COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 5: Graph Theory: Connectivity.
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Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM. Questions marked with a * may be a little harder and can be considered optional.

Problem 1 [Die25, Prob 12, page 30]
Show that every 2-connected graph contains a cycle.

Problem 2 [LLM18, Prob. 12.37]
A graph G is called *2-removable* if it contains two vertices $v \neq w$ such that $G - v$ and $G - w$ are bot connected. Prove that every connected graph with at least 2 vertices is 2-removable.

Problem 3
Given a graph $G = (V, E)$ and a minimal edge separator $F \subseteq E$, show that any cycle of G contains an even number of edges of F (this number could be 0 as well).

Problem 4
Let $\bar{G}$ be the complement of the graph G, i.e., all edges of G are non-edges of $\bar{G}$ and vice versa. Show that both G and $\bar{G}$ cannot be disconnected, i.e., at least one of them must be connected.

Problem 5 Related to [LLM18, Prob. 12.38]
The *n-Hamming cube* is a graph with $V(G) = \{0, 1\}^n$, i.e., whose vertices are vectors with n coordinates, each of which can be either 0 or 1. We put an edge between any two vertices whose vectors differ in *exactly* one coordinate.

Problem 5.1
Prove that the n-Hamming cube is a connected graph for any $n > 0$.

Problem 5.2
Find $\kappa(G)$ for the n-Hamming cube. In order to do this first prove that there are for any two vertices x, y there is a collection of n $x - y$ paths that are independent of each other (i.e. that don't share any vertex apart from x and y).

Problem 6 [LLM18, Prob. 12.42]
An edge is said to *leave* a set of vertices $A \subset V$ if only one endpoint of the edge is in A. A graph is called *k-mangled* if there is at least one edge leaving every subset of vertices of size k or smaller.

Problem 6.1 ♠
Prove that every $\lfloor |V|/2 \rfloor$-mangled graph is connected.

Problem 6.2
Is every $\lceil |V|/3 \rceil$-mangled graph connected?

Problem 7 [LLM18, Prob. 12.42]
If a graph has e edges, v vertices, and k connected components, then it has at least $e - v + k$ cycles. Prove this by induction on the number of edges e.

Problem 8
Given a graph $G = (V, E)$ such that $|V| = n$, a cut $F \subset E$ is called a *balanced cut* if $G \setminus F$ has exactly 2 components and each of these components has size at least $n/3$. Construct graphs on n vertices whose smallest balanced cut has size (a) $\theta(1)$, (b) $\theta(\sqrt{n})$ and (c) $\theta(n)$.

Problem 9 [Die25, Prob 16, page 30] *
Suppose that we have a k-edge connected graph with the property that removing any edge of this graph makes it lose its k-edge connectedness. Prove that such a graph has a vertex of degree k. Try to prove this first *without* using Menger's Theorem (c.f. Problem 10) and then see if using Menger's theorem makes it easier to prove.

Problem 10 (Menger's Theorem)
Prove that a graph G has $\lambda(G) = k$ for any $k \geq 1$ iff there are k edge-disjoint paths between any pair of vertices in G. Two paths are said to be edge-disjoint if they don't share any edges. Caution: One direction of this theorem is easy and the other is tricky.

References

[Die25] Reinhard Diestel, *Graph Theory 6ed.*, Springer, 2025.

[LLM18] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science, June 2018, MIT Open Courseware.