MTL - 101, Minor 1

Q1 (a) FALSE

$+$ is not associative in V

$$((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) = (x_1 + y_1 + z_1,\ x_2 - y_2 - z_2)$$

$$(x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) = (x_1 + y_1 + z_1,\ x_2 - y_2 + z_2)$$

And unless $z_2 = 0$, $x_2 - y_2 - z_2 \neq x_2 - y_2 + z_2$ (1)

(b) FALSE

$$(0,0) + (x_1, x_2) = (x_1, -x_2) \neq (x_1, x_2) + (0,0)$$

$\therefore$ $(0,0)$ is not an additive identity (1)

(c) TRUE

If $\alpha = 0$ then $\alpha(v+w) = 0 = \alpha v + \alpha w$

Let $\alpha \neq 0$ then

$$\begin{aligned}
\alpha v + \alpha w &= \alpha(x_1, x_2) + \alpha(y_1, y_2) \\
&= \left(\alpha x_1, \frac{x_2}{\alpha}\right) + \left(\alpha y_1, \frac{y_2}{\alpha}\right) \\
&= \left(\alpha(x_1 + y_1), \frac{x_2 - y_2}{\alpha}\right) \\
&= \alpha(x_1 + y_1, x_2 - y_2) \\
&= \alpha((x_1, x_2) + (y_1, y_2)) \\
&= \alpha(v + w).
\end{aligned}$$

(1)

(d) FALSE

If $\alpha + \beta = 0$, $\alpha \neq 0$, $\beta \neq 0$ then

$$(\alpha + \beta) v = 0$$

$$\alpha v + \beta v = \left(\alpha x_1 + \beta x_1, \frac{x_2}{\alpha} - \frac{x_2}{\beta}\right)$$

$$= \left(0, \frac{2x_2}{\alpha}\right) \neq (\alpha + \beta) v \quad \text{(in general)}$$

(1)

2. Let $S = \{v_1, v_2, \dots, v_n\} \subseteq V$

Then,

$$\text{Span}(S) := \left\{ \sum_{i=1}^{n} \alpha_i v_i \;:\; \alpha_i \in \mathbb{F} \right\}$$

- For proving Span (S) is a subspace of V. [2 Marks]

- For verifying Span (S) contains S, i.e., $S \subseteq \text{Span}(S)$. [1 Mark]

- For showing Span (S) is the smallest subspace of V that contains S in the sense that any other subspace of V that contains S must contain span (S). [1 Mark]

For a detailed proof, one can consult, for instance, Theorem 5.2.3 in the book by 'Howard Anton and Chris Rorres'.

If the answer is not very rigorous, then you cannot claim for full or partial credit.

Q3 a.) $U \cap (S+W) \subseteq (U \cap S) + (U \cap W)$

This statement is in general not true

For example

consider $V = \mathbb{R}^2$

$U =$ line $y = x = \{(x,x) : x \in \mathbb{R}\}$

$S =$ y axis $= \{(0,y) : y \in \mathbb{R}\}$ — 1

$W =$ x axis $= \{(x,0) : x \in \mathbb{R}\}$

$S+W = \mathbb{R}^2$; $U \cap (S+W) = U \cap \mathbb{R}^2 = U$

$(U \cap S) = (0,0)$ — 1

$(U \cap W) = (0,0)$ $\therefore$ $(U \cap S) + (U \cap W) = (0,0)$

and $U \not\subseteq (0,0)$

No marks will be awarded for writing true or False without counter example.

(b) $(U \cap S) + W \subseteq (U+W) \cap (S+W)$ True

Let $x \in (U \cap S) + W \Rightarrow x = x_1 + x_2$ ($x_1 \in U \cap S$ and $x_2 \in W$) —

$x_1 \in U \cap S \Rightarrow x_1 \in U$ and $x_1 \in S$

$\Rightarrow x_1 + x_2 \in U + W$ ($\because x_1 \in U$ & $x_2 \in W$)

also $x_1 + x_2 \in (S+W)$ ($\because x_1 \in S$ & $x_2 \in W$)

$\Rightarrow x = x_1 + x_2 \in (U+W) \cap (S+W)$ —

as x was arb. element of $(U \cap S) + W$

$\Rightarrow (U \cap S) + W \subseteq (U+W) \cap (S+W)$

No marks will be awarded in case you are proving this statement using a particular example.

Marking Scheme For Question 4

Part (a) 1 mark + 1 mark = 2 marks ; 1 mark for definition of subspace + 1 mark for further Proof.

Part (b) 1 mark + 1 mark = 2 marks ; 1 mark for $V = W + \tilde{W}$ + 1 mark for $W \cap \tilde{W} = \{0\}$

Q.4 Let $V = M_{2\times 2}(\mathbb{C})$ be the vector space of all 2×2 matrices with complex entries over field $F = \mathbb{R}$. Let W be the subset of V consisting of all symmetric matrices whose sum of the principal diagonal element is zero that is ①

$$W = \{X \in V \mid X^t = X,\ tr(X) = 0\}$$

ⓐ Prove that W is a subspace of V.

ⓑ Show that $V = W \oplus \tilde{W}$, where $\tilde{W}$ is the subspace of V consisting of all matrices whose first row has entries equal to zero.

Solⁿ ⓐ Given that $V = M_{2\times 2}(\mathbb{C})$ be the vector space of all 2×2 matrices with complex entries over field $F = \mathbb{R}$. Then V is of the form.

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \;\middle|\; a, b, c, d \in \mathbb{C} \right\}$$

Also Given that

$$W = \{X \in V \mid X^t = X,\ tr(X) = 0\}$$

Then W consisting matrix of the form $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$X^t = X \quad \& \quad tr(X) = 0$$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $a + d = 0$ so $\Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $a = -d$

$$\Rightarrow \quad b = c \quad \& \quad d = -a$$

Therefore $W = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix}_{2\times 2} ; a, b \in \mathbb{C} \right\}$

We want to show that W is a subspace of V. It is sufficient to show that for any $\alpha, \beta \in \mathbb{R}$ & $W_1, W_2 \in W$

$$\alpha W_1 + \beta W_2 \in W$$

Let $\alpha, \beta \in \mathbb{R}$ & $W_1 = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, $W_2 = \begin{bmatrix} c & d \\ d & -c \end{bmatrix}$ $a, b, c, d \in \mathbb{C}$ be two elements of W. Then

$$\alpha W_1 + \beta W_2 = \alpha \begin{bmatrix} a & b \\ b & -a \end{bmatrix} + \beta \begin{bmatrix} c & d \\ d & -c \end{bmatrix} = \begin{bmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \alpha b + \beta d & -(\alpha a + \beta c) \end{bmatrix}$$

$$\alpha W_1+\beta W_2 = \begin{bmatrix} \alpha a+\beta c & \alpha b+\beta d \\ \alpha b+\beta d & -(\alpha a+\beta c) \end{bmatrix}$$ so $\alpha W_1+\beta W_2$ is symmetric

matrix with trace zero. Therefore $\alpha W_1+\beta W_2 \in W$.

This $\Rightarrow$ W is subspace of V.

(b) Given $\tilde{W}$ is the subspace of V consisting of all matrices whose first row has entries equal to zero. i.e. $\tilde{W}$ is of the form

$$\tilde{W} = \left\{ \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}_{2\times 2} : x, y \in \mathbb{C} \right\}$$

We want to show that $V = W \oplus \tilde{W}$. It is sufficient to show that

(1) $W+\tilde{W} = V$ (2) $W \cap \tilde{W} = \{0\}$

(1) We know that $W+\tilde{W} = \{ w+\tilde{w} : w \in W, \tilde{w} \in \tilde{W} \}$

so $\forall\ w \in W, \tilde{w} \in \tilde{W} \Rightarrow w+\tilde{w} \in V$ $\because W \subseteq V, \tilde{W} \subseteq V$ & V is vector space

$\Rightarrow$ $\boxed{W+\tilde{W} \subseteq V}$ —(i)

Let for any $v \in V$. Then v is of the form

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad a, b, c, d \in \mathbb{C}$$

v can be written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c-b & d+a \end{bmatrix} \in W+\tilde{W}$$

$\because \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \in W$, $\begin{bmatrix} 0 & 0 \\ c-b & d+a \end{bmatrix} \in \tilde{W}$

This $\Rightarrow$ $\boxed{V \subseteq W+\tilde{W}}$ —(ii)

Combining (i) & (ii) $\boxed{W+\tilde{W} = V}$

(2) Now we prove $W \cap \tilde{W} = \{0\}$

Let $w \in W \cap \tilde{W} \Rightarrow w \in W, w \in \tilde{W}$

so $w = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ also $w = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}$ where $a, b, x, y \in \mathbb{C}$

Then $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix} \Rightarrow$ $a=0$ $b=0$ $x=0, y=0$

so $w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow W \cap \tilde{W} = \{0\}$

5. a) $\alpha_1 = (1, 1, -1, -1)$

$\alpha_2 = (0, 1, 1, 0)$

$\alpha_3 = (2, 0, 2, -2)$.

Since, $F = \mathbb{Z}_3$, so

$\alpha_1 + \alpha_3 = (1, 1, -1, -1) + (2, 0, 2, -2)$

$= (0, 1, 1, 0) = \alpha_2$.

$\longrightarrow$ 1 mark

Therefore, α_2 can be written as a linear combination of α_1 and α_3.

$\therefore \{\alpha_1, \alpha_2, \alpha_3\}$ is L.D. in $\mathbb{Z}_3$.

$\longrightarrow$ 2 mark

5. b) $\alpha_1 = 1+i$,

$\alpha_2 = 1-i$,

$\alpha_3 = 2+\sqrt{3}$,

$\alpha_4 = 2-\sqrt{3}$.

Suppose, $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$

Since, $F = \mathbb{Q}$, so we get

$c_1 + c_2 + 2c_3 + 2c_4 = 0$

$c_1 - c_2 = 0$

$c_3 - c_4 = 0$

This is a system of equation with four variables and 3 equations. So it has infinite solution in $\mathbb{R}$. $\longrightarrow$ 1 mark

i.e. $C_1 = C_2 = 2K$

$C_3 = C_4 = -K$ where $K \in \mathbb{R}$.

Therefore, the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is L.D in $\mathbb{R}$

$\longrightarrow$ 2 mark.