. if the property is evenness then the sequence will be $1, 0, 1 \dots$. Write the exponential generating function the property "evenness."

For the property evenness, we have $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

$$\hat{G}(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{K=0}^{\infty} \frac{x^{2K}}{(2K)!} = \frac{e^x + e^{-x}}{2}$$

$$\boxed{\hat{G}(x) = \frac{e^x + e^{-x}}{2}}$$

Problem 2.2 (5 marks) Using the answer of Problem 2.1 write the egf for p_n = the number of ways of partitioning a set into two parts such that one of the parts is even in size. Find an explicit formula for p_n. No egf or no use of Problem 2.1 $\Rightarrow$ 0 marks.

$P_n \to$ One of the parts is even in size

$= \text{coeff. of } x^n \text{ in } n! \left(\frac{e^x+e^{-x}}{2}\right) e^x$

Arrangement — Even partition — other partition can be even or odd

~~Thus the generating function becomes~~

So $P_n = [x^n] \; n! \left(\frac{e^{2x}+1}{2}\right) = 2^{n-1}$

Egf for P_n $\hat{P}_{(n)} = \sum_{n\geq 0} \frac{P_n}{n!} x^n = \frac{e^{2x}+1}{2}$

Problem 3.1 (3 marks) Given the set $[n]^k$, partial order $\preceq$ is defined as follows: $(x_1, \ldots, x_k) \preceq (y_1, \ldots, y_k)$ if $x_i \leq y_i$ for all $i \in [k]$. Argue that $([n]^k, \preceq)$ is a complete lattice. Recall $[n] = \{1, \ldots, n\}$.

The set $[n]^K$ has a greatest element $(\underbrace{n, n, \ldots n}_{K \text{ times}})$ as $x_i \leq n \; \forall x_i \in [n]$

Let $S = \{a_1, a_2 \ldots a_m\} \subseteq [n]^K$

$a_i = (x_{i1}, x_{i2} \ldots x_{iK})$

Define $b = (y_1, y_2 \ldots y_K)$ $\quad y_j = \min\{x_{ij}\}_{i \in [m]}$

Claim b is the glb of S

It is a lower bound as $y_j \leq x_{ij} \; \forall i \in [m], j \in [K]$

If b' is any other lower bound then,

$b' = (y_1', y_2' \ldots y_K')$ $\quad y_j' \leq x_{ij} \; \forall i \Rightarrow y_j' \leq \min\{x_{ij}\}_{i \in [m]}$

$\Rightarrow y_j' \leq y_j$

So b is the glb

Since $[n]^K$ has a $\top$ and any subset has a glb, $([n]^K, \preceq)$ is a complete lattice. [Prop^n 4.2.4]

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Problem 3.2 (5 marks) Let us suppose we are given a monotonic function $f : [n]^k \to [n]^k$, i.e.. $x \preceq y \Rightarrow f(x) \preceq f(y)$. We run the program given on the right. ($a \leftarrow b$ means "set the value of variable a to b".)

```
1: x ← (1,...,1)
2: repeat
3:    t ← x
4:    x ← f(x)
5: until x = t
6: Return x
```

Prove that the loop of lines (2)-(5) will execute at most kn times.

$([n]^k, \preceq)$ is a complete lattice and f is a monotonic function. So by Tarski's Fixed Point Theorem, f has a fixed point $\{f(x) = x$ for some $x \in [n]^k\}$.
The above ~~program~~ program finds the first fixed points of f.
In the worst case, the first fixed point will be

$$x_{max} = \bigvee\{x \in X : f(x) \geq x\} = (\underbrace{n, n, \dots n}_{k \text{ times}})$$

i.e. the largest element.
Thus, the maximum no. of steps needed to reach x_{max} will be when f increases one coordinate of $(1, 1, \dots 1)$ at a time. In that case we take

$$\underbrace{n + n + n \dots n}_{k \text{ times}} = nk \text{ steps to reach } x_{max}$$

So, max no. of iterations needed $= k \cdot n$

Problem 4 (3 + 6 + 2 = 11 marks) Variables $P_1, P_2, \dots$ can take values T or F. We use the term *literal* to denote P_i or $\neg P_i$. A *disjunctive clause of size 3* is a term of the form $L_1 \vee L_2 \vee L_3$ where L, L_2, L_3 are literals involving 3 *distinct* variables. An expression of the form $\bigwedge_{i=1}^{m}(L_{i,1} \vee L_{i,2} \vee L_{i,3})$ ia a *3-SAT expression*, where $L_{i,j}$s are all literals. A 3-SAT expression is *satisfiable* if there exists a truth value setting of $P_1, P_2, \dots$ such that the expression evaluates to T.

Problem 4.1 (3 marks) Assume for now that no disjunctive clause of size 3 is repeated in any 3-SAT expression. Show by construction that there exists a 3-SAT expression that is *not* satisfiable.

A 3-SAT expression will not be satisfiable if we perform $\wedge$ over all possible combinations of P_1, P_2, P_3

$(P_1 \vee P_2 \vee P_3) \wedge (P_1 \vee P_2 \vee \neg P_3) \wedge (P_1 \vee \neg P_2 \vee \neg P_3) \wedge (P_1 \vee \neg P_2 \vee P_3)$
$\wedge (\neg P_1 \vee P_2 \vee P_3) \wedge (\neg P_1 \vee P_2 \vee \neg P_3) \wedge (\neg P_1 \vee \neg P_2 \vee P_3) \wedge (\neg P_1 \wedge \neg P_2 \wedge \neg P_3)$

This is because $(L_1 \vee L_2 \vee L_3)$ gives value False only when all $L_1 L_2 L_3$ are false. If we do an and over all possible combinations, we can never get a true value for any combination

Problem 4.2 (6 marks) Show that there exists a setting of the truth values of $P_1, P_2, \ldots$ such that at least $\lceil 7m/8 \rceil$ of the disjunctive clauses are T for any 3-SAT expression with m disjunctive clauses of size 3. Here too assume that all the clauses are distinct. (Hint: Use the Probabilistic Method).

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Problem 4.3 (2 marks) Using the result of Problem 4.2 (even if you didn't solve it) argue that any 3-SAT expression that is not satisfiable must have at least 8 clauses.

From problem 4.2, we observe that for each $m \not< 8$ there exists a setting of truth values such that $m = \lceil 7m/8 \rceil$ of the clauses are T for any expression. So, the complete expression becomes true. (satisfiable) When $m_1 = 8$ then $\lceil 7m_1/8 \rceil = 7$ and it is possible that some setting of $P_1, P_2 \cdots$ makes the expression not satisfiable; as only 7 of them must be true (8th one may be false). For $m_1 > 8$ too $\lceil 7m_1/8 \rceil < m_1$. So, any 3-SAT exp. which isn't satisfiable has at least 8 clr.

Problem 5 (5 marks) Given a finite set of alphabets Σ prove that the set Σ^* of all finite strings with alphabets from Σ is countably infinite

Let $\Sigma = \{a_1, a_2 \cdots a_n\}$ be a finite set of alphabets

Define $\phi : \Sigma^* \longrightarrow \mathbb{N}$ as follows:

Let $s \in \Sigma^*$ $\quad s = b_1 b_2 b_3 \cdots b_K$ where $b_j \in \Sigma \;\forall j \in [K]$

and let $P = \{P_1, P_2, \cdots\}$ be the set of prime numbers

$\phi(s) = \prod_{j=1}^{K} P_j^{\,i}$ where i is taken from the index of Σ where b_j maps

(For example $s = a_1 a_2 a_2$ then $\phi(s) = P_1^1 P_2^2 P_3^2$)

Claim ϕ is a bijection

- Total – every $s \in \Sigma^*$ has a corresponding image
- Injective – Let $\phi(s_1) = \phi(s_2)$. As a number has a unique prime factorization, the sequences $b_1 b_2 \cdots b_K$ must be same. So $s_1 = s_2$
- Surjective – Corresponding to every $n \in \mathbb{N}$, we have a prime factorization, and hence, a string $b_1, b_2 \cdots b_K$

So Σ^* bij $\mathbb{N}$ hence Σ^* is countably infinite

Problem 6 (6 marks) Given a tree $T = (V, E)$ prove by induction that for every $v \in V$ there is a walk that begins and ends at v and uses every edge *exactly* twice.

Proof by induction on the number of vertices of of the tree.

$P(n)$: In a tree $T_n = (V_n, E)$, $\forall v \in V_n$ $\exists$ a walk that begins and ends at v and uses every edge exactly twice

Base case: $n = 2$

$v_1 v_2 v_1$ is a walk that begins and ends at v_1 and uses $e = v_1 v_2$ twice

Induction Hypothesis: $P(n)$ is True

Induction Step: Consider a tree T_{n+1} with $|V_{n+1}| = n+1$

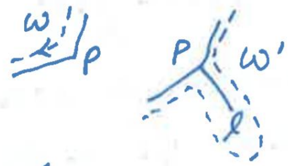
As every non-trivial tree has a leaf, and trees with $\geq$ ~~more than~~ two vertices have at least two leaves (Consider for example, the ends of a longest path), T_{n+1} has at least 2 leaves. Let v be any vertex of T_{n+1}

Case 1: v is not a leaf

Delete a leaf from T_{n+1} to obtain a tree T' with n vertices. By hypothesis, there exists a desired walk in T'. Now, consider the parent p of the deleted leaf. This is visited at some point of our walk

$w = v\, v_1\, v_2 \dots p \dots v$

Now, let us add the leaf back to T'



$w' = v\, v_1\, v_2 \dots p\, l\, p \dots v$

This walk ~~uses~~ extends w and uses pl edge exactly twice. Introduction of a leaf can add at most one edge to a tree which we have used twice. Hence by induction we are done.

Case 2: v is a leaf. Delete some other leaf (As there are ~~greater~~ ≥ 2 leaves) and proceed as in case 1

Hence proved by induction $\square$