Span :- Let V be a vector space over a field F, and $S \subseteq V$. (Here S may be an infinite set.) Then

$$\operatorname{Span}(S) := \{a_1 v_1 + \cdots + a_n v_n : n \in \mathbb{N},\ a_i \in F,\ v_i \in S\}$$

$\langle S \rangle$ denote the intersection of all subspaces of V that contains S.

Theorem :- (i) $\operatorname{Span}(S)$ & $\langle S \rangle$ both are subspaces of V (ii) $\operatorname{Span}(S)$ & $\langle S \rangle$ both are smallest subspace of V containing S (iii) $\operatorname{Span}(S) = \langle S \rangle$.

Solution :- Let $u, v \in \operatorname{Span}(S)$.

Let
$$u = a_1 u_1 + \cdots + a_p u_p$$
$$v = b_1 v_1 + \cdots + b_q v_q$$

$$au + v = (aa_1) u_1 + \cdots + (aa_p) u_p + b_1 v_1 + \cdots + b_q v_q$$

Since $aa_1, \ldots, aa_q, b_1, \ldots, b_q \in F$ & $u_1, \ldots, u_p, v_1, \ldots, v_q \in S$

(Some of vectors may be repeated), $au + v \in \text{span}(S)$

Therefore, $\text{span}(S)$ is a subspace.

For any $v \in S$, $v = 1 \cdot v \in \text{span}(S)$

Therefore, $S \subseteq \text{span}(S)$.

Let W be a subspace containing S.

Let $v \in \text{span}(S)$

$\Rightarrow v = a_1 v_1 + \cdots + a_m v_m$, where $a_i \in F$, $v_i \in S$

Since $v_i \in W$, $a_i v_i \in W$

$\Rightarrow a_1 v_1 + \cdots + a_m v_m \in W$

$\Rightarrow v \in W$

$\therefore \text{span}(S) \subseteq W$.

Thus $\text{span}(S)$ is the smallest subspace containing S.

Now we prove that $\langle S \rangle$ is a subspace.

Let $u, v \in \langle S \rangle$

Then u, v belong to all the subspaces of V containing S

Thus $au+v$ belongs to all the subspace of V containing S.

$\Rightarrow au+v \in \langle S \rangle$

Therefore $\langle S \rangle$ is a subspace of V containing S.

* Since $\operatorname{span}(S)$ is also a subspace of V containing S,

$\langle S \rangle \subseteq \operatorname{span}(S)$.

Claim. $\operatorname{span}(S) \subseteq \langle S \rangle$

Let $v \in \operatorname{span}(S)$

Then $v = a_1 v_1 + \cdots + a_n v_n$ where $a_1, \dots, a_1 \in F$
$v_1, \dots, v_n \in S$

$v_1, \cdots, v_n$ belong to any subspace of V containing S.

Then $a_1v_1, \cdots, a_nv_n$ belong to any subspace of V containing S.

Then $a_1v_1 + \cdots + a_nv_n$ belongs to any subspace of V containing S.

$$\Rightarrow v = a_1v_1 + \cdots + a_nv_n \in \langle S \rangle$$

Thus $\text{span}(S) \subseteq \langle S \rangle$.

Therefore, $\quad \text{Span}(S) = \langle S \rangle$.

Alternately, we proved that $\text{span}(S)$ is the smallest subspace of V containing S. Clearly, $\langle S \rangle$ is the smallest subspace of V containing S. Therefore,

$$\text{span}(S) = \langle S \rangle$$

Linearly dependent and linearly independent :- Let V be a vector space over F. Then a set $S \subseteq V$ is said to be linearly dependent if there exist a finite subset $\{v_1, \ldots, v_m\} \subseteq S$ and scalars $c_1, c_2, \ldots, c_m \in F$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0.$$

If S is not linearly dependent, then S is called linearly independent.

In other words, $S \subseteq V$ is linearly independent if for every finite subset $\{v_1, \ldots, v_n\} \subseteq S$

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ is true ONLY for

$$c_1 = c_2 = \cdots = c_n = 0.$$

Examples :-

(i) $V = F[x]$. Then $\{1, x, x^2, \ldots, x^n, \ldots\}$ is linearly independent set.

(ii) $V = F^{\infty}$. Then $\{e_1, e_2, e_3, \cdots\}$ is linearly independent set where $e_i = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$.

↳ i th position

Remark :-

(i) Subset of a linearly independent set is also linearly independent.

(ii) Superset of a linearly dependent set is also linearly dependent.

Basis :- Let V be a vector space over a field F. Then $S \subseteq V$ is said to be a basis of V if (i) S is linearly independent and (ii) $\operatorname{Span}(S) = V$.

Examples :-

(i) $\{1, x, x^2, \dots\}$ is a basis for $F[x]$.

(ii) $\{e_1, e_2, \dots, e_n\}$ is a basis for F^n.

(iii) $\{e_1, e_2, \dots, e_n, \dots\}$ is NOT a basis for F^∞.

Because $\operatorname{span}\{e_1, e_2, \dots, e_n, \dots\} \subsetneqq F^\infty$.

* every element of $\operatorname{span}\{e_1, e_2, \dots, e_n, \dots\}$ has only a finite number of non-zero component. For an example $(1, 1, 1, \dots) \notin \operatorname{span}\{e_1, e_2, \dots, e_n, \dots\}$

Theorem :- Every vector space has a basis.

Proof :- Not required in this course. But I give sketch.

Theorem :- A basis of a vector space V is a maximal linearly independent subset of V, i.e., a superset of a basis is linearly dependent.

Proof :- Let B be basis of V. Let $S \supsetneq B$.

Let $x \in S \setminus B$.

Then $x = c_1 v_1 + \cdots + c_n v_n$ where $v_i \in B$, $c_i \in F$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n + (-1) x = 0$

Thus $\{v_1, \ldots, v_n, x\} \subseteq S$ is linearly dependent.

Therefore, S is linearly dependent.

Theorem:- A maximal linearly independent subset of a vector space is a basis.

Proof:- Let B be a maximal linearly independent set. We claim that $V = \text{span}(B)$.

Let $v \in V$. If $v \in B$ then $v \in \text{span}(B)$.

If $v \in V \setminus B$. Then the set $B \cup \{v\}$, being a superset of B, is linearly dependent.

Therefore, there exists $\{v_1, v_2, \cdots, v_n, v\} \subseteq B \cup \{v\}$ which is linearly dependent.

Let $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n + c v = 0$ where not all $c_1, c_2, \cdots, c$ are zero. If $c = 0$ then $\{v_1, \cdots, v_n\} \subseteq B$ is linearly dependent, which is a contradiction.

Therefore, $c \neq 0$ & $v = -c^{-1} c_1 v_1 + \cdots + -c^{-1} c_n v_n \in \text{span}(B)$

Thus $V = \text{span}(B)$, i.e., B is a basis of V.

Theorem :- If B is a basis of a vector space V then every element in B is a unique linear combination of elements of B.

Proof :- Let $v \in V$ such that $v = a_1 v_1 + \cdots + a_n v_n$ and $v = b_1 v_1 + \cdots + b_n v_n$ where $v_i \neq v_j$, and $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$.

Then $(a_1 - b_1) v_1 + \cdots + (a_m - b_m) v_m = 0$ where $v_i \in B$ and not all $a_1 - b_1, \ldots, a_m - b_m$ are zero.

Therefore, $\{v_1, \ldots, v_m\}$ is linearly dependent, and hence B is linearly dependent. This is a contradiction.

Dimension :- Let V be a vector space over a field F. Let B be a basis of the vector space.

If B has an infinite number of element then we say V is infinite dimensional.

If B has a finite number of elements and $|B|=n$ then we say V is finite dimensional and V has dimension n, i.e., $\dim(V) = n$.

Example :-

(i) $V_F = F[x]$ is infinite dimensional. (over F)

(ii) $V_{\mathbb{Q}} = \mathbb{R}$ is infinite dimensional (over $\mathbb{Q}$)

(iii) $V_F = F^{\infty}$ is infinite dimensional. (over F)

(iv) $\operatorname{Dim}(V_{\mathbb{R}}) = n$, $V = \mathbb{R}^n$

(v) $\operatorname{Dim}(V_{\mathbb{R}}) = 2n$, $V = \mathbb{C}^n$ (v) $\operatorname{Dim}(V_{\mathbb{C}}) = n$, $V = \mathbb{C}^n$