Question - 1

- Using RRE determine values of a and b for which A is invertible for $b=-1$ find the inverse.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & b & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_4 \to R_4 - 2R_1$, $R_3 \to R_3 - aR_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & -2 & b & -2 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \to R_1 - R_3$
$R_4 \to R_4 + 2R_3$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & a & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 0 & b+2 & -2 & -2a & 2 & 1 \end{array}\right)$$

From above we can conclude that A is invertible for $a \in \mathbb{R}$ & $b \in \mathbb{R} - \{-2\}$.

To compute inverse for $b=-1$.

Consider, $b=-1$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & a & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -2a & 2 & 1 \end{array}\right)$$

$R_1 \to R_1 + R_4$
$R_3 \to R_3 - R_4$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -a & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & a & -1 & -1 \\ 0 & 0 & 0 & 1 & -2 & -2a & 2 & 1 \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} -1 & -a & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & a & -1 & -1 \\ -2 & -2a & 2 & 1 \end{pmatrix}$

Q2.

$$x + y + z = 3$$
$$2x + 5y + 4z = a$$
$$3x + (a^2-8)z = 12$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 5 & 4 & a \\ 3 & 0 & a^2-8 & 12 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 3 & 2 & a-6 \\ 0 & -3 & a^2-11 & 3 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 3 & 2 & a-6 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right]$$

(i) for NO solution

$\text{Rank}(A) < \text{Rank}(A|b)$

$\therefore \quad a^2 - 9 = 0$ and $a - 3 \neq 0$

$\Rightarrow \quad a = \pm 3$ and $a \neq 3$

$\therefore \quad a = -3$

(ii) for Unique solution

$\text{Rank}(A) = 3$

$\therefore \quad a^2 - 9 \neq 0$

$\Rightarrow \quad a \neq 3, -3$

(iii) for Infinitely many solutions

$$\text{Rank}(A) = \text{Rank}(A \mid b) < 3$$

$\therefore \quad a^2 - 9 = 0 \quad \text{and} \quad a - 3 = 0$

$\Rightarrow \quad a = \pm 3 \quad \text{and} \quad a = 3$

$\therefore \quad a = 3$

Solution 3 :-

(a) Let $\alpha, \beta, \gamma \in \mathbb{F}$ then we need to prove that

$$\alpha(v_1 + v_3) + \beta(3v_1 + 2v_2 + v_3) + \gamma(2v_1 + 3v_2 + v_3) = 0 \quad \text{—— (I)}$$

$$\Rightarrow \quad \alpha, \beta, \gamma = 0.$$

On solving (I)

$$\alpha v_1 + \alpha v_3 + 3\beta v_1 + 2\beta v_2 + \beta v_3 + 2\gamma v_1 + 3\gamma v_2 + \gamma v_3 = 0$$

$$\Rightarrow (\alpha + 3\beta + 2\gamma) v_1 + (2\beta + 3\gamma) v_2 + (\alpha + \beta + \gamma) v_3 = 0$$

Since $\{v_1, v_2, v_3\}$ is L.I., the coefficients of v_1, v_2, v_3 in above equations has to be zero.

Thus, we get the system of linear equations on equating the coefficients to zero as :

$$\alpha + 3\beta + 2\gamma = 0 \quad \text{—— (i)}$$

$$2\beta + 3\gamma = 0 \quad \text{—— (ii)}$$

$$\alpha + \beta + \gamma = 0 \quad \text{—— (iii)}$$

(i) – (iii), $\quad 2\beta + \gamma = 0 \quad$ —— (iv)

(ii) – (iv), $\quad 2\gamma = 0 \Rightarrow \gamma = 0$

From (iv), $\quad 2\beta + \gamma = 0 \Rightarrow 2\beta + 0 = 0 \Rightarrow \beta = 0$

From (i), $\quad \alpha + 3\beta + 2\gamma = 0 \Rightarrow \alpha + 0 + 0 = 0 \Rightarrow \alpha = 0$

$\therefore \alpha, \beta, \gamma = 0$

$\Rightarrow \{v_1 + v_3, 3v_1 + 2v_2 + v_3, 2v_1 + 3v_2 + v_3\}$ is L.I.

(b) Let $v = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1\right)$.

$v \in W$ since $v_4 = 1 \in \mathbb{Z}$

but for $\alpha = 1/2$, $\alpha v \notin W$ since

$$\alpha v = \left(\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{2}\right)$$

$$\alpha v_4 = \frac{1}{2} \notin \mathbb{Z}$$

$\therefore$ W is not a subspace since scalar multiplication doesn't hold.