Second order linear ODE :-

$$y'' + P(t)y' + q(t)y = r(t)$$

$y \equiv y(t)$. $y'' \equiv \frac{d^2y}{dt^2}$, $y' \equiv \frac{dy}{dt}$

If $r(t) = 0$ then it is called homogeneous second order linear ODE.

If $r(t) \neq 0$ then is is called non-homogeneous.

Existence and Uniqueness theorem :-

Let $y'' + P(t)y' + q(t)y = r(t)$ be a second order linear ODE. Let $P(t)$, $q(t)$ and $r(t)$ be continuous on some open interval I. Let $t_0 \in I$. Then for any $a, b \in \mathbb{R}$, the ODE with the initial conditions

$y(t_0) = a$ & $y'(t_0) = b$ has a unique solution on I.

What if one of $p(t), q(t), r(t)$ is not continuous.

$\rightarrow$ The theorem fails.

We can't say anything by that theorem.

We need a different approach

—

Now we take homogeneous second order linear ODE, i.e., $r(t) = 0$,

$$\left.\begin{array}{l} y'' + p(t)y' + q(t)y = 0. \\ y(t_0) = a, \quad y'(t_0) = b \end{array}\right\}$$

Given $p(t)$ & $q(t)$ are continuous, only one solution

* If $(a,b) = (0,0)$, $y(t) = 0$ is the only solution.

* $y'' + P(t)y' + q(t)y = 0$, $t_0 \in I$ & $P(t)$ & $q(t)$ are continuous.

Then this ODE has infinitely many solutions.
[we can chose any $(a,b) \in \mathbb{R}^2$].

* Solutions are at least second time differentiable.

<u>Solution set</u> $\subseteq C(I)$

$C(I)$: set of all continuous funcns on I.

Theorem:- The set of all solutions of $y'' + P(t)y' + q(t)y = 0$, $P(t)$ & $q(t)$ are continuous on I, forms a vector space (subspace of $C(I)$).

Proof :- Let $y_1(t)$ & $y_2(t)$ be two solutions of the ODE.

Then $c_1 y_1(t) + c_2 y_2(t)$ is also a solution of the ODE for any $c_1, c_2 \in \mathbb{R}$.

Therefore, the solution space is a subspace of $C(I)$, i.e, is a vector space.

What is the dimension of the solution space of $y'' + p(t)y' + q(t)y = 0$, $p(t)$ & $q(t)$ are continuous on I. Take $t_0 \in I$.

Take $y_1(t)$ as a solution of ODE with the initial condition $y_1(t_0) = 1$ & $y_1'(t_0) = 0$

Take $y_2(t)$ as a (unique) solution of ODE with the initial conditions $y_2(t_0) = 0$ & $y_2'(t_0) = 1$.

Let $c_1 y_1(t) + c_2 y_2(t) = 0$ — (1)

$\Rightarrow c_1 y_1'(t) + c_2 y_2'(t) = 0$ — (2)

Taking $t = t_0$, in (1) & (2), $c_1 = 0$ by (1)

$c_2 = 0$ by (2)

$\{y_1(t), y_2(t)\}$ is linearly independent.

Let $\underline{y(t)}$ be any arbitrary solution of the ODE

Define, $Y(t) = y(t_0)\, y_1(t) + y'(t_0)\, y_2(t)$

$Y(t_0) = y(t_0)\, y_1(t_0) + y'(t_0)\, y_2(t_0) = y(t_0)$

$$Y'(t) = y(t_0)\, y_1'(t) + y'(t_0)\, y_2'(t)$$

$$Y(t_0) = y'(t_0)$$

By the uniqueness theorem $Y \equiv y$

Therefore $\{y_1(t), y_2(t)\}$ spans the solution space.

The dimension of the solution space is 2.

→ one can extend these for higher order ODE.

* NOT true for non-homogeneous ODE.

* Suppose $f_1(t), f_2(t), \ldots, f_n(t)$ be functions such that each of $f_i(t)$ is differentiable $(n-1)$-times.

Let $c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = 0$

$$\Rightarrow c_1 f_1'(t) + c_2 f_2'(t) + \cdots + c_n f_n'(t) = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_1 f_1^{(n-1)}(t) + c_2 f_2^{(n-1)}(t) + \cdots + c_n f_n^{(n-1)}(t) = 0$$

$$\Rightarrow \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

Wronskian :- For n real valued functions $f_1, f_2, \cdots, f_n$ which are $(n-1)$- times differentiable we define Wronskian $W(f_1, f_2, \cdots, f_n)$ as a function,

$$W(f_1, f_2, \cdots, f_n)(t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}$$