Name: ______________________________

ID number: ______________________________

There are 2 questions for a total of 10 points.

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1. (5 points) Prove or disprove: Any strongly connected undirected graph with n vertices and $(n-1)$ edges is a tree. (*Recall that a tree is a strongly connected undirected graph without cycles.*)

Solution: We will prove the statement using mathematical induction. Let $P(n)$ denote the proposition "any strongly connected undirected graph with n vertices and $(n-1)$ edges is a tree". We will prove $\forall n, P(n)$ using induction.

Base case: $P(1)$ is true since a graph with 1 vertex and 0 edges is indeed a tree.

Inductive step: Assume that $P(1), P(2), ..., P(k)$ are true for an arbitrary $k \geq 1$. We will show that $P(k+1)$ is true. Consider any strongly connected graph G with $k+1$ vertices and k edges. Then there is a vertex v with degree exactly 1. Otherwise the sum of degrees will be $\geq 2(k+1)$ but this is not possible since we know that sum of degrees is equal to $2|E|$ which in this case is $2k$. Consider the graph G' obtained by removing the vertex v and its connecting edge. Note that G' is still strongly connected and it has k vertices and $k-1$ edges. Using the induction hypothesis, we get that G' is a tree. This implies that G is a tree.

2. (5 points) Give a closed form expression for the function $T(n)$ defined recursively as below:

$$T(n) = \begin{cases} T(n-1) & \text{if } n > 1 \text{ and } n \text{ is odd} \\ 3 \cdot T(n/2) & \text{if } n > 1 \text{ and } n \text{ is even} \\ 1 & \text{if } n = 1 \end{cases}$$

Also, argue the correctness of your answer using induction.

Solution: $T(n) = 3^{\lfloor \log_2 n \rfloor}$.

We argue using the following claim.

Claim: For all $k \geq 0$, the following holds: For all $2^k \leq n < 2^{k+1}$, $T(n) = 3^k$.

Proof. We show this by induction on k. Let $P(k)$ denote the given proposition in the claim. We need to show that $\forall k, P(k)$ is true.

Base step: Base case is trivially true since $T(1) = 1$.

Inductive step: Suppose $P(1), P(2), ..., P(i)$ are true. We will show that $P(i+1)$ is true. Consider any $2^{i+1} \leq n < 2^{i+2}$. We need to consider the case when n is even and n is odd.

If n is odd, then $T(n) = T(n-1) = 3 \cdot T(\frac{n-1}{2})$. Note that $2^{i+1} \leq n - 1 < 2^{i+2}$. So, we have $2^i \leq (n-1)/2 < 2^{i+1}$. Applying induction hypothesis, we get $T(n) = 3^{i+1}$.

If n is even, then $T(n) = 3 \cdot T(n/2)$. Since $2^{i+1} \leq n < 2^{i+2}$, we have $2^i \leq n/2 < 2^{i+1}$. Applying induction hypothesis, we get that $T(n) = 3^{i+1}$. This completes the proof of the claim. □

The remaining argument follows from the fact that $2^k \leq n < 2^{k+1}$ iff $k = \lfloor \log_2 n \rfloor$.