Lemma :- Every finite linearly independent set of vectors can be extendend to a basis of the vector space.

Proof :- Let $S = \{v_1, \cdots, v_n\}$ be a linearly independent set and B be a basis.

There exist $w_1, w_2, \cdots, w_m \in B$ such that $v_1 = c_1 w_1 + \cdots + c_m w_m$.

Since $v_1 \neq 0$, $\exists\, i \in \{1, \cdots, m\}$ such that $c_i \neq 0$

Then $c_i w_i = -c_1 w_1 - \cdots - c_{i-1} w_{i-1} + v_1 - c_{i+1} w_{i+1} - \cdots - c_m w_m$

$\Rightarrow w_i = -c_i^{-1} c_1 w_1 - \cdots - c_i^{-1} c_{i-1} w_{i-1} + c_i^{-1} v_1 - c_i^{-1} c_{i+1} w_{i+1} - \cdots - c_i^{-1} c_m w_m$

$\Rightarrow w_i \in \text{span}\{w_1, \cdots, w_{i-1}, v_1, w_{i+1}, \cdots, w_n\}$

Let $B' = \left(B \setminus \{w_i\}\right) \cup \{v_1\}$

Then $w_i \in \text{span}(B')$. Further, if $v \in B$ & $v \neq w_i$ then $v \in B'$. Thus $V = \text{span}(B) \subseteq \text{span}(B')$.

Therefore, $V = \text{span}(B')$.

claim, B' is linearly independent.

For every finite collection $u_1, \cdots, u_r \in B'$ if

$a_1u_1 + \cdots + a_r u_r = 0$ then we claim $c_i = 0$.

If $u_i \neq v_1$, then $\{u_1, \cdots, u_r\} \subseteq B$ & B is linearly independent. Therefore, $a_1 = \cdots = a_r = 0$.

If possible let $u_1 = v_1$. Then

$$a_1(c_1w_1 + \cdots + c_m w_m) + a_2u_2 + \cdots + a_r u_r = 0$$

Note that w_j may be same of u_k for some $j \in \{1, 2, \cdots, i-1, i+1, \cdots, m\}$ & $k \in \{2, \cdots, r\}$.

Here $w_1, \cdots, w_m, u_2, \cdots, u_r \in B$

Since B is linearly independent $a_1c_i = 0$, where $c_i \neq 0$

$\Rightarrow a_1 = 0$. Then $a_2u_2 + \cdots + a_r u_r = 0$ implies

$a_2 = \cdots = a_r = 0$

Thus B' is a basis of the vector space V.

Let $v_2 = p_1 v_1 + p_2 x_1 + p_3 x_2 + \cdots + p_{q+1} x_q$, where $v_1, x_1, \ldots, x_q \in B'$.

If $p_2 = \cdots = p_{q+1} = 0$ then $\{v_1, v_2\}$ is linearly dependent, which is a contradiction.

Let $p_j \neq 0$ for some $j \in \{2, \ldots, q+1\}$

Then $B'' = (B' \setminus \{x_j\}) \cup \{v_2\}$ is a new basis for V (By similar argument)

After a finite step we have a basis of V containing $\{v_1, \ldots, v_n\}$.

Corollary :- If $\dim(V) = n$ then for any linearly independent set S, $|S| \leq n$.

Corollary :- If $\dim(V) = n$ then any basis has exactly n elements.

Problem :- Extend the set $\{(1,1,1,1),(1,-1,1,-1)\}$ to a basis of $\mathbb{R}^4$.

We know that $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ is a basis of $\mathbb{R}^4$.

Let us examine if $(1,0,0,0) \in \text{span}(S)$, where $S=\{(1,1,1,1),(1,-1,1,-1)\}$

If possible let $(1,0,0,0) = c_1(1,1,1,1) + c_2(1,-1,1,-1)$

Then $c_1+c_2=1$ $\quad$ $c_1+c_2=0$

$c_1-c_2=0$ $\quad$ $c_1+c_2=-1$

The system is inconsistent (i.e., no solution).

Therefore $\{(1,0,0,0),(1,1,1,1),(1,-1,1,-1)\}$ is linearly independent.

Let us examine if $(0,1,0,0) \in \text{Span}(\{(1,0,0,0),(1,1,1,1),(1,-1,1,-1)\})$

If possible let, $(0,1,0,0) = c_1(1,0,0,0) + c_2(1,1,1,1) + c_3(1,-1,1,-1)$

$$\Rightarrow \left.\begin{aligned} c_1+c_2+c_3 &= 0 \\ c_2-c_3 &= 1 \\ c_2+c_3 &= 0 \\ c_2-c_3 &= 0 \end{aligned}\right\}$$

This is also inconsistent.

Thus, $\{(1,0,0,0),(0,1,0,0),(1,1,1,1),(1,-1,1,-1)\}$ is linearly independent, and hence a basis of $\mathbb{R}^4$.

Problem :- Find $\dim(S \cap T)$ where S and T are subspaces of vector space $\mathbb{R}^4$ given by

$$S = \{ (x, y, z, \omega) : 2x + y + 3z + \omega = 0 \}$$

$$T = \{ (x, y, z, \omega) : x + 2y + z + 3\omega = 0 \}.$$

Then $S \cap T = \{ (x, y, z, \omega) : 2x + y + 3z + \omega = 0 \text{ \& } x + 2y + z + \omega = 0 \}$

Let $(a, b, c, d) \in S \cap T$.

Then
$$2a + b + 3c + d = 0$$
$$a + 2b + c + 3d = 0$$

So, we have $a = -5b - 8d$ and $c = 3b + 5d$.

Thus, $(a, b, c, d) = (-5b - 8d, b, 3b + 5d, d)$

$$= b(-5, 1, 3, 0) + d(-8, 0, 5, 1)$$

Therefore, $S \cap T \subseteq \text{Span}(\{(-5, 1, 3, 0), (-8, 0, 5, 1)\})$.

Note that $(-5, 1, 3, 0), (-8, 0, 5, 1) \in S \cap T$.

Therefore, $\operatorname{Span}(\{(-5,1,3,0),(-8,0,5,1)\}) \subseteq S \cap T$.

Thus, $S \cap T = \operatorname{Span}(\{(-5,1,3,0),(-8,0,5,1)\})$.

Since $(-5,1,3,0)$ & $(-8,0,5,1)$ are not a scalar multiple of each other, the set $\{(-5,1,3,0),(-8,0,5,1)\}$ is linearly independent.

Therefore, $S \cap T = \operatorname{Span}(\{(-5,1,3,0),(-8,0,5,1)\})$

and $\dim(S \cap T) = 2$

Problem :- Let $\{\alpha, \beta, \gamma\}$ be a basis of a vector space V. Prove that the set $\{\alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$ is also a basis.

First we prove that the set $\{\alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$ is linearly independent.

Let $c_1(\alpha+\gamma) + c_2(\beta+\gamma) + c_3(\alpha+\beta+\gamma) = 0$

$$\Rightarrow (c_1+c_3)\alpha + (c_2+c_3)\beta + (c_1+c_2+c_3)\gamma = 0$$

Since $\{\alpha, \beta, \gamma\}$ is linearly independent,

$$c_1 + c_3 = 0 \quad \text{—} \; ①$$
$$c_2 + c_3 = 0 \quad \text{—} \; ②$$
$$c_1 + c_2 + c_3 = 0 \quad \text{—} \; ③$$

① + ② − ③ implies $c_3 = 0$, and hence $c_1 = c_2 = 0$ by ① & ②.

Since $\{\alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$ is linearly independent,

$$\operatorname{span}(\{\alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}) = V$$

Therefore, $\{\alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$ is a basis for V.

Alternately, prove

$$\operatorname{span}(\{\alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}) = \operatorname{span}(\{\alpha, \beta, \gamma\}).$$