MTL101 Tutorial 1 Solutions

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Question 1

Suppose we have a system of three linear equations in real coefficients and in two unknowns:

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$

Interpret geometrically the following statements:

1. **The system has no solutions.**
2. **The system has a unique solution.**
3. **The system has infinitely many solutions.**

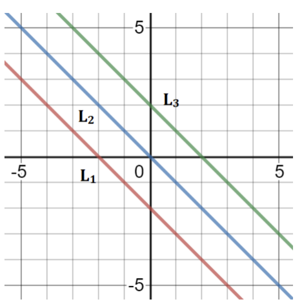
Further, provide values to $a_i, b_i, c_i \in \mathbb{R}$ $(i = 1, 2, 3)$ so that the above statements hold.

Solution

1. **The system has no solutions.**

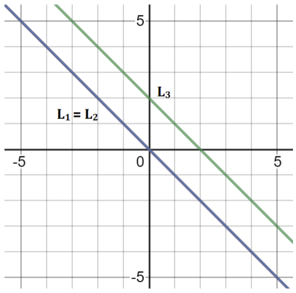
   - All three lines are parallel.

$$\begin{aligned} &L_1 : x + y = -2, \\ &L_2 : x + y = 0, \\ &L_3 : x + y = 2 \end{aligned}$$



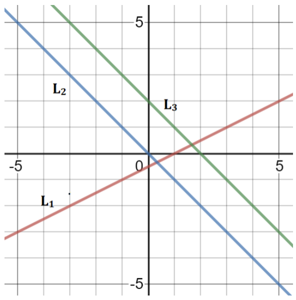
- Two lines coincide and other line is parallel to them.

$$L_1 : x + y = 0,$$
$$L_2 : x + y = 0,$$
$$L_3 : x + y = 2$$



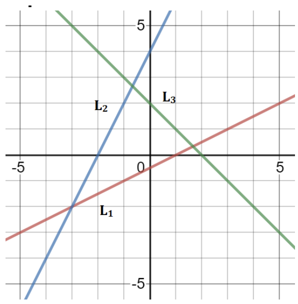
- Two lines are parallel, and the third line intersects the other two lines.

$$L_1 : x - 2y = 1,$$
$$L_2 : x + y = 0,$$
$$L_3 : x + y = 2$$



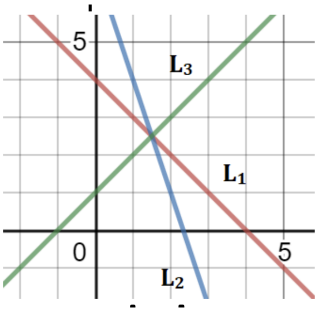
- Three lines intersect at three different points.

$$L_1 : x - 2y = 1,$$
$$L_2 : 2x - y = -4,$$
$$L_3 : x + y = 2$$



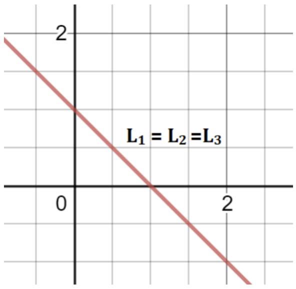
2. **The system has a unique solution.** All the three lines have only one common point of intersection.

$$L_1 : x + y = 4,$$
$$L_2 : 3x + y = 7,$$
$$L_3 : x - y = -1$$



3. **The system has infinitely many solutions.** All three lines coincide.

$$L_1 : x + y = 1,$$
$$L_2 : x + y = 1,$$
$$L_3 : x + y = 1$$



Question 2

Suppose we have a system of three linear equations in real coefficients and in three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Interpret geometrically the following statements:

1. **The system has no solutions.**
2. **The system has a unique solution.**
3. **The system has infinitely many solutions.**

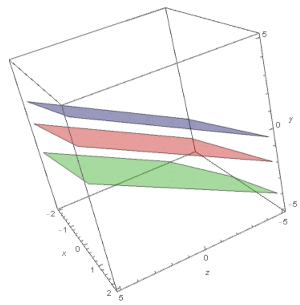
Further, provide values to $a_i, b_i, c_i, d_i \in \mathbb{R}$ $(i = 1, 2, 3)$ so that the above statements hold.

Solution

1. **The system has no solutions.**

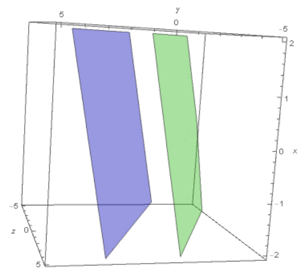
   - All three planes are parallel.

$$
\begin{aligned}
P_1 &: x - 2y + z = -2, \\
P_2 &: x - 2y + z = 1, \\
P_3 &: x - 2y + z = 5
\end{aligned}
$$



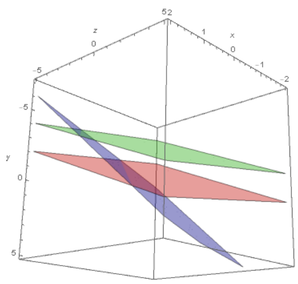
   - Two planes coincide and other plane is parallel.

$$
\begin{aligned}
P_1 &: x - 2y + z = -2, \\
P_2 &: x - 2y + z = 5, \\
P_3 &: x - 2y + z = 5
\end{aligned}
$$



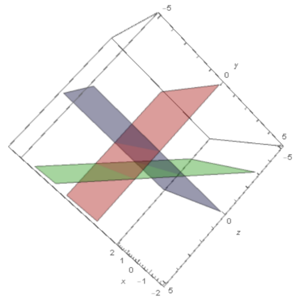
   - Two planes are parallel and the third plane intersects the other two planes.

$$
\begin{aligned}
P_1 &: x + y - z = 1, \\
P_2 &: x - 2y + z = 1, \\
P_3 &: x - 2y + z = 5
\end{aligned}
$$



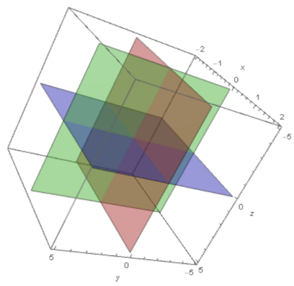
- Line of intersection of two planes is parallel to the third plane.

$$P_1 : z = 0,$$
$$P_2 : y = 0,$$
$$P_3 : y = 0$$



2. **The system has a unique solution.** Line of intersection of two planes intersects the third plane.

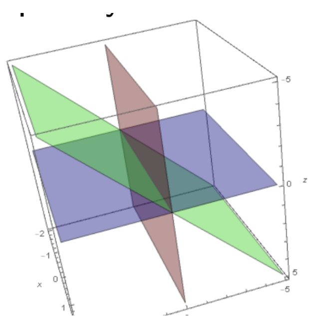
$$P_1 : x = 0,$$
$$P_2 : y = 0,$$
$$P_3 : z = 0$$



3. **The system has infinite solutions.**

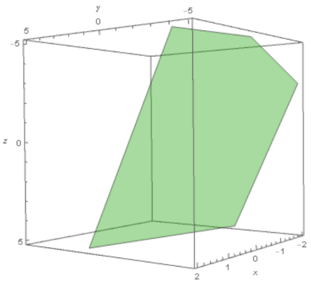
   - Line of intersection of two planes completely lies on the third plane.

$$P_1 : y + z = 0,$$
$$P_2 : y = 0,$$
$$P_3 : z = 0$$



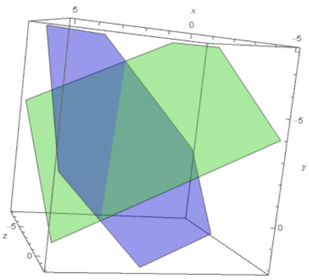
- All three planes coincide.

$$
\begin{aligned}
P_1 &: x - 2y + z = 5, \\
P_2 &: x - 2y + z = 5, \\
P_3 &: x - 2y + z = 5
\end{aligned}
$$



- Two planes coincide and the third plane intersects these planes.

$$
\begin{aligned}
P_1 &: x + y - z = 1, \\
P_2 &: x - 2y + z = 5, \\
P_3 &: x - 2y + z = 5
\end{aligned}
$$



Question 3

Suppose we have a system of two linear equations in real coefficients and in three unknowns:

$$
\begin{aligned}
a_1x + b_1y + c_1z = d_1 \\
a_2x + b_2y + c_2z = d_2
\end{aligned}
$$

Interpret geometrically the following statements:

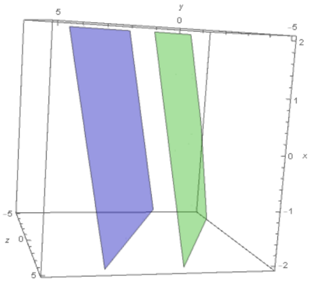
1. **The system has no solutions.**
2. **The system has infinitely many solutions.**

Further, provide values to $a_i, b_i, c_i, d_i \in \mathbb{R}$ $(i = 1, 2, 3)$ so that the above statements hold. Can you have a unique solution?

Solution

1. **The system has no solutions.** Two planes are parallel.

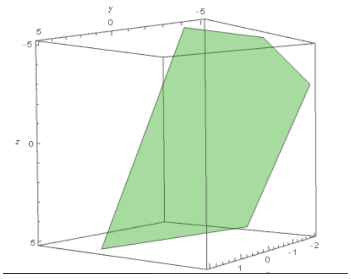
$$P_1 : x - 2y + z = -2,$$
$$P_2 : x - 2y + z = 5$$



2. **The system as infinitely many solutions.**

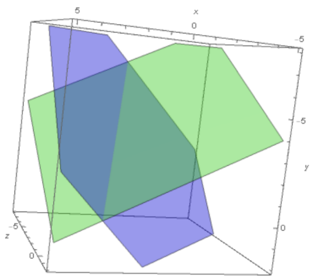
   - Two planes coincide.

$$P_1 : x - 2y + z = 5,$$
$$P_2 : x - 2y + z = 5$$



   - Two planes intersect each other.

$$P_1 : x + y - z = 1,$$
$$P_2 : x - 2y + z = 5$$



As the number of unknown variables are greater than the number of equations, the system cannot have a unique solution.

Question 4

Suppose we have a system of linear equations in complex coefficients. Can we interpret the statements in questions 1, 2, and 3 exactly the same way? Solve the following system of linear equations (for finding x, y in $\mathbb{C}$):

$$(1-i)x + (1+i)y = 2 + 3i,$$
$$(1+i)x + (1-i)y = 3 - i.$$

Solution

Interpretation in Complex Space
We can interpret this system of equations in $\mathbb{C}$ the same way as in $\mathbb{R}$ because both are **fields**. Recall that fields satisfy:

- Closure, associativity, and commutativity for addition and multiplication.
- Distributive property: $a(b + c) = ab + ac$.
- Existence of additive and multiplicative identities (0 and 1).
- Existence of additive inverses ($-a$) and multiplicative inverses (a^{-1}, $a \neq 0$).

These properties allow Gaussian elimination and row operations to be valid in $\mathbb{C}$, because we can add, subtract, and multiply by scalars from the field and divide by non-zero scalars (since multiplicative inverses exist), ensuring we can solve systems of equations in $\mathbb{C}$ using the same techniques as in $\mathbb{R}$. The distributive properties of the fields also ensures that operations like row-reduction are consistent.

For real numbers ($\mathbb{R}$), a system of linear equations represents a set of lines (or planes) in real space. Similarly, for complex numbers ($\mathbb{C}$), a system of linear equations represents a set of planes in complex space. The solutions (if they exist) are points in this space, and the operations to find these solutions (like Gaussian elimination) don't depend on whether the coefficients are real or complex—they depend on the field properties.

Why does this field structure matter? This is because fields ensure the consistency of arithmetic operations, which is the backbone of linear algebra. Without a field structure, some operations like division and row reduction may not be valid, and the interpretation of a system of equations or matrix inversion would break down.

Solution

We aim to solve the system:

$$(1-i)x + (1+i)y = 2 + 3i,$$
$$(1+i)x + (1-i)y = 3 - i.$$

We will solve using the matrix equation $A\mathbf{X} = \mathbf{B}$, and then use the standard elimination process, using the augmented matrix.

The augmented matrix is shown below. We perform the following elementary row operations:

$$\left(\begin{array}{cc|c} 1-i & 1+i & 2+3i \\ 1+i & 1-i & 3-i \end{array}\right) \xrightarrow{R_1 \to \frac{R_1}{1-i}} \left(\begin{array}{cc|c} 1 & \frac{1+i}{1-i} & \frac{(2+3i)(1+i)}{2} \\ 1+i & 1-i & 3-i \end{array}\right) = \left(\begin{array}{cc|c} 1 & i & \frac{-1+5i}{2} \\ 1+i & 1-i & 3-i \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2-(1+i)R_1} \left(\begin{array}{cc|c} 1 & i & \frac{-1+5i}{2} \\ 0 & 2 & 6-3i \end{array}\right) \xrightarrow{R_2 \to \frac{R_2}{2}} \left(\begin{array}{cc|c} 1 & i & \frac{-1+5i}{2} \\ 0 & 1 & \frac{6-3i}{2} \end{array}\right) \xrightarrow{R_1 \to R_1 - iR_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{-4-i}{2} \\ 0 & 1 & \frac{6-3i}{2} \end{array}\right)$$

Answer

From the final matrix, we have:

$$x = \frac{-4-i}{2}, \quad y = \frac{6-3i}{2}.$$

Question 5

Suppose the lines L_1 and L_2 are defined by $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z-1}{3}$. Show that $L_1 \cap L_2$ is empty. Write a system of four linear equations in three unknowns in standard form which has no solutions using the fact that $L_1 \cap L_2$ is empty.

Solution

Recall that to find the solutions of two equations in 3D space, we can write out their parametric forms and equate them. So that's what we will do here. Each line in parametric form would give us:

For L_1:

$$\begin{aligned} x &= t+1 \\ y &= 2t+2 \\ z &= 3t+3 \end{aligned}$$

For L_2:

$$\begin{aligned} x &= s+2 \\ y &= 2s+3 \\ z &= 3s+1 \end{aligned}$$

where t, s are parameters.

If the lines intersect, there must exist values of s and t where the points are equal:

$$\begin{aligned} t+1 &= s+2 \\ 2t+2 &= 2s+3 \\ 3t+3 &= 3s+1 \end{aligned}$$

From the first equation:

$$t = s+1$$

Substituting this into the second equation:

$$\begin{aligned} 2(s+1)+2 &= 2s+3 \\ 2s+2+2 &= 2s+3 \\ 2s+4 &= 2s+3 \\ 4 &= 3 \end{aligned}$$

This is a contradiction. Therefore, L_1 and L_2 do not intersect.

To write a system of four linear equations in three unknowns that has no solution, we can use the fact that if a point (x, y, z) lies on both lines, these equations must be satisfied: From L_1: The point must lie on the line, so:

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

This gives us two independent equations:

$$\begin{aligned} 2(x-1) &= y-2 \\ 3(x-1) &= z-3 \end{aligned}$$

From L_2: Similarly,

$$\frac{x-2}{1} = \frac{y-3}{2} = \frac{z-1}{3}$$

This gives us two more independent equations:

$$\begin{aligned} 2(x-2) &= y-3 \\ 3(x-2) &= z-1 \end{aligned}$$

Therefore, our system of four equations in standard form is:

$$\begin{aligned} 2x - y + 0z &= 0 \\ 3x + 0y - z &= 0 \\ 2x - y + 0z &= 1 \\ 3x + 0y - z &= 5 \end{aligned}$$

Since L_1 and L_2 don't intersect, this system has no solutions.

Question 6

Suppose A, B are square matrices of the same size. Prove the following statements:

1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ **and** $\mathrm{tr}(AB) = \mathrm{tr}(BA)$**.**
2. $\det(AB) = \det(A)\det(B)$ **(assume A and B are 2×2 matrices) and in particular,** $\det(AB) = \det(BA)$**.**
3. $\det(A+B) = \det(A) + \det(B)$ **is *false*.**

Solution

1. Recall that the **trace** of a matrix is the sum of its diagonal elements. Assume that the size of the matrix is $n \times n$. So, $\text{tr}(M) = \sum_{i=1}^{n} m_{ii}$.

   We have $\text{tr}(A + B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \text{tr}(A) + \text{tr}(B)$. Thus the first part of the question is proved.

   We also have $\text{tr}(AB) = \sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij} \cdot b_{ji}) = \sum_{i=1}^{n}(\sum_{j=1}^{n} b_{ij} \cdot a_{ji}) = \text{tr}(BA)$. Thus the second part of the question is also proved. This is because we can easily interchange the order of i and j since they are independent variables.

2. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

   Then $AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$

   First, let's calculate $\det(AB)$:

$$\begin{aligned}\det(AB) &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \\ &= a_{11}b_{11}a_{21}b_{12} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22} \\ &\quad - (a_{11}b_{12}a_{21}b_{11} + a_{11}b_{12}a_{22}b_{21} + a_{12}b_{22}a_{21}b_{11} + a_{12}b_{22}a_{22}b_{21})\end{aligned}$$

   Now, let's calculate $\det(A)\det(B)$:

$$\begin{aligned}\det(A) &= a_{11}a_{22} - a_{12}a_{21} \\ \det(B) &= b_{11}b_{22} - b_{12}b_{21}\end{aligned}$$

   Therefore,

$$\begin{aligned}\det(A)\det(B) &= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) \\ &= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}\end{aligned}$$

   After grouping and rearranging terms in $\det(AB)$, we get the same expression as $\det(A)\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$.

   As a corollary, since this is true for any 2×2 matrices A and B, we can conclude that:

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

   Thus, $\det(AB) = \det(BA)$.

   NOTE: How do we prove this for any general matrix of size $n \times n$? It is interesting to see the proof of this. For that, we will need a Lemma.

   LEMMA: For any square matrix A and elementary matrix E, $\det(EA) = \det(E)\det(A)$.

   Let us use this lemma now.

   Let A be an $n \times n$ invertible matrix. By the theory of elementary row operations, there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_n$ such that:

$$I = E_n \cdot E_{n-1} \cdot \ldots \cdot E_1 \cdot A$$

Therefore:

$$A = E_1^{-1} \cdot E_2^{-1} \cdot \ldots \cdot E_n^{-1}$$

Now, consider $\det(AB)$:

$$\begin{aligned}
\det(AB) &= \det(E_1^{-1} \cdot E_2^{-1} \cdots E_n^{-1} B) \\
&= \det(E_1^{-1}) \det(E_2^{-1} \cdots E_n^{-1} B) \quad \text{(by Lemma)} \\
&= \det(E_1^{-1}) \det(E_2^{-1}) \det(E_3^{-1} \cdots E_n^{-1} B) \quad \text{(by Lemma)} \\
&\vdots \\
&= \det(E_1^{-1}) \det(E_2^{-1}) \cdots \det(E_n^{-1}) \det(B) \\
&= \det(E_1^{-1} \cdot E_2^{-1} \cdots E_n^{-1}) \det(B) \\
&= \det(A) \det(B)
\end{aligned}$$

The above proof assumes A is invertible. For the case where A is not invertible, we know that $\det(A) = 0$. We need to show that $\det(AB) = 0$ as well.

Since A is not invertible, its rows are linearly dependent. This means that any row of AB can be expressed as a linear combination of other rows of AB, making AB singular. Therefore, $\det(AB) = 0 = \det(A) \det(B)$.

Therefore, for any $n \times n$ matrices A and B:

$$\det(AB) = \det(A) \det(B)$$

Hence,

$$\det(AB) = \det(A) \det(B) = \det(B) \det(A) = \det(BA)$$

3. Let's take two simple matrices, in this case both of them are the identity matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then:

$$A + B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{aligned}
\det(A + B) &= \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
&= (2)(2) - (0)(0) \\
&= 4
\end{aligned}$$

$$\begin{aligned}\det(A) + \det(B) &= \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= (1)(1) - (0)(0) + (1)(1) - (0)(0) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Therefore, $\det(A + B) \neq \det(A) + \det(B)$ since $4 \neq 2$.

Question 7

Which of the following matrices are row-reduced echelon matrices? Give a reason when the matrix is not row-reduced echelon.

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution

Let us recall the definition of an RRE (row - reduced echelon matrix.) A matrix is in RRE form if

- It is row-reduced, that is, the first non-zero entry (leading entry) in each non-zero row is equal to 1, and the column of R which contains the non-zero leading coefficient for a row has all other elements equal to zero.
- Any rows consisting entirely of zeroes are entirely at the bottom.
- In each non-zero row, the first non-zero entry (leading entry) is in a column to the left of any leading entry below it. That is, if rows $1, 2, \ldots, r$ are the non-zero rows of a matrix R, and if the leading non-zero entry of row i occurs in column k_i, for $i = 1, 2, \ldots, r$ then
$$k_1 < k_2 < \cdots < k_r$$

NOTE: A matrix is called row-echelon if it satisfies the last two properties only.

1. The first matrix is not row-reduced, since the leading entry of the second row is not 1, so it is not RRE.
2. The second matrix is not row-reduced, so it is not RRE.
3. The third matrix does not satisfy the third condition, so it is not RRE.
4. The row consisting entirely of zeroes is not at the bottom, so it is not RRE.
5. This matrix satisfies all the conditions, so it is an RRE matrix,

Question 8

Find RRE matrices row-equivalent to the matrices in the previous question.

Solution

Recall that a matrix B is row-equivalent to a matrix A if B can be obtained from A after a finite number of elementary row operations. Also recall that there are three kinds of elementary row operations:

- Interchange of two rows R_i and R_j. In shorthand notation, $R_i \leftrightarrow R_j$.
- Multiply a row R_i by a non-zero constant c. In shorthand notation, $R_i \to cR_i$.
- Add a multiple of a row R_j to another row R_i. In shorthand notation, $R_i \to R_i + \lambda R_j$.

Important Note: Every $m \times n$ matrix over the field $\mathbb{F}$ is row-equivalent to a row-reduced matrix. Also, every $m \times n$ matrix over the field $\mathbb{F}$ is row-equivalent to a row - reduced echelon matrix. The RRE form of the matrix can be gotten by applying a finite number of the first type of elementary row operations to the row - reduced form of the matrix.

1. We have
$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$
We can apply the elementary row transformation $R_2 \to \frac{R_2}{2}$ to get the RRE matrix which is row-equivalent to the original matrix.
$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

2. We have
$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$
We can apply two elementary row operations to get the RRE form:
$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 5R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
We get the identity matrix I which is row-equivalent to the original matrix and is in RRE form.

3. We have
$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

We can apply the row interchange $R_1 \leftrightarrow R_2$ to get

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is now in RRE form.

4. We have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can apply $R_2 \leftrightarrow R_3$ to get

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is RRE.

5. This matrix is already RRE, and **since the RRE form of a matrix is unique**, this matrix does not need to be converted to an RRE form.

Question 9

Show that every elementary matrix is invertible and the inverse is an elementary matrix.

Solution

Recall that an elementary matrix can be obtained by applying elementary row operations to the identity matrix. Suppose that E is the elementary matrix, and E' is its inverse, if it exists (which it inevitably does). Let q_k be a generic elementary row operation, then we have

$$E = q_k(q_{k-1}(\ldots q_1(I))\ldots)$$

where q is the elementary matrix. Let $\tilde{q}_k$ denote the inverse operation of q_k. For example, if q_k is $R_j \to R_j - \lambda R_i$, then $\tilde{q}_k$ is $R_j \to R_j + \lambda R_i$. Thus, we have
Since elementary row operations can always be reversed, the inverse of every elementary row operation exists, and thus the inverse of every elementary matrix exists. As can be seen, the inverse of an elementary matrix is also an elementary matrix. Thus we have:

$$E' = \tilde{q}_1(\tilde{q}_2(\ldots \tilde{q}_k(I))\ldots)$$

Hence proved.

Question 10

Compute the rank of the following matrices. Determine which are invertible.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix}$$

Solution

Recall that the RANK of a matrix is the number of non-zero rows in the row-reduced echelon form of the matrix. A square matrix is invertible if its rank is equal to its size, and the RRE form of it is the Identity Matrix.
Thus, we need to find the RREF of each matrix.

1. We can do the following elementary row operations:

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + 3R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

   This matrix is in RRE form. Thus the rank of this matrix is 2. Thus this matrix is non-invertible.

2. We have

$$\begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix} \xrightarrow[R_2 \to R_2 + R_1]{R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{R_3}{2}} \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_3} \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 4R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Thus this RREF matrix has a rank of 3 and is thus invertible.

3. We have

$$\begin{pmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - 3R_1} \begin{pmatrix} 1 & 3 & -4 \\ 0 & 2 & 3 \\ 0 & 4 & 6 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 3 & -4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \to \frac{R_2}{2}} \begin{pmatrix} 1 & 3 & -4 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{pmatrix} 1 & 0 & -\frac{17}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

   Clearly the RREF matrix has a rank of 2 and is thus non-invertible.

Question 11

Find inverse of the invertible matrices in the previous question by reducing the matrix to row-reduced echelon form (identity matrix).

Solution

There was only one matrix which was invertible in the previous question. Let us recall some things first, before we find the inverse of that second matrix.

1. Suppose R is RRE and row-equivalent to a matrix A. Then A is invertible if and only if R is invertible.
2. An $n \times n$ RRE matrix is invertible if and only if it is the identity matrix I_n.
3. Suppose the augmented matrix $(A|I)$ is row-equivalent to $(R|B)$. Then A is invertible if and only if $R = I$ and in this case $B = A^{-1}$.

Thus, we have the augmented matrix $(A|I)$ as shown below. We will now perform elementary row operations to get to the form $(I|B)$, and B will be the inverse of A.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ -1 & -1 & 5 & 0 & 1 & 0 \\ 2 & 7 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_1]{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 5 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -5 & -3 & 1 \end{array}\right) \xrightarrow{R_3 \to \frac{R_3}{2}} \left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2 + 4R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & -11 & 3 \\ 0 & 1 & 0 & \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right)$$

Thus the inverse of the matrix has been found, which is:

$$\begin{pmatrix} -16 & -11 & 3 \\ \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

IMPORTANT NOTE: In the last step, I have applied two elementary row operations together in a single step. This is NOT an elementary row operation, however, for keeping the solution short (read - I was lazy) I have done it like that.

Question 12

Write the following matrices as the product of elementary matrices (wherever possible):

$$\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}$$

Solution

The way we will go about doing this is finding the elementary row operations (and thus the matrices) to convert this to the RRE form (which should be the identity matrix if we are to write this purely as a product of elementary matrices). The answer will then be the inverses of all the elementary matrices multiplied in **reverse order.** In mathematical notation, if you can pre-multiply k elementary matrices E_i, $i \in [1, k]$ to the matrix A to get the identity matrix I, that is,

$$I = E_k(E_{k-1}(\ldots E_1(A)\ldots))$$

then A can be written as

$$A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}$$

since every elementary matrix is invertible.

1. We have

$$\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$$

It can be easily seen that applying the operations $R_2 \to R_2 + 2R_1$, $R_2 \to -\frac{R_2}{2}$, and $R_1 \to R_1 + 3R_2$ in succession gives us the identity matrix I_2. We can multiply the inverses of the elementary matrices of the elementary row operations in reverse order to get back the matrix A, which in this case is

$$\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

And this gives us the required decomposition.

2. We have

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

By applying the following elementary row operations in succession, we get the identity matrix: $R_2 \to R_2 - 4R_3$, then $R_1 \to R_1 - 3R_3$, and then $R_1 \to R_1 - 2R_2$ to get the identity matrix I_3. It is suggested that one verifies this by working it out on paper. You can then write the matrix as follows by the above theory as explained. If we call the matrix as A, then

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. We have the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{pmatrix}$$

On applying the elementary row operations $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 3R_1$, $R_3 \to R_3 - 2R_2$, and $R_1 \to R_1 - R_2$, we get the RRE form of the matrix, which is:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly the matrix is not invertible since the rank of the matrix is 2. That is, after applying a finite amount of elementary row operations, we cannot get the identity matrix I_3. Thus we cannot write this matrix as a product of elementary matrices.

Question 13

Solve the following systems of homogenous linear equations by reducing the coefficient matrix into row - reduced echelon form:

1. $$\begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ 3x_1 + 6x_2 - 7x_3 + 4x_4 &= 0 \\ 5x_1 + 10x_2 - 11x_3 + 6x_4 &= 0 \end{aligned}$$

2. $$\begin{aligned} x - 2y - 3z &= 0 \\ 2x + y + 3z &= 0 \\ 3x - 4y - 2z &= 0 \end{aligned}$$

3. $$\begin{aligned} x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 &= 0 \\ 2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 &= 0 \\ 3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 &= 0 \end{aligned}$$

Solution

Recall that we can solve a system of homogenous linear equations pretty simply - convert the coefficient matrix to the RREF of the matrix. A pivot column (the column where the leading entry of a row is present) has a dependent variable, and the non-pivot columns correspond to independent variables. So we can solve things easily.

1. We have the following system of equations:

$$\begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ 3x_1 + 6x_2 - 7x_3 + 4x_4 &= 0 \\ 5x_1 + 10x_2 - 11x_3 + 6x_4 &= 0 \end{aligned}$$

We can construct the coefficient matrix

$$\begin{pmatrix} 2 & 4 & -5 & 3 \\ 3 & 6 & -7 & 4 \\ 5 & 10 & -11 & 6 \end{pmatrix}$$

After applying a number of row operations to get to the RRE form of this matrix (which should be easy by now, I will not list the steps to get to the RRE form, will rather just show it below):

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By the theory above, the second and fourth variables are thus free variables. Call $x_2 = s$ and $x_4 = t$. Then the solution tuple for this is

$$(x_1, x_2, x_3, x_4) = (t - 2s, s, t, t)$$

2. Similarly we can construct the coefficient matrix for the second matrix as:

$$\begin{pmatrix} 1 & -2 & -3 \\ 2 & 1 & 3 \\ 3 & -4 & -2 \end{pmatrix}$$

After converting to the RRE form, we get the matrix as the identity matrix I_3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This does not have any independent variables, and also the rank of this matrix is also 3 which equals the size of the matrix, so it has the singular trivial solution:

$$(x, y, z) = (0, 0, 0)$$

3. We have the following coefficient matrix:

$$\begin{pmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{pmatrix}$$

After converting to the RRE form, we get:

$$\begin{pmatrix} 1 & 2 & 0 & -\frac{19}{2} & \frac{5}{2} \\ 0 & 0 & 1 & \frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, there are only two pivot columns. So we will have only two dependent variables, namely x_1 and x_3. Set the free variables as $x_2 = a, x_4 = b, x_5 = c$. Then, we will get the solution of the system of linear equations as:

$$(x_1, x_2, x_3, x_4, x_5) = \left(-2a + \frac{19}{2}b - \frac{5}{2}c, a, -\frac{5}{2}b - \frac{1}{2}c, b, c\right)$$

Question 14

Solve the following systems of equations by reducing the augmented matrix to the row reduced echelon form:

1.
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1, \\ 2x_1 + 2x_3 &= 1, \\ x_1 - 3x_2 + 4x_3 &= 2, \end{aligned}$$

2.
$$\begin{aligned} x_1 + 7x_2 + x_3 &= 4, \\ x_1 - 2x_2 + x_3 &= 0, \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned}$$

3.
$$\begin{aligned} x_2 + 5x_3 &= -4, \\ x_1 + 4x_2 + 3x_3 &= -2, \\ 2x_1 + 7x_2 + x_3 &= -1, \end{aligned}$$

4.
$$\begin{aligned} -2x_1 - 3x_2 + 4x_3 &= 5, \\ x_2 - x_3 &= 4, \\ x_1 + 3x_2 - x_3 &= 2. \end{aligned}$$

Solution

Recall that the system of linear equations $AX = B$ has a solution *if and only if* $\text{rank}(A) = \text{rank}(A|B)$. The system has a unique solution *if and only if* $\text{rank}(A)$ is equal to the number of unknowns. An important remark to be noted is that if a linear system has more than one solution, then it has infinitely many solutions. However, this remark is not valid over a finite field. (As an example, $\mathbb{Z}_n, \forall n \in \mathbb{P}$. This set is the set $\{0, 1, 2, \cdots, n-1\}$. This forms a field under the addition and multiplication defined under modular arithmetic. It has to be prime because when $a, b \in \mathbb{Z}_n$, a and n are co-prime for all $n \neq 0$.)

1. Using elementary row transformations on the augmented matrix $(A|B)$, we have:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + R_2]{R_2 \to R_2/2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since rank$(A) = 2$ and rank$(A|B) = 2$, and also rank$(A|B) < 3$, the above system of equations has infinite solutions.

Here, x_3 is the free variable. Let $x_3 = a$. We have:

$$x_1 - x_2 + 2x_3 = 1, \quad x_2 - x_3 = -\frac{1}{2}$$

Solving these, we get:

$$x_2 = a - \frac{1}{2}, \quad x_1 = \frac{1}{2} - a$$

Therefore,

$$(x_1, x_2, x_3) = \left(\frac{1}{2} - a,\ a - \frac{1}{2},\ a\right)$$

2. Using elementary row operations on the augmented matrix $(A|B)$ we have,

$$\left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right) \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 + 4R_1} \left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & -9 & 0 & -4 \\ 0 & 33 & 13 & 7 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{R_2}{9}} \left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 33 & 13 & 7 \end{array}\right) \xrightarrow{R_3 \to R_3 - 33R_2} \left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 13 & -\frac{23}{3} \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{R_3}{13}} \left(\begin{array}{ccc|c} 1 & 7 & 1 & 4 \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{23}{39} \end{array}\right) \xrightarrow{R_1 \to R_1 - 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{8}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{23}{39} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{173}{117} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & -\frac{23}{39} \end{array}\right)$$

Since rank$(A) =$ rank$(A|B) = 3$, the above system of equations has a unique solution. Thus, the solution is:

$$(x_1, x_2, x_3) = \left(\frac{173}{117}, \frac{4}{9}, -\frac{23}{39}\right)$$

3. We have the augmented matrix $(A|B)$ as:

$$\left(\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{array}\right)$$

We will perform the following elementary row operations on the augmented matrix to get the RRE form:

$$\left(\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-2R_1}\left(\begin{array}{ccc|c}1&4&3&-2\\0&1&5&-4\\0&-1&-5&3\end{array}\right)\xrightarrow{R_3\to R_3+R_2}\left(\begin{array}{ccc|c}1&4&3&-2\\0&1&5&-4\\0&0&0&-1\end{array}\right)$$

Clearly the we have $\text{rank}(A) = 2 \neq 3 = \text{rank}(A|B)$. Thus the system of equations has no solution. (This is also pretty obvious since making the matrix equation $AX = B$ would provide an absurd result, $0 = -1$.)

4. We have the augmented matrix $(A|B)$ as:

$$\left(\begin{array}{ccc|c}-2&-3&4&5\\0&1&-1&4\\1&3&-1&2\end{array}\right)$$

We will perform the following elementary row operations on the augmented matrix to get the RRE form:

$$\left(\begin{array}{ccc|c}-2&-3&4&5\\0&1&-1&4\\1&3&-1&2\end{array}\right)\xrightarrow{R_1\leftrightarrow R_3}\left(\begin{array}{ccc|c}1&3&-1&2\\0&1&-1&4\\-2&-3&4&5\end{array}\right)$$

$$\xrightarrow{R_3\to R_3+2R_1}\left(\begin{array}{ccc|c}1&3&-1&2\\0&1&-1&4\\0&3&2&9\end{array}\right)\xrightarrow{R_3\to R_3-3R_2}\left(\begin{array}{ccc|c}1&3&-1&2\\0&1&-1&4\\0&0&5&-3\end{array}\right)$$

$$\xrightarrow{R_3\to R_3/5}\left(\begin{array}{ccc|c}1&3&-1&2\\0&1&-1&4\\0&0&1&-\frac{3}{5}\end{array}\right)\xrightarrow{R_2\to R_2+R_3}\left(\begin{array}{ccc|c}1&3&-1&2\\0&1&0&\frac{17}{5}\\0&0&1&-\frac{3}{5}\end{array}\right)$$

$$\xrightarrow{R_1\to R_1-3R_2}\left(\begin{array}{ccc|c}1&3&-1&2\\0&1&-1&4\\0&0&1&-\frac{3}{5}\end{array}\right)\xrightarrow{R_2\to R_2+R_3}\left(\begin{array}{ccc|c}1&0&-1&-\frac{41}{5}\\0&1&0&\frac{17}{5}\\0&0&1&-\frac{3}{5}\end{array}\right)$$

$$\xrightarrow{R_1\to R_1-3R_2}\left(\begin{array}{ccc|c}1&0&0&-\frac{44}{5}\\0&1&0&\frac{17}{5}\\0&0&1&-\frac{3}{5}\end{array}\right)$$

Since $\text{rank}(A) = \text{rank}(A|B) = 3$, the above system of equations has a unique solution. Thus, the solution is:

$$(x_1, x_2, x_3) = \left(-\frac{44}{117}, \frac{17}{5}, -\frac{3}{5}\right)$$

Question 15

Consider the following system of equations:

$$\begin{aligned} x + 2y + z &= 3 \\ ay + 5z &= 10 \\ 2x + 7y + az &= b \end{aligned}$$

1. **Find all values of a for which the system of equations has a unique solution.**

2. **Find all pairs of values (a, b) for which the system has more than one solution.**

Solution

There are two ways to approach this problem. One is by using determinants, and the fact that a unique solution to the system exists **if and only if** the determinant of the coefficient matrix A, $\det(A) \neq 0$. The other method is by using elementary row operations, which is lengthier, however, both will reach the same result.

Method 1: Using determinants

We have $\det(A) = a^2 - 2a - 15 = (a-5)(a+3)$. Therefore, the system of equations has a unique solution if and only if $a \neq 5, -3$.

Method 2: Using elementary row operations

We have the augmented matrix $(A|B)$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 2 & 7 & a & b \end{array}\right)$$

Let's perform some elementary row operations. We will make some assumptions while performing the operations - these I will state while performing the operations.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 2 & 7 & a & b \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 3 & a-2 & b-6 \end{array}\right)$$

Now, we will perform the operation $R_3 \to R_3 - \frac{3}{a}R_2$. Here, we will have to assume that $a \neq 0$. So we get the following:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 0 & \frac{a^2-2a-15}{a} & \frac{ab-6a-30}{a} \end{array}\right)$$

Now, we will perform the operation $R_3 \to \frac{a}{a^2-2a-15}R_3$. Again, we are making the critical assumption that $a^2 - 2a - 15 \neq 0$, that is, $a \neq 5, -3$. Performing it, we get the following and then perform further operations as follows:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 0 & 1 & \frac{ab-6a-30}{a^2-2a-15} \end{array}\right) \xrightarrow{R_2 \to R_2 - 5R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 0 & 10 - 5\left(\frac{ab-6a-30}{a^2-2a-15}\right) \\ 0 & 0 & 1 & \frac{ab-6a-30}{a^2-2a-15} \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{R_2}{a}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & \frac{10}{a} - \frac{5}{a}\left(\frac{ab-6a-30}{a^2-2a-15}\right) \\ 0 & 0 & 1 & \frac{ab-6a-30}{a^2-2a-15} \end{array}\right) \xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 - \frac{ab-6a-30}{a^2-2a-15} \\ 0 & 1 & 0 & \frac{10}{a} - \frac{5}{a}\left(\frac{ab-6a-30}{a^2-2a-15}\right) \\ 0 & 0 & 1 & \frac{ab-6a-30}{a^2-2a-15} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 - \frac{ab-6a-30}{a^2-2a-15} - 2\left(\frac{10}{a} - \frac{5}{a}\left(\frac{ab-6a-30}{a^2-2a-15}\right)\right) \\ 0 & 1 & 0 & \frac{10}{a} - \frac{5}{a}\left(\frac{ab-6a-30}{a^2-2a-15}\right) \\ 0 & 0 & 1 & \frac{ab-6a-30}{a^2-2a-15} \end{array}\right)$$

Clearly, this matrix will have **unique** solutions if all of these assumptions, namely $a \neq 0, 5, -3$ are followed, since $\text{rank}(A) = \text{rank}(A|B)$. We need to check now what happens if these conditions are not followed? Let's check for the case $a = 0$ first.

What happens in the case $a = 0$? In this case we will have the following augmented matrix and perform the following operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 5 & 10 \\ 2 & 7 & 0 & b \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 7 & 0 & b \\ 0 & 0 & 5 & 10 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3/5} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 7 & 0 & b \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & 7 & 0 & b \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & b-2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2 \to R_2/3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & \frac{b-2}{3} \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 - 2\left(\frac{b-2}{3}\right) \\ 0 & 1 & 0 & \frac{b-2}{3} \\ 0 & 0 & 1 & 2 \end{array}\right)$$

We did not have to make any assumptions in this process to get to the RRE form of the coefficient matrix A, and clearly, $\text{rank}(A) = \text{rank}(A|B)$. Thus, for any value of b, this system will have a unique solution.

So, thus far, the values of a for which the system has unique solutions is $a \in \mathbb{R} \setminus 5, -3$, in both the methods. Let us check both of these explicitly now, as we checked $a = 0$.

Case: $a = -3$

Using elementary row operations on the augmented matrix, we get:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 2 & 7 & -3 & b \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 0 & 3 & -5 & b-6 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 5 & 10 \\ 0 & 0 & 0 & b+4 \end{array}\right)$$

We can see that $\text{rank}(A) = 2$. The above system will solutions if and only if $\text{rank}(A|B) = \text{rank}(A) = 2$. This gives us $b + 4 = 0 \implies b = -4$. Hence for the ordered pair $(a, b) = (-3, -4)$ the system has more than one solution, that is, infinite solutions.

Case: $a = 5$

Using elementary row operations on the augmented matrix, we have:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 2 & 7 & 5 & b \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 0 & 3 & 3 & b-6 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3/3]{R_2 \to R_2/5} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & \frac{b-6}{3} \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & \frac{b-12}{3} \end{array}\right)$$

We can see that rank$(A) = 2$. The above system will solutions if and only if rank$(A|B)$ = rank$(A) = 2$. This gives us $b - 12 = 0 \implies b = 12$. Hence for the ordered pair $(a, b) = (5, 12)$ the system has more than one solution, that is, infinite solutions.

Question 16

Find $a, b, c, p, q \in \mathbb{R}$ such that the following system has a solution:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & p \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Solution

Again there are two solutions - using elementary row operations and the determinant.

Method 1: Using determinants

The system will have a unique solution when $\det(A) = q \neq 0$. Therefore the system has a unique solution when $q \neq 0$ and $a, b, c, p \in \mathbb{R}$.

Now suppose that $q = 0$. Then we have the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & p & b \\ 0 & 0 & 0 & c \end{array}\right)$$

Clearly, rank$(A) = 2$. For the system to have a solution, it will have infinite solutions. In that case, we need rank$(A|B)$ = rank$(A) = 2$. For that, we need $c = 0$. Thus, the system will have infinite solutions when $a, b, p \in \mathbb{R}$ and $c, q = 0$.

In all other cases, the system will have no solutions.

Method 2: Using elementary row operations

We have the augmented matrix as shown below.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & p & b \\ 0 & 0 & 1 & c \end{array}\right)$$

We will perform the elementary row operations (subject to some conditions in between, which I will mention). The very first operation that we will perform will be $R_3 \to \frac{R_3}{q}$. For this, we make the critical assumption that $q \neq 0$. We will check the case for $q = 0$ later. We get thus and perform further operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & p & b \\ 0 & 0 & 1 & \frac{c}{q} \end{array}\right) \xrightarrow{R_2 \to R_2 - pR_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 0 & b - \frac{pc}{q} \\ 0 & 0 & 1 & \frac{c}{q} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 3R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & a - \frac{3c}{q} \\ 0 & 1 & 0 & b - \frac{pc}{q} \\ 0 & 0 & 1 & \frac{c}{q} \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a - 2b + \frac{c}{q}(2p-3) \\ 0 & 1 & 0 & b - \frac{pc}{q} \\ 0 & 0 & 1 & \frac{c}{q} \end{array}\right)$$

We have the RRE form of the coefficient matrix A in the augmented matrix. Since $\text{rank}(A) = 3$, then this system will have a unique solution (subject to the condition that $q \neq 0$). Thus the system will have a unique solution for $q \neq 0, a, b, c, p \in \mathbb{R}$.

What happens when $q = 0$? Then the same solution follows above as in Method 1, which I will replicate here:

We have the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & p & b \\ 0 & 0 & 0 & c \end{array}\right)$$

Clearly, $\text{rank}(A) = 2$. For the system to have a solution, it will have infinite solutions. In that case, we need $\text{rank}(A|B) = \text{rank}(A) = 2$. For that, we need $c = 0$. Thus, the system will have infinite solutions when $a, b, p \in \mathbb{R}$ and $c, q = 0$.