MAXIMUM MARKS: 30

Instructions: Justify all your statements. Remember that you will be graded on what you write on the answer sheet, **NOT** on what you intend to write.

Question 1: [4 marks]
For $\lambda, \mu \in \mathbb{R}$, consider the system of linear equations $AX = B$ with coefficients from $\mathbb{R}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & \lambda \\ 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 6 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, B = \begin{pmatrix} 3 \\ 5 \\ \mu \\ \mu + 3 \end{pmatrix}.$$

(a) Using row reduced echelon form of a matrix, find the values of λ and μ such that the system is consistent (i.e. the system has at least one solution).
(b) Write down all the solutions whenever the system is consistent.

Question 2: [2+2 marks]
Let $M_2(\mathbb{C})$ be the set of all 2×2 matrices with entries from $\mathbb{C}$. Observe that $M_2(\mathbb{C})$ is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$ (don't prove this). Let $\bar{d}$ denote the complex conjugate of $d \in \mathbb{C}$. Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : a + \bar{d} = 0 \right\} \subset M_2(\mathbb{C}).$$

Consider the vector spaces $V_1 = M_2(\mathbb{C})$ over $\mathbb{R}$ and $V_2 = M_2(\mathbb{C})$ over $\mathbb{C}$.
(a) Is W a subspace of V_1? Justify your answer.
(b) Is W a subspace of V_2? Justify your answer.

Queston 3: [3+1 marks]
Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z, w) = (y + 2z + 3w, x - y, x + 2z + 3w, 3x - y + 4z + 6w).$$

(a) Find a basis of the range of T.
(b) Find the nullity of T.

Question 4: [2+2 marks]

(a) Justify if there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the following:

$$T(1, 1, 0) = (3, 5),\ T(1, 0, 1) = (5, 3),\ T(2, 1, 1) = (4, 4).$$

If it exists, define one such T.
(b) Justify if there exists a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the following:

$$S(1, 0, 0) = (1, 2),\ S(0, 1, 0) = (7, 5),\ S(1, 1, 0) = (8, 7).$$

If it exists, define one such S.

Question 5: [2+2+2 marks]
Let $V = \{a_0 + a_1X + a_2X^2 + a_3X^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree less than or equal to 3 over $\mathbb{R}$. Consider the subspaces W_1 and W_2 of V defined as follows:

$$W_1 = \{a_0 + a_1X + a_2X^2 + a_3X^3 : a_0 + a_1 + a_2 + a_3 = 0, a_1 + 2a_2 + 3a_3 = 0\},$$
$$W_2 = \{a_0 + a_1X + a_2X^2 + a_3X^3 : a_0 + 2a_1 + 3a_2 + 4a_3 = 0, a_2 + 3a_3 = 0\}.$$

(a) Find $\dim(W_1 \cap W_2)$.
(b) Justify if $V = W_1 + W_2$ or not.
(c) Find a basis of $W_1 + W_2$.

Question 6: [3 marks]
Consider the vector space $\mathbb{R}^4$ over $\mathbb{R}$ and a linearly independent subset

$$S = \{(1, 2, 3, 4), (0, 1, 2, 3)\} \subset \mathbb{R}^4.$$

Extend S to a basis of $\mathbb{R}^4$. Justify your answer.

Question 7: [1+2+2 marks]
Consider the following two ordered bases of the vector space $\mathbb{R}^3$ over $\mathbb{R}$

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \text{ and } B' = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}.$$

Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x + 2y + 3z, y + 2z, -x - z).$$

(a) Find the matrix $[T]_B$ of T with respect to B.
(b) Find the matrix $[T]_{B'}$ of T with respect to B'.
(c) Find a matrix P such that $[T]_B' = P^{-1}[T]_B P$.