DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)
2023-24 SECOND SEMESTER TUTORIAL SHEET-VIII

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1. Determine the radius of convergence of the following power series:

(i) $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$ (ii) $\sum_{m=0}^{\infty} m! x^m$ (iii) $\sum_{m=0}^{\infty} \frac{(x-1)^m}{(m+1)!}$ (iv) $\sum_{m=0}^{\infty} \frac{(-1)^m}{5^m}(x+1)^{3m}$.

2. Express the sum of the following sums in terms of x^k

$$\sum_{m=0}^{\infty} \frac{(m+3)^3}{(m+2)!} x^{m+1} \quad \text{and} \quad \sum_{m=3}^{\infty} \frac{m!}{(m+4)^2} x^{m-2}.$$

3. Find the ordinary points, regular singular points and irregular singular points of the following differential equations:

(i) $(x-1)^4 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ (ii) $(x-1)^2 x^2 \frac{d^2y}{dx^2} + 3(x-1)\frac{dy}{dx} + 7y = 0$

(iii)$x(x+1)^2 \frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + 16x^3 y = 0$ (iv) $2x^4 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x+1)y = 0$.

4. Find the power series solution of the following differential equations:

(i) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ in powers of x.

(ii) $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ in powers of x.

(iii)$\frac{d^2y}{dx^2} + (x-3)\frac{dy}{dx} + y = 0$ in powers of $(x-2)$.

(iv) $(x^2 - 4x + 5)\frac{d^2y}{dx^2} + (x-2)\frac{dy}{dx} - (x-2)y = 0$ in powers of $(x-2)$.

(v) $(1-x^2)\frac{d^2y}{dx^2} + 2y = 0$, where $y(0) = 4$ and $y'(0) = 5$.

(vi) $(x^2 - 1)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + xy = 0$, where $y(2) = 4$, and $y'(2) = 6$.

5. Find the series solution of the following differential equations about $x = 0$.

(i) $2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1-x^2)y = 0$.

(ii) $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$.

(iii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0$.

(iv) $x^2 \frac{d^2y}{dx^2} + (x + x^2)\frac{dy}{dx} + (x-9)y = 0$,

(v) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$.

6. Let $J_n(x), J_{-n}(x)$ are Bessel's function of first kind of order n and $-n$ respectively and $Y_n(x)$ is Bessel's function of second kind of order n. Prove that the solution of Bessel's equation of order n,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

is written in the form:

(a) $y = C_1 J_n(x) + C_2 J_{-n}(x)$, if n is not an integer.

(b) $y = C_1 J_0(x) + C_2 Y_0(x)$, if n is zero.

(c) $y = C_1 J_n(x) + C_2 Y_n(x)$, if n is an integer.

7. Show that $J_n(x)$ and $J_{-n}(x)$ Satisfies the following properties:

(i) $J_{-n}(x) = (-1)^n J_n(x)$ (ii) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ (iii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

(iv) $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ (v) $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$

(v) $J'_n(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$ (vi) $J'_n(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x)$

(vii) $J'_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$ (viii) (v) $2\frac{n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$

8. Find the solution of following equation in terms of Bessel's function

(i) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (k^2 x^2 - n^2)y = 0$ (ii) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \frac{1}{4}xy = 0$

(iii) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 4x^2(x^2 - \frac{n^2}{x^2})y = 0$ (iv) $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \frac{1}{4})y = 0$

9. Orthogonality Relation of Bessel's Function: If α and β are the roots of $J_n(x) = 0$, then

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha = \beta \end{cases}$$

10. (i) Show that between any two positive zeros of J_n there is a zero of J_{n+1}.

(ii) Show that between any two positive zeros of J_{n+1} there is a zero of J_n.

11. Using the definition of Legendre polynomials in the form of generating function

$$(1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$

prove the following recurrence relations:

(i) $(n+1)P_{n+1}(x) - (2n+1)x P_n(x) + n P_{n-1}(x) = 0$

(ii) $x P'_n(x) - P'_{n-1}(x) = n P_n(x)$

(iii) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

(iv) $P'_{n+1}(x) - x P'_n(x) = (n+1)P_n(x)$

12. Let $P_n(x)$ is a Legendre polynomial of order n, then show that:

(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n(1) = 1$ for all n (iii) $P'_{2n}(0) = $

(iv) $P'_{2n+1}(0) = \frac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$ (v) $\int_{-1}^1 P_n(x)dx = 0$ if $n \geq 1$

(vi) $\int_{-1}^1 x^m P_n(x)dx = 0$ (m an integer $< n$) (vii) $\int_{-1}^1 x^n P_n(x)dx = 0$

13. Orthogonal property of Legendre Polynomials:

$$\int_{-1}^1 P_m(x) P_n(x)dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

14. Show that any polynomial of degree n is a linear combination of $P_0, P_1, \ldots, P_n$, where $P_n(x)$ is a Legendre polynomial of order n.

15. Let P be any polynomial of degree n, and let $P = c_0 P_0 + c_1 P_1 + \ldots + c_n P_n$, where $c_0, c_1, \ldots, c_n$ are constants. Show that

$$c_k = \frac{2k+1}{2} \int_{-1}^1 P(x) P_k(x)dx, \qquad (k = 0, 1, \ldots, n).$$

16. The Chebyshev differential equation is

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \alpha^2 y = 0, \qquad \alpha \text{ a constant.}$$

(i) Determine two linearly independent solution in powers of x for $|x| < 1$.
(ii) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n.
(iii) Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2, 3$.

17. The equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0, \qquad -\infty < x < \infty,$$

where λ is a constant, is known as the Hermite equation.
(i) Find the first four terms in each of two linearly independent solutions about $x = 0$.

(ii) Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4, 6$. Note that each polynomial is determined only up to a multiplicative constant.

(iii) The Hermite polynomial $H_n(x)$ is defines as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), H_1(x), H_2(x), H_3(x)$.

18. Show that $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$ satisfies the differential equation $xy'' + (1 - x)y' + ny = 0$.

19. Find the characteristic values and characteristic functions of each of the following Sturm-Liouville problems:
(i) $y'' + \lambda y = 0$, $y(0) = 0, y(\pi/2) = 0$.
(ii) $\frac{d}{dx}[x\frac{dy}{dx}] + \frac{\lambda}{x}y = 0$ $y'(1) = 0, y'(e^{2\pi}) = 0$.
(iii) $\frac{d}{dx}[\frac{1}{(3x^2+1)}\frac{dy}{dx}] + \lambda(3x^2 + 1)y = 0$, $y(1) = 0, y(\pi) = 0$.

20. Consider the equation $y'' + \lambda y = 0$, $0 \le x \le \pi$. Find the eigenvalues and eigenfunctions in the following cases:
(i) $y'(0) = y'(\pi) = 0$, (ii) $y(0) = 0, y'(\pi) = 0$, (iii) $y(0) = y(\pi) = 0$,
(iv) $y'(0) = y(\pi) = 0$, (v) $y(-\pi) = y(\pi), y'(-\pi) = y'(\pi)$.

21. (Orthogonality of characteristic functions) Consider the Strum-Liouville problem consisting of the differential equation
$$\frac{d}{dt}(p(t)\frac{du}{dt}) + q(t)y + \lambda r(t)y = 0$$
where p, q and r are real functions such that $p(t)$ has a continuous derivative, $q(t)$ and $r(t)$ are continuous and $p(t) > 0$ and $r(t) > 0$ for all t on a real interval $a \le t \le b$, and λ is a parameter independent of t and the conditions
$$A_1 u(a) + A_2 u'(a) = 0$$
$$B_1 u(b) + B_2 u'(b) = 0$$
where A_1, A_2, B_1, B_2 are real constants such that A_1, A_2 are not both zero and B_1, B_2 are not both zero.
Let λ_m and λ_n be any two distinct characteristic values of the problem. Let ϕ_m be the

characteristic function corresponding to λ_m and ϕ_n be the characteristic function corresponding to λ_n. Prove that the characteristic functions ϕ_m and ϕ_n are orthogonal with respect to the weight function $r(t)$ on the interval $a \leq t \leq b$.

22. Show that the infinite set of functions $\{\phi_n\}$ where $\phi_n(t) = \sin nt (0 \leq t \leq \pi)$ is an orthogonal system with respect to weight function having the constant value 1.

23. Prove that the eigenvalues of Sturm-Liouville boundary value problem are always are real.

24. Verify that the characteristic functions ϕ_n and ϕ_m are orthogonal with respect to the weight function on the given interval:
(i) $y'' + \lambda y = 0, \quad y(0) = 0, y(\frac{\pi}{2}) = 0$.
(ii) $y'' + \lambda y = 0, \quad y'(0) = 0, y'(L) = 0$, where $L > 0$.
(iii) $\frac{d}{dx}[x\frac{dy}{dx}] + \frac{\lambda}{x}y = 0 \quad y'(1) = 0, y'(e^{2\pi}) = 0$.

25. Consider the set of functions ϕ_n, where $\phi_1(t) = \frac{1}{\sqrt{\pi}}$, $\phi_{n+1}(t) = \sqrt{\frac{2}{\pi}} \cos nt, \quad n = 1, 2, 3, \ldots$ on the interval $[0, \pi]$. Show that this set is an orthonormal system with respect to the weight function $r(t) = 1$ on $[0, \pi]$.