

MTL 101
LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
MINOR EXAM

Total: 20 Marks

Time: 1:00 Hrs.

Question 1: (2 Marks) If A and B are two $n \times n$ real matrices such that $AB = 5I_{n \times n}$, then is it true that $BA = 5I_{n \times n}$? If true, give a proof else, give a counterexample.

Question 2: (2 Marks) Let A be a 3×4 real matrix of rank 3. Show that there exists 4×3 real matrix B such that $AB = I_{3 \times 3}$.

Solution 1 If $AB = 5I_{n \times n}$, then $\det(AB) = \det(A)\det(B) = 5$. Thus both $\det A, \det B$ are nonzero and hence invertible (**1 mark**). You can alternatively show that $(1/5)AB = I_{n \times n}$ and use the uniqueness of inverses to show that $B^{-1} = (1/5)A$.

Now $5I = 5BB^{-1} = B5I_{n \times n}B^{-1} = B(AB)B^{-1} = BA$ (**1 mark**).

Solution 2 Rank is the number of nonzero rows in the echelon form. Since the rank is 3, A takes the form $[I_{3 \times 3} \mid \mathbf{b}]$ in the row reduced echelon form where \mathbf{b} is a column vector. That is there is elementary matrices E_1, \dots, E_k such that $E_k \cdots E_1 A = [I_{3 \times 3} \mid \mathbf{b}]$. Let $B' = \begin{bmatrix} I_{3 \times 3} \\ \mathbf{0} \end{bmatrix}$ be a 4×3 matrix where $\mathbf{0}$ is a zero row vector (**1 mark**).

Now $E_k \cdots E_1 AB' = [I_{3 \times 3} \mid \mathbf{b}] \begin{bmatrix} I_{3 \times 3} \\ \mathbf{0} \end{bmatrix} = I_{3 \times 3}$. Now $AB' = E_1^{-1} \cdots E_k^{-1}$ and hence $A(B'E_k \cdots E_1) = I_{3 \times 3}$. Set $B = B'E_k \cdots E_1$ (**1 mark**).

Question-2 (3 Marks)

Question-2 consider the vector space $P_3(\mathbb{R})$ of polynomials of degree less than equal to three with real coefficients.

(a) Prove that $B = \{1-x, 1+x^2, 1-x^3, 1+x-x^3\}$ is a basis of $P_3(\mathbb{R})$.

Method-I B is basis iff (i) B is L.I.
(ii) $\text{Span}(B) = P_3(\mathbb{R})$. } 1 mark

(i) Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Consider,

$$c_1(1-x) + c_2(1+x^2) + c_3(1-x^3) + c_4(1+x-x^3) = 0$$

comparing coefficients both sides we get,

$$c_1 + c_2 + c_3 + c_4 = 0$$

$$-c_1 + c_4 = 0$$

$$c_2 = 0$$

$$-c_3 - c_4 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0$$

Hence, the above set B is linearly independent.

(ii) Let $P_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{R})$

Suppose $\exists c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.

$$P_1(x) = c_1(1-x) + c_2(1+x^2) + c_3(1-x^3) + c_4(1+x-x^3)$$

$$\Rightarrow c_1 + c_2 + c_3 + c_4 = a_0$$

$$-c_1 + c_4 = a_1$$

$$c_2 = a_2$$

$$-c_3 - c_4 = a_3$$

On solving, we get

$$c_1 = a_0 - a_2 + a_3$$

$$c_2 = a_2$$

$$c_3 = -a_0 - a_1 + a_2 - 2a_3$$

$$c_4 = a_0 + a_1 - a_2 + a_3$$

Since, $P_1(x)$ can be expressed in the linear combination of vectors of B .

Hence, $\text{span}(B) = P_3(\mathbb{R})$.

Method-II. Basis is the maximal L.I. set in a vector space.

Since, cardinality $(B) = 4 = \dim(P_3(\mathbb{R}))$.

(1 mark)

Hence, if B is L.I. set then we are done.

And linear independence of B is proved in (i)

(b) Find the coordinates of vector $u = 1 + x + x^2 + x^3$ with respect to ordered basis B .

Solⁿ: Step 1. $u = 1 + x + x^2 + x^3 = c_1(1-x) + c_2(1+x^2) + c_3(1-x^3) + c_4(1+x-x^3)$.

On comparing coeffⁿ—

$$\Rightarrow c_1 + c_2 + c_3 + c_4 = 1$$

$$-c_1 + c_4 = 1$$

$$c_2 = 1$$

$$-c_3 - c_4 = 1$$

(1 mark)

Step 2. Hence, $c_1 = 1$, $c_2 = 1$, $c_3 = -3$, $c_4 = 2$ on solving

$$[u]_B = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$

(1 mark)

Question 4 (4 marks)

- (a) 2-Marks Let W_1, W_2 be non-zero subspaces of a finite dimensional vector space V over \mathbb{C} . Suppose $\exists f: V \rightarrow \mathbb{R}$ such that $f(w_1) - f(w_2) < 0$ for all non-zero vectors $w_1 \in W_1$ & $w_2 \in W_2$. Prove: $\dim W_1 + \dim W_2 \leq \dim V$.

Sol: Step 1: Claiming & proving $W_1 \cap W_2 = \{0\}$

1-mark

Let if possible, $W_1 \cap W_2 \neq \{0\}$

$$\Rightarrow \exists u \neq 0 \text{ s.t. } u \in W_1 \text{ & } u \in W_2$$

$$\Rightarrow f(u) - f(u) < 0 \text{ & } u \neq 0 \Rightarrow 0 < 0$$

A Contradiction. ~~to the given condition.~~

Thus, $W_1 \cap W_2 = \{0\}$

(A)

Step 2: Using $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 \leq \dim V$ & concluding the final result.

1-mark

Note that we have:-

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad \dots \dots \dots (B)$$

Also as $W_1 + W_2$ is a subspace of V ,

$$\text{therefore } \dim(W_1 + W_2) \leq \dim(V) \quad \dots \dots \dots (C)$$

$$\text{Eq (A) gives } \dim(W_1 \cap W_2) = 0 \quad \dots \dots \dots (D)$$

Using Eq (C) & (D) in (B) to get:-

$$\dim(W_1) + \dim(W_2) \leq \dim V.$$

Question 3 (4 marks).

Consider the vector space \mathbb{C}^2 over \mathbb{C} . Find all possible linear transformations $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ such that $T^2 := T \circ T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ (the composition of T with itself) is given by

$$T^2(z_1, z_2) = (-z_1 + 2z_2, -z_2), \text{ for all } (z_1, z_2) \in \mathbb{C}^2.$$

Here is the marking scheme of the two standard solutions.

Solution 1.**Step 1.**

[1 mark] for writing the matrix of T^2 by fixing a basis, say standard basis B .

[0.5 mark] if basis is not mentioned.

Step 2.

[1 mark] for assuming $[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and using $[T^2]_B = [T]_B[T]_B$ to get 4 equations in a, b, c, d .

[0.5 mark] for any mistake in doing this.

Step 3.

[1 mark] for giving **detailed** solution of a, b, c, d .

[0.5 mark] for any mistake / inaccuracy to solve a, b, c, d .

Step 4.

[1 mark] for writing the two possible matrices followed by the two possibilities of T (invoking the basis fixed in the beginning).

[0.5 mark] for any mistake in doing so or only partially doing so.

Solution 2.**Step 1.**

[1 mark] for using that every T is given by $T(z_1, z_2) = (az_1 + bz_2, cz_1 + dz_2)$ for some $a, b, c, d \in \mathbb{C}$ and computing $T^2(z_1, z_2)$.

[0.5 mark] for any mistake in the computing $T^2(z_1, z_2)$.

Step 2.

[1 mark] for using Step 1 and $T^2(z_1, y) = (-z_1 + 2z_2, -z_2)$ to get 4 equations on a, b, c, d .

[0.5 mark] for any mistake in writing down the 4 equations.

Step 3.

[1 mark] for giving detailed solution of a, b, c, d .

[0.5 mark] for any mistake / inaccuracy to solve a, b, c, d .

Step 4..

[1 mark] for writing the two possible the two correct possibilities of T based on a, b, c, d found in Step 3.

[0.5 mark] for any mistake in doing so or doing partially.

Remark: For skipping or failing to justify any of the steps above earns [0 mark].

Question 4 (4 marks)

- ⑥ 2-marks Let $W_1 = \text{Span}\{(4, 3, 2, 1), (1, 1, 1, 2), (3, 2, 1, -1)\}$
 $W_2 = \text{Span}\{(1, 0, 3, 2), (4, 3, 2, 1)\}$
Find the dimension of $W_1 + W_2$.

Sol: Step 1: Finding Dimensions of W_1, W_2 & $W_1 \cap W_2$ Correctly. 1-mark
(with details)

(a) $\text{Dim}(W_1) = 2$.

Note: $(1, 1, 1, 2) = (4, 3, 2, 1) - (3, 2, 1, -1)$
& $\{(4, 3, 2, 1), (3, 2, 1, -1)\}$ is a linearly independent set.

Therefore, a basis(W_1) = $\{(4, 3, 2, 1), (3, 2, 1, -1)\}$

Thus, $\text{Dim}(W_1) = 2$

(b) $\text{Dim}(W_2) = 2$.

Note: $\{(1, 0, 3, 2), (4, 3, 2, 1)\}$ is a linearly independent set, thus forms a basis.

Hence $\text{Dim}(W_2) = 2$.

(c) $\text{Dim}(W_1 \cap W_2) = 1$.

Note that: $W_1 \cap W_2 = \text{Span}\{(4, 3, 2, 1)\}$

Hence, $\text{Dim}(W_1 \cap W_2) = 1$

Step 2: Writing & using the identity:-

$$\text{dim}(W_1 + W_2) = \text{dim}(W_1) + \text{dim}(W_2) - \text{dim}(W_1 \cap W_2)$$

$$\begin{aligned} \text{dim}(W_1 + W_2) &= 2 + 2 - 1 \\ &= 3 \end{aligned}$$

Question 4 (4 marks)

Sol (4)(b): Alternate Solution :-

Step 1: Using $W_1 + W_2 = \text{span}(W_1 \cup W_2)$

1 mark

As $W_1 + W_2 = \text{span}(W_1 \cup W_2)$,
Therefore the subspace $W_1 + W_2$ is spanned
by $\{(4, 3, 2, 1), (1, 1, 1, 2), (3, 2, 1, -1), (1, 0, 3, 2), (4, 3, 2, 1)\}$

Step 2: Finding ^{dimension of} ~~a basis for~~ $\text{span}(W_1 \cup W_2)$
(by removing linearly dependent vectors)

1 mark

Note: Finding ^{number of} linearly independent vectors is same
as finding non-zero rows in Row Echelon form.
Therefore, the $\dim(W_1 + W_2)$ is the dimension of
the row space of the matrix :- (or row rank)

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

The row echelon form is: (Note: $R_5 = R_1$ & $R_2 = R_1 - R_3$)

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 1/4 & 1/2 & 7/4 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, the row-rank of this matrix is 3,
we get $\dim(W_1 + W_2) = 3$

Question 5: (5 Marks) Consider the linear operator $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by

$$T((x_1, x_2, x_3, x_4)) = \left(\sum_{i=1}^4 x_i, \sum_{i=1}^4 x_i, \sum_{i=1}^4 x_i, \sum_{i=1}^4 x_i \right) \text{ for all } (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

Prove or disprove: there exists an ordered basis B of \mathbb{R}^4 such that $[T]_B$ is diagonal.

Solution 5:(Prove)

(Computing the eigenvalues): Note that for any $\lambda \in \mathbb{R}$ and a nonzero vector $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, we have $T((x_1, x_2, x_3, x_4)) = \lambda(x_1, x_2, x_3, x_4)$ if and only if

$$x_1 + x_2 + x_3 + x_4 = \lambda x_i \text{ for each } i = 1, 2, 3, 4. \quad (3)$$

This implies that $\lambda = 4$ or $\lambda = 0$. Thus 4 and 0 are two distinct eigenvalues of T ,

OR,

Let $\beta = \{e_1, e_2, e_3, e_4\}$ be the standard basis. Then $[T]_\beta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. The eigenvalues of

T and $[T]_\beta$ are same. So find the eigenvalues of $[T]_\beta$ by computing the roots of the characteristic polynomial $\det(\lambda I - [T]_\beta)$. They are 0, 0, 0, 4. **[2 mark]**

(Computing the eigenspaces): Next compute the eigenspaces, when $\lambda = 4$, we have from (3) that $E_1 = \ker(4I - T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3 = x_4\} = \text{span}(\{(1, 1, 1, 1)\})$. **[1 mark]**

Similarly, when $\lambda = 0$, we have from (3) that $E_2 = \ker(0I - T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$. Thus $E_2 = \text{span}(\{(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\})$. **[1 mark]**

(Diagonalizability check): This implies that

$$\dim(E_1) + \dim(E_2) = 1 + 3 = 4 = \dim(\mathbb{R}^4),$$

and hence T is diagonalizable. Consider $B = \{(1, 1, 1, 1), (1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\}$. Then B is a basis of \mathbb{R}^4 since eigenspaces corresponding to distinct eigenvalues are independent

and $|B| = 4$. Also $[T]_B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. **[1 mark]**