Change of basis - lecture 9

Ordered basis $\Rightarrow$ Let $V(\mathbb{F})$ be a vector space of dim n and $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ be a basis of $V(\mathbb{F})$. Then if we have an ordering on $\mathcal{B}$ we say $\mathcal{B}$ is an ordered basis.

\# u_1 is 1^{st} vector

\# u_2 " 2^{nd} vector

⋮

\# u_n " last vector

In order to diff b/w basis and ordered basis, we write ordered basis as $\mathcal{B} = (u_1, u_2, \ldots, u_n)$.

Remark :- ① $\mathcal{B}_1 = (u_1, u_2, \ldots, u_n)$

② $\mathcal{B}_2 = (u_2, u_3, \ldots, u_n, u_1)$

Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are two diff ordered basis.

Every element $u \in V$ can be written uniquely

as $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ $\alpha_i \in \mathbb{F}$

α_i's are uniquely determined.

Then $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F}^n$ is called coordinate of the vector u with respect to the basis $\mathcal{B}$; and the co-ordinate vector is given by the coloumn

$$[u]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix}$$

Examples 1

$$V = \mathbb{R}^n(\mathbb{R})$$

$$\mathcal{B} = \{ e_i \mid 1 \le i \le n \}$$

$$e_i = (0, 0, \ldots, 1, 0, \ldots, 0)$$

$u \in \mathbb{R}^n, \quad u = (x_1, x_2, \dots, x_n)$

$$u = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

Then coordinate of u with respect to B are $(x_1, x_2, \dots, x_n)$ and coordinate vector

$$[u]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Example 2 let $V = \mathbb{R}^2(\mathbb{R})$ and $B = \{(1,1), (1,-1)\}$ be an ordered basis of $\mathbb{R}^2(\mathbb{R})$.

$u \in \mathbb{R}^2(\mathbb{R})$, $u = (x_1, x_2)$

$$(x_1, x_2) = \alpha_1 (1, 1) + \alpha_2 (1, -1) \quad \alpha_1, \alpha_2 \in \mathbb{R}.$$

$$(x_1, x_2) = (\alpha_1 + \alpha_2, \ \alpha_1 - \alpha_2)$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 = x_1 \quad \text{and} \quad \alpha_1 - \alpha_2 = x_2$$

$$\Rightarrow \quad \alpha_1 = \frac{x_1 + x_2}{2} \quad \text{and} \quad \alpha_2 = \frac{x_1 - x_2}{2}$$

Then

$$[u]_B = \begin{bmatrix} \dfrac{x_1 + x_2}{2} \\ \\ \dfrac{x_1 - x_2}{2} \end{bmatrix}$$

Remark $\rightarrow$ The coordinate of a vector depends on the basis.

How does the coordinate vector change with the change of basis?

Let $V(\mathbb{F})$ be a f.d. vector space with $\dim_F(V) = n$.

Let $\mathcal{B}_1 = (u_1, u_2, \ldots, u_n)$ and $\mathcal{B}_2 = (v_1, v_2, \ldots, v_n)$

be two basis of $V(\mathbb{F})$.

Let $v \in V$. Then

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \quad \text{--- eq}^n \text{ (1)}$$

$$[v]_{B_1} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^t. \quad \text{— eqn } (2)$$

Now we write u_i, $1 \le i \le n$ as L.C of v_i, $1 \le i \le n$.

$$\boxed{u_j = \sum_{i=1}^{n} p_{ij} v_i} \quad \text{— } (*) \quad (1 \le j \le n)$$

$$v = \sum_{j=1}^{n} \alpha_j u_j = \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{n} p_{ij} v_i \right)$$

$$v = \sum_{i=1}^{n} \underline{\left(\sum_{j=1}^{n} p_{ij} \alpha_j \right)} v_i$$

$$[v]_{\mathcal{B}_2} = \begin{bmatrix} \sum_{j=1}^{n} p_{1j}\alpha_j \\ \sum_{j=1}^{n} p_{2j}\alpha_j \\ \vdots \\ \sum_{j=1}^{n} p_{nj}\alpha_j \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{1n} \\ p_{21} & p_{22} & p_{2n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$[v]_{\mathcal{B}_2} = P\,[v]_{\mathcal{B}_1}$$

$$\boxed{u_1 = p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n} \text{ --- } (*)$$

I^{st} column of the matrix P

Remark :- The matrix P occurs in the change of basis is invertible.

Proof :- Write

$$v_k = \sum_{j=1}^{n} q_{jk} u_j , \qquad 1 \le k \le n .$$

By putting value of u_j's

$$v_k = \sum_{j=1}^{n} q_{jk} \left(\sum_{i=1}^{n} p_{ij} v_i \right)$$

$$v_k = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij} q_{jk} \right) v_i$$

$$\Rightarrow \quad \sum_{j=1}^{n} p_{ij} q_{jk} = 0 \qquad \text{if } \quad i \neq k$$

and $\sum_{j=1}^{n} p_{kj} q_{jk} = 1 \qquad i=k$

$\Rightarrow \quad PQ = I$. Hence P is invertible

$\boxed{[v]_{B_1} = Q\,[v]_{B_2}}$

$$[v]_{B_2} = P\,[v]_{B_1}$$

$$Q\,[v]_{B_2} = QP\,[v]_{B_1} = [v]_{B_1}$$

$$\Rightarrow \quad [v]_{B_1} = Q\,[v]_{B_2}.$$

<u>Example</u> 1 $V = \mathbb{R}^2(\mathbb{R})$ and $\mathcal{B}_1 = \{(1,0), (0,1)\}$ and
$\mathcal{B}_2 = \{(1,1), (1,-1)\}$.

<u>Sol^n</u>: $[v]_{\mathcal{B}_1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $(x_1, x_2) \in \mathbb{R}^2$

$$(1,0) = p_{11}(1,1) + p_{21}(1,-1)$$

$$(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$$

$$\| \quad (0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$[v]_{B_2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{bmatrix}$$

<u>Example ②</u> $V = \mathbb{R}^2(\mathbb{R})$

$$B_1 = \left((1,1), \ (1,-1) \right)$$

$$B_2 = \left((1,2), \ (2,1) \right).$$

$$v = (x_1, x_2) \in \mathbb{R}^2$$

$$[v]_{B_1} = \begin{bmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1-x_2}{2} \end{bmatrix}$$ (last example)

$$(1,1) = p_{11}(1,2) + p_{21}(2,1)$$

$$(1,1) = \frac{1}{3}(1,2) + \frac{1}{3}(2,1)$$

$$(1,-1) = (-1)(1,2) + 1(2,1)$$

Hence $P = \begin{bmatrix} \frac{1}{3} & -1 \\ \frac{1}{3} & 1 \end{bmatrix}$

$$[v]_{B_2} = P[v]_{B_1}$$

Exercise ① Let $V = \mathbb{R}^3(\mathbb{R})$ and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases given by

$$\mathcal{B}_1 = ((1,0,0), (1,1,0), (1,1,1))$$

$$\mathcal{B}_2 = ((1,1,1), (1,-1,1), (1,1,0))$$

ⓐ Write $[v]_{\mathcal{B}_1}$.

ⓑ Find the matrix P

ⓒ Calculate $[v]_{\mathcal{B}_2}$ by using P.

ⓓ Prove the existance of Q. s.t $PQ = QP = I$.

Definition :- Let $A = (a_{ij})_{m\times n}$ be a matrix with $a_{ij} \in \mathbb{F}$.

Then $A = \begin{bmatrix} - R_1 - \\ - R_2 - \\ \vdots \\ - R_m - \end{bmatrix}$ with $R_i = (a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbb{F}^n$

We can see each row of A as element in $\mathbb{F}^n$.

Then linear span of $\{R_1, R_2, \dots, R_m\}$ is called row-space of A

$$\text{row-space}(A) = \text{span}(R_1, \dots, R_m) = L(R_1, \dots, R_m) \subseteq \mathbb{F}^n$$

$$\boxed{\text{row-rank}(A) = \dim(\text{row-space}(A)) \leq n.}$$

Lemma $\Rightarrow$ Let $A = (a_{ij})_{m \times n}$ and let $B = EA$ where E is an elementary matrix. Then

① row-space (A) = row-space (B)

② dim (row-space(A)) = dim (row-space(B))

Proof $\Rightarrow$ $E_{ij} = E_{ij}(c) = R_i \rightarrow R_i + cR_j$

$$\text{row-space}(B) = L(R_1, R_2, \dots, R_i + cR_j, \dots, R_m)$$

$$= \{ \alpha_1 R_1 + \alpha_2 R_2 + \dots + \alpha_i (R_i + cR_j) + \dots + \alpha_m R_m \mid \alpha_i \in F \}$$

$$= \{ \alpha_1 R_1 + \alpha_2 R_2 + \dots + (\alpha_j + c\alpha_i) R_j + \dots + \alpha_i R_i + \dots \alpha_m R_m \mid \alpha_i \in F \}$$

$$= L(R_1, R_2, \dots, R_m) = \text{row-space}(A).$$