Question -1

$$\left[\begin{array}{cccc|c} 1 & 1 & a+b & 3 & 2 \\ 0 & 1 & a+b & 0 & 1 \\ 2 & 2 & 3a+2b & a-b+6 & 6 \\ 0 & 1 & a+b & a-b & 2 \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & a+b & 3 & 2 \\ 0 & 1 & a+b & 0 & 1 \\ 0 & 0 & a & a-b & 2 \\ 0 & 1 & a+b & a-b & 2 \end{array}\right]$$

$R_1 \to R_1 - R_2$
$R_4 \to R_4 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & a+b & 0 & 1 \\ 0 & 0 & a & a-b & 2 \\ 0 & 0 & 0 & a-b & 1 \end{array}\right]$$

$R_3 \to R_3 - R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & a+b & 0 & 1 \\ 0 & 0 & a & 0 & 1 \\ 0 & 0 & 0 & a-b & 1 \end{array}\right]$$

2M

for consistency: $a \neq 0$ & $a-b \neq 0$ — 1M

then $x_4 = \frac{1}{a-b}$

$x_3 = \frac{1}{a}$

$x_2 = 1 - \frac{a+b}{a} = \frac{-b}{a}$

$x_1 = 1 - 3 \cdot \frac{1}{a-b} = \frac{a-b-3}{a-b}$

Solution: $\left(\frac{a-b-3}{a-b}, \frac{-b}{a}, \frac{1}{a}, \frac{1}{a-b} \right)$

1M

MTL101-Reminor-Question-2

Solution 2:

Let $S = \{u_1, u_2, \ldots, u_n\}$ be a minimal spanning subset of $V(\mathbb{F})$. Then, by definition we know linear span of S, that is, $L(S) = V(\mathbb{F})$. [1]

In order to show that S is a basis of V over $\mathbb{F}$, it is enough to prove that S is an linearly independent subset of $V(\mathbb{F})$.

Let us assume that S is linearly dependent. Then there exists $a_1, \cdots, a_n \in \mathbb{F}$ such that at least one $a_i \neq 0$ and

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0.$$

Without lose of generality, we may assume, $a_1 \neq 0$. Then,

$$u_1 = -(1/(a_1))(a_2u_2 + \ldots + a_nu_n).$$

This implies

$$u_1 \in L(\{u_2, u_3, \ldots, u_n\}).$$

Hence,

$$L(\{u_2, u_3, \ldots, u_n\}) = L(S) = V(\mathbb{F})$$

This contradicts the minimality of S. Hence S is a linearly independent set. [3].

Remarks.

1. Although this question was asked for any vector space but most of you answered by assuming V to be finite dimensional. Hence, the solution is written for finite dim vector space and answer copies are graded accordingly.
2. $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Q3.

Given the matrix,

$$A = \begin{pmatrix} 3 & 8 & 16 \\ 8 & 15 & 32 \\ -4 & -8 & -17 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

The characteristic equation is

$$\det(A-\lambda I) = 0,$$

which implies,

$$\begin{vmatrix} 3-\lambda & 8 & 16 \\ 8 & 15-\lambda & 32 \\ -4 & -8 & -17-\lambda \end{vmatrix} = 0 .$$

- Solving this the eigenvalues obtained are,

$$\lambda = -1, -1, 3 \quad \text{———} \quad +1$$

- Solving

$$(A-3I)x = 0 ,$$ we get

the eigen vector, which is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. +1

[Rank $(A-3I) = 2$, $\dim N(A-3I) = 1$]

- Solving $(A+I)x = 0$, for $\lambda = -1$,

the eigenvectors are

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \ \& \ \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

[Rank $(A+I) = 1$, $\dim N(A+I) = 2$]

+1

Let

$E_1 =$ eigenspace corresponding to $\lambda = 3$

& $E_2 =$ eigenspace corresponding to $\lambda = -1$.

As

$$\dim E_1 = \dim N(A-3I) = 1$$

$$\& \ \dim E_2 = \dim N(A+I) = 2$$

So,

$$\dim E_1 + \dim E_2 = 3$$

$+\frac{1}{2}$

Hence $A \in M_{3\times 3}(\mathbb{R})$ has 3 linearly independent eigenvectors.

$\therefore$ A is diagonalizable.

The matrix P, for which

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ is

given by,

$$P = \begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$ — $+\frac{1}{2}$

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Question 4

V is a finite dimensional vector space over F.
$W \subseteq V$ is a subspace.

2M $\boxed{\text{Proof of } \Rightarrow}$ Assume $\dim W > \frac{1}{2}\dim V$.

1M

Let $T: V \longrightarrow V$ be a one-to-one onto linear transformation.

Let $T|_W : W \to V$ be the restriction of T from V to W.

Then $T|_W$ is also one-to-one.

$\Rightarrow \text{nullity}(T|_W) = 0.$

Apply rank-nullity theorem for $T|_W$

$\Rightarrow \text{rank}(T|_W) = \dim W$

$\Rightarrow \dim(T(W)) = \dim W$

1M

Since $W, T(W) \subseteq V$ subspaces

$\Rightarrow W + T(W) \subseteq V$ subspace.

If $W \cap T(W) = \{0\}$ then

$$\begin{aligned} \dim(W + T(W)) &= \dim W + \dim T(W) \\ &= 2 \cdot \dim(W) \\ &> \dim V \end{aligned}$$

This is a contradiction since $W + T(W) \subseteq V$

Therefore, $W \cap T(W) \neq \{0\}$.

2M $\boxed{\text{Proof of } \Longleftarrow}$ Assume $W \cap T(W) \neq \{0\}$ for all one-one onto lin. trans. $T: V \longrightarrow V$.

1M If possible, assume $\dim W \leq \frac{1}{2} \dim V$.
Let $\{v_1, ---, v_m\}$ be a basis of W.
We extend this to a basis of V, say

$$\{v_1, ---, v_m, \omega_1, ---, \omega_m, ---\omega_k\}$$

where $\dim V = m+k$ & $k \geq m$.

Then define $T: V \longrightarrow V$ by

$$T(v_i) = \omega_i \quad \text{for } i=1, ---, m$$

$$T(\omega_i) = \begin{cases} v_i & \text{for } i=1, ---, m \\ \omega_i & \text{for } i>m \end{cases}$$

1M This T can be extended as a lin. trans. on V which is one-to-one onto.

But $W = \text{span}\{v_1, ---, v_m\}$

$$T(W) = \text{span}\{\omega_1, ---, \omega_m\}$$

$$W \cap T(W) = \{0\} \quad \text{a contradiction}$$

Therefore $\dim W \leq \frac{1}{2} \dim V$ is NOT possible

Hence, $\dim W > \frac{1}{2} \dim V$.

$$A = \begin{bmatrix} 4 & 24 & 6 \\ -1 & -7 & -2 \\ 2 & 12 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 24 & 6 \\ -1 & -7-\lambda & -2 \\ 2 & 12 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left((\lambda-3)(\lambda+7) + 24\right) + 24(-4+3-\lambda) + 6(-12+2\lambda+14) = 0$$

$$\lambda^3 - \lambda = 0 \text{ ——— } (1)$$

char eq. $A^3 - A = 0$ ——— (1)

$$A^{10} + A^{20} = (A^3)^3 \cdot A(I + A^2)$$

$$= A^2 + A^2$$

$$= 2A^2 \text{ ——— } (1)$$

$$= \begin{bmatrix} 8 & 0 & -12 \\ -2 & 2 & 4 \\ 4 & 0 & -6 \end{bmatrix} \text{ ——— } (1)$$

Q6 $\quad Mdx + Ndy = 0$

$M = \cos y\ ,\ N = \cot x \sin y.$ $\qquad y(\pi/4) = \pi/4.$

$$\frac{M_y - N_x}{N} = \frac{-\sin y + \sin y \operatorname{cosec}^2 x}{\cot x \sin y} = \frac{\sin y \cot^2 x}{\cot x \sin y}$$

$$= \cot x \qquad [\tfrac{1}{2}]$$

I.F. $\quad e^{\int \cot x\, dx} = e^{+\ln \sin x} = \sin x$
[2]

So, $\quad \sin x \cos y\, dx + \cos x \sin y\, dy = 0$ is exact.

So $\ u(x,y) = c\ $ is a soln where

$u_x = \sin x \cos y \quad \& \quad u_y = \cos x \sin y \rightarrow (*)$

integrating

$u = -\cos x \cos y + h(y)$

so that $\quad u_y = \cos x \sin y + h'(y) \qquad [\tfrac{1}{2}]$

comparing with $(*)$ $\quad h'(y) = 0 \Rightarrow h(y) = c$

Thus the general solution is

$u = \cos x \cos y = c \qquad [\tfrac{1}{2}]$

So, using initial conditions: $\frac{1}{2} = c$

Thus the soln of the IVP is

given by $\quad \cos x \cos y = \frac{1}{2} \qquad [\tfrac{1}{2}]$

or $\quad y = \cos^{-1}\left(\frac{1}{2} \sec x\right)$

Remark: part a) is of $2\frac{1}{2}$ & part b) is of $1\frac{1}{2}$.

Question 7(a): (FALSE)

Let $V = \mathbb{R}^2$ over $\mathbb{R}$.

$W = \{(x,0) : x \in \mathbb{R}\}$

$W_1 = \{(0,y) : y \in \mathbb{R}\}$

$W_2 = \{(z,z) : z \in \mathbb{R}\}$

(1M) Any counter example.

Note that $W \cap W_1 = \{0\}$ and $W \cap W_2 = \{0\}$.

(1M) Then $W \oplus W_1$ and $W \oplus W_2$ makes sense.

Verify that $W \oplus W_1 = \mathbb{R}^2 = W \oplus \mathbb{R}^2$

But, $W_1 \neq W_2$.

Question 7(b) : (FALSE)

(1M)

By Abel's theorem

$$W(y_1, y_2)(t) = c\, e^{\int_{x_0}^{t} p(x)\,dx}$$

where $c = W(y_1, y_2)(x_0)$

If $c = 0$ then $W(t) = 0 \quad \forall t \in I.$

If $c \neq 0$ then $W(t) \neq 0 \quad \forall t \in I$

(1M)

since exponential is always non-zero.

Then we cannot have $a \in I$, $b \in I$ with $W(a) = 0$ & $W(b) \neq 0$.

Question 7(c): (TRUE)

Given IVP is

$$y' = q(x) - p(x)y \; ; \; y(x_0) = y_0.$$

Write $y' = f(x,y) \; ; \; y(x_0) = y_0$

where $f(x,y) = q(x) - p(x)y$.

Since $p(x)$, $q(x)$ are continuous on $\mathbb{R}$

(1M) $f(x,y)$ is continuous on $\mathbb{R}^2$.

Then

$$\boxed{|f(x,y_1) - f(x,y_2)| = |p(x)|\,|y_1 - y_2|}$$

Consider any closed rectangle containing (x_0, y_0), say $R = \{(x,y) : |x - x_0| \leq 1, |y - y_0| \leq 1\}$

Since $p(x)$ is continuous, and $[x_0 - 1, x_0 + 1]$

(1M) is closed bounded interval, it is bounded on $[x_0 - 1, x_0 + 1]$, say by $L > 0$.

Then $f(x,y)$ is lipschitz w.r.t. y on R.

Therefore the IVP satisfies the condition of the uniqueness theorem.