Let V be a finite dimensional vector space over a field F. Let $T: V \longrightarrow V$ be a linear transform.

Recall that, $\lambda \in F$ is an eigen value of T if $\exists$ a non-zero vector $X \in V$ such that $TX = \lambda X$.

* $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad T(x, y) = (-y, x)$

$$[T]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T(1,0) = 0 \cdot (1,0) + 1 \cdot (0,1)$$
$$T(0,1) = -1 (1,0) + 0 (0,1)$$

Here field is $\mathbb{R}$

T has no eigen value as the characteristic polynomial of T has no real roots.

Characteristic Polynomial :- Let $A_{n\times n}$ be a square matrix of order n. Then $P(x) = \det(x I_{n\times n} - A)$ is called the characteric polynomial of A.

For a linear transformation $T: V \to V$, its characteristic polynomial is defined by

$$P(x) = \det(x I_{n\times n} - [T]_B).$$

Suppose, λ is an eigen value of A. $\exists$ non zero X s.t. $AX = \lambda X$

$$\Rightarrow \underbrace{(A - \lambda I_n)}_{n\times n} X = 0$$

If $(A - \lambda I_n)$ is invertible then $\underbrace{(A-\lambda I)^{-1}(A-\lambda I)X}_{=0}$

$\Rightarrow X = 0$, which is a contradiction.

$\Rightarrow \det(A - \lambda I_n) = 0$, i.e, λ satisfies $P(x)$

$I_V : V \to V$ is a linear transformation.

$I_V(v) = v$ for all $v \in V$.

Let B be a basis of V.

$$[I_V]_B = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} = I_{n\times n}$$

We know,

$$[T+S]_B = [T]_B + [S]_B$$

$$[T - \lambda I_V]_B = [T]_B - \lambda [I_V]_B$$
$$= [T]_B - \lambda I_{n\times n}$$

If $(T - \lambda I_V) X = 0$, i.e., $([T]_B - \lambda I_{n\times n})[X]_B = 0$

$\Rightarrow \det([T]_B - \lambda I_{n\times n}) = 0$ $[\because [X]_B \neq 0]$

Theorem (Cayley Hamilton) :- Every matrix A satisfies its characteristic polynomial $P(x) = \det(x I_{n\times n} - A)$, i.e., $P(A) = 0$.

If $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Then $a_0 + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$

Now, $P(0) = \det(-A) = (-1)^n \det(A) = a_0$

A is invertible if and only if $a_0 \neq 0$.

A is Not invertible if and only if 0 is an eigen value of A.

If A is invertible then $A^{-1}(a_0 + a_1 A + \cdots + a_n A^n) = 0$

$$\Rightarrow A^{-1} = -\frac{1}{a_0}\left(a_1 + a_2 A + \cdots + a_n A^{n-1}\right).$$

Example :- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Characteristic Polynomial $P(x) = \det(x I_{3\times3} - A)$

$$= \det\left(\begin{bmatrix} x-1 & 0 & 0 \\ 0 & x-1 & -1 \\ -1 & -1 & x \end{bmatrix}\right)$$

$$\Rightarrow P(x) = (x-1)\begin{vmatrix} x-1 & -1 \\ -1 & x \end{vmatrix}$$

$$= (x-1)\left[x(x-1) - 1\right]$$

$$= (x-1)(x^2 - x - 1)$$

$$= x^3 - 2x^2 + 1$$

Since $P(0) = 1$, A is invertible.

$$A^3 - 2A^2 + I = 0$$

$$\Rightarrow \quad I = +2A^2 - A^3$$

$$\Rightarrow \quad A^{-1} = 2A - A^2$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Eigen space :- Let $T : V \to V$ be a linear transformation, λ be an eigen value of T. Then $W = \{ X \in V : TX = \lambda X \}$ is called the eigen space corresponding to the eigen value λ.

A element $x \in W$ is called a eigen vector of T corresponds to the eigen value λ.

W is a Subspace :- $0 \in W$

$x_1, x_2 \in W$

$Tx_1 = \lambda x_1$

$Tx_2 = \lambda x_2$

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$$

$$= \alpha \lambda x_1 + \beta \lambda x_2$$

$$= \lambda (\alpha x_1 + \beta x_2)$$

$\Rightarrow \alpha x_1 + \beta x_2 \in W, \quad \alpha, \beta \in F$

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct eigen values of T.

let $v_1, v_2, \cdots, v_n$ are eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

Suppose $\underbrace{\{v_1, v_2, \cdots, v_n\}}_{= B}$ is a basis of V.

$$[T]_B = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

In this case T is called diagonalizable.

[Suppose, λ is repeated m times in $P(x)$. Then what is the dimension of the eigen space of λ.]