System of linear eq^n – lecture 10

Recall:– Let A be an $m \times n$ matrix. If $B = EA$

Then row-space (A) = row-space (B). Hence

row-rank (A) = row-rank (B)

Theorem:– If A and B are two row equivalent matrix. Then row-space(A) = row-space (B)

and hence row-rank (A) = row-rank(B).

Proof:– Follows from the above result.

Remark :→ ① If R is row reduced echelon form of the matrix A. Then

$$\text{row-rank}(A) = \text{row-rank}(R)$$

② Note that row-rank of a row reduced echelon matrix is equal to the no of non zero rows.

③ If we want to find the dim and a basis of row-space, then we reduce A to it's row-reduced form.

Example (1)

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow R_1 \\ \\ \\ \rightarrow R_4 \end{array}$$

Sol^n :→

$$A \simeq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \end{pmatrix}$$

$$\simeq \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{pmatrix} R_2 \rightarrow \left(-\frac{1}{2}\right) R_4 \\ R_3 \rightarrow (-1) R_3 \\ R_4 \rightarrow R_2 \left(-\frac{1}{2}\right) \end{pmatrix}$$

$$\simeq \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (R_4 \to R_4 - R_3)$$

$$\simeq \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (R_1 \to R_1 - R_2$$

$$\simeq \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (R_1 \to R_1 - R_3)$$

row-rank$(A) = 3$

and a basis is given by

$$\mathcal{B} = \{ (1,0,0,1), (0,1,0,0), (0,0,1,0) \}$$

<u>Column space</u> :→ $A = [c_1, c_2, \ldots, c_n]$ and

$$c_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, \text{ then } (a_{1i}, a_{2i}, \ldots a_{mi}) \in \mathbb{F}^m$$

$$\text{column space}(A) = L(c_1, c_2, \ldots, c_n) \subseteq \mathbb{F}^m$$

Theorem: let A be an $m \times n$ matrix and let $B = AE$ for some elementary matrix E. Then column-space (A) = column space (B). Further column-rank (A) = column-rank (B).

If A and B are column equivalent then column-space (A) = column space (B).

Theorem: Row-rank(A) = Column-rank(A)

Theorem :- Let A be an $m \times n$ matrix and $AX = 0$ be the system of hom eq^n. Then the dim of sol^n space of $AX = 0$ is $(n - r)$, where r is the row-rank of the matrix A.

Proof :- Suppose R is the row reduced echelon form of the matrix A. Then

$$\text{row-rank}(A) = \text{row-rank}(R)$$

$$\Rightarrow \quad r = \text{No of non zero rows.}$$

Hence R has $(n-r)$ zero rows.

Also we know that

$$\{X \mid AX = 0\} = \{X \mid RX = 0\}$$

Suppose for $1 \le i \le r$, the leading non zero coefficient of i^{th} row is in k_ith coloum. Then

$$k_1 < k_2 < k_3 \cdots < k_r$$

Let us assume that $x_{k_1} \cdots x_{k_r}$ are corresponding variables.

Now remaining $(n-r)$ variables are free variables

$$S = \{1, 2, \ldots n\} - \{k_1, k_2, \ldots k_r\}$$

x_j is free $\forall\ j \in S$.

We are free to choose any value for x_j's and once we have the value of x_j's $\forall j \in S$, by back substitution we can find the value of $x_{k_r}, x_{k_{r-1}} \ldots x_{k_1}$

In order to get a basis of sol^n space

choose $\boxed{x_j = s_{ij}}$ $\forall\, j \in S$.

$x_{t_1} \ldots x_{t_{n-r}}$ in S $\rightarrow$ free variable

$x_{t_1} = 1$ all other zero $\rightarrow (1, 0 \ldots, 0)$ — V_1

$x_{t_2} = 1$ and all other zero $\rightarrow (0, 1 \ldots 0)$ — V_2

$\vdots$

$x_{t_{n-r}} = 1$ " " " $\rightarrow (0, \ldots 1)$ — V_{n-r}

Then with the help of these choices we get a sol^n set

$$\overline{X} = \{ v_j \mid j \in S \}$$

v_j is obtained by taking $x_j = 1$ and all other variables (free) to be zero.

Clearly $|\overline{X}| = n - r$

Note that $\overline{X}$ is a L.I set. Further any other sol^n of $AX = 0$ will be written as

$$X = \sum_{j \in S} \lambda_j v_j \qquad \lambda_j \in \mathbb{F}.$$

Example $\Rightarrow$ Find a basis for the soln space of $AX=0$, where $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \end{bmatrix}$.

Soln :- Step (1)

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

row-rank(A) = row-rank(R) $= 2$

x_1 and x_3 are leading variables.

and x_2, x_4 are free variables.

$$x_2 = \lambda_2 \quad \text{and} \quad x_4 = \lambda_4$$

System $RX = 0$ gives us

$$x_1 + 2x_2 + x_4 = 0 \quad \& \quad x_3 + 3x_4 = 0$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$x_3 = -3\lambda_4$$

$$x_1 = -2\lambda_2 - \lambda_4$$

$$v = \left(-2\lambda_2 - \lambda_4,\ \lambda_2,\ -3\lambda_4,\ \lambda_4\right) \text{ ——— } (*)$$

In order to get a basis of sol^n space

take $x_2 = 1$ and $x_4 = 0 \Rightarrow x_2 = 1$ & $x_4 = 0$

$$v_2 = (-2, 1, 0, 0)$$

take $x_2 = 0$ and $x_4 = 1 \Rightarrow x_2 = 0$ & $x_4 = 1$

$$v_4 = (-1, 0, -3, 1)$$

$\Rightarrow \{v_2, v_4\}$ two linearly independent vectors in sol^n space and they will form a basis of sol^n space.

$$v = (-2x_2 - x_4,\ x_2,\ -3x_4,\ x_4)$$

$$= x_2(-2, 1, 0, 0) + x_4(-1, 0, -3, 1)$$

$$v = x_2 v_2 + x_4 v_4$$

dim of sol^n space $= 2 = 4 -$ rank of (A)