Q 1 $\quad \psi_\lambda : M_3(\mathbb{R}) \longrightarrow M_3(\mathbb{R})$

$$\psi_\lambda(A) = A + \lambda A^t$$

(a) $\psi_\lambda(aA+bB) = (aA+bB) + \lambda(aA+bB)^t$ for $a, b \in \mathbb{R}$.

$= aA + bB + \lambda aA^t + \lambda b B^t$

$= a(A+\lambda A^t) + b(B+\lambda B^t)$

$= a\psi_\lambda(A) + b\,\psi_\lambda(B)$

Hence ψ_λ is a linear tranformation.

(b) $\ker \psi_\lambda = \{ A \in M_3(\mathbb{R}) : A + \lambda A^t = 0 \}$

If $\lambda = 1$, $A \in \ker\psi_\lambda \Leftrightarrow A = -A^t$ a skew symmetric matrix

If $\lambda = -1$, $A \in \ker\psi_\lambda \Leftrightarrow A = A^t$ a symmetric matric.

Thus when $\lambda = \pm 1$, $\ker \psi_\lambda \neq 0$.

Hence $\ker\psi_\lambda = 0$ (i.e. ψ_λ is one to one) $\Rightarrow \lambda \neq \pm 1$.

Suppose $\lambda \neq \pm 1$, set $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then $A^t = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

If $A \in \ker\psi$, then $\begin{pmatrix} a_1+\lambda a_1 & b_1+\lambda a_2 & c_1+\lambda a_3 \\ a_2+\lambda b_1 & b_2+\lambda b_2 & c_2+\lambda b_3 \\ a_3+\lambda c_1 & b_3+\lambda c_2 & c_3+\lambda c_3 \end{pmatrix} = 0$

Hence $a_1 = b_2 = c_3 = 0$ as $1+\lambda \neq 0$ & $b_1 = -\lambda a_2$, $c_1 = -\lambda a_3$

$a_2 = -\lambda b_1$, $c_2 = -\lambda b_3$, $a_3 = -\lambda c_1$, $b_3 = -\lambda c_2$.

So $a_1 = b_2 = c_3 = 0$, $b_1 = \lambda^2 b_1$, $c_1 = \lambda^2 c_1$, $c_2 = \lambda^2 b_3$ etc.

Hence $a_1 = b_2 = c_3 = 0$ $b_1 = c_1 = c_2 = \cdots = 0$ all zero.

OR $\quad A + \lambda A^t = 0 \Rightarrow A^t + \lambda A = 0$ so $A^t = -\lambda A$. so $A + \lambda(-\lambda)A) = 0 \Rightarrow (\lambda^2-1)A = 0$

$\Rightarrow A = 0$

(c) $\mathrm{Null}(\psi_{-1}) = \{A : A - A^t = 0\}$ consists of symmetric matrices

Range $\psi_1 = \{A + A^t : A \in M_3(\mathbb{R})\}$ it is a subset of symmetric matrices in $M_3(\mathbb{R})$

But if B is a symmetric matrix $B = \frac{B+B^t}{2}$

$= \frac{1}{2}B + (\frac{1}{2}B)^t$

Thus Range ψ_1 consists of symmetric matrices.

Hence Range $\psi_1 = \mathrm{Null}(\psi_{-1})$

Q. 2

Characteristic polynomial of $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & s & t \end{pmatrix}$ is

$$\det \begin{pmatrix} x & -1 & -2 \\ -1 & x-1 & 0 \\ -1 & -s & x-t \end{pmatrix} = x(x-1)(x-t) + \{-(x-t)\} - 2\{s + (x-1)\}$$
$$= x\{x^2 - (1+t)x + t\} - x + t - 2s + 2 - 2x$$
$$= x^3 - (1+t)x^2 + (t-3)x + t - 2s + 2.$$

which is given to be $x^3 - 4x - 1$.

Comparing coefficients
$$1 + t = 0 \Rightarrow t = -1$$
$$t - 3 = -4 \Rightarrow t = -1 \text{ (same)}$$
$$t - 2s + 2 = -1 \Rightarrow -1 - 2s = -3$$
$$\Rightarrow 2s = +2$$
$$\Rightarrow \underline{s = +1}$$
$$(t, s) = (-1, 1)$$

भारतीय प्रौद्योगिकी संस्थान दिल्ली
INDIAN INSTITUTE OF TECHNOLOGY DELHI

सभी पृष्ठों पर लिखें। Write on all pages.

अनुक्रमांक Entry No. | अनुवर्ती पुस्तिका संख्या CONTINUATION BOOK No.

पाठ सं. Course No. | ग्रुप संख्या Group No. | दिनांक Date

3. (a) $\mathcal{L}(f*g) = \mathcal{L}(f)\ \mathcal{L}(g)$

$$f*g(t) = \int_0^t f(\tau)\ g(t-\tau)\,d\tau$$

$$\mathcal{L}(f*g)(s) = \int_{t=0}^{\infty}\int_{\tau=0}^{t} f(\tau)\ g(t-\tau)\,d\tau\ e^{-st}\,dt$$

$$= \int_{\tau=0}^{\infty}\int_{t=\tau}^{\infty} f(\tau)\ g(t-\tau)\ e^{-st}\,dt\,d\tau$$

$$= \int_0^{\infty} f(\tau)\ e^{-s\tau}\ \underset{\mathcal{L}(g)(s)}{G(s)}\ d\tau$$

$$= \cancel{F(s)\ G(s)}\ \mathcal{L}(f)(s)\ \mathcal{L}(g)(s).$$

(b) $y(t) = 1-\cosh t - \int_0^t e^{\tau}\,y(t-\tau)\,d\tau$

Let $g(t) = e^t$. Then $y(t) = 1-\cosh t - y*g(t)$

Taking Laplace transform, $Y(s) = \frac{1}{s} - \frac{s}{s^2-1} - \frac{Y(s)}{s-1}$

$$\left[1+\frac{1}{s-1}\right]Y(s) = \frac{1}{s} - \frac{s}{s^2-1}$$

$$\frac{s}{s-1}\ Y(s) = \frac{s-1}{s^2} - \frac{1}{s+1}$$

$$Y(s) = \frac{1}{s} - \frac{1}{s^2} - \left[\frac{1}{s+1}\right].\quad \boxed{y(t) = 1 - t - e^{-t}}$$

4. (a) $$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \quad F'(s) = \int_0^\infty \left(-t e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty e^{-st}\{-t f(t)\}\,dt$$

$$= \mathcal{L}\{-t f(t)\}$$

$$\Rightarrow \quad \mathcal{L}\{t f(t)\} = -F'(s)$$

(b) $$F(s) = \tan^{-1}\left(\frac{3}{s}\right)$$

$$\Rightarrow F'(s) = \frac{1}{1+\left(\frac{3}{s}\right)^2}\cdot\left(\frac{-3}{s^2}\right) = \frac{-3}{s^2+3^2}$$

$$\Rightarrow \mathcal{L}^{-1}\{-F'(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \sin(3t)$$

$$\Rightarrow \quad t f(t) = \sin(3t)$$

$$\Rightarrow \quad \boxed{f(t) = \frac{\sin(3t)}{t}}$$

भारतीय प्रौद्योगिकी संस्थान दिल्ली
INDIAN INSTITUTE OF TECHNOLOGY DELHI

सभी पृष्ठों पर लिखें। Write on all pages.

अनुक्रमांक Entry No. | अनुवर्ती पुस्तिका संख्या CONTINUATION BOOK No.

पाठ सं. Course No. | ग्रुप संख्या Group No. | दिनांक Date

5. $y'' + 2y' + 2y = 4e^{-x}\sec^3 x$

Char. eqn. for homog. part : $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i$

$y_h = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$ [1 mark]

$y_1 = e^{-x}\cos x \quad , \; y_2 = e^{-x}\sin x$

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= e^{-x}\cos x \cdot e^{-x}(\cos x - \sin x) - e^{-x}\sin x \cdot e^{-x}(-\sin x - \cos x)$

$= e^{-2x}$ [1 mark]

$y_p = u_1 y_1 + u_2 y_2$

where $u_1 = -\int \frac{y_2\, r(x)}{W(y_1,y_2)}\,dx$

$= -\int \frac{\cancel{e^{-x}}\sin x \cdot 4\cancel{e^{-x}}\sec^3 x}{\cancel{e^{-2x}}}\,dx$ [1]

$= -\int 4\tan x \sec^2 x\,dx = -2\tan^2 x$

and $u_2 = \int \frac{y_1\, r(x)}{W(y_1,y_2)}\,dx = \int \frac{\cancel{e^{-x}}\cos x \cdot 4\cancel{e^{-x}}\sec^3 x\,dx}{\cancel{e^{-2x}}}$

$= 4\int \sec^2 x\,dx = 4\tan x$ [1]

$\therefore \boxed{y_p = (-2\tan^2 x)\,e^{-x}\cos x + (4\tan x)\,e^{-x}\sin x}$

General soln. $\boxed{y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x + y_p}$

6. $(1-x^2)y'' - 2xy' + 2y = 2 - x \quad y(0)=0,\ y'(0)=1$

(1 mark) $y(x) = \sum_{n=0}^{\infty} a_n x^n \longleftrightarrow \boxed{\begin{array}{c} y(0) = a_0 = 0 \\ y'(0) = a_1 = 1 \end{array}}$

$xy'(x) = \sum_{n=0}^{\infty} n a_n x^n$

$y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$x^2 y'' = \sum_{n=0}^{\infty} (n-1)n\, a_n x^n$

(1 mark) $\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n-1)n a_n - 2n a_n + 2a_n\right] x^n = 2 - x$

For $n=0$: $2a_2 + 2a_0 = 2$

$a_2 = 1$

(1 mark) $n=1$ $\quad 6a_3 = -1$

$a_3 = -\frac{1}{6}$

$\boxed{\text{For } n \geq 2}$ $\quad a_{n+2} = \frac{n-1}{n+1} a_n$

$a_4 = \frac{1}{3} a_2 = \frac{1}{3}.$

$a_6 = \frac{3}{5} \cdot \frac{1}{3} = \frac{1}{5}.$

Assume $a_{2n} = \frac{1}{2n-1}$.

Then $a_{2(n+1)} = a_{2n+2} = \frac{2n-1}{2n+1} \cdot a_{2n}$

$= \frac{1}{2(n+1)-1}$

Then induction

$a_{2n} = \frac{1}{2n-1} \quad n \geq 2.$

$a_5 = \frac{2}{4} a_3 = \frac{-1}{2 \cdot 6}$

$a_7 = -\frac{4}{6} \cdot \frac{1}{2 \cdot 6} = \frac{-1}{3 \cdot 6}.$

Assume. $a_{2n+1} = \frac{-1}{6 \cdot n}$

Then $a_{2(n+1)+1} = a_{2n+1+2} = \frac{2n}{2n+2} \cdot$

$= -\frac{2n}{2n+2} \cdot \frac{1}{6 \cdot n}$

$= \frac{-1}{6(n+1)}$

$\therefore \boxed{y = x + \sum_{n=1}^{\infty} \frac{1}{2n-1} x^{2n} - \sum_{n=1}^{\infty} \frac{1}{6n} x^{2n+1}}$

7. $y'' + 5y' + 6y = \delta(t-\frac{\pi}{2}) + u(t-\pi)$; $y(0)=0$, $y'(0)=1$

$\mathcal{L}(y') = sY - y(0) = sY$

$\mathcal{L}(y'') = s^2Y - s\,y(0) - y'(0) = s^2Y - 1$

$\mathcal{L}\{\delta(t-\frac{\pi}{2})\} = e^{-\frac{\pi}{2}s}$

$\mathcal{L}\{u(t-\pi)\} = \frac{e^{-\pi s}}{s}$

$\therefore$ Taking Laplace transform, we get

$$s^2Y - 1 + 5sY + 6Y = e^{-\frac{\pi}{2}s} + \frac{e^{-\pi s}}{s}$$

$$\Rightarrow (s^2+5s+6)Y = 1 + e^{-\frac{\pi}{2}s} + \frac{e^{-\pi s}}{s}$$

$$\Rightarrow Y = \frac{1}{(s+2)(s+3)}\left[1 + e^{-\frac{\pi}{2}s} + \frac{e^{-\pi s}}{s}\right]$$

$$= \left(\frac{1}{s+2} - \frac{1}{s+3}\right) + e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) + e^{-\pi s}\cdot\frac{1}{s(s+2)(s+3)}$$

$$\frac{1}{s(s+2)(s+3)} = \frac{1/6}{s} + \frac{-1/2}{s+2} + \frac{1/3}{s+3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) = e^{-2t} - e^{-3t}$$

$$\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right)\right\} = e^{-2(t-\frac{\pi}{2})}u(t-\tfrac{\pi}{2}) - e^{-3(t-\frac{\pi}{2})}u(t-\tfrac{\pi}{2})$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s+2)(s+3)}\right\} = \left[\frac{1}{6} - \frac{1}{2}e^{-2(t-\pi)} + \frac{1}{3}e^{-3(t-\pi)}\right]u(t-\pi)$$

So,

$$\boxed{y(t) = \left(e^{-2t} - e^{-3t}\right) + \left[e^{-2(t-\frac{\pi}{2})} - e^{-3(t-\frac{\pi}{2})}\right]u(t-\tfrac{\pi}{2}) + \left[\frac{1}{6} - \frac{1}{2}e^{-2(t-\pi)} + \frac{1}{3}e^{-3(t-\pi)}\right]u(t-\pi)}$$

8. $y' = Ay + \vec{g}(t)$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix} 0 \\ \sec t \end{pmatrix}$

Eigen values of A are $\pm i$ with eigen vector $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

$$e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$\therefore$ Soln. of homog. $\vec{y}_h = c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

Fund. matrix $\tilde{Y} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$$\tilde{Y}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$\vec{y}_p = \tilde{Y}\vec{u}$, where $\vec{u}' = \tilde{Y}^{-1}\vec{g}(t)$

$$\vec{u}' = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 \\ \sec t \end{pmatrix}$$

$$= \begin{pmatrix} -\tan t \\ 1 \end{pmatrix}$$

$\Rightarrow \quad u_1 = -\int \tan t \, dt = \ln|\cos t|$

$u_2 = t$

$\therefore \quad y_p = u_1\vec{y}_1 + u_2\vec{y}_2 = \ln|\cos t|\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$$\therefore \boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (c_1 + \ln|\cos t|)\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + (c_2 + t)\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}$$

9. $y' = \dfrac{xy}{1-x^2-y^2} \qquad y(0) = \alpha \longrightarrow (1)$

Let $f(x,y) = \dfrac{xy}{1-x^2-y^2}$

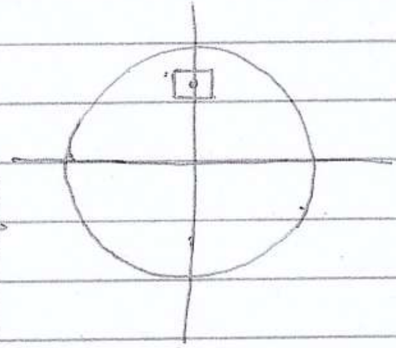
The f is cts on $\mathbb{R}^2$ except the pts (x,y) such that $x^2+y^2=1$

In particular, f is not cts at the pts $(0, y(0)) = (0, \alpha)$ with $\alpha^2 = 1$.

Hence we can't use existence and uniqueness theorem if $\alpha^2 = 1$.

Assume $\alpha^2 \neq 1$. Then

~~If $|\alpha| \leq 1$~~, we can choose a small rectangle



$$R = \left\{ (x,y) : \begin{array}{l} |x| \leq \left|\dfrac{1-|\alpha|}{4}\right| \\ |y-\alpha| \leq \left|\dfrac{1-|\alpha|}{4}\right| \end{array} \right\}$$

Such that f is cts on R.

It means that either R is ~~subset of~~ contained in $\{(x,y) : x^2+y^2$ if $|\alpha|<1$ or $R \subset \{(x,y) : x^2+y^2>1\}$ if $|\alpha|$

In both cases, $f_y = \dfrac{x(1-x^2-y^2)+2xy^2}{(1-x^2-y^2)^2}$ is cts on R.

Summary. If $\alpha^2 \neq 1$, $\exists$ R - rectangle s.t. $(0,\alpha) \in R$

(1) f is cts on R.

(2) f_y is cts on R.

then by existence & uniqueness (1) has a unique solution.