COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 11

Announced on: Apr 03 (Wed)

1. Based on Problem 15.5 in [LLM17].

   A license plate consists of either:

   - 3 letters followed by 3 digits (standard plate)
   - 5 letters (vanity plate)
   - 2 characters—letters or numbers (big shot plate)

   Let L be the set of all possible license plates.

   a) Express L in terms of
   $$\mathcal{A} = \{A, B, C, \ldots, Z\}$$
   $$\mathcal{D} = \{0, 1, \ldots, 9\}$$
   using unions ($\cup$) and set products ($\times$).

   b) Compute $|L|$, the number of different license plates, using the sum and product rules.

2. Recall that the inclusion-exclusion rule states that for n finite sets $S_1, S_2, \ldots, S_n$,
$$\begin{aligned} |S_1 \cup S_2 \cup \cdots \cup S_n| = &\sum_{i=1}^{n} |S_i| \\ &- \sum_{1 \leqslant i < j \leqslant n} |S_i \cap S_j| \\ &+ \sum_{1 \leqslant i < j < k \leqslant n} |S_i \cap S_j \cap S_k| - \ldots \\ &+ (-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n|. \end{aligned}$$
   Prove the above statement.

3. Based on Problem 15.60 in [LLM17].

   A *derangement* is a permutation $(x_1, x_2, \ldots, x_n)$ of the set $(\{1, 2, \ldots, n\}$ such that $x_i \neq i$ for all i. For example, (2, 3, 4, 5, 1) is a derangement, but (2, 1, 3, 5, 4) is not because 3 appears in the third position.

   Using inclusion-exclusion rule, show that the total number of derangements is
$$n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots + (-1)^n \frac{1}{n!}\right).$$

4. Based on Problem 15.34 in [LLM17].

   In poker, a *flush* is a hand that contains five cards all of the same suit, e.g., $K\spadesuit, 9\spadesuit, 5\spadesuit, 4\spadesuit, 2\spadesuit$. What is the total number of flush hands?

   A *straight* is a hand that contains five cards of sequential rank, not all of the same suit, e.g., $K\spadesuit, Q\heartsuit, J\diamondsuit, 10\diamondsuit, 9\clubsuit$. What is the total number of straight hands?

5. Based on Problem 15.79 in [LLM17].

   Give a combinatorial proof of

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = 2 \cdot \binom{n+1}{3}.$$

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.