DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)

2023-24 SECOND SEMESTER TUTORIAL SHEET-VI

---

1. Find all solutions of the following equations:
(a) $y'' - 4y = 0$ (b) $3y'' + 2y' = 0$ (c) $y'' + 16y = 0$
(d) $y'' = 0$ (e) $y'' - 4y' + 5y = 0$

2. Consider the equation $y'' + y' - 6y = 0$
(a) Compute the solution ϕ satisfying $\phi(0) = 1,\ \phi'(0) = 0.$
(b) Compute the solution ψ satisfying $\psi(0) = 0,\ \psi'(0) = 1.$
(c) Compute $\phi(1)$ and $\psi(1)$.

3. Find all solutions of $y'' + y = 0$ satisfying:
(a) $\phi(0) = 1,\ \phi(\frac{\pi}{2}) = 2$ (b) $\phi(0) = 0,\ \phi(\pi) = 0$
(c) $\phi(0) = 0,\ \phi'(\frac{\pi}{2}) = 0$ (d) $\phi(0) = 0,\ \phi(\frac{\pi}{2}) = 0$

4. Consider the equation $y'' + a_1 y' + a_2 y = 0$, where the constants a_1, a_2 are real. Suppose $\alpha + i\beta$ is a complex root of the characteristic polynomial, where α, β are real, $\beta \neq 0$.
(a) Show that $\alpha - i\beta$ is also a root.
(b) Show that any solution ϕ may be written in the form $\phi(x) = e^{ax}(d_1 \cos\beta x + +d_2 \sin\beta x)$, where d_1, d_2 are constants.
(c) Show that $\alpha = -\frac{a_1}{2}, \beta^2 = a_2 - \frac{a_1^2}{4}$.
(d) Show that every solution tends to zero as $x \longrightarrow +\infty$ if $a_1 > 0$.
(e) Show that the magnitude of every non-trivial solution assumes arbitrarily large values as $x \longrightarrow +\infty$ if $a_1 < 0$.

5. Show that every solution of the constant coefficient equation $y'' + a_1 y' + a_2 y = 0$ tends to zero as $x \longrightarrow \infty$ if, and only if, the real parts of the roots of the characteristic polynomial are negative.

6. Show that every solution of the constant coefficient equation $y'' + a_1 y' + a_2 y = 0$ is bounded on $0 \leq x < \infty$ if, and only if, the real parts of the roots of the characteristic polynomial are non positive and the roots with zero real part have multiplicity one.

7. Consider the equation $y'' + k^2 y = 0$, where k is a non-negative constant.
(a) For what values of k will there exist non-trivial solutions ϕ satisfying
(i) $\phi(0) = 0,\ \phi(\pi) = 0,$
(ii) $\phi'(0) = 0,\ \phi'(\pi) = 0,$
(iii) $\phi(0) = \phi(\pi),\ \phi'(0) = \phi'(\pi),$
(iv) $\phi(0) = -\phi(\pi),\ \phi'(0) = -\phi'(\pi)$?
(b) Find the non-trivial solutions for each of the cases $(i) - (iv)$ in (a).

8. Find the solutions of the following initial value problems:

(a) $y'' - 2y' - 3y = 0,\ y(0) = 0,\ y'(0) = 1$
(b) $y'' + 10y = 0,\ y(0) = \pi,\ y'(0) = \pi^2$

9. The functions ϕ_1, ϕ_2 defined below exist for $-\infty < x < \infty$. Determine whether they are linearly dependent or independent there.
(a) $\phi_1(x) = \cos x,\ \phi_2(x) = \sin x$
(b) $\phi_1(x) = x^2,\ \phi_2(x) = 5x^2$
(c) $\phi_1(x) = x,\ \phi_2(x) =\mid x \mid$

10. (a) Show that the functions $\phi_1(x) = x^2,\quad \phi_2(x) = x \mid x \mid$, are linearly independent for $-\infty < x < \infty$.
(b) Compute the Wronskian of these functions.

11. Consider the equation $y'' + a_1 y' + a_2 y = 0$, where $a_1,\ a_2$ are real constants such that $4a_2 - a_1^2 > 0$. Let $\alpha + i\beta,\quad \alpha - i\beta\quad (\alpha,\ \beta$ real) be the roots of the characteristic polynomial.
(a) Show that ϕ_1, ϕ_2 defined by $\phi_1(x) = e^{\alpha x}\cos\beta x,\quad \phi_2(x) = e^{\alpha x}\sin\beta x$ are solutions of the equation.
(b) Compute $W(\phi_1, \phi_2)$, and show that ϕ_1, ϕ_2 are linearly independent on any interval I.

12. Find all solutions of the following equations:
(a) $y'' + 4y = \cos x$ (b) $y'' + 9y = \sin 3x$
(c) $y'' + y = \tan x,\ (\frac{-\pi}{2} < x < \frac{\pi}{2})$
(d) $y'' - 4y' + 5y = 3e^{-x} + 2x^2$ (e) $y'' - 7y' + 6y = \sin x$
(f) $y'' + y = 2\sin x \sin 2x$ (g) $y'' + y = \sec x,\ (\frac{-\pi}{2} < x < \frac{\pi}{2})$
(h) $y'' - y = e^x$ (i) $6y'' + 5y' - 6y = x$

13. Consider the equation $y'' + \omega^2 y = A\cos\omega x$, where A, ω are positive constants.
(a) Find all solutions on $0 \le x < \infty$.
(b) Show that every solution ϕ is such that $\mid \phi(x) \mid$ assumes arbitrarily large as $x \longrightarrow \infty$.

14. Are the following sets of functions defined on $-\infty < x < \infty$ linearly independent or dependent there?. Why?.
(a) $\phi_1(x) = 1, \phi_2(x) = x, \phi_3(x) = x^3$ (b) $\phi_1(x) = e^{ix}, \phi_2(x) = \sin x, \phi_3(x) = 2\cos x$
(c) $\phi_1(x) = x, \phi_2(x) = e^{2x}, \phi_3(x) =\mid x \mid$

15. Find all solutions of the following equations:
(a) $y''' - 8y = 0$ (b) $y^{(4)} + 16y = 0$ (c) $y''' - 5y'' + 6y' = 0$ (d) $y^{(100)} + 100y = 0$
(d) $y^{(4)} - 16y = 0$ (e) $y^{(4)} + 5y'' + 4y = 0$ (f) $y''' - 3y' - 2y = 0$

16. (a) Compute the wronskian of four linearly independent solutions of the equation $y^{(4)} + 16y = 0$.
(b) Compute that solution ϕ of this equation which satisfies $\phi(0) = 1, \phi'(0) = 0, \phi''(0) = 0, \phi'''(0) = 0$.

17. Find four linearly independent solutions of the equation $y^{(4)} + \lambda y = 0$, in case
(a) $\lambda = 0$ (b) $\lambda > 0$ (c) $\lambda < 0$

18. Consider the equation $y''' - 4y' = 0$.
(a) Compute three linearly independent solutions.
(b) Compute the wronskian of the solutions found in (a).
(c) Find that solution ϕ satisfying $\phi(0) = 0, \phi'(0) = 1, \phi''(0) = 0$.

19. Consider the equation $y^{(5)} - y^{(4)} - y' + y = 0$.
(a) Compute five linearly independent solutions.
(b) Compute the wronskian of the solutions found in (a).
(c) Find that solution ϕ satisfying $\phi(0) = 1, \phi'(0) = \phi''(0) = 0, \phi'''(0) = \phi^{(4)} = 0$.

20. Find all real valued solutions of the following equations:
(a) $y'' + y = 0$ (b) $y'' - y = 0$ (c) $y^{(4)} - y = 0$ (d) $y^{(5)} + 2y = 0$
(d) $y^{(5)} + 2y = 0$ (e) $y^{(4)} - 5y'' + 4y = 0$.

21. Find the solution ϕ of the initial-value problem
$y''' + y = 0,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$.

22. Consider the equation $y^{(4)} - k^4 y = 0$, where k is a real constant.
(a) Show that $\cos kx, \sin kx, \cosh kx, \sinh kx$ are solutions if $k \neq 0$.
(Note $\cosh u = (e^u + e^{-u})/2, \sinh u = e^u - e^{-u})/2$).
(b) Show that there are non-trivial solutions ϕ satisfying $\phi(0) = 0, \phi'(0) = 0, \phi(1) = 0, \phi'(1) = 0$, if and only if $\cos k \cosh k = 1$ and $k \neq 0$.
(c) Compute all non-trivial solutions satisfying the conditions in (b).
(d) For what values of k will there exist non-trivial solutions satisfying
$\phi^{(j)}(0) = \phi^{(j)}(1), \quad (j = 0, 1, 2, 3)$?.
(e) Compute all non-trivial solutions satisfying the conditions in (d).

23. Find all solutions of the following equations:
(a) $y''' - y' = x$ (b) $y^{(4)} + 16y = \cos x$ (c) $y^{(4)} - y = \cos x$
(d) $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^x$

24. Find a particular solution of each of the following equations:
(a) $y'' + 4y = \cos x$ (b) $y'' + 4y = \sin 2x$ (c) $y'' - 4y = 3e^{2x} + 4e^{-x}$
(d) $y'' - y' - 2y = x^2 + \cos x$ (e) $y'' + 9y = x^2 e^{3x}$
(f) $y''' = x^2 + e^{-x} \sin x$ (g) $y''' + 3y'' + 3y' + y = x^2 e^{-x}$

25. Find all solutions of the following equations for $x > 0$:
(a) $x^2 y'' + 2xy' - 6y = 0$ (b) $2x^2 y'' + xy' - y = 0$
(c) $x^2 y'' + xy' - 4y = x$ (d) $x^2 y'' - 5xy' + 9y = x^3$
(e) $x^3 y''' + 2x^2 y'' - xy' + y = 0$

26. Find the particular solution of $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = (x^2 + x^{-3})^{-1}, \quad x > 0$, given that two solutions of the associated homogeneous equations are $y_1 = x$ and $y_2 = \frac{1}{x}$.
(Ans: $y = \frac{1}{2}(1 - x) \ln x + (x - \frac{1}{x}) \ln(1 - x)$).

27. Show that the solutions y_1 and y_2 of the linear second order differential equation $y'' +$

$a_1(x)y' + a_2(x)y = 0$ that satisfy the conditions $y_1(x_0) = 1, y_1'(x_0) = 0, y_2(x_0) = 0, y_2'(x_0) = 1$, respectively, are linearly independent.

28. Show that the two point boundary value problem $y'' + \pi^2 y = 0,\ y(0) = 0, y(1) = 1$ has no solution while the two point boundary value problem $y'' + \pi^2 y = 0,\ y(0) = 0, y(1) = 0$ has infinite number of solutions.

29. For the constant coefficient differential equation $y'' + 2\alpha y' + \beta^2 y = 0, \quad \alpha, \beta > 0$ show that the solutions are exponentially decaying if $\alpha, \beta > 0$ (decaying exponentially if $\alpha \geq \beta$ and oscillating if $\alpha < \beta$), while the solutions are purely oscillating if $\alpha = 0$.

30. Consider the constant coefficient equation $L(y) = y'' + a_1 y' + a_2 y = 0$. Let ϕ_1 be the solution satisfying $\phi_1(x_0) = 1,\ \phi_1'(x_0) = 0$, and let ϕ_2 be the solution satisfying $\phi_2(x_0) = 0,\ \phi_2'(x_0) = 1$. If ϕ is a solution satisfying $\phi(x_0) = \alpha,\ \ \phi'(x_0) = \beta$. Show that $\phi(x) = \alpha\phi_1(x) + \beta\phi_2(x)$ for all x.

31. (a) Let ϕ_n be any function satisfying the boundary value problem $y'' + n^2 y = 0, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$, where $n = 0, 1, 2, 3, \ldots$. Show that $\int_0^{2\pi} \phi_n(x)\phi_m(x)\, dx = 0$, if $m \neq n$
(b) Show that $\cos nx$ and $\sin nx$ are functions satisfying the above boundary value problem. The result of (a) then implies that
$\int_0^{2\pi} \cos nx \cos mx\, dx = 0, \qquad \int_0^{2\pi} \cos nx \sin mx\, dx = 0, \qquad \int_0^{2\pi} \sin nx \sin mx\, dx = 0, \quad (n \neq m)$

32. Are the following statements true or false?. If the statement is true, prove it; otherwise give a counterexample.
(a) "If $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent functions on an interval I, then any subset of them forms a linearly independent set of functions on I."
(b) "If $\phi_1, \phi_2, \ldots, \phi_n$ are linearly dependent functions on an interval I, then any subset of them forms a linearly dependent set of functions on I."

33. Solve the following IVPs (Initial Value Problems):
(a) $y'' - 16y = 0, y(0) = 1, y'(0) = 20\ (Ans: \quad y = 3e^{4x} - 2e^{-4x})$
(b) $y'' - 4y' + 4y = 0, y(0) = 0, y'(0) = -3\ (Ans: \quad y = -3xe^{2x})$
(c) $y'' + 6y' + 9y = 0, y(0) = -4, y'(0) = 14\ (Ans: \quad y = (2x - 4)e^{-3x})$
(d) $4y'' + 20y' + 25y = 0, y(0) = 1, y'(0) = 2\ (Ans: \quad y = (1 + \frac{9x}{2})e^{-5x/2})$
(e) $y'' - y' = 0, y(0) = -1, y'(0) = 1\ (Ans: \quad y = e^x - 2)$

34. Solve the following IVPs (Initial Value Problems):
(a) $y'' - y' - 2y = 3e^{2x}, y(0) = 0, y'(0) = 2\ \ (Ans: \quad y = \frac{1}{3}e^{2x} - \frac{1}{3}e^{-x} + xe^{2x})$
(b) $y'' + y' - 2y = -6\sin 2x - 18\cos 2x, y(0) = 0, y'(0) = 2\ \ (Ans: \quad y = 3\cos 2x + \frac{5}{3}e^{2x} - \frac{14}{3}e^x)$

35. Solve the following IVPs: $\quad (D = \frac{d}{dx})$:
(a) $(D^2 + 4D + 5)y = 0, y(0) = 0, y'(0) = -3\ \ (Ans: \quad y = -3e^{-2x}\sin x)$
(b) $(D^2 - 2D + \pi^2 + 1)y = 0, y(0) = 1, y'(0) = 1 - \pi\ \ (Ans: \quad y = e^x(\cos \pi x - \sin \pi x))$
(c) $(D^2 + 2D + 2)y = 0, y(0) = 1, y'(0) = -1\ \ (Ans: \quad y = e^{-x}\cos x)$

36. Solve the following IVPs:
(a) $x^2 y'' - 4xy' + 4y = 0, y(1) = 4, y'(1) = 13\ \ (Ans: \quad y = x + 3x^4)$
(b) $4x^2 y'' + 4xy' - y = 0, y(4) = 2, y'(4) = -\frac{1}{4}\ \ (Ans: \quad y = 4x^{-1/2} + x^{1/2})$

37. Find a particular solution of:
(a) $y'' + y = 3x^2 - 6 \quad (Ans: \quad 3x^2 - 12)$
(b) $y'' + y = 6\sin x \quad (Ans: \quad -3x\cos x)$
(c) $y'' + 4y' + 4y = 10\cosh x \quad (Ans: \quad \frac{5}{2}(\frac{e^x}{3} - e^{-x}))$
(d) $y^{(4)} - 5y'' + 4y = 10\cos x \quad (Ans: \quad \cos x)$

38. Find the general solution of the following:
(a) $y''' - y' = 0 \quad (Ans: \quad y = c_1 + c_2e^{-x} + c_3e^x)$
(b) $y''' - y'' - y' + y = 0 \quad (Ans: \quad y = (c_1 + c_2x)e^x + c_3e^{-x}$
(c) $y^{(4)} - 5y'' + 4y = 0 \quad (Ans: \quad y = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x})$
(d) $x^3y''' + x^2y'' - 2xy' + 2y = 0 \quad (Ans: \quad y = c_1x^{-1} + c_2x + c_3x^2)$

39. Find the general solution of:
(a) $y'' + 4y = 4\sec 2x \quad (Ans: \quad y = c_1\cos 2x + c_2\sin 2x + \cos 2x\ln\cos 2x + 2x\sin 2x)$
(b) $y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x} \quad (Ans: \quad y = (c_1 + c_2x + x\ln x - x)e^{2x} + \frac{3}{2})$
(c) $y'' + 2y' + y = 4e^{-x}\ln x \quad (Ans: \quad y = (c_1 + c_2x)e^{-x} + x^2e^{-x}(2\ln x - 3))$
(d) $y'' + 2y' + 2y = 2e^{-x}\sec^3 x \quad (Ans: \quad y = e^{-x}(c_1\cos x + c_2\sin x - \cos 2x\sec x))$
(e) $x^2y'' - 4xy' + 6y = 42x^{-4} \quad (Ans: \quad y = c_1x^2 + c_2x^3 + x^{-4})$
(f) $x^2y'' - 2xy' + 2y = 5x^3\cos x \quad (Ans: \quad y = c_1 + c_2x + c_3x^2 - 5x\cos x)$
(g) $xy'' - y' = x^2e^x \quad (Ans: \quad y = xc_1 + c_2 + (xe^x - e^x))$

40. In each case verify that y_1 is a solution of the given differential equation and find a general solution:
(a) $y'' - y = 3e^{2x}, y_1 = e^{2x} \quad (Ans: \quad c_1e^x + c_2e^{-x} + e^{2x})$
(b) $y'' + y' - 2y = 14 + 2x - 2x^3, y_1 = x^2 \quad (Ans: \quad c_1e^x + c_2e^{-2x} + x^2 - 1)$
(c) $y'' + 4y = 12\sin 2x, y_1 = 3x\cos 2x \quad (Ans: \quad y = c_1\cos 2x + c_2\sin 2x + 3x\cos 2x)$
(d) $y'' - 4y' + 3y = 2e^{3x}, y_1 = xe^{3x} \quad (Ans: \quad y = c_1e^{3x} + c_2e^x + xe^{3x})$

41. Find a general solution of the homogeneous linear system
$\frac{dy_1}{dx} = 3x_1 + x_2 - x_3$
$\frac{dy_2}{dx} = x_1 + 3x_2 - x_3$
$\frac{dy_3}{dx} = 3x_1 + 3x_2 - x_3$

42. Find a general solution of the nonhomogeneous linear systems using diagonalization method, undetermined coefficients method and variation of parameters method.
(a) $y_1' = 2y_2 + x, \quad y_2' = 2y_1 + 1$
(b) $y_1' = y_2 + e^3x, \quad y_2' = y_1 - 3e^3x.$

43. Solve the initial value problem
$y_1' = y_1 + 2y_2 + e^{2t} - 2t$
$y_2' = -y_2 + 1 + t$
$y_1(0) = 1, \; y_2(0) = -4$

(Ans: $y_1 = 4e^{-t} - 4e^t + e^{2t}, \quad y_2 = -4e^{-t} + t$)