DEPARTMENT OF MATHEMATICS, IIT DELHI
SEMESTER II 2024 – 25
MTL 101 (Linear Algebra and Differential Equations) - MAJOR EXAM

Date: 01/05/2025 (Thursday) Time: 3:30 pm- 5:30 pm.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

Name :		BLOCK LETTER ONLY	
Entry Number:		**Group:**	
Gradescope Id:		**Lecture Hall:**	

Question 1:

a) Let q be a continuous function on $[0, \infty)$ and

$$\lim_{x\to\infty} q(x) = 10.$$

Prove that the solution $y(x)$ of the IVP

$$y'(x) + 5y(x) = q(x), \quad y(0) = 1$$

is bounded on $[0, \infty)$.

b) Find the general solution of the non-homogeneous equation:

$$x^2y'' + 3xy' + y = \frac{1}{x}, \quad x > 0.$$

[2+4=6]

Question 2:

a) Show that if two solutions of a second order homogeneous differential equation with continuous coefficients on an interval, $I \subset \mathbb{R}$, have a common zero, then all their zeros are in common.

b) Suppose y_1 and y_2 are two linearly independent solutions of the ODE

$$y'' + a_1 y' + a_2 y = 0, \quad t \in I$$

where $a_1, a_2 \in \mathbb{R}$.

Show that $W(y_1, y_2)(t)$ is constant if and only if $a_1 = 0$.

[3+3=6]

Question 3:

(a) Let $F(s) = \mathcal{L}(f(t))(s)$ be defined for $s > 0$. Suppose that $|f(t)| \leq M$ for all $t \geq 0$. Show that

$$\lim_{s \to +\infty} F(s) = 0.$$

(b) Using Laplace transform, solve the IVP

$$y'(t) - 4 \int_0^t y(\tau)\, d\tau = 1, \quad y(0) = 0.$$

[2+3=5]

Question 4: Using the Laplace transform, solve the following system:

$$x'(t) = 3x(t) + 4y(t) + \delta(t),$$
$$y'(t) = -2x(t) + y(t) + 5\delta(t),$$

with initial conditions

$$x(0) = 1, \quad y(0) = -2,$$

where $\delta(t)$ is the Dirac delta function at $t = 0$.

[5]

Question 5: Find $A(y)$ such that equation

$$(2x + A(y))dx + 2xydy = 0$$

is an exact equation and solve it.

[4]

Question 6: Solve the following system of first-order linear ordinary differential equations using the method of variation of parameters:

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{G}(t),$$

where

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 2\sqrt{2} \end{pmatrix}, \quad \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{G}(t) = \begin{pmatrix} e^{\sqrt{2}t} \\ 0 \end{pmatrix}.$$

[4]

Question 7: Consider the vector space of 2×2 real matrices with entrywise addition and entrywise scalar multiplication:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}.$$

Consider $T : V \to V$ defined by

$$T(X) = X + X^T.$$

(a) Show that T is a linear transformation.

(b) Find bases for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

[4]

Question 8:

i) Consider the matrix

$$A = \begin{pmatrix} a & -1 & 0 \\ -1 & b & -1 \\ 0 & -1 & a \end{pmatrix}.$$

where a, b are real numbers. Determine the conditions on the pair (a, b) under which all the eigenvalues of A are positive.

ii) Consider $V = \mathbb{C}$, the 2-dimensional vector space of complex numbers over the field of real numbers, $\mathbb{R}$. Let the basis $\mathcal{B} = \{1, 1+i\}$ be ordered as it is displayed. For $\alpha \in \mathbb{C}$, define the linear transformation $M_\alpha : V \to V$ given by

$$M_\alpha(\beta) = \alpha\beta.$$

Compute the matrix $[M_\alpha]_{\mathcal{B}}$. Also, find α for which $\mathrm{Ker}(M_\alpha) \neq 0$.

[3+3=6]