Show that the set

$$S = \{1, 1 - x, 3 + 4x + x^2\}$$

is a basis of the vector space P_2 of all polynomials of degree 2 or less.

$\{1, x, x^2\}$

$\dim(P_2) = 3$

$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0$

$\Rightarrow c_1 + c_2 x + c_3 x^2 = 0_{poly}$

$\Rightarrow c_1 = c_2 = c_3 = 0$

$f(x) = a_0 + a_1 x + a_2 x^2 \in \mathrm{Span}(\{1, x, x^2\})$

$c_1 + c_2(1-x) + c_3(3 + 4x + x^2) = 0$

$\Rightarrow (c_1 + c_2 + 3c_3) + (-c_2 + 4c_3)x + c_3 x^2 = 0$

Since this is a zero polynomial, all coefficient must be zero

$\Rightarrow c_1 + c_2 + 3c_3 = 0$

$-c_2 + 4c_3 = 0$

$c_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0$

Let P_3 denote the set of polynomials of degree 3 or less with real coefficients. Consider the ordered basis

$$B = \{1 + x, 1 + x^2, x - x^2 + 2x^3, 1 - x - x^2\}.$$

Write the coordinate vector for the polynomial $f(x) = -3 + 2x^3$ in terms of the basis B.

Let $[f(x)]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$

$\Rightarrow -3 + 2x^3 = c_1(1+x) + c_2(1+x^2) + c_3(x - x^2 + 2x^3) + c_4(1 - x - x^2)$

$\Rightarrow c_1 + c_2 + c_4 = -3$

$c_1 + c_3 - c_4 = 0$

$c_2 - c_3 - c_4 = 0$

$2c_3 - c_4 = 2$

We can now create the augmented matrix which represents this equation:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -3 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right].$$

Now the equation is solved by reducing the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -3 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -3 \\ 0 & -1 & 1 & -2 & 3 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2]{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right] \xrightarrow[\frac{1}{2}R_4]{\left(\frac{-1}{3}\right)R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_2 - 2R_4]{R_1 + R_4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right].$$

Let $p_1(x), p_2(x), p_3(x), p_4(x)$ be (real) polynomials of degree at most 3. Which (if any) of the following two conditions is sufficient for the conclusion that these polynomials are linearly dependent?

(a) At 1 each of the polynomials has the value 0. Namely $p_i(1) = 0$ for $i = 1, 2, 3, 4$.

(b) At 0 each of the polynomials has the value 1. Namely $p_i(0) = 1$ for $i = 1, 2, 3, 4$.

$P_1(1) = 0 = P_2(1) = P_3(1)$
$= P_4(1)$

$P_1(0) = P_2(0) = P_3(0) = P_4(0) = 1$

$W = \{ P(x) : P(1) = 0 \}$

$f(x) = 1 \notin W$ $\quad W \subsetneq V$

$\dim(W) \leq 3$

$W = \{ P(x) : P(0) = 1 \}$ is not a subspace

$0 \notin W$

$1, \ 1+x, \ 1+x^2, \ 1+x^3$

We show that the condition (b) is not sufficient.

In fact, the following four vectors satisfy the condition (b) but they are linearly independent:

$1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3$.

(Observe that the method of the proof of (a) does not work because

$$V = \{p(x) \in P_3 \mid p(0) = 1\}$$

is not a subspace.)

Let A and B be $m \times n$ matrices.

Prove that

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

Let

$$A = [\mathbf{a}_1, \dots, \mathbf{a}_n] \text{ and } B = [\mathbf{b}_1, \dots, \mathbf{b}_n],$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are column vectors of A and B, respectively.

Then the rank of the matrix A is the dimension of the column space of A.

That is, we have

$$\operatorname{rank}(A) = \dim(\operatorname{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)).$$

Similarly, we have

$$\operatorname{rank}(B) = \dim(\operatorname{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n))$$

and

$$\operatorname{rank}(A+B) = \dim((\operatorname{Span}(\mathbf{a}_1 + \mathbf{b}_1, \dots, \mathbf{a}_n + \mathbf{b}_n))$$

since $A + B = [\mathbf{a}_1 + \mathbf{b}_1, \dots, \mathbf{a}_n + \mathbf{b}_n]$.

We claim that

$$\operatorname{Span}(\mathbf{a}_1 + \mathbf{b}_1, \dots, \mathbf{a}_n + \mathbf{b}_n) \subset \operatorname{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n) + \operatorname{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n).$$

Any vector $\mathbf{x} \in \operatorname{Span}(\mathbf{a}_1 + \mathbf{b}_1, \dots, \mathbf{a}_n + \mathbf{b}_n)$ can be written as

$$\mathbf{x} = r_1(\mathbf{a}_1 + \mathbf{b}_1) + \cdots + r_n(\mathbf{a}_n + \mathbf{b}_n)$$

Thus we have

$$\begin{aligned}\mathbf{x} &= r_1(\mathbf{a}_1 + \mathbf{b}_1) + \cdots + r_n(\mathbf{a}_n + \mathbf{b}_n) \\ &= (r_1\mathbf{a}_1 + \cdots + r_n\mathbf{a}_n) + (r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n) \\ &\in \operatorname{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_n) + \operatorname{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n),\end{aligned}$$

and hence the claim is proved.

Then we have

$$\begin{aligned}&\operatorname{rank}(A+B) \\ &= \dim(\,(\operatorname{Span}(\mathbf{a}_1 + \mathbf{b}_1, \ldots, \mathbf{a}_n + \mathbf{b}_n)\,) \\ &\leq \dim(\,\operatorname{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_n) + \operatorname{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n)\,) \\ &\leq \dim(\,\operatorname{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_n)\,) + \dim(\,\operatorname{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n)\,) \\ &= \operatorname{rank}(A) + \operatorname{rank}(B).\end{aligned}$$

Here we used the fact that for two subspaces U and V of a vector space, we have

$$\dim(U+V) \leq \dim(U) + \dim(V).$$

$$\begin{aligned}\dim(U+V) &= \dim(U) + \dim V - \dim(U \cap V) \\ &\leq \dim(U) + \dim V.\end{aligned}$$

Let V be a vector space over a field K. Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be linearly independent vectors in V. Let U be the subspace of V spanned by these vectors, that is, $U = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$.

Let $\mathbf{u}_{n+1} \in V$. Show that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{u}_{n+1}$ are linearly independent if and only if $\mathbf{u}_{n+1} \notin U$.

Suppose that the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{u}_{n+1}$ are linearly independent. If $\mathbf{u}_{n+1} \in U$, then $\mathbf{u}_{n+1}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Thus, we have

$$\mathbf{u}_{n+1} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$$

for some scalars $c_1, c_2, \dots, c_n \in K$.

However, this implies that we have a nontrivial linear combination

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n - \mathbf{u}_{n+1} = \mathbf{0}.$$

This contradicts that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{u}_{n+1}$ are linearly independent. Hence $\mathbf{u}_{n+1} \notin U$.

$(\Longleftarrow)$ Suppose now that $\mathbf{u}_{n+1} \notin U$.

If the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{u}_{n+1}$ are linearly dependent, then there exists $c_1, c_2 \dots, c_n, c_{n+1} \in K$ such that not all of them are zero and

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n + c_{n+1}\mathbf{u}_{n+1} = \mathbf{0}.$$

We claim that $c_{n+1} \neq 0$. If $c_{n+1} = 0$, then we have

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$$

and since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are linearly independent, we must have $c_1 = c_2 = \cdots = c_n = 0$. This means that all c_i are zero but this contradicts our choice of c_i. Thus $c_{n+1} \neq 0$.

Suppose $v_1, v_2, \dots, v_m$ are linearly independent in V and $w \in V$. If $v_1 + w, \dots, v_m + w$ are linearly dependent then $w \in \operatorname{span}(v_1, \dots, v_m)$

Proof :- $\exists\ c_1, c_2, \dots, c_m$ not all zero s.t.

$$c_1(v_1 + w) + c_2(v_2 + w) + \cdots + c_m(v_m + w) = 0$$

$$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = -(c_1 + c_2 + \cdots + c_m)\, w$$

If $c_1 + \cdots + c_m = 0$, $c_1 v_1 + \cdots + c_m v_m = 0$

not all c_i are zero.

This contradicts that $\{v_1, \dots, v_m\}$ is L.I

$$\Rightarrow c_1 + c_2 + \cdots + c_m \neq 0$$

$$\Rightarrow w = -\frac{c_1}{c_1 + \cdots + c_m} v_1 - \cdots - \frac{c_m}{c_1 + \cdots + c_m} v_m \in \operatorname{span}(\{v_1, \dots, v_m\})$$

$$\operatorname{span}(v_1, \dots, v_n) \overset{?}{=} \operatorname{span}(w_1, \dots, w_m)$$

$$v_i \in \operatorname{span}(w_1, \dots, w_m) \text{ for all } i = 1, 2, \dots, n$$

$$\Rightarrow \{v_1, \dots, v_n\} \subseteq \operatorname{span}(w_1, \dots, w_m)$$

$$\Rightarrow \operatorname{span}(v_1, \dots, v_n) \subseteq \operatorname{span}(w_1, \dots, w_m)$$

$$w_j \in \operatorname{span}(v_1, \dots, v_n) \text{ for all } j = 1, \dots, m$$

$\text{rank}(BA) \leq \text{rank}(A)$

$A_{m \times n} \quad B_{l \times m}$

$\dim V = n - r_A$

$\dim W = n - r_{BA}$

$\Rightarrow n - r_A \leq n - r_{BA}$

$\Rightarrow r_{BA} \leq r_A$

Solution space of $(BA)X = 0$

$X \in \mathbb{R}^n \qquad AX = 0$

Let V be the solution space of $AX = 0$

Let W be the solution space of $(BA)X = 0$

$V, W \subseteq \mathbb{R}^n$

Subspaces of $\mathbb{R}^n$

$V \subseteq W$

Let $v \in V$

$Av = 0$

$\Rightarrow B(Av) = 0$

$\Rightarrow (BA)v = 0 \Rightarrow v \in W$

$AX = b$ $\quad A_{m\times n}$ $\quad \underline{F = \mathbb{R}}$

If $b \neq 0$ then the solution space is NOT a subspace of $\mathbb{R}^n$.

$$\underline{a x_1 + b x_2}, \quad a + b = 1$$

If x_1 & x_2 are two solutions then $a x_1 + b x_2$, $a+b=1$ is also a solution.

* Let $\underline{X_{hom}}$ be a solution of $AX = 0$

Let X_{nonh} be a solution of $AX = b$

Then $\underline{X = X_{hom} + X_{nonh}}$ is also a solution of $AX = b$

$$
\begin{aligned}
AX &= AX_{hom} + AX_{nonh} \\
&= 0 + b \\
&= b
\end{aligned}
$$