

Lecture 3

Gaussian Elimination.

Ex: $x_1 + 4x_2 - 3x_3 = 2$

$$3x_1 - 2x_2 - x_3 = -1$$

$$-x_1 + 10x_2 - 5x_3 = 3.$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 3 & -2 & -1 & -1 \\ -1 & 10 & -5 & 3 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_1 \quad R_3 \rightarrow R_3 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -3 & 2 \\ 0 & -14 & 8 & -7 \\ 0 & 14 & -8 & 5 \end{array} \right]$$

$$R_3 \rightarrow R_2 + R_3$$

$$\left[\begin{array}{ccc|cc} 1 & 4 & -3 & 1 & 2 \\ 0 & -14 & 8 & 1 & -7 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right]$$

$$x_1 + 4x_2 - 3x_3 = 2$$

$$-14x_2 + 8x_3 = -7$$

No solution!

$$0 = -2$$

Remark: If at any point of solving the echelon form
 there is a row of the form $[0 \dots 0 : c]$
 then the system will always have no solution.

§ Gauß-Jordan Elimination.

- ① Multiply each non-zero row by the reciprocal of the pivot so that we end up with 1 as the leading term in each non-zero row
- ② Use row operations to introduce zeros in the entries above the pivot positions.

Ex:

$$2x_1 - 2x_2 - 6x_3 + x_4 = 3$$

$$-x_1 + x_2 + 3x_3 - x_4 = -3$$

$$x_1 - 2x_2 - x_3 + x_4 = 2$$

Aug:

$$\left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right]$$

Echelonform:

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & -3 \end{array} \right]$$

$$R_2 \rightarrow R_2(-1)$$

$$R_3 \rightarrow R_3(-1)$$

$$\left[\begin{array}{ccccccc} 1 & -2 & -1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array} \right]$$

$$R_1 \rightarrow R_1 + 2R_2$$

$$\left[\begin{array}{ccccccc} 1 & 0 & -5 & 1 & 1 & 4 \\ 0 & 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array} \right]$$

$$R_1 \rightarrow R_1 - R_3$$

$$\left[\begin{array}{ccccccc} 1 & 0 & -5 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array} \right]$$

$$x_4 = 3$$

$x_3 = s$ free parameter

$$x_2 - 2x_3 = 1 \Rightarrow x_2 = 1 + 2s$$

$$x_1 - 5x_3 = 1 \Rightarrow x_1 = 1 + 5s.$$

Dof: A matrix x is in reduced Echelon form
(or reduced row echelon form) if

- ① it is in Echelon form
- ② All the pivot positions are 1
- ③ The only nonzero pivot term in a pivot col is in the pivot position

Theorem: A given matrix is equivalent to a
UNIQUE matrix that is in reduced
Echelon form.

Ex:

$$\begin{aligned}x_1 - 2x_2 - 3x_3 &= -1 \\x_1 - x_2 - 2x_3 &= 1 \\-x_1 + 3x_2 + 5x_3 &= 2\end{aligned}$$

Augmented System

$$\left[\begin{array}{cccc|c} 1 & -2 & -3 & 1 & -1 \\ 1 & -1 & -2 & | & 1 \\ -1 & 3 & 5 & | & 2 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 + R_1$$

$$\left[\begin{array}{ccc|cc} 1 & -2 & -3 & 1 & -1 \\ 0 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Echelon form.

$$R_2 \rightarrow R_2 - R_3,$$

$$R_1 \rightarrow R_1 + 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$R_1 \rightarrow R_1 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

reduced row Echelon form

$$x_1 = 2$$

$$x_2 = 3$$

$$x_3 = -1$$

§ Homogeneous Linear System

$$A \underline{x} = 0$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

(*)

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

always have a Soln: $x_1 = x_2 = \dots = x_n = 0$

[↑] trivial solution

If there are additional solutions, they are called nontrivial solutions.

Ex:
$$\begin{aligned} 2x_1 - 6x_2 - x_3 + 8x_4 &= 0 \\ x_1 - 3x_2 - x_3 + 6x_4 &= 0 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

Aug:
$$\left[\begin{array}{cccc|c} 2 & -6 & -1 & 8 & 0 \\ 1 & -3 & -1 & 6 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{array} \right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cccc|c} 1 & -3 & -1 & 6 & 0 \\ 2 & -6 & -1 & 8 & 0 \\ -1 & 3 & -1 & 2 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 + R_1$$

$$\left[\begin{array}{cccccc} 1 & -3 & -1 & 6 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & -2 & 3 & 1 & 0 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 2R_2$$

$$\left[\begin{array}{cccccc} 1 & -3 & -1 & 6 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$R_1 \rightarrow R_1 + R_2$$

$$\left[\begin{array}{cccccc} 1 & -3 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$x_1 - 3x_2 + 2x_4 = 0$$

$$x_3 - 4x_4 = 0$$

$$x_2 = s_1, \quad x_4 = s_2$$

$$x_3 = 4s_2 \quad x_1 = 3s_2 - 2s_1$$

Proof of why a linear system has only 3 choices,

- ① no soln ② ∞ many sol ③ unique sol.

$A\mathbf{x} = \mathbf{b}$ be the linear system.

compute the augmented matrix \rightarrow echelon form.

- ⓐ The system has a row of the form $[0 \dots 0 : c]$

its corresponding equation is of the form

$$0 = c$$

\Rightarrow system has no solutions

$$\left[\begin{array}{cccc|c} 1 & \dots & 0 & \dots & 0 & : & c \\ 0 & \dots & 0 & \dots & 0 & : & 0 \end{array} \right]$$

If ⓐ does not occur, then either of the following has to happen

- ⓑ The echelon form has no free parameters \Rightarrow it has a unique sol

$$\left[\begin{array}{ccc|c} 1 & \dots & 0 & : & c \\ 0 & \dots & 0 & : & 0 \end{array} \right]$$

- ⓒ The echelon form has one or more free params
(hence ∞ many solutions)

non pivot cols.

Def: ① For a system of equations a variable that appears in the first term in at least one equation is called a leading variable

② For a system in echelon form, any variable that is not a leading variable is called a "free variable".

③ Rank of a matrix = number of nonzero rows in the echelon form.

rank of

$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2$$

← zero row.

rank

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3$$

Theorem: If $A\vec{x} = \vec{b}$ is a linear system of equations with n variables $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

then the number of free variables = $n - \text{rank } A$.

Theorem: If $A\vec{x} = 0$ is a homogeneous linear system.

If $\text{rank } A < n \Leftrightarrow A\vec{x} = 0$ has a nontrivial sol.

Theorem: $A\vec{x} = b$ is a system of linear equations with fewer equations than unknowns ($m < n$)
has at least one free variable & hence ∞ many solutions. (At least one solution)

Remark: If a System of linear equations has a $0=0$ equation then the system need not always have ∞ many solutions. It can have no solutions or unique solution.

$$A\vec{x} = b$$

Echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Unique soln

Echelon

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

No solution