## Question-1:

Soln: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$. $\quad A\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \underline{0}$ $\quad A\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \underline{0}$

$\Rightarrow \begin{bmatrix} a_{11}\\ a_{12}\\ a_{13}\\ a_{14}\end{bmatrix}, \begin{bmatrix} a_{21}\\ a_{22}\\ a_{23}\\ a_{24}\end{bmatrix},$ are solutions to

$$x+2y+3z+4w=0.$$
$$y+z+w=0.$$

1M

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ in reduced echelon form

$$\Rightarrow x + z + 2w = 0.$$
$$y + z + w = 0.$$

$z = s \quad w = t \Rightarrow y = -s-t.$
$x = -s-2t.$

1M

$\Rightarrow$ solution is $\begin{bmatrix} -s-2t \\ -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$.

$\Rightarrow A$ is $\begin{bmatrix} -1 & -1 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}$.

# MTL 101, RE-QUIZ, PROBLEM 2

2. Let $W$ be a subspace of trace zero matrices in a vector space $M_3(\mathbb{R})$ over $\mathbb{R}$.
(a) Use the rank-nullity theorem to calculcate the dimension of $W$.
Let $T : M_3(\mathbb{R}) \to \mathbb{R}$ given by $T(M) = \text{trace}(M)$.

$$T(M) = T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \text{trace}(M) = a_{11} + a_{22} + a_{33}.$$

Notice that $T$ is a linear transformation as $T(aM + bN) = \text{trace}(aM + bN) = a\text{trace}(M) + b\text{trace}(N) = aT(M) + bT(N)$.

Also, observe that the subspace $W$ is the nullspace of $T$.

By the rank-Nullity theorem, we have $\text{rank}(T) + \text{nullity}(T) = \dim M_2(\mathbb{R}) = 9$.

$\text{rank}(T)$ cannot be zero, as $T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{trace}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 1$. Since $T(M_3(\mathbb{R}) \subseteq \mathbb{R}$, we have $\text{rank}(T) = 1$.

Thus $\dim W = \text{nullity} T = \dim M_2(\mathbb{R}) - \text{rank} T = 9 - 1 = 8$.

(b) Write a basis for $W$ (no need to justify)

$W$ is the set of all matrices of the form $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, where $a_{11} + a_{22} + a_{33} = 0$, or $a_{33} = -a_{11} - a_{22}$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Ques 3 of re-quiz

Given IVP is $\frac{dy}{dx} = f(x,y)$ ; $y(0) = 0$

where $f(x,y) = x\sqrt{|y|}$

(a) $f(x,y) = x\sqrt{|y|}$ is continuous on $\mathbb{R}^2$ and hence in any rectangle containing $(x_0,y_0) = (0,0)$ [1]
So, the existence theorem is applicable and it ensures the existence of at least one solution to the IVP.

(b) For $y_1 \neq y_2$ and $x \neq 0$,

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \frac{|x|\,\left|\sqrt{|y_1|} - \sqrt{|y_2|}\right|}{|y_1 - y_2|}$$

$$= \frac{|x|\,\big|\,|y_1| - |y_2|\,\big|}{|y_1 - y_2|\left(\sqrt{|y_1|} + \sqrt{|y_2|}\right)}$$

$$= \frac{|x|}{\sqrt{y_1} + \sqrt{y_2}} \quad \text{for any } y_1>0, y_2>0, y_1\neq y_2$$

$$\to \infty \quad \text{as } y_1 \to 0+, y_2 \to 0+$$

Thus, $f(x,y)$ does not satisfy the Lipschitz condition on any rectangle containing $(x_0,y_0) = (0,0)$. [1]
$\therefore$ The uniqueness theorem is not applicable

(c) $\boxed{y \equiv 0}$ is clearly a solution to the IVP. [1]
To find another solution, assume $y>0$. Then

$$\int \frac{dy}{\sqrt{y}} = \int x\,dx \Rightarrow 2\sqrt{y} = \frac{x^2}{2} + c$$

$$y(0) = 0 \Rightarrow c = 0$$

$$\therefore \quad y = \frac{x^4}{16}$$

We can easily verify that $\boxed{y = \frac{x^4}{16}}$ is a solution to the IVP. [1]

Remark: $y = -\frac{x^4}{16}$ is NOT a solution because $\frac{dy}{dx} = -\frac{x^3}{4}$ but $x\sqrt{|y|} = x\left(\frac{x^2}{4}\right) = \frac{x^3}{4}$