Problem Set 8

Problem 1. [25 points] Find Θ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(a) [5 pts] $T(n) = 8T(\lfloor n/2 \rfloor) + n$

(b) [5 pts] $T(n) = 2T(\lfloor n/8 \rfloor + 1/n) + n$

(c) [5 pts] $T(n) = 7T(\lfloor n/20 \rfloor) + 2T(\lfloor n/8 \rfloor) + n$

(d) [5 pts] $T(n) = 2T(\lfloor n/4 \rfloor + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T(\lfloor n/9 + n^{1/9} \rfloor) + 1$

Problem 2. [30 points] It is easy to misuse induction when working with asymptotic notation.

False Claim If

$$T(1) = 1 \text{ and}$$
$$T(n) = 4T(n/2) + n$$

Then T(n) = O(n).

False Proof We show this by induction. Let $P(n)$ be the proposition that $T(n) = O(n)$.

Base Case: $P(1)$ is true because $T(1) = 1 = O(1)$.

Inductive Case: For $n \geq 1$, assume that $P(n-1), \ldots, P(1)$ are true. We then have that

$$T(n) = 4T(n/2) + n = 4O(n/2) + n = O(n)$$

And we are done.

(a) [5 pts] Identify the flaw in the above proof.

(b) [10 pts] A simple attempt to prove $T(n) \neq O(n)$ via induction ultimately fails. We assume for sake of contradiction that $T(n) = O(n)$. Then there exists positive integer n_0 and positive real number c such that for all $n \geq n_0$, $T(n) \leq cn$. We then define $P(n)$ as the proposition that $T(n) \leq cn$.

We then proceed with strong induction.

Base Case, $n = n_0$: $P(n_0)$ is true, by assumption.

Inductive Step: We assume $P(n_0)$, $P(n_0 + 1)$, ..., $P(n - 1)$ true.

Fill in the rest of this proof attempt, and explain why it doesn't work.

Note: As this problem was updated so late, the graders will be instructed to be exceedingly lenient when grading this.

(c) [5 pts] Using Akra-Bazzi theorem, find the correct asymptotic behavior of this recurrence.

(d) [10 pts] We have now seen several recurrences of the form $T(n) = aT(\lfloor n/b \rfloor) + n$. Some of them give a runtime that is $O(n)$, and some don't. Find the relationship between a and b that yields $T(n) = O(n)$, and prove that this is sufficient.

Problem 3. [15 points] Define the sequence of numbers A_i by

$A_0 = 2$

$A_{n+1} = A_n/2 + 1/A_n$ (for $n \geq 1$)

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$.

Problem 4. [30 points] Find closed-form solutions to the following linear recurrences.

(a) [15 pts] $x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3} \quad (x_0 = 3, x_1 = 4, x_2 = 14)$

(b) [15 pts] $x_n = -x_{n-1} + 2x_{n-2} + n \quad (x_0 = 5, x_1 = -4/9)$