MTL101, II Semester 2022-23, Quiz I
Answer Key and Scheme of Evaluation for Question 1

1. Suppose that $\{Av_1, Av_2, \ldots, Av_n\}$ is linearly independent. To prove that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, $0 \in M_{n\times 1}(\mathbb{R})$ and

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Then

$$A\big(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n\big) = A0 = 0.$$

By the properties of matrix multiplication, we get

$$\alpha_1 Av_1 + \alpha_2 Av_2 + \cdots + \alpha_n Av_n = 0.$$

Bearing in mind that $\{Av_1, Av_2, \ldots, Av_n\}$ is linearly independent, we have

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$

concluding that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.
The converse does not hold. For a trivial counter example, one may take $A = 0$, the null matrix. For a linearly independent set $\{v_1, v_2, \ldots, v_n\}$, the set $\{Av_1, Av_2, \ldots, Av_n\}$ reduces to $\{0\}$, and hence it is linearly dependent.

Ques 2 Let $p(x) \in W$ and write

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

for some coefficients $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$.

We first compute the values of $p(x)$ at $x = \pm 1$ and $x = \pm 2$.

We have

$$\begin{aligned} p(1) &= a_0 + a_1 + a_2 + a_3 + a_4 \\ p(1) &= a_0 - a_1 + a_2 - a_3 + a_4 \\ p(2) &= a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 \\ p(-2) &= a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4. \end{aligned}$$

Since $p(x)$ is in W, it satisfies

$$p(1) + p(-1) = 0 \text{ and } p(2) + p(-2) = 0,$$

and thus we have

$$\begin{aligned} p(1) + p(-1) &= 2a_0 + 2a_2 + 2a_4 = 0 \\ p(2) + p(-2) &= 2a_0 + 8a_2 + 32a_4 = 0. \end{aligned}$$

Dividing them by 2, we have the system of linear equations in a_0, a_2, a_4.

$$\begin{aligned} a_0 + a_2 + a_4 &= 0 \\ a_0 + 4a_2 + 16a_4 &= 0. \end{aligned}$$

We reduce the augmented matrix of the system by elementary row operations as follows.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 4 & 16 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 15 & 0 \end{array}\right] \xrightarrow{\frac{1}{3} R_2}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right].$$

Hence the solutions to the system is

$$\begin{aligned} a_0 &= 4a_4 \\ a_2 &= -5a_4 \end{aligned}$$

and a_4 is a free variable.

Substituting these relations into $p(x)$, we obtain

$$\begin{aligned} p(x) &= 4a_4 + a_1 x - 5a_4 x^2 + a_3 x^3 + a_4 x^4 \\ &= a_1 x + a_3 x^3 + a_4(4 - 5x^2 + x^4). \end{aligned}$$

Let

$$\begin{aligned} p_1(x) &= x \\ p_2(x) &= x^3 \\ p_3(x) &= 4 - 5x^2 + x^4. \end{aligned}$$

These are vectors in W.
(To see this, you may directly check the defining relations, or set $a_1 = a_3 = 0$ and $a_4 = 1$ to get $p_3(x)$. Similarly for $p_1(x)$ and $p_2(x)$.)

Then by the above computations, any vector $p(x)$ in W is a linear combination

$$p(x) = a_1 p_1(x) + a_3 p_2(x) + a_4 p_3(x).$$

Thus, $\{p_1(x), p_2(x), p_3(x)\}$ is a spanning set of W.

Also, the vectors $p_1(x), p_2(x), p_3(x)$ are linearly independent.
In fact, if we have $c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0$, then we have

$$\begin{aligned} 0 &= c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) \\ &= c_1 x + c_2 x^3 + c_3(4 - 5x^2 + x^4) \\ &= 4c_3 + c_1 x - 5c_3 x^2 + c_2 x^3 + c_3 x^4. \end{aligned}$$

Since the coefficients must be zero, we have $c_1 = c_2 = c_3 = 0$.
This proves that $p_1(x), p_2(x), p_3(x)$ are linearly independent.

Therefore the set $\{p_1(x), p_2(x), p_3(x)\}$ is linearly independent spanning set of the subspace W, hence it is a basis for W.
Thus the dimension of the subspace W is 3.

Problem 3 [6 marks] Let $S = \{A \in M_{2\times 2}(\mathbb{C}) : A = \bar{A}^T\}$, where for $A = [a_{ij}]$, $\bar{A}^T = [\overline{a_{ji}}]$, and $\overline{a_{ji}}$ is the complex conjugate of a_{ji} in $\mathbb{C}$.
(i) Is S a vector space over the field $\mathbb{R}$? If yes, find the dimension of S over $\mathbb{R}$. Justify your answer.
(ii) Is S a vector space over the field $\mathbb{C}$? If yes, find the dimension of S over $\mathbb{C}$. Justify your answer.

Solution:
(i) Let $A, B \in S$ and $\alpha, \beta \in \mathbb{R}$.
Let $C = \alpha A + \beta B$. We claim that $C \in S$.
We have $c_{ij} = \alpha a_{ij} + \beta b_{ij}$. Therefore, $\overline{c_{ji}} = \overline{\alpha a_{ji} + \beta b_{ji}} = \alpha \bar{a}_{ji} + \beta \bar{b}_{ji}$ because $\alpha, \beta \in \mathbb{R}$.
Since $A, B \in S$, we have $\bar{a}_{ji} = a_{ij}$ and $\bar{b}_{ji} = b_{ij}$.
Thus, $\overline{c_{ji}} = \alpha a_{ij} + \beta b_{ij} = c_{ij}$, which implies $C = \bar{C}^T$.
Hence, $C \in S$ implying S is a vector space over the field $\mathbb{R}$.
To determine the dimension of S over $\mathbb{R}$ we will exhibit a basis for S.

Let $\mathcal{B} = \left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}$.
If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $\bar{A}^T = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}$. Thus, $A \in S$ implies $a_{11} \in \mathbb{R}, a_{22} \in \mathbb{R}$ and $\overline{a_{12}} = a_{21}$.
Therefore, $A = \begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$. This implies $A = aA_1 + dA_2 + bA_3 + cA_4$.

Thus, $\text{span}(\mathcal{B}) = S$. Also, $\mathcal{B}$ can be shown to be linearly independent over $\mathbb{R}$ as follows.
Let $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = 0$, where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Then $\begin{pmatrix} c_1 & c_3 + ic_4 \\ c_3 - ic_4 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implying $c_1 = c_2 = c_3 = c_4 = 0$.

Hence, $\mathcal{B}$ is linearly independent over $\mathbb{R}$.
Therefore, $\mathcal{B}$ is a basis for the vector space S over $\mathbb{R}$ implying $\dim(S) = 4$.

(ii) Clearly, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$. But, $iA = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \notin S$ as $\overline{iA}^T = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix} \neq iA$.

Thus, S is not a vector space over the field $\mathbb{C}$

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Question (1) Let A be a square matrix. If $\text{Rank}(A) = \text{Rank}(A^2)$, show that the system of equations $Ax = 0$ and $A^2x = 0$ have the same solution space.

Proof. Let $V := \{x \in \mathbb{R}^n : Ax = 0\}$ and $W := \{x \in \mathbb{R}^n : A^2x = 0\}$.

Clearly, V is a subspace of W. Moreover by the given hypothesis, $\text{Rank}A = \text{Rank}A^2$, this implies dimension of Null space of A is equal dimension of Null space of A^2, i.e., $\dim V = \dim W$ hence the solution spaces $V = W$. This completes the proof. $\square$

MTL101: Solution of Question 5 in Minor Exam

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$.

Three given solutions for $(\alpha = \beta = 0,\ \alpha = 1, \beta = 0,\ \alpha = 0, \beta = 1)$ of $AX = b$ are

$$\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

We get

$$\begin{aligned} -a_{11} + a_{12} &= 2 & -a_{11} + 2a_{12} + a_{13} &= 4 & -2a_{11} + 2a_{12} + a_{13} &= 4 \\ -a_{21} + a_{22} &= 1 & -a_{21} + 2a_{22} + a_{33} &= 2 & -2a_{21} + 2a_{22} + a_{23} &= 2 \\ -a_{31} + a_{32} &= 0 & -a_{31} + 2a_{32} + a_{33} &= 0 & -2a_{31} + 2a_{32} + a_{33} &= 0 \end{aligned}$$

Giving $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and its RRE form is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Rank of $A = 1$.

Aliter:

Two scalars (or two free variables) exist in the solution set of $AX = b$. So, the dimension of the null space of A is 2, hence rank $A = 1$.

Also, $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of the null space of A.

So, $Av = 0,\ Aw = 0$ giving $a_{11} = a_{21} = a_{31} = 0$, $a_{13} = a_{23} = a_{33} = 0$.

Then, $A \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ gives $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and its RRE form is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.