Basis of a vector Space — lecture-8

Recall :- let $S = \{u_1, u_2, \dots u_n\} \subseteq V(\mathbb{F})$ be a finite set. Then is L.I $\iff$ $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$

$\Rightarrow$

$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Basis of a vector space :- A subset $\mathcal{B} \subseteq V(\mathbb{F})$ is said to be a basis of $V(\mathbb{F})$ if $\mathcal{B}$ is a linearly independent set and $L(\mathcal{B}) = V(\mathbb{F})$.

\# For Every element $u \in V(\mathbb{F})$, there exists $u_1, \dots u_m \in \mathcal{B}$ and scalars $\alpha_1, \dots \alpha_m \in \mathbb{F}$ such that $u = \alpha_1 u_1 + \dots + \alpha_m u_m$.

Example : $V = \mathbb{R}^n(\mathbb{R})$.

If we consider $B = \{ e_i \mid 1 \le i \le n \}$, where $e_i \in \mathbb{R}^n$

such that $e_i = (0, 0, \cdots, 1, 0 \cdots 0)$
$\uparrow$ i^{th} position

Suppose we have

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0 \qquad \alpha_i \in \mathbb{F} \ \forall i$$

$$\Rightarrow \alpha_1(1, 0, \cdots 0) + \alpha_2(0, 1, 0 \cdots 0) + \cdots + \alpha_n(0, \cdots, 1) = 0$$

$$\Rightarrow (\alpha_1, \alpha_2, \cdots, \alpha_n) = 0 \text{ in } \mathbb{R}^n$$

$$\Rightarrow \alpha_i = 0 \quad \forall \ 1 \le i \le n$$

Further, if we take $u = (x_1, x_2, \cdots x_n) \in \mathbb{R}^n$

$u = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in L(\mathcal{B})$

$\Rightarrow$ $L(\mathcal{B}) = V(\mathbb{F})$. Hence $\mathcal{B}$ is a basis of V.

<u>Exercise</u> :→ Take $V = \mathbb{C}^n(\mathbb{C})$. Prove that $\mathcal{B} = \{e_i \mid 1 \le i \le n\}$ is a basis set for V.

<u>Exercise</u> :→ Take $V = \mathbb{C}^n(\mathbb{R})$. Is $\mathcal{B} = \{e_i \mid 1 \le i \le n\}$ a basis of $\mathbb{C}^n(\mathbb{R})$? If not, find a basis set for $\mathbb{C}^n(\mathbb{R})$ consisting $\mathcal{B}$.

Hint :→ take $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$

write $z_i = x_i + i y_i \qquad x_i, y_i \in \mathbb{R}$.

$$(z_1, z_2, \dots, z_n) = (x_1 + i y_1, x_2 + i y_2, \dots, x_n + i y_n)$$

$$= (x_1, x_2, \dots, x_n) + i(y_1, y_2, \dots, y_n)$$

$$= x_1 e_1 + x_2 e_2 + \dots + x_n e_n + i(y_1 \dots y_n)$$

Add

$\bar{e}_1 = (i, \dots 0), (0, i, 0 \dots 0) \dots \bar{e}_n = (0, 0 \dots, i)$

$$(z_1, \dots z_n) = x_1 e_1 + \dots + x_n e_n + y_1 \bar{e}_1 + \dots + y_n \bar{e}_n$$

$\bar{B} = \{ e_1 \dots e_n, \bar{e}_1, \dots \bar{e}_n \}$ is a spanning set.

Example ③ $V = P_n(\mathbb{R}) = \{ f(x) \in \mathbb{R}[x] \mid \deg(f(x)) \leq n \}$

If we take $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$. Then it is easy to observe that $L(\mathcal{B}) = P_n(\mathbb{R})$

Consider $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ s.t

$$\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0 \text{ in } P_n(\mathbb{R})$$

$$\Downarrow$$

$$\alpha_i = 0 \quad \forall\, i \geq 0$$

Remark ↳ We can have more than one basis set.

$\mathcal{B}_1 = \{(1,0), (0,1)\}$ & $\mathcal{B}_2 = \{(1,0), (2,3)\}$. Both are basis of $\mathbb{R}^2$.

Theorem $\Rightarrow$ B is a basis of $V(F) \Leftrightarrow B$ is a maximal linearly independent subset in V.

Proof $\Rightarrow$ $\Rightarrow$ Let B be a basis of $V(F)$ and let $0 \neq u \in V - B$. Since B is a basis, this implies that $\exists$ $u_1, u_2, \ldots, u_n \in B$ and scalar $\alpha_1, \alpha_2, \ldots \alpha_n \in F$

s.t $$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

$$1 \cdot u - \alpha_1 u_1 - \alpha_2 u_2 - \cdots - \alpha_n u_n = 0$$

$\Rightarrow$ $B \cup \{u\}$ is a L.D. set.

$\Rightarrow$ B is a maximal linearly independent set.

$\Leftarrow$ Suppose $\mathcal{B}$ is a maximal linearly independent set.
Then $\mathcal{B}$ is L.I. and we need show that $L(\mathcal{B}) = V$.

Let $u \in V \setminus \mathcal{B}$. Then $\mathcal{B} \cup \{u\}$ is a L.D. set.

$\Rightarrow \exists$ $u_1, u_2, \ldots, u_m \in \mathcal{B}$ and scalars $\alpha_0, \alpha_1, \ldots \alpha_m \in \mathbb{F}$ such that

$$\alpha_0 u + \alpha_1 u_1 + \cdots + \alpha_m u_m = 0 \quad (\text{and } \alpha_0 \neq 0)$$

$$u = \frac{-1}{\alpha_0} \left(\alpha_1 u_1 + \cdots + \alpha_m u_m \right] \in L(\mathcal{B})$$

$\Rightarrow$ $L(\mathcal{B}) = V$. Hence $\mathcal{B}$ is a basis of $V(\mathbb{F})$.

Theorem $\Rightarrow$ If $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ is a basis of $V(\mathbb{F})$. Then every $u \in V$ is a unique linear combination of vectors in $\mathcal{B}$.

Proof $\Rightarrow$ Suppose $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ $\quad \alpha_i \in \mathbb{F}$

Also $u = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$ $\quad \beta_i \in \mathbb{F}$.

$$\alpha_1 u_1 + \cdots + \alpha_n u_n = \beta_1 u_1 + \cdots + \beta_n u_n$$

$$(\alpha_1 - \beta_1) u_1 + \cdots + (\alpha_n - \beta_n) u_n = 0$$

$$\Rightarrow \alpha_i - \beta_i = 0 \quad \forall\ 1 \le i \le n \quad (\text{as } \mathcal{B} \text{ is a basis})$$

$$\Rightarrow \alpha_i = \beta_i \quad \forall\ 1 \le i \le n$$

Theorem $\Rightarrow$ Every vector space has a basis.

Proof $\Rightarrow$ We will consider this statement without proof.

Lemma $\Rightarrow$ Let $V(\mathbb{F})$ be a vector space. Then any finite L.I set in V is a part of a basis of $V(\mathbb{F})$.

Proof $\Rightarrow$ Let $\mathcal{B}$ be a basis of $V(\mathbb{F})$ and let S be a linearly independent subset given by

$$S = \{v_1, v_2, \dots, v_k\}.$$

We know that

$$v_1 = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \qquad \alpha_i \in \mathbb{F}, \; u_i \in \mathcal{B}.$$

(at least one α_i will be nonzero w.l.o.g $\alpha_1 \neq 0$)

$$\frac{1}{\alpha_1} v_1 = u_1 + \frac{1}{\alpha_1}(\alpha_2 u_2 + \cdots + \alpha_n u_n)$$

$$\Rightarrow \quad u_1 = \frac{1}{\alpha_1}(v_1 - \alpha_2 u_2 + \cdots + \alpha_n u_n)$$

If we take $\mathcal{B}_1 = \mathcal{B} \setminus \{u_1\} \cup \{v_1\}$

It is easy to observe that

Exercise

1. $L(\mathcal{B}_1) = L(\mathcal{B})$
2. $\mathcal{B}_1$ is L.I.

In second step write

$$v_2 = \alpha_1 u_1' + \cdots + \alpha_n u_n' + \alpha_0 v_1$$

Observe that all α_i $1 \le i \le n$ are not identically zero

Because of so, then $v_2 = \alpha_0 v_1$

$$\Rightarrow v_2 - \alpha_0 v_1 = 0 \quad (\text{But } v_1 \, v_2 \text{ are L.I})$$

Hence $\exists$ i b/w 1 to n such that $\alpha_i \neq 0$, w.l.o.g

$\alpha_1 \neq 0$

$$\frac{1}{\alpha_1} v_2 = u_1' + \frac{1}{\alpha_1} \left(- \quad \cdots \right)$$

Replace u_1' by v_2. Keep doing it upto kth step.

Finally will be the part of a basis set.

Theorem $\Rightarrow$ Let $V(\mathbb{F})$ be a vector space with finite basis set $\mathcal{B}$, having n elements. Then any subset $S \subseteq V(\mathbb{F})$ having more than n elements is linearly dependent.

Proof $\Rightarrow$ Let $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ be a basis and $S \subseteq V(\mathbb{F})$ with $S = \{v_1, v_2, \ldots, v_m\}$, where $m > n$.

$$v_1 = \alpha_{11} u_1 + \alpha_{21} u_2 + \cdots + \alpha_{n1} u_n$$

$$\vdots$$

$$v_m = \alpha_{1m} u_1 + \alpha_{2m} u_2 + \cdots + \alpha_{nm} u_n$$

We need to prove that $\exists\ \beta_1, \beta_2, \ldots, \beta_m \in \mathbb{F}$ such that

$$\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_m v_m = 0 \quad \begin{pmatrix}\text{at least one} \\ \beta_i \text{ is nonzero}\end{pmatrix}$$

$$\Downarrow$$

$$\beta_1\left(\sum_{j=1}^{n} \alpha_{j1} u_j\right) + \cdots + \beta_m\left(\sum_{j=1}^{n} \alpha_{jm} u_j\right) = 0$$

$$\left(\sum_{i=1}^{m} \alpha_{1i} \beta_i\right) u_1 + \cdots + \left(\sum_{i=1}^{m} \alpha_{ni} \beta_i\right) u_n = 0$$

$$\Downarrow$$

$$\sum_{i=1}^{m} \alpha_{1i} \beta_i = \cdots = \sum_{i=1}^{m} \alpha_{ni} \beta_i = 0 \quad \left(\because \mathcal{B} \text{ is a basis}\right)$$

$\Rightarrow$ $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{F}^n$ is a sol^n of the hom system of eq^n $AX = 0$ where $A = (\alpha_{ij})_{n \times m}$

(n eq^n and m unknows with $m > n$)

Hence this system must have a non trivial sol^n

$\Rightarrow \exists$ $(\beta_1 \ldots, \beta_m) \in \mathbb{F}^n - (0, \ldots 0)$ s.t

$$\beta_1 v_1 + \ldots + \beta_m v_m = 0$$

$\Rightarrow$ S is L.D.

Corollary :- Let $V(F)$ be a finite dim vector space and let $\mathcal{B}_1 = \{u_1, \dots, u_n\}$ and $\mathcal{B}_2 = \{v_1, \dots v_m\}$ be two basis of V. Then

$$n = |\mathcal{B}_1| = |\mathcal{B}_2| = m.$$

Proof :- If $\mathcal{B}_1$ is a basis, then we have $m \leq n = |\mathcal{B}_1|$ — eqn (1)

If $\mathcal{B}_2$ is a basis, then

$n \leq m = |\mathcal{B}_2|$ — eqn (2)

$\Rightarrow n = m.$

Dimension of a vector space $\Rightarrow$ let $V(\mathbb{F})$ be a f.d vector space. Then the dim of $V(\mathbb{F})$, denoted by $\dim_{\mathbb{F}}(V)$, is the number of elements in a basis.

If V has no finite basis, we say it is infinite dimensional.