Problem Set 6

Problem 1. [20 points] [15] For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

(a) [5 pts] $R_n := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } x \equiv y \pmod{n}\}$

(b) [5 pts] $R := \{(x, y) \in P \times P$ s.t. x is taller than $y\}$ where P is the set of all people in the world today.

(c) [5 pts] $R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } gcd(x, y) = 1\}$

(d) [5 pts] $R_G :=$ the set of $(x, y) \in V \times V$ such that V is the set of vertices of a graph G, and there is a path $x, v_1, \ldots, v_k, y$ from x to y along the edges of G.

Problem 2. [20 points] Every function has some subset of these properties:

injective   surjective   bijective

Determine the properties of the functions below, and briefly explain your reasoning.

(a) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin(x)$.

(b) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 99x^{99}$.

(c) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $\tan^{-1}(x)$.

(d) [5 pts] The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) =$ the number of numbers that divide x. For example, $f(6) = 4$ because $1, 2, 3, 6$ all divide 6. *Note: We define here the set* $\mathbb{N}$ *to be the set of all positive integers* ($1, 2, \ldots$).

Problem 3. [20 points] In this problem we study partial orders (posets). Recall that a weak partial order $\preceq$ on a set X is reflexive ($x \preceq x$), anti-symmetric ($x \preceq y \wedge y \preceq x \to x = y$), and transitive ($x \preceq y \wedge y \preceq z \to x \preceq z$). Note that it may be the case that neither $x \preceq y$ nor $y \preceq x$. A chain is a list of *distinct* elements $x_1, \ldots, x_i$ in X for which $x_1 \preceq x_2 \preceq \cdots \preceq x_i$. An antichain is a subset S of X such that for all distinct $x, y \in S$, neither $x \preceq y$ nor $y \preceq x$.

The aim of this problem is to show that any sequence of $(n-1)(m-1)+1$ integers either contains a non-decreasing subsequence of length n or a decreasing subsequence of length m. Note that the given sequence may be out of order, so, for instance, it may have the form $1, 5, 3, 2, 4$ if $n = m = 3$. In this case the longest non-decreasing and longest decreasing subsequences have length 3 (for instance, consider $1, 2, 4$ and $5, 3, 2$).

(a) [7 pts] Label the given sequence of $(n-1)(m-1)+1$ integers $a_1, a_2, \ldots, a_{(n-1)(m-1)+1}$. Show the following relation $\preceq$ on $\{1, 2, 3, \ldots, (n-1)(m-1)+1\}$ is a weak poset: $i \preceq j$ if and only if $i \leq j$ and $a_i \leq a_j$ (as integers).

For the next part, we will need to use Dilworth's theorem, as covered in lecture. Recall that Dilworth's theorem states that if $(X, \preceq)$ is any poset whose longest chain has length n, then X can be partitioned into n disjoint antichains.

(b) [7 pts] Show that in any sequence of $(n-1)(m-1)+1$ integers, either there is a non-decreasing subsequence of length n or a decreasing subsequence of length m.

(c) [6 pts] Construct a sequence of $(n-1)(m-1)$ integers, for arbitrary n and m, that has no non-decreasing subsequence of length n and no decreasing subsequence of length m. Thus in general, the result you obtained in the previous part is best-possible.

Problem 4. [20 points] Louis Reasoner figures that, wonderful as the Beneš network may be, the butterfly network has a few advantages, namely: fewer switches, smaller diameter, and an easy way to route packets through it. So Louis designs an N-input/output network he modestly calls a *Reasoner-net* with the aim of combining the best features of both the butterfly and Beneš nets:

> The ith input switch in a Reasoner-net connects to two switches, a_i and b_i, and likewise, the jth output switch has two switches, y_j and z_j, connected to it. Then the Reasoner-net has an N-input Beneš network connected using the a_i switches as input switches and the y_j switches as its output switches. The Reasoner-net also has an N-input butterfly net connected using the b_i switches as inputs and¡ the z_j switches as outputs.

In the Reasoner-net the minimum latency routing does not have minimum congestion. The *latency for min-congestion* (LMC) of a net is the best bound on latency achievable using routings that minimize congestion. Likewise, the *congestion for min-latency* (CML) is the best bound on congestion achievable using routings that minimize latency.

Fill in the following chart for the Reasoner-net and briefly explain your answers.

diameter	**switch size(s)**	**# switches**	**congestion**	**LMC**	**CML**

Problem 5. [20 points] Let B_n denote the butterfly network with $N = 2^n$ inputs and N outputs, as defined in Notes 6.3.8. We will show that the congestion of B_n is exactly $\sqrt{N}$ when n is even.

Hints:

- For the butterfly network, there is a unique path from each input to each output, so the congestion is the maximum number of messages passing through a vertex for any matching of inputs to outputs.
- If v is a vertex at level i of the butterfly network, there is a path from exactly 2^i input vertices to v and a path from v to exactly 2^{n-i} output vertices.
- At which level of the butterfly network must the congestion be worst? What is the congestion at the node whose binary representation is all 0s at that level of the network?

(a) [10 pts] Show that the congestion of B_n is at most $\sqrt{N}$ when n is even.

(b) [10 pts] Show that the congestion achieves $\sqrt{N}$ somewhere in the network and conclude that the congestion of B_n is exactly $\sqrt{N}$ when n is even.