COL202: Discrete Mathematical Structures. I semester, 2022-23.
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Tutorial Sheet 9: Counting.
27 October 2022

Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

Problem 1
Suppose you have n men and n women and you have to seat them around a circular table so that men and women sit alternately. How many ways can you seat them?

Problem 2
Suppose there are n tennis players. How many ways can we pair them up so that everyone has a partner to play a singles match?

Problem 3
Given a set A of size m and set B of size n count the number of

1. Relations from A to B.
2. Total functions from A to B.
3. Partial functions from A to B.
4. Surjections from A to B (assume $m \geq n$). These could be partial or total functions.
5. Injections from A to B. These could be partial or total. State whatever you assume about m and n.

Problem 4 (Problem 15.7 of [1])
For integers $n, k \geq 0$ let $S_{n,k}$ be the set of non-negative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n.$$

Problem 4.1
Give a bijection between $S_{n,k}$ and the set of binary strings with n zeroes and k ones. What is $|S_{n,k}|$?

Problem 4.2 ♠
For integers $n, k, \geq 0$, let

$$L_{n,k} := \{(y_1, \ldots, y_k) \in \mathbb{N}^k : 0 \leq y_1 \leq y_2 \leq \cdots \leq y_k \leq n\}.$$

Give a bijection from $S_{n,k}$ to $L_{n,k}$.

Problem 5
Solve problem 15.8 of [1]. From there conclude that the number of trees with vertex set $[n]$ is n^{n-2}.

Problem 6
Given a plane with integer points of the type (x, y) where both x and y are integers, we define a *lattice path* from (x_1, y_1) to (x_2, y_2) to be a set of line segments that go from a point (i, j) to $(i + 1, j)$ or $(i, j + 1)$, i.e., all steps in the path either move right or up. Count the number of lattice paths between $(0, 0)$ and (m, n)?

Problem 7
We say that a function π is a *derangement of size* n if it is a bijection from $\{1, \ldots, n\}$ to itself (i.e., it is a permutation) and it has no fixed points, i.e., $\forall i : \pi(i) \neq i$. Count the number of derangements of size n. The solution you submit must use Inclusion-Exclusion. Separately also try to see if you can solve the problem without the use of Inclusion-Exclusion.

Problem 8
Prove the following identities regarding binomial coefficients by making counting arguments. Give as many different arguments as possible

Problem 8.1

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

Problem 8.2

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}.$$

Problem 8.3

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}.$$

Problem 9 (Problem 15.47 of [1])
Suppose $n+1$ numbers are selected from $\{1, 2, \ldots, 2n\}$ show using the Pigeonhole Principle that there must be two selected numbers whose quotient is a power of two.

Problem 10 (Problem 15.50 of [1])
Suppose $2n+1$ elements are selected from $[4n]$, use the Pigeonhole Principle to show that for every positive j that divides $2n$ there must be two selected numbers whose difference is j.

Problem 11
Solve Problem 15.51 of [1].

References

[1] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science June 2018, MIT Open Courseware.