COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 9

Announced on: Mar 13 (Wed)

1. Based on Problems 10.4 and 10.6 in [LLM17].

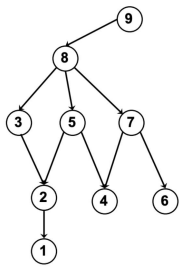
   a) Give an example of a digraph that has a closed walk including two vertices but has no cycle including those vertices.

   b) Give an example of a digraph in which a vertex v is on a positive even-length closed walk, but no vertex is on an even-length cycle.

2. Based on Problem 10.13 in [LLM17].

   A tournament digraph $G = (V, E)$ is a digraph with exactly one edge between each pair of distinct vertices. A vertex v is said to be a *king* if for any other vertex w, either $(v, w) \in E$ (i.e., v "beats" w) or there is a vertex $x \neq w$ such that $(v, x) \in E$ and $(x, w) \in E$ (i.e., v beats someone who beats w). Prove that in any tournament digraph, a vertex with the highest outdegree is a king.

3. Based on Problem 10.18 in [LLM17].



   The DAG shown above describes the prerequisites among the tasks $\{1, 2, \ldots, 9\}$. For example, in order to complete task 7, one must first finish task 8, which, in turn, depends on task 9.

   a) Suppose each task takes one hour to complete. If you could do an unlimited number of tasks in parallel, what is the minimum time it will take to complete all the tasks?

   b) If no more than two tasks can be completed in parallel, how will the answer to the above question change?

4. Prove that every tournament graph has a Hamiltonian path (i.e., a directed path passing through all vertices).

5. Based on Problem 14.11 in [LLM17].

   Use integration to find upper and lower bounds that differ by at most 0.1 for the following

sum:

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.