Systems of differential equations

We consider the system of diffeential equations

$$
\begin{aligned}
x_1'(t) &= f_1(t, x_1(t), x_2(t), ...x_n(t)) \\
x_2'(t) &= f_1(t, x_1(t), x_2(t), ...x_n(t)) \\
&\vdots \\
x_n'(t) &= f_1(t, x_1(t), x_2(t), ...x_n(t))
\end{aligned}
$$

A solution of this system is a vector valued function $x(t) : [a, b] \to \mathbb{R}^n$ deonted by $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$. We assume that $f = (f_1, f_2, \cdots, f_n)$ is continuous function in its variables t and x. We define norm (the distance of x from 0) of a vector x as

$$\|x\| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$$

Definition 1. *A vector valued function $f(t, x)$ is said to be Lipschitz continuous in x if there exists constant L such that*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \ \forall x, y, \ \forall t.$$

We have the following existence and uniqueness theorem is known as Picard's theorem:

Theorem 1. *Suppose $f(t, x)$ is Lipschitz continuous in an open set around (t_0, x_0). Then the following IVP for the system*

$$x'(t) = f(t, x), \ x(t_0) = x_0$$

admits unique solution in a neighborhood of (t_0, x_0).

Solving this IVP is equivalent to solving the following integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

From this we can define the Picard iteration:

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))dx, \ x_0(t) = x_0, \ n = 1, 2, ...$$

1 Theory of Linear systems

In this section, we study the linearly independent solutions and the dimension of the solution space of linear systems of differential equations

$$X'(t) = AX(t)$$

where $X(t) = (x_1(t), x_2(x), ..., x_n(t))^T$ and A is a $n \times n$ matrix with elements $a_{ij}(t), i, j = 1, 2...n$ are continuous functions. A solution of this system is a vector valued function $x(t) : [a, b] \to \mathbb{R}^n$, which we denote with $x(t) = (x_1(t), x_2(x), \cdots, x_n(t))^T$. Let us recall the the second order equation: Let x and y be solutions of $x'' + a_1x' + a_2x = 0$. Then the Wronskian of x, y is

$$W(x, y)(t) = \begin{vmatrix} x & y \\ x' & y' \end{vmatrix}$$

Now let us convert this equation into first order system by defining

$$x_1 = x,\ x_2 = x_1'$$

Then The second order equation become the first order system

$$\begin{aligned} &x_1' = x_2 \\ &x_2' = x_1'' = x'' = -a_1x_2 - a_2x_1. \end{aligned}$$

So in the new variables the Wronskian becomes

$$W(x, y) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Motivated from above, we define

Definition 2. *The Wronskian of n-vector valued functions, $x^1(t), x^2(t), ..., x^n(t) : [a, b] \to \mathbb{R}^n$, is defined as*

$$W(x^1, x^2, ..., x^n)(t) = \begin{vmatrix} x_1^1(t) & x_1^2(t) & ... & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & ... & x_2^n(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n^1(t) & x_n^2(t) & ... & x_n^n(t) \end{vmatrix}$$

where $x^i(t) = \left(x_1^i(t), x_2^i(t), ..., x_n^i(t)\right)^T$ for $i = 1, \cdots, n$.

Definition 3. *The vector valued functions,* $x^1(t), x^2(t), \cdots, x^n(t) : [a,b] \to \mathbb{R}^n$*, are linearly dependent if there exists* $c_1, c_2, \cdots, c_n$ *(not all zero) such that*

$$c_1 x^1(t) + c_2 x^2(t) + \cdots + c_n x^n(t) = 0.$$

That is, the following system of equations has non-trivial solution

$$\begin{pmatrix} x_1^1(t) & x_1^2(t) & \dots & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & \dots & x_2^n(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n^1(t) & x_n^2(t) & \dots & x_n^n(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.1}$$

Then as an immediate consequence, we have the following

Theorem 2. *The vector valued functions,* $x^1(t), x^2(t), \cdots, x^n(t) : [a,b] \to \mathbb{R}^n$*, are linearly dependent then* $W(x^1, x^2, \cdots, x^n)(t) = 0$ *for all* t.

However the converse is not true.

Theorem 3. *Abel's formula:* $x^1(t), x^2(t), ..., x^n(t) : [a,b] \to \mathbb{R}^n$ *be solutions of* $X' = A(t)X$*. Then their Wronskian is given by*

$$W(t) = Cexp\left(\int_{t_0}^{t} (Tr(A(s)))ds\right)$$

Proof. We give the proof for $n = 2$. In this case $x^i, i = 1, 2$ satisfies the system

$$\begin{pmatrix} (x_1^i)' \\ (x_2^i)' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix}.$$

$$\begin{aligned} \frac{d}{dt}W(t) = & \begin{vmatrix} (x_1^1)' & (x_1^2)' \\ x_2^1 & x_2^2 \end{vmatrix} + \begin{vmatrix} x_1^1 & x_1^2 \\ (x_2^1)' & (x_2^2)' \end{vmatrix} \\ & \begin{vmatrix} a_{11}x_1^1 + a_{12}x_2^1 & a_{11}x_1^2 + a_{12}x_2^2 \\ x_2^1 & x_2^2 \end{vmatrix} + \begin{vmatrix} x_1^1 & x_1^2 \\ a_{21}x_1^1 + a_{22}x_2^1 & a_{21}x_2^1 + a_{22}x_2^2 \end{vmatrix} \\ = & a_{11}W + a_{22}W = Tr(A)W. \end{aligned}$$

Integrating this, we get the required formula. □

Corollary 1. *Let $x^1(t), x^2(t), \cdots, x^n(t) : [a,b] \to \mathbb{R}^n$ be solutions of $X' = A(t)X$. Then $W(x^1, x^2, \cdots, x^n)(t_0) = 0$ for some t_0, implies $W(x^1, x^2, \cdots, x^n)(t) = 0$ for all t.*

Now we can use the uniqueness theorem to show the following:

Theorem 4. *Let $x^1(t), x^2(t), ..., x^n(t) : [a,b] \to \mathbb{R}^n$ be solutions of $X' = A(t)X$. Then $x^1(t), x^2(t), ..., x^n(t)$ are linearly dependent $\iff W(x^1, x^2, \cdots, x^n)(t) = 0$ for all t.*

Proof. $\implies$ is easy. For the converse if $W(x^1, x^2, ..., x^n)(t_0) = 0$ implies the existence of non-trivail solution $(\alpha_1, \alpha_2, ...\alpha_n)$ to the system (1.1). Now we can define $x(t) = \alpha_1 x^1 + \alpha_2 x^2 + ... + \alpha_n x^n$. Then by the linearity, $x(t)$ is a solution of $X' = AX$. We also have $x(t_0) = 0$. Therefore, by uniqueness theorem, $x(t) = \alpha_1 x^1 + \cdots + \alpha_n x^n \equiv 0$ implying $x^1, x^2, \cdots, x^n$ are linearly dependent. $\square$

Next theorem is about the "General solution"

Theorem 5. *Let $x^1(t), x^2(t), ..., x^n(t) : [a,b] \to \mathbb{R}^n$ be linearly independent solutions of $X' = A(t)X$. Then all solution of this system are in the linear span of $x^1(t), x^2(t), ..., x^n(t)$.*

Proof. Let $Y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$ be any solution of $X' = AX$. Then using the fact that $x^1, ..., x^n$ are linearly independent, we get unique solution to the system of equations

$$X(t_0)C = Y(t_0), \ t_0 \in [a,b]$$

. That is,

$$\begin{pmatrix} x_1^1(t_0) & x_1^2(t_0) & ... & x_1^n(t_0) \\ x_2^1(t_0) & x_2^2(t_0) & ... & x_2^n(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ x_n^1(t_0) & x_n^2(t_0) & ... & x_n^n(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_n(t_0) \end{pmatrix} \tag{1.2}$$

. has unique solution $C = (\alpha_1, \alpha_2, ..., \alpha_n)$ (say). Now considering the function

$$Z(t) = \alpha_1 x^1(t) + \alpha_2 x^2(t) + ... + \alpha_n x^n(t),$$

we see by (1.2) that $Z(t)$ satisfies $Z(t_0) = Y(t_0)$. Also by linearlity,

$$Y'(t) = AY, \ Z'(t) = AZ.$$

By the Uniqueness theorem for systems we get $Y(t) \equiv Z(t)$. $\square$

Corollary 2. *The dimension of solution space is n.*

2 Linear system with constant coefficients

We consider the homogeneous system

$$X' = AX$$

where (a_{ij}) are constants. In case of higher order equation, we found general solution by substituting $x(t) = e^{mt}$. This suggests that we try substituting $X(t) = e^{\lambda t}\overline{v}$ in $X' = AX$. Then we get

$$\lambda e^{\lambda t}\overline{v} = e^{\lambda t}A\overline{v}.$$

This gives rise to the equation

$$(A - \lambda I)\overline{v} = 0,$$

where I is an $n \times n$ identity matrix. So it is clear now that λ is an eigenvalue and $\overline{v}$ is the corresponding eigenvector.

We have the following cases

Case 1: A has distinct eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$.

In this case, Let v^i be the eigenvector corresponding to λ_i. We consider $x_1(t) = e^{\lambda_1 t}v^1, ..., x_n(t) = e^{\lambda_n t}v^n$. Then $x_1, x_2, ...x_n$ are linearly independent as $v^1, v^2, ...v^n$ are linearly independent. i.e.,

$$W(x_1, x_2, ...x_n)(0) = \begin{vmatrix} v_1^1 & v_1^2 & ... & v_1^n \\ v_2^1 & v_2^2 & ... & v_2^n \\ \vdots & \vdots & \vdots & \vdots \\ v_n^1 & v_n^2 & ... & v_n^n \end{vmatrix} \neq 0.$$

Case 2: A has one (or more) eigenvalues repeated. But eigenvectors form a basis of $\mathbb{R}^n$

In this case, say $\lambda_1, \lambda_2, ..., \lambda_m$ are distinct eigenvalues $(m < n)$, and let $v^1, v^2, ..., v^n$ are eigenvectors that from basis of $\mathbb{R}^n$. Then again, we can take $x_1(t) = e^{\lambda_1 t}v^1, ..., x_m(t) = e^{\lambda_m t}v^m, ..., x_n(t) = e^{\lambda_n t}v^n$. Then as in the previous case we see $W(x_1, x_2, ..., x_n)(0) \neq 0$.

Case 3: Geometric multiplicity of λ_i is not equal to algebraic multiplicity of λ_i.

Let λ be a repeated eigenvalue twice and v^1 is the only eigenvector(L.I). Then we have $x_1(t) = e^{\lambda t}v^1$ is a solution and let $x_2(t) = v^1te^{\lambda t} + ue^{\lambda t}$ and determine u such that x^1, x^2 are linearly independent. Substituting x^2 in the system, we get

$$\lambda te^{\lambda t}v^1 + e^{\lambda t}v^1 + \lambda e^{\lambda t}u = A(te^{\lambda t}v^1 + e^{\lambda t}u) = Av^1te^{\lambda t} + Aue^{\lambda t}$$

Since $Av^1 = \lambda v^1$, we have

$$\lambda t e^{\lambda t} v^1 + e^{\lambda t} v^1 + \lambda e^{\lambda t} u = \lambda v^1 t e^{\lambda t} + A u e^{\lambda t}$$

Canceling $\lambda t e^{\lambda t} v^1$, we obtain $e^{\lambda t} v^1 + \lambda e^{\lambda t} u = A u e^{\lambda t}$ and hence

$$v^1 + \lambda u = Au$$

That is, u is a solution of the system

$$(A - \lambda I)u = v^1$$

Problem 1: Find L.I. solutions of $X' = AX$, with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Solution: Eigenvalues are $\lambda = 1$ twice. The eigenvector is $v^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This yields a solution

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t}.$$

The second L.I. solutions is of the form $x^2(t) = v^1 t e^t + u$ where u is a solution of the liner system $(A - I)u = v^1$, namely,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

That is, $u_2 = 1$ and u_1 is arbitrary, say $u_1 = 0$. So $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So the second linearly independent solution is

$$x^2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t.$$

Problem 2: Find L.I. solutions of $X' = AX$, with $A = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix}$

Solution Easy to see that $\lambda = 1 \pm i$ are eigenvalues. The eigen vector associated with $\lambda = 1 + i$

is

$$\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

Therefore

$$\begin{aligned} e^{\lambda t} &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^t \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \end{aligned}$$

So we can construct the general solution as

$$x(t) = c_1 e^t \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

$\square$

Non homogeneous problems The problem

$$x' = A(t)x + f(t)$$

is called non-homogenous problem if $f(t) \not\equiv 0$. Assuming that the homogeneous part of the problem $x' = A(t)X$ is solvable, we would like to study the general solution of the non-homogenous problem.

Scalar case: Consider the equation $x' + ax = f(t)$ with constant a and $f(t)$ continuous. Then we know that $x_h(t) = ce^{-at}$ is the general solution of $x' = ax$. Now for a solution of non-homogeneous equation, we consider $x_p(t) = c(t)e^{-at}$. Then substituting this in the equation (non-homogenous)

$$f(t) = x_p' + ax_p = c'(t)e^{-at} - ac(t)e^{-at} + ac(t)e^{-at} = c'e^{-at}$$

That is, $c'(t) = e^{at}f(t)$. Therefore, $x_p(t) = e^{-at}\int e^{-as}f(s)ds$ is a solution of non-homogenous equation.

We take the first order system. i.e., let A be $n \times n$ matrix and let X be the fundamental matrix of the system $x' = Ax$. For simplicity, we take $n = 2$. i.e.,

$$x_1' = a_{11}x_1 + a_{12}x_2$$
$$x_2' = a_{21}x_1 + a_{22}x_2$$

Suppose $(x_1, x_2)^T, (y_1, y_2)^T$ are two solutions. Then we can write

$$\begin{pmatrix} x_1' & y_1' \\ x_2' & y_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

Let X be the fundamental matrix $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$. Then we can expect the solution of non-homogeneous system is of the form

$$y = Xu$$

where u is a function of t. Substituting this in the system $x' = Ax + f$, we get

$$Xu' + X'u = AXu + f$$

since, $X' = AX$, the above equation is reduced to

$$Xu' = f$$

In other words, $u' = X^{-1}f$. Therefore,

$$y = Xu = X \int X^{-1} f(t)dt$$

is a particular solution of the non-homogeneous system.

Problem 3: (Second order system): Consider the problem

$$\overline{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2t - 2 \\ -4t \end{pmatrix}$$

Solution: The general solution of homogenous part is

$$c_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{4t} \\ 3e^{4t} \end{pmatrix}$$

So $X = \begin{pmatrix} e^{2t} & e^{4t} \\ e^{2t} & 3e^{4t} \end{pmatrix}$. Then $X^{-1} = \frac{e^{-6t}}{2} \begin{pmatrix} 3e^{4t} & -e^{4t} \\ -e^{2t} & e^{2t} \end{pmatrix}$. Therefore,

$$X^{-1}\overline{f} = \frac{e^{-6t}}{2} \begin{pmatrix} 3e^{4t} & -e^{4t} \\ -e^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} 2t-2 \\ -4t \end{pmatrix} = \begin{pmatrix} 5te^{-2t} - 3e^{-2t} \\ -3te^{-4t} + e^{-4t} \end{pmatrix}$$

Hence

$$\overline{u} = \int \begin{pmatrix} 5te^{-2t} - 3e^{-2t} \\ -3te^{-4t} + e^{-4t} \end{pmatrix} = \begin{pmatrix} \frac{-5}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ \frac{3}{4}te^{-4t} - \frac{1}{16}e^{-4t} \end{pmatrix}$$

Therefore,

$$\overline{y} = X\overline{u} = \begin{pmatrix} e^{2t} & e^{4t} \\ e^{2t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} \frac{-5}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ \frac{3}{4}te^{-4t} - \frac{1}{16}e^{-4t} \end{pmatrix} = \begin{pmatrix} -\frac{7}{4}t + \frac{3}{16} \\ -\frac{1}{4}t + \frac{1}{16} \end{pmatrix}$$

□

Next we show an exaple of system for $n = 3$ and repeated eigenvalues.

Problem 4: Find all L.I. solutions of $X' = AX$, with $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Solution:

$$det(A - \lambda I) = (3-\lambda)^2(2-\lambda).$$

So the eigenvalues are $\lambda = 3$ (double) and $\lambda = 2$. The eigenvectors are

$$v^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ v^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ v^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These are linearly independent. So the linearly independent solutions are

$$x^1 = v^1e^{3t},\ v^2 = v^2e^{3t},\ x^3 = v^3e^{2t}$$

. □

Problem 5: Find all L.I.solutions of $X' = AX$ where $A = \begin{pmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{pmatrix}$.

Solution: Eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and has only two L.I. eigenvectors that are

$$e^1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \; and \; e^2 = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

So the two linearly independent solutions are $x^1 = e^1 e^{2t}, x^2 = e^2 e^{2t}$. The third linearly independent solution is of the form $(\alpha t + \beta)e^{2t}$ where α and β satisfies

$$(A - 2I)\alpha = 0 \; and \; (A - 2I)\beta = \alpha.$$

We take $\alpha = k_1 e^1 + k_2 e^2$. Then $\alpha = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix}$ and β is a solution of

$$\begin{pmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix}$$

First two equations imply $k_2 = -2k_1$. A simple non-trivial solution is $k_1 = 1, k_2 = -2$. With this choice, $\alpha = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$. With this choice of α we compute β as solution of

$$\begin{pmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

A solution of this system is $\beta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore, third L.I. solution is $x^3 = \begin{pmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{pmatrix}$

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