DEPARTMENT OF MATHEMATICS
MTL 101: MAJOR EXAM

50 Marks total

Honor Code: As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code.

Instructions: Write down all the steps of the solution clearly. No marks will be given without proper mathematical justification.

Problem 1 [6 marks] Solve the initial value problem

$$y'' - 2y' = \delta(t-1), \quad t > 0$$
$$y(0) = 1, \quad y'(0) = 0.$$

Problem 2 [7 marks] Determine the constants α, β, a and b so that $Y(s) = \frac{s}{(s+1)^2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0, \quad y(0) = a, \quad y'(0) = b.$$

Problem 3 [6 marks] Find the general solution to the following differential equation using the method of variation of parameters

$$y'' + 9y = 3\tan(3x).$$

Problem 4 [8 marks]

(i) Find Laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & t > 3. \end{cases}$$

(ii) Find inverse Laplace transform of $\ln(\frac{s+1}{s+2})$.

Problem 5 [6 marks] Consider the following initial value problem for the first order ODE

$$y' = \frac{10}{3} x y^{2/5}, \; y(x_0) = y_0.$$

(i) Show that for any x_0, y_0 in $\mathbb{R}$ the above IVP has a solution.

(ii) For $y_0 \neq 0$, and $x_0 \in \mathbb{R}$, show that the above IVP has a unique solution.

(iii) For $x_0 = 0$ and $y_0 = 0$, show that the above IVP has more than one solution.

PTO

Problem 6 [5 marks] Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = 0, \ x > 0,$$

given that $y_1(x) = e^x$ is a solution.

Problem 7 [6 marks] Let A be a 2×2 real matrix whose eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$ with eigenvectors $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, respectively. Find A^{23}.

Problem 8 [6 marks] Let S be a fixed invertible 3×3 matrix with real entries. Consider the set

$$W = \{A \in M_{3\times3}(\mathbb{R}) : S^{-1}AS \text{ is a diagonal matrix}\}.$$

Show that W is a subspace of $M_{3\times3}(\mathbb{R})$. Find a basis for W and prove your claim.