COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 3: Induction.
23 January 2025

Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM.

Problem 1 ([BSD05], Prob 13, pp 127)
Use mathematical induction to prove that the number of subsets of a set of size n is 2^n.

Problem 2 ([LLM18], Prob 5.8)
If $F(n)$ denotes the n-th Fibonacci number then prove by induction that for all $n \geq 1$,

$$F(n-1) \cdot F(n+1) - F(n)^2 = (-1)^n.$$

Problem 3 ([LLM18], Prob 5.6)
In class we discussed the problem of tiling a $2^n \times 2^n$ yard with L-shaped tiles and a central square empty (c.f. [LLM18] Sec. 5.1.5). Here we take a slightly different approach. First, prove by induction that it is possible to tile a $2^n \times 2^n$ yard with L-shaped tiles and a *corner* square empty. Then use this result to prove the original result that requires the central square to be empty.

Problem 4 ([LLM18], Prob 5.16(k))
Suppose $P(n)$ is a predicate on non-negative integers such that $\forall k : P(k) \Rightarrow P(k+2)$. (Note that since the set on which the predicate is defined is understood, it has been left out of the logical sentence). Is the following proposition valid for all P that satisfy the above condition?

$$(\exists n : P(n)) \Rightarrow (\forall n : \exists m : (m > n) \wedge P(m)).$$

Since P is not known, it may be the case that for some P the proposition is valid and for some other P it is not valid. Think through the cases and then try the problem.

Problem 5 ([LLM18], Prob 5.18)
You are given n coins numbered $0, \ldots, n-1$. The value of coin i is 2^i. Show by induction that for any number k such that $1 \leq k \leq 2^n - 1$ it is possible to find a subset of coins whose value is exactly k.

Problem 6 ([BSD05], Prob 14, pp 127)
Prove that the strong principle of mathematical induction implies the weak principle of mathematical induction, i.e., if we accept that the deduction rule of the strong principle is sound then the deduction rule of the weak principle is also sound.

Problem 7 ♠

We are given an array $A[n]$ containing 0s and 1s only. We want to sort it so that all the 0s appear before all the 1s, e.g. if $A = [1, 0, 0, 1, 0, 0]$ our output should be $A = [0, 0, 0, 0, 1, 1]$. Prove by induction that the procedure "Sort 0-1" correctly achieves this. Begin by identifying the correct induction variable.

Set $p_0 \leftarrow 0$, $p_1 \leftarrow n-1$.
while $p_0 < p_1$ **do**
    while $p_0 < n$ and $A[p_0] = 0$ **do**
        $p_0 \leftarrow p_0 + 1$
    end while
    while $p_1 > -1$ and $A[p_1] = 1$ **do**
        $p_1 \leftarrow p_1 - 1$
    end while
    if $p_0 < p_1$ **then**
        Swap $A[p_0], A[p_1]$.
    end if
end while

Problem 8
Suppose we are given a linked list with integers stored in each node and suppose that the linked list is maintained in sorted order. Write an algorithm for inserting a new element ℓ in the linked list. Prove the correctness of your algorithm using mathematical induction on the number of elements in the linked list.

Problem 9
Integer trees are a recursively defined data type. Every tree is either an empty tree, we denote it $\mathsf{EmptyTree}$ or a tuple of the form (ℓ, T_1, T_2) where ℓ is an integer and T_1 and T_2 are trees which we call the left and right subtrees respectively. Write an algorithm for finding the minimum integer in the tree. Assume for simplicity that all integers stored are non-negative. Prove the correctness of your algorithm using mathematical induction.

References

[BSD05] K. Bogart, S. Drysdale, C. Stein. Discrete Math for Computer Science Students. 2005.

[LLM18] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science, June 2018, MIT Open Courseware.