1 ~~(i)~~ To show that A is linearly independent, let

$$au+bv+cw=0 \longrightarrow (1)$$

Observe $au+bv+cw = \alpha(u+v+w)+\beta(u+2v+3w)+\gamma(u+4v+9w)$

holds if
$$\left.\begin{aligned} \alpha+\beta+\gamma &= a \\ \alpha+2\beta+4\gamma &= b \\ \alpha+3\beta+9\gamma &= c \end{aligned}\right\} \longrightarrow (2)$$

Since augment matrix $\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 4 & b \\ 1 & 3 & 9 & c \end{array}\right) \overset{R_2\to R_2-R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-a \\ 0 & 2 & 8 & c-a \end{array}\right)$

$\overset{R_3\to R_3-2R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-a \\ 0 & 0 & 2 & c-a-2(b-a) \end{array}\right)$ satisfy rank $(A|B)$ = rank (A)

i.e. rank of coeff. = rank of augmented

there exist $\alpha, \beta, \gamma \in \mathbb{R}$ s.t. (2) holds.

Choose α, β, γ satisfying (2). Then by (1)

$$\alpha(u+v+w)+\beta(u+2v+3w)+\gamma(u+4v+9w)=0$$

$\Rightarrow \alpha=\beta=\gamma=0$ (Since B is linearly independent)

$\Rightarrow a=b=c=0$ (by (2))

Hence A is linearly independent.

Q2 Given, A is invertible & $(A|I) \sim (B|C)$, with B row reduced echelon. Hence we must have $B = I$.

Let $\rho_m \cdots \rho_2 \rho_1 (A|I) = (I|C)$. Then $\rho_m \cdots \rho_2 \rho_1 (A) = I$ & $\rho_m \cdots \rho_2 \rho_1 (I) = C$. The first equality gives $\rho_m \cdots \rho_2 \rho_1 (I) \cdot A = I$; combining this with the second equality, we have $C \cdot A = I$. Since A is invertible, must have $C = A^{-1}$.

Observe

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 9 & 0 & 16 & 0 & 0 & 0 & 1 \end{array}\right) \underset{R_4 \to R_4 - 3R_2}{\overset{R_3 \to R_3 - R_1}{\sim}} \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & -3 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} R_1 \to R_1 - R_3 \\ \sim \\ R_2 \to \frac{1}{3} R_2 \\ R_3 \to \frac{1}{2} R_3 \\ R_4 \to \frac{1}{4} R_4 \end{array} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3/4 & 0 & 1/4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 4/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3/4 & 0 & 1/4 \end{array}\right)$$

Hence inverse of the given matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 4/3 & 0 & -1/3 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -3/4 & 0 & 1/4 \end{pmatrix}$$

Q3 $\text{Null}(T) = \{(x,y,z,w) : x-y+z-w=0,\ x+2y+w=0,\ y+z-w=0,\ 3x+4y+2z=0\}$

Coefficient matrix of the system of linear eq^ns defining Null(T)

is
$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 3 & 4 & 2 & 0 \end{pmatrix} \underset{R_4 \to R_4 - 3R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 7 & -5 & 3 \end{pmatrix}$$

$$\begin{array}{c} R_1 \to R_1 + R_3 \\ \sim \\ R_2 \to R_2 - 3R_3 \\ R_4 \to R_4 - 7R_3 \\ R_3 \leftrightarrow R_2 \end{array} \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & -8 & 10 \end{pmatrix} \underset{R_3 \to -\frac{1}{4}R_3}{\overset{R_4 \to R_4 - 2R_3}{\sim}} \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -5/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - 2R_3}{\sim}} \begin{pmatrix} 1 & 0 & 0 & -2+5/2 \\ 0 & 1 & 0 & -1+5/4 \\ 0 & 0 & 1 & -5/4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -5/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Number of unknowns $= 4$

rank of the coeff matrix $= 3$

dimension of the solution space is $4-3=1$.

Hence $\text{nullity}(T) = 1$

$\text{rank}(T) = \dim(\text{domain space}) - 1$

$= 4 - 1 = 3$

Since $\text{nullity}(T) = 1$, T is not injective.

Since $\text{rank}(T) = 3$, T is not surjective.

Here $U=\overline{W}$

4(a) Let $v_1, v_2 \in \overline{W}$ and $\alpha \in \mathbb{R}$.

$\overline{W} \neq \phi$ & $0 \in \overline{W}$ since $T(0)=0 \in W$.

Then $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2)$ ($\because T$ is linear)

By the definition of $\overline{W}$, $T(v_1)$, $T(v_2) \in W$.

Hence $T(\alpha v_1 + v_2) \in W$, so that

$\therefore \alpha v_1 + v_2 \in \overline{W}$.

Thus $\overline{W}$ is a subspace of V.

4(b). Let $3x^2+5x-2 = a(x-1)+b(x^2+x+1)+c\cdot 3$

Then $3x^2+5x-2 = bx^2+(a+b)x-a+b+3c$

Comparing coefficients, $-a+b+3c=-2$

$b=3, \quad a+b=5,$

Hence $a=2, \ b=3, \ \& \ c = \frac{-2+2-3}{3} = -\frac{3}{3} = -1$

Hence $[3x^2+5x-2]_B = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

where $B=\{x-1, x^2+x+1, 3\}$ an ordered basis.

5. a) Let $\alpha(e_1+e_2)+\beta(e_2+e_3)+\gamma(e_3+e_4)+\delta(e_4+e_1)=0$

Then $(\alpha+\delta)e_1+(\alpha+\beta)e_2+(\beta+\gamma)e_3+(\gamma+\delta)e_4=0$.

$\Rightarrow \alpha+\delta=0,\ \alpha+\beta=0,\ \beta+\gamma=0,\ \gamma+\delta=0.$

Hence coeff matrix is

$$\begin{pmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\overset{R_2\to R_2-R_1}{\sim}\begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&1&1&0\\0&0&1&1\end{pmatrix}$$

$$\overset{R_3\to R_3-R_2}{\sim}\begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&1&1\end{pmatrix}\overset{R_4\to R_4-R_3}{\sim}\begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{pmatrix}$$

Since rank of the coeff matrix is 3 (<4), (for instance x_4 is free), there are infinitely many solution. Hence there is a nonzero solution. Hence B is ~~not~~ linearly dependent, so that B is not a basis.

5(b) Suppose $f: V\to V$ is surjective.

Let $\dim V = n$. ~~By rank~~

Then $\operatorname{rank} f = n$.

By rank-nullity theorem $\operatorname{rank}(f)+\operatorname{nullity}(f)=n$

$\Rightarrow n+\operatorname{nullity}(f)=n$

$\Rightarrow \underline{\operatorname{nullity}(f)=0}$

$\Rightarrow \underline{f \text{ is injective}}$.