Quiz 3

● Graded

Student

Har Ashish Arora

Total Points

12 / 13 pts

Question 1

Variation of parameters

5 / 6 pts

+ 6 pts Correct

+ 0 pts Incorrect/not done

Taking $y_2(x) = v(x)e^x$ and getting $xv'' + v' = 0$

✔ **+ 1.5 pts** Correct

+ 1 pt Partly correct in differentiation

✔ **+ 1 pt** Setting $v'(x) = u(x)$ and getting $u(x) = 1/x$

✔ **+ 1 pt** Wronskian = e^{2x}/x

$u(x) = x(1 - ln(x))$ as first function with $y_1(x) = e^x$ in nonhomogeneous part

+ 1 pt correct

✔ **+ 0.5 pts** Partly correct ($r(x)$ is not correctly taken in integration or $r(x)$ correct but there is an integration issue)

Getting $v(x) = x$ as second function with $y_2(x)$ in nonhomogeneous part

+ 1 pt correct

✔ **+ 0.5 pts** Partly correct ($r(x)$ is not correctly taken in integration or $r(x)$ correct but there is an integration issue)

✔ **+ 0.5 pts** General solution $y(x) = y_c(x) + y_p(x)$

+ 4 pts The approach of y_2 is correct; a minor mistake in sign or integration gives the wrong $y_2(x)$. But thereafter, the approach is correct, and all the formulas, including $r(x) = e^x/x$, are correct. Only one early mistake in $y_2(x)$ causes the wrong answer.

+ 3 pts $y_2(x)$ is wrong and major mistake. Afterward, the Wronskian approach and formula of variation of parameter are correct. But $r(x)$ applied is wrong, but integration is done, and a general solution is computed, which results in the wrong answer.

+ 2 pts The idea for computing $y_2(x)$ is wrong. The concept of Wronskian, the formula for variation of parameters, and the concept of general solution are correct, but all calculations are wrong. Marks for attempting with some correct ideas.

Question 2

2nd Order Equation

4 / 4 pts

- ✔ **+ 4 pts** Correct
- **+ 0 pts** incorrect/ not attempting
- **+ 0.5 pts** writing characteristic equation/ auxiliary equation
- **+ 0.5 pts** writing correct solution for alpha >=0
- **+ 1 pt** writing the general solution y(x) for alpha <-1 and showing that y(x) tends to zero as x tends to infinity
- **+ 1 pt** writing the general solution y(x) for alpha =-1 and showing that y(x) tends to zero as x tends to infinity
- **+ 1 pt** writing the general solution y(x) for -1<alpha <0 and showing that y(x) tends to zero as x tends to infinity
- **+ 0.5 pts** saying few specified value of alpha without justification

Question 3

System

3 / 3 pts

- ✔ **+ 3 pts** Correct answer
- **+ 0 pts** Incorrect / didnot atempt
- **+ 1 pt** Correct eigen values.
- **+ 1 pt** correct eigen vectors
- **+ 1 pt** correct general solution
- **+ 0.5 pts** one correct eigen value
- **+ 0.5 pts** One correct eigen vector
- **+ 0.5 pts** Correct form of general solution eigen values or eigen vectors are wrong
- **+ 1.5 pts** If y_1 or y_2 is correct.

DEPARTMENT OF MATHEMATICS, IIT DELHI,
SEMESTER II 2024 – 25
MTL 101 (Linear Algebra and Differential Equations) - Quiz 3

Date: 23/04/2025 (Wednesday) Time: 7:15 PM - 8:00 PM.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

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Entry Number:	2024EE10904	Group:	13
Gradescope Id:	EE1240904	Lecture Hall:	108

Question 1: Find the general solution of the differential equation using the variation of parameters:

$$xy'' - (2x-1)y' + (x-1)y = e^x.$$

Given that $y_1 = e^x$ is one of the fundamental solution of the associated homogeneous equation.

[6]

First, we will find the other linearly independent fundamental solution by method of reduction of order.

let $y_2(x) = v(x)e^x$ $= v(x)y_1(x)$ (for the homogenous part) be the other solution. Then,

$$y_2'(x) = v'e^x + ve^x \Rightarrow y_2'' = (v''e^x + v'e^x) + (v'e^x + ve^x)$$

$$= v''e^x + 2v'e^x + ve^x$$

~~Put these into the differential equation (y_2)~~

$$\cancel{x(v''e^x + 2v'e^x + ve^x) - (2x-1)(v'e^x + ve^x) + (x-1)ve^x = e^x}$$

Convert the eq^n to the standard form (the homogenous one)

$$y'' - \left(\frac{2x-1}{x}\right)y' + \left(\frac{x-1}{x}\right)y = 0$$

(in $y'' + p(x)y' + q(x)y = 0$)

Here, $p(x) = -\frac{2x-1}{x}$

The formula for the second sol^n of the basis set is

$y_2(x) = v(x)\,y_1(x)$, where $v(x) = \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx$

So, $$V(x) = \int \frac{e^{+\int \frac{2x-1}{x}\,dx}}{(e^x)^2}\,dx$$

$$= \int \frac{e^{\int 2dx - \int \frac{1}{x}dx}}{e^{2x}}\,dx = \int \frac{e^{2x - \ln x}}{e^{2x}}\,dx = \int \frac{dx}{e^{\ln x}} = \int \frac{dx}{x}$$

$$= \ln(x)$$

$$\Rightarrow \boxed{y_2(x) = \ln(x)\, e^x}$$

Now, we have to use variation of parameters.

The sol^n of homogenous part is

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$
$$= c_1 e^x + C_2 \ln(x)\, e^x$$

let $C_1 = v_1(x)$, $C_2 = v_2(x)$, for the particular sol^n. $y_p(x) = v_1(x) e^x + v_2(x) \cdot e^x \cdot \ln x$

$$\cancel{\Rightarrow y_p(x) = v_1(x)\, e^x + v_2(x) \ln x\, e^x}$$

$$\cancel{y_p'(x) = v_1'(x) e^x + v_1(x) e^x + v_2'(x)(\ln x \cdot e^x) + v_2(x)(\ln x \cdot e^x)'}$$

~~Demand~~ $= 0$.

~~Putting it in the~~

We have to solve the following system in variation of parameters:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \end{bmatrix}$$

Here, $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^x & (\ln x)(e^x) \\ e^x & \frac{e^x}{x} + \ln x \cdot e^x \end{bmatrix}$

So, by cramer's rule,

$$v_1' = \frac{\begin{vmatrix} 0 & \ln x \cdot e^x \\ e^x & \frac{e^x}{x} + \ln x \cdot e^x \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-\ln x \cdot e^{2x}}{W(y_1, y_2)}$$

(Note: $W(y_1, y_2) = \begin{vmatrix} e^x & \ln x \cdot e^x \\ e^x & \frac{e^x}{x} + \ln x \cdot e^x \end{vmatrix} = \frac{e^{2x}}{x} + \cancel{\ln x \cdot e^{2x}} - \cancel{\ln x e^{2x}}$
$= \frac{e^{2x}}{x}$)

$$\Rightarrow v_1' = \frac{-\ln x \cdot e^{2x}}{\frac{e^{2x}}{x}} = \boxed{-x \ln x = v_1'}$$

(Also note the y_1 & y_2 we found were independent $\because W \neq 0$).

$$\text{Slly, } v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x \end{vmatrix}}{W(y_1, y_2)} = \frac{e^{2x}}{\frac{e^{2x}}{x}} = \boxed{x = v_2'}$$

$$\Rightarrow \boxed{v_2 = \frac{x^2}{2}}, \quad v_1 = \int -x \ln x \, dx$$

Use by-parts: $v_1 = -\left(x(x\ln x - x) - \int (x \ln x - x) dx \right)$

$$v_1 = -\left(x^2 \ln x - x^2 - \int x \ln x \, dx + \frac{x^2}{2} \right)$$

$$\Rightarrow 2v_1 = -x^2 \ln x + x^2 - \frac{x^2}{2} = -x^2 \ln x + \frac{x^2}{2}$$

$$\Rightarrow v_1 = \frac{x^2}{4} - \frac{x^2}{2} \ln x.$$

Thus, the final sol^n (general) calculated using the Variation of parameters method is:

$$\boxed{y(x) = y_h(x) + y_p(x) = \left(c_1 e^x + c_2 \ln x \cdot e^x\right) + \left[\left(\frac{x^2}{4} - \frac{x^2}{2}\ln x\right) e^x + \left(\frac{x^2}{2}\right)(\ln x \cdot e^x)\right]}$$

Question 2: Find the values of $\alpha \in \mathbb{R}$, for which all the solutions of the ODE

$$y''(x) + 2y'(x) - \alpha y(x) = 0$$

go to zero as $x \to +\infty$. [4]

This is a homogenous equation. Let $y = e^{mx}$ be a sol^n of this eq^n. Then,

$y' = me^{mx}$, $y'' = m^2 e^{mx}$ $\Rightarrow$ in the homogenous eq^n:

$m^2 e^{mx} + 2me^{mx} - \alpha . e^{mx} = 0$ $\because e^{mx} \neq 0 \ \forall x \in \mathbb{R}$,

$\boxed{m^2 + 2m - \alpha = 0} \longrightarrow$ Auxiliary eq^n!

Here, let us solve for the values of m:

$$m = \frac{-2 \pm \sqrt{2^2 - 4(1)(-\alpha)}}{2} = \frac{-2 \pm \sqrt{4 + 4\alpha}}{2}$$

$$\boxed{m = -1 \pm \sqrt{1+\alpha}}$$

Then, the general sol^n for this homogenous equation is:

$$\boxed{y(x) = c_1 e^{(-1+\sqrt{1+\alpha})x} + c_2 e^{(-1-\sqrt{1+\alpha})x}}.$$

$$\cancel{\Rightarrow y(x) = e^{-x}\left(c_1 e^{\sqrt{1+\alpha}\,x} + c_2 e^{-\sqrt{1+\alpha}\,x}\right)}$$

We take cases now.

Case-1: $\alpha > 0$. Then, in the boxed equation,

We get some $c_1 e^{(\text{some +ve value}).x} + c_2 e^{(\text{negative value}).x}$

This $\to \infty$ as $x \to \infty$. So not allowed.

Case-2: $\alpha = 0$. Then,

$y(x) = c_1 + c_2 e^{-2x}$ $\Rightarrow$ depends on value of c_2.

Again discarded, $\because c_2$ is arbitrary.

Case-3 : $\alpha \in (-1, 0)$.

Then, $y(x) = c_1 e^{(\text{some negative value})(x)} + c_2 e^{(\text{some other-ve value})(x)}$.

This goes to 0 $\forall$ $\alpha \in [-1, 0)$ as $x \to \infty$.

Thus, $\alpha \in (-1, 0)$ is an allowed interval.

Case-4 : $\alpha < -1$: We get complex roots of the form
$m = -1 \pm i\beta$, where $\beta = \sqrt{-\alpha - 1}$

Then, y has the type

$$y(x) = e^{-x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right).$$

As $x \to \infty$, $y(x)$ clearly goes to zero :
$e^{-x} \to 0$, $c_1 \cos\beta x + c_2 \sin \beta x$ takes some finite value

$\Rightarrow$

The interval in $\mathbb{R}$ for which the solutions of the given ODE go to zero is

$$\boxed{\alpha < 0} \longrightarrow \text{(Ans)}$$

Special note $\rightarrow$ (Case :5) In case $\alpha = -1$, then we would have repeated roots of m, $-1, -1$. Then, eq^n would be

$y(x) = c_1 e^{-x} + c_2 x e^{-x}$. This still goes to zero $\because$ e^{-x} goes to zero faster than $x \to \infty$. Thus, even $\alpha = -1$ is included here. So, $\alpha < 0$ remains valid.

Question 3: Find the general solution of the following system of differential equations:

$$\begin{cases} \frac{dy_1}{dt} = -3y_1 - 4y_2 \\ \frac{dy_2}{dt} = 5y_1 + 6y_2. \end{cases}$$

[3]

We are given the homogenous system:

$$y_1' = -3y_1 - 4y_2$$

$$y_2' = 5y_1 + 6y_2$$

$\Rightarrow$ Write it in the matrix form:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let $A := \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$. Find the eigenvalues of A:

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -3-\lambda & -4 \\ 5 & 6-\lambda \end{vmatrix} = 0 \Rightarrow (3+\lambda)(\lambda - 6) + 20 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 3\lambda - 18 + 20 = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2) = 0$. We have 2 distinct eigenvalues.

Let us find the eigenvectors X_1 & X_2 corresponding to eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ respectively. Then;

$AX_1 = \lambda_1 X_1 \Rightarrow (A - \lambda_1 I)X_1 = 0$ We need to solve this system of equations. Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

6

$\Rightarrow \begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ Pivots are encircled

Reduce to RRE: $\begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix} \xrightarrow[R_2 \to \frac{R_2}{5}]{R_1 \to \frac{-R_1}{4}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Thus, $x_1 + y_1 = 0$, $y_1 = s$, say, then $x_1 = -s$

giving one of the eigenvectors $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Similarly, for $X_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$,

$$\begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Reduce ~~to RRE~~: $\begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 \to -R_1} \begin{bmatrix} 5 & 4 \\ 0 & 0 \end{bmatrix} \Rightarrow 5x_2 + 4y_2 = 0$. Here,

y_2 is free variable, let it be $5s$. Then $x_2 = \frac{-4y_2}{5}$

$$= \frac{-4(\cancel{5}s)}{\cancel{5}} = -4s$$

$\Rightarrow$ an eigenvector $X_2 = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$.

Thus, the general sol^n $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is:

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 X_1 e^{\lambda_1 t} + c_2 X_2 e^{\lambda_2 t}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ +1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} e^{2t} \longrightarrow \text{Ans.}$$