COL202: Discrete Mathematical Structures. I semester, 2022-23.
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Tutorial Sheet 4: Graph Connectivity .
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Important: The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

Problem 1 [1, Prob 12, page 30]
Show that every 2-connected graph contains a cycle.

Problem 2
Show using only the material covered in [1, Ch 1.4] that every connected graph on n vertices has at least $n-1$ edges.

Problem 3
Generalize the result of Problem 2 to show that every graph on n vertices and m edges has at least $n-m$ components.

Problem 4
Given a graph $G=(V,E)$ and a minimal edge separator $F \subseteq E$, show that any cycle of G contains an even number of edges of F (this number could be 0 as well).

Problem 5
The *n-Hamming cube* is a graph with $V(G)=\{0,1\}^n$, i.e., whose vertices are vectors with n coordinates, each of which can be either 0 or 1. We put an edge between any two vertices whose vectors differ in *exactly* one coordinate.

Problem 5.1 ♠
Prove that the n-Hamming cube is a connected graph for any $n>0$.

Problem 5.2 *
What is the highest value k such that the n-Hamming cube is k-connected?

Problem 6
Let $\bar{G}$ be the complement of the graph G, i.e., all edges of G are non-edges of $\bar{G}$ and vice versa. Show that both G and $\bar{G}$ cannot be disconnected, i.e., at least one of them must be connected.

Problem 7
Given a graph $G=(V,E)$ such that $|V|=n$, a cut $F \subset E$ is called a *balanced cut* if $G \setminus F$ has exactly 2 components and each of these components has size at least $n/3$. Construct graphs on n vertices whose smallest balanced cut has size (a) $\theta(1)$, (b) $\theta\left(\sqrt{n}\right)$ and (c) $\theta(n)$.

Problem 8 (Menger's Theorem)
Prove that a graph G has $\lambda(G)=k$ for any $k \geq 1$ iff there are k edge-disjoint paths between any pair of vertices in G. Two paths are said to be edge-disjoint if they don't share any edges. Caution: One direction of this theorem is easy and the other is tricky.

References

[1] Reinhard Diestel, *Graph Theory 5ed.*, Springer, 2016.