MTL101:: Tutorial 3 :: Linear Algebra

Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{P}_n := \{f \in \mathbb{F}[x] : \deg f < n\}$

(1) Find a basis of the row space of the following matrices: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 1 \\ 3 & 1 & 0 & 1 \end{pmatrix}^t$.

(2) For $i \in \{1, 2, \ldots, m\}$, define $p_i : \mathbb{F}^m \to \mathbb{F}$ by $p_i(x_1, x_2, \ldots, x_m) = x_i$ (the i-th projection).
  (a) Show that it is a linear transformation.
  (b) If $T : \mathbb{F}^m \to \mathbb{F}$ is a linear transformation then it is an $\mathbb{F}$-linear combination of the projections, that is, $T = a_1 p_1 + a_2 p_2 \cdots + a_m p_m$ for $a_1, \ldots, a_m \in \mathbb{F}$.
  (c) Further, show that $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation if and only if for each $i \in \{1, 2, \ldots, n\}$, the composition $S \circ p_i : \mathbb{F}^m \to \mathbb{F}$ is a linear transformation.
  (d) If $S : \mathbb{F}^m \to \mathbb{F}^n$ is a linear transformation then $S(x_1, x_2, \ldots, x_m) = (y_1, y_2, \ldots, y_n)$ where $y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m$ for $a_{ij} \in \mathbb{F}$ with $(1 \leq i, j \leq m)$.

(3) Find the rank and nullity of the following linear transformations. Also write a basis of the range space in each case.
  (a) $T : \mathbb{F}^3 \to \mathbb{F}^3$ defined by $T(x, y, z) = (x + y + z, x - y + z, x + z)$.
  (b) Assume that $0 \leq m \leq n$. $T : \mathbb{F}^n \to \mathbb{F}^m$ defined by $T(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_m)$.

(4) Write the matrix representations of the linear operators with respect to the ordered basis $\mathcal{B}$.
  (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x, y)$, $\mathcal{B} = \{(1, 1), (1, -1)\}$.
  (b) $\mathcal{D} : \mathcal{P}_{n+1} \to \mathcal{P}_{n+1}$ such that $\mathcal{D}(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$, $\mathcal{B} = \{1, x, \ldots, x^n\}$.
  (c) $T : M_2(\mathbb{F}) \to M_2(\mathbb{F})$, $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \longmapsto \begin{pmatrix} x + w & z \\ z + w & x \end{pmatrix}$, $\mathcal{B} = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$.

(5) Suppose $\dim V = \dim W < \infty$ and $T : V \to W$ is a linear transformation. Show that the following statements are equivalent
  (a) T is an isomorphism.
  (b) T is injective (i.e., one to one).
  (c) $\ker T = 0$.
  (d) T is surjective (i.e., onto).

(6) Suppose $m > n$. Justify the following statements:
  (a) There is no one to one (injective) $\mathbb{R}$-linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.
  (b) There is no onto (surjective) $\mathbb{R}$-linear tranformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

(7) Find the eigenvalues, eigenvectors and dimension of eigen-spaces of the following operators.
  (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (x + y, x)$,
  (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (y, x)$,
  (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (y, -x)$
  (d) $T : \mathbb{C}^2(\mathbb{C}) \to \mathbb{C}^2(\mathbb{C})$ with $T(x, y) = (y, -x)$.
  (e) $\mathbb{C}^n \to \mathbb{C}^n$ defined by $(x_1, x_2, \ldots, x_n) \mapsto (x_n, x_1, \ldots, x_{n-1})$.
  (f) $\mathbb{C}^2 \to \mathbb{C}^2$ defined by $(z_1, z_2) \mapsto (z_1 - 2z_2, z_1 + 2z_2)$.

(8) (To be discussed in the lecture class) Suppose $\{\lambda_1, \ldots, \lambda_t\}$ is the complete set of distinct eigenvalues of a linear operator $T : V \to V$. Denote by V_i the eigenspace of λ_i. If B_i is a basis of V_i then show that $\bigcup_{i=1}^{t} B_i$ is a basis of the sum $\sum_{i=1}^{t} V_i$. Conclude that it is a direct sum.

Further, show that T is diagonalizable if and only if $\sum_{i=1}^{t} V_i = V$. Equivalently, show that T is diagonalizable if and only if $\dim V = \dim V_1 + \dim V_2 + \cdots + \dim V_t$.

(9) Find a basis B such that $[T]_B$ is a diagonal matrix in case T is diagonalizable. Find P such that $[T]_B = P[T]_S P^{-1}$ where S is the standard basis in each case.
  (a) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (y, -x)$.
  (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (5x - 6y - 6z, -x + 4y + 2z, 3x - 6y - 4z)$.
  (c) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$.

(10) Characteristic polynomial of a matrix is satisfied by the matrix (Cayley-Hamilton). Use it to find (invertibility and) the inverse of the following linear operators.
i) $(x, y, z) \mapsto (x + y + z, x + z, -x + y)$.
ii) $(x, y, z) \mapsto (x, x + 2y, x + 2y + 3z)$.

(11) Which of the following is an inner product.
(a) $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1x_2 + y_1y_2 + 3$ on $\mathbb{R}^2$ over $\mathbb{R}$.
(b) $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1x_2 - y_1y_2$ on $\mathbb{R}^2$ over $\mathbb{R}$.
(c) $\langle (x_1, y_1), (x_2, y_2)\rangle = y_1(x_1 + 2x_2) + y_2(2x_1 + 5x_2)$ on $\mathbb{R}^2$ over $\mathbb{R}$.
(d) $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1x_2 + y_1y_2$ on $\mathbb{C}^2$ over $\mathbb{C}$.
(e) $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1\bar{x}_2 - y_1\bar{y}_2$ on $\mathbb{C}^2$ over $\mathbb{C}$.
(f) If $A, B \in M_n(\mathbb{C})$ define $\langle A, B\rangle = \text{Trace}(A\bar{B})$.
(g) Suppose $\mathcal{C}[0, 1]$ is the space of continuous complex valued functions on the interval $[0, 1]$ and for $f, g \in \mathcal{C}[0, 1]$, $\langle f, g\rangle := \int_0^1 f(t)\bar{g}(t)\, dt$.

(12) Suppose $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in M_2(\mathbb{R})$ is such that $a > 0$ and $det(A) = ad - b^2 > 0$. Show that $\langle X, Y\rangle = X^tAY$ is an inner product on $\mathbb{R}^2$.

(13) Suppose V is an inner product space. Define $||v|| := \sqrt{\langle v, v\rangle}$. Show the following statements.

(a) $||v|| = 0$ if and only if $v = 0$
(b) For $a \in F$, $||av|| = |a|||v||$.
(c) $||v + w|| \leq ||v|| + ||w||$.
(d) $|||v|| - ||w||| \leq ||v - w||$.
(e) $\langle v, w\rangle = 0 \iff ||v + w||^2 = ||v||^2 + ||w||^2$.

(14) Use standard inner product on $\mathbb{R}^2$ over $\mathbb{R}$ to prove the following statement: "A parallelogram is a rhombus if and only if its diagonals are perpendicular to each other."

(15) Find with respect to the standard inner product of $\mathbb{R}^3$, an orthonormal basis containing $(1, 1, 1)$.

(16) Find an orthonormal basis of $\mathcal{P}_3 := \{f(x) \in \mathbb{R}[x] : \deg f(x) < 3\}$ with respect to the inner product defined by $\langle f, g\rangle := \int_0^1 f(t)g(t)dt$.

(17) Suppose W is a subspace of the finite dimensional inner product space V. Define $W^\perp := \{v \in V : \langle w, v\rangle = 0 \ \forall\ w \in W\}$. Show the following statements.

(a) $W^\perp$ is a subspace of V.
(b) $W \cap W^\perp = 0$
(c) $V = W \oplus W^\perp$.
(d) $(W^\perp)^\perp = W$.

(18) Suppose $W = \{(x, y) \in \mathbb{R}^2 \ : \ x+y = 0\}$. Find the shortest distance of $(a, b) \in \mathbb{R}^2$ from W with respect to i) the standard inner product, ii) the inner product defined by $\langle (x_1, y_1), (x_2, y_2)\rangle = 2x_1x_2 + y_1y_2$.

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