

COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 28

QUIZ 3

MAR 20, 2024

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PROBLEM 1

Problem 1 [16 points]

Let $G = (V, E)$ be a simple undirected graph such that $|V| \geq 3$. Prove or disprove:

If every vertex of G has degree at least $|V|/2$, then G has a Hamiltonian cycle.

PROBLEM 1

Proof: (by a "path extension" argument)

Claim 1 : G is connected

Proof of Claim 1 : If G is not connected , then \exists vertex sets V_0 and V_1 s.t. $V_0 \cup V_1 = V$, $V_0 \cap V_1 = \emptyset$ and no vertex in V_0 is connected to a vertex in V_1 .

If $|V_0| \leq \frac{n}{2}$, then $\deg(v) \geq \frac{n}{2}$ for $v \in V_0$ is not possible.

Otherwise $|V_1| \leq \frac{n}{2}$, $\overbrace{\hspace{10em}}^{\forall v \in V_1} \overbrace{\hspace{10em}}^{\square}$.

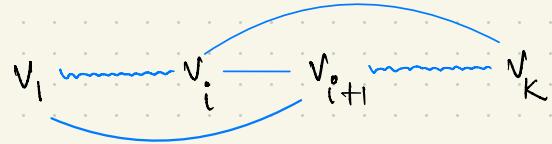
PROBLEM 1

Let $P = v_1, v_2, \dots, v_k$ be the longest path in G .

Then, all neighbors of v_1 and v_k must belong to P
(otherwise P can be made longer)

We will show that Hamiltonian cycle can be derived from P .

PROBLEM 1



Claim 2: $\exists i \in \{1, 2, \dots, k\}$ s.t. $\{v_i, v_k\}$ and $\{v_1, v_{i+1}\}$ are edges.

Proof of Claim 2: (by contradiction)

vertices in P that are adjacent to v_1 = $\deg(v_1)$

vertices in P that are immediately to
the left of a vertex adjacent to v_1 = $\deg(v_1)$

None of these vertices should be adjacent to v_k .

\Rightarrow # vertices in $P \geq \deg(v_1) + \deg(v_k) + 1$ $\leftarrow v_k \text{ itself}$
 $\geq \frac{n}{2} + \frac{n}{2} + 1 \geq n$

PROBLEM 1

Claim 3 : The cycle $C = (v_1, \dots, v_i, v_k, v_{k+1}, \dots, v_{i+1}, v_1)$ is Hamiltonian.

Proof of Claim 3: (by contradiction)

Suppose $G \setminus C$ is nonempty

G is connected (Claim 1) $\Rightarrow \exists v \in G \setminus C$ adjacent to some v_j .



Then, the path from v to v_j along with the path around C is longer than P . ◻

Claim 3 proves the theorem. ◻

PROBLEM 1 [16 points]

Mentioning the proof technique _____ [1 pt]

Proving that G is connected _____ [3 pts]

Defining longest path P _____ [1 pt]

Deriving cyclicity condition from P _____ [8 pts]

Demonstrating Hamiltonian cycle _____ [3 pts]

PROBLEM 2

Problem 2 [4x2=8 points]

For each of the following, draw an example of the object described or explain why such an object cannot exist.

- (a) A simple undirected graph where no two vertices have the same degree.
- (b) A simple undirected graph that has an Euler tour but no Hamiltonian cycle.
- (c) A simple undirected graph that has a Hamiltonian cycle but no Euler tour.
- (d) A simple undirected graph $G = (V, E)$ where $|V| \geq 3$ and every vertex has degree at least $\lfloor |V|/2 \rfloor$ such that G does not have a Hamiltonian cycle.

PROBLEM 2

(a) Thm: In any simple graph, at least two vertices must have the same degree.

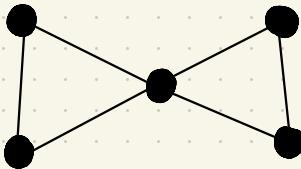
Proof : (by Pigeonhole principle)

Degree can take values in $\{0, 1, 2, \dots, n-1\}$.

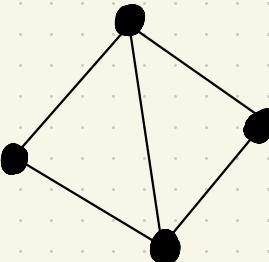
- * If $k \geq 2$ vertices have degree 0, we are done.
- * If $k=1$ vertices have degree 0, then the remaining $n-1$ vertices have degree in $\{1, 2, \dots, n-2, \cancel{n-1}\}$.
- * If $k=0$ vertices have degree 0, then the remaining n vertices have degree in $\{1, 2, \dots, n-2, n-1\}$.

PROBLEM 2

(b) Simple graph with Euler tour but no Hamiltonian cycle.



(c) Simple graph with Hamiltonian cycle but no Euler tour.



PROBLEM 2

(d) Simple graph $G = (V, E)$ with $|V| \geq 3$ and each vertex has degree at least $\lfloor |V|/2 \rfloor$ but no Hamiltonian cycle.



PROBLEM 2 [8 points]

Each part worth 2 points

Binary grading.