Q. (a) Let q be a cts. fn. on $[0,\infty)$ &

$$\lim_{x\to\infty} q(x) = 10.$$

Prove that the solⁿ $y(x)$ of IVP

$$y' + 5y = q(x), \quad y(0) = 1$$

is bdd. on $[0,\infty)$

(b) Find the general solⁿ of non-hom. eqⁿ.

$$x^2 y'' + 3xy' + y = \frac{1}{x}$$

Solⁿ. (a)

$$y' + 5y = q(x)$$

$$\text{I.f.} = e^{\int 5\,dx} = e^{5x}$$

$$\Rightarrow y \cdot e^{5x} = \int q(x) \cdot e^{5x}\,dx + c$$

$$\Rightarrow y = e^{-5x}\int q(x)\,e^{5x}\,dx + c\,e^{-5x} \quad (*)$$

$q(x)$ is bdd. $\Rightarrow |q(x)| \le M$

for some $M > 0$

$$\Rightarrow |y| \le e^{-5x}\int M\cdot e^{5x}\,dx + C\cdot e^{-5x}$$

$$\le e^{-5x}\cdot \frac{Me^{5x}}{5} + C\cdot e^{-5x}$$

$$\le \underbrace{\frac{M}{5} + C\cdot e^{-5x}}_{\text{bdd. in } [0,\infty)}$$

as $e^{-5x} \in (0,1]$ in $[0,\infty)$

$\Rightarrow$ y is bdd.

Or. $(*) = y = \dfrac{\int e^{5x} q(x)\,dx}{e^{5x}} + \underbrace{ce^{-5x}}_{\text{bdd. in } [0,\infty)}$

$q(x), e^{5x}$ is cts.

$\Rightarrow$ using L-Hopitals rule

$$\lim_{x\to\infty} \frac{\int_0^x e^{5t} q(t)\,dt}{e^{5x}} = \lim_{x\to\infty} \frac{e^{5x}\cdot q(x)}{5\cdot e^{5x}}$$

$$= \frac{10}{5} = 2$$

$\Rightarrow$ y is bdd.

(for $[0,n]$ $e^{5x}q(x)$ is bdd &
when $n \to \infty$, then y is again bdd.

(b) $x^2y'' + 3xy' + y = \frac{1}{x}$, $x > 0$ — (1)

Let $y = x^m$ is a solⁿ of

$$\boxed{x^2y'' + 3xy' + y = 0}$$

$$m(m-1)x^m + 3mx^m + x^m = 0$$

$$\Rightarrow (m^2 - m + 3m + 1)x^m = 0$$

$$\Rightarrow (m^2 + 2m + 1)\underset{\neq 0}{x^m} = 0$$

$$\Rightarrow m = -1, -1$$

$$y_{hom.} = C_1 \cdot \frac{1}{x} + C_2 \frac{\ln x}{x}$$

$y_1 = \frac{1}{x}$, $y_2 = \frac{\ln x}{x}$

$y_1' = -\frac{1}{x^2}$, $y_2' = \frac{1 - \ln x}{x^2}$

$$W(y_1, y_2) = \begin{vmatrix} \frac{1}{x} & \frac{\ln x}{x} \\ -\frac{1}{x^2} & \frac{1-\ln x}{x^2} \end{vmatrix}$$

$$= \frac{1}{x^3}(1-\ln x) + \frac{1}{x^3}\ln x$$

$$= \frac{1}{x^3}$$

Convert the ODE into standard form

$$y'' + \frac{3y'}{x} + \frac{y}{x^2} = \frac{1}{x^3}$$

$$y_p = y_1 \underbrace{\int \frac{-y_2 \cdot \frac{1}{x^3}}{W(y_1, y_2)}\,dx}_{u_1} + y_2 \underbrace{\int \frac{y_1 \cdot \frac{1}{x^3}}{W(y_1, y_2)}\,dx}_{u_2}$$

$$u_1 = \int \frac{-\frac{\ln x}{x} \cdot \frac{1}{x^3}}{\frac{1}{x^3}}\,dx = -\frac{(\ln x)^2}{2}$$

$$u_2 = \int \frac{\frac{1}{x} \cdot \frac{1}{x^3}}{\frac{1}{x^3}}\,dx = \ln x$$

$$\Rightarrow y_p = -\frac{1}{x} \cdot \frac{(\ln x)^2}{2} + \frac{(\ln x)^2}{x} = \frac{(\ln x)^2}{2x}$$

$$\Rightarrow \boxed{y(x) = \frac{c_1}{x} + c_2 \cdot \frac{\ln x}{x} + \frac{(\ln x)^2}{2x}}$$

Q-2 (a).

let y_1, y_2 be two solutions on $I \subset \mathbb{R}$.

let $x_0 \in I$ be such that $y_1(x_0) = 0 = y_2(x_0)$.

$$\Rightarrow \quad W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0.$$

$\Rightarrow$ y_1, y_2 are L.D.

$\Rightarrow$ $y_2(x) = c \cdot y_1(x)$ for some constant $c \in \mathbb{R}$.

$\Rightarrow$ y_1 & y_2 have all zeros in common. //.

(b). Given. $y'' + a_1 y' + a_2 y = 0$, $t \in I$, $a_1, a_2 \in \mathbb{R}$.

auxiliary eq^n: $m^2 + a_1 m + a_2 = 0$

$$\Rightarrow \quad m = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$\Rightarrow \quad m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad m_2 = -\frac{a_1}{2} - \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

and $y_1 = e^{m_1 t}$ and $y_2 = e^{m_2 t}$ are two sol^n.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$= e^{\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \cdot t} \cdot \left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}\right) \cdot e^{\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \cdot t}$$

$$- \left(\frac{-a_1 + \sqrt{a_1^2 + 4a_2}}{2}\right) \cdot e^{\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} t} \cdot e^{\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \cdot t}$$

$$= \left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}\right) \cdot e^{-a_1 t} - \left(\frac{-a_1 + \sqrt{a_1^2 + 4a_2}}{2}\right) \cdot e^{-a_1 t}$$

$$= -\sqrt{a_1^2 - 4a_2} \cdot e^{-a_1 t} \quad \Rightarrow \quad W(y_1, y_2)(t) \text{ is constant } \Leftrightarrow a_1 = 0. \; //.$$

Question 3:(a) Let $F(s) = \mathcal{L}(f(t))(s)$, defined for $s > 0$, and suppose that $|f(t)| \le M$ for all $t \ge 0$. Show that

$$\lim_{s\to\infty} F(s) = 0$$

(b) Using Laplace transform, solve the IVP:

$$y'(t) - 4\int_0^t y(\tau)\,d\tau = 1, \quad y(0) = 0.$$

Solution:

(a)

Since $|f(t)| \le M$, we have

$$|F(s)| = \left|\int_0^\infty e^{-st} f(t)\,dt\right| \le \int_0^\infty e^{-st}|f(t)|\,dt \le \int_0^\infty e^{-st} M\,dt = M\int_0^\infty e^{-st}\,dt = \frac{M}{s}.$$

So for any $s > 0$, we have:

$$-\frac{M}{s} \le F(s) \le \frac{M}{s}.$$

Applying the Sandwich (Squeeze) Theorem we get as $s \to \infty$,

$$\lim_{s\to\infty} F(s) = 0,$$

since

$$\lim_{s\to\infty} \frac{M}{s} = 0.$$

(b)

Let $Y(s) = \mathcal{L}[y(t)]$. Take the Laplace transform :

$$\mathcal{L}[y'(t)] - 4\mathcal{L}\left[\int_0^t y(\tau)\,d\tau\right] = \mathcal{L}[1].$$

We know:

$$\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s),$$

$$\mathcal{L}\left[\int_0^t y(\tau)\,d\tau\right] = \frac{1}{s}Y(s), \quad \text{and} \quad \mathcal{L}[1] = \frac{1}{s}.$$

So the equation becomes:

$$sY(s) - 4\cdot\frac{1}{s}Y(s) = \frac{1}{s}.$$

$$\left(s - \frac{4}{s}\right)Y(s) = \frac{1}{s}.$$

$$Y(s) = \frac{1}{s^2 - 4}.$$

Now take the inverse Laplace transform:

$$Y(s) = \frac{1}{s^2 - 2^2} = \frac{1}{4}\left(\frac{1}{s-2} - \frac{1}{s+2}\right).$$

So,

$$y(t) = \frac{1}{4}\left(e^{2t} - e^{-2t}\right) = \frac{1}{2}\sinh(2t).$$

Ques 4) Using the Laplace Transform, solve the following system

$$x'(t) = 3x(t) + 4y(t) + \delta(t)$$
$$y'(t) = -2x(t) + y(t) + 5\delta(t)$$
$$x(0) = 1, \quad y(0) = -2$$

Solution) We know that
$$L(x'(t)) = sL(x) - x(0)$$
$$L(y'(t)) = sL(y) - y(0)$$

$$\Rightarrow L(x'(t)) = sX(s) - 1 \qquad \text{where } L(x) = X(s)$$
$$L(y'(t)) = sY(s) + 2 \qquad L(y) = Y(s)$$

Also, $\mathcal{L}(\delta(t)) = 1$

Taking Laplace Transform for the given system of equations we obtain

$$sX(s) - 1 = 3X(s) + 4Y(s) + 1$$
$$sY(s) + 2 = -2X(s) + Y(s) + 5$$

$$\Rightarrow (s-3)X(s) - 4Y(s) = 2 \quad \text{------} \quad (0.5 \text{ marks})$$
$$2X(s) + (s-1)Y(s) = 3 \quad \text{------} \quad (0.5 \text{ marks})$$

The above system of equations can be written as

$$\begin{bmatrix} s-3 & -4 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} s-3 & -4 \\ 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{(s-3)(s-1)+8} \begin{bmatrix} (s-1) & 4 \\ -2 & (s-3) \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{s^2-4s+11} \begin{bmatrix} 2s+10 \\ 3s-13 \end{bmatrix}$$

$$\Rightarrow X(s) = \frac{2s+10}{s^2-4s+11} \quad \text{------} \quad (1 \text{ mark})$$

$$Y(s) = \frac{3s-13}{s^2-4s+11} \quad \text{------} \quad (1 \text{ mark})$$

Then $x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{2s+10}{s^2-4s+11}\right)$

$$= \mathcal{L}^{-1}\left(\frac{2(s-2)+14}{(s-2)^2+7}\right)$$

$$= 2\mathcal{L}^{-1}\left(\frac{(s-2)}{(s-2)^2+(\sqrt{7})^2}\right) + 2\sqrt{7}\,\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{(s-2)^2+(\sqrt{7})^2}\right)$$

$$= 2e^{2t}\cos(\sqrt{7}t) + 2\sqrt{7}e^{2t}\sin(\sqrt{7}t) \quad \text{-----} \quad (1\text{ mark})$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{3s-13}{s^2-4s+11}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{3(s-2)-7}{(s-2)^2+(\sqrt{7})^2}\right)$$

$$= 3\mathcal{L}^{-1}\left(\frac{(s-2)}{(s-2)^2+(\sqrt{7})^2}\right) - \sqrt{7}\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{(s-2)^2+(\sqrt{7})^2}\right)$$

$$= 3e^{2t}\cos(\sqrt{7}t) - \sqrt{7}e^{2t}\sin(\sqrt{7}t) \quad \text{-------} \quad (1\text{ mark})$$

Final solution,

$$x(t) = 2e^{2t}\left[\cos(\sqrt{7}t) + \sqrt{7}\sin(\sqrt{7}t)\right]$$

$$y(t) = e^{2t}\left[3\cos(\sqrt{7}t) - \sqrt{7}\sin(\sqrt{7}t)\right]$$

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Q5 Find $A(y)$ such that the equation
$$(2x + A(y))\,dx + 2xy\,dy = 0 \rightarrow (1)$$
is exact and solve it.

Sol

Taking $M(x,y) = 2x + A(y)$
and $N(x,y) = 2xy$.

$\Rightarrow$ (1) becomes : $M(x,y)\,dx + N(x,y)\,dy = 0 \rightarrow (II)$
Equation (II) will be exact if and only if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
Now, if A is differentiable, then
$$\frac{\partial M}{\partial y} = A'(y) \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y.$$

$\Rightarrow$ (1) will be exact iff
$$A'(y) = 2y$$
$$\Rightarrow A(y) = y^2 + C_0.$$

Thus, the required exact equation is given by :
$$(2x + y^2 + C_0)\,dx + 2xy\,dy = 0 \rightarrow (III)$$

General solution of (III) is given by
$$F(x,y) = C.$$
where F is such that
$$\frac{\partial F}{\partial x} = M = 2x + y^2 + C_0 \rightarrow (IV)$$
and $$\frac{\partial F}{\partial y} = N = 2xy. \rightarrow (V)$$

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Integrating (iv) with respect to x, we get:

$$F = x^2 + xy^2 + C_0 x + g(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = 2xy + g'(y)$$

$\Rightarrow$ By (v)

$$2xy + g'(y) = 2xy.$$

$$\Rightarrow g(y) = C_1$$

$$\therefore F(x,y) = x^2 + xy^2 + C_0 x + C_1$$

$\Rightarrow$ General sol: $x^2 + xy^2 + C_0 x = \bar{C}$

$[\bar{C} = C - C_1]$.

One can consider $C_0 = 0$ & hence get

$$x^2 + xy^2 = \bar{C}$$

Question 6: Solve the following system of first-order linear ordinary differential equations using the method of variation of parameters:

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{G}(t),$$

where

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 2\sqrt{2} \end{pmatrix}, \quad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{G}(t) = \begin{pmatrix} e^{\sqrt{2}t} \\ 0 \end{pmatrix}.$$

[4]

Solⁿ:- The characteristic polynomial of A is

$$p(x) = \det(xI - A)$$
$$= x^2 - 2\sqrt{2}x + 2$$
$$= (x - \sqrt{2})^2$$

Eigenvalues of A are $\sqrt{2}, \sqrt{2}$

$\lambda = \sqrt{2}$.

The eigenspace corr. to $\lambda = \sqrt{2}$ is

$$E_\lambda = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, (A - \lambda I)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, \begin{bmatrix} -\sqrt{2} & -2 \\ 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow \left. \begin{aligned} -\sqrt{2}a - 2b &= 0 \\ a + \sqrt{2}b &= 0 \end{aligned} \right\} \Rightarrow a + \sqrt{2}b = 0$$
$$\Rightarrow a = -\sqrt{2}b$$

$$E_\lambda = \left\{ \begin{bmatrix} -\sqrt{2}b \\ b \end{bmatrix} \,\middle|\, b \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \right\}$$

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Eigenvector corr. to $\lambda = \sqrt{2}$ is $\vec{v} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$

Now, we have to find $\vec{u}$ satisfying

$$(A - \lambda I)\vec{u} = \vec{v}$$

$$\begin{bmatrix} -\sqrt{2} & -2 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} -\sqrt{2}\, u_1 - 2u_2 &= -\sqrt{2} \\ u_1 + \sqrt{2}\, u_2 &= 1 \end{aligned}\right\} \Rightarrow u_1 + \sqrt{2}\, u_2 = 1$$

Take $u_2 = 0$, then $u_1 = 1$

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The solution of homogeneous system $y'(t) = Ay(t)$ is given by

$$\vec{y}_c = c_1 \underbrace{\vec{v} e^{\lambda t}}_{\vec{y}^{(1)}} + c_2 \underbrace{\left(\vec{v} t e^{\lambda t} + \vec{u} e^{\lambda t} \right)}_{\vec{y}^{(2)}}$$

where
$$\vec{y}^{(1)} = \begin{bmatrix} -\sqrt{2}\, e^{\sqrt{2}t} \\ e^{\sqrt{2}t} \end{bmatrix}$$

$$\vec{y}^{(2)} = \begin{bmatrix} -\sqrt{2}t\, e^{\sqrt{2}t} + e^{\sqrt{2}t} \\ t e^{\sqrt{2}t} \end{bmatrix}$$

Fundamental matrix is given by

$$\phi(t) = \begin{bmatrix} \vec{y}^{(1)} & \vec{y}^{(2)} \end{bmatrix} = \begin{bmatrix} -\sqrt{2}\, e^{\sqrt{2}t} & (-\sqrt{2}t + 1) e^{\sqrt{2}t} \\ e^{\sqrt{2}t} & t e^{\sqrt{2}t} \end{bmatrix}$$

$$\det \phi(t) = |\phi(t)| = -e^{2\sqrt{2}t}$$

$$\Rightarrow \phi(t)^{-1} = \frac{1}{-e^{2\sqrt{2}t}} \begin{bmatrix} t e^{\sqrt{2}t} & (-1 + \sqrt{2}t) e^{\sqrt{2}t} \\ -e^{\sqrt{2}t} & -\sqrt{2}\, e^{\sqrt{2}t} \end{bmatrix}$$

$$\phi(t)^{-1} = \begin{bmatrix} -t\,e^{-\sqrt{2}t} & (\sqrt{2}t+1)\,e^{-\sqrt{2}t} \\ e^{-\sqrt{2}t} & \sqrt{2}\,e^{-\sqrt{2}t} \end{bmatrix}$$

Now, we use the method of variation of parameters to find the particular solution

Let $\vec{y}_p = \phi(t)\,\vec{\omega}(t)$

where $\vec{\omega}'(t) = \phi(t)^{-1}\,G$

$$= \begin{bmatrix} -t\,e^{-\sqrt{2}t} & (1-\sqrt{2}t)\,e^{-\sqrt{2}t} \\ e^{-\sqrt{2}t} & \sqrt{2}\,e^{-\sqrt{2}t} \end{bmatrix}\begin{bmatrix} e^{\sqrt{2}t} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -t \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{\omega}(t) = \int \begin{bmatrix} -t \\ 1 \end{bmatrix} dt = \begin{bmatrix} -t^2/2 \\ t \end{bmatrix}$$

The general solution is given by

$$\vec{y} = \vec{y}_c + \vec{y}_p$$

where $\vec{y}_p = \phi(t)\,\vec{\omega}(t)$

$$= \begin{bmatrix} e^{\sqrt{2}t}\left(t - \frac{1}{\sqrt{2}}t^2\right) \\ e^{\sqrt{2}t}\cdot t^2/2 \end{bmatrix}$$

Q7 $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$

$T : V \to V$ defined by

$T(X) = X + X^T$

(a) Show that T is a linear transformation.

(b) Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

(a) Let $c \in \mathbb{R}$, $X, Y \in V$ then

$$T(cX+Y) = (cX+Y) + (cX+Y)^T$$
$$= (cX+Y) + (cX^T + Y^T)$$
$$= c(X+X^T) + Y+Y^T$$
$$= cT(X) + T(Y)$$

(b) Basis for $\mathrm{Ker}(T)$

$T(X) = 0$

$\Rightarrow X + X^T = 0 \quad \Rightarrow X^T = -X$ (skew symmetric)

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

$\Rightarrow a = 0, \; b = -c, \; d = 0$

$\Rightarrow X = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}$ where $c \in \mathbb{R}$

$\Rightarrow \mathrm{Ker}(T) = \left\{ c \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$

Thus, $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is the basis for $\ker(T)$

<u>Basis for Im(T)</u>

$Im(T) = \{ T(X) \mid X \in V \}$

let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$T(X) = X + X^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

$$= 2a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b+c) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad Im(T) \subseteq \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

where the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is

linearly independent

Also, it is easy to see

$$\text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq Im(T) \quad — (*)$$

(Aliter

$\dim(Im(T)) = \dim V - \dim(\ker(T)) = 4 - 1 = 3$

This implies $(*)$ as well)

Thus, basis for $Im(T)$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Question 8 :→

(i) Consider the matrix

$$A = \begin{pmatrix} a & -1 & 0 \\ -1 & b & -1 \\ 0 & -1 & a \end{pmatrix}$$

where a, b are real numbers. Determine the conditions on the pair (a, b) under which all the eigenvalues of A are positive.

(ii) Consider $V = \mathbb{C}$, the 2-dimensional vector space of complex numbers over the field of real numbers, $\mathbb{R}$. Let the basis $\mathcal{B} = \{1, 1+i\}$ be ordered as it is displayed. For $\alpha \in \mathbb{C}$, define the linear transformation $M_\alpha : V \to V$ given by $M_\alpha(\beta) = \alpha\beta$.

Compute the matrix $[M_\alpha]_\mathcal{B}$. Also find α for which $Ker(M_\alpha) \neq 0$

Ans :→

(i). For eigenvalue

$$\begin{vmatrix} a-\lambda & -1 & 0 \\ -1 & b-\lambda & -1 \\ 0 & -1 & a-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)\left[\lambda^2 - \lambda(a+b) + ab - 1\right] + (\lambda - a) = 0$$

$$\Rightarrow (a-\lambda)\left[\lambda^2 - \lambda(a+b) + ab - 2\right] = 0$$

So, either. $\lambda = a$, or $\lambda^2 - \lambda(a+b) + (ab-2) = 0$

$$\Rightarrow \lambda_i = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4(ab-2)}}{2}$$

$$= \frac{(a+b) \pm \sqrt{(a-b)^2+8}}{2}.$$

Since all the eigenvalues are positive so, $a > 0$
and $\lambda_i > 0 \ \forall i = 1, 2$.

Now, Since. $(a-b)^2 + 8 > 0 \ \forall a, b$

So, λ_i's are ~~are~~ all real numbers.

So, $(a+b) \pm \sqrt{(a-b)^2+8} > 0$

$\Rightarrow (a+b)^2 > (a-b)^2 + 8$

$\Rightarrow 4ab > 8$

$\Rightarrow ab > 2$

So, all eigen values are positive if $a > 0$ and $ab > 2$.

(ii). Given $V = \mathbb{C}$ over $\mathbb{R}$.

and basis $B = \{1, 1+i\}$

Let, $\alpha = a+ib$

So, $M_\alpha(1) = (a+ib).1$

$$= (a-b).1 + b.(1+i)$$

$$M_\alpha(1+i) = (a+ib)(1+i)$$

$$= (a-b) + i(b+a)$$

$$= -2b + (a+b)(1+i)$$

So, $[M_\alpha]_\beta = \begin{pmatrix} a-b & -2b \\ b & a+b \end{pmatrix}$

$$\left|[M_\alpha]_\beta\right| = (a-b)(a+b) + 2b^2$$

$$= a^2 - b^2 + 2b^2$$

$$= a^2 + b^2$$

So, For $\mathrm{Ker}([M_\alpha]_\beta) \neq 0$

$$\Rightarrow a^2 + b^2 = 0$$

$$\Rightarrow a = 0 \text{ and } b = 0$$

$$\Rightarrow \alpha = 0 + 0.i = 0$$

So, For $\alpha = 0$, $\mathrm{Ker}([M_\alpha]_\beta) \neq 0$

$\square$.