DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)

2023-24 SECOND SEMESTER TUTORIAL SHEET-I

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1. Let F be a field. Prove the following.
(i) If $a, b \in F$, with $a \neq 0$ and $b \neq 0$, then $ab \neq 0$.
(ii) For all $a \in F$, $-(-a) = a$.
(iii) For all $a, b \in F$, $(a)(-b) = (-a)(b) = -ab$.
(iv) For all $a, b \in F$, $(-a)(-b) = ab$
(v) For all $a \in F$, $a \neq 0$, $(a^{-1})^{-1} = a$ and $(-a)^{-1} = -a^{-1}$

2. Prove that each subfield of the complex numbers contain every rational number. (A subfield F of C is a field contained in C)

3. Show that the set of all matrices of same order form an abelian group with respect to matrix addition.

4. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$
$$c(x, y) = (cx, y).$$

Is V, with these operations, a vector space over the field of real numbers?.

5. Let V be the set of pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0).$$

Is V, with these operations, a vector space ?.

6. On R^n define two operations

$$x \oplus y = x - y$$
$$c.x = -cx.$$

The operations on the right are the usual ones. Which of the axioms from vector space are satisfied by $(R^n, \oplus, .)$?

7. Show that the real field R, is a vector space over the rational field. Is it finite dimensional?

8. Let $l_2 = \{$ Collection of all infinite sequences with real elements $(a_1, a_2, a_3, \ldots\ldots)$ such that $\sqrt{a_1^2 + a_2^2 + \ldots\ldots + a_n^2 + \ldots\ldots} < \infty$ $\}$. Define vector addition as a component wise addition and $c(a_1, a_2, a_3, \ldots\ldots) = (ca_1, ca_2, ca_3, \ldots\ldots)\ \forall\ c \in R$. Show that l_2 is a vector space.

9. Show that the set of all polynomials in an indeterminate x, over a field of real numbers form a vector space relative to addition of polynomials and multiplication of a polynomial by a real number.

10. If W is any subspace of a vector space $V(F)$, then show that the set $\frac{V}{W}$ of all cosets $W+x$ where x is any vector in $V(F)$ forms a vector space over F, under the operations defined by

$$(W+x)+(W+y)=W+(x+y), \qquad x,y\in V$$
$$\alpha(W+x)=W+\alpha x, \qquad \alpha\in F.$$

(The coset $W+x$ is defined as: $W+x=\{w+x \ : \ w\in W\}$
Also prove

$$dim(\frac{V}{W})=dimV-dimW.$$

11. Prove that the set V of all real valued continuous functions defined on $[0,1]$ forms a vector space over the field R of real numbers under addition and scalar multiplication defined by

$$(f+g)(x)=f(x)+g(x) \qquad f,g\in V$$
$$(\alpha f)(x)=\alpha f(x) \qquad \alpha\in R$$

for all $x\in[0,1]$.

12. Show that $\{1,i\}$ is linearly dependent in $C(C)$, but linearly independent in $C(R)$?

13. Let V be a vector space over a subfield F of the complex numbers. Suppose x,y,z are linearly independent vectors in V. Prove that $(x+y),(y+z)$ and $(z+x)$ are linearly independent.

14. If none of the elements appearing along the principal diagonal of a lower triangular matrix is zero, show that row(column) vectors are linearly independent.

15. Show that the set $\{(1,0,0),(1,1,0),(1,1,1,)\}$ is a basis of $C^3(C)$. Is it a basis of $C^3(R)$ also?. Determine the coordinates of $(2i,3+2i,5i)$ with respect to the given basis.

16. Show that the set of all real valued continuous functions $y=f(x)$ satisfying the differential equation

$$\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$$

is a vector space over R. Find a basis of this.

17. Which of the following are subspaces of (a) $C^n(R)$ $\qquad (b)$ $C^n(C)$
(i) $\{(z_1,z_2,\ldots\ldots,z_n) \ : \ z_1$ is real $\}$
(ii) $\{(z_1,z_2,\ldots\ldots,z_n) \ : \ z_1+z_2=z_3\}$
(iii) $\{(z_1,z_2,\ldots\ldots,z_n) \ : \ |z_1|=|z_2| \ \}$

18. Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

19. Let

$$M = \{(x_1, x_2, x_3) : x_1 + x_2 + 4x_3 = 0\}$$
$$N = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}.$$

Determine bases for $M + N$ and $M \cap N$ as subspaces of R^3.

20. Let W_1 and W_2 be subspaces of R^8 and dim $(W_1) = 6$, dim $(W_2) = 5$. What are possible dimensions of $W_1 \cap W_2$?.

21. Does there exist subspaces M, N of R^7 such that dim $M = 4 =$dim N and dim $(M \cap N) = 0$.

22. Show that the union of two subspaces of a vector space is a subspace if and only if one is contained in the other.

23. Let V be the vector space of problem 9. Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of the matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.
(b) Find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

24. Let V be set of all 2×2 matrices A with complex entries which satisfy $A_{11} + A_{22} = 0$.
(a) Show that V is a vector space over the field of real numbers, with usual operations of matrix addition and multiplication of a matrix by a scalar.
(b) Find a basis for this vector space.
(c) Let W be the set of all matrices A in V such that $A_{21} = -\overline{A_{12}}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.

25. Let $V(F)$ be a vector space. Let $W_1, W_2, \ldots\ldots, W_n$ be subspaces of V. Suppose

$$V = W_1 + W_2 + \ldots\ldots + W_n$$

and

$$W_i \cap \{ \sum_{j=1,\ j \neq i}^{n} W_j\} = \{0\}, \quad 1 \leq i \leq n.$$

Prove that

$$V = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_n.$$

26. Let V be the vector space of all functions from R into R; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.
(a) Prove that V_e and V_o are subspaces of V.

(b) Prove that $V_e + V_o = V$.
(c) Prove that $V_e \cap V_o = \{0\}$.

27. Find the coordinates of the vector $(1, 0, 1)$ in the basis of $C^3(C)$ consisting of vectors $(2i, 1, 0), (2, -1, 1), (0, 1 + i, 1 - i)$, in that order.

28. Let W be the subspace of $C^3(C)$ spanned by $x_1 = (1, 0, i)$ and $x_2 = (1 + i, 1, -1)$
(a) Show that x_1 and x_2 form a basis for W.
(b) Show that the vectors $y_1 = (1, 1, 0)$ and $y_2 = (1, i, 1 + i)$ are in W and form another basis for W.
(c) What are the coordinates of x_1 and x_2 in the ordered basis $\{y_1, y_2\}$ for W?

29. If

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

Find all solutions of $AX = 0$ by row reducing A.

30. Find a row reduced matrix which is row equivalent to

$$A = \begin{bmatrix} 1 & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}.$$

31. Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which triples (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

32. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix}.$$

(a) Find an invertible matrix P such that PA is a row reduced echelon matrix R.
(b) Find a basis for row space W of A.
(c) Say for which vectors $\{b_1, b_2, b_3, b_4\}$ are in W.
(d) Find the coordinate matrix of each vector $\{b_1, b_2, b_3, b_4\}$ in W in the ordered basis chosen in (b).
(e) Write each vector $\{b_1, b_2, b_3, b_4\}$ in W as a linear combination of the rows of A.
(f) Give explicit definition of the vector space V of 4×1 column vector X such that $AX = 0$.
(g) Find a basis for V?
(h) For what 4×1 vector the vector Y does the equation $AX = Y$ have a solution?