

Name: \_\_\_\_\_

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There are 2 questions for a total of 10 points.

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1. Answer the following questions related to set equality. For this question we will use the set identity  $A - B = A \cap \overline{B}$  that holds for all sets  $A, B$ . We will call this “set-difference law”.

- (a) (2  $\frac{1}{2}$  points) For all sets  $A, B$  show that  $(A - B) \cup (B - A) = (A \cup B) \cap (\overline{A \cap B})$  using set identities given in the lecture slides along with the “set-difference law”.

**Solution:** The equality follows from the following chain of equalities:

$$\begin{aligned}
 (A - B) \cup (B - A) &= (A \cap \overline{B}) \cup (B \cap \overline{A}) && \text{(set-difference law)} \\
 &= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) && \text{(Distributive law)} \\
 &= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) && \text{(Distributive law)} \\
 &= ((A \cup B) \cap U) \cap (U \cap (\overline{B} \cup \overline{A})) && \text{(Domination law)} \\
 &= (A \cup B) \cap (\overline{B} \cup \overline{A}) && \text{(Identity law)} \\
 &= (A \cup B) \cap \overline{(B \cap A)} && \text{(De Morgan's law)} \\
 &= (A \cup B) \cap \overline{(A \cap B)} && \text{(Commutative law)}
 \end{aligned}$$

- (b) (2  $\frac{1}{2}$  points) For all sets  $A, B, C$  show that  $\overline{(A - B) - (B - C)} = \overline{A} \cup B$  using set identities given in the lecture slides along with the “set difference law”.

**Solution:** The equality follows from the following chain of equalities:

$$\begin{aligned}
 \overline{(A - B) - (B - C)} &= \overline{(A \cap \overline{B}) \cap \overline{(B \cap \overline{C})}} && \text{(set-difference law)} \\
 &= \overline{(A \cap \overline{B})} \cup \overline{\overline{(B \cap \overline{C})}} && \text{(De Morgan law)} \\
 &= \overline{(A \cap \overline{B})} \cup (B \cap \overline{C}) && \text{(Complementation law)} \\
 &= (\overline{A} \cup \overline{\overline{B}}) \cup (B \cap \overline{C}) && \text{(De Morgan law)} \\
 &= (\overline{A} \cup B) \cup (B \cap \overline{C}) && \text{(Complementation law)} \\
 &= (\overline{A} \cup B \cup B) \cap (\overline{A} \cup B \cup \overline{C}) && \text{(Distributive law)} \\
 &= (\overline{A} \cup B) \cap (\overline{A} \cup B \cup \overline{C}) && \text{(Idempotent law)} \\
 &= (\overline{A} \cup B) && \text{(Absorption law)}
 \end{aligned}$$

2. (5 points) Argue that the set  $\mathbb{Z}^+ \times \mathbb{Z}^+$  is countable.

**Solution:** An infinite set is countable if and only if its elements can be listed in a sequence. We list the elements of the set  $\mathbb{Z}^+ \times \mathbb{Z}^+$  in a sequence by first listing those tuples  $(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$  such that  $p + q = 2$  (there is only one such tuple), then list those tuples  $(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$  such that  $p + q = 3$  (there are two such tuples), and so on. So, the first few elements of the sequence is  $(1, 1), (1, 2), (2, 1), (3, 1), (2, 2), (1, 3), \dots$ . Since all the elements of the the set  $\mathbb{Z}^+ \times \mathbb{Z}^+$  are in the list, we can conclude that the set is countable.