COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 9: Counting.
3 April 2025

Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM. Questions marked with a * may be a little harder and can be considered optional.

Problem 1
Suppose you have n men and n women and you have to seat them around a circular table so that men and women sit alternately. How many ways can you seat them?

Problem 2
Suppose there are n tennis players participating in a tournament. In each round the players who won the previous rounds are paired up for the next round. In case the number of players is odd then one of them gets a bye (let's say, to avoid randomness, that the remaining player with the lowest id gets a bye in each round).

Problem 2.1
How many ways can we pair them up so that everyone has a partner to play a match in the first round?

Problem 2.2
How many matches played in the entire tournament?

Problem 2.3
What is the total number of possible tournaments?

Problem 3
Given a set A of size m and set B of size n count the number of

1. Relations from A to B.
2. Total functions from A to B.
3. Partial functions from A to B.
4. Surjections from A to B (assume $m \geq n$). These could be partial or total functions.
5. Injections from A to B. These could be partial or total. State whatever you assume about m and n.

Problem 4 [LLM18, Prob. 15.7]
For integers $n, k \geq 0$ let $S_{n,k}$ be the set of non-negative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n.$$

Problem 4.1
Give a bijection between $S_{n,k}$ and the set of binary strings with n zeroes and k ones. What is $|S_{n,k}|$?

Problem 4.2 ♠
For integers $n, k, \geq 0$, let

$$L_{n,k} := \{(y_1, \ldots, y_k) \in \mathbb{N}^k : 0 \leq y_1 \leq y_2 \leq \cdots \leq y_k \leq n\}.$$

Give a bijection from $S_{n,k}$ to $L_{n,k}$.

Problem 5 [LLM18, Prob. 15.8]
Suppose that we have a numbered tree on n vertices, each vertex with a distinct id taken from $[n]$. We define an $n-2$ length code of the tree algorithmically as follows: While there are more than 2 vertices left, write down the id of the parent of the leaf with the largest id, then remove that leaf and repeat recursively on the remaining tree. (See the book for an example).

Problem 5.1
What is the code associated with the unique numbered tree on 2 vertices?

Problem 5.2
Construct a bijection between the numbered trees and the sequences of length $n-2$ drawn with each element taken from $[n]$. From there conclude that the number of trees with vertex set $[n]$ is n^{n-2}. (Hint. The book suggests an algorithmic route to finding such a bijection, c.f., 15.8(a)).

Problem 6
Given a plane with integer points of the type (x, y) where both x and y are integers, we define a *lattice path* from (x_1, y_1) to (x_2, y_2) to be a set of line segments that go from a point (i, j) to $(i+1, j)$ or $(i, j+1)$, i.e., all steps in the path either move right or up. Count the number of lattice paths between $(0, 0)$ and (m, n)?

Problem 7
We say that a function π is a *derangement of size n* if it is a bijection from $\{1, \ldots, n\}$ to itself (i.e., it is a permutation) and it has no fixed points, i.e., $\forall i : \pi(i) \neq i$. Count the number of derangements of size n. The solution you submit must use Inclusion-Exclusion. Separately also try to see if you can solve the problem without the use of Inclusion-Exclusion.

Problem 8
Prove the following identities regarding binomial coefficients by making counting arguments. Give as many different arguments as possible

Problem 8.1

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

Problem 8.2

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}.$$

Problem 8.3

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}.$$

Problem 9 [LLM18, Prob. 15.47]
Suppose $n+1$ numbers are selected from $\{1, 2, \ldots, 2n\}$ show using the Pigeonhole Principle that there must be two selected numbers whose quotient is a power of two.

Problem 10 [LLM18, Prob. 15.50]
Suppose $2n+1$ elements are selected from $[4n]$, use the Pigeonhole Principle to show that for every positive j that divides $2n$ there must be two selected numbers whose difference is j.

Problem 11
Suppose that $f(x)$ is a polynomial with integral coefficients and $f(x) = 2$ for three different integers. Prove that $f(x)$ cannot be equal to 3 for any integer x.

References

[LLM18] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science, June 2018, MIT Open Courseware.