COL202 - Tutorial 4

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Q2.

Theorem: A water level of x L is possible to create in the given game if and only if x is a multiple of gcd (a, b, c).

Observation: The amount of water in each jug is an integer linear combination of a, b and c.

Corollary: The amount of water in each jug is an integer multiple of gcd (a, b, c)

Lemma 1: $\gcd(a,b,c) = \gcd(\gcd(a,b), c)$

Proof of Lemma 1: Let $g_1 = \gcd(a,b,c)$

$\Rightarrow g_1 | a, \; g_1 | b, \; g_1 | c$

$\Rightarrow g_1 | \gcd(a,b)$ and $g_1 | c$

$\Rightarrow g_1 | \gcd(\gcd(a,b), c)$ — ①

Also, let $\gcd(\gcd(a,b),c) = g_2$

$\Rightarrow g_2 | \gcd(a,b)$ and $g_2 | c$

$\Rightarrow g_2 | a, \; g_2 | b, \; g_2 | c$

$\Rightarrow g_2 | \gcd(a,b,c)$ — ②

From ① and ②: $g_1 | g_2$ and $g_2 | g_1$ and since g_1 and g_2 are positive (both are gcd), $g_1 = g_2$

Hence proved.

Lemma 2: $\gcd(a,b,c) = \text{spc}(a,b,c)$

Proof of Lemma 2: Let $S :=$ set of positive integers of the form $\lambda a + \mu b + \eta c$ $(\lambda, \mu, \eta \in \mathbb{Z})$ and we can see that the set S is non-empty (can choose $\lambda = \mu = \eta = 1$)

$\therefore$ By Well Ordering Principle, S has a least element $f = spc(a,b,c)$

$\Rightarrow f = \lambda_0 a + \mu_0 b + \eta_0 c$

To show: $f | b$. Assume contradiction.

$\therefore$ By division theorem, $b = qf + r$, $0 < r < f$

$\Rightarrow r = b - qf$

$= b(1 - q\mu_0) - qa\lambda_0 - qb\eta_0$

$\Rightarrow$ r is a linear combination of a, b, c and $r < f$

which is a contradiction since $f =$ ~~spf~~ $spc(a,b,c)$

$\therefore f | b$ and $r = 0$.

Similarly, $f | a$ and $f | c$

$\therefore$ f is a common divisor of a, b and c

$\Rightarrow f \leq \gcd(a,b,c)$ — (3)

Now, let $g = \gcd(a,b,c)$

$\Rightarrow g | a$, $g | b$, $g | c$

$\Rightarrow g | (\lambda_0 a + \mu_0 b + \eta_0 c)$

$\Rightarrow g | f$

$\Rightarrow g \leq f$ — (4)

From (3) and (4), $\gcd(a,b,c) = spc(a,b,c)$

Proof of theorem :

($\Rightarrow$) Follows from corollary

($\Leftarrow$) Suppose x is a multiple of $\gcd(a,b,c)$

Then x is a multiple of $spc(a,b,c)$ (From Lemma 2, $\gcd = spc$)

W.l.o.g. assume $a \le b \le c$

Feasability: $0 \le x \le c$

If $x = c$, it is trivial.

Hence, assume $0 \le x < c$

Claim: We can replace jugs a and b with a single jug of capacity $\gcd(a,b)$.

We have, $x = s(\alpha a + \beta b) + tc$, where $\gcd(a,b) = \alpha a + \beta b = g$

with $s > 0$ and $\alpha > 0$ $\longrightarrow$ (if not, we can always adjust s, t, α, β to achieve this)

$\{x$ is a linear combination of $a, b, c\}$

Repeat the procedure s times:

* Repeat α times:
    - Fill aL jug and pour it into bL jug.
    - If bL jug is full, empty it and pour rest of a into b
* Pour bL jug into cL jug.
* If cL jug is full, empty it and pour rest of b into c.

We had already proved for 2 jug case that the above procedure:

* at each iteration, after α repeats of 1st step, bL jug will have exactly gL ($= \gcd(a,b) = \alpha a + \beta b$) of water left.

* Hence, 1st and 2nd step combined is equivalent to replacing aL and bL jugs with a single jug of gL.

* Hence, again, after s iterations, we will have exactly $x = [s(\alpha a + \beta b) + tc]L$ of water in jug c {from the two jug case proved in class}

Now, from Lemma 1, $\gcd(\gcd(a,b), c) = \gcd(a,b,c)$

Hence, x is an integer multiple of $\gcd(a,b,c)$ as well as $\gcd(\gcd(a,b),c)$

Hence, if x is an integer multiple of $\gcd(a,b,c)$, then xL is possible to fill.

Hence proved.