Name: ______________________________

ID number: ______________________________

There are 2 questions for a total of 10 points.

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1. (5 points) What is the expected number of bins that are empty when m balls are distributed into n bins uniformly at random?

Solution: Let X_i denote the random variable that is 1 if the i^{th} bin is empty and 0 otherwise. The following holds:

$$\forall i, \mathbf{Pr}[X_i] = \left(1 - \frac{1}{n}\right)^m$$

This is because the probability that a randomly thrown ball does not enter the i^{th} bin is $(1 - 1/n)$ and so the probability that all the m balls miss the i^{th} bin is $(1-1/n)^m$. The total number of empty bins is given by $\sum_{i=1}^{n} X_i$. So, we have

$$\begin{aligned}
\mathbf{E}[\sum_{i=1}^{n} X_i] &= \sum_{i=1}^{n} \mathbf{E}[X_i] \qquad \text{(using linearity of expectation)} \\
&= \sum_{i=1}^{n} (1 \cdot \mathbf{Pr}[X_i = 1] + 0 \cdot \mathbf{Pr}[X_i = 0]) \\
&= n \cdot \left(1 - \frac{1}{n}\right)^m .
\end{aligned}$$

2. Recall the Longest Increasing Subsequence problem discussed in class. Consider the sequence of numbers in array $A = [14, 8, 2, 7, 4, 10, 6, 0, 1, 16, 5, 13, 3, 11, 12, 15]$. As in the class discussion, let $L(i)$ denote the length of the longest increasing subsequence of $A[1...n]$ that ends with $A[i]$.

   (a) (3 points) Fill the table for L below.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L[i]$	1	1	1	2	2	3	3	1	2	4	3	4	3	4	5	6

   (b) (2 points) Give a longest increasing subsequence.

(b) $(0, 1, 5, 11, 12, 15)$