

**Important:** The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a \* are optional challenge problems and are not to be discussed in the tutorial.

**Problem 1**

Prove that any finite lattice is complete.

**Problem 2**

In [1] on page 122 it is stated that a finite poset is a lattice iff it has a greatest element and a least element. However this is not true. Prove that (i) a finite lattice has a greatest element and a least element and (ii) there is a finite poset with a greatest element and a least element which is not a lattice.

**Problem 3**

Prove that the meet and join of a complete lattice are commutative, associative and idempotent and have the absorption property, i.e., prove Proposition 4.2.2 of [1].

**Problem 4**

Given a set  $X$ , let  $S \subseteq 2^X$  be a collection of subsets of  $X$  such that

1.  $X \in S$ , and
2. if  $A_x \in S$  for all  $x \in I$  where  $I$  is some index set, then  $\cap_{x \in I} A_x$  is also in  $S$ .

Prove that  $(S, \subseteq)$  is a complete lattice.

**Problem 5**

Write out the complete proof of step 2 of Tarski's Fixed Point Theorem (Theorem 4.2.6 of [1].)

**Problem 6**

Suppose that  $(X, \preceq)$  is a lattice. For any  $k > 1$ , we define a relation  $\preceq_k$  on  $X^k$  as follows:  $(x_1, \dots, x_k) \preceq_k (y_1, \dots, y_k)$  if  $\forall i \in [k] : x_i \preceq y_i$ .

**Problem 6.1 ♠**

Prove that  $(X^k, \preceq_k)$  is a lattice. Beging by showing that it is a poset.

**Problem 6.2 \***

If  $(X, \preceq)$  is a complete lattice, is  $(X^k, \preceq_k)$  also a complete lattice?

**Problem 7**

A lattice  $X$  is called *distributive* if for all  $x, y, z \in X$ ,

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ , and
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ .

Give an example of a distributive lattice and a lattice which is not distributive.

**Problem 8**

Suppose that  $(X, \leq)$  is a finite distributive lattice and there is an  $a \in X$  such that  $a$  is minimal in  $X \setminus \{\perp\}$ . Let  $S_1 = \{x \in X : a \not\leq x\}$  and  $S_2 = \{x \in X : x = x' \vee a \text{ for some } x' \in S_1\}$ . Show that  $S_1$  and  $S_2$  form distributive lattices.

**Problem 9**

Wisden decides to publish a book with exactly 100 pages containing biographies and photos of the top 100 cricketers of all time. The cricketers are ranked from 1 to 100. Although the text can be broken over multiple pages, each photo has to appear on a single page. Multiple photos can appear on the same page but the photos must appear in order of rank, i.e., the photo of cricketer ranked  $i$  must appear before the photo of cricketer ranked  $j$  whenever  $i < j$ .

**Problem 9.1**

Prove using Tarski's fixed point theorem that there is at least one cricketer whose photo appears on a page number equal to their rank (e.g. the photo of the 47th greatest cricketer appears on page 47).

**Problem 9.2 \***

Give another proof using induction.

## References

- [1] J. Gallier. Discrete Mathematics for Computer Science: Some Notes arXiv:0805.0585, 2008.