Differential equation

A differential equation is an equation involving one or more dependent variables and its derivatives with respect to one or more independent variables.

If a differential equation has only one independent variable and its derivatives then it is called an ordinary differential equation (ODE). If we have more than one function in one independent variable, then we call it a system of ordinary differential equations.

If a differential equation has more than one independent variables with their derivatives then it is called a partial differential equation.

Examples ① Let $y = f(x)$.

$$\boxed{\frac{dy}{dx} = 2y}.\quad \text{ODE}$$

② Let $y = f(x)$.

$$\boxed{\frac{dy}{dx} + 5y = 7e^{x}} \quad \text{ODE}$$

③ $$\boxed{\begin{aligned}\frac{dx}{dt} &= 4x+3y+\sin t\\ \frac{dy}{dt} &= -5x+y+e^{t}\end{aligned}}$$ a system of ODE.

In general, a general form of an ODE is $f(t, y(t), \ldots, y^{(n)}(t)) = 0$. (n)

④ Let $\omega = f(x,y,z)$.

$$\boxed{\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}+\frac{\partial^2\omega}{\partial z^2}=0}$$ PDE

Order of a ODE

The order of an ODE is the order of the highest derivative appearing in the equation.

Examples

① $$\boxed{\frac{d^2y}{dx^2}+xy\frac{dy}{dx}=0}$$ Order 2 ODE

② $$\boxed{\frac{d^4y}{dx^4}+5\frac{d^2y}{dx^2}+7x=0}$$ Order 4 ODE

A general form of first order ODE is $f(x,y,y')=0$ and a general form of second order ODE is $f(x,y,y',y'')=0$.

Linear ODE

If f is a linear function of $y, \dots, y^{(n)}$, then $f(x, y, \dots, y^{(n)}) = 0$ is called a linear ODE.

A general form of linear ODE of order n is of the form

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_0(x) y = r(x)$$

where $a_n(x) \neq 0$.

We standardly use the form of linear ODE of order n as

$$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_0(x)\, y = r(x).$$

If $r(x) \equiv 0$ for all x considered, then we call this linear ODE a homogeneous linear ODE and otherwise a non-homogeneous linear ODE.

Examples

① $\boxed{y^{(2)} + 5y^{(1)} + 6y = 0}$ Second order homogeneous linear ODE

② $\boxed{y^{(4)} + 2\left(y^{(3)}\right)^2 + x^3 y^{(1)} = x e^{x}}$

4th order non-homogeneous non-linear ODE.

(3) $$\boxed{\frac{d^2y}{dx^2} + xy\frac{dy}{dx} = 0}$$

2nd order non-linear ODE

Degree of an ODE

When an ODE involves polynomials in all the derivatives involved, the exponent of the higest order derivative is known as the degree.

Examples

(1) $$\boxed{y^{(2)} + 5y^{(1)} + 6y = 0}$$

Degree 1 order 2 linear homogeneous ODE.

(2) $$\boxed{(y')^2 + y'' + y = 0}$$

Degree 1 order 2 non-linear homogeneous ODE.

(3) $$\boxed{(y'')^3 + y' = \sin x}$$

Degree 3 Order 2 non-linear non-homogeneous ODE.

(4) $$\boxed{(y'')^{\frac{2}{3}} = 2 + 3y'}$$

Degree 2 order 2 non-linear non-homogeneous ODE

$$(y'')^2 = (2+3y')^3$$

(5) $$\boxed{y'' = 3(y')^{\frac{1}{3}} + x^2}$$

Degree 3 order 2 non-linear non-homogeneous ODE

$$(y'' - x^2)^3 = 3y'$$

⑥ $\boxed{y''' = 3y' + \sin y'}$ Degree not defined.

⑦ $\boxed{y''' = 3y' + \sin y}$ Degree not defined.

⑧ $\boxed{y''' = 3y' + \sin x}$ Degree 1 order 3 linear non-homogeneous ODE.

Initial value problem (IVP)

Consider $\frac{dy}{dx} = f(x)$. We solve it by considering $y(x) = \int f(x) + c$.

To determine the solution we need the starting point of the integral say x_0.

To determine the constant we need to know the initial value $y(x_0) = y_0$ (Say).

The problem can be written as

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x) \\ y(x_0) &= y_0 \end{aligned}\right\}$$

or generally,

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x, y), \\ y(x_0) &= y_0 . \end{aligned}\right\}$$

This kind of problems are called initial value problems.

Suppose we have,

$$\frac{d^2y}{dx^2} = f(x, y(x), y'(x)).$$

Write $y_1(x) = y(x)$ and $y_2(x) = y'(x)$.

So rewriting the equation we have,

$$\left.\begin{aligned} &y_1'(x) = y_2(x) \\ \text{and} \quad &y_2'(x) = f(x, y_1(x), y_2(x)). \end{aligned}\right\}$$

So to solve this equation we need two initial conditions one each for y_1 and y_2 respectively. So an order 2 IVP is of the form

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

In general, an order n IVP is of the form,

$$\begin{cases} y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \\ y(x_0) = y_0, \\ y'(x_0) = y_1, \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1}. \end{cases}$$

Separable ODE

$$\boxed{g(y)y' = f(x)}$$

A solution of this form can be obtained by writing $\int g(y)\,dy = \int f(x)\,dx + c$, where f, g are integrable functions. Then one can obtain solution $y = h(x)$. This method of solving ODE is called the method of separating variables.

Examples

① $y' = 1 + y^2$

so we have,

$$\int \frac{dy}{1+y^2} = \int dx + C \quad \text{i.e. } \tan^{-1} y = x + C$$

$$\text{i.e. } y = \tan(x + C).$$

② $y' = (x+1)e^{-x}y^2.$

so we have,

$$\int \frac{dy}{y^2} = \int e^{-x}(x+1)\,dx + C$$

$$\text{i.e. } -\frac{1}{y} = -(x+1)e^{-x} + \int e^{-x}\,dx + C$$

$$= -(x+2)e^{-x} + C$$

So, $y = \dfrac{1}{(x+2)e^{-x} - c}$.

③ $y' = -2xy$, $y(0) = 1\cdot 8$

We have, $\int \dfrac{dy}{y} = \int_0^x (-2x)\,dx + C$.

So, $\log y = -x^2 + C$.

$y(0) = 1\cdot 8$ gives $C = \log(1\cdot 8)$.

Hence $\log y = -x^2 + \log(1\cdot 8)$.

So, $y = 1\cdot 8\, e^{-x^2}$.

Reduction to separable ODE

$y' = f\left(\dfrac{y}{x}\right)$ where f is a differentiable function in $\dfrac{y}{x}$.

We can take $u = \dfrac{y}{x}$.

So, $y = ux$.

So, $y' = u'x + u$.

So, $u'(x) + u = f(u)$.

i.e. $\dfrac{du}{dx} = f(u) - u$.

So, $\int \dfrac{du}{f(u) - u} = \int \dfrac{dx}{x} + C$.

<u>Example</u> $2xyy' = y^2 - x^2$.

Write $\dfrac{y^2-x^2}{2xy} = \dfrac{y}{2x} - \dfrac{x}{2y}$

Take $u = \dfrac{y}{x}$. So, $y = ux$.

So, $y' = u'x + u$

$\quad\ \ \|$

$\quad \frac{1}{2}u - \frac{1}{2u}$

So, $u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right)$

$\qquad = -\frac{1}{2}\,\dfrac{u^2+1}{u}$.

So, $\displaystyle\int \frac{2u\,du}{u^2+1} = -\int \frac{dx}{x} + c$.

Hence, $\log(1+u^2) = -\log x + c$ for $x>0$

So, $1+u^2 = \frac{1}{x}c_0$ for $x>0$

So, $1 + \dfrac{y^2}{x^2} = \dfrac{c_0}{x}$ for $x>0$

So, $x^2+y^2 = c_0 x$ for $x>0$

$\Rightarrow \left(x - \frac{c_0}{2}\right)^2 + y^2 = \dfrac{c^2}{4}$.

Exact ODEs

Idea

Let $u(x,y)$ be a function having continuous partial derivatives.

From calculus we know the total differential $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$.

So if u is a constant function i.e. $u(x,y) = c$, then $du = 0$.

If we can read our differential equation $y' = f(x,y)$ in the form $du = 0$, then we can have a solution $u(x,y) = c$.

For example consider $y' = -\frac{1+2xy^3}{3x^2y^2}$.

Then,

$$\underbrace{(1+2xy^3)\,dx + 3x^2y^2\,dy}_{du \text{ where } u(x,y) = x + x^2y^3} = 0.$$

So, $u(x,y) = c$ i.e. $x + x^2y^3 = c$ is a solution of this ODE.

Definition A first order ODE $P(x,y) + Q(x,y)\,y' = 0$, written as $P(x,y)\,dx + Q(x,y)\,dy = 0$ is said to be in exact form if there is a function $u(x,y)$ such that the differential equation can be read as $du = 0$ i.e. $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$. In this case $u(x,y) = c$ is a general solution of this ODE.

Observation

$$0 = P(x,y)\,dx + Q(x,y)\,dy$$

$$\parallel$$

$$0 = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du$$

So when we are reading the given ODE in the form $du = 0$, then

$$\boxed{\frac{\partial u}{\partial x} = P(x,y)} \text{ and } \boxed{\frac{\partial u}{\partial y} = Q(x,y)}.$$

If P, Q are continuous and have continuous first order partial derivatives (in required domain), then $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial P}{\partial y}$, and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}$ gives $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

This is because of the following:

In several variable calculus we learnt, for a function g if g_x, g_y and at least one of g_{xy}, g_{yx} exist and continuous in an open set then $g_{xy} = g_{yx}$.

So we have got a necessary condition for the differential equation $P\,dx + Q\,dy = 0$ to be exact which is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Next we see that this condition is sufficient too. Assume $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Also $$P(x,y) = \frac{\partial u}{\partial x}.$$

So, $$u = \int P(x,y)\,dx + k(y).$$

a function of y

So, $\frac{\partial u}{\partial y}$ involves $\frac{dk}{dy}$.

Also $$\frac{\partial u}{\partial y} = Q(x,y).$$

So we have $\frac{dk}{dy}$ interms of P and Q, then by integrating we get $k(y)$ and hence we have u.

Remark Not all equations of the form $P\,dx + Q\,dy = 0$ is exact. For example, $-y\,dx + x\,dy = 0$ is not exact as here $P = -y$ and $Q = x$ and $P_y \neq Q_x$.

Example

$$\cos(x+y)\,dx + \left(3y^2 + 2y + \cos(x+y)\right)dy = 0.$$

$$P = \cos(x+y)$$

$$Q = 3y^2 + 2y + \cos(x+y)$$

Note that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. So this is an exact ODE.

So the general solution is

$$u(x,y) = \int P\,dx + k(y)$$

$$= \sin(x+y) + k(y).$$

So, $\dfrac{\partial u}{\partial y} = \cos(x+y) + \dfrac{dk}{dy}$.

$\underset{Q}{\shortparallel}$

So, $\dfrac{dk}{dy} = 3y^2 + 2y$.

So, $k(y) = y^3 + y^2 + c_0$.

So, $u(x,y) = \sin(x+y) + y^3 + y^2 = c$ is a general solution.

Reduction to exact form

$\boxed{-y\,dx + x\,dy = 0}$ is not exact.

But $\boxed{-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0}$ is exact

as here $P = -\frac{y}{x^2}$, $Q = \frac{1}{x}$ and

so, $P_y = Q_x$.

We can write $\frac{1}{x^2}(-y\,dx + x\,dy) = 0$

as $\left(\frac{y}{x}\right)' = 0$. So, $y = xc$ is a general solution.

Integrating factor Suppose we have an ODE $P(x,y)dx + Q(x,y)dy = 0$ which is not exact, but $\exists$ a non-zero function $F(x,y)$ such that $FP\,dx + FQ\,dy = 0$ is exact. Then such a function F is called an integrating factor of $P\,dx + Q\,dy = 0$.

Remark Integrating factor need not be unique. For $-y\,dx + x\,dy = 0$ $\frac{1}{xy}$ is also an integrating factor.

How to find integrating factor?

Suppose $P\,dx + Q\,dy = 0$ is not exact but $FP\,dx + FQ\,dy = 0$ is exact.

Then $\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$.

So, $F_y P + F P_y = F_x Q + F Q_x$.

In general finding such an F may be difficult. We shall try to look for F in one variable so that $F_y = 0$.

So, $F \cdot P_y = F' \cdot Q + F \cdot Q_x$.

So, $\boxed{\frac{F'}{F} = \frac{1}{Q}(P_y - Q_x)}$, assuming F, Q are non-vanishing.

Now, $\frac{1}{Q}(P_y - Q_x)$ is a function of x only as $\frac{F'}{F}$ is a function of only x.

By integrating we get a choice of $F = e^{\int R\,dx}$.

Similarly one can try to find F as a function of y only. In that case $F = e^{\int \tilde{R}\,dy}$ where $\tilde{R} = \frac{1}{P}(Q_x - P_y)$.

Example $(e^{x+y} - y)\,dx + (xe^{x+y} + 1)\,dy = 0$.

$P_y = e^{x+y} - 1$, $Q_x = (x+1)e^{x+y}$

Here $P_y \neq Q_x$. $\frac{P_y - Q_x}{Q} = -1$

So, $\frac{F'}{F} = -1$. So, $F(x) = e^{-x}$.

First order linear ODE

A first order ODE is called linear if it is of the form $\boxed{y' + p(x)y = r(x)}$ where $p(x), r(x)$ are functions of x.

This is the standard form as if we have a co-efficient of y', then upon diving the equation by that co-efficient we can bring that equation in this form. For example,

$y'\cos x + y\sin x = x$ is a linear ODE whose standard form is $y' + y\tan x = x\sec x$.

A first order linear ODE is called homogeneous if it is of the form $\boxed{y' + p(x)y = 0}$.

To solve this ODE, we write, $\frac{dy}{y} = -p(x)\,dx$.

Then integrating we have, for $y>0$,

$$\log y = -\int p(x)dx + c_0 \quad , c_0 \text{ constant}.$$

$$\text{So,} \quad y = ce^{-\int p(x)dx} \quad , c = e^{c_0}.$$

If $c=0$ then $\underbrace{y=0}_{}$.

trivial solution of any homogeneous linear ODE.

A first order ODE is called non-homogeneous if it is of the form $y' + a(x)y = r(x)$ where $r(x) \not\equiv 0$.

Recall If $P(x,y)\,dx + Q(x,y)\,dy = 0$ is not exact but $F(x,y)P(x,y)\,dx + F(x,y)Q(x,y)\,dy = 0$ is exact, then $F(x,y)$ is called an integrating factor. Also by the necessary and sufficient condition for exactness we have,

$$F_y P + F P_y = F_x Q + F Q_x .$$

If F is of one variable, say for example of x only, then $F_y = 0$. So,

$$F P_y = F' Q + F Q_x .$$

So, $$\frac{F'}{F} = \underbrace{\frac{1}{Q}(P_y - Q_x)}_{\text{a function of } x \text{ only as } F'/F \text{ is a function of } x \text{ only.}},$$ if F, Q are non-vanishing.

By integrating we have,

$$F = e^{\int R(x)\,dx} \quad \text{where } R(x) = \frac{1}{Q}(P_y - Q_x).$$

Similarly if F is a function of y only then we have an integrating factor

$$F = e^{\int R(y)\,dy} \quad \text{where } R(y) = \frac{1}{P}(Q_x - P_y).$$

Now in case of $y' + p(x)y = r(x)$ with $r(x) \neq 0$, write $(p(x)y - r(x))\,dx + dy = 0$.

consider $P(x,y) = p(x)y - r(x)$

$Q(x,y) = 1$.

So, $P_y = p(x)$ and $Q_x = 0$.

So, $\frac{1}{Q}(P_y - Q_x) = p(x)$ is a function of one variable.

Integrating factor (IF), $F(x) = e^{\int p(x)dx}$.

So,

$$F(x)\left(p(x)y - r(x)\right)dx + F(x)dy = 0.$$

i.e. $F(x)p(x)y + F(x)y' = F(x)r(x)$.

Now, $F(x) = e^{\int p(x)dx}$.

So, $F'(x) = e^{\int p(x)dx}\, p(x)$

$= p(x)F(x)$.

So, $\left(F(x)y\right)' = F(x)r(x)$.

$\Rightarrow \quad F(x)y = \int F(x)r(x)\,dx + c$

So, $y = \frac{1}{F(x)}\left(\int F(x)r(x)dx + c\right)$.

<u>Example</u> Solve the IVP $\boxed{y' + y\tan x = \sin 2x}$, $y(0) = 1$.

First order non-homogeneous linear ODE

IF $F(x) = e^{\int \tan x\, dx}$

$= |\sec x|$

So, $F(x) = \sec x$ or $-\sec x$

So, $y = \cos x\left(\int \sec x \sin 2x\, dx + c\right)$

$= \cos x\left(2\int \sin x\, dx + c\right)$

$= c\cos x - 2\cos^2 x$

$y(0) = 1 \quad \Rightarrow \quad c = 3.$

So, $y = 3\cos x - 2\cos^2 x$.

Reduction to linear form

Sometimes non-linear ODEs can be reduced to linear ODEs. We consider the case of Bernoulli equation $\boxed{y' + p(x)y = g(x)y^a \text{ where } a \in \mathbb{R}.}$

Note that, this equation is linear if and only if $a = 0$ or 1.

Let $a \neq 0, 1$.

Write, $u(x) = y^{1-a}$.

So, $u'(x) = (1-a)\, y^{-a}\, y'$.

We write the ODE $y' + p(x)y = g(x)y^a$ as

$$(1-a)\, y^{-a}\, y' + (1-a)\, p(x)\, y^{1-a} = (1-a)\, g(x).$$

So, $u' + (1-a)\, p\, u = (1-a)\, g$.

This is a first order non-homogeneous linear ODE in u. Solving this we get u and hence y.

Example: Solve the IVP $y' + xy = \frac{x}{y}$, $y(0) = 3$.

Write $u = y^2$.

So, $u' = 2yy'$.

So the given ODE is

$$u' + 2xu = 2x.$$

IF $e^{\int 2x\,dx} = e^{x^2}$.

So, $u = \frac{1}{e^{x^2}}\left(\int 2x e^{x^2} + c_0\right)$

$= 1 + ce^{-x^2}$.

So, $u' = 2x(1-u)$.

So, $\dfrac{u'}{1-u} = 2x$.

$\Rightarrow -\log(1-u) = x^2 + c$.

$\Rightarrow 1-u = c_0 e^{-x^2}$

$\Rightarrow u = 1 - c_0 e^{-x^2}$.

Now, $y(0) = 3 \Rightarrow u(0) = 9$.

So, $c_0 = -8$.

So, $y^2 = 1 + 8e^{-x^2}$.

Existence and uniqueness of solutions of IVPs

An IVP may not have solution, may have unique solution, may have more than one solution.

Examples

① $|y'| + |y| = 0$, $y(0) = 1$.

Does not have any solution as $|y'| = -|y|$ implies $y \equiv 0$ and $y \equiv 0$ does not satisfy the initial condition.

② $y' = 2x$, $y(0) = 1$.

Solving this we get $y = x^2 + 1$. This is unique solution.

③ $xy' = y - 1, \; y(0) = 1.$

So, $\displaystyle\int \frac{dy}{y-1} = \int \frac{dx}{x} + c.$

$\Rightarrow \log|y-1| = \log|x| + c_0.$

$\Rightarrow \quad |y-1| = c|x|.$

$y - 1 = cx \quad \text{or} \quad -cx.$

i.e. $y = 1 + cx$ or $1 - cx$.

These satisfy $y(0) = 1$.

So, $y = 1 + cx, \; c \in \mathbb{R}$ are all the solutions of this IVP.

Now we want to understand the following two questions:

1) Is there a sufficient condition for an IVP to have a solution?

2) Is there a sufficient condition for an IVP to have a unique solution?

Existence theorem

Consider the IVP $y' = f(x,y), \; y(x_0) = y_0$. Suppose $f(x,y)$ is continuous and bounded in a rectangle $R = \{(x,y) \in \mathbb{R}^2 : |x - x_0| < a, |y - y_0| < b\}$ centered at (x_0, y_0). Assume $|f(x,y)| \leq K$ $\forall\, (x,y) \in R$. Then the IVP has at least

One solution y in the interval $(x_0-\alpha, x_0+\alpha)$ where $\alpha = \min\{a, b/K\}$.

Remark This is just a sufficient condition for the existence of a solution of an IVP. This condition is not necessary.

consider $y' = f(x,y), y(0)=0$ where

$$f(x,y) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 . \end{cases}$$

Note that f is not continuous at $(0,0)$

$$\begin{cases} x_n = \frac{1}{n\pi} \longrightarrow 0 \text{ as } n \to \infty \\ f(x_n, 0) = \frac{2}{n\pi} \sin n\pi - \cos n\pi \\ \qquad = (-1)^{n+1} \not\longrightarrow 0 = f(0,0). \end{cases}$$

So we can not find any rectangle around $(0,0)$ where f is continuous.

But $y = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

is a solution of this IVP.

Uniqueness theorem

Consider the IVP $y' = f(x,y)$, $y(x_0) = y_0$. Suppose $f(x,y)$ is continuous and bounded in a rectangle $R = \{(x,y) \in \mathbb{R}^2 : |x - x_0| < a, |y - y_0| < b\}$ centered at (x_0, y_0). Assume $|f(x,y)| \leq K$ $\forall (x,y) \in R$. Further assume that the partial derivative $f_y(x,y)$ exists and is countinuous and bounded in R. Then this IVP has a unique solution in $(x_0 - \alpha, x_0 + \alpha)$ where $\alpha = \min\{a, b/K\}$.

Remark

The condition that $f_y(x,y)$ exists and is countinuous and bounded in R is not necessary, this is only a sufficient condition. Under the weaker condition that f is Lipschitz with respect to y in R (i.e. $\exists$ $M > 0$ such that

$$|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|$$

$\forall (x,y_1), (x,y_2) \in R$), one has the uniqueness of the solution of IVP in $(x_0 - \alpha, x_0 + \alpha)$.

Why are we saying $f(x,y)$ is Lipschitz in y is weaker condition than $f_y(x,y)$ is continuous and bounded in R?

Clearly $f_y(x,y)$ is continuous and bounded in $R \Rightarrow f$ is Lipschitz with respect to y in R

$$\left|\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2}\right| = |f_y(x,\tilde{y})| \leq M$$

$\tilde{y} \in (y_1,y_2)$ or (y_2,y_1)

But f is Lipschitz with respect to y in R $\not\Rightarrow f_y(x,y)$ exists in R.

For example consider,

$f(x,y) = x^2|y|$ for $|x|<1$, $|y|<1$.

Note that

$$|f(x,y_1)-f(x,y_2)| = x^2\,\big||y_1|-|y_2|\big| \leq 1|y_1-y_2|.$$

So f is Lipschitz in y. But f_y does not exist at any point $(x,0)$ where $|x|<1$.

$$f_y(x,0) = \lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k}$$

$$= x^2 \lim_{k\to 0}\frac{|k|}{k}$$ does not exist

Example (f(x, y) is not Lipschitz in y but the IVP $y' = f(x,y)$, $y(x_0) = y_0$ has a solution)

Consider the IVP $y' = \frac{1}{y^2}$, $y(1) = 0$.

So, $\int y^2 dy = \int dx + c_0$.

So, $y^3 = 3x + c$

Now $y(1) = 0 \Rightarrow c = -3$.

So, $y^3 = 3x - 3$.

So, $y = (3x-3)^{1/3}$ is the unique solution of this IVP.

Now note that $f(x,y) = \frac{1}{y^2}$ can not be made continuous at $(1,0)$ as $\lim\limits_{(x,y)\to(1,0)} f(x,y) = \lim\limits_{y\to 0} \frac{1}{y^2}$ does not exist. So no matter what value we define for $f(1,0)$, the function cannot be Lipschitz in y.

Exercise Discuss the existence and the uniqueness of solution of IVP $y' = y + y^2$, $y(\frac{\pi}{2}) = 1$ for the domain $R = \{(x,y) \in \mathbb{R}^2 : |x - \frac{\pi}{2}| \leq 3, |y-1| \leq 1\}$.

Solution Here $f(x,y) = y + y^2$.

Note that $f(x,y)$ is continuous in R. Also $f(x,y)$ is bounded in R.

$|f(x,y)| = |y + y^2| \leq |y| + |y|^2 \leq 6$ in R.

so by the existence theorem
$y' = f(x,y)$ has a solution in
$\left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right)$ where

$$\alpha = \min\left\{3, \frac{1}{6}\right\} = \frac{1}{6}.$$

Now we shall check for the uniqueness of this solution in $\left(\frac{\pi}{2} - \frac{1}{6}, \frac{\pi}{2} + \frac{1}{6}\right)$.

$f_y(x,y) = 1 + 2y$.

Note that $f_y(x,y)$ is also continuous and bounded in R. Hence by the uniqueness theorem this solution is unique.

Picard's iteration method

Heuristic Suppose we have the IVP $y' = f(x,y)$, $y(x_0) = y_0$, has unique solution $y = \int_{x_0}^{x} f(x,y)\,dx + c$, $c \in \mathbb{R}$.

Putting $y(x_0) = y_0$, one gets $c = y_0$.

So, $y = \int_{x_0}^{x} f(x,y)\,dx + y_0$.

Define for $n \in \mathbb{N}$,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx.$$

Thm (Emile Picard)

Suppose the IVP $y' = f(x,y)$, $y(x_0) = y_0$ satisfies the hypotheses of the existence and uniqueness theorems, then y_n converges to the unique solution of this IVP.

Example consider $y' = x + y$, $y(0) = 0$.

write $y' = x + y$ as

$$\frac{d}{dx}(x+y+1) = x+y+1.$$

So, $\displaystyle\int \frac{d(x+y+1)}{x+y+1} = \int dx + c.$

i.e. $\log|x+y+1| = x + c.$

$y(0)=0$ gives $c=0$.

So, $|x+y+1| = e^{x}$

$$x+y+1 = e^{x} \quad \text{or} \quad x+y+1 = -e^{x}$$

$$y = e^{x}-x-1 \quad \text{or} \quad y = -e^{x}-x-1$$

As $y(0)=0$, we have $y = e^{x}-x-1$ is the unique solution of this IVP.

Now by Picard's iteration method,

$$y_1 = \int_0^x f(x, y_0)\,dx = \int_0^x x\,dx = \frac{x^2}{2}.$$

$$y_2 = \int_0^x f(x, y_1)\,dx = \int_0^x \left(x + \frac{x^2}{2}\right) dx$$

$$= \frac{x^2}{2} + \frac{x^3}{6} = \frac{x^2}{2} + \frac{x^3}{3!}$$

Proceeding this way we get,

$$y_n = \frac{x^2}{2!} + \cdots + \frac{x^{n+1}}{(n+1)!} \quad \forall\ n \geq 1.$$

Note that $\underbrace{y_n \to e^{x}-x-1}$ as $n \to \infty$.

$y_n(x) \to e^{x}-x-1$

A sequence of functions $f_n : (a,b) \to \mathbb{R}$ converges to a function $f : (a,b) \to \mathbb{R}$ if $f_n(x) \to f(x)\ \forall\ x \in (a,b)$.

Orthogonal trajectories

What is the idea?

Let $C = (x, \phi(x))$ be a curve in xy-plane. The slope of this curve at (x,y) be $f(x,y)$.

$$\text{so,} \quad f(x,y) = \phi'(x).$$

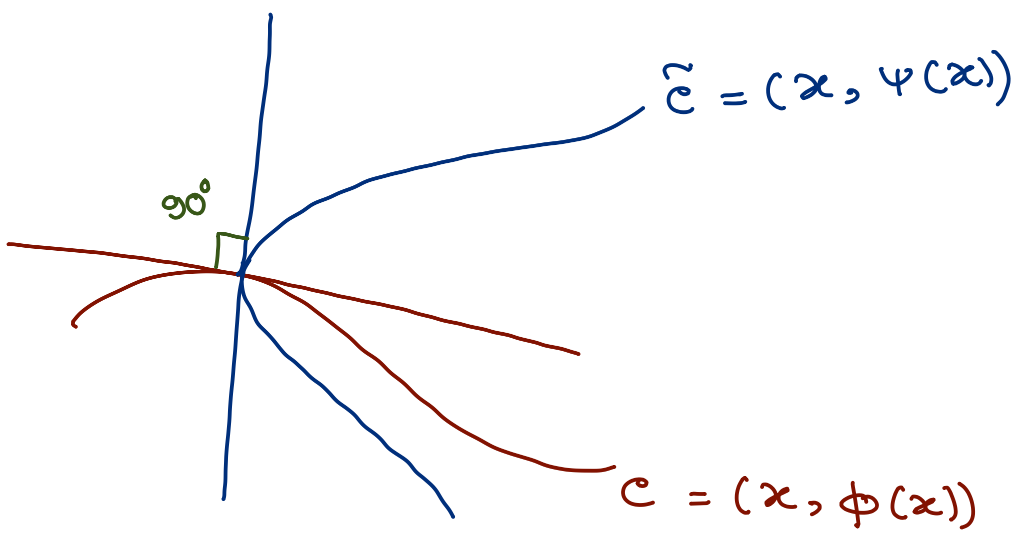
We want to find another curve $\tilde{C} = (x, \psi(x))$ passing through (x,y) such that the tangent of $\tilde{C}$ at (x,y) is perpendicular to the tangent of C at (x,y).

The slope of $\tilde{C}$ at $(x,y) = \psi'(x)$.

$$\phi'(x)\psi'(x) = -1 \quad \text{gives}$$

$$\psi'(x) = -\frac{1}{f(x,y)}.$$

So to find out $\tilde{C}$ we need to solve $\quad y' = -\dfrac{1}{f(x,y)}.$



Let S be a region in the xy-plane and $\mathcal{F}$ be a family of curves in S such that any point $(x,y) \in S$ can lie only on one curve of $\mathcal{F}$. Let $c \in \mathcal{F}$. Let $f(x,y)$ be the slope of the tangent of c at (x,y). If $c = (x, \phi(x))$, then $\phi(x)$ is a solution of $y' = f(x,y)$. Let $\psi(x)$ be a solution of the differential equation $y' = -\dfrac{1}{f(x,y)}$. Then the tangent at (x,y) of the curve $\tilde{c} = (x, \psi(x))$ is perpendicular to the tangent of c at (x,y). The family of curves $\tilde{\mathcal{F}} = (\tilde{c})$ obtained from $\mathcal{F} = (c)$ is called the set of Orthogonal trajectories of $\mathcal{F}$.

Examples ① Find out the orthogonal trajectories of $\mathcal{F} = \left\{ \frac{x^2}{2} + y^2 = c,\ c>0 \right\}$.

$$\frac{x^2}{2} + y^2 = c \quad , c > 0$$

The ODE of such curve is

$$x + 2yy' = 0$$

i.e. $y' = -\dfrac{x}{2y}$.

So, we need to solve the ODE

$$y' = \frac{2y}{x}.$$

$$\text{So, } \log|y| = \log x^2 + c_0$$

$$\text{So, } y = a x^2$$

So $\tilde{\mathcal{F}}_1 = \{ y = a x^2 , a > 0 \}$ is the set of orthogonal trajectories of $\mathcal{F}_1$.

② Find the orthogonal trajectories of the family of curves $r = a(1+\cos\theta)$, $a \in \mathbb{R}$.

Method for curves given in polar coordinates

From the curve $f(r, \theta, a) = 0$, find out $\frac{dr}{d\theta}$. So one has an equation of the form $F(r, \theta, \frac{dr}{d\theta}) = 0$.

Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$.

Solve, $F(r, \theta, -r^2 \frac{d\theta}{dr}) = 0$ to get the orthogonal trajectories of $f(r, \theta, a) = 0$, $a \in \mathbb{R}$.

The slope of $f(r, \theta, a) = 0$ at (r, θ) is $r\dfrac{d\theta}{dr}$.

So the slope of the desired curve is $-\dfrac{1}{r}\dfrac{dr}{d\theta}$

So we would have

$$\tan\psi = -\frac{1}{r}\frac{dr}{d\theta}.$$

So, in the differential equation $r\dfrac{d\theta}{dr}$ has to be replaced by $-\dfrac{1}{r}\dfrac{dr}{d\theta}$.

So, $\dfrac{dr}{d\theta}$ has to be replaced by $-r^2\dfrac{d\theta}{dr}$.

Given $r = a(1+\cos\theta)$.

$$\text{So, } \frac{dr}{d\theta} = -a\sin\theta.$$

$$= -\frac{r\sin\theta}{1+\cos\theta}$$

Consider, $-r^2\dfrac{d\theta}{dr} = -\dfrac{r\sin\theta}{1+\cos\theta}$.

$$\Rightarrow \int \frac{1+\cos\theta}{\sin\theta}\,d\theta = \int \frac{1}{r}\,dr.$$

$$\Rightarrow \int \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\,d\theta = \int \frac{1}{r}\,dr$$

$$\Rightarrow \int \cot\frac{\theta}{2}\,d\theta = \log r + c_0$$

$$\Rightarrow 2\log\left|\sin\frac{\theta}{2}\right| = \log r + C_0 \qquad \left\{\begin{array}{l} \sin\frac{\theta}{2} = u \\ \cos\frac{\theta}{2}\,d\theta = 2du \end{array}\right.$$

$$\Rightarrow r = c\sin^2\frac{\theta}{2} = c\left(1-\cos^2\frac{\theta}{2}\right)$$

$$= \frac{c}{2}\left(2-2\cos^2\frac{\theta}{2}\right)$$

$$= \frac{c}{2}\left(1-\cos\theta\right)$$

So, $\{r = b(1-\cos\theta) : b \in \mathbb{R}\}$ is the set of orthogonal trajectories of $\{r = a(1+\cos\theta) : a \in \mathbb{R}\}$.

Second order linear ODE

Standard form : $y'' + p(x)y' + q(x)y = r(x)$.
If $r(x) \equiv 0$, we call it 2nd order homogeneous linear ODE and otherwise 2nd order non-homogeneous linear ODE.

Examples

1. $y'' + y' = e^x$ non-homogeneous linear
2. $xy'' + y' + xy = 0$ homogeneous linear
3. $yy'' - (y')^2 = 0$ homogeneous non-linear
4. $y'' + yy' = e^x$
5. $y'' + xy' + y^2 = x^3$

$\left.\right\}$ non-homogeneous non-linear

Thm Let $p(x), q(x), r(x)$ be continuous functions on some open interval I. Let $x_0 \in I$. Consider the initial value problem

$$y'' + p(x)y' + q(x)y = r(x),$$
$$y(x_0) = y_0,$$
$$y'(x_0) = y_1.$$

Then this IVP has a unique solution on I.

Remark This is a sufficient condition for the existence and uniqueness of solution of a second order linear homogeneous ODE, not a necessary condition.

Second order homogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = 0$$

Let $y_1(x)$ and $y_2(x)$ be two solutions of this ODE. Let $c \in \mathbb{R}$. Note that $cy_1(x) + y_2(x)$ is also a solution of this ODE as

$$(cy_1 + y_2)'' + p(cy_1 + y_2)' + q(cy_1 + y_2)$$
$$= c(y_1'' + py_1' + qy_1) + (y_2'' + py_2' + qy_2).$$

Remark The set of all solutions of a second order homogeneous linear ODE is a vector space over $\mathbb{R}$. But this is ingeneral not true in case of a non-homogeneous or a non-linear ODE.

For example, consider the non-homogeneous linear ODE, $y'' + y' = e^x$.

Note that $y(x) = \frac{1}{2}e^x$ is a solution but $2y(x)$ is not a solution.

Another example,

$$y''y - (y')^2 = 0.$$

This is a homogeneous non-linear ODE.

Note that $y_1(x) = 1$, $y_2(x) = e^x$ are solutions.

But $y_1(x) + y_2(x) = 1 + e^x$ is not a solution.

Dimension of the solution space of a second order homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0$$

Suppose $p(x), q(x)$ are continuous on an open interval I. Let $x_0 \in I$.

Let $y_1(x)$ be the unique solution with initial conditions $y_1(x_0) = 1$,
$y_1'(x_0) = 0$.

Let $y_2(x)$ be the unique solution with initial conditions $y_2(x_0)=0$,

$y_2'(x_0)=1$.

<u>Claim</u> y_1, y_2 are l.I.

Let $c_1 y_1(x)+c_2 y_2(x)=0 \;\forall\; x\in I$.

so, $c_1 y_1'(x)+c_2 y_2'(x)=0 \;\forall\; x\in I$.

$$c_1 y_1(x_0)+c_2 y_2(x_0)=0 \Rightarrow c_1=0$$
$$c_1 y_1'(x_0)+c_2 y_2'(x_0)=0 \Rightarrow c_2=0.$$

so, $\dim \geq 2$.

<u>Claim</u> $\dim = 2$.

Let $y(x)$ be any solution of this ODE.

Denote $\hat{y}(x) = y(x_0)y_1(x)+y'(x_0)y_2(x)$.

clearly $\hat{y}(x)$ is also a solution as $y_1(x), y_2(x)$ are solutions.

$$\hat{y}(x_0) = y(x_0)y_1(x_0)+y'(x_0)y_2(x_0)$$
$$= y(x_0)$$
$$\hat{y}'(x_0) = y(x_0)y_1'(x_0)+y'(x_0)y_2'(x_0)$$
$$= y'(x_0)$$

By uniqueness of the solution $\hat{y}(x)=y(x)$.

<u>Thm</u> If $p(x), q(x)$ are continuous on an open interval I, then the solution space of $y''+p(x)y'+q(x)=0$ is of dimension 2.

Let $y_1(x), y_2(x)$ be any two l.I solutions of $y'' + p(x)y' + q(x) = 0$, $p(x), q(x)$ are continuous on I.

Then any solution $y(x)$ can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ for some } c_1, c_2 \in \mathbb{R}.$$

This is called a general solution of this ODE and $y_1(x), y_2(x)$ are called a basis or a fundamental system of solutions of this ODE on I.

If we assign specific values to c_1, c_2, then we call $y(x)$ to be a particular solution of this ODE on I.

Examples

① $y'' + y = 0$

$y_1(x) = \sin x$,
$y_2(x) = \cos x$ are solutions of this ODE.

Note that $y_1(x), y_2(x)$ are l. I as $\sin x \neq c \cos x$ for any $c \in \mathbb{R}$.

So $\{y_1, y_2\}$ is a basis of the solution space.

Any general solution of this ODE is of the form $y(x) = a\cos x + b \sin x$ where $a, b \in \mathbb{R}$.

$y(x) = 3\cos x - 0.5\sin x$ is a particular solution of this ODE or in other words it is the unique solution of the IVP

$$\left.\begin{aligned} &y'' + y = 0, \\ &y(0) = 3, \\ &y'(0) = -0.5. \end{aligned}\right\}$$

(2) $y'' - y = 0 \qquad y(0)=6,\ y'(0)=-2$.

Note that $y_1(x) = e^{x}$
$y_2(x) = e^{-x}$ are solutions.

In fact, $y_1(x), y_2(x)$ are l.I as e^{2x} is not constant.

So, $\{y_1, y_2\}$ is a basis of the solution space.

Now a general solution is of the form
$$y(x) = c_1 e^{x} + c_2 e^{-x}.$$

$$y(0) = 6 \Rightarrow c_1 + c_2 = 6.$$
$$y'(0) = -2 \Rightarrow c_1 - c_2 = -2.$$
$$\text{So, } c_1 = 2,\ c_2 = 4.$$

So $y(x) = 2e^{x} + 4e^{-x}$ is a particular solution.

To find a basis if one non-zero solution is known

We first discuss the following example:
$$(x^2 - x)y'' - xy' + y = 0.$$
Observe that $y_1(x) = x$ is a solution.

Put $y = uy_1 = ux$.

So, $y' = u + u'x$,
$y'' = 2u' + u''x$.

Substituting this in the ODE we get,
$$(x^2 - x)(2u' + u''x) - x(u + u'x) + ux = 0.$$

$$2x^2u' - 2xu' + x^3u'' - x^2u'' - xu - x^2u' + xu = 0.$$

$$x^2u' - 2xu' + x^3u'' - x^2u'' = 0$$

$$\Rightarrow xu' - 2u' + x^2u'' - xu'' = 0$$

$$\Rightarrow (x-2)u' + (x^2-x)u'' = 0$$

Write $v = u'$.

So, $(x^2-x)v' + (x-2)v = 0.$

So this is a first order homogeneous linear ODE.

So, $$\frac{dv}{v} = -\frac{x-2}{x^2-x}\,dx$$

$x^2 - x = x(x-1)$

$$= -\frac{2x-2-x}{x(x-1)}\,dx$$

$$= -\frac{2dx}{x} + \frac{dx}{x-1}$$

$$\log|v| = -2\log|x| + \log|x-1|, \quad x \in (0,1).$$

$$\Rightarrow |v| = \left|\frac{x-1}{x^2}\right|.$$

consider $v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}.$

$v = u'$

$$u = \log x + \frac{1}{x}.$$

Take $y_2(x) = x\log x + 1.$

Note that $\{y_1, y_2\}$ are l.I.

So we have obtained a basis of the solution space on $(0,1)$.

General method (Reduction of order)

We shall start with second order homogeneous linear ODE in standard form;

$$y'' + p(x) y' + q(x) y = 0.$$

Let $y_1(x)$ be a non-zero solution of this ODE on an open interval I.

Put $y = y_2 = u y_1$.

So, $y' = u' y_1 + u y_1'$,

$$y'' = 2u' y_1' + u'' y_1 + u y_1''.$$

So, $u'' y_1 + 2u' y_1' + u y_1'' + p(x) u' y_1 + p(x) u y_1'$
$+ q(x) u y_1 = 0.$

$$u'' y_1 + u'(2y_1' + p y_1) + u \underbrace{(y_1'' + y_1' p + q y_1)}_{=0 \text{ as } y_1 \text{ is a solution.}} = 0$$

So, $u'' y_1 + u'(2y_1' + p y_1) = 0$

$$u'' + u' \left(\frac{2y_1' + p y_1}{y_1} \right) = 0$$

Put $v = u'$.

So, $$v' + v \left(\frac{2y_1' + p y_1}{y_1} \right) = 0.$$

$$\frac{dv}{v} = - \frac{2y_1'}{y_1} dx - p\, dx.$$

$$\log|v| = -2 \log|y_1| - \int p\, dx.$$

$$v = \frac{1}{y_1^2} e^{-\int p\,dx}$$

$$u = \int v\,dx.$$

So, the desired solution is

$$y_2(x) = y_1(x) \int v\,dx.$$

Note that,

$$\frac{y_2(x)}{y_1(x)} = \int v\,dx \quad , v > 0$$

non-constant function of x.

So, y_2, y_1 are l.I.

Second order homogeneous linear ODEs with constant coefficients

$$y'' + ay' + by = 0 \qquad a, b \in \mathbb{R}$$

Recall In case of $y' + ay = 0$, $a \in \mathbb{R}$, we have $y = ce^{-ax}$ is a general solution.

For $y'' + ay' + by = 0$, we guess the solution might be of the form $y = e^{mx}$.

Note that $y = e^{mx}$ is a solution iff

$$m^2 + am + b = 0.$$

The equation $x^2 + ax + b = 0$ is called the characteristic equation of the ODE $y'' + ay' + by = 0$.

We know the solutions of $m^2 + am + b = 0$ are $\dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$.

Case-I If $a^2 - 4b > 0$, then we have two distinct real solutions of the characteristic equation, say m_1, m_2.

Consider $y_1(x) = e^{m_1 x}$, $y_2(x) = e^{m_2 x}$.

They are l.I as $e^{(m_1 - m_2)x}$ is not constant function. So a general solution of the ODE in this case can be written in the form

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case-II If $a^2 - 4b = 0$, then we have a double root $m = -\frac{a}{2}$ of $x^2 + ax + b = 0$.

So we get a solution $y_1(x) = e^{-\frac{a}{2}x}$ of the ODE.

we find another solution $y_2(x)$ which is l.I to $y_1(x)$.

From the reduction of order method we know,

$$y_2(x) = y_1(x) \int v \, dx \quad \text{where}$$

$$v = \frac{e^{-\int a \, dx}}{y_1^2} = \frac{e^{-ax}}{y_1^2} = 1.$$

So, $y_2(x) = x\, y_1(x) = x e^{-\frac{a}{2}x}$.

Now, y_1, y_2 are l.I as y_1/y_2 is not a constant function.

So a general solution of the ODE in this case can be written as

$$y(x) = c_1 e^{-a/2\, x} + c_2 x e^{-a/2\, x}$$

Case - III If $a^2 - 4b < 0$, then we have two distinct non-real roots $m, \bar{m}$. ($\bar{m}$ denotes the complex conjugate of m)

Write $m = -\frac{a}{2} + it$.

$$\bar{m} = -\frac{a}{2} - it.$$

Then we have two l.I solutions

$$y_1(x) = e^{mx}, \; y_2(x) = e^{\bar{m}x}.$$

But these are complex valued functions.

We want to find two l.I real solutions. So we write,

$$y_1(x) = e^{-\frac{a}{2}x} \cdot e^{it}$$

$$= e^{-\frac{a}{2}x}(\cos t + i \sin t),$$

$$y_2(x) = e^{-\frac{a}{2}x} \cdot e^{-it}$$

$$= e^{-\frac{a}{2}x}(\cos t - i \sin t).$$

Denote $\hat{y}_1(x) = \dfrac{y_1(x) + y_2(x)}{2}$,

$$\hat{y}_2(x) = \frac{y_1(x) - y_2(x)}{2i}.$$

Clearly $\hat{y}_1(x), \hat{y}_2(x)$ are solutions of the ODE being linear combinations of $y_1(x), y_2(x)$. Note that,

$$\hat{y}_1(x) = e^{-\frac{a}{2}x} \cos tx, \quad \hat{y}_2(x) = e^{-\frac{a}{2}x} \sin tx$$

are l.I as $\tan t$ is not a constant function.

A general solution of the ODE in this case can be written as

$$y(x) = e^{-\frac{a}{2}x}(c_1 \cos tx + c_2 \sin tx).$$

Summary

If we have $y'' + ay + b = 0$, $a, b \in \mathbb{R}$, then

① $a^2 - 4b > 0 \Rightarrow$ a general solution is of the form $c_1 e^{m_1 x} + c_2 e^{m_2 x}$ where

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2},$$

$$m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

② $a^2 - 4b = 0 \Rightarrow$ a general solution is of the form $(c_1 + c_2 x)e^{mx}$ where $m = -\frac{a}{2}$.

③ $a^2-4b=0 \Rightarrow$ a general solution is of the form
$$e^{-\frac{ax}{2}}(c_1 \cos tx + c_2 \sin tx)$$
where $m_1 = -\frac{a}{2} + it$,
$m_2 = -\frac{a}{2} - it$.

Examples

① $y'' - y = 0$

characteristic equation $x^2 - 1 = 0$.

so a general solution is of the form $y(x) = c_1 e^{x} + c_2 e^{-x}$.

② $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$.

characteristic equation $x^2 + x - 2 = 0$.

So, $m = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$
$= 1, -2$.

so a general solution is of the form $y(x) = c_1 e^{x} + c_2 e^{-2x}$.

$$\left.\begin{aligned} y(0) = 4 &\Rightarrow c_1 + c_2 = 4. \\ y'(0) = -5 &\Rightarrow c_1 - 2c_2 = -5. \end{aligned}\right\} \begin{aligned} c_2 &= 3, \\ c_1 &= 1. \end{aligned}$$

So the particular solution is $e^{x} + 3e^{-2x}$.

(3) $y'' + 2y' + 10y = 0$, $y(0) = 1$, $y'(0) = 1$.

Characteristic equation

$$x^2 + 2x + 10 = 0.$$

so, $$m = \frac{-2 \pm \sqrt{4-40}}{2}$$

$$= -1 \pm 3i$$

so a general solution is of the form $y(x) = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$.

$y(0) = 1 \Rightarrow c_1 = 1$

$y'(0) = 1 \Rightarrow -c_1 + 3c_2 = 1$.

so, $c_2 = \frac{2}{3}$.

so the particular solution is

$$e^{-x}\left(\cos 3x + \frac{2}{3}\sin 3x\right).$$

(4) $y'' - 10y' + 25y = 0$.

characteristic equation $x^2 - 10x + 25 = 0$.

$m = 5, 5$.

so a general solution is

$$e^{5x}(c_1 + c_2 x).$$

Euler-Cauchy equation

Consider an ODE of the form

$$x^2 y'' + a x y' + b y = 0 \quad , \; x>0, \; a,b \in \mathbb{R} .$$

Natural guess: $y = x^m$ could be a solution

$$\left(m(m-1) + a m + b\right) x^m = 0 .$$

So, $y = x^m$ is a solution of the given ODE $\Leftrightarrow$ $\underbrace{m(m-1) + am + b = 0}_{m^2 + m(a-1) + b = 0}$.

So, $m = \dfrac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$

$$m_1 = \frac{1-a}{2} + \sqrt{\frac{1}{4}(1-a)^2 - b}$$

$$m_2 = \frac{1-a}{2} - \sqrt{\frac{1}{4}(1-a)^2 - b}$$

Case-I If $m_1 \neq m_2$ are real, then $y_1(x) = x^{m_1}$, $y_2(x) = x^{m_2}$ are two L.I solutions. So in this case a general solution is of the form $y(x) = c_1 x^{m_1} + c_2 x^{m_2}$.

Case-II If $m_1 = m_2 = m = \frac{1-a}{2}$.

Then $y_1(x) = x^m$ is a solution.

By the reduction of order method, we can find $y_2(x) = x^m \int v \, dx$,

where $v = \frac{e^{-\int \frac{a}{x} dx}}{y_1^2} = x^{-2m} e^{-a \log x}$

$= x^{-a-2m}$

$= x^{-a-1+a}$

$= \frac{1}{x}$

So, $y_2(x) = x^m \log x = x^{\frac{1-a}{2}} \log x$.

y_1, y_2 are l.I.

A general solution in this case is

$$y(x) = x^{\frac{1-a}{2}} (c_1 + c_2 \log x).$$

Case-III If m_1, m_2 are non-real.

Then write, $m_1 = \frac{1-a}{2} + it$,

$m_2 = \frac{1-a}{2} - it$.

$$x^{m_1} = x^{\frac{1-a}{2}} x^{it} = x^{\frac{1-a}{2}} e^{it \log x}$$

$$= x^{\frac{1-a}{2}} (\cos(t \log x) + i \sin(t \log x))$$

$$x^{m_2} = x^{\frac{1-a}{2}} x^{-it}$$

$$= x^{\frac{1-a}{2}} e^{-it\log x}$$

$$= x^{\frac{1-a}{2}} \left(\cos(t\log x) - i\sin(t\log x)\right)$$

$\hat{y}_1(x) = x^{\frac{1-a}{2}} \cot(t\log x)$,

$\hat{y}_2(x) = x^{\frac{1-a}{2}} \sin(t\log x)$ are two l.I solutions.

So a general solution in this case is $x^{\frac{1-a}{2}}\left(c_1\cos(t\log x) + c_2\sin(t\log x)\right)$.

Examples

(1) $2x^2y'' + 4xy' + 3y = 0$.

Write it as $x^2y'' + 2xy' + \frac{3}{2}y = 0$.

characteristic equation

$$x^2 + x + \frac{3}{2} = 0 \quad \text{i.e.}$$

$$2x^2 + 2x + 3 = 0.$$

So, $$m = \frac{-2 \pm \sqrt{4 - 24}}{4}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{5}}{2} i.$$

so, the general solution is

$$y(x) = x^{-\frac{1}{2}}\left(c_1 \cos\left(\frac{\sqrt{5}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{5}}{2}\log x\right)\right).$$

② $2x^2 y'' + 3x y' - 15y = 0, \; y(1)=0, \; y'(1)=1.$

char. equation

$$x^2 + \frac{x}{2} - \frac{15}{2} = 0$$

i.e. $\quad 2x^2 + x - 15 = 0.$

$$m = \frac{-1 \pm \sqrt{1+120}}{4}$$

$$= \frac{-1 \pm 11}{4} = -3, \frac{5}{2}.$$

So the general solution is

$$y(x) = c_1 x^{5/2} + \frac{c_2}{x^3}.$$

$$y(1) = 0 \Rightarrow c_1 + c_2 = 0$$

$$y'(1) = 1 \Rightarrow \frac{5}{2}c_1 - 3c_2 = 1$$

$$\frac{5}{2}c_1 + 3c_1 = 1$$

$$\Rightarrow c_1 = \frac{2}{11}. \quad \text{So, } c_2 = -\frac{2}{11}.$$

So the desired solution is

$$y(x) = \frac{2}{11}\left(x^{5/2} - \frac{1}{x^3}\right).$$

3. $x^2 y'' + 5xy' + 4y = 0.$

char. equation $x^2 + 4x + 4 = 0.$

So, $m = -2, -2.$

So the general solution is

$$y(x) = x^{-2}(c_1 + c_2 \log x).$$

Recall The solutions of a second order homogeneous linear ODE $y'' + p(x)y' + q(x)y = 0$ form a vector space. We have seen that if $p(x), q(x)$ are continuous on an open interval I, then the dimension of the solution space is 2. Infact we learnt to find out basis of the solution space in case of second order homogeneous linear ODEs with constant coefficients and of Euler-Cauchy equations. In order to find out a basis we needed to find out two linearly independent solutions y_1, y_2, which is nothing but checking whether y_1/y_2 is not a constant function. But if we want to find out basis for the solution space of an order $n (> 2)$ homogeneous linear ODE, then we need to deal with n functions.

$y_1, \dots, y_n$ are l. I

$\Leftrightarrow c_1 y_1 + \dots + c_n y_n = 0 \Rightarrow c_i = 0 \ \forall \ 1 \leq i \leq n.$

Checking this might not be easy always. To deal with it, we have the notion of Wronskian named after Polish mathematician Wronski.

Wronskian

Let $y_1, \dots, y_n$ be $(n-1)$ times differentiable linearly dependent functions.

So $\exists$ constants $c_1, \dots, c_n$ not all 0 such that $c_1 y_1 + \dots + c_n y_n = 0$.

So,
$$c_1 y_1' + \dots + c_n y_n' = 0,$$
$$c_1 y_1'' + \dots + c_n y_n'' = 0,$$
$$\vdots \qquad\qquad \vdots$$
$$c_1 y_1^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0.$$

So,
$$\underbrace{\begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}}_{\substack{\parallel \\ A}} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now the system $Ax = 0$ has a non-trivial solution as not all c_i's are 0. So $\det A = 0$.

The determinant of the matrix A denoted by $W(y_1, \dots, y_n)$ for any $(n-1)$ times differentiable functions $y_1, \dots, y_n$ is called the Wronskian of $y_1, \dots, y_n$.

<u>Thm</u> Let $y_1, \dots, y_n$ be $(n-1)$ times differentiable, linearly dependent functions. Then $W(y_1, \dots, y_n) = 0$.

<u>Corollary</u> If $\exists$ $x_0 \in I$ such that $W(y_1, \dots, y_n)(x_0) \neq 0$, then $y_1, \dots, y_n$ are linearly independent.

<u>Remark</u> $W(y_1, \dots, y_n) = 0 \not\Rightarrow y_1, \dots, y_n$ are linearly dependent.

For example let

$y_1(x) = x|x|$ and

$y_2(x) = x^2$ on $(-1, 1)$.

So, $y_1'(x) = 2|x|$ and $y_2'(x) = 2x$.

So, $W(y_1, y_2) = \det \begin{pmatrix} x|x| & x^2 \\ 2|x| & 2x \end{pmatrix}$

$= 2x^2|x| - 2x^2|x| = 0$.

But y_1, y_2 are not linearly dependent as $\dfrac{y_1(x)}{y_2(x)} = \dfrac{|x|}{x}$ is not a constant function on $(-1, 1)$.

For linear dependence of y_1, y_2 we would have needed non-zero scalars c_1, c_2 such that $c_1y_1 + c_2y_2 = 0$.

Here $W(y_1, y_2) = 0 \Rightarrow AX = 0$ has non-trivial solution made of non-constant functions.

Note that,

$$\begin{pmatrix} x|x| & x^2 \\ 2|x| & 2x \end{pmatrix} \begin{pmatrix} |x| \\ -x \end{pmatrix} = 0.$$

To check y_1, y_2 are linearly independent or not, we need to check $W(y_1, y_2)(x_0)$ is non-zero or not at some point $x_0 \in I$. The next theorem tells us that it is enough to check at any random point on I.

<u>Thm</u> Consider $y'' + p(x)y' + q(x)y = 0$, $p(x)$, $q(x)$ are continuous on an open interval I. Let y_1, y_2 be two solutions of it. Then either $W(y_1, y_2) \equiv 0$ or never zero on I.

Proof It is enough to show that if $W(y_1, y_2)$ is 0 at some point then it is identically zero function on the given domain.

Let $W(y_1, y_2)(x_0) = 0$, i.e.

$$\det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = 0.$$

So, $\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}$ is not invertible and hence the columns are linearly dependent. So $\exists$ c_1, c_2 constants not all 0 such that

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 .$$

consider $Y(x) = c_1 y_1(x) + c_2 y_2(x)$.

$$Y(x_0) = 0,$$
$$Y'(x_0) = 0.$$

Clearly $Y(x)$ is a solution of the IVP

$$\begin{cases} y'' + p(x) y' + q(x) y = 0, \\ y(x_0) = 0, \\ y'(x_0) = 0. \end{cases}$$

Now $y(x) \equiv 0$ is also a solution.

So by the uniqueness of solution

$$Y(x) = y(x) \equiv 0.$$

So, y_1, y_2 are linearly dependent.

So, $W(y_1, y_2) = 0$.

Corollary of the above proof

Let y_1, y_2 be two linearly independent solutions of a second order homogeneous linear ODE. Then $W(y_1, y_2) \neq 0$. (never vanishing)

Pf If $W(y_1, y_2)(x_0) = 0$, then following the above proof we have y_1, y_2 are linearly dependent. But we are given y_1, y_2 are linearly independent. So, $W(y_1, y_2)(x_0) \neq 0$.
So, $W(y_1, y_2)$ is never 0.

Observation If y_1, y_2 are two solutions of $y'' + p(x)y' + q(x)y = 0$ then y_1, y_2 are l.I $\Leftrightarrow W(y_1, y_2) \neq 0$.

So the example we had,
$y_1(x) = x|x|$, $y_2(x) = x^2$ are never solutions of a second order homogeneous linear ODE.

<u>Remark</u> The above theorem and its proof extends to higher order homogeneous linear ODES. So to have the linear independence of solutions we need to just have the non-vanishing of the Wronskian at a point.

Non-homogeneous linear ODE

$$\boxed{a_n(x)\,y^{(n)} + \cdots + a_0(x)\,y = r(x) \quad \text{where } r(x) \not\equiv 0} \quad (*)$$

Let y_1, y_2 be two solutions of it.
Then $y_1 - y_2$ is a solution of
$$a_n(x)\,y^{(n)} + \cdots + a_0(x)\,y = 0.$$

Now, $y_1 = (y_1 - y_2) + y_2$.

So a solution of the non-homogeneous equation can be written as a sum of a solution of homogeneous part and a particular solution of non-homogeneous equation.

We know, a general solution of
$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$$ is
of the form $c_1y_1 + \cdots + c_ny_n$ where
$y_1, \ldots, y_n$ are linearly independent.

So a general solution of
$$\boxed{a_n(x)y^{(n)} + \cdots + a_0(x)y = r(x)}$$ is of
$r(x) \neq 0$
the form
$c_1y_1 + \cdots + c_ny_n + y_p$ where y_p is a particular solution of $(*)$.

We come back to the specific case of second order. For homogeneous case, we dealt with equations with constant coefficients and Euler-Cauchy equation. So for the non-homogeneous second order linear ODE we start with the case of constant co-efficients. We need to find a particular solution of such an equation. This method is called the method of undetermined coefficients. We will make guesses looking at the equation.

Examples

(1) $y'' - 3y' - 4y = 4x^2 + 2x - 9$.

For the homogeneous part

$$y'' - 3y' - 4y = 0.$$

Char. equation $m^2 - 3m - 4 = 0$

$$m = \frac{3 \pm \sqrt{9+16}}{2}$$

$$= \frac{3 \pm 5}{2} = 4, -1.$$

So a general solution is of the form $c_1 e^{4x} + c_2 e^{-x}$.

Particular solution for non-homogeneous

Suppose y_p is of the form $ax^2 + bx + c$.

$$y_p' = 2ax + b, \quad y_p'' = 2a.$$

So, $2a - 3(2ax + b) - 4(ax^2 + bx + c) = 4x^2 + 2x - 9$

So, $-4ax^2 - x(6a + 4b) + (2a - 3b - 4c) = 4x^2 + 2x - 9$

So, $a = -1$.

$6a + 4b = -2$

$\Rightarrow 4b = -2 + 6 = 4$

$\Rightarrow b = 1$.

$2a - 3b - 4c = -9$

$\Rightarrow 4c = -2 - 3 + 9 = 4$

$\Rightarrow c = 1$.

So $y_p = -x^2 + x + 1$.

Hence a general solution of the given ODE is $c_1 e^{4x} + c_2 e^{-x} - x^2 + x + 1$.

(2) $y'' - y = 16e^{3x}$.

Homogeneous part $\quad y'' - y = 0$

Char. eq. $m^2 - 1 = 0$

$m = \pm 1$.

a general solution is of the form $c_1 e^{x} + c_2 e^{-x}$.

Non-homogeneous part

Guess $y_p = ae^{3x}$.

$y_p' = 3ae^{3x}, \quad y_p'' = 9ae^{3x}$.

So, $9ae^{3x} - ae^{3x} = 16e^{3x}$.

So, $a = 2$.

So, $y_p = 2e^{3x}$.

So a general solution of the given ODE is $c_1 e^{x} + c_2 e^{-x} + 2e^{3x}$.

③ $y'' - y' - 2y = 10 \sin x$

Homogeneous part

$$y'' - y' - 2y = 0.$$

char. equation $m^2 - m - 2 = 0$

$$m = \frac{1 \pm \sqrt{1+8}}{2}$$

$$= \frac{1 \pm 3}{2} = 2, -1.$$

a general solution is of the form $c_1 e^{2x} + c_2 e^{-x}$.

Non-homogeneous part

Guess $y_p = a \sin x + b \cos x$

$$y_p' = a \cos x - b \sin x$$

$$y_p'' = -a \sin x - b \cos x = -y_p$$

$$-3y_p - a \cos x - b \sin x = 10 \sin x$$

i.e. $(b - 3a) \sin x + (-a - 3b) \cos x = 10 \sin x$.

$$\left.\begin{aligned} b-3a&=10 \\ -3b-a&=0 \end{aligned}\right\} \Rightarrow \begin{aligned} a&=-3, \\ b&=1. \end{aligned}$$

So a general solution of the given ODE is $c_1e^{2x}+c_2e^{-x}-3\sin x+\cos x$.

4. $y''-2y'-3y=3xe^{2x}$.

Homogeneous part

$$y''-2y'-3y=0$$

Char. equation $\quad m^2-2m-3=0$

i.e. $\quad (m-3)(m+1)=0$

So, $m=3,-1$

So the general solution of the homogeneous part is of the form $c_1e^{-x}+c_2e^{3x}$.

Non-homogeneous part

Guess $\quad y_p=(ax+b)e^{2x}$.

$$y_p'=2(ax+b)e^{2x}+ae^{2x}$$
$$=(2ax+a+2b)e^{2x}$$

$$y_p''=2ae^{2x}+2(2ax+a+2b)e^{2x}$$
$$=(4ax+4a+2b)e^{2x}$$

So, $y_p'' - 2y_p' - 3y_p = 3xe^{2x}$ gives

$$(-3ax + 2a - 3b)e^{2x} = 3xe^{2x}.$$

i.e. $-3a = 3$ and $2a - 3b = 0$.

i.e. $a = -1$ and $b = -\frac{2}{3}$.

So, $y_p = (-x - \frac{2}{3})e^{2x}$.

So, the general solution of the given ODE is $c_1 e^{-x} + c_2 e^{3x} - (x + \frac{2}{3})e^{2x}$.

(5) $y'' - y = 16e^{-x}$

Homogeneous part $\quad c_1 e^{x} + c_2 e^{-x}$.

Non-homogeneous part

Guess $\quad y_p = ae^{-x}$.

$y_p' = -ae^{-x}$, $y_p'' = ae^{-x} = y_p$

So, $y_p'' - y_p = 0$.

So this initial guess doesnot work.

Take $\quad y_p = axe^{-x}$

$y_p' = ae^{-x} - axe^{-x}$

$y_p'' = -2ae^{-x} + axe^{-x}$.

So, $-2ae^{-x} + axe^{-x} - axe^{-x} = 16e^{-x}$

$\Rightarrow \quad a = -8$

$\therefore \quad y_p = -8xe^{-x}$.

So a general solution of the given ODE is $c_1e^x + c_2e^{-x} - 8xe^{-x}$

⑥ $y'' - y' = 4x$.

<u>Homogeneous part</u>

char. eq. $m^2 - m = 0$ $\quad m = 0, 1$

general solution $c_1 + c_2e^x$.

<u>Non-homogeneous part</u>

<u>Guess</u> $y_p = ax + b$

$y_p' = a, \; y_p'' = 0$.

So, $-a = 4x$ Not possible.

<u>Take</u> $y_p = x(ax+b)$

$y_p' = 2ax + b, \; y_p'' = 2a$.

So, $2a - (2ax+b) = 4x$

$\Rightarrow -2ax + (2a-b) = 4x$.

$b = 2a, \quad -2a = 4$

$\Rightarrow a = -2$

So $a = -2, \; b = -4$.

So a general solution of the given ODE is $c_1 + c_2e^x - 2x(x+2)$.

Table (First guess)

$r(x)$	$y_p(x)$
$ce^{\alpha x}$	$ae^{\alpha x}$
cx^n	$a_0 + a_1 x + \cdots + a_n x^n$
$cx^n e^{\alpha n}$	$(a_0 + a_1 x + \cdots + a_n x^n) e^{\alpha x}$
$c\cos\alpha x$ $c\sin\alpha x$	$a\cos\alpha x + b\sin\alpha x$
$ce^{\alpha x}\cos\beta x$ $ce^{\alpha x}\sin\beta x$	$e^{\alpha x}(a\cos\beta x + b\sin\beta x)$
$cx^n e^{\alpha x}\cos\beta x$ $cx^n e^{\alpha x}\sin\beta x$	$e^{\alpha x}\cos\beta x\,(a_0 + a_1 x + \cdots + a_n x^n)$ $+ e^{\alpha x}\sin\beta x\,(b_0 + b_1 x + \cdots + b_n x^n)$

If first guess does not work, if we see that a term in our choice of y_p is there as a term in the general solution of the homogeneous part, then we need to multiply y_p by x, and consider guess as xy_p. If still in xy_p we get such a common term we need to multiply by x^2 and consider $x^2 y_p$ and so on.

If $r(x) = r_1(x) + r_2(x)$ and $r_i(x)$ for $i=1,2$ have particular solutions $y_{p_i}(x)$ then for $r(x)$ we need to work with $y_{p_1}(x) + y_{p_2}(x)$.

Examples (1) $y'' + 3y' + \frac{9}{4}y = -10e^{-3/2 x}$

$y(0)=1, \; y'(0)=0.$

Homogeneous part

$$m^2 + 3m + \frac{9}{4} = 0$$

$$4m^2 + 12m + 9 = 0$$

$$(2m+3)^2 = 0. \qquad m = -\frac{3}{2}, -\frac{3}{2}$$

general solution $(c_1 + c_2 x)e^{-\frac{3}{2}x}$.

Non-homogeneous part

Initial guess $ae^{-3/2 x}$

Modified guess $y_p = ax^2e^{-3/2 x}$

$$y_p' = a\left(2 - \frac{3}{2}x\right)x e^{-3/2 x}$$

$$y_p'' = a\left(2 - 6x + \frac{9}{4}x^2\right)e^{-3/2 x}$$

So, $a\left(2-6x+\frac{9}{4}x^2\right)+3a\left(2-\frac{3}{2}x\right)x$

$+\frac{9}{4}ax^2=-10$

$\Rightarrow 2a=-10 \Rightarrow a=-5.$

So, $y_p=-5x^2e^{-3/2x}.$

So a general solution of the given ODE is $(c_1+c_2x)e^{-3/2x}-5x^2e^{-3/2x}.$

$=(c_1+c_2x-5x^2)e^{-3/2x}$

$y(0)=1 \Rightarrow c_1=1.$

$y'(x)=(c_2-10x)e^{-3/2x}-\frac{3}{2}e^{-3/2x}(c_1+c_2x-5x^2)$

$y'(0)=0$

$\Rightarrow -\frac{3}{2}c_1+c_2=0$

$\Rightarrow c_2=\frac{3}{2}.$

So the desired solution is

$y(x)=\left(1+\frac{3}{2}x-5x^2\right)e^{-3/2x}.$

<u>Recall</u> Euler-Cauchy equation

$$x^2 y'' + axy' + by = 0.$$

We took $y = x^m$

char. eq. $m^2 - (a-1)m + b = 0$

Note that Euler-Cauchy equation can be viewed as homogeneous linear ODE with constant coefficients.

Put $x = e^t$.

Now $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = e^t \dfrac{dy}{dx} = x\dfrac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{1}{x}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

So, $\left(\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\right) + a\dfrac{dy}{dt} + by = 0.$

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0.$$

Exercise Write down a particular solution for

$x^2 y'' + a x y' + b y = P(x)$ where $P(x)$ is a polynomial in x.

Solution by variation of parameters

$$y'' + p(x) y' + q(x) y = r(x).$$

We have discussed how to solve non-homogeneous second order linear differential equations in some specific cases of $r(x)$ and with $p(x), q(x)$ as constants. We would now like to deal with a larger class of problems.

Assume $y'' + p(x) y' + q(x) y = r(x)$ in the standard form

$p(x), q(x), r(x)$ are continuous on given domain

Suppose, y_1, y_2 are two linearly independent solutions.

Then we know,

$y_h = c_1 y_1 + c_2 y_2$ and a general solution of the given ODE is $c_1 y_1 + c_2 y_2 + y_p$ where y_p is a particular solution of the given ODE.

Variation of parameters (Lagrange)

A method to find y_p.

$y_p = u y_1 + v y_2$, u, v are particular functions

Formula : $$u = -\int \frac{y_2\, r}{W(y_1, y_2)}\, dx \qquad W(y_1,y_2) \neq 0$$

as y_1, y_2 are l.I solutions

Remember to take ODE in standard form

$$v = \int \frac{y_1\, r}{W(y_1, y_2)}\, dx$$

Examples

① $y'' + y = \dfrac{1}{\cos x}$

Homogeneous part

$$\left\{ \begin{array}{l} y'' + y = 0 \\ \text{char. equation } \; x^2 + 1 = 0 \\ \qquad m = \pm i \\ \text{a general solution } \; c_1 \cos x + c_2 \sin x. \end{array} \right.$$

$$W(\cos x, \sin x) = \det \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

$$= 1$$

$$u = -\int \frac{\sin x}{\cos x}\, dx = \log|\cos x|$$

$$v = \int \frac{\cos x}{\cos x}\, dx = x$$

so $y_p = (\cos x)(\log|\cos x|) + x \sin x$.

A general solution of the ODE is

$$(c_1 - \log|\cos x|)\cos x + (c_2 + x)\sin x.$$

② $y'' + 4y' + 4y = e^{-2x}\log x$.

<u>Homogeneous part</u> $\quad y'' + 4y' + 4y = 0$

char. equation $m^2 + 4m + 4 = 0$

$m = -2, -2$.

$$y_h = (c_1 + c_2 x)e^{-2x}$$

$$y_1 = e^{-2x}, \quad y_2 = x e^{-2x}$$

$$W(y_1, y_2) = \det\begin{pmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & e^{-2x} - 2x e^{-2x} \end{pmatrix}$$

$$= e^{-4x}$$

$$u = -\int \frac{x e^{-2x}\; e^{-2x}\log x}{e^{-4x}}\, dx$$

$$= -\int x \log x\, dx$$

$$= -\frac{x^2}{2}\log x + \frac{1}{4}x^2$$

$$v = \int \frac{e^{-2x} e^{-2x} \log x}{e^{-4x}} dx$$

$$= \int \log x \, dx = x \log x - x$$

A general solution of the given ODE is

$$c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2} x^2 e^{-2x} \log x - \frac{3}{4} x^2 e^{-2x}.$$

Proving the formulae

$y_p = u y_1 + v y_2$ where

$$u = -\int \frac{y_2 r}{W(y_1, y_2)} dx \quad \text{and}$$

$$v = \int \frac{y_1 r}{W(y_1, y_2)} dx.$$

so, $y_p' = u' y_1 + u y_1' + v' y_2 + v y_2'$

$$= (u' y_1 + v' y_2) + (u y_1' + v y_2')$$

$$y_p'' = (u' y_1 + v' y_2)' + u' y_1' + u y_1'' + v' y_2' + v y_2''$$

As $y_p'' + p y_p' + q y_p = r$, we have

$$(u' y_1 + v' y_2)' + u' y_1' + \underline{\underline{u y_1''}} + v' y_2' + v y_2''$$

$$+ p(u' y_1 + v' y_2) + \underline{\underline{pu}} y_1' + p v y_2' + \underline{\underline{q u}} y_1$$

$$+ q v y_2 = r.$$

so,

$$(u'y_1+v'y_2)' + u\underbrace{(y_1''+py_1'+qy_1)}_{=0} + v\underbrace{(y_2''+py_2'+qy_2)}_{=0} + p(u'y_1+v'y_2) + (u'y_1'+v'y_2') = r$$

$$(u'y_1+v'y_2)' + p(u'y_1+v'y_2) + (u'y_1'+v'y_2') = r$$

Set $u'y_1+v'y_2=0$

and $u'y_1'+v'y_2'=r$.

so,
$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}.$$

so,
$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W(y_1,y_2)}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ r \end{pmatrix}$$

so,
$$u' = \frac{-y_2 r}{W(y_1,y_2)}$$

$$v' = \frac{y_1 r}{W(y_1,y_2)}$$

so,
$$u = -\int \frac{y_2 r}{W(y_1,y_2)}\,dx,$$

$$v = \int \frac{y_1 r}{W(y_1,y_2)}\,dx.$$

Differential operator

Consider S = Set of n times diff. function.

S is a vector space.

Denote $y' = Dy$, $y'' = D^2 y$ and so on $y^{(n)} = D^n y$.

So we can write,

$a_n(x) y^{(n)} + \cdots + a_0(x) y = r(x)$ as

$$\left(a_n(x) D^n + \cdots + a_0(x) I\right) y = r(x).$$

D is a linear transformation on S.

This helps us to apply the theory of linear transformations to study linear ODEs.

For example,

$y'' + a_1 y' + a_0 y = 0$ can be viewed as

$$(D^2 + a_1 D + a_0 I) y = 0.$$

If $x^2 + a_1 x + a_0 = 0$ has two distinct solutions λ_1, λ_2 then

$$(D - \lambda_1 I)(D - \lambda_2 I) y = 0.$$

Solving $(D - \lambda_1 I) y = 0$ and $(D - \lambda_2 I) y = 0$ separately

we get $y = e^{\lambda_1 x}, e^{\lambda_2 x}$.

Note that the linear combinations of $e^{\lambda_1 x}$, $e^{\lambda_2 x}$ are in the kernel of $(D-\lambda_1 I)(D-\lambda_2 I)$. Recollect that in the case of $y'' + a_1 y' + a_2 y = 0$ where $x^2 + a_1 x + a_2 = 0$ has two distinct roots we get the solutions in the form $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Higher order linear ODE

Standard form

$$y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_0(x) y = r(x)$$

Non-homogeneous if $r(x) \not\equiv 0$

Homogeneous if $r(x) \equiv 0$

If $r(x) \equiv 0$

1. The solutions form a vector space.
2. If $a_{n-1}(x), \ldots, a_0(x)$ are continuous on the given open interval, then the IVP has a unique solution.
3. The dimension of the solution space is n, given that $a_{n-1}(x), \ldots, a_0(x)$ are continuous. So a general

solution is of the form $c_1y_1 + \cdots + c_ny_n$ where $y_1, \ldots, y_n$ are l.I solutions.

④ How to check n-many solutions $y_1, \ldots, y_n$ are l.I or not.

Find out $W(y_1, \ldots, y_n)$.

$y_1, \ldots, y_n$ are l.I $\Leftrightarrow W(y_1, \ldots, y_n) \neq 0$.

⑤ For $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$, the characteristic equation is

$$x^n + a_{n-1}x + \cdots + a_0 = 0.$$

Let $m_1, \ldots, m_n$ be zeros.

If $m_1 \neq m_2 \neq \cdots \neq m_n$ are real, then $e^{m_1 x}, \ldots, e^{m_n x}$ are solutions. They are l.I too as

$$W(e^{m_1 x}, \ldots, e^{m_n x}) = \det \begin{pmatrix} e^{m_1 x} & \cdots & e^{m_n x} \\ m_1 e^{m_1 x} & \cdots & m_n e^{m_n x} \\ \vdots & & \\ m_1^{n-1} e^{m_1 x} & \cdots & m_n^{n-1} e^{m_n x} \end{pmatrix}$$

$$= e^{(m_1 x + \cdots + m_n x)} \det \begin{pmatrix} 1 & \cdots & 1 \\ m_1 & \cdots & m_n \\ \vdots & & \vdots \\ m_1^{n-1} & \cdots & m_n^{n-1} \end{pmatrix}$$

Vander Monde matrix

So, $W(e^{m_1x}, \cdots, e^{m_nx}) = e^{(m_1x + \cdots + m_nx)} \prod_{1 \le i < j \le n} (m_j - m_i)$

If $\lambda_1, \cdots, \lambda_n$ are real but there are some repeated roots, then we have the following:

If λ_1 is repeated k times $k \le n$, then the corresponding to λ_1 solutions are

$$e^{\lambda_1 x}, x e^{\lambda_1 x}, \cdots, x^{k-1} e^{\lambda_1 x}.$$

Similarly one shall write the solutions corresponding to other λ_i's. Then by the Wronskian one can check that all these solutions are linearly independent.

If some λ_j's are complex, then they appear in pairs. If

$$\lambda_1 = \alpha + i\beta \text{ and } \lambda_2 = \alpha - i\beta.$$

Then we take $e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$.

If $\alpha \pm i\beta$ appear k-times then we take $e^{\alpha x} \cos \beta x, x e^{\alpha x} \cos \beta x, \cdots, x^{k-1} e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x, x e^{\alpha x} \sin \beta x, \cdots, x^{k-1} e^{\alpha x} \sin \beta x$.

Examples

① $y^{(4)} - 5y^{(2)} + 4y = 0$.

char. equation $x^4 - 5x^2 + 4 = 0$.

$m = \pm 1, \pm 2$.

A general solution is of the form $c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$.

② $y^{(4)} - y = 0$.

char. equation $x^4 - 1 = 0$.

so, $m = \pm 1, \pm i$.

$y_1(x) = e^{x}$

$y_2(x) = e^{-x}$

$y_3(x) = \cos x$

$y_4(x) = \sin x$

So a general solution is of the form $c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$.

③ $y^{(5)} + 2y = 0$

Char. equation $x^5 + 2 = 0$

$$m = (-2)^{\frac{1}{5}}, \underline{\underline{(-2)^{\frac{1}{5}}\eta}}, \underline{(-2)^{\frac{1}{5}}\eta^2}, \underline{(-2)^{\frac{1}{5}}\eta^3}, \underline{\underline{(-2)^{\frac{1}{5}}\eta^4}}$$

where $\eta = e^{2\pi i/5}$
$= \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$$y_1(x) = e^{-\sqrt[5]{2}\,x}$$

$$y_2(x) = e^{-x\sqrt[5]{2}\cos\frac{2\pi}{5}} \cos\left(x\sin\frac{2\pi}{5}\right)$$

$$y_3(x) = e^{-x\sqrt[5]{2}\cos\frac{2\pi}{5}} \sin\left(x\sin\frac{2\pi}{5}\right)$$

$$y_4(x) = e^{-x\sqrt[5]{2}\cos\frac{4\pi}{5}} \cos\left(x\sin\frac{4\pi}{5}\right)$$

$$y_5(x) = e^{-x\sqrt[5]{2}\cos\frac{4\pi}{5}} \sin\left(x\sin\frac{4\pi}{5}\right)$$

So a general solution is of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + c_4 y_4(x) + c_5 y_5(x).$$

④ $y^{(4)} + 2y^{(2)} + y = 0$

char. equation $x^4 + 2x^2 + 1 = 0$.

So, $(x^2+1)^2 = 0$

So, $x = \pm i, \pm i$

$y_1(x) = \cos x$

$y_2(x) = \sin x$

$y_3(x) = x\cos x$

$y_4(x) = x\sin x$

A general solution is of the form

$(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$.

⑤ $y^{(5)} - 3y^{(4)} + 3y^{(3)} - y^{(2)} = 0$.

char. equation $x^5 - 3x^4 + 3x^3 - x^2 = 0$.

So, $x^2(x^3-1) - 3x^3(x-1) = 0$.

So, $x^2(x-1)(x^2+x+1) - 3x^3(x-1) = 0$.

So, $x^2(x-1)(x^2+x+1-3x) = 0$

So, $x^2(x-1)^3 = 0$

$x = 0, 0, 1, 1, 1$.

$y_1(x) = 1$

$y_2(x) = x$

$y_3(x) = e^x$

$y_4(x) = xe^x$

$y_5(x) = x^2 e^x$

A general solution is of the form

$(c_1 + c_2 x) + (c_3 + x c_4 + x^2 c_5)e^x$.

(6) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$

Put $y = x^m$

$y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$,

$y''' = m(m-1)(m-2)x^{m-3}$.

So, $m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0$

$\Rightarrow (m-1)(m^2 - 2m - 3m + 6) = 0$

$\Rightarrow (m-1)(m-3)(m-2) = 0$.

So $m = 1, 2, 3$

$y_1(x) = x$, $y_2(x) = x^2$, $y_3(x) = x^3$.

So, a general solution is of the form $y(x) = c_1 x + c_2 x^2 + c_3 x^3$.

(5) <u>Method of undetermined co-efficients</u>

$y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_0(x) y = r(x)$ (*)

$r(x) \not\equiv 0$.

Consider $y^{(n)} + \dots + a_0(x) y = 0$.
A general solution is of the form
$c_1 y_1 + \dots + c_n y_n$ where $y_1, \dots, y_n$ are l.I solutions.

Let y_p be a particular solution of (*)

Then a general solution of (*) is

$c_1 y_1 + \dots + c_n y_n + y_p$.

We here consider $a_0(x), \dots, a_{n-1}(x)$ to be constant functions. Then we find out y_p by determining the coefficients of y_p. This is called solving by undetermined coefficients.

Example $\quad y''' + 3y'' + 3y' + y = 30e^{-x}$.

Consider $y''' + 3y'' + 3y' + y = 0$

$$m^3 + 3m^2 + 3m + 1 = 0 \Rightarrow (m+1)^3 = 0$$

$$\Rightarrow m = -1, -1, -1.$$

So, $y_h = (c_1 + c_2 x + c_3 x^2) e^{-x}$.

Take $\quad y_p = a x^3 e^{-x}$

$$y_p' = a(3x^2 - x^3)e^{-x}$$

$$y_p'' = a(6x - 6x^2 + x^3)e^{-x}$$

$$y_p''' = a(6 - 18x + 9x^2 - x^3)e^{-x}.$$

So, $a(6 - 18x + 9x^2 - x^3) + 3a(6x - 6x^2 + x^3)$
$+ 3a(3x^2 - x^3) + a x^3 = 30$.

So, $6a = 30$. So $a = 5$.

So, $\quad y_p = 5x^3 e^{-x}$.

So a general solution of the given ODE is $\quad (c_1 + c_2 x + c_3 x^2 + 5x^3)e^{-x}$.

⑥ Method of variation of parameters

$$y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \cdots + a_0(x)y = r(x),$$
$$r(x) \not\equiv 0.$$

Let $y_1, \ldots, y_n$ be l.I solutions of $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$.

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(y_1,\ldots,y_n)}\, r(x)\,dx,$$

where $W_k(x)$ is the determinant of

$$\begin{pmatrix} y_1 & \cdots & y_{k-1} & 0 & \cdots & y_n \\ \vdots & & \vdots & \ddots & & \vdots \\ y_1^{(n-2)} & \cdots & y_{k-1}^{(n-2)} & 0 & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & & y_{k-1}^{(n-1)} & 1 & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Example $\quad x^3y''' - 3x^2y'' + 6xy' - 6y = x^4\log x.$

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0.$$

$$y_h = c_1x + c_2x^2 + c_3x^3.$$

$$W(y_1, y_2, y_3) = \det\begin{pmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{pmatrix}$$

$$= 2x^3$$

$$W_1 = \det\begin{pmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{pmatrix} = x^4.$$

$$W_2 = \det\begin{pmatrix} x & 0 & 3x^2 \\ 1 & 0 & 6x \\ 0 & 1 & 6 \end{pmatrix} = -2x^3.$$

$$W_3 = \det\begin{pmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{pmatrix} = x^2.$$

$r(x) = x\log x$ as the ODE is to be taken in the standard form.

$$\text{So, } y_p(x) = y_1(x)\int \frac{x^4}{2x^3}\,x\log x\,dx + y_2(x)\int \frac{-2x^3}{2x^3}\,x\log x\,dx + y_3(x)\int \frac{x^2}{2x^3}\,x\log x\,dx.$$

$$= \frac{x}{2}\int x^2\log x\,dx - x^2\int x\log x\,dx + \frac{x^3}{2}\int \log x\,dx.$$

$$= \frac{x}{2}\left(\frac{x^3}{3}\log x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x\log x - x).$$

Abel's formula

Let $y_1, \dots, y_n$ be n-many solutions of $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = 0$, $a_0, \dots, a_{n-1}$ are constants.

Then for any x_0 in the given domain,

$$W(y_1, \dots, y_n) = e^{-a_{n-1}(x-x_0)} W(y_1, \dots, y_n)(x_0).$$

The above result helps to determine l.I of solutions $y_1, \dots, y_n$ as calculating $W(y_1, \dots, y_n)(x_0)$ is at times easier than calculating $W(y_1, \dots, y_n)$.