DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI
MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)
2023-24 SECOND SEMESTER TUTORIAL SHEET-V

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1. Find all solutions of the following equations :
(a) $y' + 2xy = x$ (b) $xy' + y = 3x^3 - 1$ (for $x > 0$) (c) $y' + e^x y = 3e^x$
(d) $y' - (\tan x)y = e^{\sin x}$ (for $0 < x < \frac{\pi}{2}$) (e) $y' + 2xy = xe^{-x^2}$

2. Consider the equation $y' + (\cos x)y = e^{-\sin x}$
(a) Find the solution ϕ which satisfies $\phi(\pi) = \pi$.
(b) Show that any solution ϕ has the property that $\phi(\pi k) - \phi(0) = \pi k$, where k is any integer.

3. Consider the equation $x^2 y' + 2xy = 1$ on $0 < x < \infty$.
(a) Show that every solution tends to zero as $x \longrightarrow \infty$.
(b) Find that the solution ϕ which satisfies $\phi(2) = 2\phi(1)$.

4. Find all the real-valued solutions of the following :
(a) $y' = x^2 y$ (b) $yy' = x$ (c) $y' = x^2 y^2 - 4x^2$
(d) $y' = \frac{x+x^2}{y-y^2}$ (d) $y' = \frac{e^{x-y}}{1+e^x}$

5. Solve the following equations:
(a) $y' = \frac{x-y+2}{x+y-1}$ (b) $y' = \frac{2x+3y+1}{x-2y-1}$ (c) $y' = \frac{x+y+1}{2x+2y-1}$

6. Suppose there is a family F of curves in a region S in the plane with the property that through each point (x,y) of S there passes one, and only one, curve C of F, and that the slope of the tangent of C at (x, y) is given by $f(x, y)$, where f is continuous. If a curve in F can be written as $(x, \phi(x))$, where x runs over some interval I, then ϕ is a solution of $y' = f(x, y)$. If ψ is any solution of the equation of the equation $y' = \frac{-1}{f(x,y)}$, then the curve $C^{\perp}$ given by the points $(x, \psi(x))$ will have a tangent at each of its points (x, y) which is perpendicular to the curve in F passing through (x, y). The set G of all curves $C^{\perp}$ is called the set of *orthogonal trajectories* to the family F.
The following relations determine a family of curves, one curve for each value of the constant c. Find the orthogonal trajectories of these families.
(a) $x^2 + y^2 = c$, $(c > 0)$ (b) $y = cx$ (c) $y = cx^2$
(d) $e^x - e^{-y} = c$; (Ans: $e^{-x} + e^y = c'$) (e) $r = c(\sec\theta + \tan\theta)$ (Ans: $r = ce^{-\sin\theta}$)
(f) $y^2 = 4c(c + x)$; (Ans: $y^2 = 4c(c + x)$)
(g) $\cos x \sinh y = c$; (Ans: $\cosh y \sin x = c$) (h) $r = ce^{\theta}$; (Ans: $r = ke^{-\theta}$)

7. The equations below are written in the form $M(x, y)\ dx + N(x, y)\ dy = 0$, where M, N exist on the whole plane. Determine which equations are exact there, and solve these.
(a) $2xy\ dx + (x^2 + 3y^2)\ dy = 0$
(b) $(x^2 + xy)\ dx + xy\ dy = 0$
(c) $e^x\ dx + (e^y(y + 1))\ dy = 0$
(d) $\cos x \cos^2 y\ dx - \sin x \sin 2y\ dy = 0$

(e) $x^2y^3\ dx - x^3y^2\ dy = 0$
(f) $(x+y)\ dx + (x-y)\ dy = 0$
(g) $(2ye^{2x} + 2x\cos y)\ dx + (e^{2x} - x^2\sin y)\ dy = 0$

8. Find an integrating factor for each of the following and solve them .
(a) $(2y^2+2)\ dx + 3xy^2\ dy = 0$
(b) $\cos x\cos y\ dx - 2\sin x\sin y\ dy = 0$
(c) $(5x^3y^2 + 2y)\ dx + (3x^4y + 2x)\ dy = 0$
(d) $(e^x + xe^y)\ dx + xe^y\ dy = 0$

9. Consider the equation $M(x,y)\ dx + N(x,y)\ dy = 0$, where M, N have continuous first partial derivatives on some rectangle R, where $R = \{\ (x,y) : \quad |\ x-x_0\ | \leq a, \quad |\ y-y_0\ | \leq b\ \}$. Prove that a function u on R, having continuous first partial derivatives, is an integrating factor if and only if, $u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}$ on R.

10. Consider the equation $M(x,y)\ dx + N(x,y)\ dy = 0$, where M, N have continuous first partial derivatives on some rectangle R, where $R = \{\ (x,y) : \quad |\ x-x_0\ | \leq a, \quad |\ y-y_0\ | \leq b\ \}$.
(a) Show that if the equation $M(x,y)\ dx + N(x,y)\ dy = 0$, has an integrating factor u, which is a function of x alone, then $p = \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is a continuous function of x alone.
(b) If p is continuous and independent of y, show that an integrating factor is given by $u(x) = e^{P(x)}$, where P is any function satisfying $P' = p$.

11. Consider the equation $M(x,y)\ dx + N(x,y)\ dy = 0$, where M, N have continuous first partial derivatives on some rectangle R, where $R = \{\ (x,y) : \quad |\ x-x_0\ | \leq a, \quad |\ y-y_0\ | \leq b\ \}$.
(a) Show that if the equation $M(x,y)\ dx + N(x,y)\ dy = 0$, has an integrating factor u, which is a function of y alone, then $q = \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ is a continuous function of y alone.
(b) If q is continuous and independent of x, show that an integrating factor is given by $u(y) = e^{Q(x)}$, where Q is any function satisfying $Q' = q$.

12. Use the results of the above to find the solutions of the following differential equations.
(a) $(-x^2+y)\ dx + (x^2y - x)\ dy = 0$; (Ans: $x - \frac{y^2}{2} + \frac{y}{x} = c$)
(b) $(2xy^2 + y)\ dx + (2y^3 - x)\ dy = 0$ (Ans: $x^2 + y^2 + \frac{x}{y} = c$)

13. Find an integrating factor of the form x^py^q and solve.
(a) $(4xy^2 + 6y)\ dx + (5x^2y + 8x)\ dy = 0$ (Ans: $p = 2, q = 3$; $x^4y^5 + 2x^3y^4 = c$)
(b) $(5x + 2y + 1)\ dx + (2x + y + 1)\ dy = 0$ (Ans: $p = q = 0$; $5x^2 + y^2 + 4xy + 2x + 2y = c$)

14. Find the most general function $N(x,y)$ such that the following equations become exact differential equations.
(a) $(x^3 + xy^2)\ dx + N(x,y)\ dy = 0$; Ans: $N(x,y) = yx^2 + g(y)$)
(b) $(x^{-2}y^{-2} + xy^{-3})\ dx + N(x,y) = 0$; Ans: $N(x,y) = 2x^{-1}y^{-3} - \frac{3}{2}x^2y^{-4} + f(y)$)

15. Solve the following differential equations:
(a) $y' + y\tan x = \sec x$; (Ans: $y\sec x = \tan x + c$)
(b) $\sin\theta\frac{dr}{d\theta} + 2r\cos\theta + 1 = 0$; (Ans: $r\sin^2\theta = \cos\theta + c$)
(c) $y' + y = e^x$; (Ans: $2ye^x = e^{2x} + c$)

(d) $(x^2 + y)y' = 6x$; (Ans: $(x^2 + y + 3)e^{-y/3} = c$)
(e) $y\frac{dy}{dx} + x = x^2 \ln x$; (Ans: $\frac{x^2+y^2}{2} - (\ln x - \frac{1}{3})\frac{x^3}{3} = c$)
(f) $xy' + y = y^2 \ln x$; (Ans: $\frac{1}{xy} - \frac{1+\ln x}{x} = c$)
(g) $e^{3y}\frac{dx}{dy} + 3(xe^{3y} + y) = 0$; (Ans: $2xe^{3y} + 3y^2 = c$)
(h) $y\sqrt{x^2 - a^2}\frac{dy}{dx} + y^2 = \sqrt{x^2 - a^2} - x$

16. Reduce to the following second order differential equations to the first order and solve
(a) $y'' + 9y' = 0$; (Ans: $c_1 e^{-9x} + c_2$)
(b) $xy'' + y' = 0$; (Ans: $y = c_1 \ln x + c_2$)
(c) $y'' + e^{2y}y'^3 = 0$; (Ans: $x = \frac{1}{4}e^{2y} + c_1 y + c_2$)
(d) $y'' - (1 - y^{-1})y'^2 = 0$; (Ans: $(y - 1)e^y = c_1 x + c_2$)

17. By computing appropriate Lipschitz constants, show that the following functions satisfy Lipschitz conditions on the sets S indicted:
(a) $f(x, y) = 4x^2 + y^2$, on $S: \quad |x| \leq 1, \quad |y| \leq 1$
(b) $f(x, y) = x^2 \cos^2 y + y \sin^2 x$, on $S: \quad |x| \leq 1, \quad |y| < \infty$
(c) $f(x, y) = x^3 e^{-xy^2}$, on $S: \quad 0 \leq x \leq a, \quad |y| < \infty, \quad (a > 0)$

18. (a). Show that the function f given by $f(x, y) = y^{1/2}$ does not satisfy a Lipschitz condition on $R: |x| \leq 1, \quad 0 \leq y \leq 1$.
(b). Show that this f satisfies a Lipschitz condition on any rectangle R of the form $R: \quad |x| \leq a, \quad b \leq y \leq c, \quad (a, b, c > 0)$

19. For each of the following problems compute the Picard's first three successive approximations
(a) $y' = x^2 + y^2, \quad y(0) = 0$ (b) $y' = 1 + xy, \quad y(0) = 1$
(c) $y' = y^2 \quad y(0) = 0,$ (d) $y' = y^2 \quad y(0) = 1$
(d) $y' = x + y, \quad y(1) = 2$ (Ans: $2, \ \frac{x^2}{2} + 2x - \frac{1}{2}, \ \frac{5}{6} - \frac{x}{2} + \frac{3x^2}{2} + \frac{x^3}{6}$)
(e) $y' = 2xy + 1, \quad y(0) = 0,$ (Ans: $0, \ x, \ \frac{2x^3}{3} + 1$)
(f) $y' = -2xy, \ y(0) = 1,$ (Ans: $1, \ 1 - x^2, \ 1 - x^2 + x^4 - \frac{x^6}{3}$)

20. Discuss the existence and uniqueness of solution of IVP $y' = y + y^2, \ y(\frac{\pi}{2}) = 1$ for the domain $R: |x - \frac{\pi}{2}| \leq 3, \quad |y - 1| \leq 1$. Obtain the solution with the help of standard method as well as by Picard's iterates. Discuss the validity of the solution in the neighborhood of $\frac{\pi}{2}$. (Ans: Unique solution in $[\frac{\pi}{2} - \frac{1}{6}, \frac{\pi}{2} + \frac{1}{6}]$)