COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 1

Announced on: Jan 4 (Thurs)

1. For each of the following statements, discuss why or why not it is a proposition?

    a) This statement is `FALSE`.

    b) This statement is `TRUE`.

    c) Give me an A grade!

2. Based on Problems 1.3(a) and 1.4 in [LLM17].

    Identify whether the following claims and their proofs are correct or bogus (and, in case of the latter, point out the bug in the claim and/or its proof).

    a) Claim: ${}^1\!/_8 > {}^1\!/_4$.

    Proof:

$$\begin{aligned} 3 &> 2 \\ 3\log_{10}({}^1\!/_2) &> 2\log_{10}({}^1\!/_2) \\ \log_{10}({}^1\!/_2)^3 &> \log_{10}({}^1\!/_2)^2 \\ \log_{10} {}^1\!/_8 &> \log_{10} {}^1\!/_4 \\ {}^1\!/_8 &> {}^1\!/_4 \end{aligned}$$

$\square$

    b) Claim: For all non-negative real numbers a and b, $\frac{a+b}{2} \geqslant \sqrt{ab}$.

    Proof:

$$\begin{aligned} (a-b)^2 &\geqslant 0 \\ a^2 + b^2 - 2ab &\geqslant 0 \\ a^2 + b^2 + 2ab &\geqslant 4ab \\ (a+b)^2 &\geqslant 4ab \\ a + b &\geqslant 2\sqrt{ab} \\ \frac{(a+b)}{2} &\geqslant \sqrt{ab} \end{aligned}$$

$\square$

3. a) The Twin Prime conjecture is one of the most famous open problems in mathematics. It says that there are infinitely many pairs of "nearby" prime numbers, i.e., primes that differ by 2 (e.g., 3 and 5, 11 and 13, and so on).

   Write the statement of the twin prime conjecture in formal notation. You may use quantifiers such as $\forall$ ("for all") and $\exists$ ("there exists"), and use the notation $\mathbb{N}$ for the set of natural numbers. You may also use the proposition $p(x)$ to denote "x is prime".

   Suppose, in the future, someone proves that the twin prime conjecture is false. What would be the correct statement of the result in that case? Write it in formal notation.

   b) Another well-known problem is Bertrand's postulate which says that for any natural number n, there is always a prime number between n and $2n$. Write the statement of Bertrand's postulate in formal notation.

   c) Here's another fact about primes: There are infinitely many prime numbers that do not have the digit 7 in their decimal expansion.[1] Write this statement in formal notation.

4. Prove or disprove that the following pairs of propositions are logically equivalent:

   a) $\neg(p \vee q \vee r) \vee s$ and $(\neg p \vee s) \wedge (\neg q \vee s) \wedge (\neg r \vee s)$

   b) $(p \wedge q) \Rightarrow r$ and $(p \Rightarrow r) \vee (q \Rightarrow r)$

   c) $(p \Rightarrow q) \Rightarrow r$ and $p \Rightarrow (q \Rightarrow r)$

5. Prove or disprove the following:

   a) Every natural number can be written as either the sum of two perfect squares or the difference of two perfect squares or both. (You may include 0^2 if needed.)

   b) There exist irrational numbers x and y such that x^y is rational.

   c) $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.

[1] Actually, the result is true for any choice of missing digit in $\{0,1,\ldots,9\}$.