Minor Exam

● Graded

Student

Har Ashish Arora

Total Points

30 / 30 pts

Question 1

Maximal Linear Independent

4 / 4 pts

- **+ 0 pts** Not done or completely incorrect
- ✔ **+ 4 pts** Completely Correct (for a general vector space V)
- **+ 3 pts** Took a vector in V and mentioned SU{v} is LD. But wrongly took the combination as infinite and then applied the correct approach. Deduct -1 to take linear combination as infinite but else approach and arguments are correct
- **+ 3 pts** Took a vector in V and mentioned SU{v} is LD. But took S to be a finite set and then applied the correct approach. Deduct -1 to take maximal LI set to be finite but else approach and arguments are correct
- **+ 3 pts** Took a vector in V and mentioned SU{v} is LD. It is not said that S is finite but also did not explain why v is a finite linear combination of vectors of S or scalar of v non zero . Deduct -1 to take linear combination as finite, but with no explanation on why finite, although the argument thereafter is correct
- **+ 2 pts** Took maximal LI set S to be finite only. Then added a few (one or more than one) vectors in S to make it LD and then showed one of these added vectors is a linear combination of vectors of S and other added elements. No explanation is given on the scalar attached to the added vector to be non-zero.
- **+ 2 pts** Took maximal LI set to be general. Added a vector v to it to make it LD, wrote v as infinite linear combination, and also not explain why scalar attached with v is non-zero
- **+ 2 pts** Took maxima LI as general S. Took span(S) not equal to V. Then argued that SU{v} is LI but the argument of LI and LD not properly used for general S.
- **+ 1 pt** Only mentioned that S is maximal LI so adding a vector to it makes it a dependent set in V. . But the rest is wrong or not justified. Here S may be finite or general S.
- **+ 0.5 pts** Only and only a basis definition is given or any one statement relevant to the question that is correct, but nothing else is done or explained or all is incorrect.

Question 2

Rank Nullity

3 / 3 pts

✔ **+ 3 pts** Correct

+ 0 pts Not correct/ not attempted

+ 1 pt Step-1: For correctly finding the T^2 or ToT

+ 1 pt Step-2: For correctly finding a basis for the null space/ range space

+ 1 pt Step-3: For correctly finding the nullity and rank either by using the rank-nullity theorem or by finding a basis for range space and null space

+ 0.5 pts For partially correct step-1

+ 0.5 pts For partially correct step-2

+ 0.5 pts For partially correct step-3

Question 3

Matrix of Linear Transformation

4 / 4 pts

✔ **+ 4 pts** Full Corect

+ 0 pts Incorrect/ Not Attempted

+ 1 pt For Partial Mark: If one of the co-ordinate vector is right and rest of the calculation is wrong.

+ 2 pts For Partial Mark: If two of the co-ordinate vector are right and rest of the calculation is wrong.

+ 3 pts For Partial Mark: If three co-ordinate vector are right and matrix representation is wrong.

+ 1 pt For Partial Mark: Only if Matrix representation is right and previous calculation is not showing/or wrong.

Question 4

Eigenvalue and Eigenvector

3 / 3 pts

+ 0 pts Incorrect Answer or if Didnot Attempt

✔ **+ 3 pts** Correct

+ 1.5 pts Calculated all the three correct eigenvalues: $\lambda = 2, 2+\sqrt{2}, 2-\sqrt{2}$

+ 0.5 pts Only one correct eigen value

+ 1 pt Only two correct eigen values

+ 1 pt Computed correct Eigen vector corresponding to eigenvalue that is integer, i.e., for $\lambda = 2.$

+ 0.5 pts Obtained **all the eigenvectors** , i.e., the eigen space correspoding to a integer eigen value

Question 5

Injective Linera transformation

4 / 4 pts

✔ **+ 3 pts** Correct part i)

✔ **+ 1 pt** part ii): correct with justification

+ 0.5 pts p1-i): Writing the conditions for showing linear independent of the set { T(v1), T(v2),..., T(vk)}

+ 0.5 pts Final answer with justification

+ 0.5 pts proper use of linearly independent property of {v1, v2, ..vk}

+ 0.5 pts part ii): correct final answer without justification

+ 1.5 pts Use linearity of T and Ker (T)={0} to show the linear combination of {v1, v2,..,vk} equals to zero

+ 0 pts incorrect/not attempted

+ 0 pts Click here to replace this description.

Question 6

Subspace Dimension

4 / 4 pts

✔ **+ 4 pts** Correct

+ 0.5 pts To compute the subspace $W_1 \cap W_2$ correctly.

+ 0.5 pts To compute dim($W_1 \cap W_2$).

+ 1 pt To compute the dimension of W_1.

+ 1 pt To compute the dimension of W_2.

+ 1 pt To show $W_1 + W_2 \neq R^5$.

+ 0 pts Not done or completely incorrect

Question 7

Prove or Disprove

8 / 8 pts

- ✔ **+ 8 pts** Correct
- **+ 0 pts** all incorrect
- ✔ **+ 2 pts** If (i) is correct.
- **+ 0.5 pts** If only decision for (i) is correct. False.
- ✔ **+ 2 pts** If (ii) is correct.
- **+ 0.5 pts** If only decision for (ii) is correct. TRUE
- ✔ **+ 2 pts** If (iii) is correct. Whether decision is correct or not the main idea is correct: $a_1Y_1 + a_2Y_2$ is a solution if $a_1 + a_2 = 1$. (The statement is false)
- **+ 0.5 pts** If only decision for (iii) is correct. False
- ✔ **+ 2 pts** ****If (iv) is correct. E.g., $(1,1,1,1) = (1,0,1,0) - (0,-1,0,-1)$. But their images fail to satisfy this equality: $(1,0,0,0) \neq (0,1,0,0) - (0,0,1,0)$.
- **+ 0.5 pts** If only decision for (iv) is correct.

DEPARTMENT OF MATHEMATICS, IIT DELHI
SEMESTER II 2024 – 25
MTL 101 (Linear Algebra and Differential Equations) - Minor Exam

Date: 27/02/2025 (Thursday) Time: 1:00 pm- 3:00 pm.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

Name :	HAR ASHISH ARORA	BLOCK LETTER ONLY	
Entry Number:	2024EE10904	Group:	13
Gradescope Id:	EE1240904	Lecture Hall:	408

Question 1: Prove that any maximal linearly independent set of vectors in a vector space, V (not necessarily finite dimensional), is a basis of V. [4]

Suppose we are given a set S of vectors in a vector space, which is maximally linearly independent. The two necessary and sufficient conditions are (i) S should be a linearly independent set
(ii) S should span the vector space V.

We are given that S is a linearly independent set (in fact it is a maximal linearly independent set). It is enough to prove that S spans the vector space V.

Claim : S spans the vector space V. ~~Pro~~
We will prove this by contradiction.
Proof :
Suppose S does not span the vector space V.
Then there exists a $v_0 \in V$ such that
$v_0 \in V$ but $v_0 \notin \text{span}(S)$.

If $v_0 \notin \text{span}(S)$, a new set $S' = S \cup \{v_0\}$ can be created such that all elements in S' are linearly independent.

(We can do this because if $\exists$ a vector ~~not belonging~~ to span of a set, the L.I. vectors spanning the set and the new vector are all linearly independent).

Thus S' is a linearly independent set with one more element in it than S, which was also a linearly independent set. However, S was a maximal linearly independent set. Thus S' cannot exist $\Rightarrow$ contradiction.

Thus, our assumption that S does not span the vector space V was incorrect $\Rightarrow$

S spans the vector space V. And S is a linearly independent set. $\Rightarrow$ S forms a basis of V.

In our proof, nowhere we assumed that V was finite dimensional.

$\Rightarrow$ $\boxed{\text{Any maximal linearly Independent set of vectors in a vector space } V \text{ (not necessarily finite dimensional) is a basis of } V.}$

Hence proved.

Question 2: Let $T : \mathbb{R}^4 \to \mathbb{R}^4$, be the linear transformation given by

$$T(x, y, z, w) = (x + y, z, w, w).$$

Compute the rank and nullity of T^2, where $T^2 = T \circ T$. [3]

We need $\text{rank}(T^2)$ & $\text{nullity}(T^2)$.

We will employ the rank nullity theorem to find ~~rank(T^2)~~ nullity (T^2).

First, find $T^2(x,y,z,w) = T(T(x,y,z,w))$

$= T(x+y, z, w, w)$

$= (x+y+z, w, w, w)$

We can write range $(T^2) = \{T^2(v) : v \in V = \mathbb{R}^4\}$

$= \{(x+y+z, w, w, w) : x, y, z, w \in \mathbb{R}\}$.

$= \{(x+y+z)(1,0,0,0) + w(0,1,1,1) : x, y, z, w \in \mathbb{R}\}$.

We can write $x+y+z = \alpha \in \mathbb{R}$ (say) and $w = \beta \in \mathbb{R}$ (say).

<u>NB:</u> α covers all reals since x, y, z cover all reals.

$\Rightarrow$ range $(T^2) = \{\alpha(1,0,0,0) + \beta(0,1,1,1) : \alpha, \beta \in \mathbb{R}\}$.

Thus, the vectors $(1,0,0,0)$ & $(0,1,1,1)$ span range(T^2). Also, $(1,0,0,0)$ & $(0,1,1,1)$ are L.I.:

$0 = a(1,0,0,0) + b(0,1,1,1) \Rightarrow (a,b,b,b) = (0,0,0,0)$

$\Rightarrow a = b = 0$. Thus the set $\beta = \{(1,0,0,0), (0,1,1,1)\}$ forms a basis of range(T^2).

[P.T.O.]

Hence, $\dim(\text{range}(T^2)) \overset{\xrightarrow{= \text{rank}(T^2)}}{=} \text{cardinality}(\beta) = 2$.

Also, by rank - nullity theorem:

$$\text{rank}(T^2) + \text{nullity}(T^2) = \dim(\overset{\to V, \text{ here}}{\mathbb{R}^4})$$

$\because T^2: \mathbb{R}^4 \to \mathbb{R}^4$.

$\Rightarrow 2 + \text{nullity}(T^2) = 4$

$\Rightarrow \text{nullity}(T^2) = 4 - 2 = 2$.

$\Rightarrow \boxed{\text{rank}(T^2) = \text{nullity}(T^2) = 2} \to$ (Ans)

<u>ALITER</u>: We could've found nullity (T^2) first.

$\ker(T) = \{(x,y,z,w) : (x+y+z, w, w, w) = 0\}$

$\Rightarrow \begin{array}{l} x+y+z+0w=0 \\ 0x+0y+0z+w=0 \end{array} \Rightarrow$ coefficient matrix for this homogenous system $= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$ pivots $= 2$

$\Rightarrow$ free variables $= 4 - 2 = 2 \Rightarrow \dim(\ker(T))$ = no. of free variables in spaces of $\ker(T) = 2$

$\Rightarrow \text{nullity}(T^2) = 2$

$\Rightarrow$ using rank - nullity theorem,

$\text{rank}(T^2) + \text{nullity}(T^2) = 4 \Rightarrow \text{rank}(T^2) = 2$.

Question 3: Let B and B' be the following standard ordered bases of $P_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$, respectively:

$$B = \{1, x, x^2\},$$

and

$$B' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, be the linear transformation given by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}.$$

Compute the matrix of linear transformation $[T]_B^{B'}$.

[4]

A matrix of a linear transformation is written as $[T]_B^{B'}$, where each j^{th} column represents $T(v_j)$, where $v_j \in B$, written in terms of elements w_i of B'.

So, $\bullet\; T(\underbrace{1}_{f(x)}) = \begin{pmatrix} f'(1)|_{x=0} & 2f(x)|_{x=1} \\ 0 & f''(x)|_{x=3} \end{pmatrix}$

$$= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \boxed{0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = T(1)}$$

$\bullet\; T(\underbrace{x}_{f(x)}) = \begin{pmatrix} f'(x)|_{x=0} & 2f(x)|_{x=1} \\ 0 & f''(x)|_{x=3} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow$

$$\boxed{T(x) = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

$\bullet\; T(\underbrace{x^2}_{f(x)}) = \begin{pmatrix} f'(x)|_{x=0} & 2f(x)|_{x=1} \\ 0 & f''(x)|_{x=3} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$

$$\Rightarrow \boxed{T(x^2) = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

Thus the matrix of the linear transformation

$[T]_B^{B'}$ is given by (arranging column-wise):

$$[T]_B^{B'} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \text{(Ans.)}$$

Question 4: Find the eigenvalues of the following 3×3 matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Also find eigenvectors corresponding to eigenvalues that are integers. [3]

We find eigen values by the corresponding characteristic polynomial of the matrix, $\det(xI - M) = 0$

$$\Rightarrow \det\begin{pmatrix} x-2 & 1 & 0 \\ +1 & x-2 & 1 \\ 0 & 1 & x-2 \end{pmatrix} = (x-2)\left((x-2)^2 - 1\right) - 1\left((x-2) - 0\right) + 0$$

$$= (x-2)^3 - (x-2) - (x-2)$$

$$\Rightarrow (x-2)^3 - 2(x-2) = 0$$

$$\Rightarrow (x-2)\left((x-2)^2 - 2\right) = 0$$

$$\cancel{(x-2)(x^2+4-4x-2) = 0 \Rightarrow (x-2)(}$$

either $\boxed{x=2 \quad \text{or} \quad x = 2 \pm \sqrt{2}}$.

Thus the eigenvalues are $\boxed{\lambda = 2,\ \lambda = 2+\sqrt{2},\ \lambda = 2-\sqrt{2}.}$

The eigenvalue which is an integer is $\lambda = 2$, so we only need to find the eigenvector for this eigenvalue (given in the question). Thus, eigenvector eq":

$Mx = \lambda x$. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$\Rightarrow (M - \lambda I)x = 0$

We need to solve ~~M-λ~~ the homogenous system $(M - \lambda I)x = 0$, consider here $M - \lambda I$ is the coefficient matrix. $M - \lambda I = M - 2I$:

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow[R_2 \to R_2]{R_3 \to R_3 - R_1} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to -R_2 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ → RRE form.

Pivot variables are encircled.

Clearly, x_3 is a free variable,

say $x_3 = t \in \mathbb{R}$. then

$x_1 + x_3 = 0 \Rightarrow x_1 = -t, \quad x_2 = 0$

$\Rightarrow$ the X which have eigenvalue $\lambda = 2$ are :

$$X = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \; t \in \mathbb{R}.$$

These are the eigenvectors corresponding to the eigenvalues which are integers.

Question 5: Let V and W be two finite dimensional vector spaces over the field $\mathbb{F}$ and $T : V \to W$ be a linear transformation.

i) Prove that, if $\operatorname{Ker}(T) = \{0\}$, then T sends linearly independent set to linearly independent set, i.e., if $\{v_1, \dots, v_k\}$ is linearly independent in V, then $\{T(v_1), \dots, T(v_k)\}$ is linearly independent in W.

ii) Is the above result true if we drop the condition $\operatorname{Ker}(T) = \{0\}$? Justify.

[3+1=4]

(i) Suppose (given that) $\{v_1, \dots, v_K\}$ is an L.I. set.

Suppose $\alpha_1 T(v_1) + \dots + \alpha_K T(v_K) = 0$, $\alpha_1, \dots, \alpha_K \in \mathbb{F}$.

~~Apply linear Transformation on both sides (note that $T(0) = 0$):~~

$T(\alpha_1 v_1 + \dots + \alpha_K v_K) = 0$ (using linearity)

~~$\Rightarrow \alpha_1 T(v_1) + \dots + \alpha_K T(v_K) = 0$ (using linearity)~~

$\Rightarrow \alpha_1 v_1 + \dots + \alpha_K v_K \in \operatorname{Ker}(T) = \{0\}$

$\Rightarrow \alpha_1 v_1 + \dots + \alpha_K v_K = 0 \Rightarrow \alpha_i = 0 \ \forall \ 1 \le i \le K$

since $\{v_1, \dots, v_K\}$ is L.I.

Thus the set $\{T(v_1), \dots, T(v_K)\}$ is L.I. in W.

(ii) The above result need not be true if we drop the condition $\operatorname{Ker}(T) = \{0\}$.

Suppose $\operatorname{Ker}(T)$ includes a non-zero element $v_0 \in V$.

Then $\alpha_1 v_1 + \dots + \alpha_K v_K$ may equal v_0

$\Rightarrow \alpha_1 v_1 + \dots + \alpha_K v_K - v_0 = 0$. However, this does not directly imply that $\alpha_i = 0 \ \forall \ 1 \le i \le K$. As an example, consider $V = \mathbb{R}^2$, $W = \mathbb{R}$, $T(x,y) = x+y$

$\operatorname{Ker}(T) = \{(x,y) : y+x = 0\} = \{(-y, y) : y \in \mathbb{R}\} \supset \{0\}$.

Choose 2 L.I. elements in V:

say ~~{0,1}~~ ~~{1,1} a~~ (3,2) & (2,3).

$T(3,2) = 5, \quad T(2,3) = 5$

Clearly, (3,2) & (2,3) are L.I.

But $\alpha T(3,2) + \beta T(2,3) = 0$

$\Rightarrow (\alpha+\beta) 5 = 0 \Rightarrow \alpha = -\beta$, (say $\alpha = 2, \beta = -2$).

not necessary that $\alpha = \beta = 0$.

Thus if $\ker(T) \neq \{0\}$, not necessary for the above condition to hold.

~~$T(x,y) = x+y$~~

~~$\ker(T) = \{(-y,y)\}$~~

~~3,2 & 2,3~~

Question 6 Let W_1 and W_2 be subspaces of $\mathbb{R}^5$ given by

$$W_1 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 + x_2 + 2x_3 = 0, \quad 2x_4 + x_5 = 0\}$$

and

$$W_2 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_2 + 2x_3 = 0, \quad x_1 + 2x_4 + x_5 = 0\}.$$

Find $\dim(W_1 \cap W_2)$. Is $W_1 + W_2 = \mathbb{R}^5$? Justify. [4]

We will find the general forms of W_1, W_2, $W_1 \cap W_2$ (rather we will find the number of free variables in space of W_1, W_2, $W_1 \cap W_2$; and no. of free variables of a space gives the dimension of that space).

To find elements of W_1: Solve the given system of equations in W_1 to obtain restraints & thus the general form of elements in W_1. We have

$$\cancel{x_1 + x_2 + 2x_3 = 0}$$

$$x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 = 0$$
$$0x_1 + 0x_2 + 0x_3 + 2x_4 + x_5 = 0$$

$\Rightarrow$ Coeff matrix for homogenous sys. of eq^{n}s:

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 \to \frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \end{pmatrix} \text{ (RRE)}$$

Pivots are encircled. $\Rightarrow$ no. of free variables $= 5 - 2 = 3$

$\Rightarrow \boxed{\dim(W_1) = 3}$

To find dim of W_2: Similarly for W_2:

$$0x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 = 0$$
$$x_1 + 0x_2 + 0x_3 + 2x_4 + x_5 = 0$$

$$\Rightarrow A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \end{pmatrix} \text{ (RRE)}.$$ Pivots are encircled. Free variables $= 5 - 2 = 3 \Rightarrow \boxed{\dim(W_2) = 3}$.

To find $\dim(W_1 \cap W_2)$:

We will have all of the constraints above, leading to 4 equations $\Rightarrow$ 4 rows in the coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 \end{pmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 0 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -2 & -1 \end{pmatrix} \xrightarrow[R_1 \to R_1 - R_3]{R_4 \to R_4 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -2 & -1 \end{pmatrix} \xrightarrow{R_4 \to R_4 + R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to (\text{RREF})$$

$\Rightarrow$ free variables are $5 - 3 = 2$.

$\Rightarrow \boxed{\dim(W_1 \cap W_2) = 2}$.

using $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$$= 3 + 3 - 2$$

$$= 6 - 2 = 4.$$

$\Rightarrow$ $W_1 + W_2$ has 4 basis elements in a basis, and $V = \mathbb{R}^5$ has 5 basis elements.

Clearly, $\boxed{W_1 + W_2 \neq V = \mathbb{R}^5}$, since $W_1 + W_2$ would need to have the same dimension as $\mathbb{R}^5$.

Hence proved.

Question 7 Prove (if true) or disprove (if false) the following statements.

i) If W_1, W_2 and W are subspaces of a vector space V such that

$$W_1 \oplus W = W_2 \oplus W,$$

then $W_1 = W_2$.

ii) The span of the set $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 4\}$ over $\mathbb{R}$ is $\mathbb{R}^3$.

iii) If Y_1 and Y_2 are solutions of the system of linear equations

$$AX = B, \text{ where } B \neq \mathbf{0},$$

then for $a_1 \neq 0 \neq a_2$, $a_1Y_1 + a_2Y_2$ is not a solution of $AX = B$.

iv) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\begin{aligned} T(1,1,1,1) &= (1,0,0,0), \\ T(1,0,1,0) &= (0,1,0,0), \\ T(0,-1,0,-1) &= (0,0,1,0), \\ T(0,0,0,1) &= (0,0,0,0). \end{aligned}$$

[2+2+2+2=8]

(i) It need not be necessary that $W_1 = W_2$. As a counterexample, consider $V = \mathbb{R}^2$. Also, consider the following subspaces of $V = \mathbb{R}^2$:

(a) $W = \{(x,0) : x \in \mathbb{R}\}$. (c) $W_2 = \{(z,z) : z \in \mathbb{R}\}$.

(b) $W_1 = \{(0,y) : y \in \mathbb{R}\}$

It is clear that $W \cap W_1 = \{0\} = W \cap W_2$.

Also, $W + W_1 = \{(x,y) : x,y \in \mathbb{R}\} = \mathbb{R}^2$.

And $W + W_2 = \{(x+z, z) : x,z \in \mathbb{R}\} = \mathbb{R}^2$

(we can let x vary freely to create the first coordinate, $x+z$).

So $W \oplus W_1 = W \oplus W_2 = \mathbb{R}^2$.

But $W_1 \neq W_2$. <u>Hence disproved</u>.

P.T.O.

(ii) We have $S := \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=4\}$.

We need to check if $\text{Span}(S)=\mathbb{R}^3$.

We know $\dim(\mathbb{R}^3)=3 \Rightarrow$ any basis of $\mathbb{R}^3$ has 3 linearly independent elements, which makes it a maximal L.I. set, which makes it a basis.

If we can select three linearly independent elements from S, it will be a maximal L.I. set. That would imply that those elements will form a basis of $\mathbb{R}^3$ which spans $\mathbb{R}^3$. The rest of the elements would be L.D. on these 3 elements.

Consider $\beta = \{(2,0,0),(0,2,0),(\sqrt{2},\sqrt{2},\;)\}$.
All elements of $\beta \in S$. This set is L.I. as well.

$$\alpha(2,0,0)+\beta(0,2,0)+\gamma(\sqrt{2},\sqrt{2},\;)=(0,0,0)$$

$$\Rightarrow (2\alpha+\sqrt{2}\gamma,\ 2\beta+\gamma\sqrt{2},\;)=(0,0,0)$$

$$\Rightarrow \gamma=0 \Rightarrow \beta=0,\ \alpha=0 \Rightarrow \beta \text{ is L.I. set}$$

which is maximal L.I. set of $\mathbb{R}^3 \Rightarrow$ forms a basis of $\mathbb{R}^3 \Rightarrow \text{span}(S)=\mathbb{R}^3$ $\because \beta\subset S$, and all other elements in S can be created by linear combinations of elements in β.

(iii) we have $AY_1 = B$, $AY_2 = B$. $(B \neq 0)$

$a_1, a_2 \neq 0$, $a_1, a_2 \in \mathbb{F}$, $\mathbb{F}$ is a field.

For $a_1 Y_1 + a_2 Y_2$ to be a solution,

$$A(a_1Y_1 + a_2Y_2) = B.$$

But $A(a_1Y_1 + a_2Y_2) = a_1 AY_1 + a_2 AY_2 = a_1 B + a_2 B = (a_1 + a_2) B$.

In the general case, $a_1Y_1 + a_2Y_2$ ~~are~~ is not a solution of $AX = B$. However, they can be solutions if $a_1 + a_2 = 1$. $\boxed{\text{PLEASE SEE NEXT PAGE FOR EXAMPLE / COUNTEREXAMPLE}}$

(iv) We have $(1,0,1,0) = (1,1,1,1) + (0,-1,0,-1)$

~~If T~~ Suppose T is a linear transformation satisfying these relations.

Then $T(1,0,1,0) = T((1,1,1,1) + (0,-1,0,-1))$

$$= T(1,1,1,1) + T(0,-1,0,-1)$$
$$= (1,0,0,0) + (0,0,1,0)$$
$$= (1,0,1,0)$$

But it is given that $T(1,0,1,0) = (0,1,0,0)$.

$\Rightarrow$ contradiction $\Rightarrow$ our assumption was wrong!

~~T cannot be a linear~~ there cannot exist

such a linear transformation since it violates the basic constraints of linearity.

~~Counterexample for (iii):~~

~~Consider~~ $\cancel{A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$

Example/Counterexample for (iii):

suppose $Y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $Y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$AY_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad AY_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Consider $a_1 = 1, a_2 = 1$.

$$\Rightarrow a_1Y_1 + a_2Y_2 = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A(a_1Y_1 + a_2Y_2) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} = B.$$

Thus, it $(a_1Y_1 + a_2Y_2)$ is not a solution of $AX = B$.