COL202: Discrete Mathematical Structures. I semester, 2020-21.
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Tutorial Sheet 7: Lattices.
19 November 2020

Important: The question marked with a ♠ is to be submitted via gradescope by 11:59PM on the day that you have your tutorial.

Problem 1
Prove that any finite lattice is complete.

Problem 2
In [1] on page 122 it is stated that a finite poset is a lattice iff it has a greatest element and a least element. However this is not true. Prove that (i) a finite lattice has a greatest element and a least element and (ii) there is a finite poset with a greatest element and a least element which is not a lattice.

Problem 3
Prove that the meet and join of a complete lattice are commutative, associative and idempotent and have the absorption property, i.e., prove Proposition 4.2.2 of [1].

Problem 4 ♠
Prove that a set with two binary operations with the four properties–commutativity, associativity, idempotence and absorption–forms a lattice, i.e., prove Proposition 4.2.3 of [1].

Problem 5
Given a set X, let $S \subseteq 2^X$ be a collection of subsets of X such that

1. $X \in S$, and
2. if $A_x \in S$ for all $x \in I$ where I is some index set, then $\cap_{x \in I} A_x$ is also in S.

Prove that $(S, \subseteq)$ is a complete lattice.

Problem 6
Write out the complete proof of step 2 of Tarski's Fixed Point Theorem (Theorem 4.2.6 of [1].)

Problem 7
A lattice X is called *distributive* if for all $x, y, z \in X$,

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Give an example of a distributive lattice and a lattice which is not distributive.

Problem 8
Suppose that $(X, \leq)$ is a finite distributive lattice and there is an $a \in X$ such that a is minimal in $X \setminus \{\perp\}$. Let $S_1 = \{x \in X : a \not\leq x\}$ and $S_2 = \{x \in X : x = x' \vee a \text{ for some } x' \in S_1\}$. Show that S_1 and S_2 form distributive lattices.

References

[1] J. Gallier. Discrete Mathematics for Computer Science: Some Notes arXiv:0805.0585, 2008.