Question 1: (5Marks) Let $u, v \in \mathbb{R}^n$ be two fixed vectors such that $\langle u|v\rangle \neq 0$, where $\langle \cdot|\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = \langle x|v\rangle u$. Then
(a). Show that T is a linear operator on $\mathbb{R}^n$.
(b). Prove/disprove: T is diagonalizable.

Solution 1: (a) This follows by using the inner product property, for any $\alpha \in \mathbb{R}$ and $w, z \in \mathbb{R}^n$, we have $T(\alpha w + z) = \langle \alpha w + z|v\rangle u = \langle \alpha w|v\rangle u + \langle z|v\rangle u = \alpha T(w) + T(z)$.
(b) **(Prove)** First observe that $T(x) = \langle x|v\rangle u = 0$ for any vector x orthogonal to v, and $\langle u|v\rangle \neq 0$ implies that $u \neq 0$ and $v \neq 0$. Now extend v to a basis of $\mathbb{R}^n$, so let $S = \{v, u_2, \ldots, u_n\}$ be a basis of $\mathbb{R}^n$. Now apply Gram-Schmidt orthogonalization process on S to obtain an orthogonal set $\{v, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$. This implies that for each $i = 2, \ldots, n$, $\langle v_i|v\rangle = 0$ and we have $T(v_i) = \langle v_i|v\rangle u = 0 = 0 \cdot v_i$. This implies that $\lambda_1 = 0$ is an eigenvalue of T, and $v_2, \ldots, v_n$ are the corresponding linearly independent eigenvectors. This implies that $\dim(\ker(\lambda_1 I - T)) \geq n - 1$. Also note that $T(u) = \langle u|v\rangle u$, which implies $\lambda_2 = \langle u|v\rangle$ is a nonzero eigenvalue of T. This implies that $\dim(\ker(\lambda_2 I - T)) \geq 1$. Since eigenvectors corresponding to distinct eigenvalues are linearly independent, we have that $\dim(\ker(\lambda_1 I - T)) = n - 1$ and $\dim(\ker(\lambda_2 I - T)) = 1$. This shows that $\dim(\ker(\lambda_1 I - T)) + \dim(\ker(\lambda_2 I - T)) = \dim(\mathbb{R}^n)$. Hence T is diagonalizable.

Marking scheme:
- checking linearity **1 mark**
- finding both the eignevalues correctly **1 mark**
- finding the eigenspace corresponding to nonzero eigenvalue **1 mark**
- finding the eigenspace corresponding to zero eigenvalue **1 mark**
- diagonalizability check **1 mark**

Question: Let V be a vector space over $\mathbb{C}$ with $\dim(V) = 2$. Suppose $T : V \to V$ has eigenvalues 1 and 2. For all positive integer n, show that

$$T^n = 2^n(T - I) - (T - 2I).$$

Here I is the identity operator on V.

Solution-1:

Step 1 (2 mark): (For $T^2 - 3T + 2I = 0$).

Since $\dim V = 2$ and $T : V \to V$, the characteristic polynomial is of degree 2 (which is monic). The two eigenvalues of T are 1 and 2 which are the roots of the characteristic polynomial implies that the characteristic polynomial must be [1 mark]

$$(X - 1)(X - 2) = X^2 - 3X + 2.$$

By Cayley-Hamilton theorem we have [1 mark]

$$T^2 - 3T + 2I = 0.$$

In particular, we have

$$T^2 - T = 2(T - I) \ \text{ and } \ T^2 - 2T = T - 2I. \tag{A}$$

Step 2 (1 mark):

Now we use principle of mathematical induction on n to prove the statement

$$P(n) \quad : \quad T^n = 2^n(T - I) - (T - 2I).$$

Base case $n = 1$:

$$RHS = 2^1(T - I) - (T - 2I) = 2T - 2I - T + 2I = T = LHS.$$

Therefore $P(1)$ is true.

Step 3 (2 mark):

Induction Step: If $P(k)$ is true the $P(k + 1)$ is true.

To verify the induction step, we assume that $P(k)$ is true for $k \geq 1$ ($k > 1$ is not correct), i.e.

$$T^k = 2^k(T - I) - (T - 2I) \qquad \text{(induction hypothesis).}$$

Claim: $P(k + 1)$ is true.

$$\begin{aligned} T^{k+1} &= T \cdot T^k \\ &= T \cdot \big(2^k(T - I) - (T - 2I)\big) \quad \text{(by induction hypothesis)} \\ &= 2^k T(T - I) - T(T - 2I) \\ &= 2^k(T^2 - T) - (T^2 - 2T) \\ &= 2^k\, 2(T - I) - (T - 2I) \qquad \text{(by (A) above)} \\ &= 2^{k+1}(T - I) - (T - 2I). \end{aligned}$$

This proves the claim and thus we have verified the induction step for $P(n)$.

Therefore $P(n)$ holds for all positive integers n.

Solution-2:
((1 Mark) Let v, w be eigenvectors of T corresponding to the eigenvalues 1 and 2 respectively. Since v and w are the eigenvectors for distinct eigenvalues the set $B = \{v, w\}$ is linearly independent. Since $\dim V = 2$, B is an ordered basis of V and we get

$$[T]_B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(1 Mark) Note that after fixing a basis the matrix corresponding to a linear transformation is unique. Then $T^n = 2^n(T - I) - (T - 2I)$ is equivalent

$$[T^n]_B = [2^n(T - I) - (T - 2I)]_B. \tag{A}$$

(2 Mark) Use induction and prove that $[T^n]_B = [T]_B^n = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$ for all positive integers.

(1 Mark) Compute separately the RHS and prove that

$$[2^n(T - I) - (T - 2I)]_B = 2^n([T]_B - [I]_B) - ([T]_B - 2[I]_B) = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}.$$

Solution-3:
(1 Mark) Let v_1, v_2 be eigenvectors of T corresponding to the eigenvalues 1 and 2 respectively. We get that $B = \{v_1, v_2\}$ is a basis of V (see Solution-2).
(1 Mark) Since we want to prove the identity $T^n = 2^n(T - I) - (T - 2I)$, it is enough to prove that for all $v \in V$ we have

$$T^n(v) = (2^n(T - I) - (T - 2I))(v).$$

For any $v \in V$ there exists $a, b \in \mathbb{C}$ such that $v = av_1 + bv_2$.
(2 Mark) Use induction and prove $T^n(v) = T^n(av_1 + bv_2) = aT^n(v_1) + bT^n(v_2) = av_1 + 2^n bv_2$.
(1 Mark) Further prove by direct computation $(2^n(T - I) - (T - 2I))(v) = av_1 + 2^n bv_2$.

Solution-4:
(1 mark) Let v_1, v_2 be eigenvectors of T corresponding to the eigenvalues 1 and 2 respectively. We get that $B = \{v_1, v_2\}$ a basis of V (see Solution-2).
(1 Mark) Verify the following two relation

$$T^2(v_1) = T(T(v_1)) = T(v_1) = v_1 = 3v_1 - 2v_1 = 3T(v_1) - I(v_1) = (3T - 2I)(v_1). \tag{1}$$

$$T^2(v_2) = T(T(v_2)) = T(2v_2) = 4v_2 = 6v_2 - 2v_2 = 3T(v_2) - 2I(v_2) = (3T - 2I)(v_2). \tag{2}$$

Since B is a basis, we get $T^2(v) = (3T - 2I)(v)$ for all $v \in V$, ie. $T^2 = 3T - 2I$.
(3 Marks) Now complete the proof by induction.

Solution-5:

(1 Mark) Let v, w be eigenvectors of T corresponding to the eigenvalues 1 and 2 respectively. We get that $B = \{v, w\}$ is linearly independent, because v and w are the eigenvectors for distinct eigenvalues. Since $\dim V = 2$, B is an ordered basis of V and we get

$$[T]_B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

(1 Mark) We want to prove $T^n = 2^n(T - I) - (T - 2I)$. It is enough to prove that

$$[T^n]_B = [2^n(T - I) - (T - 2I)]_B. \tag{A}$$

(1 Mark) Write $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then

$$[T]_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = I + B.$$

Use induction to prove $B^n = B$ for all integers $n \geq 1$.

(2 Mark) Since I and B commute, using binomial theorem for $n \geq 1$ we have

$$\begin{aligned} [T]_B^n = (I + B)^n &= {}^nC_0 I +^n C_1 B + \cdots +^n C_n B^n \\ &= I + ({}^nC_1 B + \cdots +^n C_n B) \\ &= I + ({}^nC_1 + \cdots +^n C_n) B \\ &= I + (2^n - 1)B \\ &= I + (2^n - 1)([T]_B - I) \\ &= 2^n([T]_B - I) - ([T]_B - I) + I \\ &= 2^n([T]_B - I) - ([T]_B - 2I). \end{aligned}$$

Problem. Prove that the set $\{I, A, \ldots, A^n\}$ can not be a part of basis of $M_n(\mathbb{R})$.

Proof:- Let $0 \neq A \in M_n(\mathbb{R})$ and let $f(x)$ be the char poly of A.

$$f(x) = \det(xI - A)$$
$$= x^n + a_1 x^{n-1} + \cdots + a_n \quad \text{--- (1)}$$

Now by Cayley Hamilton theorem we have $f(A) = 0$

$$\Rightarrow A^n + a_1 A^{n-1} + \cdots + a_n I = 0 \quad \text{--- (2)}$$

From above eq^n we can see

that the elements $I, A, A^2, \ldots A^n$ are L.D as all the coefficients are not identically zero. Indeed the coefficient of A^n is 1. — (2)

Hence the set $\{I, A, \ldots, A^n\}$ can not be a part of a basis.

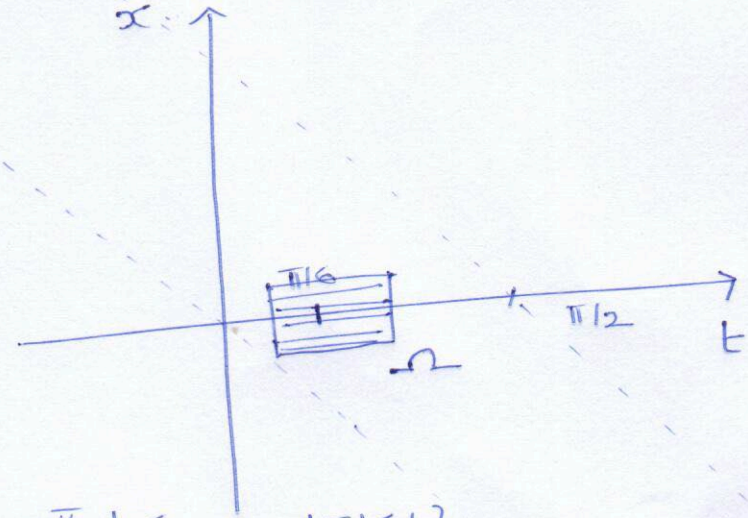
4

(a) Here
$$\left.\begin{aligned} f(t, x) &= t^2 + \sqrt{\tan(t+x)} \\ (t_0, x_0) &= (\pi/6, 0) \end{aligned}\right\}$$

For $f = f(t, x)$ to be a well-defined continuous function, we need $\tan(\cdot) \geq 0$.

Since we need a "region" containing $(\pi/6, 0)$, consider $0 < t + x < \frac{\pi}{2}$

Choose $a > 0$, $b > 0$ sufficiently small so that the closed and bounded "rectangle"



$$\Omega := \left\{ (t, x) : \left| t - \tfrac{\pi}{6} \right| \leq a, \ |x| \leq b \right\}$$

lies in $0 < t + x < \pi/2$.

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{\tan(t+x)}} \sec^2(t+x)$$

Note that $\frac{\partial f}{\partial x}$ is continuous on Ω, and hence bounded. Consequently, f is Lipschitz in x-variable on Ω.

By the existence and uniqueness theorem, the given IVP has a unique solution $\phi(t)$ in an interval $|t - \pi/6| \leq \alpha$ for a suitable α.

$\square$

4

(b) If $x = x(t)$ is a continuous function, then given conditions imply $x \equiv 0$.

Let x be a continuous nonnegative function.

$$u(t) := \int_{t_0}^{t} x(s)\, ds.$$

By the fundamental theorem of calculus, u is differentiable (Here we use the assumption, x is continuous).

$$\frac{du}{dt} = x(t) \leq L \int_{t_0}^{t} x(s)\, ds = Lu(t$$

ie,
$$\frac{du}{dt} - L\,u(t) \leq 0.$$

This implies,

$$\frac{d}{dt}\left(u \cdot e^{-L(t-t_0)} \right) \leq 0.$$

consequently,

$$e^{-L(t-t_0)}\, u(t) \leq u(t_0) = 0.$$

ie
$$u(t) \leq 0 \quad \forall \quad t \geq t_0.$$

This implies $u(t) = 0$ and hence $x \equiv 0$ ($\because$ x being nonnegative continuous fn

If the additional assumption of continu is not there, then it is not true, in gene that $x \equiv 0$.; give a counter example.

$\square$

Q5

$f_1, f_2 \in C'(I)$

Given $W(f_1, f_2)(t) = 0 \quad \forall \ t \in I$

To show that there exists an interval $I_0 \subseteq I$ s.t. f_1 & f_2 are linearly dependent on I_0.

If either f_1 or f_2 is identically zero on I,

say $f_1(t) = 0 \quad \forall \ t \in I$

1 mark

then $c_1 f_1(t) + c_2 f_2(t) = 0 \quad \forall t \in I$ holds for any $c_1 \in \mathbb{R}/\{0\}$

& $c_2 = 0$.

$\Rightarrow$ f_1 & f_2 are linearly dependent on I.

4 marks

If f_1 is not identically zero on I

$\Rightarrow \ \exists \ t_0 \in I$ such that $f_1(t_0) \neq 0$

w.l.o.g suppose $f_1(t_0) > 0$.

Since f_1 is continuous on I

$\Rightarrow \ \exists$ an interval $I_0 \subseteq I$ s.t. $f_1(t) > 0 \ \forall \ t \in I_0$

Now $W(f_1, f_2)(t) = 0 \quad \forall \ t \in I$

$\Rightarrow W(f_1, f_2)(t) = 0 \quad \forall \ t \in I_0$

$f_1(t) f_2'(t) - f_2(t) f_1'(t) = 0 \quad \forall \ t \in I_0$

Upon dividing by f_1^2 (Keeping in mind $f_1(t) > 0 \ \forall t \in I_0$)

we get $\dfrac{f_1(t) f_2'(t) - f_2(t) f_1'(t)}{f_1^2(t)} = 0 \quad \forall \ t \in I_0$

$\Rightarrow \left(\dfrac{f_2(t)}{f(t)}\right)' = 0 \quad \forall t \in I_0 \quad \Rightarrow \quad f_2(t) = c f_1(t) \ \forall t \in I_0$, $c \in \mathbb{R}$

$\Rightarrow f_1$ & f_2 are linearly independent on I_0.

MTL 101 SAMPLE QUESTIONS

For $\alpha \in \mathbb{R}$, let

$$L(x(t)) = t^3x''' + 3(1-\alpha)t^2x'' + (3\alpha^2 - 3\alpha + 1)t\alpha' - \alpha^3 x, t > 0 \tag{1}$$

Reduction of (1) *into DE with constant coefficients:* Set $t = e^s$, $\frac{dt}{ds} = e^s$, $\frac{d^2t}{ds^2} = e^s$. Now we have

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = x't$$
$$\frac{d^2x}{ds^2} = \frac{d^2x}{dt^2}\left(\frac{dt}{ds}\right)^2 + x't = x''t^2 + x't$$

The above two equations leads to $t^2x'' = \frac{d^2x}{ds^2} - \frac{dx}{ds}$. Now

$$\frac{d^3x}{ds^3} = x'''t^3 + 3x''t^2 + x't$$
$$t^3x''' = \frac{d^3x}{ds^3} - 3\left[\frac{d^2x}{ds^3} - \frac{dx}{ds}\right] - \frac{dx}{ds}$$
$$= \frac{d^3x}{ds^3} - 3\frac{d^2x}{ds^3} + 2\frac{dx}{ds}$$

Therefore $L(x(t)) = 0$ reduces to

$$\frac{d^3x}{ds^3} - 3\frac{d^2s}{dx^2} + 2\frac{dx}{ds} + 3(1-\alpha)\left[\frac{d^2x}{ds^2} - \frac{dx}{ds}\right] + (3\alpha^2 - 3\alpha + 1)\frac{dx}{ds} - \alpha^3 x = 0$$

Further simplication leads to $\mathbf{\frac{d^3x}{ds^3} - 3\alpha\frac{d^2x}{ds^2} + 3\alpha^2\frac{dx}{ds} - \alpha^3x = 0}$ (**1 mark**).

General solution to DE: Let $x = e^{ms}$. The characteristic polynomial is $m^3 - 3\alpha m^2 + 3\alpha^2 m - \alpha^3 = 0$ or $(m-\alpha)^3 = 0$. The roots of this equation are $m = \alpha, \alpha, \alpha$. Thus the general solution is $x(s) = c_1x_1(s) + c_2x_2(s) + c_3x_3(s)$ where $x_1(s)e^{\alpha s}, x_2(s) = se^{\alpha s}, x_3(s) = s^2e^{\alpha s}$. Thus the general solution is

$$x(s) = c_1e^{\alpha s} + c_2se^{\alpha s} + c_3s^2e^{\alpha s}$$
$$x(t) = c_1t^\alpha + c_2t^\alpha ln(t) + c_3t^\alpha(ln(t))^2 (\textbf{1 mark})$$

General solution of $L(x(t)) = 1$ *for* $\alpha = 1$: It suffices to find the general solution of $M(x(s)) = \frac{d^3x}{ds^3} - 3\frac{d^2x}{ds^2} + 3\frac{dx}{ds} - x = 1$ (with $t = e^s$). The general solution to homogeneous DE $M(x(s)) = 0$ is $x_h(s) = c_1x_1(s) + c_2x_2(s) + c_3x_3(s)$ where $x_1(s)e^{\alpha s} = e^s, x_2(s) = se^{\alpha s} = se^s, x_3(s) = s^2e^{\alpha s} = s^2e^s$.

We try the particular solution as

$$x_p(s) = u_1(s)x_1(s) + u_2(s)x_2(s) + u_3(s)x_3(s)$$
$$x_p'(s) = u_1(s)x_1'(s) + u_2(s)x_2'(s) + u_3(s)x_3'(s) + u_1'(s)x_1(s) + u_2'(s)x_2(s) + u_3'(s)x_3(s)$$

Set

$$u_1'(s)x_1(s) + u_2'(s)x_2(s) + u_3'(s)x_3(s) = 0 \tag{2}$$

Now

$$x_p''(s) = u_1(s)x_1''(s) + u_2(s)x_2''(s) + u_3(s)x_3''(s) + u_1'(s)x_1'(s) + u_2'(s)x_2'(s) + u_3'(s)x_3'(s)$$

Set

$$u_1'(s)x_1'(s) + u_2'(s)x_2'(s) + u_3'(s)x_3'(s) = 0 \tag{3}$$

Then

$$x_p'''(s) = u_1(s)x_1'''(s) + u_2(s)x_2'''(s) + u_3(s)x_3'''(s) + u_1'(s)x_1''(s) + u_2'(s)x_2''(s) + u_3'(s)x_3''(s)$$

Upon subtituting $x_p(s)$ in $M(x(s)) = 1$ and simplifying we get

$$u_1'(s)x_1''(s) + u_2'(s)x_2''(s) + u_3'(s)x_3''(s) = 1 \tag{4}$$

Thus from (2),(3),(4)we have

$$u_1'(s)e^s + u_2'(s)se^s + u_3'(s)s^2e^s = 0$$
$$u_1'(s)e^s + u_2'(s)(s+1)e^s + u_3'(s)(2s+s^2)e^s = 0$$
$$u_1'(s)e^s + u_2'(s)(s+2)e^s + u_3'(s)(s^2+4s+2)e^s = 1$$

Using Cramer's rule we get

$$W(x_1,x_2,x_3) = \begin{vmatrix} e^s & se^s & s^2e^s \\ e^s & (s+1)e^s & (2s+s^2)e^s \\ e^s & (s+2)e^s & (s^2+4s+2)e^s \end{vmatrix} = 2e^{3s}$$

$$u_1'(s) = \frac{\begin{vmatrix} 0 & se^s & s^2e^s \\ 0 & (s+1)e^s & (s^2+2s)e^s \\ 1 & (s+2)e^s & (s^2+4s+2)e^s \end{vmatrix}}{W(x_1,x_2,x_3)} = \frac{s^2e^{2s}}{2e^{3s}} = \frac{1}{2}s^2e^{-s}$$

$$u_2'(s) = \frac{\begin{vmatrix} e^s & 0 & s^2e^s \\ e^s & 0 & (s^2+2s)e^s \\ e^s & 1 & (s^2+4s+2)e^s \end{vmatrix}}{W(x_1,x_2,x_3)} = \frac{-2se^{2s}}{2e^{3s}} = -se^{-s}$$

$$u_3'(s) = \frac{\begin{vmatrix} e^s & se^s & 0 \\ e^s & (s+1)e^s & 0 \\ e^s & (s+2)e^s & 1 \end{vmatrix}}{W(x_1,x_2,x_3)} = \frac{e^{2s}}{2e^{3s}} = \frac{1}{2}e^{-s}$$

Thus we have

$$u_1(s) = 1/2\int s^2e^{-s} = \frac{-s^2}{2}e^{-s} - se^{-s} - e^{-s}$$
$$u_2(s) = se^{-s} + e^{-s}$$
$$u_3(s) = \frac{-1}{2}e^{-s}$$

Thus the particular solution is

$$\begin{aligned} x_p(s) &= u_1x_1 + u_2x_2 + u_3 + x_3 \\ &= e^s\left(\frac{-s^2}{2}e^{-s} - se^{-s} - e^{-s}\right) + (se^{-s} + e^{-s})se^s + \left(\frac{-1}{2}e^{-s}\right)s^2e^s \\ &= -\frac{-s^2}{2} - s - 1 + s^2 + s - \frac{-s^2}{2} = -1(\textbf{ 3 marks}) \end{aligned}$$

The general solution of $M(x(s)) = 1$ is $x_h(s) + x_p(s)$. The general solution of $L(x(t)) = 1$ with $\alpha = 1$ is

$$c_1t + c_2t\ln t + c_3t(\ln t)^2 - 1$$

Q7

$$A = \begin{bmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{bmatrix} \qquad \det(A - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & 12 & -2 \\ -3 & -4-\lambda & 1 \\ -1 & -2 & 2-\lambda \end{bmatrix}\right)$$

$$= (8-\lambda)\left[(-4-\lambda)(2-\lambda) + 2\right] - 12\left[-3(2-\lambda) + 1\right] - 2\left[6 - (4+\lambda)\right]$$

$$= 0 \implies -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = 0$$

$$\implies \lambda = 2, 2, 2 \qquad 1$$

First eigenvector $(A - 2I)\vec{u} = 0$

$$\implies \begin{bmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$6u_1 + 12u_2 - 2u_3 = 0$

$\implies -u_1 - 2u_2 = 0$

$\implies -2u_3 = 0 \implies u_3 = 0$

$u_1 = -2u_2$

take $u_2 = t = 1$

$$\implies \vec{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{x}^{(1)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{2t} \qquad 1$$

$$(A-2I)\,\vec{u} = \vec{u} \Rightarrow \begin{bmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad u_1 = -2u_2 \Rightarrow u_3 = 1 \Rightarrow \vec{u} = \begin{bmatrix} -2k \\ k \\ 1 \end{bmatrix} \quad \text{taking } k=1$$

we get $\vec{u} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{x}^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} t e^{2t} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{2t} \qquad (1)$$

$$[A-2I](\vec{w}) = \vec{u} \Rightarrow \begin{bmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad w_1 = -2w_2 - 1, \quad 3 + w_3 = 1 \Rightarrow w_3 = -4$$

$\Rightarrow$ taking $w_2 = 1$ we get $\vec{w} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$

$$\vec{x}^{(3)}(t) = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} -3 \\ 1 \\ -4 \end{pmatrix} e^{2t} \quad (1)$$

General Solution

$$\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + c_3 \vec{x}^{(3)}$$

$$= c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} e^{2t} \right]$$

$$+ c_3 \left[\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} -3 \\ 1 \\ -4 \end{pmatrix} e^{2t} \right] \quad (1)$$

No Details = No Marks

(8) $y'' + xy' + y = 0$

Let $y = \sum c_k x^k$ Then $y' = \sum k c_k x^{k-1}$ and

$$y'' = \sum k(k-1) c_k x^{k-2}$$

Substituting in the Equation we get

$$\sum \left[(k+2)(k+1) c_{k+2} + k c_k + c_k\right] x^k = 0$$

$$\therefore \quad (k+2)(k+1) c_{k+2} + (k+1) c_k = 0$$

$\Rightarrow$ The recurrence relation

$$c_{k+2} = -\frac{c_k}{(k+2)} \qquad k = 0, 1, 2, \ldots$$

(2 marks)

Therefore,

$$c_2 = \frac{-c_0}{2}, \quad c_3 = \frac{-c_1}{3}$$

$$c_4 = -\frac{c_2}{4} = \frac{c_0}{4\cdot 2} = \frac{c_0}{2^2\, 2!}\ ; \quad c_5 = \frac{-c_3}{5} = \frac{c_1}{3\times 5}.$$

$$c_6 = -\frac{c_0}{6\cdot 4\cdot 2} = -\frac{c_0}{2^3\, 3!}$$ 1 mark

Hence $$c_{2m} = \frac{(-1)^m c_0}{2^m m!}, \quad c_{2m+1} = \frac{(-1)^n c_1}{3\cdot 5\cdot 7\cdots 2m+1}$$

$$\phi_1 = \sum c_{2m} x^{2m} + \sum c_{2m+1} x^{2m+1}$$

is the general solution. (2 marks)