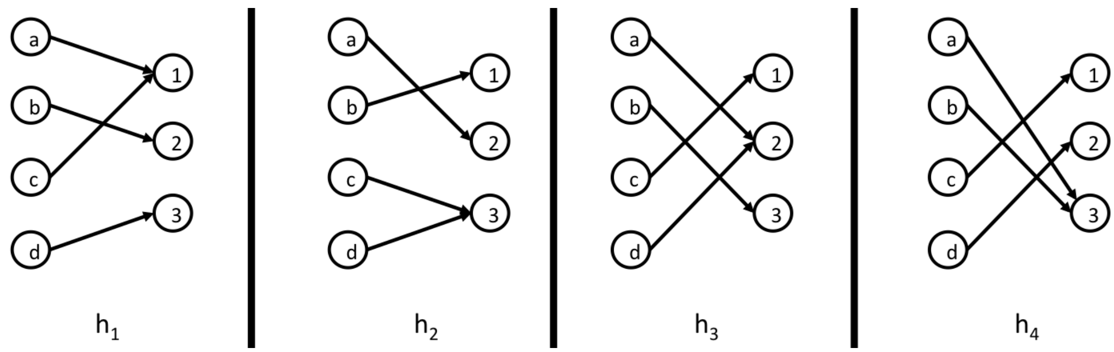
Name: ______________________

ID number: ______________________

There are 2 questions for a total of 10 points.

---

1. Consider the hash function family $H = \{h_1, h_2, h_3, h_4\}$ where h_i's are defined below and answer the questions that follow.



   (a) (1 point) State true or false: H is 2-universal.

   (a) **True**

   (b) (2 points) Give reasons for your answer to part (a).

   Solution: We have :

   1. $\mathbf{Pr}_{h \leftarrow H}[h(a) = h(b)] = 1/4$
   2. $\mathbf{Pr}_{h \leftarrow H}[h(a) = h(c)] = 1/4$
   3. $\mathbf{Pr}_{h \leftarrow H}[h(a) = h(d)] = 1/4$
   4. $\mathbf{Pr}_{h \leftarrow H}[h(b) = h(c)] = 0$
   5. $\mathbf{Pr}_{h \leftarrow H}[h(b) = h(d)] = 0$
   6. $\mathbf{Pr}_{h \leftarrow H}[h(c) = h(d)] = 1/4$

   Since all the probabilities are $\leq 1/3$, H is a 2-universal hash function family.

2. (7 points) Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.

Solution: Let $G(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + ...$ Then we have:

$$\begin{array}{rcl} G(x) & = a_0 + & a_1x^1 + a_2x^2 + a_3x^3 + ... \\ x \cdot G(x) & = & a_0x^1 + a_1x^2 + a_2x^3 + ... \\ 2x^2 \cdot G(x) & = & \qquad 2a_0x^2 + 2a_1x^3 + 2a_2x^4 + ... \end{array}$$

This gives:

$$\begin{aligned} (1 - x - 2x^2) \cdot G(x) &= a_0 + (a_1 - a_0)x + \sum_{k=2}^{\infty}(a_k - a_{k-1} - 2a_{k-2})x^k \\ &= 4 + 8x + \sum_{k=2}^{\infty} 2^k x^k \quad (\text{since } a_k = a_{k-1} + 2a_{k-2} + 2^k) \\ &= 4 + 8x + 4x^2 \cdot \big(1 + (2x) + (2x)^2 + ...\big) \\ &= 4 + 8x + \frac{4x^2}{1 - 2x} \end{aligned}$$

This implies that:

$$G(x) = \frac{4 - 12x^2}{(1 + x)(1 - 2x)^2} = \frac{a}{1 + x} + \frac{bx + c}{(1 - 2x)^2}.$$

We solve for constants a, b, c. We get the following equations:

$$\begin{aligned} 4a + b &= -12 \\ -4a + b + c &= 0 \\ a + c &= 4 \end{aligned}$$

which gives $a = -\frac{8}{9}$, $b = -\frac{76}{9}$, and $c = \frac{44}{9}$. Collecting the coefficients of x^k, we get that:

$$\begin{aligned} a_k &= -\frac{8}{9}(-1)^k + \frac{44}{9}(k + 1)2^k + (-\frac{76}{9})k2^{k-1} \\ &= -\frac{8}{9}(-1)^k + \frac{2}{3}k2^k + \frac{44}{9}2^k \end{aligned}$$