Problems for Recitation 8

1 Build-up error

Recall a graph is ***connected*** iff there is a path between every pair of its vertices.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

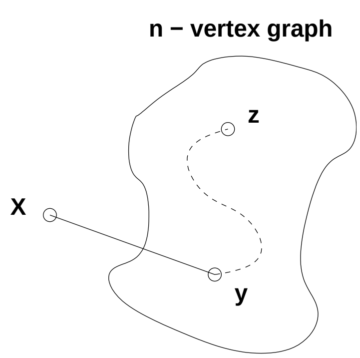
1. Prove that this Claim is indeed false by providing a counterexample.
2. Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

   Proof. We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

   Base cases: $(n \leq 2)$. In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

   $P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

   Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an n-vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$-vertex graph:

   All that remains is to check that there is a path from x to every other vertex z. Since x has positive degree, there is an edge from x to some other vertex, y. Thus, we can obtain a path from x to z by going from x to y and then following the path from y to z. This proves $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim.

□

2 The Grow Algorithm

Yesterday in lecture, we saw the following algorithm for constructing a minimum-weight spanning tree (MST) from an edge-weighted N-vertex graph G.

ALG-GROW:

1. Label the edges of the graph $e_1, e_2, \ldots, e_t$ so that $wt(e_1) \leq wt(e_2) \ldots \leq wt(e_t)$.
2. Let S be the empty set.
3. For $i = 1 \ldots t$, if $S \cup \{e_i\}$ does not contain a cycle, then extend S with the edge e_i.
4. Output S.

2.1 Analysis of ALG-GROW

In this problem you may assume the following lemma from the problem set:

Lemma 1. *Suppose that* $T = (V, E)$ *is a simple, connected graph. Then* T *is a tree iff* $|E| = |V| - 1$.

In this exercise you will prove the following theorem.

Theorem. *For any connected, weighted graph* G*, ALG-GROW produces an MST of* G*.*

(a) Prove the following lemma.

Lemma 2. *Let* $T = (V, E)$ *be a tree and let* e *be an edge not in* E*. Then,* $G = (V, E \cup \{e\})$ *contains a cycle.*

(Hint: Suppose G does *not* contain a cycle. Is G a tree?)

(b) Prove the following lemma.

Lemma 3. *Let $T = (V, E)$ be a spanning tree of G and let e be an edge not in E. Then there exists an edge $e' \neq e$ in E such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G.*

(Hint: Adding e to E introduces a cycle in $(V, E \cup \{e\})$.)

(c) Prove the following lemma.

Lemma 4. *Let $T = (V, E)$ be a spanning tree of G, let e be an edge not in E and let $S \subseteq E$ such that $S \cup \{e\}$ does not contain a cycle. Then there exists an edge $e' \neq e$ in $E - S$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of G.*

(Hint: Modify your proof to part (b). Of all possible edges $e' \neq e$ that can be removed to construct T^*, at least one is not in S.)

(d) Prove the following lemma.

Lemma 5. *Define S_m to be the set consisting of the first m edges selected by ALG-GROW from a connected graph G. Let $P(m)$ be the predicate that if $m \leq |V|$ then $S_m \subseteq E$ for some MST $T = (V, E)$ of G. Then $\forall m \,.\, P(m)$.*

(Hint: Use induction. There are two cases: $m + 1 > |V|$ and $m + 1 \leq |V|$. In the second case, there are two subcases.)

(e) Prove the theorem. (Hint: Lemma 5 says there exists an MST $T = (V, E)$ for G such that $S \subseteq E$. Use contradiction to rule out the case in which S is a proper subset of E.)