Question 1 :

Consider the ordered basis $B = \{(1,1,1),(0,2,1),(2,1,0)\}$ of $\mathbb{R}^3$. Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is such that the matrix representation of T with respect to the ordered basis B is,

$$[T]_B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find $T(x,y,z)$.

Sol. Consider the standard basis of $\mathbb{R}^3$, the matrix representation of T in that basis will be

$$T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{(-1)}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 8 \\ -4 & 8 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

So,

$$T(x,y,z) = (2x - 4y + 8z, -4x + 8y + 2z, -x + 2y + 5z)$$

Problem [5 marks] Let V be the vector space of all polynomials of degree less than equal to 3. Let $T : V \to V$ be the derivative operator, i.e., $T(p(x)) = p'(x)$.

(i) Determine the matrix representation of T with respect to the ordered basis $B = \{1, x, x^2, x^3\}$.

(ii) Find the eigenvalues of T.

(iii) Is the operator T diagonalizable? Justify your answer.

Solution:

(i) We have

$$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3$$
$$T(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3$$

Therefore,

$$[T]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) The characteristic polynomial of T is $p(t) = \det(tI - T) = t^4$.

Since the eigenvalues of T are the roots of the characteristic polynomial, 0 is the only eigenvalue of T.

(iii) Since the eigenspace corresponding to the eigenvalue 0 is nothing but the null space of T, the dimension of the eigenspace is the nullity of T.

Since $\text{rank}(T) = \text{rank}([T]_B) = 3$, $\text{nullity}(T) = 4 - 3 = 1$.

Therefore, the sum of the dimension of eigenspaces of T is equal to 1, which is less than the dimension of V. This implies that the operator T is not diagonalizable.

Question 3:[**5 marks**] Let $W_1 = \{(x, y, z) \in \mathbb{R}^3 : 2x + z = 0, y + 2z = 0\}$ and $W_2 = \text{span}\{(1, 1, 2), (2, 5, 0)\}$. Find a basis of $W_1 + W_2$.

1. Note that

$$W_1 = \{(x, y, z) : 2x + z = 0, y + 2z = 0\} = \text{span}\{(-1, -4, 2)\}.$$

Given

$$W_2 = \text{span}\{(1, 1, 2), (2, 5, 0)\}.$$

By the definition, $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$, and, here, clearly

$$W_1 + W_2 = \text{span}\{(-1, -4, 2), (1, 1, 2), (2, 5, 0)\} = \text{span}\{(-1, -4, 2), (1, 1, 2)\},$$

as $(2, 5, 0) = -1(-1, -4, 2) + (1, 1, 2)$. Furthermore, $\{(-1, -4, 2), (1, 1, 2)\}$ is linearly independent. Consequently, $\{(-1, -4, 2), (1, 1, 2)\}$ is a basis for $W_1 + W_2$.

Question:- Let $A, B \in M_{n\times n}(\mathbb{R})$. Prove that AB and BA have same eigenvalues.

Proof:- If $0 \in \mathbb{R}$ is an eigenvalue of AB then $\exists$ a non-zero $X \in M_{n\times 1}(\mathbb{R})$ such that $(AB)X = 0X = O_{n\times 1}$ (zero matrix). If $\det(AB) \neq 0$ then (AB) is invertible and hence $X = O_{n\times 1}$ which is a contradiction.

Therefore $\det(AB) = 0$

$$\Rightarrow \det(A)\det(B) = 0$$

$$\Rightarrow \det(B)\det(A) = 0$$

$$\Rightarrow \det(BA) = 0$$

Then there exists a non-zero $X \in M_{n\times 1}(\mathbb{R})$ such that $(BA)X = O_{n\times 1} = 0X$. Then 0 is an eigenvalue of BA.

Reversing the role of A & B we can conclude that 0 is an eigenvalue of AB iff 0 is an eigenvalue of BA.

Now, let $\lambda \neq 0$ be an eigenvalue of AB.

Then there exists a non-zero $X \in M_{n\times 1}(\mathbb{R})$ such that

$(AB)X = \lambda X$

$\Rightarrow B(AB)X = B(\lambda X)$

$\Rightarrow (BA)(BX) = \lambda(BX)$

If $BX = O_{n\times1} \Rightarrow A(BX) = O_{n\times1} \Rightarrow (AB)X = O_{n\times1} \Rightarrow \lambda X = 0.$

Since $\lambda \neq 0$, $X = O_{n\times1}$, which is a contradiction.

Therefore, $BX \neq 0$. Thus, λ is an eigenvalue of BA.

Reversing the role of A & B we can conclude that λ is an eigenvalue of AB iff λ is an eigenvalue of BA.

Thus, AB and BA have same eigenvalues.