2202-MTL101B Minor Exam

Tatsam

TOTAL POINTS

23 / 30

QUESTION 1

1 Linear Independence 2 / 6

+ 0 pts Not attempted

+ 0 pts Completely wrong/deserves no partial marks

+ 6 pts Correct

+ 4 pts Showing that $$\{Av_1, Av_2, \dots, Av_n\}$$ is LI implies $$\{v_1,v_2\dots, v_n\}$$ is LI.

✓ **+ 2 pts** *Showing that converse is not true with a counter example.*

+ 1 pts For providing the definition of LI precisely and applying it correctly.

QUESTION 2

2 Basis of Polynomial Space 6 / 6

✓ **- 0 pts** *Correct [{x,x^3, 4 - 5 x^2 + x^4 } is a basis, and Dim 3]*

- 1 pts Conclude the basis correctly but did not mention the dimension/ basis is not written in polynomial form/or some minor mistake

- 2 pts Conclude dimension is 3 with the correct procedure but the basis is not correct.

- 4 pts Only conclude that Dimension is 3 but the procedure is not correct/ or procedure is correct but get dimension NOT 3

- 6 pts Wrong Solution

QUESTION 3

3 Basis of Matrix Space 6 / 6

✓ **- 0 pts** *Correct*

- 6 pts Unattempted/completely incorrect

- 2 pts In Part (i) did not find the dimension or completely incorrect

- 2 pts Part (ii) completely incorrect or not attempted

- 1 pts Part (ii) partially correct

- 2 pts Part (i) did not show that it is a subspace

- 1 pts In part (i) some mistakes in finding the dimension/Incorrect argument/argument missing

- 1 pts In part (i) incomplete argument to show the subspace

QUESTION 4

4 Solution Space 3 / 6

- 0 pts Correct

- 6 pts Not correct/RRE corresponding to a matrix is unique/not attempted

- 4 pts This just a particular case of $$2\times 2$$ Matrix/partial marks awarded/second argument is not correct/

✓ **- 3 pts** *Partially correct/ arguments missing (not correct) if $$A$$ is not invertible/last part is not clear*

- 5 pts Major arguments missing

- 1 pts minor arguments missing

QUESTION 5

5 Row Reduced Echelon Form **6 / 6**

✓ **+ 6 pts** *Correct*

+ 5 pts RRE form is correct and Rank is 1 but there is some minor mistake

+ 2 pts Rank is 1 & proof is correct

+ 1 pts Tried but did not converge anywhere

+ 2 pts RRE is not correct but is very close to the correct one and contains (0,1,0)

+ 0 pts Either totally wrong or not attempted

Prob 1) $M_{n\times n}(\mathbb{R})$.

If $\{AV_1, AV_2 ---- AV_n\}$ is linearly ind. in $M_{n\times 1}(\mathbb{R})$

this means $\alpha_1 AV_1 + \alpha_2 AV_2 + \alpha_3 AV_3 + ---- \alpha_n AV_n = 0$
is True if & only if
$\alpha_1 = \alpha_2 = \alpha_3 = ---- = \alpha_n = 0$

assume $\alpha_1 A = \beta_1 \quad \alpha_2 A = \beta_2 \quad ------ \quad \alpha_n A = \beta_n$

$\beta_1 V_1 + \beta_2 V_2 + ---------- \beta_n V_n = 0$ —— (1)
holds true for : ~~αA~~

$$\left\{\begin{array}{l} \alpha_1 A \Rightarrow (0)A = 0 = \beta_1 \\ \alpha_2 A \Rightarrow (0)A = 0 = \beta_2 \\ \vdots \\ \alpha_n A \Rightarrow (0)A = 0 = \beta_n \end{array}\right\}$$ this is an only if statement.

$\therefore$ (1) holds true if and only if
$\beta_1 = \beta_2 = \beta_3 ---------- = \beta_n = 0$

$\therefore \{V_1 V_2 ----- V_n\}$ are linearly independent.

Converse :
given $\alpha_1 V_1 + \alpha_2 V_2 + ---- + \alpha_n V_n = 0$
holds true if & only if $\alpha_1 = \alpha_2 --- = \alpha_n = 0$

in order to show $\{AV_1, AV_2 ------ AV_n\}$ to be LI we must show
$\alpha_1 AV_1 + \alpha_2 AV_2 + ---- + \alpha_n AV_n = 0$ holds if and only if $\alpha_1 = \alpha_2 ---- = \alpha_n = 0$

In order to disprove, we can take any contradicting example.

say $n = 2$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

given : v_1 v_2 are L.I which true

$\therefore$ $\alpha v_1 \neq v_2$ for any α.

$$\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \neq \begin{pmatrix} 4 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{matrix} \alpha = 4 \\ \alpha = 0 \end{matrix}\right\} \text{not possible.}$$

Now, $AV_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$AV_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

we can easily see $AV_2 = 4\,(AV_1)$ which means $\{AV_1, AV_2\}$ are not linearly independent as $\exists\ \alpha, \beta$ both of which are not 0, such that

$$\alpha AV_1 + \beta AV_2 = 0$$

$\alpha = 4,\ \beta = -1$

Hence the converse is not true.

1 Linear Independence **2 / 6**

+ 0 pts Not attempted

+ 0 pts Completely wrong/deserves no partial marks

+ 6 pts Correct

+ 4 pts Showing that $\{Av_1, Av_2, \dots, Av_n\}$ is LI implies $\{v_1,v_2\dots, v_n\}$ is LI.

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सभी पृष्ठों पर लिखें। Write on all pages.

अनुक्रमांक Entry No. 2022MT61969

अनुवर्ती पुस्तिका संख्या CONTINUATION BOOK No. 1

पाठ सं. Course No. MTL101

ग्रुप संख्या Group No. 11

दिनांक Date 18 04 2023

Problem 2) $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

this vector space is governed by 5 variables $a_0\ a_1\ a_2\ a_3\ a_4$.

basis of V : $\{1, x, x^2, x^3, x^4\}$

Now, $W = \{p(x) \in V \mid p(1) = -p(-1)\ \&\ p(2) = -p(-2)\}$

$p(1) = -p(-1)$.

$\Rightarrow \cancel{a_0} + a_1 + \cancel{a_2} + a_3 + \cancel{a_4} = \cancel{a_0} - a_1 + \cancel{a_2} - a_3 + \cancel{a_4}$

$2a_1 = -2a_3$

$\boxed{a_1 = -a_3}$ — 1

$p(2) = -p(-2)$

$\cancel{a_0} + 2a_1 + 4\cancel{a_2} + 8a_3 + 16a_4 = \cancel{a_0} - 2a_1 + 4\cancel{a_2} - 8a_3 + 16\cancel{a_4}$

$4a_1 = -16a_3$

$\boxed{a_1 = -4a_3}$ — 2

Eq^n ① and ② give us that

$$a_1 = a_3 = 0.$$

and there is no restriction on a_0, a_2, a_4

$\therefore$ W contains $p(x)$ of the form

$$a_0 + a_2 x^2 + a_4 x^4.$$

$p(1) = -p(-1)$

$$a_0 + \cancel{a_1} + a_2 + \cancel{a_3} + a_4 = -a_0 + \cancel{a_1} - a_2 + \cancel{a_3} - a_4$$

$$\boxed{a_0 + a_2 + a_4 = 0} \quad — ①$$

$p(2) = -p(-2)$

$$a_0 + 2\cancel{a_1} + 4a_2 + 8\cancel{a_3} + 16a_4 = -a_0 + 2\cancel{a_1} - 4a_2 + 8\cancel{a_3} - 16a_4$$

$$\boxed{a_0 + 4a_2 + 16a_4 = 0} \quad — ②$$

take $a_0 = \lambda_0$

eq^n ① $\Rightarrow$ $a_2 + a_4 = -\lambda_0$ — ③

eq^n (2) $4a_2 + 16a_4 = -\lambda_0$ — (4)

(4) $- 4 \times$ (3)

$$12a_4 = 3\lambda_0$$

$$a_4 = \frac{1}{4}\lambda_0$$

$$a_2 = -\lambda_0 - a_4 = \frac{-5}{4}\lambda_0$$

There are no restrictions on a_1 & a_3

let them be $\lambda_1 = a_1$ and $\lambda_2 = a_3$

$$\left(\begin{array}{l} a_0 = \lambda_0 \\ a_1 = \lambda_1 \\ a_2 = -5/4\,\lambda_0 \\ a_3 = \lambda_2 \\ a_4 = 1/4\,\lambda_0 \end{array} \right)$$

These are the constraints.

$\therefore$ W has the $p(x)$ of the form

$$p(x) = \lambda_0 + \lambda_1 x + \left(\frac{-5}{2}\lambda_0\right)x^2 + \lambda_2 x^3 + \left(\frac{\lambda_0}{4}\right)x^4$$

with 3 free variables λ_0 λ_1 & λ_2.

$\therefore$ basis of $W \equiv \left\{ x, x^3, \left(1 - \frac{5x^2}{2} + \frac{x^4}{4}\right) \right\}$

$\boxed{\text{dimension} = 3}$

2 Basis of Polynomial Space 6 / 6

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Problem 3:

$$A = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$\bar{A}^T = \begin{pmatrix} \bar{c}_1 & \bar{c}_3 \\ \bar{c}_2 & \bar{c}_4 \end{pmatrix}$$

$A = \bar{A}^T \implies c_1 = \bar{c}_1$

$c_2 = \bar{c}_3$

$\bar{c}_2 = c_3$

$c_4 = \bar{c}_4$

$\therefore$ A must be of the form $\begin{pmatrix} \alpha & \gamma + i\delta \\ \gamma - i\delta & \beta \end{pmatrix}$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

(i) if we take $\alpha = \beta = \gamma = \delta = 0$ $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ Additive identity $0 \in S$

Now, if we prove
a. commutative $(A_1 + A_2 = A_2 + A_1)$
b. associative $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$, $(\alpha\beta)A_1 = \alpha(\beta A_1)$
c. distributive $\alpha(A_1 + A_2) = \alpha A_1 + \alpha A_2$, $(\alpha + \beta)A_1 = \alpha A_1 + \beta A_1$
d. closure over $+$ and $\cdot$
e. additive inverse
f. mult. identity

then S becomes a vector space.

(a), (b), (c), (d) hold true as they are properties of $F=\mathbb{R}$ and $a_{ij} \in \mathbb{C}$.

e. take $\alpha' = -\alpha$ $\beta' = -\beta$ $\gamma' = -\gamma$ $\delta' = -\delta$
we get A'
s.t $A' + A = 0$

f. take $\gamma = \delta = 0$ and $\alpha = \beta = 1$
$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ represents mult. identity.

dimension of S is 4 as we need 4 different variables to span S.

basis : $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

(ii) No. Let's check closure over :
(taking any random example to disprove)

$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$\bar{A}^T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$\therefore A \in S$

take $c_1 = i \in \mathbb{C}$
then $c_1 A$ must also belong to S.

$$B = C_1 A = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

$$\bar{B}^T = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

$$B \neq \bar{B}^T \quad \therefore \quad B \notin S$$

Hence S is not a vector space over field $\mathbb{C}$.

Problem 4) $AX = 0$
in order to solve this, we try to reduce A in the row reduced echlon form by a sequence of elementary operations. $(A \,\vdots\, 0) \longrightarrow (A' \,\vdots\, 0)$
$\hookrightarrow$ RRE form.

We know $f(A) = f(I)\,A$
where f is any elemantary row op.

Also $f(AB) = f(I)(AB)$
$= (f(I)(A))(B)$
$= f(A)\,B$

Now, $f(A^2) = f(A)\,A$

3 Basis of Matrix Space 6 / 6

✓ **- 0 pts** *Correct*

- 6 pts Unattempted/completely incorrect

- 2 pts In Part (i) did not find the dimension or completely incorrect

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$$B = C_1 A = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

$$\bar{B}^T = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

$B \neq \bar{B}^T \quad \therefore \quad B \notin S$

Hence S is <u>not</u> a vector space over field $\mathbb{C}$.

<u>Problem 4)</u> $AX = O$

in order to solve this, we try to reduce A in the row reduced echlon form by a sequence of elementary operations. $(A \,\vdots\, O) \longrightarrow (A' \,\vdots\, O)$

$\quad\quad \hookrightarrow$ RRE form.

We know $\quad f(A) = f(\underline{I})\,A$

where f is any elemantary row op.

Also $\quad f(AB) = f(I)(AB)$

$\quad\quad = (f(I)(A))(B)$

$\quad\quad = f(A)\,B$

Now, $\quad f(A^2) = f(A)\,A$

To show both of them have the same solution space, we need to show all the solutions of $AX = 0$ also satisfy $A^2X = 0$ and vice versa.

$A^2X = 0 \Rightarrow A(AX) = 0 \Rightarrow A(0) = 0$

$\therefore$ All sol^n of $AX = 0$ satisfy $A^2X = 0$.

We need to prove the converse now.

case (I) $\quad \text{rank}(A) = n \Rightarrow |A| \neq 0 \Rightarrow A^{-1}$ exists.

$A^2X = 0$

$A^{-1}A \cdot AX = A^{-1} \cdot 0$

$AX = 0$ satisfies.

case II $\quad \text{rank}(A) < n = m$

given $\text{rank}(A^2) = m$.

This means A and A^2 have the same no. of non-zero rows in the RRE form.

While reducing A^2 into its RRE form, let us first use the same sequence of row operations that converted A into its RRE form.

$P_1 P_2 P_3 \text{----} P_K (A) \Rightarrow \text{RRE} = \text{say } (A_R)$

Now $P(A^2) = P(A)A$ {proved above}

$$P_1 P_2 P_3 \cdots P_K (A^2) = A_R A$$

Now, since we are given rank (A) = rank (A^2)

After calculating $A_R A$ and again reducing to RRE form, we are sure to arrive at the same A_R!

The zero rows in A_R would reduce the rows of A to 0 which became 0 while calculating A_R; and the other rows would return the non zero rows as it is.

$\therefore$ RRE of A = RRE of A^2

Hence the solution space of $AX = 0$ and $A^2 X = 0$ is exactly the same.

4 Solution Space 3 / 6

- 0 pts Correct

- 6 pts Not correct/RRE corresponding to a matrix is unique/not attempted

- 4 pts This just a particular case of $$2\times 2$$ Matrix/partial marks awarded/second argument is not correct/

✓ **- 3 pts** *Partially correct/ arguments missing (not correct) if $$A$$ is not invertible/last part is not clear*

- 5 pts Major arguments missing

- 1 pts minor arguments missing

Problem 5) $AX = b$

$$X = \begin{pmatrix} -2+\alpha \\ 2 \\ \alpha+\beta \end{pmatrix}$$

As we can see the no. of unknows were 3 and there exist ∞ soln $\therefore$ rank(A) = rank (A | b) < 3

let's take variables as x_1 x_2 x_3

$x_1 = \alpha - 2$

$\therefore$ this True for $\forall$ $\alpha \in \mathbb{R}$

$\alpha - 2 = t \in \mathbb{R}$

$x_1 = t$

$x_2 = 2$

$x_3 = \alpha + \beta = t + \beta + 2$, again this takes all $\mathbb{R}$ values for diff β. $\therefore$ $t + \beta + 2 = \omega \in \mathbb{R}$

$x_3 = \omega$

So, basically the solution set is $(t, 2, \omega)$ $t, \omega \in \mathbb{R}$.

which means x_1 ~~&~~, and ~~&~~ x_3 were free variables ~~and~~ ~~none~~ ~~of~~ ~~them~~ ~~had~~ ~~a~~ ~~leading~~ also $x_2 = 2$ fixed.

Hence, we are sure that the RRE form of (A|b) must look something like :

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

given $b = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$, we can estimate A could be $\left(\begin{array}{ccc|c} 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\left\{ \text{applying } \begin{array}{l} R_2 \to R_2 + R_1 \\ R_1 \to 2R_1 \end{array} \right\} \longrightarrow$

RRE form of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

rank(A) = no. of non zero rows = $\boxed{1}$.

5 Row Reduced Echelon Form 6 / 6

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लघु परीक्षा उत्तर पुस्तिका
MINOR TEST ANSWER BOOK

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पाठ्यक्रम सं.
Course No. MTL101

ग्रुप संख्या
Group No. 11

पाठ्यक्रम शीर्षक
Course Title LINEAR ALGEBRA & DIFFERENTIAL EQUATIONS

लघु परीक्षा सं.
Minor Test No. 1

दिनांक
Date 18-04-2023

प्रयोग किए गए अनुवर्ती पृष्ठों की संख्या
No. of continuation sheets used 1

प्रश्न सं. Q.No.	प्राप्त अंक Marks
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पाठ्यक्रम निर्धारक के हस्ताक्षर और दिनांक
Signature of Course Co-ordinator and date