

* If $y_1(t)$ & $u(t)y_1(t)$ are two solutions then

$$u'(t) = \frac{c}{[y_1(t)]^2} \in (-\int P(x) dx)$$

Homogeneous linear ODE with constant coefficient

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R} \quad a \neq 0.$$

* Let m_1, m_2 be solutions of $am^2 + bm + c = 0$.

- (i) If $m_1 \neq m_2$ real then the general solution is given by $c_1 e^{m_1 t} + c_2 e^{m_2 t}$ for some $c_1, c_2 \in \mathbb{R}$.
- (ii) If $m_1 = m_2 = m$ then the general solution is given by $(c_1 + c_2 t) e^{m t}$ for some $c_1, c_2 \in \mathbb{R}$.
- (iii) If $m_1 \neq m_2$ are complex say $m_1 = \lambda + i\mu$ & $m_2 = \lambda - i\mu$
Then the general solution is given by $e^{\lambda t} (c_1 \cos \mu t + c_2 \sin \mu t)$ for some $c_1, c_2 \in \mathbb{R}$.

Consider the ODE $a t^v y'' + b t y' + c y = 0$, $t > 0$ with $a \neq 0$.

$$\text{let } t = e^x. \quad \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \cdot e^x = t \frac{dy}{dt} = t y'$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dt} \right) = \frac{d}{dx} \left(t \frac{dy}{dt} \right) = \frac{d}{dt} \left(t \frac{dy}{dt} \right) \frac{dt}{dx} \\ &= \left(\frac{dy}{dt} + t \frac{d^2 y}{dt^2} \right) e^x \\ &= t^v \frac{d^2 y}{dt^2} + t \frac{dy}{dt} \end{aligned}$$

$$\therefore a \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + b \frac{dy}{dt} + c y = 0$$

$$\Rightarrow a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + c y = 0$$

Let m_1, m_2 be two solutions of $a m^2 + (b-a)m + c = 0$

(i) If $m_1 \neq m_2$ real then the general solution is given by $c_1 e^{m_1 x} + c_2 e^{m_2 x}$ for some $c_1, c_2 \in \mathbb{R}$.
" $c_1 t^{m_1} + c_2 t^{m_2}$

(ii) If $m_1 = m_2 = m$ then the general solution is given by $(c_1 + c_2 x) e^{mx}$ for some $c_1, c_2 \in \mathbb{R}$.

$$\parallel$$

$$(c_1 + c_2 \ln(x)) x^m, \quad x > 0$$

(iii) If $m_1 \neq m_2$ are complex say $m_1 = \lambda + i\mu$ & $m_2 = \lambda - i\mu$
 Then the general solution is given by $e^{\lambda x} (c_1 \cos \mu x + c_2 \sin \mu x)$
 for some $c_1, c_2 \in \mathbb{R}$.

$$\parallel$$

$$x^\lambda [c_1 \cos(\mu \ln(x)) + c_2 \sin(\mu \ln(x))] \quad x > 0.$$

One can define for $x < 0$ as well by
 replacing x by $-x$.

* Homogeneous linear ODE (Second order).
with constant coefficient.

$$a y''(t) + b y'(t) + c y(t) = 0 \text{ --- } (1)$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$.

Let $y = e^{mt}$ be a solution of (1)

$$\Rightarrow (am^2 + bm + c) e^{mt} = 0$$

$$\Rightarrow am^2 + bm + c = 0 \quad (\because e^{mt} \neq 0)$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2)$$

Case 1, $m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ & $m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are

two different real solution of (2)

The the general solution of ① will be

$$y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}, \quad c_1, c_2 \in \mathbb{R}.$$

Case 2 :- Let $m_1 = m_2 = m = -\frac{b}{2a}$

Then e^{mt} is a solution.

Then let $u(t) e^{mt}$ be another solution

$$\text{then } u'(t) = \frac{1}{[y_1(t)]^2} e^{(-\int P(t) dt)} \quad P(t) = \frac{b}{a}$$

$$= \frac{1}{e^{2mt}} e^{-\frac{b}{a} \int dt}$$

$$= e^{-2(-\frac{b}{2a}) \cdot t} e^{-\frac{b}{a} t}$$

$$= e^{\frac{b}{a} t} \cdot e^{-\frac{b}{a} t} = 1$$

$$u(t) = t$$

The other solution will be $u(t) e^{mt} = t e^{mt}$

The general solution of (1) is

$$y(t) = c_1 e^{mt} + c_2 t e^{mt} \\ = (c_1 + c_2 t) e^{mt}.$$

Case 3 :- $m_1 = 1 + i\mu$ & $m_2 = 1 - i\mu$

Then $e^{(1+i\mu)t}$ & $e^{(1-i\mu)t}$ are solutions of (1)
" " " "

$$e^{1t} [\cos \mu t + i \sin \mu t] \quad e^{1t} [\cos \mu t - i \sin \mu t].$$

We know that if y_1 & y_2 are two linearly independent solutions, then $\frac{1}{2}(y_1 + y_2)$ & $\frac{1}{2}i(y_1 - y_2)$ are " " sol.
" " " "
 $e^{1t} \cos \mu t$ $e^{1t} \sin \mu t$

A general solution will be

$$c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t.$$

* **Euler-Cauchy** second order ODE.

$$a t^2 y''(t) + b t y'(t) + c y(t) = 0$$

$$\text{let } t = e^x$$

$$t > 0$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \cdot e^x = t y'(t)$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d}{dt} (t y'(t)) \frac{dt}{dx}$$

$$= (t y''(t) + y'(t)) t$$

$$= t^2 y''(t) + t y'(t)$$

$$\therefore t^2 y''(t) = \frac{d^2 y}{dx^2} - \frac{dy}{dx}$$

$$a \left(\frac{d^2 y}{dx^2} - \frac{dy}{dx} \right) + b \frac{dy}{dx} + cy = 0$$

$$\Rightarrow a \frac{d^2 y}{dx^2} + (b-a) \frac{dy}{dx} + cy = 0$$

$$(i) \quad m_1 \neq m_2 \in \mathbb{R} \quad y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \\ = c_1 t^{m_1} + c_2 t^{m_2}$$

$$(ii) \quad m_1 = m_2 = m, \quad y = (c_1 + c_2 x) e^{mx} \\ = (c_1 + c_2 \ln(t)) t^m$$

$$(iii) \quad m_1, m_2 \in \mathbb{C}, \quad y = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x. \\ = c_1 t^{\lambda} \cos(\mu \ln(t)) + c_2 t^{\lambda} \sin(\mu \ln(t)) \\ t > 0$$

If $t < 0$, then proceed with $-t$.

Look at non-homogeneous ODE of second order.

$$\text{Let } y'' + p(t)y' + q(t)y = r(t), \quad \underline{r(t) \neq 0.} \quad (1)$$

If y_1 & y_2 are two solutions of (1)

then $y_1 - y_2$ is a solution of

$$y'' + p(t)y' + q(t)y = 0.$$

If you know the general solution of $c_1 y_1 + c_2 y_2$

$y'' + p(t)y' + q(t)y = 0$ and a particular solution y_p of $y'' + p(t)y' + q(t)y = r(t)$,

then the general solution of (1) will be

$$c_1 y_1 + c_2 y_2 + y_p.$$

Method of Variation of Parameters :-

Let $y'' + p(x)y' + q(x)y = r(x)$ be an ODE.

Let y_1 and y_2 be two solution of

$$y'' + p(x)y' + q(x)y = 0.$$

Then $W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

$$\text{Then } y_p = -y_1 \int \frac{y_2(x) r(x)}{W(y_1, y_2)(x)} dx + y_2 \int \frac{y_1(x) r(x)}{W(y_1, y_2)(x)} dx$$

gives a Particular solution of

$$y'' + p(x)y' + q(x)y = r(x).$$

Solve:- $y'' - 2y' + y = \frac{e^t}{t^2 + 1}$ ——— ①

To find the general solution of $y'' - 2y' + y = 0$
let $y = e^{mx}$ is a solution. ——— ②

$$\text{Then } m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1$$

Thus e^t and $t e^t$ are two solutions of — ②

$$\text{let } y_1 = e^t \quad y_2 = t e^t$$

let y_p be a particular solution of ①

$$\begin{aligned} \text{Then } W(y_1, y_2)(t) &= \begin{vmatrix} e^t & t e^t \\ e^t & e^t + t e^t \end{vmatrix} \\ &= e^{2t} \end{aligned}$$

$$\begin{aligned}
 \text{Then } y_p &= -y_1 \int \frac{y_2(t) \gamma(t)}{W(y_1, y_2)(t)} dt + y_2 \int \frac{y_1(t) \gamma(t)}{W(y_1, y_2)(t)} dt \\
 &= -e^t \int \frac{t e^t \cdot e^t}{(1+t^2) e^{2t}} dt + t e^t \int \frac{e^t e^t}{(1+t^2) e^{2t}} dt \\
 &= -\frac{e^t}{2} \int \frac{2t dt}{1+t^2} + t e^t \int \frac{dt}{1+t^2} \\
 &= -\frac{e^t}{2} \ln(1+t^2) + t e^t \tan^{-1}(t).
 \end{aligned}$$

The general solution of ① is

$$c_1 e^t + c_2 t e^t - \frac{e^t}{2} \ln(1+t^2) + t e^t \tan^{-1}(t).$$