Tutorial

COL202

August 2021

1 Tut 1

1. We prove the required claim using a proof by contradiction.

   Suppose that there are only finitely many primes of the form $4k+3$. Let them be $q_1 < q_2 < \cdots < q_r$. Then consider the integer $m = 4q_1q_2 \dots q_r - 1$. Note that $q_1 = 3$, so $m \geq 4 \times 3 - 1 = 11 > 1$, so $m > 1$.

   Note that $m \equiv -1 \not\equiv 1 \pmod 4$.

   Suppose $p \in \{2, q_1, q_2, \dots, q_r\}$ and $p|m$. Then $p|1$, which is a contradiction. So all prime factors of m are $1 \pmod 4$. Then $m \equiv 1 \pmod 4$, which is a contradiction.

   Hence proved.

2. Call two people adjacent if they gave fist bumps to each other, and non-adjacent otherwise. Call a set of k people a k-clique if they're all pairwise adjacent to each other. Call a set of k people a k-coclique if they're all pairwise non-adjacent to each other.We need to prove that among 10 people, there exists a 3-coclique or a 4-clique. We'll use the following lemma proven in class -

   Lemma - Among any 6 people, their exists a 3-clique or a 3-coclique.

   We prove the required result by taking cases.

   Consider any one person P. If P is non-adjacent to Q, R and Q, R are non-adjacent to each other, then P, Q, R form a 3-coclique, and we're done. Now consider the case where the people non-adjacent to P form a clique. Then if there are ≥ 4 people non-adjacent to P, we can choose 4 of them to get a 4-clique. Now consider the case where there are at most 3 people non-adjacent to P. Then ≥ 6 people are adjacent to P. Then, using our lemma, there exists a 3-coclique or a 3-clique among them. In the former case, we're done. In the latter case, we can add P to the 3-clique to get a 4-clique, and we're done.

3. We use the obvious interpretation in terms of edge-coloring of the complete graph and Ramsey numbers. We are required to find an upper bound on $R(3,3,3)$. We'll use the fact that $R(3,3) \leq 6$, which was proven in class.

We'll prove that $R(3,3,3) \leq 17$. We prove it using cases.

Let the colour set be c_1, c_2, c_3. Consider a vertex v. Let v be connected to exactly a_i number of vertices by an edge colored c_i, $i \in \{1,2,3\}$. WLOG, a_1 is the largest of a_1, a_2, a_3. Then $a_1 \geq 16/3 > 5$, so $a_1 \geq 6$. If some two vertices u, w among the a_1 neighbours connected to v by c_1-colored edges are connected to each other by a c_1-colored edge, we have a c_1-colored triangle (u, v, w), and we're done. Otherwise, all edges amongst them are colored by c_2 or c_3. Then, using $R(3,3) \leq 6 \leq a_1$, we can say that a monochromatic triangle must exist amongst these a_1 vertices, and we're done.

4. Denote $P(n,1)$ by $Q(n)$.

   Claim 1: $Q(n)$ is true for all $n \in \mathbb{N}$.

   Proof by induction:

   Base case - $Q(1) \iff P(1,1)$, $P(1,1)$ is true (given).

   Induction step - For all $m \in \mathbb{N}$ and $m > 1$, $P(m-1,1) \Rightarrow P(m,1)$. So, for all $m \in \mathbb{N}$ and $m > 1$, $Q(m-1) \Rightarrow Q(m)$. Hence proved.

   Denote $P(m,n)$ by $P_m(n)$.

   Claim 2: For any fixed natural number m, $P_m(n)$ is true for all $n \in \mathbb{N}$.

   Proof by induction:

   Base case - $P_m(1) \iff Q(m)$, $Q(m)$ is true using Claim 1.

   Induction step - For all $m, n \in \mathbb{N}$ and $n > 1$, $P(m, n-1) \Rightarrow P(m,n)$, So, for all $n \in \mathbb{N}$ and $n > 1$, $P_m(n-1) \Rightarrow P_m(n)$. Hence proved.

   Claim 3: $P(m,n)$ is true for all $(m,n) \in \mathbb{N} \times \mathbb{N}$.

   Proof: For any $(m,n) \in \mathbb{N} \times \mathbb{N}$, $P(m,n) \iff P_m(n)$, $P_m(n)$ is true using Claim 2. Hence proved.

5. We claim that $P(n)$: We can always find a consistent subset T with $|T| \geq \lceil \log_2(n+1) \rceil$, is true for all $n \in \mathbb{N}$. We prove it using strong induction on n.

   Base case: $P(1)$ is easy to check.

   Induction step: Suppose that $P(n)$ is true for all $n \leq k$, $k \geq 1$. We claim that $P(k+1)$ must also be true.

   If $(k+1)$ is not a power of 2, $\lceil \log_2((k+1)+1) \rceil = \lceil \log_2(k+1) \rceil$, so we can just choose a large enough consistent subset T from among k teams and be done.

   Now assume $(k+1) = 2^r, r \geq 1$. Then, pick a team v. Suppose v defeated d_v teams, and lost to l_v teams. $d_v + l_v = 2^r - 1$, so at least one of $d_v, l_v \geq 2^{r-1}$. WLOG, $d_v \geq 2^{r-1}$. Then we can choose a consistent subset T' of size $\geq r$ from among these d_v teams. Let $T = T' \cup \{v\}$, then T is consistent (v is its strongest team), and $|T| \geq r + 1 = \lceil \log_2((k+1)+1) \rceil$, so we're done.

6. The logical flow in the last line of the proof is incorrect. l'Hopital's rule provides that $\lim \frac{f}{g}$ exists and equals $\lim \frac{f'}{g'}$, only if the latter exists. So, we first need to check for the existence of $\lim \frac{f'}{g'}$, and only then can we say that $\lim \frac{f}{g} = \lim \frac{f'}{g'}$.

7. WOP $\Rightarrow$ PMI: Suppose there exists a set S such that $1 \in S$, and for any $n \in \mathbb{N}$, $n \in \mathbb{N} \implies (n+1) \in \mathbb{N}$. We will prove that $S = \mathbb{N}$, by contradiction. Suppose $S \neq \mathbb{N}$, then $T = \mathbb{N} \setminus S$ is non-empty. So T contains a smallest element n_0 using WOP. $n_0 \neq 1$, as $1 \in S$. So $(n_0 - 1) \in \mathbb{N}$, so $(n_0 - 1) \in S$. However, $(n_0 - 1) \in S \Rightarrow n_0 \in S$, which is a contradiction.

   PMI $\Rightarrow$ WOP: Suppose $S \subseteq N$, and S doesn't have a smallest element. We will prove $S = \phi$. Let $T = \mathbb{N} \setminus S$. Define $P(n) : \{1, ..., n\} \subseteq T$. Note that $1 \notin S$, as otherwise it would be its smallest element. So $P(1)$ is true. We claim $P(n) \Rightarrow P(n+1)$ for any $n \in \mathbb{N}$. Note that if $P(n)$ is true, and $(n+1) \in S$, then $(n+1)$ is the smallest element of S, which is contradiction, so $(n+1) \in T$, so $\{1, ..., (n+1)\} \in T$, so $P(n+1)$ is true. Then by PMI, $P(m)$ is true for all $m \in \mathbb{N}$. We claim that S must be empty. If $x \in S$, then $P(x)$ is false, which is a contradiction. Hence proved.

8. We prove the statement for $n = 2m - 1$ for all $m \in \mathbb{N}$ by induction on m.

   Base case: $m = 1$ is easily checked.

   Induction step: Suppose the statement is true for $m = k, k \geq 1$. We prove it for $(m+1)$. Suppose the i^{th} person gets colour thrown on them by x_i number of people. Suppose $x_i \geq 1$ for all i. $\sum x_i = (2m+1)$ by double counting, forcing $x_i = 1$ for all i. Due to finiteness, there exists a smallest distance over all distances between pairs. Let a, b be located at this smallest distance. Then they throw colour on each other, and nobody else throws colour on them. So the rest of the $(2m-1)$ people are independent from them, so by induction hypothesis at least one of them doesn't have colour thrown at them.

   For even number of people the claim is false. Just pair off people, and have people in a pair stand closest to each other.

   Claim ain't true for best friend relationships. Consider the case where the best friend of i is $(i+1) \mod n$.

9. If $x = y = 0$, then $a = b = 1$ works.

   Otherwise, consider the set $S = \{ax + by : a, b \in \mathbb{Z}, ax + by > 0\}$. It is non-empty because at least one of $x, -x, y, -y$ lie in it. Let its smallest element (which exists by WOP) be $s_0 = a_0 x + b_0 y$. Let $s = ax + by$, for some $a, b \in \mathbb{Z}$. Then, by Euclidean division, we can write $s = s_0 q + r, 0 \leq r < s_0$. Note that $(s - s_0 q) \in S$ if it is > 0. So, if $r > 0$, then $r < s$ and $r \in S$, a contradiction. So $r = 0$, so $s_0 | s$. Therefore $s_0 | x$ and $s_0 | y$. So $s_0 \leq \gcd(x, y)$. But $\gcd(x, y) | s_0$, so $\gcd(x, y) \leq s_0$. So $x = \gcd(x_0, y_0)$.

10. Let S be nice.

    If S has no element with non-zero absolute value, then $S = \{0\}$, so $S = \{a \times 0 | a \in \mathbb{Z}\}$.

    Otherwise, $\{a | a > 0, \exists b \in S, a = |b|\}$ is non-empty. Let a_0 be it's smallest element, with $a_0 = |x_0|, x_0 \in S$. Then any multiple $a_0 t$ of a_0 is in S, as $a_0 t = a_0 \times t + a_0 \times 0$. Suppose $c \in S$, s.t. $c = a_0 q + r$, where $0 < r < a_0$. Then $r = c \times 1 - a_0 \times q$ so $r \in S$, but $0 < |r| < a_0$, a contradiction. So $r = 0$, so $a_0 | c$. Therefore S is exactly the set of multiples of a_0.

11. $(a) \Rightarrow (b)$: Suppose $b \in B'$ and $f^{-1}(b) = a$ exists. Note $b = (f \circ g)^k(b^*)$ for some $b^* \in B$ s.t. $b^* \notin \text{Im}(f)$. Note that $k \geq 1$. Let $c = (f \circ g)^{k-1}(b^*)$, then $c \in B'$, and $a = g(c)$, so $a \in A'$.

    $(b) \Rightarrow (c)$: If $f^{-1}(b)$ doesn't exist, then $b \in B'$. If $f^{-1}(b) = a \in A'$, then by Claim 1 $b = f(a) \in B'$. So $b \in B'$. So $g(b) \in A'$.

    $(c) \Rightarrow (a)$: Let $g(b) = a \in A'$. Then their exists $b' \in B$ s.t. $a = g(b')$. By injectivity of g $b = b'$ so $b \in B'$.

12. a) Forward direction : Suppose there is an injection f from N to S, and S has finite cardinality n, Then consider the image I of $\{1, ..., n+1\}$ under f. $|I| = n + 1 > n = |S|$. But $I \subseteq S$, so $|I| \leq |S|$. So we get a contradiction.

    Reverse direction: Consider the function f defined inductively as follows. We claim that this function is injective:

    Base case: As S is infinite, S in non-empty, so there is $x_1 \in S$. Let $f(1) = x_1$.

    Inductive step: Suppose f has been defined for $N_n = \{1, 2, ..., n\}$. Let $A = \text{Im}(N_n)$. Then A is finite, so $S \backslash A$ is non-empty, so choose $x_{n+1} \in S \backslash A$ and let $f(n+1) = x_{n+1}$. Note that $f(n+1) \neq f(i)$ for any $i < (n+1)$.

    b)Forward direction: Let f be an injection from S to A, where A is a proper subset of S. Let S be finite. Then note that $|A| < |S|$, which is a contradiction, as for finite sets, an injective function exists only if cardinality of domain $\leq$ cardinality of co-domain.

    Reverse direction: Let f be an injection from $\mathbb{N}$ to S. Define A to be the $\text{Im}(\mathbb{N})$ under f. Then define $h : S \to S \setminus \{f(1)\}$ such that: if $x \in A$, $h(x) = f(f^{-1}(x) + 1)$, otherwise $h(x) = x$. Note that $h(x) = x \neq f(1)$ if $x \notin A$, and $h(x) = f(f^{-1}(x) + 1) \neq f(1)$ if $x \in A$, because f is injective and $(f^{-1}(x) + 1) \neq 1$, so h is well-defined.

    We claim that h is injective. Suppose $h(a) = h(b)$. If $a, b \notin A$, $a = h(a) = h(b) = b$. If $a, b \in A$, then $f(f^{-1}(a) + 1) = f(f^{-1}(b) + 1) \Rightarrow f^{-1}(a) = f^{-1}(b) \Rightarrow a = b$. If $a \in A, b \notin A$, $h(a) \in A$ but $h(b) \notin A$, so $h(a) \neq h(b)$.

13. a) Let f be an injection from A to N, g be an injection from B to N. Define $h : A \cup B \to Z$ as follows: $h(x) = f(x)$ if $x \in A, h(x) = -g(x)$ otherwise. It is easy to check that h is injective. As Z is countable, $A \cup B$ is countable. Define $i : A \times B \to N \times N$ as follows: $i(x, y) = (f(x), g(y))$.

It's easy to check that i is injective. As $N \times N$ is countable, $A \times B$ is countable.

b) 1) Proof by strong induction. Base case: $n = 1$ - A_1 is countable. $n = 2$ - Proved before. Induction step: $S = A_1 \cup \cdots \cup A_n$ is countable, so $S \cup A_{n+1}$ is countable by our hypothesis for $n = 2$. So $A_1 \cup \cdots \cup A_{n+1}$ is countable. 2) Proof by strong induction. Base case: $n = 1$ - A_1 is countable. $n = 2$ - Proved before. Induction step: $S = A_1 \times \cdots \times A_n$. So $S \times A_{n+1}$ is countable by hypothesis for $n = 2$. There is an obvious bijection between $S \times A_{n+1}$ and $A_1 \times \cdots \times A_{n+1}$, so the latter is countable.

14. *a*) True. In general, if X is uncountable, then so is any superset of X (consider the identity injection).

    b) False. Consider $A = \mathbb{R}_{\geq 0}$ and $B = \mathbb{R}_{\leq 0}$.

    c) False. Take $A = B$.

    d) True. Let b be any element of B. The set, $\{(x, b) | x \in A\}$ is a subset with obvious bijection to A.

    e) True. If not, then $S = (A \setminus B) \cup B$ is countable, and $A \subseteq S$.

15. Set of finite strings from a countable character set is countable. But the set of subsets of $\mathbb{N}$ is uncountable. So there cannot exist a surjection.

16. We prove there is no surjection in the opposite direction. Consider the set $T = \{x : x \notin f(x)\}$.

17. We induct on the number of edges e. Base case $e = 0$ is easy to check. For the induction step, note that adding an edge increases sum of degrees by 2, and number of edges by 1.

18. (*ii*) No. (Path of length 5), (Triangle and path of length 2)

19. 10, 48

20. Lemma: A connected graph on n vertices has atleast $n - 1$ edges.

    Proof: Consider a graph G' on n vertices such that G is connected and has minimal number of edges. Then if G' has a cycle, we can remove an edge from the cycle and the graph will still remain connected. So G' is acyclic, so it's a tree, so it has $n - 1$ edges.

    Now let G have k connected components, such that the i^{th} component has n_i vertices and e_i edges. $e_i \geq n_i - 1$. Taking a sum over i from 1 to k gives $e \geq n - k$, so $k \geq n - e$.

21. Tree $\implies$ acyclic $\implies$ ≤ 1 path.

    Tree $\implies$ connectec $\implies$ ≥ 1 path.

22. Since the connected components are trees, if there are k connected components, then number of edges $= n - k$. So $k = 2$.

    Note that the new graph is connected and has $n - 1$ edges, so it's a tree.

23. If there's a cycle, then removing an edge of the cycle still leaves only one connected component. So the graph is acyclic, hence a tree.

24. Suppose there are two distinct cycles. Then both of them must have the edge $\{u, v\}$ in them. Then removing the edge $\{u, v\}$ from them both gives two distinct paths between u, v in the original tree, a contradiction.

    Still connected, $n - 1$ edges

25. Adding and edge between two vertices in different connected components can't cause a cycle to be created, so the graph is connected, hence a tree.

26. if - obvious

    only if - Consider the connected subgraph G' of G, containing all vertices of G, such that G' has the minimal possible number of edges. Then G' must be acyclic, so it's a tree.

27. Euler walk which ain't an Euler tour - Iff there exist exactly 2 vertices with odd degree Only if - Obvious If - If v, u have odd degree, then add a new vertex w and edges $\{u, w\}, \{v, w\}$. You get an Euler tour, which is easily changed to an Euler walk in the original graph.

28. If $|M| < |M^*|/2$, then there exists an edge $e \in M^*$ s.t. neither end-point of e belongs to M. So we can add e to M and still have a matching, so M isn't maximal.

    Tightness - Path of length 3, middle edge is maximal matching of size half that of maximum-size matching.

    Ans. $\frac{l-1}{l+1}$. Is tight if we consider a path of length l, and take M to be the $(l-1)/2$ even position edges.

    Proof: Take the graph $(V, M \cup M^*)$. Then for any connected component C, either it has no edges, or $\frac{|C \cap M|}{|C \cap M^*|} \geq \frac{l-1}{l+1}$.

29. Let V_1 be a vertex cover, and M_1 be any matching. Let $f : M_1 \to V_1$ be s.t. for any edge $e \in M_1$, $f(e)$ is an endpoint of e. Such an f exists because V_1 is a vertex cover.

    Since two distinct edges in a matching cannot have a common endpoint, f is injective. So $|V_1| \geq |M_1|$.

    Example for which this isn't tight: Union of n disjoint triangles, $n \in \mathbb{N}$. A maximum-size matching has n edges, and a minimum-size vertex cover has $2n$ vertices.

30. Let (V_1, V_2) be a bipartition of G, V' be a minimum-sized vertex cover of G, $V_1' = V_1 \cap V', V_2' = V_2 \cap V', V_1'' = V_1 - V_1', V_2'' = V_2 - V_2'$. Then consider the subgraph of V formed by the vertices $V_1' \cup V_2''$ and the edges between them. Let $S \subseteq V_1'$. Then note that $W = N(S) \cup (V_1' - S) \cup V_2'$ also forms a vertex cover. So $|N(S)| \geq |S|$. So by Hall's Theorem, we can get a matching of size $|V'|$.

Alternate proof - S as defined in hint. $(S^C \cap V_1) \cup (S \cap V_2)$ is a vertex cover of size equalling maximum-size matching.

Bipartiteness isn't necessary to achieve equality. Cycle of length 3 $v_1v_2v_3$ along with edge v_1v_4 gives an example.

31. Consider the smallest length closed walk. If it isn't a cycle, we can get a smaller length closed walk.

32. (a) Consider the longest path. Endpoint is a sink. Starting point is a source.

    (b) If - Suppose a cycle exists. Then the vertex with the smallest index in it cannot have an incoming edge.

    Only if - Induct on the number of vertices to prove that a topological sort exists. For the induction step, remove a source vertex, then add it at the beginning of a topological sort of the remaining vertices.

33. $\binom{m+n}{n}$

34. Both the sides count the number of ways to choose k objects out of $(m+n)$ distinct objects. The RHS takes a sum over the mutually exclusive and exhaustive cases conditioned on the number i of objects chosen out of the first m objects, the ways for each case equalling $\binom{m}{i}\binom{n}{k-i}$.

35. $n\Pi_{i=1}^{k}(1-\frac{1}{p_i})$. Inclusion-exclusion.

36. a) Same initial values, same recurrence.

    b) Reflection principle

    c) $\binom{2n}{n} - \binom{2n}{n-1}$

37. a) $f(x) = \frac{x}{x^2-3x+1}$. $a_n = \frac{\beta}{1-\beta^2}\alpha^n + \frac{\alpha}{1-\alpha^2}\beta^n$ where α, β are roots of denominator.

    b) $f(x) = \frac{x^2}{x^3-x^2-x+1}$. $b_n = \frac{2n-1+(-1)^n}{4}$

    c) $f(x) = \frac{x}{(1-x)^3}$. $c_n = \binom{n+1}{2}$.

    d) Let $f(x) = \sum_{i=0}^{\infty} d_i x^i$. Then $\int f(x)dx = f(x) - e^x + c$. Differentiating both sides we get $f(x) = f'(x) - e^x$. The solution to this DE satisfying $f(0) = 1$ is $(x+1)e^x$. $d_n = \frac{n+1}{n!}$

38. $\Pi_{t\in T}\frac{1}{1-x^t}$

39. $\int_1^n \ln(x)dx \le \sum_{i=1}^{n} \ln(i) \le \int_1^{n+1} \ln(x)dx$

40.

41. Not true for $l = k = 1$. Otherwise obvious for $k = 1$, so assume $k \geq 2$.

    Suppose every graph with n vertices, with edges coloured using l colours has a monochromatic clique of size k. Count in two ways the number m of pairs (G, c_k) where G is a graph with n vertices with edges coloured using l colours and c_k is a monochromatic clique of size k in it. Then $m \geq l^{\binom{n}{2}}$, and $m = \binom{n}{k} l^{1+\binom{n}{2}-\binom{k}{2}}$. So $\binom{n}{k} \geq l^{\binom{k}{2}-1}$, so $\frac{n^k}{k!} > l^{\binom{k}{2}-1}$, which gives $n > (k!)^{\frac{1}{k}} l^{\frac{k-1}{2}-\frac{1}{k}}$.

42. Reflexive: Use identity. Symmetric: Use inverse. Transitive: Use composition.

43. Obvious

44. Lemma 1: For $x, y \in \Sigma^*$ s.t. xRy, $x \in L \iff y \in L$.

    Proof: Consider z to be the empty string.

    Lemma 2: For $x, y, z \in \Sigma^*$ s.t. xRy, we have $(xz)R(yz)$.

    Proof: Let $z' \in \Sigma$ be any string. Then $x(zz') \in L \iff y(zz') \in L$, so $(xz)z' \in L \iff (yz)z' \in L$.

    If $C_1, ..., C_k$ are the equivalence classes of R_L, then define a corresponding state set $Q = \{s_1, ..., s_k\}$.

    If the empty string is in class C_j, $q_0 = s_j$.

    Define f so that if a string x is in the class corresponding to the state s, then $f(s, b)$ is the state corresponding to the class of x"b".This is well-defined using Lemma 2 (i.e. doesn't depend on the specific x chosen).

    Finally, note that using Lemma 1, we know that $L = \cup_{i \in T} C_i$, where T is some fixed subset of $\{1, ..., k\}$. Then define $A = \{s_i : i \in T\}$.

    Lemma 3: If the machine has received a string x as input up till now, and x is in class C_i, then the machine is in state s_i.

    Proof: Induct on the size of x (denote it by $|x|$).

    Base case: $|x| = 0$ - Follows from the definition of q_0.

    Induction step: Follows from the way f is defined.

    Finally, note that the machine gives the correct output due to Lemma 3 and the definition of A.

45. Reflexive, symmetric - Obvious. Transitive - Use $\epsilon/2, \epsilon/2$.

    $\sqrt{2}$ - Odd and even length decimal approximations of $\sqrt{2}$.

    Less than - There exists an $n_0 \in \mathbb{N}$ s.t. $x_n < y_n$ for all $n \geq n_0$.

    Add / multiply - Term-wise add / multiply

46. Claim 1: The set of minimum elements of a poset forms an antichain.

Claim 2: If all chains in a poset S have finite length, and M is the set of minimum elements of S, then (length of a longest chain in $(S \setminus M)$) = (length of a longest chain in S) -1.

For the original claim, use induction on the size of a longest chain in the poset. In the induction step, remove the set of minimum elements as one antichain.

47. Suppose $(x, y) \in H$, but $(x, y) \notin H'$. Since (x, y) is in the reflexive transitive closure of H', there is an $n \in \mathbb{N} \cup \{0\}$ and $x = x_0, \ldots, x_n = y$ s.t. $x_{i-1} H' x_i$. Using WOP, consider the smallest such n. Note that $n \geq 2$. Then xRx_1 and x_1Ry implies $x_1 = x$ or $x_1 = y$, which contradicts the minimality of n. Contradiction.
48. Closure, identity, inverse, associativity - obvious
49. Closure, identity, inverse, associativity - obvious
50. Take intersection of all subgroups containing S
51. Check reflexive symmetric transitive. Left and right cosets
52. $g * h * g^{-1}$ defines a group isomorphism from H to $g * H * g^{-1}$.

    Check reflexive, symmetric, transitive.