Department of Mathematics

Indian Institute of Technology Delhi

MTL-101: Linear Algebra and Differential Equations - Minor II

Total marks: 20 Time: 60 Minutes

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(1) All the questions are compulsory.

(2) No marks will be provided for answers without proper justification.

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(1) Determine (with justification) whether the following statements are true or false. $[2 \times 2 = 4]$

(a) Let V be a finite dimensional vector space over $\mathbb{R}$. Let $T_1, T_2 : V \to V$ be linear transformations having the same characteristic polynomial. If T_1 is diagonalizable then T_2 is also diagonalizable.

(b) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. Assume that $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then $1 + \lambda + 2\lambda^2$ is an eigenvalue for $I + T + 2T^2$, where I denotes the identity transformation.

(2) Consider the initial value problem (IVP)

$$\frac{dy}{dx} = \cos x + y \text{ and } y(0) = 1.$$

(a) Solve the given IVP. [2]

(b) Using Picard's iteration method, find the first three iterates $y_1(x), y_2(x)$ and $y_3(x)$. [3]

(3) Define $f(x, y) = e^{-\cos(x^2+y^2)} \tan(xy)$ whenever $xy \neq (2n+1)\pi/2$ for $n \in \mathbb{Z}$. Consider the initial value problem (IVP): $\frac{dy}{dx} = f(x, y)$; $y(0) = 0$. $[2 \times 3 = 6]$

(a) Find a rectangle R such that the conditions of the "existence theorem" are satisfied.

(b) Find an $\alpha > 0$ such that there exists a solution in the interval $I = (-\alpha, \alpha)$.

(c) Does the given IVP have a unique solution?

(4) Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$. Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be a linear transformation given by

$$T(X) = BX \text{ for all } X \in M_2(\mathbb{R}).$$

Find the matrix $[T]_S$ of the transformation T with respect to the ordered basis [3]

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

(5) Consider the ordinary differential equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$. Let $\mu(x, y) = x$ and $\nu(x, y) = y$ be integrating factors of this ODE (i.e. the ODE becomes exact when we multiply by the integrating factor). Prove that the given ODE is separable. [2]