COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 7: Countable and Uncountable Sets.
20 March 2025

Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM. Questions marked with a * may be a little harder and can be considered optional.

Note: For the purposes of this sheet we will assume that $\mathbb{N}$ includes 0.

Problem 1
In class we discussed the proof of $\mathbb{N}$ bij $\mathbb{N}^2$. Write down the exact bijection between the two sets.

Problem 2
For a fixed $k \geq 2$, write down the exact bijection that proves $\mathbb{N}$ bij $\mathbb{N}^k$.

Problem 3 [LLM18, Prob. 8.30]
We call a real number *quadratic* when it is the root of a degree two polynomial with integer coefficients. Prove that the set of quadratic reals is countable.

Problem 4 ♠
Show that $\{0,1\}^{\mathbb{N}}$ bij $\{0,1,2\}^{\mathbb{N}}$.

Problem 5
In class we tried to show $C^{\mathbb{N}}$ bij $\mathbb{R}$ where $C = \{0,1,2,\ldots,9\}$ and $C^{\mathbb{N}}$ denotes the set of vectors with coordinates indexed by $\mathbb{N}$, each of whose coordinates are drawn from C. Here we complete that proof.

Problem 5.1
We define $f : C^{\mathbb{N}} \to \mathbb{R}$ as follows: For $\boldsymbol{x} \in C^{\mathbb{N}}$,

$$f(\boldsymbol{x}) = \sum_{i \geq 0} \boldsymbol{x}(i) \cdot 10^{-(i+1)}.$$

Show that f is a total surjection from $C^{\mathbb{N}}$ to $[0,1)$.

Problem 5.2
Argue that f as defined above is not an injection. Either using f or through some other route prove $C^{\mathbb{N}}$ bij $[0,1)$.

Problem 5.3
Show $\mathbb{N} \times [0,1)$ bij $\mathbb{R}$.

Problem 5.4
Using all the subproblems above show that $C^{\mathbb{N}}$ bij $\mathbb{R}$.

Problem 6 [LLM18, Prob. 8.26(b)]
Prove by a diagonalization argument that $\{1,2,3\}^{\mathbb{N}}$ is uncountable.

Problem 7 [LLM18, Prob. 8.17]
A vector $\boldsymbol{x} \in \{0,1\}^{\mathbb{N}}$ is called *lonely* if $\boldsymbol{x}(i)$ and $\boldsymbol{x}(i+1)$ are *not* both 1 for any $i \in \mathbb{N}$. Show that the set of lonely vectors is uncountable. (Note: The book contains a hint.)

Problem 8 [LLM18, Prob. 8.22]
Suppose $\mathcal{S} = \{S_0, S_1, \ldots\}$ is a countable set whose elements are infinite subsets of the non-negative integers. Show using diagonalization that there is an infinite set of non-negative integers U such that $U \notin \mathcal{S}$ and also no subset of U is in $\mathcal{S}$. (Note: The book contains a hint.)

References

[LLM18] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science, June 2018, MIT Open Courseware.