Name: ______________________

ID number: ______________________

There are 2 questions for a total of 10 points.

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1. Are the following valid argument forms? Explain your answer.

   (a) (2 ½ points)
   $$\begin{array}{l} \exists x\ (P(x) \wedge Q(x)) \\ \forall x\ (P(x) \rightarrow R(x)) \\ \hline \therefore \exists x\ (R(x) \wedge \neg Q(x)) \end{array}$$

   (b) (2 ½ points)
   $$\begin{array}{l} \forall x\ \forall y\ (S(x) \wedge A(x,y) \rightarrow \neg A(y,x)) \\ \hline \therefore \forall x\ (A(x,x) \rightarrow \neg S(x)) \end{array}$$

Solution:

(a) We will show that the given argument form **is not valid**. Let the domain be a set $\{a, b\}$ and P, Q, R over this domain is as follows: $P(a) = True; P(b) = False; Q(a) = True; Q(b) = False; R(a) = True; R(b) = False$. The hypothesis holds for the above but the conclusion does not. Hence this is not a valid argument form.

(b) We will argue that the given argument for **is valid** using rules of inference.

(1) $\forall x\ \forall y\ (S(x) \wedge A(x,y) \rightarrow \neg A(y,x))$ (Hypothesis)

(2) $S(a) \wedge A(a,a) \rightarrow \neg A(a,a)$ for an arbitrary a in domain (Universal instantiation)

(3) $\neg S(a) \vee \neg A(a,a) \vee \neg A(a,a)$ (since $p \rightarrow q \equiv \neg p \vee q$)

(4) $\neg S(a) \vee \neg A(a,a)$ (Idempotent law)

(5) $\neg A(a,a) \vee \neg S(a)$ (Commutative law)

(6) $A(a,a) \rightarrow \neg S(a)$ (since $p \rightarrow q \equiv \neg p \vee q$)

(7) $\forall x\ (A(x,x) \rightarrow \neg S(x))$ (Universal generalization)

2. (5 points) Prove or disprove: For any positive integer n, if n leaves a remainder 3 on being divided by 4, then n cannot be the sum of squares of two integers.

Solution: We will prove the statement. For the sake of contradiction, assume that there exists integers p and q such that $n = p^2 + q^2$. We will consider the following four exhaustive cases:

1. Both p and q are even: In this case there are integers x, y such that $p = 2k$ and $q = 2y$. This gives $p^2 + q^2 = 4(x^2 + y^2)$. This is a number that is divisible by 4. This is a contradiction since $p^2 + q^2 = n$ that leaves remainder 3 when divided by 4.
2. Exactly one of p or q is odd: WLOG assume that p is even and q is odd. So, there are integers x, y such that $p = 2x$ and $q = 2y + 1$. This gives $p^2 + q^2 = 4(x^2 + y^2 + y) + 1$ which is a number that leaves remainder 1 when divided by 4. This again contradicts with the fact that $p^2 + q^2 = n$ leaves a remainder 3 when divided by 4.
3. Both p and q are odd: In this case there are integers x, y such that $p = 2x + 1$ and $q = 2y + 1$. This gives $p^2 + q^2 = 4(x^2 + y^2 + x + y) + 2$ which is a number that leaves remainder 2 when divided by 4. This again contradicts with the fact that $p^2 + q^2 = n$ leaves a remainder 3 when divided by 4.

Thus we can conclude that there does not exist integers p and q such that $n = p^2 + q^2$.