COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 8: Partial Orders and Lattices.
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Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM. Questions marked with a * may be a little harder and can be considered optional.

Note: For the purposes of this sheet we will assume that $\mathbb{N}$ includes 0.

Problem 1
Suppose $\preceq$ is a binary relation on a set X which is reflexive and transititive but not anti-symmetric. In this problem we will show how to derive a partial order from $\preceq$.

Problem 1.1
Let $\sim$ be a binary relation on X such that $x \sim y$ if $x \preceq y$ and $y \preceq x$. Prove that $\sim$ is an equivalence relation.

Problem 1.2
Let $\widehat{X} \subseteq 2^X$ be the set of equivalence classes of $\sim$, i.e., each $U \in \widehat{X}$ has the property that for every $x, y \in U$, $x \sim y$. Prove that for every $U_1 \neq U_2 \in \widehat{X}$, either $\forall x \in U_1 : \forall y \in U_2 : x \preceq y$ or $\forall x \in U_1 : \forall y \in U_2 : y \preceq x$ or $\forall x \in U_1 : \forall y \in U_2 : y \not\preceq x \wedge x \not\preceq y$.

Problem 1.3
Define a binary relation $\sqsubseteq$ on $\widehat{X}$ as follows: for $U_1, U_2 \in \widehat{X}$, $U_1 \sqsubseteq U_2$ if $\forall x \in U_1 : \forall y \in U_2 : x \preceq y$. Show that $(\widehat{X}, \sqsubseteq)$ is a poset.

Problem 2
Suppose that $G = (V, E)$ is a directed graph. We say that $U \subseteq V$ spans a *strongly connected component* (SCC) if U is maximal for the property: for every $u, v \in U$ there is a path from u to v *and* and path from v to u in $G[U]$. Define the SCC graph of G has $\widehat{G} = (\widehat{V}, \widehat{E})$ where $\widehat{V}$ is the set of strongly connected components of G and $(U, V) \in \widehat{E}$ if there is a $u \in U$ and $v \in V$ such that $(u, v) \in E$. In this problem we will prove in a roundabout way that $\widehat{G}$ contains no directed cycles.

Problem 2.1
Given a directed graph $G = (V, E)$ we define a binary relation on V as follows: $u \preceq v$ if there is a directed path from u to v in V. Prove that $\preceq$ is a partial order iff G is a Directed Acyclic Graph (DAG), i.e., that G contains no directed cycles.

Problem 2.2
Define a binary relation $\sim$ on V as follows: $u \sim v$ if there is a path from u to v and a path from v to u in G. Prove that $\sim$ is an equivalence relation.

Problem 2.3
Use the results of Problem **??** and Problem **??** to prove that $\widehat{G}$ is a DAG.

Problem 2.4
Give a direct proof of the fact that $\widehat{G}$ is a DAG.

Problem 3
Given an alphabet $\Sigma = \{a_1, \ldots, a_n\}$ with a total order $\leq$ defined on it, we define the *lexicographical order* $\preceq$ on the set Σ^* of all finite strings on Σ. Given two strings $u, v \in \Sigma^*$, we say that $u \preceq v$ if u is the empty string. Otherwise we can say $u = u_1\hat{u}$ and if $v = v_1\hat{v}$ where u_1, v_1 are in Σ and $\hat{u}, \hat{v}$ are in Σ^*, we say that $u \preceq v$ if $u_1 \leq v_1$ or if $u_1 = v_1$ and $\hat{u} \preceq \hat{v}$. Prove that the lexicographical order is a partial order. Also show that it is a total order.

Problem 4
Let Σ^+ be the set of all strings on Σ of length at least 1. Show that the prefix order $u \preceq v$ if $v = uz$ for some $z \in \Sigma^+$ is a strict partial order.

Problem 5 ♠
Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two posets defined on disjoint sets S, T. The *linear sum* $S \oplus T$ of the two posets is $(S \cup T, \preceq)$ where for $x, y \in S \cup T$ we say $x \preceq y$ if either $x \preceq_S y$ or $x \preceq_T y$ or if $x \in S$ and $y \in T$. Show that $\preceq$ is a partial order on $S \cup T$. Draw an example of a graph for which a normal spanning tree's tree order can be represented as the linear sum of two tree orders.

Problem 6
Prove that any finite lattice is complete.

Problem 7
In [**?**] on page 122 it is stated that a finite poset is a lattice iff it has a greatest element and a least element. However this is not true. Prove that (i) a finite lattice has a greatest element and a least element and (ii) there is a finite poset with a greatest element and a least element which is not a lattice.

Problem 8
Prove that the meet and join of a complete lattice are commutative, associative and idempotent and have the absorption property, i.e., prove Proposition 4.2.2 of [**?**].

Problem 9
Given a set X, let $S \subseteq 2^X$ be a collection of subsets of X such that

1. $X \in S$, and
2. if $A_x \in S$ for all $x \in I$ where I is some index set, then $\cap_{x \in I} A_x$ is also in S.

Prove that $(S, \subseteq)$ is a complete lattice.

Problem 10
Wisden decides to publish a book with exactly 100 pages containing biographies and photos of the top 100 cricketers of all time. The cricketers are ranked from 1 to 100. Although the text can be broken over multiple pages, each photo has to appear on a single page. Multiple photos can appear on the same page but the photos must appear in order of rank, i.e., the photo of cricketer ranked i must appear before the photo of cricketer ranked j whenever $i < j$.

Problem 10.1
Prove using Tarski's fixed point theorem that there is at least one cricketer whose photo appears on a page number equal to their rank (e.g. the photo of the 47th greatest cricketer appears on page 47).

Problem 10.2 *
Give another proof using induction.

References

[Gallier08] J. Gallier. Discrete Mathematics for Computer Science: Some Notes arXiv:0805.0585, 2008.