Problem :- For what value of K the planes

$$x+y+z=2,\quad 3x+y-2z=k,\quad 2x+4y+7z=k+2$$

intersect in a line ? Find the solution set. (6)

Solution :- The system of linear equation can be expressed as $AX=B$ where

$$A=\begin{bmatrix}1&1&1\\3&1&-2\\2&4&7\end{bmatrix}\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}\quad B=\begin{bmatrix}2\\k\\k+2\end{bmatrix}$$

$$[A\,|\,B]=\begin{bmatrix}1&1&1&2\\3&1&-2&k\\2&4&7&k+2\end{bmatrix}\xrightarrow[R_2-3R_1]{R_3-2R_1}\begin{bmatrix}1&1&1&2\\0&-2&-5&k-6\\0&2&5&k-2\end{bmatrix}$$

$$\xrightarrow{R_3+R_2}\begin{bmatrix}1&1&1&2\\0&-2&-5&k-6\\0&0&0&2k-8\end{bmatrix}\xrightarrow{-\frac{1}{2}R_2}\begin{bmatrix}1&1&1&2\\0&1&\frac{5}{2}&\frac{6-k}{2}\\0&0&0&2(k-4)\end{bmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -\frac{3}{2} & \frac{K-2}{2} \\ 0 & 1 & \frac{5}{2} & \frac{6-K}{2} \\ 0 & 0 & 0 & 2(K-4) \end{bmatrix}$$

upto this

+2

(Total 2)

The planes intersect in a line if

$$\text{rank}(A) = \text{rank}(A \mid B) = 2$$

If someone finds it by any other way.

Therefore, $2(K-4) = 0$

$\Rightarrow K = 4$ ✓ +2 (Total 4)

Then the system of linear equation is equivalent to

$$x - \frac{3}{2} z = 1$$

$$y + \frac{5}{2} z = 1$$

Let $z = \lambda$ then $y = 1 - \frac{5}{2}\lambda$ and $x = 1 + \frac{3}{2}\lambda$

Therefore, the solution set is $\left\{ \left(1 + \frac{3}{2}\lambda,\ 1 - \frac{5}{2}\lambda,\ \lambda\right) : \lambda \in \mathbb{R} \right\}$

✓ +2

Total (6)

Q2 Using elementary row operations find the inverse of the matrix (if exists)

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

$\det(A) \neq 0 \Rightarrow A$ is invertible — ① mark

$RREF([A \mid I]) = [R \mid B]$ with $R = I$ (as A is invertible)

(row reduced echelon form)

Let $R = I = \underbrace{E_K E_{K-1} \cdots E_1}_{\text{elementary matrices}} A = EA$ where $E = E_K \cdots E_1$

Then

$B = EI = E$

① mark

i.e. we have $BA = I$. Since A is invertible, we have $B = A^{-1}$.

We have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_2/2} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & -1/2 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/2 & -1/2 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 \times 1/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right]$$

$$R_2 \to R_2 + R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right]$$

$I \qquad A^{-1}$

Thus $$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

④ marks

MTL101, II Semester 2022-23, Quiz I
Answer Key and Scheme of Evaluation for Question 3

3. The set $\mathbb{R}^3$ is not a vector space over $\mathbb{R}$ under the operations defined in the question. To this end, identifying an axiom which fails to hold and giving the requisite details [4 Marks].

For instance,

- For $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, we have

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + 2y_1, x_2 + y_2, x_3 + y_3),\\ \mathbf{y}+\mathbf{x} &= (y_1, y_2, y_3) + (x_1, x_2, x_3) = (y_1 + 2x_1, y_2 + x_2, y_3 + x_3).\end{aligned}$$

  Therefore, $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ holds if and only if $x_1 = y_1$, which reveals that the *commutativity axiom* $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ fails to hold for an arbitrary pair of elements $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^3$.

- For $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$ in $\mathbb{R}^3$,

$$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= (x_1 + 2y_1 + 2z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3),\\ \mathbf{x}+(\mathbf{y}+\mathbf{z}) &= (x_1 + 2y_1 + 4z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3).\end{aligned}$$

  Thus, the *associativity axiom* does not hold, in general.

- There does not exists a fixed $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ such that $\theta + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^3$. Thus a (left) identity (zero element) does not exist. It makes no sense to talk about left inverse. However, $\mathbf{0} = (0, 0, 0)$ acts as a right identity, that is, $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^3$.

- For $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$, the axiom

$$(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$$

  does not hold, in general.

Some additional comments

1. If a few axioms are verified correctly, but no correct conclusion is drawn, then only 1 mark may be given.
2. Note that for an arbitrary $(x_1, x_2, x_3) \in \mathbb{R}^3$, $(0, 0, 0) + (x_1, x_2, x_3) \neq (x_1, x_2, x_3)$. However, this observation is not helpful to prove that $\mathbb{R}^3$ is not a vector space over $\mathbb{R}$ under the given operations. It says that $(0, 0, 0)$ is not a zero element, but does not show the nonexistence of a zero element. No marks will be given.

3. A common wrong/inadequate argument observed during evaluation: For $\mathbb{R}^3$ to be a vector space over $\mathbb{R}$, the vector addition $(x_1, x_2, x_3)+(y_1, y_2, y_3)$ should be defined as $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$, but here it is defined as $(x_1 + 2y_1, x_2 + y_2, x_3 + y_3)$. Therefore, $\mathbb{R}^3$ is not a vector space over $\mathbb{R}$ under the given operations.

   We know that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$ under the so-called *standard operations* given by:

$$\begin{aligned}(x_1, x_2, x_3) + (y_1, y_2, y_3) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3),\\ \alpha(x_1, x_2, x_3) &= (\alpha x_1, \alpha x_2, \alpha x_3).\end{aligned}$$

   However, it may be possible to define some exotic-looking operations to make $\mathbb{R}^3$ a vector space over $\mathbb{R}$. Interested reader may refer "M.A. Carchidi, Generating exotic-looking vector spaces, The College Mathematics Journal, 29, 1998, 5 pp."

4. Suppose that V with given operations is already known to be a vector space over $\mathbb{F}$. One can prove that a nonempty subset W of V is a subspace of V (and hence W itself is a vector space) by verifying that $\alpha\mathbf{u} + \mathbf{v} \in W$ whenever $\mathbf{u}, \mathbf{v} \in W$ and $\alpha \in \mathbb{F}$.

   However, for a set W with given operations to be a vector space, it is not enough to prove that $\alpha\mathbf{u} + \mathbf{v} \in W$ whenever $\mathbf{u}, \mathbf{v} \in W$ and $\alpha \in \mathbb{F}$. The vector space axioms need to be checked.

Problem 4 [4 marks] Let $V = C([0,1])$ be the vector space (over $\mathbb{R}$) of all real-valued continuous functions defined on $[0,1]$. Let W be the set of all real polynomials of odd degrees more than one. Verify whether $W \cup \{0\}$ is a subspace of V.

Solution: Let $p(x) = 1 + x^3$ and $q(x) = 1 - x^3$.
Then, both $p(x)$ and $q(x)$ belong to the set W.
But, $p(x) + q(x) = 2$ does not belong to the set $W \cup \{0\}$ since it is a nonzero polynomial of degree 0.
Hence, $W \cup \{0\}$ is not a subspace of V.