COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 35

QUIZ 3 DISCUSSION & PROBABILITY (CONTD.)

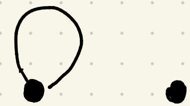
APR 18, 2023 | ROHIT VAISH

PROBLEM 1

Given an undirected graph $G = (V, E)$ with $|E| \geq 1$.

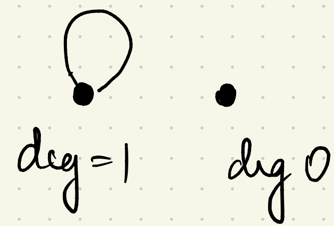
Prove (or disprove) that it is impossible for all vertices in G to have different degrees.

For undirected graphs with self-loops

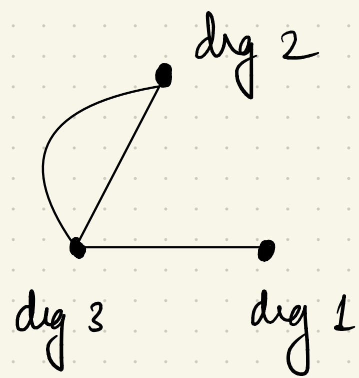


deg = 1   deg 0

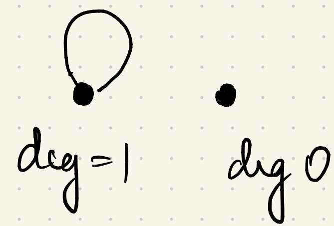
For undirected graphs with self-loops



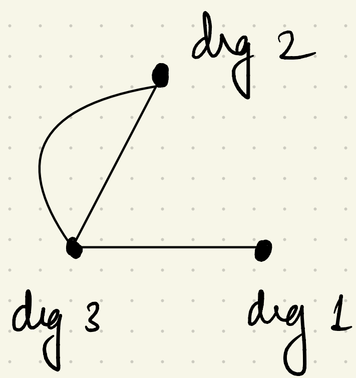
For undirected multigraphs



For undirected graphs with self-loops



For undirected multi graphs



For the rest of the proof, assume simple graphs.

Case I Suppose G has no isolated (degree 0) vertices.

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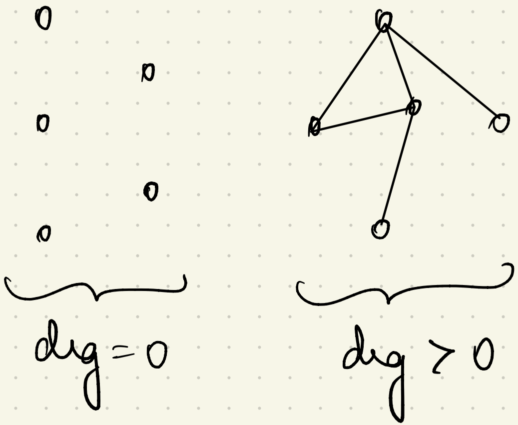
By Pigeonhole principle, at least two vertices must have the same degree.

Case II Suppose G has ≥ 1 isolated (degree 0) vertex.



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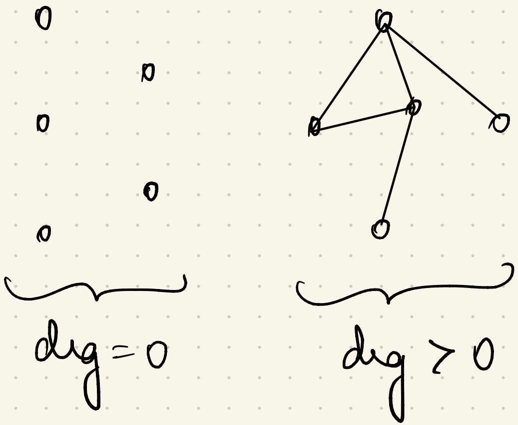

Possible values of degrees are

$0, 1, \dots, n-2$

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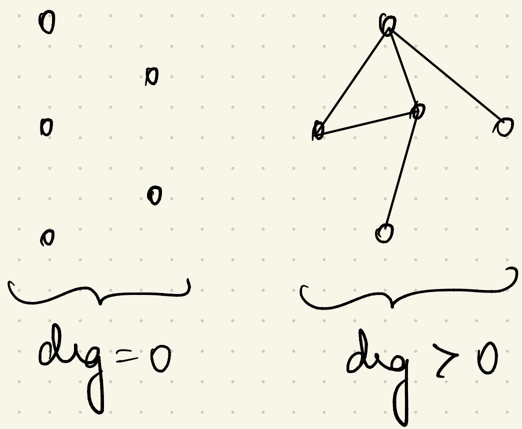

Possible values of degrees are

$0, 1, \cdots, n-2$

Again, by pigeonhole principle, at least two vertices should have the same degree.

$\boxtimes$

PROBLEM 1

Total = 15 points

Observing that statement fails for non-simple graphs and assuming simple graphs for the proof. [0 pts]*

* But you have my admiration!

Identifying two cases depending on the number of isolated vertices [5 pts]

Correct application of Pigeonhole principle in each case. [5 + 5 pts]

PROBLEM 2

Prove the inclusion - exclusion principle.

$$
\begin{aligned}
|S_1 \cup S_2 \cup \cdots \cup S_n| &= \sum_{i=1}^{n} |S_i| \\
&- \sum_{1 \le i < j \le n} |S_i \cap S_j| \\
&+ \sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k| \\
&\ \ \vdots \\
&+ (-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n|
\end{aligned}
$$

Consider any element $x \in S_1 \cup S_2 \cup \cdots \cup S_n$.

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Suppose x belongs to exactly k of the n sets $S_1, S_2, \dots, S_n$.

Then, $1 \leq k \leq n$.

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Suppose x belongs to exactly k of the n sets $S_1, S_2, \dots, S_n$.

Then, $1 \leq k \leq n$.

Let us calculate how many times x is counted by the different terms in RHS.

Contribution

$$\sum_{i=1}^{n} |S_i| \longrightarrow$$

$$- \sum_{1 \le i < j \le n} |S_i \cap S_j| \longrightarrow$$

$$+ \sum_{1 \le i < j < l \le n} |S_i \cap S_j \cap S_l| \longrightarrow$$

$$\vdots$$

$$+ (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}| \longrightarrow$$

$$\left.\begin{array}{l} \vdots \\ + (-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n| \end{array}\right\} \longrightarrow$$

Contribution

$$\sum_{i=1}^{n} |S_i| \longrightarrow +k = {}^{k}C_1$$

$$- \sum_{1 \le i < j \le n} |S_i \cap S_j| \longrightarrow$$

$$+ \sum_{1 \le i < j < l \le n} |S_i \cap S_j \cap S_l| \longrightarrow$$

$$\vdots$$

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$$\left. \begin{array}{l} \vdots \\ + (-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n| \end{array} \right\} \longrightarrow$$

Contribution

$$\sum_{i=1}^{n} |S_i| \longrightarrow +k = {}^{k}C_1$$

$$- \sum_{1 \le i < j \le n} |S_i \cap S_j| \longrightarrow -{}^{k}C_2$$

$$+ \sum_{1 \le i < j < l \le n} |S_i \cap S_j \cap S_l| \longrightarrow$$

$$\vdots$$

$$+ (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}| \longrightarrow$$

$$\vdots$$

$$\left. + (-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n| \right\} \longrightarrow$$

Contribution

$$\sum_{i=1}^{n} |S_i| \longrightarrow +k = {}^{k}C_1$$

$$-\sum_{1 \le i < j \le n} |S_i \cap S_j| \longrightarrow -{}^{k}C_2$$

$$+\sum_{1 \le i < j < l \le n} |S_i \cap S_j \cap S_l| \longrightarrow +{}^{k}C_3$$

$$\vdots$$

$$+(-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}| \longrightarrow$$

$$\left.\begin{array}{l} \vdots \\ +(-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n| \end{array}\right\} \longrightarrow$$

Contribution

$$\sum_{i=1}^{n} |S_i| \longrightarrow +k = {}^{k}C_1$$

$$-\sum_{1\le i<j\le n} |S_i \cap S_j| \longrightarrow -{}^{k}C_2$$

$$+\sum_{1\le i<j<l\le n} |S_i \cap S_j \cap S_l| \longrightarrow +{}^{k}C_3$$

$$\vdots$$

$$+(-1)^{k+1}\sum_{1\le i_1<i_2<\dots<i_k\le n} |S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_k}| \longrightarrow (-1)^{k+1}\,{}^{k}C_k$$

$$\left.\begin{array}{l} \vdots \\ +(-1)^{n+1} |S_1 \cap S_2 \cap \dots \cap S_n| \end{array}\right\} \longrightarrow$$

Contribution

$$\sum_{i=1}^{n} |S_i| \longrightarrow +k = {}^kC_1$$

$$- \sum_{1 \le i < j \le n} |S_i \cap S_j| \longrightarrow - {}^kC_2$$

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$$+ (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}| \longrightarrow (-1)^{k+1} \, {}^kC_k$$

$$\left.\begin{array}{l} \vdots \\ + (-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n| \end{array}\right\} \longrightarrow 0$$

Number of times x is counted in RHS

$$= {}^{k}C_1 - {}^{k}C_2 + {}^{k}C_3 - \cdots + (-1)^{k+1}\,{}^{k}C_k$$

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$$= {}^{k}C_1 - {}^{k}C_2 + {}^{k}C_3 - \text{———} + (-1)^{k+1}\, {}^{k}C_k + 1 - 1$$

Number of times x is counted in RHS

$$= {}^kC_1 - {}^kC_2 + {}^kC_3 - \text{———} + (-1)^{k+1}\,{}^kC_k + 1 - 1$$

$$= 1 - \left({}^kC_0 - {}^kC_1 + {}^kC_2 - \cdots - (-1)^{k+1}\,{}^kC_k\right)$$

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$$= {}^kC_1 - {}^kC_2 + {}^kC_3 - \text{———} + (-1)^{k+1}\,{}^kC_k + 1 - 1$$

$$= 1 - \left({}^kC_0 - {}^kC_1 + {}^kC_2 - \cdots - (-1)^{k+1}\,{}^kC_k\right)$$

Binomial Thm : $(a+b)^n = \sum_{l=0}^{n} {}^nC_l\, a^l\, b^{n-l}$

$a = -1$

$b = 1$

$n = k$

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$$= 1 - (-1+1)^k$$

$$= 1$$

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$a = -1$

$b = 1$

$n = k$

$$= 1 - (-1+1)^k$$

$$= 1 \implies x \text{ is counted exactly once in RHS.} \qquad \blacksquare$$

PROBLEM 2

Total = 20 points

Identifying proof by counting contribution of an element x on both sides [7 pts]

Correctly counting contribution for each term [8 pts]

Correct application of binomial theorem [5 pts]