Ordered Basis :- A basis of a vector space V with an ordering of the elements of B is called an ordered basis.

Now we consider finite dimensional vectorspace V.

Let $B = \{v_1, \cdots, v_n\}$ be a basis of a vector space V over a field F. We fix an ordering say $v_1, v_2, \cdots, v_n$

Let $v = c_1 v_1 + \cdots + c_n v_n \in V$. Then $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Example :- We know $B = \{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$ over $\mathbb{R}$.

If we choose ordering e_1, e_3, e_2 of elements of B, then $v = (2, 3, 5)$ can be written as $2e_1 + 5e_3 + 3e_2$

Thus $[v]_B = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$

$[v]_B$ is called the coordinate vector of v with respect to the basis B.

Change of Basis :-

* $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$. Let $B=\{(1,0),(0,1)\}$
* $\{(1,1),(1,-1)\}$ is a basis of $\mathbb{R}^2$. Let $B'=\{(1,1),(1,-1)\}$

Then $[(a,b)]_B = \begin{bmatrix} a \\ b \end{bmatrix}$ and $[(a,b)]_{B'} = \begin{bmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{bmatrix}$

Note that, $[(a,b)]_{B'} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$$(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$$

$$(0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$$

$B = \{v_1, \dots, v_n\}$ and $B' = \{w_1, \dots, w_n\}$ be two ordered basis of V. Let $v_j = \sum_{i=1}^{n} P_{ij} w_i$

Suppose for $v \in V$, $[v]_B = (a_1 \; a_2 \; \cdots \; a_n)^T$

Then

$$v = \sum_{j=1}^{n} a_j v_j$$

$$= \sum_{j=1}^{n} a_j \sum_{i=1}^{n} P_{ij} w_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} a_j \right) w_i$$

$$v_1 = P_{11} w_1 + P_{21} w_2 + \cdots + P_{n1} w_n$$

$$v_2 = P_{12} w_1 + P_{22} w_2 + \cdots + P_{n2} w_n$$

$$\cdot \quad \cdot \quad \cdot$$

$$v_n = P_{1n} w_1 + P_{2n} w_2 + \cdots + P_{nn} w_n$$

Thus,

$$[v]_{B'} = \begin{bmatrix} \sum_{j=1}^{n} P_{1j} a_j \\ \vdots \\ \sum_{j=1}^{n} P_{nj} a_j \end{bmatrix} = \begin{bmatrix} P_{11} a_1 + P_{12} a_2 + \cdots + P_{1n} a_n \\ P_{21} a_1 + P_{22} a_2 + \cdots + P_{2n} a_n \\ \vdots \\ P_{n1} a_1 + P_{n2} a_2 + \cdots + P_{nn} a_n \end{bmatrix}$$

$$= (P_{ij})_{n \times n} [v]_B$$

NOTE :- $[v]_B = (P_{ij})^{-1} [v]_{B'}$

Problem :- Find the coordinate vector of $\alpha = (1,3,1)$ relative to the ordered basis $B = (\alpha_1, \alpha_2, \alpha_3)$ of $\mathbb{R}^3$ where $\alpha_1 = (1,1,1)$, $\alpha_2 = (1,1,0)$, $\alpha_3 = (1,0,0)$.

Solution : Let us find scalars c_1, c_2, c_3 such that

$$(1,3,1) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$$

$$\Rightarrow (1,3,1) = (c_1 + c_2 + c_3,\ c_1 + c_2,\ c_1)$$

Then
$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 + c_2 &= 3 \\ c_1 &= 1 \end{aligned}$$

$$\Rightarrow c_1 = 1,\ c_2 = 2 \ \& \ c_3 = -2$$

Therefore, $[\alpha]_B = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Problem :- Find the vector α in $\mathbb{R}^3$ s.t. $[\alpha]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$
$B = \{(1,1,1), (1,1,0), (1,0,0)\}$

Solution: $\alpha = 3(1,1,1) + 2(1,1,0) + (1,0,0)$
$= (6,5,3)$

Problem:- Let $\alpha = \{x^2+x+1,\ x^2+1,\ x-1\}$ and $\beta = \{2x^2+3x+1,\ 2x^2+2x+1,\ -x^2-2\}$ be two basis of $P_3 := \{f \in R[x],\ \deg f < 3\}$ For a vector $v \in P_3$ if $[v]_\beta = \begin{bmatrix}1\\2\\3\end{bmatrix}$, then what is $[v]_\alpha$?

Solution:-
$$2x^2+3x+1 = 2(x^2+x+1) + 0\cdot(x^2+1) + (x-1)$$
$$2x^2+2x+1 = (x^2+x+1) + (x^2+1) + (x-1)$$
$$-x^2-2 = -(x^2+x+1) + 0(x^2+1) + (x-1)$$

Here $P = \begin{bmatrix}2 & 1 & -1\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix}$. Thus, $[v]_\alpha = P[v]_\beta$

$$= \begin{bmatrix}2 & 1 & -1\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$= \begin{bmatrix}1\\2\\6\end{bmatrix}$$

NOTE: (i) $[v]_\beta = P^{-1}[v]_\alpha$

(ii) We need to express old basis element in terms of new basis to get the matrix P.

Problem:- Find a basis of the solution space of $AX = 0$ where $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Solution:- $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \end{bmatrix}$

$$\xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the system of linear equations is equivalent to

$$x_1 + 2x_2 + x_4 = 0$$
$$x_3 + 3x_4 = 0$$

Let $x_2 = a$ and $x_4 = b$ then $x_1 = -2a - b$ and $x_3 = -3b$.

Therefore, $(x_1, x_2, x_3, x_4) = (-2a-b, a, -3b, b)$
$= a(-2, 1, 0, 0) + b(-1, 0, -3, 1)$

Since $\{(-2,1,0,0), (-1,0,-3,1)\}$ is linearly independent (as it is not a scalar multiplication of other), that set is a basis.

Note that, number of unknown is 4, rank of A is 2

Thus dim of solution space is $4-2=2$.

Theorem:- The dimension of the solution space of $AX=0$ is $n-r$, where n is the number of unknown and r is the rank of the matrix A.

Proof:- The solution space of $AX=0$ is same as the solution space of $RX=0$ where R is the row reduced echelon matrix form of A.

Since $\text{rank}(A)=r$, R has r non-zero rows and $R_1, R_2, \ldots, R_r$ has leading elements 1. The leading 1 are is the k_i-th column where $k_1 < k_2 < \cdots < k_r$.

We can assume $\{x_1, x_2, \ldots, x_n\} \setminus \{x_{k_1}, \ldots, x_{k_r}\}$ as arbitrary elements $d_1, d_2, \ldots, d_{n-r}$, then each x_{k_i}

can be expressed in terms of $\alpha_1, \alpha_2, \cdots, \alpha_{n-r}$.

Let $(x_1, x_2, x_3, \cdots, x_n) = \alpha_1 S_1 + \alpha_2 S_2 + \cdots + \alpha_{n-r} S_{n-r}$.

Then the solution space is Span $\{S_1, \cdots, S_{n-r}\}$.

Further, $\{S_1, \cdots, S_{n-r}\}$ is linearly independent.

as $c_1 S_1 + \cdots + c_{n-r} S_{n-r} = (0, \cdots, 0)$ implies

$$(\cdots, c_1, \cdots, c_2, \cdots, \cdots, c_{n-r}, \cdots) = (0, 0, \cdots, 0)$$

$$\Rightarrow c_1 = c_2 = \cdots = c_{n-r} = 0.$$

Thus $\{S_1, \cdots, S_{n-r}\}$ is a basis of the solution space of $AX = 0$, and hence it has dimension $n-r$.

Row space :- Let $A_{m\times n} \in M_{m\times n}(F)$. Let $R_1, R_2, \cdots, R_m$ be rows of $A_{m\times n}$. Then $R_1, \cdots, R_m \in F^n$.

The vector space span by $\{R_1, \ldots, R_m\}$ is called the row space of A.

Note that, $\{R_1, \ldots, R_m\}$ may not be linearly independent and it is a subspace of F^n.

Therefore the dimension of the row space of $A \leq \min\{m, n\}$.

If A has rank r and $R_1', \ldots, R_r'$ are the non-zero row of the row reduced echelon form of A then $\text{span}\{R_1, \ldots, R_m\} = \text{span}\{R_1', \ldots, R_r'\}$ (as R_i' is linear combination of R_i and vice verse)

Also, $\{R_1', \ldots, R_r'\}$ is linearly independent.

Therefore rank(A) = dimension of the row space of A.