Inner product space

Let V be a vector space over F ($F = \mathbb{R}$ or $\mathbb{C}$). An inner product on V is a function $\langle\, , \rangle : V \times V \longrightarrow F$ such that (for $(v_1, v_2) \in F \times F$, $\langle v_1, v_2 \rangle \in F$)

1. $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle \;\forall v_1, v_2, v_3 \in V$
2. $\langle c v_1, v_2 \rangle = c \langle v_1, v_2 \rangle \;\forall c \in F, \forall v_1, v_2 \in V$
3. $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle} \;\forall v_1, v_2 \in V$ ($\overline{\langle v_2, v_1 \rangle}$ denotes complex conjugate of $\langle v_2, v_1 \rangle$.)
4. $\langle v, v \rangle > 0 \;\forall v (\neq 0) \in V$

A vector space over $\mathbb{R}$ or $\mathbb{C}$, equipped with an inner product is called an inner product space.

Observe

$$\begin{aligned}
\langle c v_1 + v_2, v_3 \rangle &= \langle c v_1, v_3 \rangle + \langle v_2, v_3 \rangle \\
&= c \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle
\end{aligned}$$

$$\begin{aligned}
\langle v_1, c v_2 + v_3 \rangle &= \overline{\langle c v_2 + v_3, v_1 \rangle} \\
&= \overline{\langle c v_2, v_1 \rangle} + \overline{\langle v_3, v_1 \rangle} \\
&= \bar{c} \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle
\end{aligned}$$

We say an inner product is linear is first component and conjugate linear in second component. If $F = \mathbb{R}$, then it is linear in both components.

Remark In condition ③ we do not have symmetry if $F = \mathbb{C}$.

Let $v \neq 0$. $\langle iv, iv \rangle = i \underbrace{\langle v, iv \rangle}_{\shortparallel}$

Not valid step $= i \langle iv, v \rangle$ if we had symmetry

$= i^2 \langle v, v \rangle$

$= -1 \langle v, v \rangle$

So simultaneously $\langle v, v \rangle$, $\langle iv, iv \rangle$ could have not been positive. That violates condition ④.

Observe $\langle 0_V, v \rangle$

$= \langle 0_V + 0_V, v \rangle$

$= \langle 0_V, v \rangle + \langle 0_V, v \rangle$

So, $\langle 0_V, v \rangle = 0 \quad \forall v \in V$.

So, $\langle v, 0_V \rangle = \overline{\langle 0_V, v \rangle} = 0 \quad \forall v \in V$.

Examples ① consider $\mathbb{R}^2$ over $\mathbb{R}$,

define $\langle (x_1, x_2), (y_1, y_2) \rangle := x_1 y_1 + x_2 y_2$

Check This is an inner product

If we define the same map on $\mathbb{C}^2$ over $\mathbb{C}$, then it is not an inner product.

$\langle (0, i), (0, i) \rangle = 0 + i^2 = -1$, not positive

② consider $\mathbb{C}^2$ over $\mathbb{C}$, define

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$$

Check This is an inner product.

③ consider F^n over F. ($F = \mathbb{R}$ or $\mathbb{C}$)

Define

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := \sum_{i=1}^{n} x_i \bar{y}_i .$$

Check This is an inner product on F^n.

This inner product we refer as the standard inner product on F^n where $F = \mathbb{R}$ or $\mathbb{C}$.

Note that $\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$= (\bar{y}_1 \cdots \bar{y}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= y^* \cdot x$$

where $*$ denotes conjugate transpose

④ On F^n (where $F = \mathbb{R}$ or $\mathbb{C}$), define

$\langle x, y \rangle := y^* A^* A x$ for some fixed $n \times n$ invertible matrix A

Check This is an inner product on F^n.

$$\langle x, x \rangle = x^* A^* A x$$
$$= (Ax)^* (Ax)$$

If $x \neq 0$, then $Ax \neq 0$ as A is invertible.

So if $Ax = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, then $c_i \neq 0$ for some $1 \leq i \leq n$.

So, $\langle x, x \rangle = \sum_{i=1}^{n} |c_i|^2 > 0$

Note

For $A = Id$, we get back the inner product of example (3).

(5) On $\mathbb{R}^2$, define

$$\langle (x_1, x_2), (y_1, y_2) \rangle := x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$$

Note that,

$$\langle (x_1, x_2), (y_1, y_2) \rangle = (y_1 \;\; y_2) \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (y_1 \;\; y_2) \begin{pmatrix} 1 & 0 \\ -1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Check This is an inner product on $\mathbb{R}^2$.

(6) Consider $C[0,1] = \{ f : [0,1] \to \mathbb{R} : f \text{ is continuous} \}$

This is an $\mathbb{R}$-vector space.

Define, $\langle f, g \rangle := \int_0^1 f(t) g(t) \, dt$.

Check This is an inner product on $C[0,1]$.

Let $f \neq 0$.

$$\langle f, f \rangle = \int_0^1 f(t)^2 \, dt.$$

Now $f(t)^2$ is non-negative function.

As $f \neq 0$, $\exists$ $c \in [0,1]$ such that $f(c) \neq 0$.

So $f(c)^2 > 0$.

So, $\int_0^1 f(t)^2 \, dt > 0$.

So, $\langle f, f \rangle > 0 \;\forall\; f \neq 0$

g non-negative continuous on $[0,1]$.
If $g(c) \neq 0$ for some $c \in [0,1]$, then take $\epsilon = g(c) > 0$.
As g is cont. at c, for $\epsilon/2 > 0$ $\exists$ $\delta > 0$ such that
$g(x) \in (g(c) - \epsilon/2, g(c) + \epsilon/2)$ $\forall$ $x \in (c-\delta, c+\delta)$.
So, $g(x) \geq \epsilon/2$ $\forall$ $x \in (c-\delta, c+\delta)$.
So, $\int_0^1 g(x)\,dx \geq \int_{c-\delta}^{c+\delta} g(x)\,dx > \epsilon\delta > 0.$

(7) Let V, W be two vector spaces over F $(= \mathbb{R}$ or $\mathbb{C})$. Suppose W is an inner product space and $T : V \to W$ is an one-one linear map. Then we have an inner product on V. Define

$$\langle v_1, v_2 \rangle_V := \langle T v_1, T v_2 \rangle_W$$

<u>check</u> $\langle v_1, v_2 \rangle_V$ is an inner product on V.

$$\langle v, v \rangle_V = \langle T(v), T(v) \rangle_W$$

Now $v = 0 \Leftrightarrow T(v) = 0$ as T is one-one.

So, $\langle v, v \rangle_V = \langle T(v), T(v) \rangle_W > 0 \;\forall\; v \neq 0$.

Norm of a vector Let V be an inner product space. Define $\|\cdot\|: V \longrightarrow \mathbb{R}_{\geq 0}$, by $\|v\| := \sqrt{\langle v, v\rangle}$ $\forall\, v \in V$. This map is referred as the norm map on V and $\|v\|$ is referred as the norm of the vector v.

$\mathbb{R}_{\geq 0}$ is the set of all non-negative real numbers

Observe $\|v\| = 0 \Leftrightarrow v = 0$. Because $\langle v, v\rangle > 0$ $\forall\, v \neq 0$ and $\langle 0_V, 0_V\rangle = 0$.

Parallelogram identity

Let $v, w \in V$. Then, $\|v + w\|^2$

$$= \langle v + w, v + w\rangle$$
$$= \langle v, v + w\rangle + \langle w, v + w\rangle$$
$$= \overline{\langle v + w, v\rangle} + \overline{\langle v + w, w\rangle}$$
$$= \overline{\langle v, v\rangle} + \overline{\langle w, v\rangle} + \overline{\langle v, w\rangle} + \overline{\langle w, w\rangle}$$
$$= \|v\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|^2.$$

Also, $\|v - w\|^2 = \langle v - w, v - w\rangle$

$$= \langle v, v - w\rangle - \langle w, v - w\rangle$$
$$= \overline{\langle v - w, v\rangle} - \overline{\langle v - w, w\rangle}$$
$$= \overline{\langle v, v\rangle} - \overline{\langle w, v\rangle} - \overline{\langle v, w\rangle} + \overline{\langle w, w\rangle}$$
$$= \|v\|^2 - \langle v, w\rangle - \overline{\langle v, w\rangle} + \|w\|^2$$

So, $\boxed{\|v + w\|^2 + \|v - w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right)}$ $\forall\, v, w \in V$.

Observe

If the underlying field F is $\mathbb{R}$, then

$$\|v+w\|^2 = \|v\|^2 + 2\langle v,w\rangle + \|w\|^2 \quad \text{and}$$

$$\|v-w\|^2 = \|v\|^2 - 2\langle v,w\rangle + \|w\|^2 \quad \forall\, v,w \in V$$

So, $$4\langle v,w\rangle = \|v+w\|^2 - \|v-w\|^2 \quad \forall\, v,w \in V.$$

So, $$\boxed{\langle v,w\rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)} \quad \forall\, v,w \in V$$

This is referred as the real polarization identity

If the underlying field $F = \mathbb{C}$, then

$$\|v+w\|^2 = \|v\|^2 + 2\,\mathrm{Re}\langle v,w\rangle + \|w\|^2 \quad \text{and}$$

$$\|v-w\|^2 = \|v\|^2 - 2\,\mathrm{Re}\langle v,w\rangle + \|w\|^2 \quad \forall\, v,w \in V.$$

So, $$4\,\mathrm{Re}\langle v,w\rangle = \|v+w\|^2 + \|v-w\|^2 \quad \forall\, v,w \in V.$$

So, $$\boxed{\mathrm{Re}\langle v,w\rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)} \quad \forall\, v,w \in V.$$

Also,
$$\begin{aligned}
\|v+iw\|^2 &= \langle v+iw, v+iw\rangle \\
&= \langle v, v+iw\rangle + i\langle w, v+iw\rangle \\
&= \overline{\langle v+iw, v\rangle} + i\overline{\langle v+iw, w\rangle} \\
&= \overline{\langle v,v\rangle} - i\overline{\langle w,v\rangle} + i\overline{\langle v,w\rangle} + \overline{\langle w,w\rangle} \\
&= \|v\|^2 - i\left(\langle v,w\rangle - \overline{\langle v,w\rangle}\right) + \|w\|^2 \\
&= \|v\|^2 + 2\,\mathrm{Im}\langle v,w\rangle + \|w\|^2.
\end{aligned}$$

$$\|v - iw\|^2 = \langle v - iw, v - iw\rangle$$
$$= \langle v, v - iw\rangle - i\langle w, v - iw\rangle$$
$$= \overline{\langle v - iw, v\rangle} - i\,\overline{\langle v - iw, w\rangle}$$
$$= \|v\|^2 + i\langle v, w\rangle - i\,\overline{\langle v, w\rangle} + \|w\|^2$$
$$= \|v\|^2 - 2\,\mathrm{Im}\langle v, w\rangle + \|w\|^2$$

So, $$4\,\mathrm{Im}\langle v, w\rangle = \|v + iw\|^2 - \|v - iw\|^2 \quad \forall\, v, w \in V$$

i.e. $$\boxed{\mathrm{Im}\langle v, w\rangle = \frac{1}{4}\left(\|v + iw\|^2 - \|v - iw\|^2\right)}$$

Now,
$$\langle v, w\rangle = \mathrm{Re}\langle v, w\rangle + i\,\mathrm{Im}\langle v, w\rangle$$

So, $$\boxed{\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2\right)}$$

This is referred as the complex polarization identity.

Remark: So one can express an inner product via the norm defined by it.

Matrix of an inner product

Let V be a finite dimensional inner product space and $B = \{v_1, \dots, v_n\}$ be an ordered basis. The matrix $A = (c_{ij}) := \langle v_j, v_i\rangle$ is called the matrix of the inner product with respect to the basis B.

Observe

① $c_{ii} = \langle v_i, v_i \rangle > 0$.

② For $v, w \in V$, write $v = \sum_{i=1}^{n} a_i v_i$,

$$w = \sum_{i=1}^{n} b_i v_i, \quad a_i, b_i \in F \ \forall\, 1 \leq i \leq n.$$

Then
$$\langle v, w \rangle = \left\langle \sum_{j=1}^{n} a_j v_j, \sum_{i=1}^{n} b_i v_i \right\rangle$$

$$= \sum_{j=1}^{n} a_j \left\langle v_j, \sum_{i=1}^{n} b_i v_i \right\rangle$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_j \overline{b_i} \langle v_j, v_i \rangle$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} a_j (c_{ij}) \overline{b_i}$$

$$= (\overline{b_1} \ \overline{b_2} \cdots \overline{b_n}) \, A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$= [w]_B^* \, A \, [v]_B$$

<u>Thm</u> Let V be a finite dimensional inner product space and A be the matrix of the inner product with respect to an ordered basis $B = \{v_1, \ldots, v_n\}$.

Then, ① $A^* = A$ i.e. A is a Hermitian matrix.

② $[v]_B^* A [v]_B > 0$ for every $v \in V$ and $v \neq 0$.

③ A is invertible.

Pf ① $A = (a_{ij}) = \langle v_j, v_i \rangle$

$$= \overline{\langle v_i, v_j \rangle}$$

$$= \overline{a_{ji}}$$

$$= A^*$$

② We know $\langle v, v \rangle > 0 \ \forall v \neq 0$.

Also we know,

$$\langle v, v \rangle = [v]_B^* \, A \, [v]_B .$$

Hence $[v]_B^* A [v]_B > 0 \ \forall \ v \neq 0$.

③ Consider the homogeneous system of linear equations $Ax = 0$.

Note that, A is invertible

$\Longleftrightarrow$ $AX = 0$ has only trivial solution

Let $Ax = 0$ for $x \in F^n$. So $x^* A x = 0$.

Let $v = x_1 v_1 + \cdots + x_n v_n \in V$,

where $x = (x_1, \ldots, x_n) \in F^n$.

Now, $x^* A x = 0 \Rightarrow [v]_B^* \, A \, [v]_B = 0$.

But we know, $[v]_B^* A [v]_B > 0 \ \forall v \neq 0$.

So, $v = 0$. So $x_i = 0 \ \forall \ 1 \leq i \leq n$. So, $x = 0$.

Exercise Let A be an invertible Hermitian matrix such that $x^* A x > 0 \ \forall \ x (\neq 0) \in F^n$, $(F = \mathbb{C}$ or $\mathbb{R})$. Let V be an n-dimensional vector space over F. For any ordered basis B of V, define $\langle v, w \rangle := [w]_B^* A^* [v]_B \ \forall \ v, w \in V$. Show that it is an inner product on V and the matrix of this with respect to B is A.

Thm Let V be an inner product space. Then for $v, w \in V$ and $c \in F$, we have,

1. $\|cv\| = |c|\,\|v\|$
2. $\|v\| > 0$ for all $v \neq 0$
3. $|\langle v, w\rangle| \leq \|v\|\,\|w\| \quad \forall\, v, w \in V$

   This is referred as the Cauchy-Schwarz inequality.
4. $\|v + w\| \leq \|v\| + \|w\| \quad \forall\, v, w \in V$

   This is referred as the triangle inequality.

Pf ① $\|cv\|^2 = \langle cv, cv\rangle$

$$= c\langle v, cv\rangle = c\,\overline{\langle cv, v\rangle}$$
$$= c\bar{c}\,\overline{\langle v, v\rangle}$$
$$= |c|^2\,\|v\|^2$$

So, $\|cv\| = |c|\,\|v\|$.

② $\|v\|^2 = \langle v, v\rangle > 0$ for $v \neq 0$.

Also $\|\cdot\|$ is a map to $\mathbb{R}_{\geq 0}$. So $\|v\| > 0 \ \forall\, v \neq 0$.

③ <u>To show</u> $|\langle v, w\rangle| \leq \|v\|\,\|w\| \ \forall\, v, w \in V$.

If $v = 0$ or $w = 0$, then the inequality holds. We assume $w \neq 0$.

Let $u = v - \dfrac{\langle v, w\rangle}{\|w\|^2}\, w$.

So, $\langle u, w\rangle = \langle v, w\rangle - \dfrac{\langle v, w\rangle}{\|w\|^2}\, \underbrace{\langle w, w\rangle}_{= \|w\|^2}$.

$= 0$

So, $\|u\|^2 = \langle u, u\rangle$

$$= \left\langle u,\ v - \frac{\langle v, w\rangle}{\|w\|^2} w\right\rangle$$

$$= \overline{\left\langle v - \frac{\langle v, w\rangle}{\|w\|^2} w,\ u\right\rangle}$$

$$= \overline{\langle v, u\rangle} - \frac{\overline{\langle v, w\rangle}}{\|w\|^2} \overline{\underbrace{\langle w, u\rangle}_{\substack{\| \\ \langle u, w\rangle \\ \| \\ 0}}}$$

$$= \langle u, v\rangle$$

$$= \left\langle v - \frac{\langle v, w\rangle}{\|w\|^2} w,\ v\right\rangle$$

$$= \underbrace{\langle v, v\rangle}_{\substack{\| \\ \|v\|^2}} - \frac{\langle v, w\rangle}{\|w\|^2} \underbrace{\langle w, v\rangle}_{\substack{\| \\ \overline{\langle v, w\rangle}}}$$

$$= \|v\|^2 - \frac{|\langle v, w\rangle|^2}{\|w\|^2}$$

As $\|u\|^2 \geq 0$, we get $\|v\|^2 \|w\|^2 - |\langle v, w\rangle|^2 \geq 0$

i.e. $|\langle v, w\rangle|^2 \leq \|v\|^2 \|w\|^2$

i.e. $|\langle v, w\rangle| \leq \|v\| \, \|w\|$.

Remarks (1) Why such choice of u is considered? Note that if $v = cw$, then $\langle v, w\rangle = \langle cw, w\rangle = c\|w\|^2$

So, $c = \dfrac{\langle v, w\rangle}{\|w\|^2}$.

So, $v = \dfrac{\langle v, w\rangle}{\|w\|^2} w$.

Hence if $v \neq c\omega$, we work with such

$$u = v - \frac{\langle v, \omega \rangle}{\|\omega\|^2}\,\omega.$$

② Note that, $\|v\|\|\omega\| = |\langle v, \omega\rangle| \Leftrightarrow u = 0$

$\Leftrightarrow v, \omega$ are linearly dependent.

So, equality holds in Cauchy-Schwarz inequality iff both the vectors are linearly dependent.

Cauchy-Schwarz inequality

For $\mathbb{R}^n$ with standard inner product is $\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$ for $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$

For $\mathbb{C}^n$ with standard inner product is

$$\left|\sum_{i=1}^{n} x_i \overline{y_i}\right| \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}$$

for $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{C}^n$.

(4) <u>To show</u> $\|v+w\| \leq \|v\| + \|w\|$.

We have, $\|v+w\|^2$

$$= \langle v+w, v+w\rangle$$
$$= \langle v, v+w\rangle + \langle w, v+w\rangle$$
$$= \langle \overline{v+w, v}\rangle + \langle \overline{v+w, w}\rangle$$
$$= \langle \overline{v,v}\rangle + \underbrace{\langle \overline{w,v}\rangle}_{\substack{\| \\ \langle v,w\rangle}} + \langle \overline{v,w}\rangle + \langle \overline{w,w}\rangle$$
$$= \|v\|^2 + 2\,\mathrm{Re}\,\langle v,w\rangle + \|w\|^2$$
$$\leq \|v\|^2 + 2\underbrace{|\langle v,w\rangle|}_{\leq \|v\|\,\|w\| \text{ by Cauchy-Schwarz inequality}} + \|w\|^2$$
$$\leq (\|v\| + \|w\|)^2.$$

So, $\|v+w\| \leq \|v\| + \|w\|$

<u>Remarks</u> (1) If $F = \mathbb{R}$, then note that,

$$\|v\|^2 + 2\,\mathrm{Re}\,\langle v,w\rangle + \|w\|^2$$
$$= \|v\|^2 + 2\underbrace{\langle v,w\rangle}_{\substack{\| \\ \|v\|\,\|w\|}} + \|w\|^2$$

iff v, w are linearly dependent.

So the equality holds in triangle inequality for $F = \mathbb{R}$ iff both the vectors are linearly dependent.

② Consider $\mathbb{C}$ over $\mathbb{C}$ with the standard inner product. Note that, $1, 1+i$ are linearly dependent but

$$\| 2+i \| < \| 1 \| + \| (1+i) \| .$$

$$\| 2+i \|^2 = (2+i)(2-i) = 5$$

and $\| 1 \|^2 = 1$, $\| (1+i) \|^2 = 2$

Now, $\sqrt{5} < 1 + \sqrt{2}$.

So if the underlying field is $\mathbb{C}$, then even if two vectors are linearly dependent, equality may not hold in triangle inequality.

Orthogonality

Let V be an inner product space.

Let $v, w \in V$. We say that v, w are orthogonal to each other if $\langle v, w \rangle = 0$.

Examples ① 0 is orthogonal to every vector $v \in V$ as $\langle 0, v \rangle = 0 \ \forall\, v \in V$.

② On $\mathbb{R}^2$ over $\mathbb{R}$ with standard inner product, $(2,0), (0,5)$ are orthogonal to each other as $\langle (2,0), (0,5) \rangle = 0$. Also, $(1,0), (0,1)$ are orthogonal to each other.

Note If v_1, v_2 are orthogonal then $c_1 v_1, c_2 v_2$ for $c_1, c_2 \in F$ are orthogonal.

Let $S \subseteq V$. We say S is an orthogonal set if $\langle v, w \rangle = 0 \;\forall\; v \neq w \in S$.

For example, $\{(1,1), (-1,1)\}$ is an orthogonal subset of $\mathbb{R}^2$. The empty set and the singleton set is considered as orthogonal set in every inner product space.

Let $S \subseteq V$. We say S is an orthonormal set if it is orthogonal and every element of S has norm 1.

If S is orthonormal set then $0 \notin S$.

For example, $\{(1,0), (0,1)\}$ is an orthonormal subset of $\mathbb{R}^2$ but $\{(1,1), (1,-1)\}$ is not an orthonormal subset of $\mathbb{R}^2$.

Consider, $\dfrac{(1,1)}{\underbrace{\|(1,1)\|}_{=\sqrt{2}}}, \dfrac{(1,-1)}{\underbrace{\|(1,-1)\|}_{=\sqrt{2}}} \in \mathbb{R}^2$. Note that both of them are of norm 1 and are orthogonal. So, $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}$ is a orthonormal subset of $\mathbb{R}^2$.

<u>Remark</u> If S is an orthogonal subset of V, then $S' = \left\{\frac{v}{\|v\|} : v \in S\right\}$ is an orthonormal subset of V. So from an orthogonal set of vectors we always can make an orthonormal set of vectors. Moreover, span of S = span of S'.

<u>Example</u> Consider $\langle f, g\rangle := \int_0^1 f(t)g(t)\,dt$ on $C[0,1]$ over $\mathbb{R}$.

$S := \{1, \cos(2n\pi t), \sin(2n\pi t)\}$ is an orthogonal subset of $C[0,1]$

Because, $\int_0^1 \cos(2n\pi t)\,dt = 0$, $\int_0^1 \sin(2n\pi t)\,dt = 0$

and $\int_0^1 \cos(2n\pi t)\sin(2n\pi t)\,dt$

$= \frac{1}{2}\int_0^1 \sin(4n\pi t)\,dt = 0.$

$$\|\cos(2n\pi t)\|^2 = \int_0^1 \cos^2(2\pi n t)\,dt$$

$$= \frac{1}{2}\int_0^1 \left(\cos(4\pi n t) + 1\right) dt$$

$$= \frac{1}{2}$$

$$\|\sin(2\pi n t)\|^2 = \int_0^1 \sin^2(2\pi n t)\,dt$$

$$= \frac{1}{2}\int_0^1 (1 - \cos(4\pi n t))\,dt$$

$$= \frac{1}{2}$$

So, S is not an orthonormal set.

But $S' := \{1, \sqrt{2}\cos(2\pi n t), \sqrt{2}\sin(2\pi n t)\}$ is an orthonormal set.

Thm Let S be an orthogonal set of non-zero vectors in an inner product space V. Then S is linearly independent subset of V.

Pf Let $\sum_{i=1}^{n} c_i v_i = 0$ for some $v_1, \dots, v_n \in S$ and $c_1, \dots, c_n \in F$.

We know, $v_1, \dots, v_n$ are orthogonal to each other i.e. $\langle v_i, v_j \rangle = 0 \ \forall \ i \neq j$.

Now, for $1 \leq j \leq n$,

$$\underbrace{\left\langle \sum_{i=1}^{n} c_i v_i, v_j \right\rangle}_{\substack{\| \\ \langle 0, v_j \rangle \\ \| \\ 0}} = \sum_{i=1}^{n} c_i \langle v_i, v_j \rangle$$

$$= c_j \langle v_j, v_j \rangle$$

Now, $c_j \langle v_j, v_j \rangle = 0 \Rightarrow c_j = 0$ as $\langle v_j, v_j \rangle > 0$ being $v_j \neq 0$.

Hence S is linearly independent.

Corollary Any orthonormal subset of an inner product space V is linearly independent.

Let S be a set of non-zero orthogonal vectors in an inner product space V. Then we know, S is linearly independent.

Let $v \in$ span of S.

So, $v = c_1 v_1 + \cdots + c_n v_n$ for $c_1, \dots, c_n \in F$ and $v_1, \dots, v_n \in S$.

we get, $\langle v, v_i \rangle = c_i \|v_i\|^2 \quad \forall\ 1 \leq i \leq n$.

So, $$v = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i .$$

So if S is orthonormal, then
$$v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i .$$

Remark This indicates us the importance of an orthogonal / orthogonal basis of an inner product space as we can express the co-ordinates with respect to that basis via the inner product. This brings us to the Gram-Schmidt orthogonalisation process. In this process a linearly independent set transforms into an orthogonal set and hence to an orthonormal set. In fact we shall see that the spanning is preserved.

Thm Let V be an inner product space and $v_1, \dots, v_n \in V$ be any linearly independent vectors in V. Then $\exists$ orthogonal vectors $w_1, \dots, w_n \in V$ such that

$$\text{span of } \{v_1, \dots, v_k\} = \text{span of } \{w_1, \dots, w_k\} \quad \forall\ 1 \leq k \leq n.$$

Pf We prove it inductively.

Take $w_1 = v_1$. So $\{w_1\}$ is orthogonal and span of v_1 = span of w_1.

Suppose $w_1, \dots, w_k$ are orthonormal and obtained from $v_1, \dots, v_k$ such that span of $\{w_1, \dots, w_k\}$ = span of $\{v_1, \dots, v_k\}$.

Define,
$$w_{k+1} := v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} w_i.$$

Observe, ① $w_{k+1} \neq 0$.

If $w_{k+1} = 0$, then $v_{k+1} \in$ Span of $\{w_1, \dots, w_k\}$
$= $ span of $\{v_1, \dots, v_k\}$

This is not possible as $v_1, \dots, v_n$ are linearly independent.

② w_{k+1} is orthogonal to each w_j $\forall\, 1 \le j \le k$.

$$\langle w_{k+1}, w_j \rangle = \langle v_{k+1}, w_j \rangle - \sum_{i=1}^{k} \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} \langle w_i, w_j \rangle$$

$$= \langle v_{k+1}, w_j \rangle - \langle v_{k+1}, w_j \rangle$$

as $\langle w_i, w_j \rangle = 0$ $\forall\, 1 \le i \ne j \le k$.

$= 0$.

③ span of $\{w_1, \dots, w_{k+1}\}$ = Span $\{v_1, \dots, v_{k+1}\}$.

Because $w_{k+1} \in$ Span of $\{w_1, \dots, w_k, v_{k+1}\}$
$=$ Span of $\{v_1, \dots, v_k, v_{k+1}\}$

Remarks ① So from a linearly independent set $\{v_1, \dots, v_n\}$ we get an orthonormal set of vectors $\left\{ \frac{w_1}{\|w_1\|}, \dots, \frac{w_n}{\|w_n\|} \right\}$ such that span of $\{v_1, \dots, v_n\}$

$$= \text{span of } \left\{ \frac{w_1}{\|w_1\|}, \dots, \frac{w_n}{\|w_n\|} \right\}.$$

we have,

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$$

$$w_{k+1} = v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} w_i$$

Recipe to get orthogonal $w_1, \dots, w_n$ from $v_1, \dots, v_n$

② If to start with we have $v_1, \dots, v_n$ linearly dependent, then in the Gram-Schmidt process we will encounter $w_{k+1} = 0$ for some $1 \leq k \leq n-1$.

If we remove those w_i's, then with the remaining w_i's we get a set of non-zero orthogonal vectors and hence an orthonormal set.

③ Every finite dimensional inner product space has an orthogonal and hence an orthonormal basis.

Example consider $\mathbb{R}^3$ over $\mathbb{R}$ with standard inner product.

consider $S = \{ \underbrace{(3,0,4)}_{v_1}, \underbrace{(-1,0,7)}_{v_2}, \underbrace{(2,9,11)}_{v_3} \}$.

So, $w_1 = (3,0,4)$

$$w_2 = (-1,0,7) - \frac{\langle (-1,0,7), (3,0,4) \rangle}{\| (3,0,4) \|^2} (3,0,4)$$

$$= (-1,0,7) - \frac{25}{25} (3,0,4)$$

$$= (-4, 0, 3)$$

$$w_3 = (2,9,11) - \frac{\langle (2,9,11), (3,0,4) \rangle}{\| (3,0,4) \|^2} (3,0,4) - \frac{\langle (2,9,11), (-4,0,3) \rangle}{\| (-4,0,3) \|^2} (-4,0,3)$$

$$= (2,9,11) - 2(3,0,4) - (-4,0,3)$$

$$= (0,9,0)$$

Bessel's inequality

Let V be an inner product space and $\{v_1, \dots, v_n\}$ be an orthogonal set of non-zero vectors in V. Then for $v \in V$,

$$\sum_{i=1}^{n} \frac{|\langle v, v_i\rangle|^2}{\|v_i\|^2} \leq \|v\|^2.$$

Equality occurs

$\Leftrightarrow v \in$ span of $\{v_1, \dots, v_n\}$

$\Leftrightarrow v = \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i.$

Pf Let $\omega = \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i$

$$\underbrace{\|\omega\|^2}_{\langle \omega, \omega\rangle} = \left\langle \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i, \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i \right\rangle$$

$$= \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\|v_i\|^4} \overline{\langle v, v_i\rangle} \langle v_i, v_i\rangle$$

$$= \sum_{i=1}^{n} \frac{|\langle v, v_i\rangle|^2}{\|v_i\|^4} \|v_i\|^2$$

$$= \sum_{i=1}^{n} \frac{|\langle v, v_i\rangle|^2}{\|v_i\|^2} = \left\langle v, \sum_{i=1}^{n} \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i \right\rangle$$

$$= \langle v, \omega\rangle$$

Now, $\|v\|^2 = \|v - \omega + \omega\|^2$

$$= \langle (v-\omega)+\omega, (v-\omega)+\omega\rangle$$

$$= \langle v-\omega, (v-\omega)+\omega\rangle + \underbrace{\langle \omega, (v-\omega)+\omega\rangle}_{=0}$$

$$= \langle v-\omega, v-\omega\rangle + \underbrace{\langle v-\omega, \omega\rangle}_{=0} + \langle \omega, v-\omega\rangle + \langle \omega, \omega\rangle$$

$$\begin{cases} \langle v-w, w\rangle = \langle v, w\rangle - \langle w, w\rangle = 0 \\ \text{so, } \langle w, v-w\rangle = 0 \end{cases}$$

So, $\|v\|^2 = \langle v-w, v-w\rangle + \langle w, w\rangle$

$= \|v-w\|^2 + \|w\|^2$

So, $\|v\|^2 \geq \|w\|^2$.

Observe that, $\|v\|^2 = \|w\|^2$
$\Leftrightarrow v-w=0$ i.e. $v=w$
$\Leftrightarrow v \in$ span of $\{v_1, \dots, v_n\}$

Exercises

① If $B=\{v_1, \dots, v_n\}$ is an orthonormal basis of an inner product space V, then show that for $v \in V$,

$$[v]_B = \begin{pmatrix} \langle v, v_1\rangle \\ \vdots \\ \langle v, v_n\rangle \end{pmatrix}.$$

Solution write

$v = a_1 v_1 + \dots + a_n v_n$, $a_i \in F$ $\forall 1 \leq i \leq n$.

So, $[v]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

We show, $a_j = \langle v, v_j\rangle$ for each $1 \leq j \leq n$.

$$\langle v, v_j\rangle = \langle \sum_{i=1}^{n} a_i v_i, v_j\rangle$$

$$= \sum_{i=1}^{n} a_i \langle v_i, v_j\rangle$$

$= a_j \langle v_j, v_j\rangle$ as $\langle v_i, v_j\rangle = 0$ $\forall i \neq j$

$= a_j \|v_j\|^2 = a_j$ as $\|v_j\| = 1$ $\forall j$.

(2) Consider $\mathbb{C}^3$ over $\mathbb{C}$ with the standard inner product. Find an orthonormal basis for the subspace spanned by $(1,0,i)$, $(2,1,1+i)$.

Solution Denote $v_1 = (1,0,i)$

$v_2 = (2,1,1+i)$.

First we construct ω_1, ω_2 orthogonal vectors such that span of $\{v_1, v_2\}$ $=$ span of $\{\omega_1, \omega_2\}$.

By Gram-Schmidt orthogonalization process we get,

$$\omega_1 = v_1 = (1,0,i)$$

and
$$\omega_2 = v_2 - \frac{\langle v_2, \omega_1 \rangle}{\|\omega_1\|^2} \omega_1$$

$$= (2,1,1+i) - \frac{\langle (2,1,1+i), (1,0,i) \rangle}{\|(1,0,i)\|^2} (1,0,i)$$

$$= (2,1,1+i) - \left(\frac{3-i}{2}, 0, \frac{3i+1}{2} \right)$$

$$= \left(\frac{1+i}{2}, 1, \frac{1-i}{2} \right)$$

Now, $\|\omega_1\| = \sqrt{2}$, $\|\omega_2\| = \sqrt{2}$

So, $\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{i}{\sqrt{2}} \right), \left(\frac{1+i}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1-i}{2\sqrt{2}} \right) \right\}$ is an orthonormal basis for the subspace spanned by $\{(1,0,i), (2,1,1+i)\}$.

③ Let W be a subspace of a finite dimensional inner product space V over F. Define,

$$W^\perp := \{v \in V : \langle v, \omega \rangle = 0 \ \forall\ \omega \in W\}.$$

Show that $W^\perp$ is a subspace of V and $V = W \oplus W^\perp$. (The subspace $W^\perp$ is called the orthogonal complement of the subspace W).

<u>Solution</u> $W^\perp \neq \phi$ as $0 \in W^\perp$.

Let $\alpha, \beta \in W^\perp$ and $c \in F$. Let $\omega \in W$.

$$\text{Now, } \langle c\alpha + \beta, \omega \rangle = c\langle \alpha, \omega \rangle + \langle \beta, \omega \rangle$$
$$= c \cdot 0 + 0 = 0$$

So, $c\alpha + \beta \in W^\perp$. So $W^\perp$ is a subspace of V.

Note that $W \cap W^\perp = \{0\}$ because $\forall\ \omega(\neq 0) \in W$, we have $\langle \omega, \omega \rangle > 0$ and hence $\omega \notin W^\perp$.

Let $v \in V$. <u>To show</u> $v \in W + W^\perp$.

Let $\{v_1, \dots, v_m\}$ be a basis of W and then we extend it to a basis $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ of V.

Now by Gram-Schmidt orthogonalization process we get an orthogonal basis $\{\omega_1, \dots, \omega_n\}$ of V such that

$$\text{span of } \{v_1, \dots, v_k\} = \text{span}\{\omega_1, \dots, \omega_k\}$$
$$\forall\ 1 \leq k \leq n.$$

We have, $v \in \text{span}\{\omega_1, \dots, \omega_n\}$.

So write $v = \sum_{i=1}^{n} a_i w_i$

$$= \underbrace{\sum_{i=1}^{m} a_i w_i}_{= w} + \underbrace{\sum_{j=m+1}^{n} a_j w_j}_{= w'}$$

Note that, $w = \sum_{i=1}^{m} a_i w_i \in$ span of $\{w_1, \dots, w_m\}$

$= \text{span of } \{v_1, \dots, v_m\}$

$= W.$

Also, $w' = \sum_{j=m+1}^{n} a_j w_j \in W^{\perp}$ as $\langle \sum_{j=m+1}^{n} a_j, w_j, w_i \rangle$

$$= \sum_{j=m+1}^{n} a_j \langle w_j, w_i \rangle$$

$$= 0 \quad \forall \quad 1 \leq i \leq m.$$

So, $v = w + w'$ where $w \in W$, $w' \in W^{\perp}$.

Conclusions ① $\dim V = \dim W + \dim W^{\perp}$.

② The maximum number of linearly independent vectors that are orthogonal to every vector of W is $\dim W^{\perp}$ i.e. $\dim V - \dim W$.

④ Let $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Determine a basis of $W^{\perp}$.

Solution Note that $W = \{(\alpha, \beta, -\alpha-\beta) : \alpha, \beta \in \mathbb{R}\}$

$= \text{span of } \{(1, 0, -1), (0, 1, -1)\}.$

We extend $\{(1,0,-1), (0,1,-1)\}$ to a basis of $\mathbb{R}^3$ over $\mathbb{R}$. In fact $\{\underbrace{(1,0,-1)}_{v_1}, \underbrace{(0,1,-1)}_{v_2}, \underbrace{(1,0,0)}_{v_3}\}$ is a basis of $\mathbb{R}^3$.

By Gram-Schmidt orthogonalization process we get an orthogonal basis $\{\omega_1, \omega_2, \omega_3\}$ from $\{v_1, v_2, v_3\}$ where span of $\{\omega_1, \omega_2\}$ $=$ span of $\{v_1, v_2\}$.

$$\omega_1 = v_1 = (1,0,-1)$$

$$\omega_2 = v_2 - \frac{\langle v_2, \omega_1\rangle}{\|\omega_1\|^2}\omega_1$$

$$= (0,1,-1) - \frac{\langle (0,1,-1), (1,0,-1)\rangle}{2}(1,0,-1)$$

$$= (0,1,-1) - \left(\frac{1}{2}, 0, -\frac{1}{2}\right)$$

$$= \left(-\frac{1}{2}, 1, -\frac{1}{2}\right)$$

$$\omega_3 = v_3 - \frac{\langle v_3, \omega_1\rangle}{\|\omega_1\|^2}\omega_1 - \frac{\langle v_3, \omega_2\rangle}{\|\omega_2\|^2}\omega_2$$

$$= (1,0,0) - \frac{\langle (1,0,0), (1,0,-1)\rangle}{2}(1,0,-1)$$

$$- \frac{\langle (1,0,0), (-\frac{1}{2}, 1, -\frac{1}{2})\rangle}{3/2}\left(-\frac{1}{2}, 1, -\frac{1}{2}\right)$$

$$= (1,0,0) - \left(\frac{1}{2}, 0, -\frac{1}{2}\right) + \left(-\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}\right)$$

$$= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Now, $3\omega_3 \in W^\perp$ and $\dim W^\perp = 1$. So $\{(1,1,1)\}$ is a basis of $W^\perp$.

⑤ Consider $V = \{ f : [-1,1] \longrightarrow \mathbb{R} : f \text{ is continuous} \}$ over $\mathbb{R}$ with $\langle f, g \rangle = \int_{-1}^{1} f(t) g(t) dt$.

Let $W = \{ f \in V : f(-t) = -f(t) \}$. Then show that W is a subspace of V and find out $W^\perp$.

<u>Pf</u> Clearly $W \neq \Phi$ as $0 \in W$.

Let $f, g \in W$ and $c \in \mathbb{R}$.

Then
$$(cf+g)(-t) = cf(-t) + g(-t)$$
$$= -cf(t) - g(t)$$
$$= -(cf+g)(t).$$

So, $cf + g \in W$.

To find $W^\perp$, first we note that,

$S = \{ f \in V : f(-t) = f(t) \} \subseteq W^\perp$.

Let $f \in S, g \in W$. So, $\langle f, g \rangle$
$$= \int_{-1}^{1} f(t) g(t) dt$$
$$= \int_{-1}^{0} f(t) g(t) + \int_{0}^{1} f(t) g(t)$$
$$= -\int_{0}^{1} f(t) g(t) + \int_{0}^{1} f(t) g(t) = 0$$

Put $t = -u$

So $f(t) g(t)$

$= f(-u) g(-u)$

$= -f(u) g(u)$

So, $S \subseteq W^\perp$.

<u>Claim</u> $W^\perp = S$.

Let $f \in W^\perp$. so $\langle f, g \rangle = 0 \ \forall \ g \in W$.

Suppose $f \notin S$.

Note that any function f can be written as follows:

$$f(t) = \underbrace{\frac{f(t)+f(-t)}{2}}_{\text{Denote } f_{even}} + \underbrace{\frac{f(t)-f(-t)}{2}}_{f_{odd}}$$

f_{even} is even function and f_{odd} is odd function

So, $\langle f_{even} + f_{odd}, g \rangle = 0 \ \forall \ g \in W$

$\Rightarrow \langle f_{even}, g \rangle + \langle f_{odd}, g \rangle = 0 \ \forall \ g \in W$

$\Rightarrow \langle f_{odd}, g \rangle = 0 \ \forall \ g \in W$ as $f_{even} \in S \subseteq W^\perp$.

$\Rightarrow \langle f_{odd}, f_{odd} \rangle = 0$, a contradiction as $f \notin S$, $f_{odd} \neq 0$ and so $\langle f_{odd}, f_{odd} \rangle > 0$.

So if $f \in W^\perp$ then $f \in S$.

Hence $W^\perp = S$.

⑥ Let W be a subspace of an inner product space V. Let $v \in V$ be such that $\langle v, w\rangle + \langle w, v\rangle \leq \langle w, w\rangle \ \forall\, w \in W$. Show that $\langle v, w\rangle = 0 \ \forall\, w \in W$.

Pf We show, $\mathrm{Re}\langle v, w\rangle = 0$ and $\mathrm{Im}\langle v, w\rangle = 0$ and hence $\langle v, w\rangle = 0 \ \forall\, w \in W$. Let $w (\neq 0) \in W$.

Given $\mathrm{Re}\langle v, w\rangle \leq \|w\|^2$.

Claim $\mathrm{Re}\langle v, w\rangle = 0$.

If $\mathrm{Re}\langle v, w\rangle > 0$, then for $n \in \mathbb{N}$,

$$0 < \frac{1}{n}\mathrm{Re}\langle v, w\rangle = \mathrm{Re}\langle v, \frac{w}{n}\rangle \leq \frac{1}{n^2}\|w\|^2.$$

So, $0 < \mathrm{Re}\langle v, w\rangle \leq \dfrac{\|w\|^2}{n}$.

Taking $n \to \infty$ by Sandwich theorem we have $\mathrm{Re}\langle v, w\rangle = 0$.

If $\mathrm{Re}\langle v, w\rangle < 0$, then $\mathrm{Re}\langle v, -w\rangle > 0$.

As, $0 < \mathrm{Re}\langle v, -w\rangle \leq \dfrac{\|w\|^2}{n}$ we have $\mathrm{Re}\langle v, -w\rangle = 0$, hence

$$\mathrm{Re}\langle v, w\rangle = 0.$$

Claim $\mathrm{Im}\langle v, w\rangle = 0$.

If $\mathrm{Im}\langle v, w\rangle > 0$ then for $n \in \mathbb{N}$,

$$0 < \frac{1}{n}\mathrm{Im}\langle v, w\rangle = \mathrm{Im}\langle v, \frac{w}{n}\rangle$$
$$= \mathrm{Re}\langle v, \frac{i}{n}w\rangle \leq \frac{1}{n^2}\|w\|^2.$$

So, $0 < \mathrm{Im}\langle v, w\rangle \leq \frac{1}{n}\|w\|^2$.

By taking $n \to \infty$ we get, $\text{Im}\langle v, w\rangle = 0$.

If $\text{Im}\langle v, w\rangle < 0$, then $\text{Im}\langle v, -w\rangle > 0$.

So, $$0 < \text{Im}\langle v, -\frac{w}{n}\rangle = \text{Re}\langle v, -\frac{i}{n} w\rangle$$
$$\leq \frac{1}{n^2}\|w\|^2.$$

So, $$0 < \text{Im}\langle v, -w\rangle \leq \frac{1}{n}\|w\|^2.$$

Taking $n \to \infty$ we get, $\text{Im}\langle v, -w\rangle = 0$.

So, $\text{Im}\langle v, w\rangle = 0$.