Q1 Find the general solution of differential equation using the variation of parameters

$$x y'' - (2x-1)y' + (x-1)y = e^x \quad — \quad (*)$$

Given that $y_1 = e^x$ is one of the fundamental solution of the associated homogeneous equation.

Sol^n We assume that the 2^{nd} linearly independent solution of the associated homogeneous equation is of the form

$$y_2(x) = v(x)\, y_1(x)$$

where $y_1(x) = e^x$ and $v(x)$ is some unknown f^n.

$$y_2'(x) = v'(x) e^x + v(x) e^x$$

$$y_2''(x) = v''(x) e^x + 2v' e^x + v e^x$$

Subsituting these values in homogeneous eq^n, we obtain

$$x\left(v'' e^x + 2v' e^x + v e^x\right) - (2x-1)\left(v' e^x + v e^x\right)$$

$$+ (x-1)(v e^x) = 0$$

which is equivalent to

$$x v'' + v' = 0$$

Let $v' = u \Rightarrow x u' + u = 0$

$$\Rightarrow u(x) = \frac{c}{x} \qquad \left(\text{By Seperation of Variables}\right)$$

$$\Rightarrow \qquad v(x) = c\log(x) + c_1$$

Thus, $\quad y_2(x) = c_1 e^x + c e^x \log x$

WLOG, we can drop the constants

$$\Rightarrow \qquad \boxed{y_2(x) = e^x \log x}$$

Now, we convert the ODE $(*)$ into standard form, i.e.,

$$y'' - \frac{(2x-1)}{x} y' + \frac{(x-1)}{x} y = \frac{e^x}{x} \qquad \text{— } (**)$$

Moreover, the homogeneous solutions of equation $(**)$ are same as of equation $(*)$ provided $x=0$ is not in the domain.

Wronskian is given by

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & \log x \, e^x \\ e^x & \frac{1}{x} e^x + \log x \, e^x \end{vmatrix} = \frac{e^{2x}}{x} \quad \text{for } x \neq 0$$

By Variation of Parameters, particular solution y_p is given by

$$y_p = u_1 y_1 + u_2 y_2$$

where

$$u_1(x) = \int \frac{W_1(y_1, y_2)(x)\, dx}{W(y_1, y_2)(x)}$$

and

$$u_2(x) = \int \frac{W_2(y_1, y_2)(x)}{W(y_1, y_2)(x)} \, dx$$

where $W_1(y_1, y_2)(x) = \begin{vmatrix} 0 & \log x \, e^x \\ e^x/x & \frac{1}{x} e^x + \log x \, e^x \end{vmatrix}$

$$= \frac{-e^{2x} \log x}{x}$$

and

$$W_2(y_1, y_2)(x) = \begin{vmatrix} e^x & 0 \\ e^x & e^x/x \end{vmatrix} = \frac{e^{2x}}{x}$$

Thus,

$$u_1(x) = \int \frac{W_1}{W} = \int \frac{-e^{2x} \log x \cdot x}{x \quad e^{2x}} = \int -\log(x)$$

$$= x(1 - \log x)$$

$$u_2(x) = \int \frac{W_2}{x} = \int \frac{e^{2x}/x}{e^{2x}/x} = x$$

Therefore

$$\boxed{y_p(x) = u_1 y_1 + u_2 y_2 = x e^x}$$

General solution y_g is given by

$$y_g = y_p + y_h$$

$$\boxed{y_g = x e^{x} + c_1 e^{x} + c_2 e^{x} \log x}$$

2. Find the values of $\alpha \in \mathbb{R}$, for which all the solutions of the O.D.E

$$y''(x) + 2y'(x) - \alpha y(x) = 0$$

goes to zero as $x \to \infty$.

Solution

0.5

Auxiliary / characteristic equation corresponding to the given equation :

$$m^2 + 2m - \alpha = 0 \quad \to (1)$$

Case I

$\alpha \geq 0$

0.5

For $\alpha = 0$, the roots of (1) are 0 and -2

hence $y(x) = C_1 + C_2 e^{-2x}$.

this y need not converge to zero as $x \to \infty$ for $C_1 \neq 0$

For $\alpha > 0$, (1) will have real and distinct roots given by

$$m = -1 \pm \sqrt{1+\alpha}$$

hence

$$y(x) = C_1 e^{(-1+\sqrt{1+\alpha})x} + C_2 e^{(-1-\sqrt{1+\alpha})x}$$

For $C_1 > 0$

$$C_1 e^{(-1+\sqrt{1+\alpha})x} \longrightarrow \infty \text{ as } x \to \infty$$

$[\because -1 + \sqrt{1+\alpha} > 0]$

Hence, for $\alpha \geq 0$,
all solutions need not converge to zero as $x \to \infty$.

Case II

$-1 < \alpha < 0$

1

In this case, (1) will have, real, distinct and negative roots, and the general solution will be

$$y(x) = C_1 e^{(-1+\sqrt{1+\alpha})x} + C_2 e^{(-1-\sqrt{1+\alpha})x}.$$

Now $\because$ $-1+\sqrt{1+\alpha} < 0$ and $-1-\sqrt{1+\alpha} < 0$
for $0 < \alpha < 1$

$\Rightarrow$ $y(x)$ goes to zero as $x \to \infty$.

$\therefore$ $\alpha \in (-1, 0)$

Case III $\alpha = -1$

for $\alpha = -1$

$0 = m^2 + 2m - \alpha = (m+1)^2$

$\Rightarrow$ $m = -1, -1$

Hence $y(x) = (C_1 + C_2 x)\, e^{-x}$

$\longrightarrow 0$ as $x \to \infty$

[$\because$ e^{-x} goes to zero, faster than any polynomial]

$\therefore$ $\alpha = -1$ is also a possibility.

Case IV $\alpha < -1$

In this case, (1) will have complex roots

$m = -1 \pm i\sqrt{1+\alpha}$

& hence

$y(x) = e^{-x}\left[\, C_1 \sin((\sqrt{1+\alpha})\, x) + C_2 \cos((\sqrt{1+\alpha})\, x)\right]$

$\longrightarrow 0$ as $x \to \infty$

[$\because$ $\sin(t)$ and $\cos(t)$ are bounded functions]

$\therefore$ $\alpha \in (-\infty, -1)$

Final conclusion:

For $\alpha \in (-\infty, 0)$ or $\alpha < 0$, all solutions of the given differential equation will converge to zero as $x \to \infty$.

Q3. Find the general solution of the following system of differential equations:

$$\frac{dy_1}{dt} = -3y_1 - 4y_2$$

$$\frac{dy_2}{dt} = 5y_1 + 6y_2$$

Answer: Let $A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$

First, we find eigen values of A.

$$P(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -3-\lambda & -4 \\ 5 & 6-\lambda \end{pmatrix}$$

$$= (-3-\lambda)(6-\lambda) + 20$$

$$= -18 - 3\lambda + \lambda^2 + 20 = \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 1)(\lambda - 2)$$

$\therefore$ Eigen values of A are 1, 2.

Eigen vector for $\lambda = 1$

$(A - I)x = 0$ where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left[\begin{array}{cc|c} -4 & -4 & 0 \\ 5 & 5 & 0 \end{array}\right]$$

$-4x_1 - 4x_2 = 0 = 5x_1 + 5x_2$

$\Rightarrow x_1 = -x_2$

$\therefore \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 1$.

Eigen vector corresponding to $\lambda = 2$.

$(A-2I)x = 0$; $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left[\begin{array}{cc|c} -5 & -4 & 0 \\ 5 & 4 & 0 \end{array}\right] \Rightarrow -5x_1 - 4x_2 = 0$$

$$\Rightarrow x_1 = -\frac{4}{5}x_2$$

So, $\begin{pmatrix} -4/5 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 2$.

General solution :-

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -4/5 \\ 1 \end{pmatrix} e^{2t}$$