1 Q. Find the Picard's nth iterate for the following IVP $y' = x^2+y, \quad y(0)=0.$

Soln: Here $y_0 = 0, \quad y_1 = 0 + \int_0^x (t^2+0)\,dt = \frac{x^3}{3}$

$$y_2 = 0 + \int_0^x \left(t^2 + \frac{t^3}{3}\right) dt = 2\frac{x^3}{3!} + 2\frac{x^4}{4!}$$

$$y_3 = \int_0^x \left(t^2 + 2\cdot\left(\frac{t^3}{3!} + \frac{t^4}{4!}\right)\right) dt = 2\left\{\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right\}$$

Induction hypothesis $y_{n-1} = 2\left\{\frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}\right\}$

$$y_n = \int_0^x t^2 + 2\cdot\left(\frac{t^3}{3!} + \cdots + \frac{t^{n+1}}{(n+1)!}\right) dt$$

$$= 2\left\{\frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^{n+1}}{(n+1)!}\right\}.$$ completed.

$$\boxed{\text{Thus } y(x) = \lim_{n\to\infty} y_n = 2\left\{\sum_{i=3}^{\infty} \frac{x^i}{i!}\right\} = 2\left\{\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1 - x - \frac{x^2}{2!}\right\} = 2\left(e^x - 1 - x - \frac{x^2}{2}\right)}$$

② ~~1.~~ Find existence & uniqueness of the following IVP on a nbhd of $x=1$.

~~$\sin(x+y)\,dy = (x+y)\,dx$~~, ~~$y(1)=-1$~~

Solution: ~~Here~~ $\frac{dy}{dx} = f(x,y) \quad y(1) = -1$

where $f(x,y) = \begin{cases} \frac{x+y}{\sin(x+y)} & \text{if } x+y \neq 0 \\ 1 & \text{if } x+y = 0. \end{cases}$

Solution:
We know, for $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|t| < \delta \implies \left| \frac{t}{\sin t} - 1 \right| < \varepsilon$$

Now for any $a \in \mathbb{R}$,

$$|x-a| < \delta/2 \ \& \ |y+a| < \delta/2 \implies |x+y| = |(x-a)+(y+a)| \leq |x-a| + |y+a| < \delta$$

Thus, $|x-a| < \delta/2$ & $|y+a| < \delta/2 \implies \left| \frac{x+y}{\sin(x+y)} - 1 \right| < \varepsilon.$

Therefore $f(x,y)$ is continuous at $(a,-a)$ $\forall a \in \mathbb{R}$.

Since $x+y$ & $\sin(x+y)$ are continuous & the denominator does not vanish for (a,b) $(b \neq -a)$, $f(x,y)$ is continuous at (a,b) $(b \neq -a)$. The

For $x+y \neq 0$ $\quad f_y = \dfrac{\sin(x+y) - (x+y)\cos(x+y)}{\sin^2(x+y)}$

and for $f_y(a,-a) = \lim_{h \to 0} \dfrac{\left\{ \frac{(a+h)+(-a+h)}{\sin(a+h+(-a)+h)} - 1 \right\}}{h}$

$$= \lim_{h \to 0} \frac{\frac{2h}{\sin 2h} - 1}{h} = 0$$

Then observe, $\lim_{\substack{x \to a \\ y \to -a}} f_y(x,y) = 0$

Since f is continuous on a closed rectangle R around $(1,-1)$ and f_y exists & continuous on R, by existence & uniqueness theorem, the IVP has a unique solution on a nbhd of $x=1$.

3 Q. Consider the following inner product on $\mathbb{R}^4$

$$\langle (x,y,z,w) \mid (x',y',z',w') \rangle = xx' + yy' + x'y + 2yy' + zz' + ww'.$$

Suppose $W = \{ (s,t,s,t) \mid s,t \in \mathbb{R} \}$.

(i) Find an orthogonal basis of W.

(ii) Use the basis of part (i) to find the best approximation of $(1,1,2,2)$ by a vector in W.

Solution: Since $(s,t,s,t) = s(1,0,1,0) + t(0,1,0,1)$

$\{ (1,0,1,0), (0,1,0,1) \}$ is a basis of W.

Now we apply Gram Schmidt process

$v_1 = (1,0,1,0)$.

$$v_2 = (0,1,0,1) - \frac{\langle (0,1,0,1) \mid (1,0,1,0) \rangle}{\|(1,0,1,0)\|^2} (1,0,1,0)$$

$$= (0,1,0,1) - \frac{0+0+1+0+0+0}{1+0+0+0+1+0} (1,0,1,0)$$

$$= (0,1,0,1) - (\tfrac{1}{2},0,\tfrac{1}{2},0) = (-\tfrac{1}{2},1,-\tfrac{1}{2},1).$$

Thus $\{ (1,0,1,0), (-\frac{1}{2},1,-\frac{1}{2},1) \}$ is an orthogonal basis of W.

Therefore, the best approximation of $(1,1,2,2)$ is

$$\frac{\langle (1,1,2,2) \mid (1,0,1,0) \rangle}{\|(1,0,1,0)\|^2} (1,0,1,0) + \frac{\langle (1,1,2,2) \mid (-\frac{1}{2},1,-\frac{1}{2},1) \rangle}{\|(-\frac{1}{2},1,-\frac{1}{2},1)\|^2} (-\tfrac{1}{2},1,-\tfrac{1}{2},1)$$

$$= \frac{1+0+1+0+2+0}{1+0+0+0+1+0} (1,0,1,0) + \frac{-\frac{1}{2}+1-\frac{1}{2}+2-1+2}{\frac{1}{4}-\frac{1}{2}-\frac{1}{2}+2+\frac{1}{4}+1} (-\tfrac{1}{2},1,-\tfrac{1}{2},1)$$

$$= \frac{4}{2} (1,0,1,0) + \frac{3}{5/2} (-\tfrac{1}{2},1,-\tfrac{1}{2},1) = (2,0,2,0) + (-\tfrac{3}{5},\tfrac{6}{5},-\tfrac{3}{5},\tfrac{6}{5})$$

$$= \left(\tfrac{7}{5},\tfrac{6}{5},\tfrac{7}{5},\tfrac{6}{5}\right).$$

4 ~~3~~. Consider the linear operator $T: M_{3\times 3}(\mathbb{R}^3) \to M_{3\times 3}(\mathbb{R})$ defined by $T(A) = A - A^t$.

(a) Find nullity and null space of T

(b) Find all eigenvalues of T from definition

(c) Find dimension of each eigen space

(d) Is T diagonalizable?

Solution: (a) The null ~~space~~ of T is $\{A \in M_3(\mathbb{R}) : A = A^t\}$

$= \{A \in M_3(\mathbb{R}) : a_{ij} = a_{ji} \text{ for } 1 \le i, j \le 3\}$

$= \{A \in M_3(\mathbb{R}) : a_{ij} = a_{ji} \text{ for } 1 \le i < j \le 3\}$

Which is defined by there ~~eqs~~ independent eqns, namely $a_{12} = a_{21}$, $a_{13} = a_{31}$ & $a_{23} = a_{32}$

The there are $a_{11}, a_{22}, a_{33}, a_{21}, a_{31}, a_{32}$ are free unknowns. so that dimension of the null space is 6

(this is the space of 3×3 symmetric matrices over $\mathbb{R}$)

(b) Have to find $\lambda \in \mathbb{R}$ s.t. $A - A^t = \lambda A$ for some $A \neq 0$

$\Leftrightarrow$ ~~$(\lambda = 1)$~~ $A^t = (1-\lambda) A \Leftrightarrow A = (1-\lambda) A^t$ (since $(A^t)^t = A$)

Thus $A = (1-\lambda)(1-\lambda) \cdot A$ (using $A^t = (1-\lambda)A$)

$\Leftrightarrow A = (1 - 2\lambda + \lambda^2) A \Leftrightarrow \lambda(\lambda - 2) A = 0$

Thus $\lambda = 0$ or 2.

(c) For $\lambda = 0$ we have the eigen space of T = null space of T whose dimension is 6.

For $\lambda = 2$, $A - A^t = 2A \Leftrightarrow A^t = -A$ 3×3 skew symmetric matrices.

$\{A \in M_3(\mathbb{R}) : a_{ij} = -a_{ji} \ 1 \le i, j < 3\}$

$= \{A \in M_3(\mathbb{R}) : a_{11} = 0, a_{22} = 0, a_{33} = 0, a_{12} = -a_{21}$
$a_{13} = -a_{31}, a_{23} = -a_{32}\}$

~~Defined by~~ There are only three free unknowns

Thus dimension of the eigen space of T corresponding to 2 is 3.

(d) Now since $6 + 3 = 9 = \dim(M_3(\mathbb{R}))$, the sum of the dimensions of eigen spaces of T is the dimension of the whole space, T is diagonalizable.

5 Q. Suppose $V = P_5(x) = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$

Let $W_1 = \{f(x) \in P_5(x) : x^4 f(\frac{1}{x}) = f(x)\}$

$W_2 = \{f(x) \in P_5(x) : f(-x) = f(x)\}$.

Find $\dim W_1$, $\dim W_2$, $\dim(W_1 + W_2)$.

Solution: ~~$W_1 = \{f(x) \in P_5(x)$~~ Set $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

$$x^4 f(\tfrac{1}{x}) = f(x)$$

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

$\Leftrightarrow$ $a_0 = a_4$, $a_1 = a_3$, Thus $\dim W_1 = 3$ (three free unknowns) namely, a_2, a_3, a_4.

And $f(-x) = f(x)$

$\Leftrightarrow$ $a_0 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 = a_0 + a_1x + a_2x^2 + a_3x^4 + a_4x^4$

$\Leftrightarrow$ $a_1 = 0$, $a_3 = 0$.

Thus $\dim W_2 = 3$ there are three free unknowns namely, a_0, a_2, a_4.

If $f(x) \in W_1 \cap W_2$, $\Leftrightarrow$ $x^4 f(\frac{1}{x}) = f(x)$ & $f(-x) = f(x)$

$\Leftrightarrow$ $a_1 = 0$, $a_3 = 0$, $a_0 = a_4$,

Thus there two free unknowns, namely a_2, a_4. Hence $\dim(W_1 \cap W_2) = 2$.

Finally
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
$$= 3 + 3 - 2$$
$$= 4$$