DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI
MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)
2023-24 SECOND SEMESTER TUTORIAL SHEET-II

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1. Determine which of the following equations defines a linear transformation in the space P of polynomials.
(i) $T(P) = (P(x))^2$ (ii) T(P)=xP(x) (iii)T(P)=P(x+1)-P(0)
(iv) $T(P) = P(x^2)$ (v) $T(P) = P''(x) + 5P'(x)$ (vi)$T(P) = P(|x|)$

2. Let $T : R^3 \longrightarrow R^3$. Determine which of the transformations defined below are linear transformations.
(i) $T(x_1, x_2, x_3) = (x_1, x_2, x_3^2)$
(ii) $T(x_1, x_2, x_3) = (x_3, x_1, x_2 - 1)$
(iii) $T(x_1, x_2, x_3) = (x_1 + 2x_2 - 3x_3, 3x_1 - x_2 + 3x_3, 3x_1 + 3x_2 + 2x_3)$

3. Determine the range and null space of the linear transformation

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_2 - x_3, x_1)$$

4. Let $T : R^3 \longrightarrow R^2$ be a linear transformation defined by

$$T(x, y, z) = (x + y, y + z)$$

Find a basis, dimension of each of the range and null space of T.

5. Determine a linear transformation $T : R^3 \longrightarrow R^3$ whose range space is spanned by $\{(1, 2, 0), (0, 1, 1)\}$.

6. Let V be a vector space of all $n \times n$ matrices over a field F, and let B be any fixed $n \times n$ matrix. If $T(A) = AB - BA$, verify that T is a linear transformation from V into V.

7. Determine explicitly the linear transformation $T : R^2 \longrightarrow R^2$ such that $T(2, 3) = (4, 5)$ and $T(1, 0) = (0, 1)$.

8. Let V be n-dimensional vector space over the field F and let T be a linear transformation from V into V such that the range and null space of T are identical. Prove that n is even.

9. Let V be a finite dimensional vector space and let T be a linear operator on V. Suppose that rank (T^2)=rank (T). Prove that the range and null space of T are disjoint. i.e. have only the zero vector in common.

10. Let V and W be vector spaces over the field F. Let T and U be linear transformations from V into W. Show that the function $(T + U)$ is defined by

$$(T + U)(x) = T(x) + U(x)$$

is a linear transformation from V into W. Also show that If α is any element of F, the function (αT) defined by

$$(\alpha T)(x) = \alpha T(x)$$

is also a linear transformation from V into W. Prove that the set of all linear transformations from V into W, together with vector addition and scalar multiplication defined above, is a vector space over the field F.

12. Denote the space of linear transformations from V into W by $L(V, W)$. Let V be an $n-$dimensional vector space over the field F, and let W be an $m-$dimensional vector space over F. Then prove that $L(V, W)$ is finite-dimensional and has dimension mn.

13. Let V and W be vector spaces over the field F. Prove that V and W are isomorphic if and only if dim $V =$ dim W.

14. Let T be a linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so find T^{-1}.

15. Let $V(F)$ be a vector space and let T be a linear transformation from V into V. Prove that the following two statements about T are equivalent
(a) The intersection of range of T and Null space of T is the zero subspace of V.
(b) If $T(T(x)) = 0$, then $T(x) = 0$

16. Let T be a linear operator on FDVS $V(F)$. Suppose that there is a linear operator U on V such that $UT = I$. Show that T is invertible and $T^{-1} = U$.

17. Show that the conclusion of the previous problem fails if V is not finite dimensional.

18. Let T be a linear operator on FDVS $V(F)$. Prove that T is one-one if and only if T is onto.

19. Let $T(x_1, x_2, x_3) = (0, x_1, x_2)$ for all $(x_1, x_2, x_3) \in R^3$. Find the matrix representation of T with respect to the bases.
(i) $B_1 = \{(1, 0, 0,), (0, 1, 0), (0, 0, 1)\}$
(ii) $B_2 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$
(iii) $B_3 = \{(1, 2, 0), (2, 1, 0), (0, 0, 1)\}$

20. Let $T : P^3 \longrightarrow P^3$ be defined by

$$T[P(x)] = \frac{d}{dx}P(x).$$

Determine the matrix of T with respect to the basis $\{1, x, x^2, x^3\}$. Also determine the transformation whose matrix with respect to above basis is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

21. Let T be the linear operator on R^2 defined by

$$T(x_1, x_2) = (-x_2, x_1)$$

(a). Prove that for every real number c the operator $(T - cI)$ is invertible.
(b). Prove that if B is any ordered basis for R^2 and $[T]_B = A$, then $a_{12}a_{21} \neq 0$, where $A = (a_{ij})$

22. Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

(a) What is the matrix of T in the standard ordered basis for R^2?
(b) Prove that T is invertible and give a rule for T^{-1} like the one which defines T?

23. Let T be the linear operator on R^3, the matrix of which in the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Find a basis for the range of T and a basis for the Null space of T.