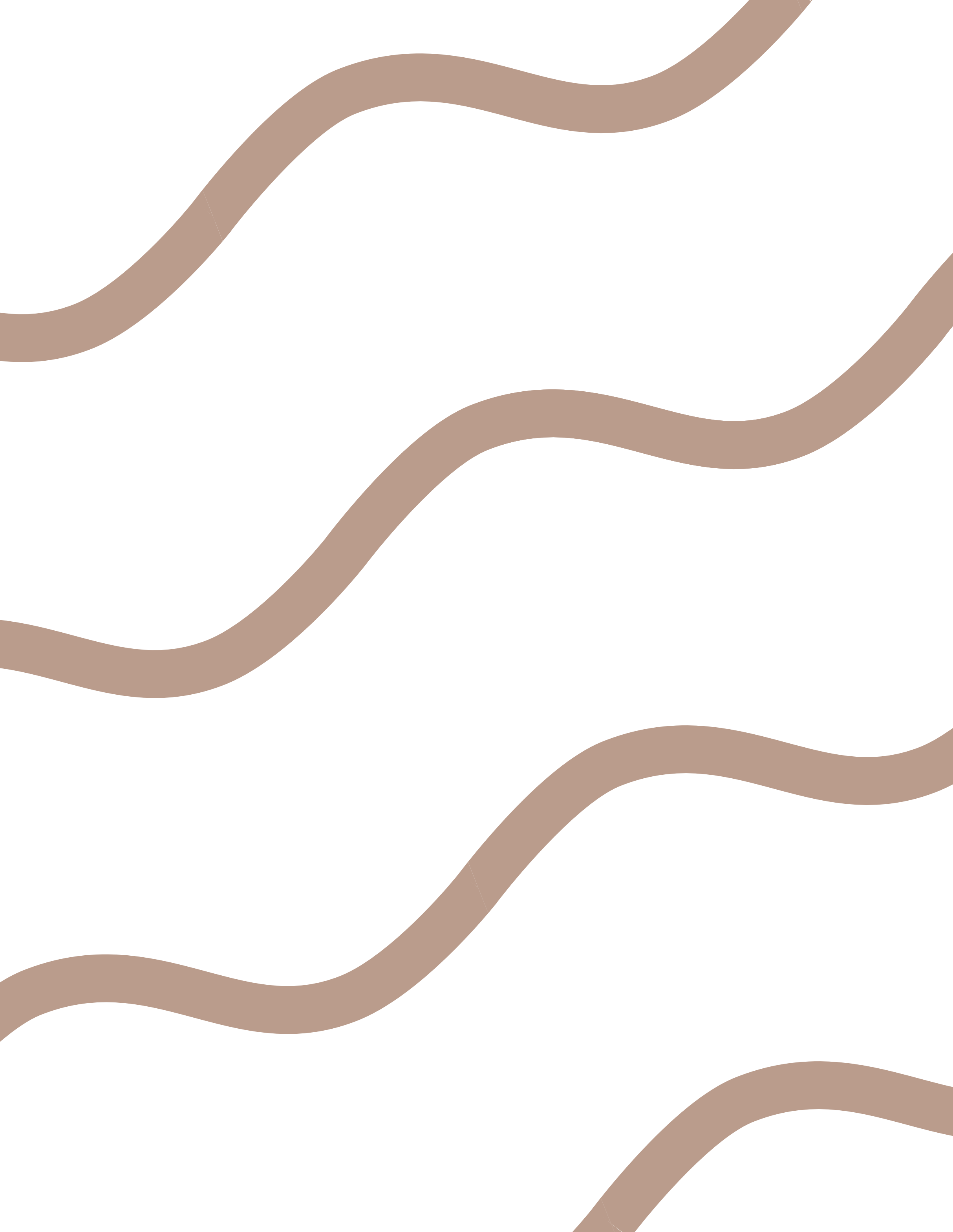

1)

For $\lambda, \mu \in \mathbb{R}$, consider the system of linear equations $AX = B$ with coefficients from $\mathbb{R}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & \lambda \\ 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 6 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, B = \begin{pmatrix} 3 \\ 5 \\ \mu \\ \mu+3 \end{pmatrix}.$$

(a) Using row reduced echelon form of a matrix, find the values of λ and μ such that the system is consistent (i.e. the system has at least one solution).

(b) Write down all the solutions whenever the system is consistent.

Solⁿ

MTL101-Minor-Question-1

Solution 1. For $\lambda, \mu \in \mathbb{R}$, consider the augmented matrix (A, B)

$$(A,B) = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & \lambda & 5 \\ 1 & 1 & 2 & 4 & \mu \\ 2 & 2 & 3 & 6 & \mu+3 \end{bmatrix}$$

Step 1 We apply row operations to form its row reduced echleon form.

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & \lambda & 5 \\ 1 & 1 & 2 & 4 & \mu \\ 2 & 2 & 3 & 6 & \mu+3 \end{pmatrix} \xrightarrow{(R_3 \to R_3 - R_1), (R_2 \to R_2 - R_1), (R_4 \to R_4 - 2R_1)}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 1 & 2 & \mu-3 \\ 0 & 0 & 1 & 2 & \mu-3 \end{pmatrix} \xrightarrow{(R_4 \to R_4 - R_3)} \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 1 & 2 & \mu-3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(R_3 \to R_3 - R_2)}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 4-\lambda & \mu-5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[1]

Step 2 System is consistent for: 1) $\lambda = 4$ and $\mu = 5$ [0.5]

2) $\lambda \neq 4$ and $\mu \in \mathbb{R}$ [0.5]

Step 3

1. When $\lambda = 4$ and $\mu = 5$, we have

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying row operations again to obtain the solution set;

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(R_1 \to R_1 - R_2)} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have x_2, x_4 as free variables and $x_1 + x_2 = 1, x_3 + 2x_4 = 2$. Hence, solution set is given as:

$\{(1-a, a, 2-2b, b) | a, b \in \mathbb{R}\}$ [1]

Step 4

2. When $\lambda \neq 4$ and $\mu \in \mathbb{R}$ we have the augmented matrix as;

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 1 & ((\mu-5)/(4-\lambda)) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Applying row operations again to obtain the solution set;

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 1 & ((\mu-5)/(4-\lambda)) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(R_1 \to R_1 - R_2)} \begin{pmatrix} 1 & 1 & 0 & 4-\lambda & 1 \\ 0 & 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 0 & 1 & ((\mu-5)/(4-\lambda)) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case, x_2 is the free variable, $x_4 = (\mu-5)/(4-\lambda)$, $x_3 = 2 - (\lambda-2)((\mu-5)/(4-\lambda))$ and $x_1 + x_2 + (4-\lambda)x_4 = 1$ i.e $x_1 = 6 - x_2 - \mu$. Hence, the solution set is given as :

$\{(6-a-\mu, a, 2-(\lambda-2)((\mu-5)/(4-\lambda)), (\mu-5)/(4-\lambda)) | a \in \mathbb{R}\}$ [1]

2)

Let $M_2(\mathbb{C})$ be the set of all 2×2 matrices with entries from $\mathbb{C}$. Observe that $M_2(\mathbb{C})$ is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$ (don't prove this). Let $\bar{d}$ denote the complex conjugate of $d \in \mathbb{C}$. Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : a + \bar{d} = 0 \right\} \subset M_2(\mathbb{C}).$$

Consider the vector spaces $V_1 = M_2(\mathbb{C})$ over $\mathbb{R}$ and $V_2 = M_2(\mathbb{C})$ over $\mathbb{C}$.

(a) Is W a subspace of V_1? Justify your answer.

(b) Is W a subspace of V_2? Justify your answer.

Sol:

2. [2+2] Let $M_2(\mathbb{C})$ be the set of all 2×2 matrices with entries in $\mathbb{C}$. Observer that $M_2(\mathbb{C})$ is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$. Let $\overline{d}$ denote the complex conjugate of $d \in \mathbb{C}$. Let

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) \ : \ a + \overline{d} = 0 \right\} \subseteq M_2(\mathbb{C})$$

Consider $V_1 = M_2(\mathbb{C})$ over $\mathbb{R}$ and $V_2 = M_2(\mathbb{C})$ over $\mathbb{C}$.

(a) Is W a subspace of V_1? Justify your answer.

Let $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, v = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in W$. Then $a + \overline{d} = 0 = e + \overline{h}$. For $c_1, c_2 \in \mathbb{R}$, we get

$$c_1 u + c_2 v = c_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} + c_2 \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} c_1 a + c_2 e & c_1 b + c_2 f \\ c_1 c + c_2 g & c_1 d + c_2 h \end{bmatrix}$$

Notice that $c_1 a + c_2 e + \overline{c_1 d + c_2 h} = c_1 a + c_2 e + c_1 \overline{d} + c_2 \overline{h} = c_1(a + \overline{d}) + c_2(e + \overline{h}) = 0$ (as $c_1, c_2 \in \mathbb{R}$). Thus $c_1 u + c_2 v \in W$ and hence W is a subspace of V_1.

(b) Is W a subspace of V_2? Justify your answer.

For $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & d_1 + id_2 \end{bmatrix} \in W$, since $a + \overline{d} = 0$, we have $a_1 + ia_2 + \overline{d_1 + id_2} = a_1 + ia_2 + d_1 - id_2 = (a_1 + d_1) + i(a_2 - d_2) = 0$. Thus $d_1 = -a_1, d_2 = a_2$. Thus

$$W = \left\{ \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & -a_1 + ia_2 \end{bmatrix} \right\} \tag{0.1}$$

For $u \in W$, notice that

$$iu = i \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & -a_1 + ia_2 \end{bmatrix} = \begin{bmatrix} ia_1 - a_2 & ib_1 - b_2 \\ ic_1 - c_2 & -ia_1 - a_2 \end{bmatrix}$$

But $ia_1 - a_2 + \overline{-ia_1 - a_2} = (-a_2 - a_2) + i(a_1 + a_1) \neq 0$ (if either of a_1, a_2 is nonzero). Thus $iu \notin W$ if either of a_1, a_2 is nonzero. Thus W is NOT subspace of V_2.

3)

Let $V = \{a_0 + a_1X + a_2X^2 + a_3X^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree less than or equal to 3 over $\mathbb{R}$. Consider the subspaces W_1 and W_2 of V defined as follows:

$$W_1 = \{a_0 + a_1X + a_2X^2 + a_3X^3 : a_0 + a_1 + a_2 + a_3 = 0, a_1 + 2a_2 + 3a_3 = 0\},$$
$$W_2 = \{a_0 + a_1X + a_2X^2 + a_3X^3 : a_0 + 2a_1 + 3a_2 + 4a_3 = 0, a_2 + 3a_3 = 0\}.$$

(a) Find $\dim(W_1 \cap W_2)$.
(b) Justify if $V = W_1 + W_2$ or not.
(c) Find a basis of $W_1 + W_2$.

Sol:

5a) $W_1 \cap W_2 = \left\{ a_0 + a_1x + a_2x^2 + a_3x^3 : \begin{array}{l} a_0+a_1+a_2+a_3=0 \\ a_1+2a_2+3a_3=0 \\ a_0+2a_1+3a_2+4a_3=0 \\ a_2+3a_3=0 \end{array} \right\}$

The coeff. matrix is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is RRE. Its rank is 3 so that $\dim W_1 \cap W_2$ is $4-3=1$

5b) The coeff. matrix for W_1 is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ which is RRE & its rank is 2 so that $\dim W_1 = 4-2 = 2$

The coeff. matrix of W_2 is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ which is RRE & its rank is 2 so that $\dim W_2 = 4-2=2$.

By dimension formula $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$
$= 2+2-1 = 3 \neq 4 = \dim V$ (since $\{1, x, x^2, x^3\}$ is a basis of V.

5c) By using 5a) $\{1-3x+3x^2-1\}$ is a basis of $W_1 \cap W_2$.
Using RRE form of the matrix defining W_1, $(2,-3,0,1)$ i.e. $2-3x+x^3 \in W_1$ which is not a multiple of $1-3x+3x^2-1$
so that $\{1-3x+3x^2-1, 2-3x+x^3\}$ is a basis of W_1.
Use the same argument for W_2 to see
$\{1-3x+3x^2-1, 2-x\}$ is a basis of W_2
So, $\{1-3x+3x^2-1, 2-3x+x^3, 2-x\}$ is a basis of W_1+W_2

Alternatively, pick any bases B_1 & B_2 of $W_1 \cap W_2$.
If $B_1 \cap B_2 = \phi$, $B_1 \cup B_2$ is not LI (although $B_1 \cup B_2$ spans W_1+W_2). Find a basis of $W_1 \cup W_2$ from $B_1 \cup B_2$
You form matrix whose rows are coordinate vectors of vectors in $B_1 \cup B_2$ & find its RRE form. The non zero rows gives a basis of W_1+W_2.

4)

Question 6: [3 marks]
Consider the vector space $\mathbb{R}^4$ over $\mathbb{R}$ and a linearly independent subset

$$S = \{(1,2,3,4),(0,1,2,3)\} \subset \mathbb{R}^4.$$

Extend S to a basis of $\mathbb{R}^4$. Justify your answer.

Sol^n

Ques 6

$S = \{(1,2,3,4), (0,1,2,3)\} \subseteq \mathbb{R}^4$

Extend S to a basis of $\mathbb{R}^4$

Solution: Since $\dim(\mathbb{R}^4) = 4$, we need to add two more vectors to S in order to get a basis of $\mathbb{R}^4$.

It is enough to find $v_3, v_4 \in \mathbb{R}^4$ such that $\{(1,2,3,4),(0,1,2,3), v_3, v_4\}$ is linearly indep.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow[\sim]{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

If we take $v_3 = (0,0,1,0)$ and $v_4 = (0,0,0,1)$, then the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is of rank 4 which implies the rows are linearly indep.

Hence, $\mathcal{B} = S \cup \{(0,0,1,0),(0,0,0,1)\}$ is a basis of $\mathbb{R}^4$.

5)

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z, w) = (y + 2z + 3w, x - y, x + 2z + 3w, 3x - y + 4z + 6w)$$

(a) Find a basis of the range of T.

(b) Find the nullity of T.

Sol^n

a) Let $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ denote the standard basis for $\mathbb{R}^4$. Then

$$T(e_1) = (0, 1, 1, 3)$$
$$T(e_2) = (1, -1, 0, -1)$$
$$T(e_3) = (2, 0, 2, 4)$$
$$T(e_4) = (3, 0, 3\ 6) \qquad (1)$$

Now,
$$\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & -1 & 0 & -1 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 3 & 6 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$
$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 3 & 6 \end{pmatrix}$$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$
$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \\ 0 & 3 & 3 & 9 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$
$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (1\tfrac{1}{2})$$

A basis of the range of T is $\{(1,0,1,2), (0,1,1,3)\}$.
$(1/2)$.

b) By a) rank $(T) = 2$. Now, using rank-nullity theorem,

$$\text{nullity}(T) = \dim(\mathbb{R}^4) - \text{rank}(T)$$
$$= 4 - 2 = 2. \qquad (1).$$

6)

(a) Justify if there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the following:

$$T(1,1,0) = (3,5),\ T(1,0,1) = (5,3),\ T(2,1,1) = (4,4).$$

If it exists, define one such T.

(b) Justify if there exists a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the following:

$$S(1,0,0) = (1,2),\ S(0,1,0) = (7,5),\ S(1,1,0) = (8,7).$$

If it exists, define one such S.

Sol^n

Suppose there exists a linear transformation T such that

$$T(1,1,0) = (3,5), \quad T(1,0,1) = (5,3), \quad T(2,1,1) = (4,4).$$

Since

$$(2,1,1) = (1,1,0) + (1,0,1), \qquad \text{[0.5 marks]}$$

then using the linearity of T, we have

$$T(2,1,1) = T(1,1,0)+T(1,0,1) = (3,5)+(5,3) = (8,8) \neq T(2,1,1). \quad \text{[1 marks]}$$

Therefore T doesn't satisfy linearity for $u = (1,1,0)$ and $v = (1,0,1)$. This contradicts our assumption. Hence there doesn't exists any such linear transformation. [0.5 marks]

Question 4(b):[2 marks]

Method I:

Since $\{(1,0,0),(0,1,0)\}$ is a linearly independent set, therefore it can be extended to a basis of $\mathbb{R}^3$. Let $\{(1,0,0),(0,1,0),v\}$ be the basis of $\mathbb{R}^3$ (one such choice of v is $(0,0,1)$). [1 marks]

Now, corresponding to each $\gamma \in \mathbb{R}^2$, we can define a linear transformation by fixing the image of v equal to γ, therefore we can construct infinitely many such linear transformations. For simplicity, we choose $\gamma = (0,0)$ and set $S(0,0,1) = (0,0)$. [0.5 marks]

For $(x,y,z) \in \mathbb{R}^3$, we have $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$. Now we can define

$$\begin{aligned} S(x,y,z) &= S(1,0,0) + S(0,1,0) + S(0,0,1) \\ &= x(1,2) + y(7,5) + z(0,0) \\ &= (x+7y, 2x+5y) \quad \text{[0.5 marks]} \end{aligned}$$

Method II:

Since

$$(1,1,0) = (1,0,0) + (0,1,0),$$

observe that

$$S(1,0,0) + S(0,1,0) = (1,2) + (7,5) = (8,7) = S(1,1,0).$$

This gives the guarantee for existence of such a linear transformation. [1 marks]

Now, a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$ must be of the form

$$S(x,y,z) = (ax+by+cz, dx+ey+fz), \quad a,b,c,d,e,f \in \mathbb{R}. \qquad (1)$$

Now using the given conditions, we get

$$\begin{aligned} S(1,0,0) &= (a,d) = (1,2) \\ S(0,1,0) &= (b,e) = (7,5) \\ S(1,1,0) &= (a+b, d+e) = (8,7) \end{aligned} \qquad (2)$$

this gives $a = 1, b = 7, d = 2, e = 5$ and c, f can be any real number. Therefore, for different choice of c, d we can define a linear transformation satisfying our given conditions. For example take $c = d = 0$, then

$$S(x,y,z) = (x+7y, 2x+5y), \quad \text{[1 marks]}.$$

7)

Consider the following two ordered bases of the vector space $\mathbb{R}^3$ over $\mathbb{R}$

$$B = \{(1,0,0), (0,1,0), (0,0,1)\} \text{ and } B' = \{(1,0,0), (1,1,0), (1,1,1)\}.$$

Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x + 2y + 3z, y + 2z, -x - z).$$

(a) Find the matrix $[T]_B$ of T with respect to B.
(b) Find the matrix $[T]_{B'}$ of T with respect to B'.
(c) Find a matrix P such that $[T]'_B = P^{-1}[T]_B P$.

Sol^n:

7.a

$T(x,y,z) = (x+2y+3z, y+2z, -x-z)$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\frac{1}{2}$ Mark
$$\left\{\begin{aligned}
T(1,0,0) &= (1,0,-1) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1)\\
T(0,1,0) &= (2,1,0) = 2(1,0,0) + 1(0,1,0) + 0(0,1,0)\\
T(0,0,1) &= (3,2,-1) = 3(1,0,0) + 2(0,1,0) - 1(0,1,0)
\end{aligned}\right.$$

$\frac{1}{2}$ mark
$$[T]_B = \begin{pmatrix} 1 & 2 & 3\\ 0 & 1 & 2\\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1\\ 2 & 1 & 0\\ 3 & 2 & -1 \end{pmatrix}^T$$

7.b

$B' = \{(1,0,0), (1,1,0), (1,1,1)\}$

1 mark
$$\left\{\begin{aligned}
T(1,0,0) &= (1,0,-1) = 1(1,0,0) + 1(1,1,0) - 1(1,1,1)\\
T(1,1,0) &= (3,1,-1) = 2(1,0,0) + 2(1,1,0) - 1(1,1,1)\\
T(1,1,1) &= (6,3,-2) = 3(1,0,0) + 5(1,1,0) - 2(1,1,1)
\end{aligned}\right.$$

1 mark
$$[T]_{B'} = \begin{pmatrix} 1 & 2 & 3\\ 1 & 2 & 5\\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1\\ 2 & 2 & -1\\ 3 & 5 & -2 \end{pmatrix}^T$$

7.c

$$\left.\begin{aligned}
(1,0,0) &= 1(1,0,0) + 0(0,1,0) + 0(0,0,1)\\
(1,1,0) &= 1(1,0,0) + 1(0,1,0) + 0(0,0,1)\\
(1,1,1) &= 1(1,0,0) + 1(0,1,0) + 1(0,0,1)
\end{aligned}\right\} \text{1 mark}$$

$$\left.\begin{aligned}
&P = \begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{pmatrix} \text{ satisfying}\\
&P[T]_{B'} = [T]_B P
\end{aligned}\right\} \text{1 mark}$$

N.B. If anyone construct other P satisfying the relation $P[T]_{B'} = [T]_B P$, we are giving 2 marks to them.

8)

Consider the following system of linear equations:

$$\begin{pmatrix} 1 & 1 & a+b & 3 \\ 0 & 1 & a+b & 0 \\ 2 & 2 & 3a+2b & a-b+6 \\ 0 & 1 & a+b & a-b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 6 \\ 2 \end{pmatrix}.$$

(a) For what $a, b \in \mathbb{R}$ the system has at least one solution.

(b) Whenever the solutions exist, write all the solutions.

Sol^n

Question -1

$$\left[\begin{array}{cccc|c} 1 & 1 & a+b & 3 & 2 \\ 0 & 1 & a+b & 0 & 1 \\ 2 & 2 & 3a+2b & a-b+6 & 6 \\ 0 & 1 & a+b & a-b & 2 \end{array}\right].$$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & a+b & 3 & 2 \\ 0 & 1 & a+b & 0 & 1 \\ 0 & 0 & a & a-b & 2 \\ 0 & 1 & a+b & a-b & 2 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$R_4 \to R_4 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & a+b & 0 & 1 \\ 0 & 0 & a & a-b & 2 \\ 0 & 0 & 0 & a-b & 1 \end{array}\right]$$

$R_3 \to R_3 - R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & a+b & 0 & 1 \\ 0 & 0 & a & 0 & 1 \\ 0 & 0 & 0 & a-b & 1 \end{array}\right]$$

2M

for consistency: $a \neq 0$ & $a-b \neq 0$ 1M

then $x_4 = \frac{1}{a-b}$

$x_3 = \frac{1}{a}$

$x_2 = 1 - \frac{a+b}{a} = \frac{-b}{a}$

$x_1 = 1 - 3 \cdot \frac{1}{a-b} = \frac{a-b-3}{a-b}$

Solutions: $\left(\frac{a-b-3}{a-b}, \frac{-b}{a}, \frac{1}{a}, \frac{1}{a-b} \right)$

1M

9) Recall that a subset of a vector space is called minimal spanning subset if it spans the vector space but any of its proper subset does not span the vector space. Prove that a minimal spanning subset of a vector space is a basis of the vector space.

Solⁿ

Let $S = \{u_1, u_2, \ldots, u_n\}$ be a minimal spanning subset of $V(\mathbb{F})$. Then, by definition we know linear span of S, that is, $L(S) = V(\mathbb{F})$. [1]

In order to show that S is a basis of V over $\mathbb{F}$, it is enough to prove that S is an linearly independent subset of $V(\mathbb{F})$.

Let us assume that S is linearly dependent. Then there exists $a_1, \cdots, a_n \in \mathbb{F}$ such that at least one $a_i \neq 0$ and

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0.$$

Without lose of generality, we may assume, $a_1 \neq 0$. Then,

$$u_1 = -(1/(a_1))(a_2u_2 + \ldots + a_nu_n).$$

This implies

$$u_1 \in L(\{u_2, u_3, \ldots, u_n\}).$$

Hence,

$$L(\{u_2, u_3, \ldots, u_n\}) = L(S) = V(\mathbb{F})$$

This contradicts the minimality of S. Hence S is a linearly independent set. [3].

Remarks.

1. Although this question was asked for any vector space but most of you answered by assuming V to be finite dimensional. Hence, the solution is written for finite dim vector space and answer copies are graded accordingly.
2. $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

10)

Let $A = \begin{pmatrix} 3 & 8 & 16 \\ 8 & 15 & 32 \\ -4 & -8 & -17 \end{pmatrix} \in M_3(\mathbb{R})$.

(a) Prove that A is similar to a diagonal matrix D.

(b) Find an invertible matrix P such that $P^{-1}AP = D$.

Sol^n

Given the matrix,

$$A = \begin{pmatrix} 3 & 8 & 16 \\ 8 & 15 & 32 \\ -4 & -8 & -17 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

The characteristic equation is

$$\det(A - \lambda I) = 0,$$

which implies,

$$\begin{vmatrix} 3-\lambda & 8 & 16 \\ 8 & 15-\lambda & 32 \\ -4 & -8 & -17-\lambda \end{vmatrix} = 0.$$

• Solving this the eigenvalues obtained are,

$\lambda = -1, -1, 3$ —— (+1)

• Solving $(A-3I)x = 0$, we get the eigen vector, which is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. (+1)

[Rank $(A-3I) = 2$, $\dim N(A-3I) = 1$]

• Solving $(A+I)x = 0$, for $\lambda = -1$, the eigenvectors are

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \;\&\; \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

(+1)

[Rank $(A+I) = 1$, $\dim N(A+I) = 2$]

Let E_1 = eigenspace corresponding to $\lambda = 3$

& E_2 = eigenspace corresponding to $\lambda = -1$.

As $\dim E_1 = \dim N(A-3I) = 1$

& $\dim E_2 = \dim N(A+I) = 2$

so, $\dim E_1 + \dim E_2 = 3$

(+½)

Hence $A \in M_{3\times 3}(\mathbb{R})$ has 3 linearly independent eigenvectors.

$\therefore$ A is diagonalizable.

The matrix P, for which $P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is

given by,

$$P = \begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$ —— (+½)

11)

Question 4: [4 marks]
Let W be a subspace of a finite dimensional vector space V over a field F. Then prove that:
$\boxed{\dim W > \frac{1}{2}\dim V} \Leftrightarrow \boxed{W \cap T(W) \neq \{0\} \text{ for all one-one and onto linear transformations } T: V \to V}$.

Sol^n

V is a finite dimensional vector space over F.
$W \subseteq V$ is a subspace.

2M $\boxed{\text{Proof of } \Rightarrow}$ Assume $\dim W > \frac{1}{2}\dim V$.

Let $T: V \longrightarrow V$ be a one-to-one onto linear transformation.
Let $T|_W : W \to V$ be the restriction of T from V to W.

1M Then $T|_W$ is also one-to-one.
$\Rightarrow$ nullity $(T|_W) = 0$.
Apply rank-nullity theorem for $T|_W$
$\Rightarrow$ rank $(T_W) = \dim W$
$\Rightarrow$ $\dim(T(W)) = \dim W$

Since $W, T(W) \subseteq V$ subspaces
$\Rightarrow$ $W + T(W) \subseteq V$ subspace.
If $W \cap T(W) = \{0\}$ then
$\dim(W + T(W)) = \dim W + \dim T(W)$
$= 2 \cdot \dim(W)$
$> \dim V$

1M This is a contradiction, since $W + T(W) \subseteq V$
Therefore, $W \cap T(W) \neq \{0\}$.

2M $\boxed{\text{Proof of } \Leftarrow}$ Assume $W \cap T(W) \neq \{0\}$ for all one-one onto lin. trans. $T: V \to V$.

If possible, assume $\dim W \leq \frac{1}{2}\dim V$.
Let $\{v_1, \dots, v_m\}$ be a basis of W.

1M We extend this to a basis of V, say
$\{v_1, \dots, v_m, \omega_1, \dots, \omega_m, \dots \omega_k\}$
where $\dim V = m + k$ & $k \geq m$.

Then define $T: V \to V$ by

$T(v_i) = \omega_i$ for $i = 1, \dots, m$

$T(\omega_i) = \begin{cases} v_i & \text{for } i = 1, \dots, m \\ \omega_i & \text{for } i > m \end{cases}$

1M This T can be extended as a lin. trans. on V which is one-to-one onto.
But $W = \text{span}\{v_1, \dots, v_m\}$
$T(W) = \text{span}\{\omega_1, \dots, \omega_m\}$

$W \cap T(W) = \{0\}$ a contradiction

Therefore $\dim W \leq \frac{1}{2}\dim V$ is NOT possible
Hence, $\dim W > \frac{1}{2}\dim V$.

12) Let $A = \begin{pmatrix} 4 & 24 & 6 \\ -1 & -7 & -2 \\ 2 & 12 & 3 \end{pmatrix} \in M_3(\mathbb{R})$. Use Cayley-Hamilton theorem to find $A^{10} + A^{20}$.

Sol^n

$$A = \begin{bmatrix} 4 & 24 & 6 \\ -1 & -7 & -2 \\ 2 & 12 & 3 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 24 & 6 \\ -1 & -7-\lambda & -2 \\ 2 & 12 & 3-\lambda \end{vmatrix} = 0$$

$(4-\lambda)((\lambda-3)(\lambda+7)+24) + 24(-4+3-\lambda) + 6(-12+2\lambda+14) = 0$

$\lambda^3 - \lambda = 0$ ——— (1)

charc eq. $A^3 - A = 0$ ——— (1)

$$A^{10} + A^{20} = (A^3)^3 \cdot A(I + A^2)$$

$$= A^2 + A^2$$

$$= 2A^2 \text{ ——— (1)}$$

$$= \begin{bmatrix} 8 & 0 & -12 \\ -2 & 2 & 4 \\ 4 & 0 & -6 \end{bmatrix} \text{ ——— (1)}$$

(3)

Question 7: [...]
Justify whether the following statements are true or false.

(a) Let W, W_1, W_2 be subspaces of a vector space. Then,

$$W \oplus W_1 = W \oplus W_2 \implies W_1 = W_2.$$

Sol:

Question 7 (a): (FALSE)

Let $V = \mathbb{R}^2$ over $\mathbb{R}$.

$W = \{(x,0) : x \in \mathbb{R}\}$

$W_1 = \{(0,y) : y \in \mathbb{R}\}$

$W_2 = \{(z,z) : z \in \mathbb{R}\}$

(1M) Any counter example

Note that $W \cap W_1 = \{0\}$ and $W \cap W_2 = \{0\}$.

(1M) Then $W \oplus W_1$ and $W \oplus W_2$ makes sense.

Verify that $W \oplus W_1 = \mathbb{R}^2 = W \oplus \mathbb{R}^2$

But, $W_1 \neq W_2$.

14)

1. Let $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$, and the field $F = \mathbb{R}$. Define two operations on V as follows:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$$

$$\alpha(x_1, x_2) = \begin{cases} (0,0) & \text{if } \alpha = 0 \\ \left(\alpha x_1, \dfrac{x_2}{\alpha}\right) & \text{if } \alpha \neq 0 \end{cases} \qquad \alpha \in F.$$

State whether each of the following statements is **TRUE** or **FALSE**. Give reason to support your answers. No marks will be awarded if the answer of the statement is not justified.

(a) Vector addition (defined by $+$) in V is associative but not commutative.

(b) $(0,0)$ is the additive identity in V.

(c) $\alpha(v + w) = \alpha v + \alpha w, \quad \forall \alpha \in F, \quad \forall v, w \in V.$

(d) $(\alpha + \beta) v = \alpha v + \beta v, \quad \forall \alpha, \beta \in F, \quad \forall v \in V.$ [4]

Sol^n

(a) FALSE

$+$ is not associative in V

$((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) = (x_1 + y_1 + z_1, \; x_2 - y_2 - z_2)$

$(x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) = (x_1 + y_1 + z_1, \; x_2 - y_2 + z_2)$

And unless $z_2 = 0$, $x_2 - y_2 - z_2 \neq x_2 - y_2 + z_2$ (1)

(b) FALSE

$(0,0) + (x_1, x_2) = (x_1, -x_2) \neq (x_1, x_2) + (0,0)$

$\therefore$ $(0,0)$ is not an additive identity (1)

(c) TRUE

If $\alpha = 0$ then $\alpha(v+w) = 0 = \alpha v + \alpha w$

Let $\alpha \neq 0$ then

$\alpha v + \alpha w = \alpha(x_1, x_2) + \alpha(y_1, y_2)$

$= \left(\alpha x_1, \frac{x_2}{\alpha}\right) + \left(\alpha y_1, \frac{y_2}{\alpha}\right)$

$= \left(\alpha(x_1 + y_1), \; \frac{x_2 - y_2}{\alpha}\right)$

$= \alpha(x_1 + y_1, \; x_2 - y_2)$

$= \alpha((x_1, x_2) + (y_1, y_2))$

$= \alpha(v + w)$. (1)

(d) FALSE

If $\alpha + \beta = 0$, $\alpha \neq 0$, $\beta \neq 0$ then

$(\alpha + \beta) v = 0$

$\alpha v + \beta v = \left(\alpha x_1 + \beta x_1, \; \frac{x_2}{\alpha} - \frac{x_2}{\beta}\right)$

$= \left(0, \frac{2x_2}{\alpha}\right) \neq (\alpha + \beta) v$ (in general)

(1)

15)

Sol^n

2. Prove that the linear span of a non-empty finite subset S of a vector space V over the field F is the smallest subspace of V containing S. [4]

2.

Let $S = \{v_1, v_2, \ldots, v_n\} \subseteq V$

Then,

$$\text{Span}(S) := \left\{ \sum_{i=1}^{n} \alpha_i v_i \ : \ \alpha_i \in \mathbb{F} \right\}$$

- For proving Span (S) is a subspace of V. [2 Marks]

- For verifying Span (S) contains S, i.e., $S \subseteq$ Span (S). [1 Mark]

- For showing Span (S) is the smallest subspace of V that contains S in the sense that any other subspace of V that contains S must contain span (S). [1 Mark]

For a detailed proof, one can consult, for instance, Theorem 5.2.3 in the book by 'Howard Anton and Chris Rorres'.

If the answer is not very rigorous, then you cannot claim for full or partial credit.

16)

Let U, S, W be any three subspaces of a vector space. Prove or disprove (with counter-example) the following:

(a) $U \bigcap (S+W) \subseteq (U \bigcap S) + (U \bigcap W)$.

(b) $(U \bigcap S) + W \subseteq (U+W) \bigcap (S+W)$. [4]

Sol^n

a.) $U \cap (S+W) \subseteq (U \cap S) + (U \cap W)$

This statement is in general not true

For example

consider $V = \mathbb{R}^2$

$U =$ line $y = x = \{(x,x) : x \in \mathbb{R}\}$

$S =$ y axis $= \{(0,y) : y \in \mathbb{R}\}$ — ①

$W =$ x axis $= \{(x,0) : x \in \mathbb{R}\}$

$S+W = \mathbb{R}^2$; $U \cap (S+W) = U \cap \mathbb{R}^2 = U$

$(U \cap S) = (0,0)$ — ①

$(U \cap W) = (0,0)$ $\therefore$ $(U \cap S) + (U \cap W) = (0,0)$

and $U \nsubseteq (0,0)$

No marks will be awarded for writing true or False without counter example.

ⓑ $(U \cap S) + W \subseteq (U+W) \cap (S+W)$ True

Let $x \in (U \cap S) + W \Rightarrow x = x_1 + x_2$ ($x_1 \in U \cap S$ and $x_2 \in W$)

$x_1 \in U \cap S \Rightarrow x_1 \in U$ and $x_1 \in S$

$\Rightarrow x_1 + x_2 \in U + W$ ($\because x_1 \in U$ & $x_2 \in W$)

also $x_1 + x_2 \in (S+W)$ ($\because x_1 \in S$ & $x_2 \in W$) -

$\Rightarrow x = x_1 + x_2 \in (U+W) \cap (S+W)$

as x was arb element of $(U \cap S) + W$

$\Rightarrow (U \cap S) + W \subseteq (U+W) \cap (S+W)$

No marks will be awarded in case you are proving this statement using a particular example.

17)

4. Let $V = M_{2\times 2}(\mathbb{C})$ be the vector space of all 2×2 matrices with complex entries over the field $F = \mathbb{R}$. Let W be a subset of V consisting of all symmetric matrices whose sum of the principal diagonal elements is zero, that is,

$$W = \{X \in V \mid X^t = X,\ \operatorname{trace}(X) = 0\}.$$

(a) Prove that W is a subspace of V.

(b) Show that $V = W \bigoplus \widetilde{W}$, where $\widetilde{W}$ is the subspace of V consisting of all matrices whose first row has entries equal to zero. [4]

Solⁿ

Q.4 Let $V = M_{2\times2}(\mathbb{C})$ be the vector space of all 2×2 matrices with complex entries over field $F = \mathbb{R}$. Let W be the subset of V consisting of all symmetric matrices whose sum of the principal diagonal element is zero that is ①

$$W = \{X \in V \mid X^t = X,\ tr(X) = 0\}$$

(a) Prove that W is a subspace of V.

(b) Show that $V = W \oplus \tilde{W}$, where $\tilde{W}$ is the subspace of V consisting of all matrices whose first row has entries equal to zero.

Solⁿ (a) Given that $V = M_{2\times2}(\mathbb{C})$ be the vector space of all 2×2 matrices with complex entries over field $F = \mathbb{R}$. Then V is of the form.

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2} \middle| \ a, b, c, d \in \mathbb{C} \right\}$$

Also Given that

$$W = \{X \in V \mid X^t = X,\ tr(X) = 0\}$$

Then W consisting matrix of the form $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$X^t = X \ \& \ tr(X) = 0$$

so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $a + d = 0$ so $\Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $a = -d$

$$\Rightarrow \quad b = c \ \& \ d = -a$$

Therefore $W = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix}_{2\times2} : a, b \in \mathbb{C} \right\}$

We want to show that W is a subspace of V. It is sufficient to show that for any $\alpha, \beta \in \mathbb{R}$ & $W_1, W_2 \in W$

$$\alpha W_1 + \beta W_2 \in W$$

Let $\alpha, \beta \in \mathbb{R}$ & $W_1 = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, $W_2 = \begin{bmatrix} c & d \\ d & -c \end{bmatrix}$ $a, b, c, d \in \mathbb{C}$ be two elements of W. Then

$$\alpha W_1 + \beta W_2 = \alpha \begin{bmatrix} a & b \\ b & -a \end{bmatrix} + \beta \begin{bmatrix} c & d \\ d & -c \end{bmatrix} = \begin{bmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \alpha b + \beta d & -(\alpha a + \beta c) \end{bmatrix}$$

②

$$\alpha W_1 + \beta W_2 = \begin{bmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \alpha b + \beta d & -(\alpha a + \beta c) \end{bmatrix}$$

so $\alpha W_1 + \beta W_2$ is symmetric matrix with trace zero. Therefore $\alpha W_1 + \beta W_2 \in W$.

This $\Rightarrow$ W is subspace of V.

(b) Given $\tilde{W}$ is the subspace of V consisting of all matrices whose first row has entries equal to zero. i.e. $\tilde{W}$ is of the form

$$\tilde{W} = \left\{ \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}_{2\times2} : x, y \in \mathbb{C} \right\}$$

We want to show that $V = W \oplus \tilde{W}$. It is sufficient to show that

① $W + \tilde{W} = V$ ② $W \cap \tilde{W} = \{0\}$

① We know that $W + \tilde{W} = \{w + \tilde{w} : w \in W, \tilde{w} \in \tilde{W}\}$

so $\forall$ $w \in W, \tilde{w} \in \tilde{W} \Rightarrow w + \tilde{w} \in V$ $\because W \subseteq V, \tilde{W} \subseteq V$ & V is vector space

$$\Rightarrow \boxed{W + \tilde{W} \subseteq V} \ \text{-(i)}$$

Let for any $v \in V$. Then v is of the form

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad a, b, c, d \in \mathbb{C}$$

v can be written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c-b & d+a \end{bmatrix} \in W + \tilde{W}$$

$\because \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \in W$, $\begin{bmatrix} 0 & 0 \\ c-b & d+a \end{bmatrix} \in \tilde{W}$

This $\Rightarrow$ $\boxed{V \subseteq W + \tilde{W}}$ -(ii)

Combining (i) & (ii) $\boxed{W + \tilde{W} = V}$

② Now we prove $W \cap \tilde{W} = \{0\}$

Let $w \in W \cap \tilde{W} \Rightarrow w \in W, w \in \tilde{W}$

so $w = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ also $w = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}$ where $a, b, x, y \in \mathbb{C}$

Then $\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix} \Rightarrow a = 0 \ b = 0$, $x = 0, y = 0$

so $w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow W \cap \tilde{W} = \{0\}$

18)

5. Which of the following sets of vectors are linearly independent/dependent over the specified field?

(a) $\{(1,1,-1,-1),\ (0,1,1,0),\ (2,0,2,-2)\} \subset (\mathbb{Z}_3)^4$, $F = \mathbb{Z}_3$, the field of integer modulo 3.

(b) $\{1+i,\ 1-i,\ 2+\sqrt{3},\ 2-\sqrt{3}\} \subset \mathbb{C}$, $F = \mathbb{Q}$, the field of rational numbers. [4]

Sol^n

5. a) $\alpha_1 = (1,1,-1,-1)$,

$\alpha_2 = (0,1,1,0)$,

$\alpha_3 = (2,0,2,-2)$.

Since $F = \mathbb{Z}_3$, so

$\alpha_1 + \alpha_3 = (1,1,-1,-1) + (2,0,2,-2)$

$= (0,1,1,0) = \alpha_2$.

$\rightarrow$ 1 mark

Therefore, α_2 can be written as a linear combination of α_1 and α_3.

$\therefore \{\alpha_1, \alpha_2, \alpha_3\}$ is L.D. in $\mathbb{Z}_3$.

$\rightarrow$ 2 mark

5. b) $\alpha_1 = 1+i$,

$\alpha_2 = 1-i$,

$\alpha_3 = 2+\sqrt{3}$,

$\alpha_4 = 2-\sqrt{3}$.

Suppose, $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$

Since, $F = \mathbb{Q}$, so we get

$c_1 + c_2 + 2c_3 + 2c_4 = 0$

$c_1 - c_2 = 0$

$c_3 - c_4 = 0$

(9) Sol^n

Question 2 (**3 Marks**) Consider the vector space $P_3(\mathbb{R})$ of polynomials of degree less than or equal to three with real coefficients.

a) Prove that $\mathcal{B} = \{1-x, 1+x^2, 1-x^3, 1+x-x^3\}$ is a basis for $P_3(\mathbb{R})$.

b) Find the coordinates of the vector $u = 1+x+x^2+x^3$ with respect to ordered basis $\mathcal{B}$.

$\boxed{\text{Method-I}}$ $\mathcal{B}$ is basis iff (i) $\mathcal{B}$ is L.I.
(ii) $\text{span}(\mathcal{B}) = P_3(\mathbb{R})$. } .... 1 mark

(i) Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Consider,
$$c_1(1-x)+c_2(1+x^2)+c_3(1-x^3)+c_4(1+x-x^3)=0$$
comparing coefficients both sides we get,
$$c_1+c_2+c_3+c_4=0$$
$$-c_1+c_4=0$$
$$c_2=0$$
$$-c_3-c_4=0$$
$$\Rightarrow c_1=c_2=c_3=c_4=0$$
Hence, the above set $\mathcal{B}$ is linearly independent.

(ii) Let $P_1(x) = a_0+a_1x+a_2x^2+a_3x^3 \in P_3(\mathbb{R})$

Suppose $\exists\ c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.
$$P_1(x) = c_1(1-x)+c_2(1+x^2)+c_3(1-x^3)+c_4(1+x-x^3)$$
$$\Rightarrow c_1+c_2+c_3+c_4=a_0$$
$$-c_1+c_4=a_1$$
$$c_2=a_2$$
$$-c_3-c_4=a_3$$
On solving, we get
$$c_1 = a_0-a_2+a_3$$
$$c_2 = a_2$$
$$c_3 = -a_0-a_1+a_2-2a_3$$
$$c_4 = a_0+a_1-a_2+a_3$$
since, $P_1(x)$ can be expressed in the linear combination of vectors of $\mathcal{B}$.
Hence, $\text{span}(\mathcal{B}) = P_3(\mathbb{R})$.

$\boxed{\text{Method-II.}}$ Basis is the maximal L.I. set in a vector space.

Since, cardinality $(\mathcal{B}) = 4 = \dim(P_3(\mathbb{R}))$. (1 mark)

Hence, if $\mathcal{B}$ is L.I. set then we are done.

And linear independence of $\mathcal{B}$, is proved in (i)

Sol^n: Step 1. $u = 1+x+x^2+x^3 = c_1(1-x)+c_2(1+x^2)+c_3(1-x^3) + c_4(1+x-x^3)$.

On comparing coeff^n —
$$\Rightarrow \left.\begin{aligned} c_1+c_2+c_3+c_4 &= 1 \\ -c_1+c_4 &= 1 \\ c_2 &= 1 \\ -c_3-c_4 &= 1 \end{aligned}\right\}$$
(1 mark)

Step 2. Hence, on solving $c_1=1$, $c_2=1$, $c_3=-3$, $c_4=2$
$$[u]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$
(1 mark)

20)

Question 3 (**4 Marks**) Consider the vector space $\mathbb{C}^2(\mathbb{C})$. Find all possible linear transformations $T : \mathbb{C}^2 \to \mathbb{C}^2$ such that $T^2 := T \circ T : \mathbb{C}^2 \to \mathbb{C}^2$ (the composition of T with itself) is given by

$$T^2(z_1, z_2) = (-z_1 + 2z_2, -z_2) \text{ for all } (z_1, z_2) \in \mathbb{C}^2.$$

Sol^n

Solution 1.
Step 1.
[1 *mark*] for writing the matrix of T^2 by fixing a basis, say standard basis B.
[0.5 *mark*] if basis is not mentioned.

Step 2.
[1 *mark*] for assuming $[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and using $[T^2]_B = [T]_B[T]_B$ to get 4 equations in a, b, c, d.
[0.5 *mark*] for any mistake in doing this.

Step 3.
[1 *mark*] for giving **detailed** solution of a, b, c, d.
[0.5 *mark*] for any mistake / inaccuracy to solve a, b, c, d.

Step 4.
[1 *mark*] for writing the two possible matrices followed by the two possibilities of T (invoking the basis fixed in the beginning).
[0.5 *mark*] for any mistake in doing so or only partially doing so.

Solution 2.
Step 1.
[1 *mark*] for using that every T is given by $T(z_1, z_2) = (az_1 + bz_2, cz_1 + dz_2)$ for some $a, b, c, d \in \mathbb{C}$ and computing $T^2(z_1, z_2)$.
[0.5 *mark*] for any mistake in the computing $T^2(z_1, z_2)$.

Step 2.
[1 *mark*] for using Step 1 and $T^2(z_1, y) = (-z_1 + 2z_2, -z_2)$ to get 4 equations on a, b, c, d.
[0.5 *mark*] for any mistake in writing down the 4 equations.

Step 3.
[1 *mark*] for giving detailed solution of a, b, c, d.
[0.5 *mark*] for any mistake / inaccuracy to solve a, b, c, d.

Step 4..
[1 *mark*] for writing the two possible the two correct possibilities of T based on a, b, c, d found in Step 3.
[0.5 *mark*] for any mistake in doing so or doing partially.

Remark: For skipping or failing to justify any of the steps above earns [0 *mark*].

21)

Question 4 (4 Marks)

a) Let W_1, W_2 be non-zero subspaces of a finite dimensional vector space V over $\mathbb{C}$. Suppose that there exists $f : V \to \mathbb{R}$ such that $f(w_1) - f(w_2) < 0$ for all non-zero vectors $w_1 \in W_1$ and $w_2 \in W_2$. Prove that $\dim W_1 + \dim W_2 \leq \dim V$.

b) Let W_1, W_2 be subspaces of the vector space $\mathbb{R}^4$ over $\mathbb{R}$ given by

$$\begin{aligned} W_1 &= \text{span}\{(4,3,2,1),(1,1,1,2),(3,2,1,-1)\} \\ W_2 &= \text{span}\{(1,0,3,2),(4,3,2,1)\} \end{aligned}$$

Find the dimension of $W_1 + W_2$.

Sol^n

Sol: Step 1: Claiming & proving $W_1 \cap W_2 = \{0\}$ $\boxed{\text{1- mark}}$

Let if possible, $W_1 \cap W_2 \neq \{0\}$

$\Rightarrow \exists\ u \neq 0$ s.t. $u \in W_1$ & $u \in W_2$

$\Rightarrow f(u) - f(u) < 0$ & $u \neq 0 \Rightarrow 0 < 0$

A Contradiction. ~~to the given condition.~~

Thus, $W_1 \cap W_2 = \{0\}$ ------ (A)

Step 2: Using $\dim(W_1+W_2) = \dim W_1 + \dim W_2 \leq \dim V$ & concluding the final result. $\boxed{\text{1- mark}}$

Note that we have :-

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad \text{------ (B)}$$

Also as $W_1 + W_2$ is a subspace of V, therefore $\dim(W_1+W_2) \leq \dim(V)$ ------ (C)

Eq (A) gives $\dim(W_1 \cap W_2) = 0$ ------ (D)

Using Eq (C) & (D) in (B) to get:

$$\dim(W_1) + \dim(W_2) \leq \dim V.$$

Sol: Step 1: Finding Dimensions of W_1, W_2 & $W_1 \cap W_2$ Correctly. (with details) $\boxed{\text{1- mark}}$

(a) $\text{Dim}(W_1) = 2$.

Note: $(1,1,1,2) = (4,3,2,1) - (3,2,1,-1)$ & $\{(4,3,2,1),(3,2,1,-1)\}$ is a linearly independent set.

Therefore, basis$(W_1) = \{(4,3,2,1),(3,2,1,-1)\}$

Thus, $\text{Dim}(W_1) = 2$

(b) $\text{Dim}(W_2) = 2$.

Note: $\{(1,0,3,2),(4,3,2,1)\}$ is a Linearly independent set, thus forms a basis.

Hence $\text{Dim}(W_2) = 2$.

(c) $\text{Dim}(W_1 \cap W_2) = 1$

Note that: $W_1 \cap W_2 = \text{Span}\{(4,3,2,1)\}$

Hence, $\text{Dim}(W_1 \cap W_2) = 1$

Step 2: Writing & using the identity:- $\boxed{\text{1- mark}}$

$$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$\dim(W_1+W_2) = 2 + 2 - 1 = 3$$

Sol 4(b): Alternate Solution :-

Step 1: Using $W_1 + W_2 = \text{Span}(W_1 \cup W_2)$ $\boxed{\text{1 mark}}$

As $W_1 + W_2 = \text{Span}(W_1 \cup W_2)$, Therefore the subspace $W_1 + W_2$ is spanned by $\{(4,3,2,1),(1,1,1,2),(3,2,1,-1),(1,0,3,2),(4,3,2,1)\}$

Step 2: Finding dimension of ~~a basis for~~ $\text{Span}(W_1 \cup W_2)$ (by removing linearly dependent vectors) $\boxed{\text{1 mark}}$

Note: Finding number of linearly independent vectors is same as finding non-zero rows in Row Echelon form. Therefore, the $\dim(W_1+W_2)$ is the dimension of the row space of the matrix :- (or row rank)

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

The row echelon form is: (Note: $R_5 = R_1$ & $R_2 = R_1 - R_3$)

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 1/4 & 1/2 & 7/4 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, the row-rank of this matrix is 3, we get $\dim(W_1 + W_2) = 3$

22)

Question 5: (5 Marks) Consider the linear operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T((x_1, x_2, x_3, x_4)) = \Big(\sum_{i=1}^{4} x_i, \sum_{i=1}^{4} x_i, \sum_{i=1}^{4} x_i, \sum_{i=1}^{4} x_i\Big) \text{ for all } (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

Prove or disprove: there exists an ordered basis B of $\mathbb{R}^4$ such that $[T]_B$ is diagonal

Sol^n

Solution 5:(Prove)

(Computing the eigenvalues): Note that for any $\lambda \in \mathbb{R}$ and a nonzero vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, we have $T((x_1, x_2, x_3, x_4)) = \lambda(x_1, x_2, x_3, x_4)$ if and only if

$$x_1 + x_2 + x_3 + x_4 = \lambda x_i \quad \text{for each } i = 1, 2, 3, 4. \tag{3}$$

This implies that $\lambda = 4$ or $\lambda = 0$. Thus 4 and 0 are two distinct eigenvalues of T,

OR,

Let $\beta = \{e_1, e_2, 2_3, e_4\}$ be the standard basis. Then $[T]_\beta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. The eigenvalues of T and $[T]_\beta$ are same. So find the eigenvalues of $[T]_\beta$ by computing the roots of the characteristic polynnomial $\det(\lambda I - [T]_\beta)$. They are $0, 0, 0, 4$. **[2 mark]**

(Computing the eigenspaces): Next compute the eigenspaces, when $\lambda = 4$, we have from (3) that $E_1 = \ker(4I - T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3 = x_4\} = \operatorname{span}(\{(1, 1, 1, 1)\})$. **[1 mark]**

Similarly, when $\lambda = 0$, we have from (3) that $E_2 = \ker(0I - T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$. Thus $E_2 = \operatorname{span}(\{(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\})$[**1 mark**].

(Diagonalizability check): This implies that

$$\dim(E_1) + \dim(E_2) = 1 + 3 = 4 = \dim(\mathbb{R}^4),$$

and hence T is diagonalizable. Consider $B = \{(1, 1, 1, 1), (1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\}$. Then B is a basis of $\mathbb{R}^4$ since eigenspaces corresponding to distinct eigenvalues are independent and $|B| = 4$. Also $[T]_B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. **[1 mark]**

23)

1. **(4 Marks)** Consider the linear system of equations in three unknowns (x, y, z):

$$\begin{aligned} 3x + 2y + az &= 2, \\ 9x + 2y + 3z &= b, \\ 6x + 8y + 5z &= 5. \end{aligned}$$

For what values of $a, b \in \mathbb{R}$, the system has:
(a) No solution.
(b) Unique solution.
(c) Infinitely many solutions.
Also, when solution(s) exists, derive the expression for solution(s).

Submission

The given system can be represented as:

$$A\vec{X} = \vec{b}$$

where $A = \begin{bmatrix} 3 & 2 & a \\ 9 & 2 & 3 \\ 6 & 8 & 5 \end{bmatrix}$, $\vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ b \\ 5 \end{bmatrix}$

Converting the augmented matrix $[A \mid b]$ into echelon form:

$$\left[\begin{array}{ccc|c} 3 & 2 & a & 2 \\ 9 & 2 & 3 & b \\ 6 & 8 & 5 & 5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & a & 2 \\ 0 & -4 & 3-3a & b-6 \\ 0 & 4 & 5-2a & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & a & 2 \\ 0 & -4 & 3-3a & b-6 \\ 0 & 0 & 8-5a & b-5 \end{array}\right]$$

Now,

a) System has no solution when rank $(A) \neq$ rank $(A|b)$

i.e. here when $8-5a = 0$ and $b-5 \neq 0$

rank $(A) = 2$ & rank $(A|b) = 3 \neq 2$

and the system has no solution.

$\therefore$ $a = \frac{8}{5}$ and $b \neq 5$ or $b \in \mathbb{R} - \{5\}$

b) For unique solution, rank $(A) =$ rank $(A|b) = 3$

$\therefore$ $8-5a \neq 0$, $b \in \mathbb{R}$

$\Rightarrow$ $a \neq \frac{8}{5}$, $b \in \mathbb{R}$

$\Rightarrow$ $a \in \mathbb{R} \setminus \{8/5\}$, $b \in \mathbb{R}$

In this case:

$(8-5a)z = b-5$; $-4y + (3-3a)z = b-6$

$\Rightarrow$ $z = \frac{(b-5)}{(8-5a)}$; $\Rightarrow$ $4y = \frac{(3-3a)(b-5)}{(8-5a)} - (b-6)$

$\Rightarrow$ $y = \frac{(3-3a)(b-5) - (b-6)(8-5a)}{4(8-5a)}$

$\Rightarrow$ $y = \frac{3b - 3ab - 15 + 15a - 8b + 48 + 5ab - 30a}{4(8-5a)}$

$= \frac{2ab - 15a - 5b + 33}{4(8-5a)}$

& $3x + 2y + az = 2$ $\Rightarrow$ $3x = 2 - 2y - az$

$= 2 - \frac{(2ab - 15a - 5b + 33)}{2(8-5a)} - \frac{a(b-5)}{(8-5a)}$

$\Rightarrow$ $x = \frac{4(8-5a) - 2ab + 15a + 5b - 33 - 2ab + 10a}{6(8-5a)}$

$= \frac{5a + 5b - 4ab - 1}{6(8-5a)}$

$\therefore$ $x = \frac{5a + 5b - 4ab - 1}{6(8-5a)}$; $y = \frac{2ab - 15a - 5b + 33}{4(8-5a)}$;

$z = \frac{(b-5)}{(8-5a)}$

c) For infinitely many solutions, rank $(A) =$ rank $(A|b) < 3$

$\therefore$ $8-5a = 0$ and $b-5 = 0$

$\Rightarrow$ $a = \frac{8}{5}$, $b = 5$

$\therefore$ In this case,

Let $z = s$ (free variable)

$-4y + 3\left(\frac{-3}{5}\right)z = -1$

$\Rightarrow$ $-4y - \frac{9s}{5} = -1$

$\Rightarrow$ $4y = 1 - \frac{9s}{5}$ $\Rightarrow$ $y = \frac{5-9s}{20}$

and $3x + 2y + \frac{8z}{5} = 2$

$\Rightarrow$ $3x = 2 - \frac{(5-9s)}{10} - \frac{8s}{5}$

$= \frac{20 - 5 + 9s - 16s}{10} = \frac{15-7s}{10}$

$\Rightarrow$ $x = \frac{15-7s}{30}$

$\therefore$ $x = \frac{15-7s}{30}$; $y = \frac{5-9s}{20}$, $z = s$

24)

Submission

2. **(4 Marks)** Find the inverse of the following 3×3 matrix using Gauss-Jordan Method:

$$\begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 5 \end{pmatrix}.$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$; $\quad R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -4 & -4 & -1 & 1 \end{array}\right]$$

$R_2 \to \frac{R_2}{2}$; $\quad R_3 \to \frac{R_3}{(-4)}$; $\quad R_1 \to (-1)R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 1/4 & -1/4 \end{array}\right]$$

$R_2 \to R_2 - (7/2)R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -3/8 & 7/8 \\ 0 & 0 & 1 & 1 & 1/4 & -1/4 \end{array}\right]$$

$R_1 \to R_1 + R_2$, $\quad R_1 \to R_1 + 2R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1/8 & 3/8 \\ 0 & 1 & 0 & -2 & -3/8 & 7/8 \\ 0 & 0 & 1 & 1 & 1/4 & -1/A \end{array}\right] = [I \mid A^{-1}]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1 & 1/8 & 3/8 \\ -2 & -3/8 & 7/8 \\ 1 & 1/4 & -1/4 \end{bmatrix}$$

25)

4. **(4 Marks)** Consider $\mathbb{P}_3$, the vector space of all polynomials of degree atmost 3, over $\mathbb{R}$. Expand the set

$$S = \{x + x^2, x + x^2 + x^3\},$$

to form a basis of $\mathbb{P}_3$.

Submission

4. Claim: $S = \{1, x, x+x^2, x+x^2+x^3\}$ is the basis of $\mathbb{P}_3$

Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

Then consider $\alpha + \beta x + \gamma(x+x^2) + \delta(x+x^2+x^3) = 0$

$\Rightarrow \alpha + (\beta+\gamma+\delta)x + (\gamma+\delta)x^2 + \delta x^3 = 0$

which is true for all $x \in \mathbb{R}$ iff all coeff.s are zero

i.e. $\alpha = 0$, $\beta+\gamma+\delta = 0$, $\gamma+\delta = 0$, $\delta = 0$

$\Rightarrow \alpha = \beta = \gamma = \delta = 0$

$\therefore$ S is linearly independent.

Now, let, $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{P}_3$ be any arbitrary polynomial.

For $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Let $a_0 + a_1x + a_2x^2 + a_3x^3 = \alpha + \beta x + \gamma(x+x^2) + \delta(x+x^2+x^3)$

$= \alpha + (\beta+\gamma+\delta)x + (\gamma+\delta)x^2 + \delta x^3$

Comparing coefficients on both sides,

$\alpha = a_0$; $\beta+\gamma+\delta = a_1$; $\gamma+\delta = a_2$; $\delta = a_3$

Solving : $\delta = a_3 \Rightarrow \gamma = a_2 - a_3$

& $\beta = a_1 - a_2$

& $\alpha = a_0$

$\therefore (\alpha, \beta, \gamma, \delta) = (a_0, a_1 - a_2, a_2 - a_3, a_3)$

$\therefore$ Any polynomial in $\mathbb{P}_3$ can be expressed as a linear combination of vector in $S = \{1, x, x+x^2, x+x^2+x^3\}$

$\therefore S = \{1, x, x+x^2, x+x^2+x^3\}$ is a basis of $\mathbb{P}_3$

2 6)

5. (4 Marks) Prove that the set

$$S = \{e^x, \sin(x), \cos(x)\}$$

is Linearly Independent in vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Sol^n

Let $\alpha, \beta, \gamma \in \mathbb{R}$

s.t.

$$\alpha e^x + \beta \sin x + \gamma \cos x = 0$$

Let $\beta = \rho \cos\theta$, $\gamma = \rho \sin\theta$, $\rho, \theta \in \mathbb{R}$

$$\therefore \alpha e^x + \rho(\sin x \cos\theta + \cos x \sin\theta) = 0$$

$$\Rightarrow \alpha e^x + \rho \sin(x+\theta) = 0$$

Now, this eq^n to be true for all $x \in \mathbb{R}$ for any particular arbitrary $\alpha, \rho, \theta \in \mathbb{R}$

the only possibility is $\alpha_0 = \rho = 0 \Rightarrow \alpha = \beta = \gamma = 0$

since e^x & $\sin(x+\theta)$ are continuous functions and are ~~~~ $e^x > 0 \ \forall x \in \mathbb{R}$ and $\sin(x+\theta)$ is not always constantly 0.

Hence, $\{e^x, \sin x, \cos x\}$ is linearly independent.

27)

(a) Let V be a finite dimensional vector space over a field F and, let $T : V \to V$ be a linear transformation. Let $\mathrm{Im}(T) = \{T(v) : v \in V\}$. Prove that

$$\mathrm{Im}(T) = \mathrm{Im}(T^2) \quad \text{if and only if} \quad \ker(T) + \mathrm{Im}(T) = V.$$

(b) Consider $\mathbb{R}^4$ as a vector space over $\mathbb{R}$. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that the rank of T is 1 and $T^2 \neq 0$. Then calculate the nullity of T^2.

Solⁿ

Solution: Part (a)

[1.5 marks] Proof of $(\Rightarrow)$: Assume $\mathrm{Im}(T) = \mathrm{Im}(T^2)$.
Since $\ker(T), \mathrm{Im}(T)$ are subspaces of V, $\ker(T) + \mathrm{Im}(T) \subset V$.
Enough to prove: $V \subset \ker(T) + \mathrm{Im}(T)$.
Let $v \in V$. Then

$$\begin{aligned} & T(v) \in \mathrm{Im}(T) = \mathrm{Im}(T^2) \\ \Rightarrow\ & \exists w \in V \text{ such that } T(v) = T^2(w) \\ \Rightarrow\ & T(v - T(w)) = 0 \\ \Rightarrow\ & v - T(w) \in ker(T). \end{aligned}$$

Write $v - T(w) = u$, where $u \in \ker(T)$. Then

$$v = u + T(w) \in \ker(T) + \mathrm{Im}(T) \Rightarrow V \subset \ker(T) + \mathrm{Im}(T).$$

[1.5 marks] Proof of $(\Leftarrow)$: Assume $V = \ker(T) + \mathrm{Im}(T)$. Note that $\mathrm{Im}(T^2) \subset \mathrm{Im}(T)$ for any T.
Enough to prove: $\mathrm{Im}(T) \subset \mathrm{Im}(T^2)$.
Let $T(v) \in \mathrm{Im}(T)$ for some $v \in V$.
Since $V = \ker(T) + \mathrm{Im}(T), \exists u \in \ker(T)$ & $T(w) \in \mathrm{Im}(T)$ for some $w \in V$ such that $v = u + T(w)$.
Then $T(v) = T(u) + T^2(w) = 0 + T^2(w) \in \mathrm{Im}(T^2)$.
Therefore, $\mathrm{Im}(T) \subset \mathrm{Im}(T^2)$. Done!

Solution: Part (b):

Step-1: [1 mark] For any T, we have $T^2(V) \subset T(V)$. Then $0 \leq rank(T^2) \leq rank(T) = 1$.
Now $T^2 \neq 0 \Rightarrow rank(T^2) \neq 0$. Therefore, $rank(T^2) = 1$.
Step-2: [1 mark] Apply rank-nullity theorem for $T^2 : \mathbb{R}^4 \to \mathbb{R}^4$,

$$rank(T^2) + nullity(T^2) = 4 \Longrightarrow 1 + nullity(T^2) = 4 \Longrightarrow nullity(T^2) = 3.$$

28)

Question 2: [3+3 marks]

(a) Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -2 & -4 \end{pmatrix}$. Find A^{-1} (the inverse of A) by using Cayley-Hamilton theorem.

(b) Consider $\mathbb{R}^3, \mathbb{R}^4$ as vector spaces over $\mathbb{R}$. Let W be the subspace of $\mathbb{R}^3$ spanned by the subset $\{(1,2,1),(0,1,1),(1,3,2)\}$. Construct a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that the range of T equals W.

Sol^n

a) Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -2 & -4 \end{pmatrix}$. Find A^{-1} (the inverse of A) by using Cayley-Hamilton theorem.

Solution. The given matrix is

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -2 & -4 \end{pmatrix}.$$

The characteristic polynomial of A is

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 & -3 \\ -1 & \lambda-3 & 3 \\ 2 & 2 & \lambda+4 \end{vmatrix}$$

$$= \lambda^3 - 14\lambda - 4 \longrightarrow ①$$

By Cayley-Hamilton theorem, A satisfies p, i.e,

$p(A) = 0$

i.e., $A^3 - 14A - 4I = 0$

$\Rightarrow A^2 - 14I - 4A^{-1} = 0$ (multiplying by A^{-1})

$\Rightarrow A^{-1} = \frac{1}{4}A^2 - \frac{14}{4}I \longrightarrow ①$

$$= \frac{1}{4}\begin{pmatrix} -4 & -2 & -12 \\ 10 & 16 & 6 \\ 4 & 0 & 16 \end{pmatrix} - \frac{7}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1/2 & -3 \\ 5/2 & 4 & 3/2 \\ 1 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 7/2 & 0 & 0 \\ 0 & 7/2 & 0 \\ 0 & 0 & 7/2 \end{pmatrix}$$

$$= \begin{pmatrix} -9/2 & -1/2 & -3 \\ 5/2 & 1/2 & 3/2 \\ 1 & 0 & 1/2 \end{pmatrix}. \longrightarrow ①$$

b) Consider $\mathbb{R}^3, \mathbb{R}^4$ as vector spaces over $\mathbb{R}$. Let W be the subspace of $\mathbb{R}^3$ spanned by the subset $\{(1,2,1),(0,1,1),(1,3,2)\}$. Construct a linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^3$ such that the range of T equals W.

Solution. Observe that

$$\dim(W) = 2 \longrightarrow ①$$

Let $\{w_1, w_2\}$ be a basis for W & let w_3 & w_4 be any two vectors from W. Also, let $\{v_1, v_2, v_3, v_4\}$ be a basis for $\mathbb{R}^4$. Define T as

$$\left.\begin{aligned} T(v_1) &= w_1, \quad T(v_2) = w_2 \\ T(v_3) &= w_3 \;\&\; T(v_4) = w_4 \end{aligned}\right\} \longrightarrow ②$$

Then extend it linearly to whole of $\mathbb{R}^4$. This T will satisfy the requirements.

29) Sol^n

Let V be a vector space over $\mathbb{R}$ and let $A = \{u, v, w\}$ be a subset of V. Suppose $B = \{u+v+w, u+2v+3w, u+4v+9w\}$. Show, by using the definition of linear independence, that A is linearly independent if B is linearly independent. [4]

To show that A is linearly independent, let

$$au + bv + cw = 0 \longrightarrow (1)$$

Observe $au + bv + cw = \alpha(u+v+w) + \beta(u+2v+3w) + \gamma(u+4v+9w)$

holds if

$$\left.\begin{aligned} \alpha + \beta + \gamma &= a \\ \alpha + 2\beta + 4\gamma &= b \\ \alpha + 3\beta + 9\gamma &= c \end{aligned}\right\} \longrightarrow (2)$$

Since augment matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 4 & b \\ 1 & 3 & 9 & c \end{array}\right) \overset{R_2 \to R_2 - R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-a \\ 0 & 2 & 8 & c-a \end{array}\right)$$

$$\overset{R_3 \to R_3 - 2R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-a \\ 0 & 0 & 2 & c-a-2(b-a) \end{array}\right)$$

satisfy rank $(A|B) = $ rank(A)

i.e. rank of coeff. = rank of aug matrix

there exist $\alpha, \beta, \gamma \in \mathbb{R}$ s.t. (2) holds.

Choose α, β, γ satisfying (2). Then by (1)

$$\alpha(u+v+w) + \beta(u+2v+3w) + \gamma(u+4v+9w) = 0$$

$\Rightarrow \alpha = \beta = \gamma = 0$ (Since B is linearly independent)

$\Rightarrow a = b = c = 0$ (by (2))

Hence A is linearly independent.

30)

Let $A \in M_{n\times n}(\mathbb{R})$ and let I be the identity matrix of size n. Suppose the augmented matrix $(A|I)$ is row equivalent to $(B|C)$ where $B \in M_{n\times n}(\mathbb{R})$ is row reduced echelon matrix. Show that if A is invertible, then $C = A^{-1}$.

Find the inverse of the following matrix performing elementary row operations on a suitable matrix. $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 1 & 0 & 4 & 0 \\ 0 & 9 & 0 & 16 \end{pmatrix}$ $[1+3=4]$

Sol^n

Q2 Given, A is invertible & $(A|I) \sim (B|C)$, with B row reduced echelon. Hence we must have $B = I$.

Let $P_m \cdots P_2 P_1 (A|I) = (I|C)$. Then $P_m \cdots P_2 P_1 (A) = I$ & $P_m \cdots P_2 P_1 (I) = C$. The first equality gives $P_m \cdots P_2 P_1 (I) \cdot A = I$; combining this with the second equality, we have $C \cdot A = I$. Since A is invertible, must have $C = A^{-1}$.

Observe.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 9 & 0 & 16 & 0 & 0 & 0 & 1 \end{array}\right) \underset{R_4 \to R_4 - 3R_2}{\overset{R_3 \to R_3 - R_1}{\sim}} \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & -3 & 0 & 1 \end{array}\right)$$

$R_1 \to R_1 - R_3$, $R_2 \to \frac{1}{3} R_2$, $R_3 \to \frac{1}{2} R_3$, $R_4 \to \frac{1}{4} R_4$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3/4 & 0 & 1/4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 4/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3/4 & 0 & 1/4 \end{array}\right)$$

Hence inverse of the given matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 4/3 & 0 & -1/3 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -3/4 & 0 & 1/4 \end{pmatrix}$$

31)

Suppose the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$T(x, y, z, w) = (x - y + z - w, x + 2y + w, y + z - w, 3x + 4y + 2z).$$

(a) Find rank(T) and nullity(T).

(b) Use these values (rank(T) and nullity(T)) to find whether T is (i) one to one (injective), (ii) onto (surjective). [2 + 2 = 4]

Sol^n

Q3 $\text{Null}(T) = \{(x,y,z,w) : x-y+z-w=0,\ x+2y+w=0,\ y+z-w=0,\ 3x+4y+2z=0\}$

Coeffient matrix of the system of linear eq^{ns} defining $T\text{Null}(T)$ is

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 3 & 4 & 2 & 0 \end{pmatrix} \underset{R_4 \to R_4 - 3R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 7 & -1 & 3 \end{pmatrix}$$

$$\underset{\substack{R_2 \to R_2 - 3R_3 \\ R_4 \to R_4 - 7R_3 \\ R_3 \leftrightarrow R_2}}{\overset{R_1 \to R_1 + R_3}{\sim}} \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & 5 \\ 0 & 0 & -8 & 10 \end{pmatrix} \underset{R_3 \to -\frac{1}{4}R_3}{\overset{R_4 \to R_4 - 2R_3}{\sim}} \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -5/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{R_2 \to R_2 - R_3}{\overset{R_1 \to R_1 - 2R_3}{\sim}} \begin{pmatrix} 1 & 0 & 0 & -2 + 5/2 \\ 0 & 1 & 0 & -1 + 5/4 \\ 0 & 0 & 1 & -5/4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -5/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Number of unknowns $= 4$

rank of the coeff matrix $= 3$

dimension of the solution space is $4 - 3 = 1$.

Hence nullity$(T) = 1$

rank$(T) = \dim(\text{domain space}) - 1$

$= 4 - 1 = 3$

Since nullity$(T) = 1$, T is not injective.

Since rank$(T) = 3$, T is not surjective.

32)

(a) Let V be a vector space over $\mathbb{R}$ and let W be a subspace of V. Let $T : V \to V$ be a linear transformation. Show that U, defined by

$$U = \{v \in V : T(v) \in W\},$$

is a subspace of V.

(b) Let $\mathcal{P}_3 = \{f(x) \in \mathbb{R}[x] : \deg(f(x)) \leq 2\}$. Find the coordinate vector $[v]_B$ of $v = 3x^2 + 5x - 2$ with respect to the ordered basis $B = \{x - 1, x^2 + x + 1, 3\}$ (the order is as elements are written). [$2 \times 2 = 4$]

Soln

Here $U = \overline{W}$

4(a) Let $v_1, v_2 \in \overline{W}$ and $\alpha \in \mathbb{R}$.

$\overline{W} \neq \phi$ & $0 \in \overline{W}$ since $T(0) = 0 \in W$.

Then $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2)$ ($\because T$ is linear)

By the definition of $\overline{W}$, $T(v_1), T(v_2) \in W$.

Hence $T(\alpha v_1 + v_2) \in W$, so that

$\alpha v_1 + v_2 \in \overline{W}$.

Thus $\overline{W}$ is a subspace of V.

4(b). Let $3x^2 + 5x - 2 = a(x-1) + b(x^2+x+1) + c \cdot 3$

Then $3x^2 + 5x - 2 = bx^2 + (a+b)x - a + b + 3c$

Comparing coefficients, $-a + b + 3c = -2$

$b = 3$, $a + b = 5$, $c = \dfrac{-2 + 2 - 3}{3} = -\dfrac{3}{3} = -1$

Hence $a = 2$, $b = 3$, $c = -1$

Hence $[3x^2 + 5x - 2]_B = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

ordered basis.

33)

Find whether the following statements are True or False. Justify in each case.

(a) Let a subset B of $\mathbb{R}^4$ be defined by $B = \{e_1+e_2, e_2+e_3, e_3+e_4, e_4+e_1\}$ where e_i denotes a 4-tuple with i-th entry equal to 1 and other entries 0. Then B is a basis of $\mathbb{R}^4$.

(b) Let V be a finite dimensional vector space over $\mathbb{R}$. Every onto (surjective) linear transformation from V to V is one to one (injective). $[2 \times 2 = 4]$

Sol^n

5. a) Let $\alpha(e_1+e_2)+\beta(e_2+e_3)+\gamma(e_3+e_4)+\delta(e_4+e_1)=0$

Then $(\alpha+\delta)e_1+(\alpha+\beta)e_2+(\beta+\gamma)e_3+(\gamma+\delta)e_4=0$.

$\Rightarrow \alpha+\delta=0, \quad \alpha+\beta=0, \quad \beta+\gamma=0, \quad \gamma+\delta=0.$

Hence coeffient matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_3} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since rank of the coeffient matrix is 3 (<4), (for instance x_4 is free), there are infinitely many solution. Hence there is a non zero solution. Hence B is ~~not~~ linearly dependent so that B is not a basis.

5(b) Suppose $f: V \to V$ is surjective.

Let $\dim V = n$. ~~By rank~~

Then rank $f = n$.

By rank-nullity theorem $\operatorname{rank}(f) + \operatorname{nullity}(f) = n$

$\Rightarrow n + \operatorname{nullity}(f) = n$

$\Rightarrow \operatorname{nullity}(f) = 0$

$\Rightarrow f$ is injective.

34)

Let α be a positive real number. Suppose $A = \begin{pmatrix} \alpha & -2 & 1 \\ 4 & -\alpha & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find all values of α (> 0) for which A is **not** diagonalizable. [$4 = 1 + 3$]

Sol^n

① $\alpha > 0$, $\alpha \in \mathbb{R}$

$$A = \begin{pmatrix} \alpha & -2 & 1 \\ 4 & -\alpha & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Characteristic poly. of A $= \det(\lambda I - A)$

$= \lambda^3 - \lambda^2 + (8-\alpha^2)\lambda - (8-\alpha^2)$] 1 marks

$= (\lambda - 1)(\lambda^2 + 8 - \alpha^2)$.

$= (\lambda-1)(\lambda - \sqrt{\alpha^2-8})(\lambda + \sqrt{\alpha^2-8})$

(b) since we know that if E. values are distinct then A is diagonalizable so for A not to be diagonalizable we must have repeated E. values.

which is possible in 2 cases

(i) $\alpha = \pm 2\sqrt{2}$; then E. values are $1, 0, 0$] 1 mark

(ii) $\alpha = \pm 3$; " " " $1, 1, -1$.

we discard here -ve values of α.

Case (i)

$\alpha = 2\sqrt{2}$, $\lambda = 0$

$\dim W_0 = 1$; $\dim W_1 = 1$

$\Rightarrow$ A is not diagonalisable.

Case (ii)

$\alpha = 3$ $\lambda = 1$

$\dim W_1 = 2$; $\dim W_{-1} = 1$

$\Rightarrow$ A is diagonalisable.

so The value of $\alpha (>0)$ for which A is not diagonalizable is $2\sqrt{2}$ only.

Students should show $\dim W_0 = 1$ (when $\alpha = 2\sqrt{2}$) & $\dim W_1 = 2$ when $\alpha = 3$.

35)

2. (a) Show that if λ is an eigenvalue of a matrix B, then λ^2 is an eigenvalue of B^2.

   (b) Suppose that the characteristic polynomial of B is $x^3+\mu x^2+1$. Write B^{-1} as a polynomial in B.

   (c) For B in part (b), find the trace of B^{-1}. (You may use that for any matrix, its trace is the sum of its eigenvalues.) [4 = 1 + 1 + 2]

Sol^n

2 (a) Given : 'λ' is an e-value of B.

Prove : λ^2 is an e-value of B^2

Proof :

$BX = \lambda X, \quad X \neq 0$

pre-multiply by B

$B^2X = B(\lambda X)$

$B^2X = \lambda(BX)$

$B^2X = \lambda(\lambda X) = \lambda^2 X.$

hence λ^2 is an e-value of B^2.

Aliter:

$|\lambda^2 I - B^2|$

$= \underbrace{|\lambda I - B|}_{0}\, |\lambda I + B|$

$= 0 \cdot |\lambda I + B|$

$= 0$

(b) Char poly^n of B $= x^3 + \mu x^2 + 1 . = p(x)$ (say) —(1)

By Cayley-Hamilton Thm.

$B^3 + \mu B^2 + I = 0$

pre-multiply by B^{-1} (B^{-1} exists coz $p(0) = 1$)

$B^2 + \mu B + B^{-1} = 0$

$$\boxed{B^{-1} = -(\mu B + B^2)}$$

(c) find $tr(B^{-1})$.

(★) $tr(B^{-1}) = tr[-(\mu B + B^2)]$

$= -tr(\mu B) - tr(B^2)$

From (1), $tr(B) = -\mu$ —— (2)

$\alpha_1 + \alpha_2 + \alpha_3 = -\mu$

$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = 0$

$\alpha_1\alpha_2\alpha_3 = -1$

So $tr(B^2) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$

$= (\alpha_1+\alpha_2+\alpha_3)^2 - 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)$

$= \mu^2$ —— (3)

Use (2) + (3) in (★)

$tr(B^{-1}) = -\mu(-\mu) - \mu^2$

$= \mu^2 - \mu^2$

$$\boxed{tr(B^{-1}) = 0}$$

36)

Show that the following system of linear equations have infinitely many solutions and find all solutions. [4]

$$\begin{aligned} x + 2y + 3z + w &= 4 \\ 2x + 3y + z - w &= 4 \\ 3x + 4y - z + 2w &= 5 \\ 2x + 3y + z + 4w &= 5 \end{aligned}$$

Sol^n

Major Solution Q7

The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 2 & 3 & 1 & -1 & 4 \\ 3 & 4 & -1 & 2 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{array}\right) \underset{R_4 \to R_4 - 2R_1}{\overset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 0 & -1 & -5 & -3 & -4 \\ 0 & -2 & -10 & -1 & -7 \\ 0 & -1 & -5 & 2 & -3 \end{array}\right)$$

$$\underset{\substack{R_4 \to R_4 - R_2 \\ R_2 \to (-1) \times R_2}}{\overset{R_3 \to R_3 - 2R_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 4 \\ 0 & 1 & 5 & 3 & 4 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 & 1 \end{array}\right) \underset{R_1 \to R_1 - 2R_2}{\overset{\substack{R_4 \to R_4 - R_3 \\ R_3 \to \frac{1}{5} \times R_3}}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & -7 & -5 & -4 \\ 0 & 1 & 5 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{R_2 \to R_2 - 3R_3}{\overset{R_1 \to R_1 + 5R_3}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & -7 & 0 & -3 \\ 0 & 1 & 5 & 0 & 4 - \frac{3}{5} \\ 0 & 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \simeq \left(\begin{array}{cccc|c} 1 & 0 & -7 & 0 & -3 \\ 0 & 1 & 5 & 0 & \frac{17}{5} \\ 0 & 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Where the coefficient matrix is an RRE. The unknown z is free. Assign $z = \lambda \in \mathbb{R}$. Then

$x - 7\lambda = -3$, $y + 5\lambda = \frac{17}{5}$, $w = 1/5$. Hence

the general solution is

$x = 3 + 7\lambda$, $y = \frac{17}{5} - 5\lambda$, $z = \lambda$, $w = \frac{1}{5}$

for $\lambda \in \mathbb{R}$. This is an infinitely many solution case.

37)

9. Let

$$W_1 = \{(x, y, z, w) \in \mathbb{R}^4 : x = 0, y + z + w = 0\}$$

and

$$W_2 = \{(x, y, z, w) \in \mathbb{R}^4 : x - y = 0, x + y + z + w = 0\}.$$

Find a basis B_1 of $W_1 \cap W_2$ and find a basis B_2 of $W_1 + W_2$ containing B_1. [4]

Sol^n

By inspection, $W_1 = \{(0, \lambda, \mu, -\lambda-\mu) : \lambda, \mu \in \mathbb{R}\}$

$W_2 = \{(\alpha, \alpha, \beta, -2\alpha-\beta) : \alpha, \beta \in \mathbb{R}\}$

& $W_1 \cap W_2 = \{(x, y, z, w) \in \mathbb{R}^4 : x = 0, x - y = 0, y + z + w = 0, x + y + z + w = 0\}$ (by definition of intersection)

$= \{(x, y, z, w) \in \mathbb{R}^4 : x = y = 0, z + w = 0\}$

$= \{(0, 0, \gamma, -\gamma) \in \mathbb{R}^4 : \gamma \in \mathbb{R}\}$.

Since $\gamma(0, 0, 1, -1) = (0, 0, \gamma, -\gamma)$ & $\{(0, 0, 1, -1)\}$ is linearly independent,

$B_1 = \{(0, 0, 1, -1)\}$ is a basis of $W_1 \cap W_2$.

To find a basis of $W_1 + W_2$, we extend B_1 to a basis A of W_1 & C of W_2. Then $B_2 := A \cup C$ is a basis $W_1 + W_2$.

$\lambda = 0, \mu = 1$ gives $(0, 0, 1, -1)$ in W_1 &

& $\lambda = 1, \mu = 0$ gives $(0, 1, 0, -1)$ in W_1 so that

$(0, \lambda, \mu, -\lambda-\mu) = \lambda(0, 1, 0, -1) + \mu(0, 0, 1, -1)$

$A = \{(0, 0, 1, -1), (0, 1, 0, -1)\}$ is a basis of W_1

$\alpha = 1, \beta = 0$ gives $(1, 1, 0, -2)$ in W_2 & $(0, 0, 1, -1) \in W_2$.

& $(\alpha, \alpha, \beta, -2\alpha-\beta) = \alpha(1, 1, 0, -2) + \beta(0, 0, 1, -1)$ so that

$C = \{(0, 0, 1, -1), (1, 1, 0, -2)\}$ is a basis of W_2

Hence $B_2 = A \cup C = \{(0, 0, 1, -1), (0, 1, 0, -1), (1, 1, 0, -2)\}$ is a basis of $W_1 + W_2$. (Using dimension formula one can verify that $\dim(W_1 + W_2)$ is 3).

38)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose [4]

$$\begin{aligned} T(1,1,1) &= (1,-1,0), \\ T(1,-1,1) &= (0,1,-1), \\ T(0,1,-1) &= (-1,0,1). \end{aligned}$$

Find the nullity of T and $T(0,0,1)$.

Sol^n

Major, Solution, Q9: It is given that $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $T(1,1,1) = (1,-1,0)$, $T(1,-1,1) = (0,1,-1)$, $T(0,1,-1) = (-1,0,1)$.

First, we ~~obj~~ observe that $B = \{(1,1,1), (1,-1,1), (0,1,-1)\}$ is a basis of $\mathbb{R}^3$. For instance, the matrix (obtained from ~~the ba~~ B)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \underset{R_3 \to R_3 - R_2}{\overset{R_2 \to -\frac{1}{2} \times R_2}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \underset{R_1 \to R_1 - R_2 - R_3}{\overset{R_3 \to R_3 \times (-1)}{\sim}} I_3$$

has rank 3 so that the row space has dimension 3.

Therefore span $\{(1,-1,0), (0,1,-1), (-1,0,1)\}$ is the range space of T. ~~To find~~ The rank of T is the rank of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \overset{R_3 \to R_3 + R_1}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\overset{R_3 \to R_3 + R_2}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1 \to R_1 + R_2}{\sim} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

whose rank is 2. Hence nullity of T is $3 - \text{rank } T = 3 - 2 = 1$ (by using rank-nullity theorem)

$\frac{1}{2}\{(1,1,1) - (1,-1,1)\} = (0,1,0)$

Hence $\frac{1}{2}\{(1,1,1) - (1,-1,1)\} - (0,1,-1) = (0,0,1)$. Now apply T

$T(0,0,1) = \frac{1}{2} T(1,1,1) - \frac{1}{2} T(1,-1,1) - T(0,1,-1) = \left(\frac{1}{2}, -\frac{1}{2}, 0\right) - \left(0, \frac{1}{2}, -\frac{1}{2}\right) - (-1,0,1)$

$= \left(\frac{3}{2}, -1, -\frac{1}{2}\right)$ ~~$= \left(\frac{1}{2}, 1, \frac{1}{2}\right)$~~

39) Find the span of the set $\{(t, t^2, t^3) \in \mathbb{R}^3 : t \in \mathbb{R}\}$ in $\mathbb{R}^3$.

Sol^n

Let $\mathcal{A} = \left\{ \begin{pmatrix} 1 & t \\ t^2 & t^3 \end{pmatrix} : t \in \mathbb{R} \right\}$ & let $A_t = \begin{pmatrix} 1 & t \\ t^2 & t^3 \end{pmatrix}$.

Span $\mathcal{A}$ is a subspace of $M_2(\mathbb{R})$. Then

$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$,

$A_2 = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ and so on. (Here we expect that four different values of t ~~with~~ produces a set of linearly independent set of vectors. In fact,

$A_1 + A_{-1} = 2\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ & $P = \frac{1}{2}(A_1 + A_{-1}) - A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \text{Span}(\mathcal{A})$

Now ~~$Q = A_1 - A_0 - P =$~~ $Q = \frac{1}{2}(A_1 - A_{-1}) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \text{span}(\mathcal{A})$

$A_2 - A_0 - 4P - 2Q = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \in \text{Span}\,\mathcal{A}$.

Thus $R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{6}(A_2 - A_0 - 4P - 2Q) \in \text{span}(\mathcal{A})$

Lastly, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = Q - R \in \text{span}(\mathcal{A})$

We have seen that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are in span$(\mathcal{A})$. Hence $\mathcal{A}$ spans entire $M_2(\mathbb{R})$.

40)

Let $A = \begin{pmatrix} 3/2 & -1/2 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & -1/2 & 3/2 \end{pmatrix}$.

(a) Find the characteristic polynomial of A and its roots.

(b) Use part (a) to show that A is diagonalizable.

(c) Find $P \in M_3(\mathbb{R})$ such that $P^{-1}AP$ is a diagonal matrix. [$2+2+2=6$]

Sol^n

1. (a) char. poly. $p(x) = \det(xI - A)$

$$= \begin{vmatrix} x-3/2 & 1/2 & -1/2 \\ 0 & x-1 & 0 \\ -1/2 & 1/2 & x-\frac{3}{2} \end{vmatrix}$$

$$= (x-1)\left[(x-\tfrac{3}{2})^2 - \tfrac{1}{4}\right]$$

$$= (x-1)\left[x^2 - 3x + 2\right]$$

$$= (x-1)(x-1)(x-2)$$

Roots of $p(x)$ are $1, 1, 2$.

(b) From (a), the eigenvalues are 1 (repeated twice) & 2.

Let's find the dimension of the eigenspace corresponding to eigenvalue 1.

$$1.I - A = \begin{pmatrix} -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 1/2 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{—} (*)$$

$\Rightarrow$ rank$(I-A) = 1$ $\Rightarrow$ nullity$(I-A) = 3-1 = 2$

$\therefore$ $\dim E_1 = 2$

Also, $\dim E_2 = 1$ ($\because$ multiplicity of eigenvalue 2 is 1)

Since $\dim E_1 + \dim E_2 = 3 = \dim \mathbb{R}^3$, A is diagonalizable.

(c) To find P we need to find a basis consisting of eigenvectors.

We see from $(*)$, $\{(1,1,0), (-1,0,1)\}$ forms a basis for E_1.

Also, $\{(1,0,1)\}$ forms a basis for E_2.

$\therefore$ for $P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ we have

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D.$$

41)

(a) Find a basis for $W_1 \cap W_2$, where [3]

$$W_1 = \{(x, y, z, w) \in \mathbb{R}^4 : x - y + z - w = 0, 5x - 4y + 3z - 2w = 0\}$$
$$W_2 = \{(x, y, z, w) \in \mathbb{R}^4 : x - 2y + 3z - 4w = 0, x - 2y + 2z - w = 0\}.$$

(b) Find a basis for span(S), where [2]

$$S = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + z = 2, x + y + z = 1\}.$$

Sol^n

1. (a) $(x,y,z,w) \in W_1 \cap W_2 \iff \begin{cases} x-y+z-w=0 \\ 5x-4y+3z-2w=0 \\ x-2y+3z-4w=0 \\ x-2y+2z-w=0 \end{cases}$

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 5 & -4 & 3 & -2 \\ 1 & -2 & 3 & -4 \\ 1 & -2 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank $= 3$ $\quad \therefore$ dim (soln. space) $= 4 - 3 = 1$

$W_1 \cap W_2 = \{(\lambda, 3\lambda, 3\lambda, \lambda) : \lambda \in \mathbb{R}\}$
$= \text{span}\{(1,3,3,1)\}$

$\therefore$ $\{(1,3,3,1)\}$ is a basis for $W_1 \cap W_2$.

(b) $S = \{(x,y,z) \in \mathbb{R}^3 : 3x+2y+z=2, \ x+y+z=1\}$ is a straight line in $\mathbb{R}^3$ not passing through the origin.

So, any two points on S spans span(S).

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$S = \{(\lambda, 1-2\lambda, \lambda) : \lambda \in \mathbb{R}\}$

Putting $\lambda = 0$ & $\lambda = 1$, we get $(0,1,0), (1,-1,1) \in S$

$\therefore$ $\{(0,1,0), (1,-1,1)\}$ is a basis for span(S).

42) Find a condition on $\alpha, \beta, \gamma, \delta$ so that

$$\{(0,1,0,1),(1,0,1,0),(2,1,1,1),(\alpha,\beta,\gamma,\delta)\}$$

is a linearly dependent set in $\mathbb{R}^4$. [3]

Sol^n

(2)

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ \alpha & \beta & \gamma & \delta \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \alpha & \beta & \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta-\beta \end{pmatrix}$$

Rank of the matrix is 4 if $\delta-\beta \neq 0$ and 3 if $\delta-\beta=0$.

For the set to be linearly dependent the rank must be less than 4.

$\therefore$ The given set is L.D. iff. $\boxed{\beta-\delta=0}$

43)

Let V be the vector space over the field $\mathbb{R}$ of all real-valued functions on $\mathbb{R}$.

(a) Let $S_a = \{\sin 2x, \sin ax\}$, $a \in \mathbb{R}$. Find all a such that S_a is linearly independent. [2]

(b) Let $\mathcal{B} = \{e^{3x}, \cos 2x, \sin 2x\}$ and $W = \operatorname{span}(\mathcal{B})$. Let $f(x) = e^{3x} + \sin 2x - \cos 2x$.

(i) Show that $f'(x) \in W$, where $f'(x)$ denotes the derivative of $f(x)$. [1]

(ii) Find the coordinate vector $[f'(x)]_{\mathcal{B}}$ by treating $\mathcal{B}$ as an ordered basis for W. [1]

Sol^n

(3). (a) $S_a = \{\sin 2x, \sin ax\}$

If $a = 0$ then S_a is L.D. as $0 \in S_a$.

Assume $a \neq 0$.

S_a is L.D. $\Leftrightarrow$ $\sin ax = \lambda \sin 2x \quad \forall x \in \mathbb{R}$ for some $\lambda \in \mathbb{R}$.

Now $\sin ax = \lambda \sin 2x \quad \forall x \in \mathbb{R}$

$\Rightarrow a \cos ax = 2\lambda \cos 2x$

$\Rightarrow -a^2 \sin ax = -4\lambda \sin 2x$

$\Rightarrow a^2 \sin ax = 4\lambda \sin 2x = 4 \sin ax$

$\Rightarrow (a^2 - 4) \sin ax = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow a^2 - 4 = 0 \quad (\because a \neq 0)$

$\Rightarrow a = \pm 2$.

$\therefore S_a$ is L.I. if $a \notin \{0, \pm 2\}$

Also if $a = \pm 2$, $S_a = \{\sin 2x, \pm \sin 2x\}$ is L.D.

$\therefore S_a$ is L.I. iff $a \in \mathbb{R} \setminus \{0, 2, -2\}$.

(b) $\mathcal{B} = \{e^{3x}, \cos 2x, \sin 2x\}$, $W = \operatorname{span}(\mathcal{B})$

$f(x) = e^{3x} + \sin 2x - \cos 2x$

(i) $f'(x) = 3e^{3x} + 2\cos 2x + 2\sin 2x \in \operatorname{span}(\mathcal{B}) = W$

(ii) $[f'(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

44)

Prove or disprove the following statements. [$6 = 3 \times 2$]

(a) Let X and Y be nonempty subsets of a vector space V. Then

$$\text{span}(X) \cup \text{span}(Y) = \text{span}(X \cup Y).$$

(b) If $\{u, v, w\}$ is linearly independent, then $\{u-3v, 3v-w, w-u\}$ is linearly independent.

(c) For any $A \in M_{n\times n}(\mathbb{R})$, $W = \{X \in M_{n\times n}(\mathbb{R}) : AX = XA^t\}$ is a subspace of $M_{n\times n}(\mathbb{R})$, where A^t denotes the transpose of A.

Sol^n

(4) (a) Take $V = \mathbb{R}^2$, $X = \{(1,0)\}$, $Y = \{(0,1)\}$

Then $\text{span}(X) = \{(x,0) : x \in \mathbb{R}\}$

$\text{span}(Y) = \{(0,y) : y \in \mathbb{R}\}$

$\text{span}(X \cup Y) = \mathbb{R}^2$

$\therefore$ $\text{span}(X) \cup \text{span}(Y) \neq \text{span}(X \cup Y)$

Thus the statement is FALSE.

(b) FALSE: $(u-3v) + (3v-w) + (w-u) = 0$

$\therefore$ $\{u-3v, 3v-w, w-u\}$ is L.D.

(c) Since $AO = OA^t$, $O \in W$ (O denotes the ~~zero~~ matrix of size $n\times n$)

$\therefore$ $W \neq \phi$.

If $X, Y \in W$ & $\lambda \in \mathbb{R}$, then

$AX = XA^t$; $AY = YA^t$

$\therefore$ $A(X+\lambda Y) = AX + \lambda AY = XA^t + \lambda YA^t = (X+\lambda Y)A^t$

$\Rightarrow$ $X + \lambda Y \in W$

$\therefore$ W is a subspace.

45) Write down all possible 3×3 real RRE matrices of rank 2.

Sol^n

(1) White

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \mu & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\alpha, \beta \in \mathbb{R}$ $\qquad$ $\mu \in \mathbb{R}$

46) Write down all possible 2×3 real RRE matrices of rank 2. [2]

Sol^n

(5) Green

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{pmatrix}, \quad \begin{pmatrix} 1 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\alpha, \beta \in \mathbb{R}$ $\qquad$ $\mu \in \mathbb{R}$

47)

Prove or disprove the following statements. [$6 = 3 \times 2$]

(a) For any $A, B \in M_{n\times n}(\mathbb{R})$, $W = \{X \in M_{n\times n}(\mathbb{R}) : AXB = BXA\}$ is a subspace of $M_{n\times n}(\mathbb{R})$.

(b) Let A and B be nonempty subsets of a vector space V. Then

$$\operatorname{span}(A) \cap \operatorname{span}(B) = \operatorname{span}(A \cap B).$$

(c) If $\{u, v, w\}$ is linearly independent, then $\{u-2v, 2v-w, w-u\}$ is linearly independent.

Sol^n

2. (a) Since $AOB = O = BOA$, $O \in W$

(O denotes the zero matrix of size $n\times n$)

$\therefore$ $W \neq \phi$

If $X, Y \in W$ and $\lambda \in \mathbb{R}$, then

$AXB = BXA$ and $AYB = BYA$

$\therefore$ $A(X+\lambda Y)B = AXB + \lambda AYB$

$= BXA + \lambda BYA = B(X+\lambda Y)A$

$\Rightarrow$ $X + \lambda Y \in W$

Hence, W is a subspace.

(b) FALSE: Take $V = \mathbb{R}$, $A = \{1\}$, $B = \{2\}$

Then $\operatorname{span}(A) = \mathbb{R}$, $\operatorname{span}(B) = \mathbb{R}$

But $\operatorname{span}(A \cap B) = \{0\}$.

(c) FALSE: $(u-2v) + (2v-w) + (w-u) = 0$

$\Rightarrow$ $\{u-2v, 2v-w, w-u\}$ is linearly dependent.

48)

(a) Find a basis for $W_1 \cap W_2$, where [3]

$$W_1 = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0, 2x + 3y + 4z + 5w = 0\}$$
$$W_2 = \{(x, y, z, w) \in \mathbb{R}^4 : 4x + 3y + 2z + w = 0, x + 2y + 2z + w = 0\}.$$

(b) Find a basis for span(S), where [2]

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1, x + 2y + 3z = 2\}.$$

Sol^n

3. (a) $(x,y,z,w) \in W_1 \cap W_2 \iff \begin{cases} x+y+z+w=0 \\ 2x+3y+4z+5w=0 \\ 4x+3y+2z+w=0 \\ x+2y+2z+w=0 \end{cases}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \underset{\substack{R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - R_1}}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\underset{R_3 \to R_3 + R_2}{\overset{R_1 \to R_1 - R_4}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R_3 \to R_3 - R_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_2 \to R_2 - R_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 3 \implies \dim(\text{soln. space}) = 4 - 3 = 1$

$W_1 \cap W_2 = \{(-\lambda, 3\lambda, -3\lambda, \lambda) : \lambda \in \mathbb{R}\}$

$= \text{span}\{(-1, 3, -3, 1)\}$

$\therefore$ A basis for $W_1 \cap W_2$ is $\{(-1, 3, -3, 1)\}$

---

3. (b) $S = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=1, x+2y+3z=2\}$ is a straight line in $\mathbb{R}^3$ not passing through $(0,0,0)$.

So, any two points on S spans span(S).

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$S = \{(\lambda, 1-2\lambda, \lambda) : \lambda \in \mathbb{R}\}$

Putting $\lambda = 0$ & $\lambda = 1$, we get $(0,1,0), (1,-1,1) \in S$.

$\therefore \{(0,1,0), (1,-1,1)\}$ is a basis for span(S).

49)

Let V be the vector space over the field $\mathbb{R}$ of all real-valued functions on $\mathbb{R}$.

(a) Let $S_a = \{\sin 3x, \sin ax\}$, $a \in \mathbb{R}$. Find all a such that S_a is linearly independent. [2]

(b) Let $\mathcal{B} = \{e^{2x}, \sin 3x, \cos 3x\}$ and $W = \text{span}(\mathcal{B})$. Let $f(x) = e^{2x} + \sin 3x - \cos 3x$.

(i) Show that $f'(x) \in W$, where $f'(x)$ denotes the derivative of $f(x)$. [1]

(ii) Find the coordinate vector $[f'(x)]_{\mathcal{B}}$ by treating $\mathcal{B}$ as an ordered basis for W. [1]

Sol^n

5. (a) $S_a = \{\sin 3x, \sin ax\}$

If $a = 0$, S_a is L.D. because $0 \in S_a$.

Assume $a \neq 0$.

S_a is L.D. $\Leftrightarrow$ $\sin ax = \lambda \sin 3x \quad \forall x \in \mathbb{R}$ for some $\lambda \in \mathbb{R}$.

Now $\sin ax = \lambda \sin 3x \Rightarrow a \cos ax = 3\lambda \cos 3x$

$\Rightarrow -a^2 \sin ax = -9\lambda \sin 3x$

$\Rightarrow a^2 \sin ax = 9\lambda \sin 3x$

$= 9 \sin ax$

$\Rightarrow (a^2 - 9) \sin ax = 0 \quad \forall x \in \mathbb{R}.$

$\Rightarrow a^2 = 9 \quad (\because a \neq 0)$

Clearly if $a = \pm 3$, S_a is L.D.

$\therefore$ S_a is L.I. iff $a \in \mathbb{R} \setminus \{0, 3, -3\}$.

(b) $\mathcal{B} = \{e^{2x}, \sin 3x, \cos 3x\}$, $W = \text{span}(\mathcal{B})$

$f(x) = e^{2x} + \sin 3x - \cos 3x$

(i) $f'(x) = 2e^{2x} + 3\cos 3x + 3\sin 3x \in \text{span}(\mathcal{B}) = W$

(ii) $[f'(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

50) Find a condition on a, b, c, d so that

$$\{(1,1,1,1),(1,0,1,0),(1,1,0,1),(a,b,c,d)\}$$

is a linearly dependent set in $\mathbb{R}^4$. [3]

Sol^n

(4) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ a & b & c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ a & b & c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \overset{R_4 \to R_4 - aR_1 - bR_2 - cR_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d-b \end{pmatrix}$

Rank of the matrix is 4 if $d \neq b$ and 3 if $d = b$.

For linearly dependent the rank must be less than 4.

$\therefore$ The given set is L.D. iff $\boxed{b - d = 0}$.

51)

Consider the following linear operator on $M_{3\times 3}(\mathbb{R})$. [2+2+2+1=7]

$$T(A) = A - A^t \quad \text{for } A \in M_{3\times 3}(\mathbb{R}).$$

(a) Find the nullity and the null space of T.

(b) Find all the eigenvalues of T from the definition (without computing the characteristic polynomial of T), that is, find all $\lambda \in \mathbb{R}$ such that $T(A) = \lambda A$ for some $A \neq 0$.

(c) Find the dimension of each eigenspace.

(d) Is T diagonalizable?

Sol^n

4 ~~8~~. Consider the linear operator $T : M_{3\times3}(\mathbb{R}^?) \to M_{3\times3}(\mathbb{R})$ defined by $T(A) = A - A^t$.

(a) Find nullity and null space of T

(b) Find all eigen values of T from definition

(c) Find dimension of each eigen space

(d) Is T diagonalizable?

Solution: (a) The null space of T is $\{A \in M_3(\mathbb{R}) : A = A^t\}$

$= \{A \in M_3(\mathbb{R}) : a_{ij} = a_{ji} \text{ for } 1 \le i, j \le 3\}$

$= \{A \in M_3(\mathbb{R}) : a_{ij} = a_{ji} \text{ for } 1 \le i < j \le 3\}$

Which is defined by three ~~eq~~ independent eq^{ns}, namely $a_{12} = a_{21}$, $a_{13} = a_{31}$ & $a_{23} = a_{32}$

The there are $a_{11}, a_{22}, a_{33}, a_{21}, a_{31}, a_{32}$ are free unknowns. So that dimension of the null space is 6

(this is the space of 3×3 symmetric matrices over $\mathbb{R}$)

(b) Have to find $\lambda \in \mathbb{R}$ s.t. $A - A^t = \lambda A$ for some $A \neq 0$

$\Leftrightarrow$ ~~$(\lambda=1)$~~ $A^t = (1-\lambda) A \Leftrightarrow A = (1-\lambda)A^t$ (since $(A^t)^t = A$)

Thus $A = (1-\lambda)(1-\lambda)\cdot A$ (using $A^t = (1-\lambda)A$)

$\Leftrightarrow$ $A = (1 - 2\lambda + \lambda^2) A \Leftrightarrow \lambda(\lambda - 2) A = 0$

Thus $\lambda = 0$ or 2.

(c) For $\lambda = 0$ we have the eigenspace of T = null space of T whose dimension is 6.

For $\lambda = 2$, $A - A^t = 2A \Leftrightarrow A^t = -A$ 3×3 skew symmetric matrices.

$\{A \in M_3(\mathbb{R}) : a_{ij} = -a_{ji} \ 1 \le i, j < 3\}$

$= \{A \in M_3(\mathbb{R}) : a_{11} = 0, a_{22} = 0, a_{33} = 0, a_{12} = -a_{21}, a_{13} = -a_{31}, a_{23} = -a_{32}\}$

~~Defined by~~ There are only three free unknowns

Thus dimension of the eigenspace of T corresponding to 2 is 3.

(d) Now since $6 + 3 = 9 = \dim(M_3(\mathbb{R}))$, the sum of the dimensions of eigen spaces of T is the dimension of the whole space, T is diagonalizable.

52)

Suppose $\mathcal{P}_5 = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$. Let [3]

$$W_1 = \{f(x) \in \mathcal{P}_5 : x^4 f(1/x) = f(x)\}$$
$$W_2 = \{f(x) \in \mathcal{P}_5 : f(-x) = f(x)\}.$$

Find $\dim(W_1)$, $\dim(W_2)$ and $\dim(W_1 + W_2)$.

Sol^n

Q. Suppose $V = P_5(x) = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$

Let $W_1 = \{f(x) \in P_5(x) : x^4 f(\frac{1}{x}) = f(x)\}$

$W_2 = \{f(x) \in P_5(x) : f(-x) = f(x)\}$.

Find $\dim W_1$, $\dim W_2$, $\dim(W_1 + W_2)$.

Solution: ~~$W_1 = \{f(x) \in P_5(x)$~~ Set $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

$x^4 f(\frac{1}{x}) = f(x)$

$\Leftrightarrow a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

$\Leftrightarrow a_0 = a_4,\ a_1 = a_3,$ Thus $\dim W_1 = 3$ (three free unknowns) namely, a_2, a_3, a_4.

And $f(-x) = f(x)$

$\Leftrightarrow a_0 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 = a_0 + a_1x + a_2x^2 + a_3x^4 + a_4x^4$

$\Leftrightarrow a_1 = 0,\ a_3 = 0.$

Thus $\dim W_2 = 3$ there are three free unknowns namely, a_0, a_2, a_4.

If $f(x) \in W_1 \cap W_2 \Leftrightarrow x^4 f(\frac{1}{x}) = f(x)$ & $f(-x) = f(x)$

$\Leftrightarrow a_1 = 0,\ a_3 = 0,\ a_0 = a_4,$

Thus there two free unknowns, namely a_2, a_4. Hence $\dim(W_1 \cap W_2) = 2$.

Finally $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$= 3 + 3 - 2$

$= 4$

53)

Let $\lambda \in \mathbb{R}$ (an arbitrary scalar). The space of 3×3 matrices with real entries is denoted by $M_3(\mathbb{R})$. Define $\psi_\lambda : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ by $\psi_\lambda(A) = A + \lambda A^t$, where A^t is the transpose of A. $[6 = 2\times 3]$

(a) Show that ψ_λ is a linear transformation for every λ.

(b) Show that ψ_λ is one-to-one if and only if $\lambda \neq \pm 1$.

(c) Show that the range space of ψ_1 is the same as the null space of ψ_{-1}.

Sol^n

Q 1 $\quad \psi_\lambda : M_3(R) \longrightarrow M_3(R)$

$\psi_\lambda(A) = A + \lambda A^t$

(a) $\psi_\lambda(aA+bB) = (aA+bB) + \lambda(aA+bB)^t$ for $a, b \in \mathbb{R}$.

$= aA + bB + \lambda a A + \lambda b B^t$

$= a(A+\lambda A^t) + b(B+\lambda B^t)$

$= a\,\psi_\lambda(A) + b\,\psi_\lambda(B)$

Hence ψ_λ is a linear transformation.

(b) $\ker \psi_\lambda = \{A \in M_3(R) : A + \lambda A^t = 0\}$

If $\lambda = 1$, $A \in \ker\psi_\lambda \Leftrightarrow A = -A^t$ a skew symmetric matrix

If $\lambda = -1$, $A \in \ker\psi_\lambda \Leftrightarrow A = A^t$ a symmetric matrix.

Thus when $\lambda = \pm 1$, $\ker\psi_\lambda \neq 0$.

Hence $\ker\psi_\lambda = 0$ (i.e. ψ_λ is one to one) $\Rightarrow \lambda \neq \pm 1$.

Suppose $\lambda \neq \pm 1$, set $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then $A^t = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

If $A \in \ker\psi$, then $\begin{pmatrix} a_1+\lambda a_1 & b_1+\lambda a_2 & c_1+\lambda a_3 \\ a_2+\lambda b_1 & b_2+\lambda b_2 & c_2+\lambda b_3 \\ a_3+\lambda c_1 & b_3+\lambda c_2 & c_3+\lambda c_3 \end{pmatrix} = 0$

Hence $a_1 = b_2 = c_3 = 0$ as $1+\lambda \neq 0$ & $b_1 = -\lambda a_2$, $c_1 = -\lambda a_3$

$a_2 = -\lambda b_1$, $c_2 = -\lambda b_3$, $a_3 = -\lambda c_1$, $b_3 = -\lambda c_2$.

So $a_1 = b_2 = c_3 = 0$, $b_1 = \lambda^2 b_1$, $c_1 = \lambda^2 c_1$, $c_2 = \lambda^2 b_3$ etc.

Hence $a_1 = b_2 = c_3 = 0$ $b_1 = c_1 = c_2 = \cdots = 0$ all zero.

OR $\quad A + \lambda A^t = 0 \Rightarrow A^t + \lambda A = 0$ so $A^t = -\lambda A$. so $A + \lambda(-\lambda)A) = 0 \Rightarrow (\lambda^2-1)A = 0$

$\Rightarrow A = 0$

(c) $\mathrm{Null}(\psi_{-1}) = \{A : A - A^t = 0\}$ consists of symmetric matrices

$\mathrm{Range}\ \psi_1 = \{A + A^t : A \in M_3(R)\}$ ~~such~~ it is a subset of symmetric matrices in $M_2(R)$

But if B is a symmetric matrix $B = \frac{B+B^t}{2}$

$= \frac{1}{2}B + (\frac{1}{2}B)^t$

Thus Range ψ_1 consists of symmetric matrices.

Hence Range $\psi_1 = \mathrm{Null}(\psi_{-1})$

54) Let $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & s & t \end{pmatrix}$ where $s, t \in \mathbb{R}$. Find s and t, if the characteristic polynomial of A is $x^3 - 4x - 1$. [2]

Sol^n

Characteristic polynomial of $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & s & t \end{pmatrix}$ is

$$\det \begin{pmatrix} x & -1 & -2 \\ -1 & x-1 & 0 \\ -1 & -s & x-t \end{pmatrix} = x(x-1)(x-t) + \{-(x-t)\} - 2\{s + (x-1)\}$$

$$= x\{x^2 - (1+t)x + t\} - x + t - 2s + 2 - 2x$$

$$= x^3 - (1+t)x^2 + (t-3)x + t - 2s + 2.$$

which is given to be $x^3 - 4x - 1$.

Comparing coefficients $1+t = 0 \Rightarrow t = -1$

$t - 3 = -4 \Rightarrow t = -1$ (same)

$t - 2s + 2 = -1 \Rightarrow -1 - 2s = -3$

$\Rightarrow 2s = +2$

$\Rightarrow \underline{s = +1}$

$(t, s) = (-1, 1)$

55)

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x, y, z) = (7x + 2y + 3z, 8y, x - 2y + 5z).$$

Find a basis of $\mathbb{R}^3$ with respect to which the matrix of T is a diagonal matrix. [4]

Sol^n

The matrix of T w.r.t. to the standard basis is

$$A = \begin{pmatrix} 7 & 2 & 3 \\ 0 & 8 & 0 \\ 1 & -2 & 5 \end{pmatrix}$$

$\therefore$ Characteristic polynomial of T is

$$p(x) = \det(xI - A) = \begin{vmatrix} x-7 & -2 & -3 \\ 0 & x-8 & 0 \\ -1 & 2 & x-5 \end{vmatrix}$$

$$= (x-8)\left[(x-7)(x-5) - 3\right]$$

$$= (x-8)(x^2 - 12x + 32)$$

$$= (x-8)(x-8)(x-4)$$

[1 mark]

Thus the eigenvalues are 4, 8, 8.

For $\lambda = 4$: $4I - A = \begin{pmatrix} -3 & -2 & -3 \\ 0 & -4 & 0 \\ -1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ $(1, 0, -1)$ is an eigenvector with evalue 4. [1 mark]

For $\lambda = 8$: $8I - A = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ -1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ $\{(2, 1, 0), (3, 0, 1)\}$ is a basis for the eigenspace corresponding to eigenvalue 8. [1 mark]

If we take $\boxed{\mathcal{B} = \{(1, 0, -1), (2, 1, 0), (3, 0, 1)\}}$, [1 mark]

then $[T]_{\mathcal{B}}$ is the diagonal matrix $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.


That's all Folks!