DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)

2023-24 SECOND SEMESTER TUTORIAL SHEET-III

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1. If $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$, show that
$< x, y >= 10x_1y_1 + 3x_1y_2 + 3x_2y_1 + 2x_2y_2 + x_2y_3 + x_3y_2 + x_3y_3$ defines an inner product in $R^3(R)$.

2. Let $V(C)$ be the vector space of all complex valued functions on the unit interval, $0 \leq t \leq 1$. If $f(t), g(t) \in V$, define

$$< f, g >= \int_{-1}^{1} f(t)\overline{g(t)}dt.$$

Prove that this is an inner product.

3. Prove that the set of all $n \times n$ matrices forms an inner product space over F where F is either real or complex, the inner product being defined as

$$< A, B >= trace \ \ (AB^*),$$

where B^* is conjugate transpose of B.

4. Let $<, >$ be the standard inner product on R^2.
(a) Let $x = (1, 2)$, $y = (-1, 1)$. If z is a vector such that $< x, z >= -1$ and $< y, z >= 3$, find z.
(b) Show that for any x in R^2 we have $x =< x, e_1 > e_1 + < x, e_2 > e_2$.

5. In $R^n(R)$, prove the following:
(a) $< x, y >= 0 \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2$
(b) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$
(c) $\|x\| = \|y\| \iff < x + y, x - y >= 0$
Are the results valid in $C^n(C)$?

6. In $C^n(C)$ prove that
(a) $4 < x, y >= \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$
(b) If $x \neq 0$, then $\|x + ix\|^2 = \|x\|^2 + \|ix\|^2$, but $< x, ix >\neq 0$
(c) If $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ and $\|x + iy\|^2 = \|x\|^2 + \|iy\|^2$ then $< x, y >= 0$

7. Let V be an inner product space. The distance between two vectors x and y in V is defined by

$$d(x, y) = \|x - y\|.$$

Show that
(a) $d(x, y) \geq 0$;
(b) $d(x, y) = 0$ if and only if $x = y$;
(c) $d(x, y) = d(y, x)$;
(d) $d(x, y) \leq d(x, z) + d(z, y)$.

8. Consider C^3, with standard inner product. Find an orthonormal basis for the subspace spanned by $x_1 = (1, 0, i)$ and $x_2 = (2, 1, 1 + i)$.

9. Suppose $\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ is an orthonormal basis of an inner product space V. Show that for each $u \in V$, the numbers $< u, v_i >, i = 1, 2, \ldots, n$ are coordinates of u with respect to the basis.

10. Let M be a subspace of R^n and dim $M = m$. A vector $x \in R^n$ is said to be orthogonal to M if $< x, y >= 0, \forall y \in M$. How many vectors can be orthogonal to M?
If $M = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$, determine a maximal set of linearly independent vectors orthogonal to M.

11. If V is a finite dimensional inner product space and W is a subspace of V, then $V = W \oplus W^{\perp}$.

12. Let V be the real inner product space consisting of the space of real-valued continuous functions on the interval, $-1 \leq t \leq 1$, with the inner product

$$< f, g >= \int_{-1}^{1} f(t)g(t)dt.$$

Let W be the subspace of odd functions, i.e., functions satisfying $f(-t) = -f(t)$. Find the orthogonal complement of W.

13. If W is a subspace of an inner product space V and $v \in V$ satisfies
$< v, w > + < w, v > \quad \leq \quad < w, w > \forall w \in W$, then prove that $< v, w >= 0$ for all $w \in W$.

14. Let x and y be two vectors in an inner product space V. Prove that Cauchy- Schwarz inequality (i.e. $|< x, y >| = \|x\|\|y\|$) becomes equality if and only if x and y are linearly dependent.

15. Suppose $B = \{x_1, \ldots\ldots, x_n\}$ be any othonormal set of the vectors in an inner product space V. Let Y be any vector in V. Prove that Bessel's inequality becomes equality if and only if $Y \in L(B)$.

16. Let an inner product in P^2 over the real field be defined as

$$< p(x), q(x) >= \int_{0}^{1} p(t)g(t)dt.$$

Starting from the basis $\{1, x, x^2\}$ of P^2 obtain an orthonormal basis.