DEPARTMENT OF MATHEMATICS, IIT DELHI
SEMESTER II 2024 – 25
MTL 101 (Linear Algebra and Differential Equations) - Reminor Exam

Date: 11/04/2025 (Friday) Time: 5:30 PM - 7:30 PM.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

[redacted]

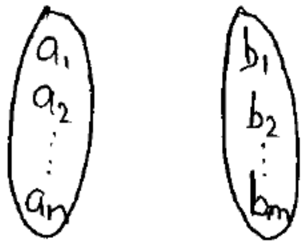
Question 1: State and prove Rank-Nullity Theorem for linear transformation. [4]

Rank-Nullity Theorem states that for linear transformation $T: V \to W$

T, $\text{rank}(T) + \text{nullity}(T) = \dim V$

where rank(T) is the dimension of T(V) and nullity(T) is the dimension of T(v) = 0 and dim V is dimension of V.

Proof Let $T: A \to B$



i) $n \geq m$

$\dim(A) = n$

Mapping occurs till a_m after that i.e ~~a_{m+1}~~

$a_{m+1}, a_{m+2}, \dots a_n$ ~~the~~ give 0

$\Rightarrow \text{nullity}(A) = n - m$ and $\text{rank}(A) = m$

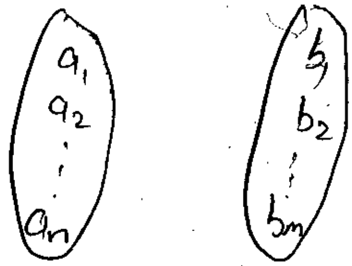
Hence $\text{rank}(A) + \text{nullity}(A) = \dim(A)$

ii) ~~if~~ $n<m$

$\dim(A) = n$

$\text{rank}(A) = n$

$\text{nullity}(A) = 0$



Question 2: Consider the system:

$$x + y + z = 1$$
$$2x + ay + 3z = b$$
$$x + 2y + cz = 3$$

Using RRE method:

(i) Find the values of a, b, c for which the system has a unique solution.

(ii) Find for which values does it have infinitely many solutions.

(iii) Show when it is inconsistent.

[4]

$AX = B$

$$\left|\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & a & 3 & b \\ 1 & 2 & c & 3 \end{array}\right| \quad R_3 \to R_3 - R_1 \quad R_2 \to R_2 - 2R_1$$

$$\left|\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & a-2 & 1 & b-2 \\ 0 & 1 & c-1 & 2 \end{array}\right| \xrightarrow{R_1 \to R_1 - R_3} \left|\begin{array}{ccc|c} 1 & 0 & -c & -1 \\ 0 & a-2 & 1 & b-2 \\ 0 & 1 & c-1 & 2 \end{array}\right|$$

$$\left|\begin{array}{ccc|c} 1 & 0 & -c & -1 \\ 0 & 1 & c-1 & 2 \\ 0 & a-2 & 1 & b-2 \end{array}\right| \to \text{This is in RRE form}$$

ii) For $a=3$, ~~and~~ $b=4$, $c=2$ ~~the~~ it has ~~infinct~~ infinitely many solutions.

iii) For $a \neq 3$ and $b=4, c=2$
$b \neq 4$ and $a=3, c=2$
$c \neq 2$ and $a=3, b=4$
it is inconsistent

i) for rest of the condition it has unique solution.
~~i.e. $a=3, b\neq 4, c\neq 2$ and $a\neq 3, b$~~

Question 3: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

i) Find the eigenvalues and corresponding eigenspaces of A and show that A is diagonalizable.

ii) Write down an ordered basis $\mathcal{B} = \{v_1; v_2, v_3\}$ of $\mathbb{R}^3$ consisting of eigenvectors of A. Using this, find an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix.

iii) Find the coordinates of

$$\begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$$

with respect to the ordered basis $\mathcal{B}$.

iv) Compute:

$$A^{10} \begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$$

[2+2+1+1=6]

i) for ~~eigenst~~ eigenvalues, $\det|\lambda I - A| = 0$

$$\begin{bmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{bmatrix} = 0$$

$$(\lambda-1)[(\lambda-1)^2-1] - (-1)((1-\lambda)-1) + (-1)(1-(1-\lambda)) = 0$$

$$(\lambda-1)(\lambda^2-2\lambda) - \lambda - \lambda = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$

$$\lambda = 0, 0, 3$$

$\therefore$ Eigenvalues are $0, 0, 3$.

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i) $\lambda = 0$

$AV = \lambda V$

$\Rightarrow AV = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x + y + z = 0 \Rightarrow \boxed{\text{dimension} = 2}$

i.e ~~H~~ Eigenspace of A ~~and its dimension = 2~~

$(a, -a, 0)$ and $(a, 0, -a)$

$\Rightarrow$ Eigenvectors are $(1, -1, 0)$ and $(1, 0, -1)$

ii) $\lambda = \cancel{8}3$

$AV = \lambda V$

$\Rightarrow (A - \lambda I) V = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x + y + z = 0$

$x - 2y + z = 0$

$x + y - 2z = 0$

$\Rightarrow x = y = z$

$\Rightarrow \boxed{\text{dimension} = 1}$

Eigenspace of ~~a~~ A is $(a, a, a) \Rightarrow$ Eigenvector is $(1, 1, 1)$

As the dimension $= 2 + 1 = 3$ $\therefore$ A is diagonalizable

ii) Order basis is ~~B = {v_1, v_2, v_3}~~ is $B = \{v_1, v_2, v_3\}$

$v_1 = (1,-1,0)$
$v_2 = (1,0,-1)$
$v_3 = (1,1,1)$

$\therefore S = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ and as $\det|S| \neq 0$ it is invertible.

iii) ~~6 =~~ For coordinates of $\begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$

$(6,-1,-2)\cancel{6} = a_{11}(1,-1,0) + a_{21}(1,0,-1) + a_{31}(1,1,1)$

$\cancel{-1 = a_{12}(1,-1,0) + a_{22}(1,0,-1) + a_{32}(1,1,1)}$

$\cancel{-2 = a_{13}(1,-1,0) + a_{23}(1,0,-1) + a_{33}(1,1,1)}$

$a_{11} + a_{21} + a_{31} = 6$
$-a_{11} + a_{31} = -1$ $\Rightarrow \boxed{a_{31} = 1 \quad a_{11} = 2 \quad a_{21} = 3}$
$-a_{21} + a_{31} = -2$

$\therefore$ Coordinates are $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

iv) $A^{10}\begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix} = 3^9 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix}$ (By observing pattern)

$A^n = 3^{n-1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$= 3^9 \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3^{10}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Question 4: Let $V(\mathbb{F})$ be a finite dimensional vector space with $\dim V = 6$. Let W be a subspace of V such that $\dim W = 3$. Find a linear operator $T : V \to V$ such that

$$\mathrm{R}(T) = W \quad \text{and} \quad \mathrm{Null}(T) = W,$$

where $\mathrm{R}(T)$ denotes the range of T and $\mathrm{Null}(T)$ denotes the null space or kernel of T. [4]

$T : V \to V$

$\cancel{T(x, y, z) = (x^2 y^2 z^2)\ (x^2}$

$T(a, b, c, d, e, f) = (a-b,\ c-d,\ e-f)$

$\Rightarrow \dim(V) = 6$

null $(T) = 3$ as $a = b$
$c = d$
$e = f$

and range(T) i.e $R(T) = 3$

Question 5: Let $V(\mathbb{F})$ be a vector space and $T : V \to V$ be a linear operator such that

$$T^2 = T.$$

Prove that

$$V = \mathrm{Null(T)} \oplus \mathrm{R}(T),$$

where $\mathrm{R}(T)$ denotes the range of T and $\mathrm{Null}(T)$ denotes the null space or kernel of T. [4]

$T : V \to V$ $\quad T^2 = T$ this implies

$T : (x, y) = (0, 0)$ or

~~and~~ $T : (x, y) = (x, y)$

i) $T : (x, y) = (0, 0)$

$Null(T) = 2 \quad R(T) = 0$

$\Rightarrow$ ~~$V = Nu$~~ $dim(V) = 2$

Hence $V = Null(T) + R(T)$

ii) $T : (x, y) = (x, y)$

$\Rightarrow dim(V) = 2 \quad Null(T) = 0 \quad R(T) = 2$

Hence $V = Null(T) + R(T)$

Question 6:

a) Let $V(\mathbb{F})$ be a finite dimensional vector space over the field $\mathbb{F}$ and $T : V \to V$ be a linear operator such that

$$T^k = 0, \quad \text{for some } k \geq 2, \ k \in \mathbb{N}.$$

Find the eigen values of T.

b) Let $V = \{x \in \mathbb{R} \mid x > 0\}$, and define a binary operation $x \boxplus y = x \cdot y$ (i.e., usual multiplication of real numbers) for $x, y \in V$.

Find a scaler multiplication $\boxtimes : \mathbb{Q} \times V \to V$ such that V, equipped with $\boxplus$ and $\boxtimes$ becomes a vector space over $\mathbb{Q}$. Justify your answer. [2+2=4]

a) $T^k V = \lambda V$

as $T^k = 0$ $\Rightarrow \boxed{\lambda = 0}$

$\Rightarrow$ Eigen value of T is zero.

as for $\lambda = 0$ for some $k \geq 2$ $k \in N$

$T^k = 0$

Question 7: Prove or disprove the following statements with proper justification or counterexample.

i) If V be a finite dimensional vector space over the field $\mathbb{F}$ with $\dim V > 1$, then there exists a linear functional

$$T : V \to \mathbb{F}$$

such that

$$T(v_1) = 0, T(v_2) = 1,$$

where v_1, v_2 are linearly independent.

ii) If S_1 and S_2 are two linearly independent subsets of a vector space V such that $S_1 \cap S_2 = \emptyset$, then

$$\dim(\text{Span}(S_1 \cup S_2)) = | S_1 \cup S_2 |,$$

where $| S_1 \cup S_2 |$ denotes the number of elements of $S_1 \cup S_2$.

[2+2=4]

ii) $\dim(\text{Span}(S_1 \cup S_2)) = |S_1 \cup S_2|$ is true

$S_1 = \{u_1, u_2, \dots u_n\}$ $S_2 = \{v_1, v_2, \dots v_n\}$

As $S_1 \cap S_2 = \phi$, the elements spanned through the elements of S_1 $(a_1u_1 + a_2u_2 \dots + a_nu_n)$ can't be ~~$\{S_1 = \{u_1, u_2 \dots u_n\}$, $S_2 = \{v_1, v_2 \dots v_n\}\}$~~ ~~for~~ spanned through the elements of S_2 and the elements spanned through the elements of S_2 $(b_1v_1 + b_2v_2 + \dots + b_nv_n)$ can't be spanned through the elements of S_1

$\therefore \dim(\text{Span}(S_1 \cup S_2)) = |S_1 \cup S_2|$ as $\text{Span}(S_1 \cup S_2)$ includes elements of both S_1 and S_2

i) There exists a linear functional $T : V \to F$ s.t $T(v_1) = 0$, $T(v_2) = 1$ v_1, v_2 are linearly independent

$T : (x, y, z) = (xyz)$

$2x + 3y + (c+1)z = 4$

$2x + ay + 3z = b$

$\boxed{b=4}$

$\boxed{c=2}$

$x + y + z = 1$

$2x + ay + 3z = b$

$x + 2y + cz = 3$

$x + (a-2) + (3-c) = b-3$

$$\begin{array}{ccc} & & b \\ \end{array}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & a & 3 \\ 1 & 2 & c \end{vmatrix} = (ac-6) - (2c-3) + (4-a) = 0$$

$ac - 2c - a + 1 = 0$

$(a-1)(a-2) = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}}_{\to A^2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{pmatrix}}_{A^3}$$

$A^3 \to 9 \times 3 = 3^3$