Quiz 1

● Graded

Student

Har Ashish Arora

Total Points

13 / 13 pts

Question 1

Inverse

4 / 4 pts

✔ **+ 4 pts** Correct

+ 0 pts Completely incorrect/ Not attempted.

+ 0.5 pts For considering $(A|I)$

+ 1.5 pts Correctly applying elementary row operations to get a the form $(B|C)$ where B is upper triangular with one of the diagonal entry function of b (namely, $b+2$).

+ 0.5 pts To say that A is invertible if $b \neq -2$ and any a

+ 1.5 pts For $b=-1$, getting $(I|D)$, where rows of D are $(-1,-a,1,1)$; $(0,1,0,0)$; $(2,a,-1,-1)$; $(-2,-2a,2,1)$.

+ 1 pt Applying elementary row operations on the given matrix and then finding invertibility condition.

+ 1 pt For applying the same operations on I for $b=-1$.

+ 2 pts For getting the correct inverse finally.

+ 0.5 pts If procedure is correct but the inverse is calculated incorrectly.

Question 2

Solutions of System

5 / 5 pts

✔ **+ 5 pts** Correct answer

+ 0.5 pts To write the augmented matrix of the system correctly.

+ 1.5 pts To obtain the row reduced form where the last row of the matrix is $[0,0,a^2-9,a-3]$

+ 1 pt If $a^2-9 \neq 0$, then row-rank(A) = row-rank([A, b])= No of variables and the system has the unique solution.

+ 0.5 pts If $a^2-9 \neq 0$, then the system has a unique solution.

+ 1 pt If $a^2-9=0$ and $a=3$, then row-rank(A) = row-rank([A, b])< No of variables and the system has infinitely many solutions.

+ 0.5 pts If $a^2-9=0$ and $a=3$, then the system has infinitely many solutions.

+ 1 pt If $a^2-9=0$ and $a=-3$, then 2 =row-rank(A) $\neq$ row-rank([A, b])=3, and the system has no solution.

+ 0.5 pts If $a^2-9=0$ and $a=-3$, then system has no solution.

+ 0 pts Not attempted /Completely wrong solution

Question 3

Vector Space

4 / 4 pts

+ 0.5 pts p1-a): writing the appropriate linear combination of the vectors from the second set equals to zero

✔ **+ 3 pts** Correct part a)

✔ **+ 1 pt** correct part b) with justification

+ 1.5 pts p2-a): use of l.i. of the vectors v_1, v_2, v_3 to find the system of linear equations for coefficients

+ 0.5 pts wrote the system of linear equations for coefficients with major mistake

+ 1 pt wrote the system of linear equations for coefficients with minor mistake

+ 1 pt p3-a): remaining part to show linearly independent with justification

+ 2.5 pts Minor mistake while finding the solution of system. Rest is correct.

+ 0.5 pts correct ans b) without proper justification

+ 0 pts incorrect/ not attempting

+ 0 pts part b is wrong/not attempted

+ 0 pts Part a is wrong/not attempted

DEPARTMENT OF MATHEMATICS, IIT DELHI

SEMESTER II 2024 – 25

MTL 101 (Linear Algebra and Differential Equations) - Quiz 1

Date: 29/01/2025 (Wednesday) Time: 6:30 PM - 7:15 PM.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

Name :	HAR ASHISH ARORA	BLOCK LETTER ONLY	
Entry Number:	2024EE10904	**Group:**	13
Gradescope Id:	EE1240904	**Lecture Hall:**	108

Question 1: Using RRE method of finding the inverse, determine all values of a and b for which the matrix A is invertible and compute the inverse for $b = -1$:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 1 \\ 2 & 0 & 0 & b \end{pmatrix}$$

[4]

Using the augmented matrix $[A|I]$; we will go to $[I|B]$. B will be A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & b & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4 \to R_4 - 2R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & b & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_4 \to -\frac{R_4}{2}]{R_3 \to R_3 - aR_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & -\frac{b}{2} & 1 & 0 & 0 & -\frac{1}{2} \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 0 & -\frac{b}{2}-1 & 1 & a & -1 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_4 \to \frac{R_4}{-\frac{b}{2}-1}} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{-1}{\frac{b}{2}+1} & \frac{-a}{\frac{b}{2}+1} & \frac{1}{\frac{b}{2}+1} & \frac{1}{b+2} \end{array}\right] \xrightarrow{R_3 \to R_3 - R_4}$$

(this step assumes $-\frac{b}{2} - 1 \neq 0 \Rightarrow \boxed{b \neq -2}$)

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{b+2} & -a + \frac{2a}{b+2} & 1 - \frac{2}{b+2} & -\frac{1}{b+2} \\ 0 & 0 & 0 & 1 & \frac{-2}{b+2} & -\frac{2a}{b+2} & \frac{2}{b+2} & \frac{1}{b+2} \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1-\frac{2}{b+2} & a-\frac{2a}{b+2} & \frac{2}{b+2}-1 & \frac{1}{b+2} \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{b+2} & \frac{2a}{b+2}-a & 1-\frac{2}{b+2} & \frac{-1}{b+2} \\ 0 & 0 & 0 & 1 & \frac{-2}{b+2} & \frac{-2a}{b+2} & \frac{2}{b+2} & \frac{1}{b+2} \end{array}\right]$$

$$\Rightarrow$$

$$B = A^{-1} = \begin{bmatrix} \frac{b}{b+2} & \frac{ab}{b+2} & \frac{-b}{b+2} & \frac{1}{b+2} \\ 0 & 1 & 0 & 0 \\ \frac{2}{b+2} & \frac{-ab}{b+2} & \frac{b}{b+2} & \frac{-1}{b+2} \\ \frac{-2}{b+2} & \frac{-2a}{b+2} & \frac{2}{b+2} & \frac{1}{b+2} \end{bmatrix}$$

The values of a & b for which the matrix A is invertible are $\boxed{a \in \mathbb{R}}$ and $\boxed{b \in \mathbb{R} - \{-2\}, \text{ or } b \neq -2.}$ (as shown in the steps).

The value of the inverse for $b=-1$ is: $(b+2 = 2-1 = 1)$.

$$B = A^{-1} = \begin{bmatrix} -1 & -a & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & a & -1 & -1 \\ -2 & -2a & 2 & 1 \end{bmatrix}. \quad \text{— (Ans).}$$

NOTE: In case $b=-2$, then the rank of the matrix on the left would become less than 4. This would imply that A can't be invertible ∵ A would no longer belong to the equivalence class of I_4, ∵ all matrices in the same equivalence class have the same rank, and $\text{rank}(I_4)=4$. Thus we could not get to the form $[I|B]$.

Question 2: Using elementary row operations find for what values of a, the following system has i) no solution, ii) a unique solution and iii) infinite number of solutions.

$$x + y + z = 3$$
$$2x + 5y + 4z = a$$
$$3x + (a^2 - 8)z = 12$$

[5]

We will use the augmented matrix $[A|b]$, and then use the ranks of $[A|b]$ and $[A]$, and by comparing them, we can get the required answers & values for a.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 5 & 4 & a \\ 3 & 0 & a^2-8 & 12 \end{array}\right]$$

(A is the coefficient matrix, b is the constant matrix).

Convert to RREF:

$$[A|b] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 3 & 2 & a-6 \\ 0 & -3 & a^2-11 & 3 \end{array}\right] \xrightarrow[R_3 \to \frac{R_3}{-3}]{R_2 \to \frac{R_2}{3}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & \frac{2}{3} & \frac{a-6}{3} \\ 0 & 1 & \frac{11-a^2}{3} & -1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2/3 & \frac{a-6}{3} \\ 0 & 0 & \frac{9-a^2}{3} & \frac{3-a}{3} \end{array}\right]$$

$$\xrightarrow[\cancel{R_2 \to 3R_3}]{R_3 \to 3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2/3 & (a-6)/3 \\ 0 & 0 & 9-a^2 & 3-a \end{array}\right]$$

→ (this can be further converted to RREF, but we stop here for simplicity ∵ we are only dealing w/ the last row now)

(i) The system has no solution if $\text{rank}([A|b]) \neq \text{rank}([A])$. For this to be true, $9 - a^2 = 0$ but $3 - a \neq 0 \Rightarrow \boxed{a = -3}$. Thus for the value of $a = -3$, the system has no solutions.

$\Rightarrow$ for no solutions, $a = -3$.

(ii) The system has a unique solⁿ if

rank([A|b]) = rank([A]) = 3.

Thus, for this, $9-a^2 \neq 0 \Rightarrow \boxed{a \neq \pm 3}$.

$\Rightarrow \boxed{a \in \mathbb{R} - \{3, -3\}}$.

(iii) For an infinite number of solⁿs,

rank([A|b]) = rank([A]) < 3. This is possible only in the case of $\boxed{a=3}$ $\because$ for this $9-a^2=0$ and $3-a=0$ (we can only tell for this row $\because$ looking at the above row, we see that rank([A|b]) = rank([A]) > 2, definitely.)

Question 3: (a) Let V be a vector space over $\mathbb{R}$ and $\{v_1, v_2, v_3\}$ be a set of linearly independent vectors in V. Show that the set $\{v_1 + v_3, 3v_1 + 2v_2 + v_3, 2v_1 + 3v_2 + v_3\}$ is linearly independent in V. [3]

(b) Consider the vector space

$$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbb{R}, 1 \leq i \leq 4\}.$$

Is

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 \in \mathbb{Z}\}$$

a subspace of $\mathbb{R}^4(\mathbb{R})$? Justify. [1]

(a) Consider the equation

$$\alpha(v_1+v_3) + \beta(3v_1+2v_2+v_3) + \gamma(2v_1+3v_2+v_3) = 0,$$

where $v_1, v_2, v_3 \in V$ and $\alpha, \beta, \gamma \in \mathbb{R}$. In order to show that the set $\{v_1+v_3, 3v_1+2v_2+v_3, 2v_1+3v_2+v_3\}$ is linearly independent, we need to show that $\alpha=\beta=\gamma=0$ for this equation.

Simplifying the above eq^n, we get:

$$(v_1)(\alpha+3\beta+2\gamma) + (v_2)(2\beta+3\gamma) + (v_3)(\alpha+\beta+\gamma) = 0.$$

We know that $\{v_1, v_2, v_3\}$ is a set of linearly independent vectors in V. Thus,

$$\alpha + 3\beta + 2\gamma = 0 \quad \text{——} (*)$$
$$2\beta + 3\gamma = 0 \quad \text{——} (**)$$
$$\alpha + \beta + \gamma = 0 \quad \text{——} (***)$$

Subtract ~~(*) from (***) to get~~ (***) from (*) to get:-

$2\beta + \gamma = 0$. Now, use $(**) \Rightarrow$ ~~2~~ $\gamma = 3\gamma \Rightarrow \boxed{\gamma = 0}$.

In (**), implies $\boxed{\beta = 0}$. In (*), implies $\boxed{\alpha = 0}$.

Thus, α, β, γ are all zero together, for that equation to be valid. Hence, the given set of vectors is linearly independent. (Hence proved).

(b) In order to show that W is NOT a subspace of $\mathbb{R}^4$, we will show that scalar multiplication is not closed (wrt the given field, which is $\mathbb{R}$).

Consider the vector $v = (1, 1, 1, 1)$. This $v \in W$ $\because$ $x_i \in \mathbb{Z}$, $i \in [1,4]$. Now, consider one of the axioms of a vector space $(V(\mathbb{F}), +, \cdot)$:-

(**) $\forall \lambda \in \mathbb{F}, \ \forall v \in V, \ \lambda \cdot v \in V$.

Since here $\mathbb{F} = \mathbb{R}$ (vector space of $\mathbb{R}^4(\mathbb{R})$), choose $\lambda = \frac{1}{2}$. Then,

$$\lambda \cdot v = \frac{1}{2} \cdot v = \frac{1}{2} \cdot (1,1,1,1) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

Now, $x_1, x_2, x_3, x_4 \notin \mathbb{Z} \Rightarrow \lambda \cdot v \notin W \Rightarrow$ W is not closed under scalar multiplication $\Rightarrow$ $\boxed{W \text{ cannot be a subspace of } \mathbb{R}^4(\mathbb{R})}$ $\therefore$ it is not a vector space over the given field, $\mathbb{R}$.

(Ans).