

MTL 101 Major Brief Solutions

1(a). $\{x_1 - x, \dots, x_n - x\}$ is L.D. if there exist scalars $\alpha_1, \dots, \alpha_n$ not all zero such that $\sum_{i=1}^n \alpha_i(x_i - x) = 0$
 i.e. $\sum_{i=1}^n \alpha_i x_i - (\sum_{i=1}^n \alpha_i)x = 0 \rightarrow ①$

Claim: $\sum_{i=1}^n \alpha_i \neq 0$

If $\sum_{i=1}^n \alpha_i = 0$, then from ①, $\alpha_i = 0, \forall i$, since x_1, \dots, x_n are L.I.
 So we discard this case.

If $\sum_{i=1}^n \alpha_i \neq 0$, then from ①, $x = \frac{\sum_{i=1}^n \alpha_i x_i}{\sum_{i=1}^n \alpha_i}$

$$\Rightarrow \frac{\alpha_i}{\sum_{i=1}^n \alpha_i} = \beta_i$$

$$\Rightarrow \sum_{i=1}^n \beta_i = 1$$

1(b). $T\left(\sum_{i=0}^3 \alpha_i x^i\right) = \sum_{i=0}^3 \alpha_i (x+1)^i$

$$\text{let } B = \{v_1 = 1, v_2 = 1+x, v_3 = 1+x^2, v_4 = 1+x^3\}$$

$$T(v_1) = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4$$

$$T(v_2) = 1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4$$

$$T(v_3) = -1 \cdot v_1 + 2 \cdot v_2 + 1 \cdot v_3 + 0 \cdot v_4$$

$$T(v_4) = -5 \cdot v_1 + 3 \cdot v_2 + 3 \cdot v_3 + 1 \cdot v_4$$

$$[T]_B = \begin{bmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2(a). $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 4 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$Tx = 0 \Rightarrow Ax = 0 \Rightarrow x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \quad \therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ Means Null space of } T$$

Since $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is L.I.

Therefore this set forms a basis of Null space of T.

$$2(b) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & k \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & k-10 \end{array} \right)$$

For unique solution: $P(A)=3=P(AB)$. This is possible if $\lambda-3 \neq 0$ or $\lambda \neq 3$.

For infinitely many solutions. $P(A)=P(AB)<3$
This is possible if $\lambda=3$ & $k=10$.

For NO solution: $P(A) \neq P(AB)$. This is possible if $\lambda=3$ and $k \neq 10$.

3. $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. Eigenvalues of A are: $\lambda = -3, -1$

An eigenvector corresponding to $\lambda = -3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

An eigenvector corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $x = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow \bar{x} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Define $z = \bar{x}'y$ or $y = \bar{x}z$. Then

$$y' = A\bar{x} + b \Rightarrow z' = Dz + h, \text{ where}$$

$$D = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, h = \bar{x}'b = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2e^{-x} \\ 3x \end{pmatrix}$$

$$z_j' = \lambda_j z_j + h_j, \quad j=1, 2$$

$$\therefore z_j = c_j e^{\lambda_j x} + e^{\lambda_j x} \int e^{-\lambda_j x} h_j(x) dx, \quad j=1, 2$$

$$z_1 = c_1 e^{-3x} + \frac{1}{2} e^{-x} - \frac{1}{2} x + \frac{1}{6}$$

$$z_2 = c_2 e^{-x} + x e^{-x} + \frac{3}{2} x - \frac{3}{2}$$

$$\therefore y = xz$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + z_2 \\ -z_1 + z_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-3x} + c_2 e^{-x} + x e^{-x} + \frac{1}{2} e^{-x} + x - \frac{4}{3} \\ -c_1 e^{-3x} + c_2 e^{-x} + x e^{-x} - \frac{1}{2} e^{-x} + 2x - \frac{5}{3} \end{pmatrix}.$$

$$4\text{m}. \quad xy'' + y' + y = 0 \Leftrightarrow x^2y'' + xy' + y = 0$$

Let $y_1(x) = x^r \sum_{m=0}^{\infty} c_m x^m$, $c_0 \neq 0$. Substitute $y_1(x)$ and its derivatives in the given D.E. This gives

$$\sum_{m=0}^{\infty} \left\{ (m+r)(m+r-1) c_m + (m+r)c_{m-1} \right\} x^{m+r} + c_m x^{m+r+1} = 0$$

$$\Rightarrow (m+r)^2 c_m + c_{m-1} = 0 \quad (\text{coeff. of } x^{m+r}, c_{-1} = 0)$$

$m=0 \Rightarrow r^2 = 0$ (indicial equation) $\Rightarrow r=0$ is double root.

$$\Rightarrow c_m = -\frac{1}{m^2} c_{m-1}$$

$$\Rightarrow y_1(x) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2}, \quad (\text{with } c_0 = 1).$$

Since it is the case of double root, we have

$$y_2(x) = (\log x) y_1 + x^0 \sum_{m=1}^{\infty} b_m x^m.$$

Substitute $y_2(x)$ and its derivatives in the given D.E. This leads to

$$0 = 2xy_1' + \sum_{m=1}^{\infty} \left[b_m x^{m+1} + m b_m x^m + m(m-1) b_m x^m \right]$$

$$\Rightarrow \frac{2(-1)^m m}{(m)^2} + b_{m-1} + m^2 b_m = 0, \quad m=1, 3, \dots$$

$$\text{where } b_0 = 0 \Rightarrow b_1 = 2$$

$$\Rightarrow b_m = -\frac{b_{m-1}}{m^2} - \frac{2(-1)^m}{m(m)^2}$$

$$\Rightarrow b_m = -\frac{2(-1)^m}{(m)^2} h_m, \quad \text{where } h_m = 1 + \frac{1}{2} + \dots + \frac{1}{m}.$$

$$\therefore y_2(x) = (\log x) \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{(m)^2} h_m x^m.$$

Hence the complete solution is given by

$$y(x) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} + c_1 \left[\log x \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m h_m x^m}{(m)^2} \right].$$

$$4\text{(b).} \quad \text{(i)} \quad \frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x). \quad \text{(ii)} \quad \frac{d}{dx} [\bar{x}^n J_n(x)] = -\bar{x}^n J_{n+1}(x).$$

Let α, β be two consecutive positive zeros of $J_{n+1}(x)$. Using $\frac{d}{dx} [x^{n+1} J_{n+1}(x)] = x^{n+1} J_n(x)$ (part (i) with $n=n+1$) By Rolle's theorem, there is a ' r' such that $\alpha < r < \beta$ and $J_n(r)=0$. If there were more than one zero, say $\alpha < r < \delta < \beta$ using $\frac{d}{dx} [\bar{x}^n J_n(x)] = -\bar{x}^n J_{n+1}(x)$ (part (ii)) Rolle's theorem would imply a zero of $J_{n+1}(x)$ in between r and δ , which would contradict consecutiveness of α and β . This completes the proof.

5(a). $((1+x^2)y')' + \lambda(x^2+1)^{-1}y = 0$, where prime denotes differentiation with respect to x .

$$\left((1+x^2) \frac{dy}{dx} \right)' + \lambda(x^2+1)^{-1}y = 0 \rightarrow \text{eqn}$$

Substitution $\frac{dy}{dx} = x^2+1$

$$\int \frac{dx}{x^2+1} = dt \Rightarrow t = \arctan x \text{ or } x = \tan t$$

Substituting in ~~eqn~~, $\Rightarrow \frac{d^2y}{dt^2} + \lambda y = 0$, whose general solution is

$$y = C_1 \cos \sqrt{\lambda} t + C_2 \sin \sqrt{\lambda} t$$

$$0 = y(x=0) = y(t=0) \Rightarrow C_1 = 0, \text{ therefore } y = C_2 \sin \sqrt{\lambda} t$$

$$0 = y(x=1) = y(t=\frac{\pi}{4}) \Rightarrow \sin \sqrt{\lambda} \frac{\pi}{4} = 0 = \sin n\pi \quad (\text{for nontrivial solution } C_2 \neq 0)$$

Therefore, the eigenvalues are $\lambda_n = 16n^2$

$$\text{and the corresponding eigenfunctions are } y_n = \sin(4nt) \\ = \sin(4n \arctan x) \\ n = 1, 2, 3, \dots$$

5(b). Let $\phi_i(x) = \cos \frac{i\pi x}{L}$, $\phi_j(x) = \cos \frac{j\pi x}{L}$

$$\text{For } i \neq j, \text{ consider } \langle \phi_i(x), \phi_j(x) \rangle = \int_{-L}^L \phi_i(x) \phi_j(x) dx = \int_{-L}^L \cos \frac{i\pi x}{L} \cos \frac{j\pi x}{L} dx = 0$$

This shows that the given set of functions are orthogonal.

$$\text{Now, the norm of } \phi_i = \|\cos \frac{i\pi x}{L}\| = \langle \phi_i(x), \phi_i(x) \rangle^{1/2} = \left(\int_{-L}^L \cos^2 \frac{i\pi x}{L} dx \right)^{1/2} = \sqrt{L}$$

$$\therefore \text{Orthonormal set is } \frac{\phi_i}{\|\phi_i\|} = \left\{ \frac{1}{\sqrt{L}} \cos \frac{i\pi x}{L} \right\}_{i=1}^{\infty}$$

$$\text{For } i=0, \|\phi_0\| = \langle \phi_0(x), \phi_0(x) \rangle^{1/2} = \left(\int_{-L}^L dx \right)^{1/2} = \sqrt{2L}$$

$$\therefore \text{Orthonormal set is } \left\{ \frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}} \cos \frac{i\pi x}{L} \right\}_{i=1}^{\infty}$$

6(a). Prove existence of Laplace transform

$$\underline{\text{Sol. }} \mathcal{L}\{\tilde{t}^{\frac{1}{2}}\} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}} ; \quad \mathcal{L}\{\tilde{e}^t \tilde{t}^{-\frac{1}{2}}\} = \sqrt{\frac{\pi}{s+1}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{\sqrt{\pi}} \int_0^t \tilde{e}^{-\tau} \tau^{-\frac{1}{2}} d\tau = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \tilde{e}^{-x^2} x^{-1/2} dx = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \tilde{e}^{-x^2} dx = -\text{erf}(\sqrt{t}).$$

(∴ $\frac{F(s)}{s} = \mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\}$)