DEPARTMENT OF MATHEMATICS
MTL 101: MINOR EXAM

30 Marks total

<u>**Instructions:**</u> Write down all the steps of the solution clearly.

Problem 1 [6 marks] Let $M_{n\times n}(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of all $n \times n$ matrices with real entries and $A \in M_{n\times n}(\mathbb{R})$. Show that if $\{Av_1, Av_2, ..., Av_n\}$ is linearly independent in $M_{n\times 1}(\mathbb{R})$, then $\{v_1, v_2, ..., v_n\}$ is linearly independent in $M_{n\times 1}(\mathbb{R})$. Is the converse true? Justify your answer.

Problem 2 [6 marks] Let V be the vector space (over $\mathbb{R}$) of all real polynomials of degree less than or equal to four. Let W be the subspace of V defined by $W = \{p(x) \in V \mid p(1) + p(-1) = 0 \text{ and } p(2) + p(-2) = 0\}$. Find a basis of the subspace W, and determine the dimension of W.

Problem 3 [6 marks] Let $S = \{A \in M_{2\times 2}(\mathbb{C}) : A = \bar{A}^T\}$, where for $A = [a_{ij}]$, $\bar{A}^T = [\bar{a}_{ji}]$, and $\bar{a}_{ji}$ is the complex conjugate of a_{ji} in $\mathbb{C}$.

(i) Is S a vector space over the field $\mathbb{R}$? If yes, find the dimension of S over $\mathbb{R}$. Justify your answer.

(ii) Is S a vector space over the field $\mathbb{C}$? If yes, find the dimension of S over $\mathbb{C}$. Justify your answer.

Problem 4 [6 marks] Let $A \in M_{n\times n}(\mathbb{R})$ such that $\text{Rank}(A) = \text{Rank}(A^2)$. Show that the systems of linear equations $AX = 0$ and $A^2X = 0$ have the same solution space.

Problem 5 [6 marks] Let $\alpha, \beta \in \mathbb{R}$. Let

$$X = 2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

be all solutions of the system of linear equations $AX = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$. Determine the row reduced echelon form of A and find the rank of A.