Ordinary differential equation (ODE)

Lecture 1

* First order ODE
* Initial Value Problem (IVP)

First order ODE :- A first order ODE is an equation involving an independent x, a dependent $y(x)$ and $y'(x)$ (i.e. $\frac{dy}{dx}$) on some open interval I.

Implicit form : $F(x, y(x), y'(x)) = 0$

Explicit form : $y' = f(x, y)$

A solution to an ODE is some function $y = h(x)$ which differentiable on an open interval I and satisfy the equation, i.e.,

$$F(x, h(x), h'(x)) = 0 \quad \text{or} \quad h'(x) = f(x, h(x)).$$

IVP (first order):- A first order initial value problem is an equation of the form

$$F(x, y(x), y'(x)) = 0 \quad \text{or}$$

$$y'(x) = f(x, y(x)) \quad \text{where} \quad y(x_0) = y_0.$$

A solution to an IVP is some function $y = h(x)$ which is differentiable on an open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$.

Question :- (i) Does every first order ODE / IVP have a solution ?

(ii) If the solution exists then it is unique.

In general, answer is NO.

Under some condition, we can have the positive answer.

Example :- $(y')^2 + 1 = 0$

Here there is no real solution.

Example :- $x y' = y - 1, \quad y(0) = 1$

$y = 1 + cx$ is a solution of the given IVP for any $c \in \mathbb{R}$. Solution is not unique.

Example :- $(y')^2 + y^2 = 0, \quad y(0) = c, \quad c \neq 0$

This has no solution.

Example :- $(y')^2 + y^2 = 0 \ , \ y(0) = 0$

This has an unique solution, $y(x) = 0$.

Existence theorem :- Consider the IVP

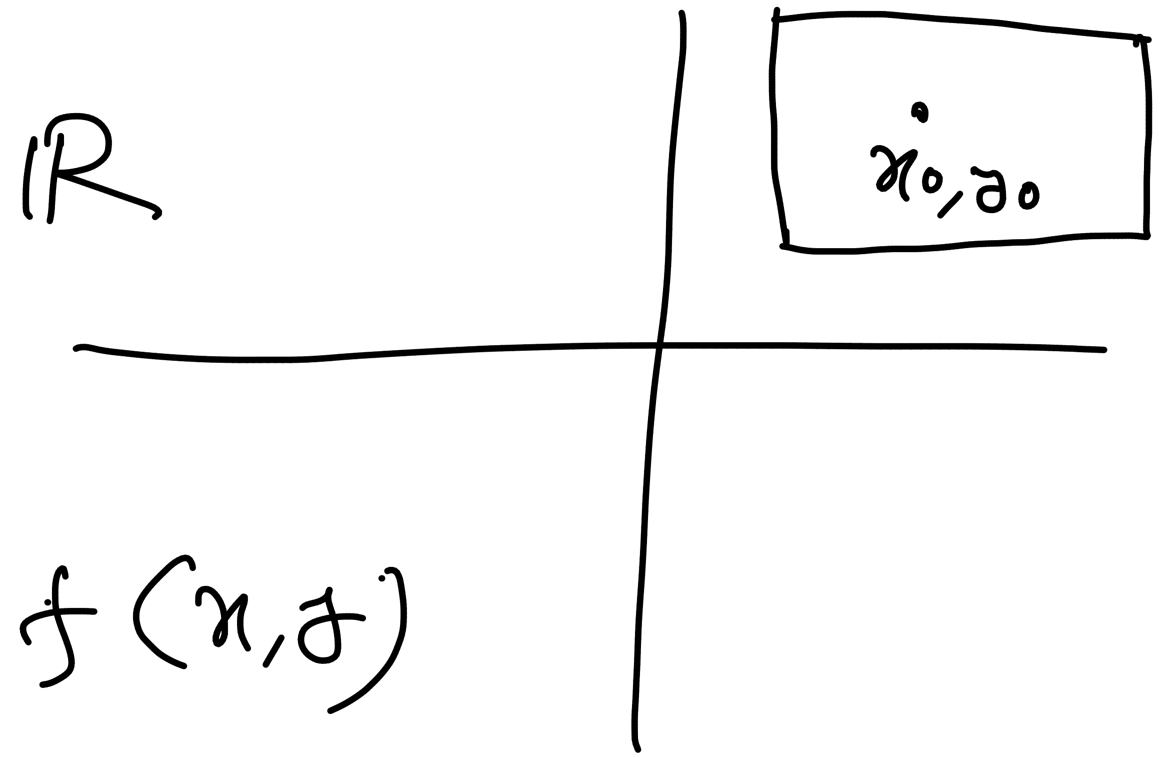
$$y' = f(x, y(x)), \quad y(x_0) = y_0 .$$

Suppose $f(x, y(x))$ is continuous on some closed rectural region $R = \{(x, y) \in \mathbb{R}^2 \mid |x - x_0| \leq a \ \& \ |y - y_0| \leq b\}$ for some $a, b > 0 \in \mathbb{R}$.

Then the IVP must have at least one solution on some interval $(x_0 - \alpha, x_0 + \alpha)$.

If $|f(x, y(x))| \leq K \quad \forall (x, y) \in R$

$\alpha \leq \min \left\{ a, \frac{b}{K} \right\}$ where $K > 0$



* This theorem gives only the sufficient condition NOT the necessary condition.

There exists a discontinuous function $f(x, y)$ on R, for which $y' = f(x, y)$, $y(x_0) = y_0$ has a solution.

$$y' = \frac{y-1}{x}, \quad y(0) = 1$$

Here $f(x, y(x)) = \dfrac{y-1}{x}$, $\quad x_0 = 0$, $y_0 = 1$

f is not continuous in any rectangular neighborhood around $(0, 1)$.

If the solution exists then the solution may exist outside of $(x_0-\alpha, x_0+\alpha)$ as well.

Example :- $y' = 1+y^2$, $y(0)=0$

$$f(x,y) = 1+y^2$$

f is condition on an closed rectangle

$$R = \{(x,y) : |x| \le a, \ |y| \le b\}$$ for any $a,b \in \mathbb{R}$

$$|f(x,y)| \le 1+b^2$$

Then $\alpha = \min\left\{a, \frac{b}{K}\right\} = \min\left\{a, \frac{b}{1+b^2}\right\} \le \frac{1}{2}$

Solution exists on $(-\alpha, \alpha)$

Aclutal solution is $y = \tan(x)$. This solution exists in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

A function $f(x,y)$ is said to satisfy the Lipchitz condition on a region R if there exists a constant $L>0$ such that

$$|f(x,y_1)-f(x,y_2)| \le L|y_1-y_2|$$

$$\forall\ (x,y_1),(x,y_2) \in R.$$

Uniqueness theorem :- Suppose $f(x,y)$ be continuous on a closed rectangle $R=\{(x,y): |x-x_0|\le a, |y-y_0|\le b\}$. Also, $f(x,y)$ satisfy the Lipchitz condition (in y variable) then the

IVP $y'=f(x,y)$, $y(x_0)=y_0$ has an unique solution on an interval $(x_0-\alpha, x_0+\alpha)$.