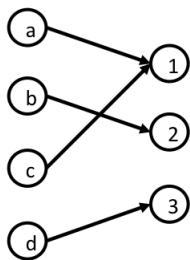
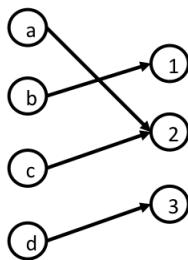
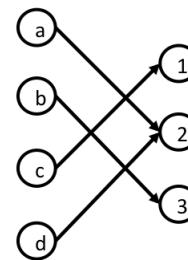
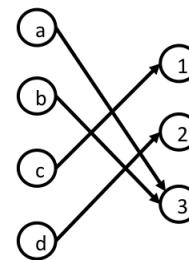


Name: _____

ID number: _____

There are 2 questions for a total of 10 points.

1. Consider the hash function family $H = \{h_1, h_2, h_3, h_4\}$ where h_i 's are defined below and answer the questions that follow.

 h_1  h_2  h_3  h_4

- (a) (1 point) State true or false: H is 2-universal.

(a) _____ **False** _____

- (b) (2 points) Give reasons for your answer to part (a).

Solution: We have :

1. $\Pr_{h \leftarrow H}[h(a) = h(b)] = 1/4$
2. $\Pr_{h \leftarrow H}[h(a) = h(c)] = 1/2$
3. $\Pr_{h \leftarrow H}[h(a) = h(d)] = 1/4$
4. $\Pr_{h \leftarrow H}[h(b) = h(c)] = 0$
5. $\Pr_{h \leftarrow H}[h(b) = h(d)] = 0$
6. $\Pr_{h \leftarrow H}[h(c) = h(d)] = 0$

Since $\Pr_{h \leftarrow H}[h(a) = h(c)] = 1/2 > 1/3$, H is not a 2-universal hash function family.

2. (7 points) Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.

Solution: Let $G(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots$. Then we have:

$$\begin{aligned} G(x) &= a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots \\ x \cdot G(x) &= a_0x^1 + a_1x^2 + a_2x^3 + \dots \\ 2x^2 \cdot G(x) &= 2a_0x^2 + 2a_1x^3 + 2a_2x^4 + \dots \end{aligned}$$

This gives:

$$\begin{aligned} (1 - x - 2x^2) \cdot G(x) &= a_0 + (a_1 - a_0)x + \sum_{k=2}^{\infty} (a_k - a_{k-1} - 2a_{k-2})x^k \\ &= 4 + 8x + \sum_{k=2}^{\infty} 2^k x^k \quad (\text{since } a_k = a_{k-1} + 2a_{k-2} + 2^k) \\ &= 4 + 8x + 4x^2 \cdot (1 + (2x) + (2x)^2 + \dots) \\ &= 4 + 8x + \frac{4x^2}{1 - 2x} \end{aligned}$$

This implies that:

$$G(x) = \frac{4 - 12x^2}{(1+x)(1-2x)^2} = \frac{a}{1+x} + \frac{bx+c}{(1-2x)^2}.$$

We solve for constants a, b, c . We get the following equations:

$$\begin{aligned} 4a + b &= -12 \\ -4a + b + c &= 0 \\ a + c &= 4 \end{aligned}$$

which gives $a = -\frac{8}{9}$, $b = -\frac{76}{9}$, and $c = \frac{44}{9}$. Collecting the coefficients of x^k , we get that:

$$\begin{aligned} a_k &= -\frac{8}{9}(-1)^k + \frac{44}{9}(k+1)2^k + \left(-\frac{76}{9}\right)k2^{k-1} \\ &= -\frac{8}{9}(-1)^k + \frac{2}{3}k2^k + \frac{44}{9}2^k \end{aligned}$$