COL202 : Discrete Mathematical Structures

Name: T Pratham
Entry no.: 2020EE10559
Group : 2

Tutorial Sheet - 2

2

3.

a) Theorem: The number of subsets of an n-element set is 2^n.

Proof: By induction.

$P(n)$: The number of subsets of n-element set is 2^n.

Base Case: $n=0$:- $P(0)$ is true since a 0-element set is an empty set whose number of subsets is just 1 ($=2^0$), the empty set itself.

Inductive step: Let's assume for some $n \geq 0$, $P(n)$ is true. Hypothesis
i.e. the number of subsets of n-element set is 2^n.

Now, Consider any set of $n+1$ elements:

$$S = \{a_1, a_2, \ldots, a_n, a_{n+1}\}$$

So, $S \setminus \{a_{n+1}\}$ is an n-element set and by induction hypothesis its number of subsets is 2^n.

Now, the extra new element a_{n+1} is being added into the set. So, while making subsets of set S, consider all 2^n subsets of $S \setminus \{a_{n+1}\}$. We can either include a_{n+1} in a subset of $S \setminus \{a_{n+1}\}$ or not include it. So, from each subset of $S \setminus \{a_{n+1}\}$, we can make two subsets of S, one which includes a_{n+1} and one which does not.

Hence, the total number of subsets of S is $2 \times 2^n = 2^{n+1}$.

Hence $P(n+1)$ is true.

Thus, $\forall n \in \mathbb{N} \cup \{0\}$, $P(n) \Rightarrow P(n+1)$ is true.

Therefore, by induction, $P(n)$ is true for all $n \in \mathbb{N} \cup \{0\}$.

b) Theorem: The number of ways of ranking n different objects is $n!$

Proof: By induction

$P(n)$: The number of ways of ranking n different objects is $n!$

Base case: For $n=0$: the number of ways of ranking 0 objects is exactly 1, which is $0!$.

Similarly, for $n=1$, the number of ways of ranking 1 object is exactly 1, which is $1!$

Induction step: Let's assume for some $n \geq 0$, $P(n)$ is true, i.e. the number of ways of ranking n different objects is $n!$.

Now, we introduce a new object which is different from the previous n objects. Now, by our induction hypothesis, we can rank our previous n objects in $n!$ ways. After ranking, let's arrange these n objects in a line in the order of their ranks. Now the new $(n+1)^{th}$ object can only be placed between two objects or at the edges in this line to create a new ranking of $n+1$ objects. This object thus has $n+1$ ways to be included in the ranking created by n objects ($n-1$ between the objects and 2 edges in the line). After inserting it in the line, it will create a new ranking. Hence, the number of ways of ranking $n+1$ distinct objects is

Write less

$$(n+1) \text{ ways to insert new object} \times n! \text{ ways to arrange } n \text{ objects}$$
$$= (n+1) \times n! = (n+1)!$$

Hence, $P(n+1)$ is true. Thus, $\forall n \in \mathbb{N} \cup \{0\}$, $P(n) \Rightarrow P(n+1)$ is true.

Therefore, by induction, $P(n)$ is true $\forall n \in \mathbb{N} \cup \{0\}$.