Quiz 2

Graded

Student

Har Ashish Arora

Total Points

8.5 / 13 pts

Question 1

Diagonalizable

6 / 6 pts

- ✔ **+ 6 pts** Correct
- **+ 0 pts** Not correct or not attempted

Part-(i)

- **+ 1 pt** Step-1: Finding the eigenvalues 0,0,1,1 correctly.
- **+ 0.5 pts** For partially correct step-1
- **+ 1 pt** Step-2: For eigenvalue $\lambda = 1$, showing dim(null(T-I))=2 for all $t \in \mathbb{R}$
- **+ 0.5 pts** For partially correct step-2
- **+ 1 pt** Step-3: For eigenvalue $\lambda = 0$, showing dim(Null(T-0I))=2 if and only if $s = 0$.
- **+ 0.5 pts** For partially correct step-3
- **+ 1 pt** Step-4: Concluding with justification that T is diagonalizable when $s = 0$ and $t \in \mathbb{R}$
- **+ 0.5 pts** For partially correct step-4

Part-(ii)

- **+ 1 pt** Step-1: For $s = 0$, and $t = 0$, finding a basis of eigenvectors.
- **+ 0.5 pts** For partially correct step-1
- **+ 1 pt** Step-2: $s = 0$, and $t \neq 0$, finding a basis of eigenvectors.
- **+ 0.5 pts** For partially correct step-2

Question 2

ODE

0.5 / 4 pts

+ 1 pt The function $f(x,y)=\frac{xy}{x^2+y^2}$ is not continuous at $(0,0)$. For $x_0=0$, the IVP cannot decide the existence of a solution around $x_0=0$ according to the theorem.

+ 1 pt For $x_0\neq 0$, choose any $0<a\leq|x_0|$ and $b>0$. Then on the rectangle centered at $(x_0,0)$ and sides $2a$ (along x-axis) and $2b$ (along y-axis), the function $f(x,y)$ is continuous being a ratio of polynomials with no zeros for its denominator.

+ 0 pts For not able to choose a and b for $x_0\neq 0$. If part 2 is correct this becomes irrelevant.

✓ **+ 0.5 pts** $K=Sup\{f(x,y)\}=0.5$.

+ 0.5 pts $\alpha=Min\{|x_0|,b/K\}=Min\{|x_0|,2b\}$.

+ 0.5 pts Choose $b>|x_0|/2$ (to conclude that the largest value of $\alpha=|x_0|$)

+ 0.5 pts For saying the largest α is $|x_0|$.

+ 0 pts Not attempting or incorrect solution.

+ 4 pts Everything is correct.

Question 3

Cayley-Hamilton

2 / 3 pts

+ 3 pts Correct

+ 0 pts Completely Incorrect/ Didn't attempt

✓ **+ 1 pt** To compute characteristic polynomial correctly

✓ **+ 0.5 pts** To conclude, A is invertible using Cayley Hamilton

+ 0.5 pts To compute A^2 correctly.

+ 1 pt To compute the inverse correctly.

+ 0.5 pts Calculation error

1 wrong

2 wrong

DEPARTMENT OF MATHEMATICS, IIT DELHI
SEMESTER II 2024 − 25
MTL 101 (Linear Algebra and Differential Equations) - Quiz 2

Date: 27/03/2025 (Thursday) Time: 6:45 PM - 7:30 PM.

"As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

Name:	HAR ASHISH ARORA	BLOCK LETTER ONLY	
Entry Number:	2024EE10904	Group:	13
Gradescope Id:	EE1240904	Lecture Hall:	527

Question 1: Let T be the linear operator on $\mathbb{R}^4$ given by the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with respect to the standard basis.

(i) Find all values of s and t for which T is diagonalizable.

(ii) Find an ordered basis $\mathcal{B}$ of $\mathbb{R}^4$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix, whenever T is diagonalizable.

[4+2=6]

(i) T is diagonalisable if and only if

$\dim(\mathbb{R}^4) = 4$ = sum of dimensions of eigenspaces.

We first need to find eigenvalues.

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & 0 & 0 \\ -s & \lambda & 0 & 0 \\ 0 & -t & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = \lambda \begin{vmatrix} \lambda & 0 & 0 \\ -t & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix}$$

$$= \lambda\left(\lambda \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix}\right) = \lambda^2(\lambda-1)^2$$

So, the eigenvalues are 0, 1. (no dependence on s and t!!!).

Eigenspace corresponding to $\lambda = 0$:

$Av = \lambda v = 0 \Rightarrow Av = 0$

$\Rightarrow$ using homogenous system of equations,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

On hold here, for now.

For corresponding to $\lambda = 1$:

$Av = \cancel{1}\, v \Rightarrow (A - I)v = 0$ so matrix for $A - I$:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - sR_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - tR_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, take it case wise. Suppose $s \neq 0$ $t \neq 0$.

Then nullity $(A) = 1$ (∵ rank $(A) = 3$, then rank-nullity) $\Rightarrow$ ~~eigenspace of dimension~~ dimension of eigenspace corresponding to $\lambda = 0$ is 1. (no. of free variables)

S'lly, ~~nullity(A) =~~ nullity$(A - I) = 2$ (does not depend on s or t, as I have shown). ~~(I can write do $R_2 \to R_2 - sR_1$~~

∴ $4 \neq 1 + 2$, in ~~(s, t)~~ $= s \neq 0, t \neq 0$, not diagonalisable.

<u>If $s \neq 0, t = 0$:</u>

Again, nullity $(A) = 1$, $\because$ rank$(A) = 3$, and nullity $(A - I) = 2$ (always).

$\because$ nullity(A) corresponds to dim of eigenspace corresponding to $\lambda = 0$ & nullity$(A-I)$ corresponds to dim of eigenspace corresponding to $\lambda - I$, again we get t not diagonalisable.

<u>If $s = 0, t \neq 0$ or $s = 0, t = 0$</u>

<u>nullity$(A - I) = 2$, nullity$(A) = 2$. So here, T is diagonalisable.</u>

So, $s = 0$, $t \in \mathbb{R}$ is the condition for which T is diagonalisable.

(ii) We just have to find the eigenspaces when $s = 0$, $t \in \mathbb{R}$.

<u>For $\lambda = 0$:</u>

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{(ii) } R_2 \leftrightarrow R_3]{\text{(i) } R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

we have free variables x & z when $t \neq 0$.

$x = a$, $z = b$.

$ty + z = 0 \Rightarrow y = -\frac{b}{t}$, $w = 0$

<u>when $t \neq 0$</u>

$\Rightarrow$ eigenspace: $\{a(1,0,0,0)^T + b(0, -\frac{1}{t}, 1, 0)^T\}$ $(a, b \in \mathbb{R})$

and when <u>$t = 0$</u>: free variables are $\underset{=c}{x}$ & $\underset{=d}{y}$, so eigenspace becomes $\{c(1,0,0,0)^T + d(0,1,0,0)^T\}$ $\{c, d \in \mathbb{R}\}$.

<u>$\lambda = 1$</u>: Free variables are clearly $\underset{=e}{z}$ & $\underset{=f}{w}$. So, eigenspace becomes $\{e(0,0,1,0)^T + f(0,0,0,1)^T\}$, $(e, f \in \mathbb{R})$.

<u>When $t = 0$</u>: $B = \{(1,0,0,0)^T, (0,1,0,0)^T, (0,0,1,0)^T, (0,0,0,1)^T\}$

<u>When $t \neq 0$</u>: $B = \{(1,0,0,0)^T, (0, -\frac{1}{t}, 1, 0)^T, (1,0,0,0)^T, (0,1,0,0)^T\}$

Question 2: Consider the initial value problem (IVP):

$$y' = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ \frac{1}{2}, & (x,y) = (0,0) \end{cases}$$

with initial condition:

$$y(x_0) = 0.$$

(i) For what values of x_0 does this IVP have a solution according to the existence theorem?

(ii) Additionally, for such, x_0 find the largest positive α such that the solution exists in the interval $(x_0 - \alpha, x_0 + \alpha)$.

[2+2=4]

(i) We have $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ \frac{1}{2}, & (x,y) = (0,0) \end{cases}$

Notice that $f(x,y)$ has been made continuous at $(0,0)$. $\left(\because \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{1}{2} \right)$.

Existence theorem: Take a rectangle

$$R = \{ (x,y) \mid |x - x_0| \le a, |y| \le b \}.$$

We can see that for all values of x & y, in the rectangle $f(x,y)$ is a continuous function. Also, it is bounded.

There is no constraint on x_0. So, for all real values of x_0, as per the existence theorem, this IVP will have a solution in the neighbourhood of $(x_0, 0)$.

(ii) For the solution to exist, and we need it to exist in the interval $(x_0 - \alpha, x_0 + \alpha)$, $\alpha = \min\{a, \frac{b}{K}\}$,

$K = \sup|f(x,y)|$, where a, b were defined in the rectangular region on the previous page.

$$\left|\frac{xy}{x^2+y^2}\right| = \cancel{\frac{|x|\,|y|}{|x^2+y^2|}} \quad \left|\frac{1}{\frac{x}{y}+\frac{y}{x}}\right|.$$

$(x \neq 0, y \neq 0)$.
If $x = 0$, $f(x,y) = 0$
$y = 0$, $f(x,y) = 0$
both zero, $f(x,y) = \frac{1}{2}$

By Arithmetic mean – geometric mean inequality,

$$\frac{\left(\frac{x}{y}+\frac{y}{x}\right)}{2} \geq \sqrt{\frac{x}{y}\frac{y}{x}} = 1 \Rightarrow \left|\frac{x}{y}+\frac{y}{x}\right| \geq 2$$

($\because \frac{x}{y}$ & $\frac{y}{x}$ will be of the same sign).

$$\Rightarrow K = \sup|f(x,y)| = \frac{1}{2}.$$

So, $\alpha = \min\{a, \frac{b}{\frac{1}{2}}\} = \min\{a, 2b\}$.

So, it depends on the size of the rectangle that we take. If we make the rectangle arbitrarily large, then the solution will exist over all real numbers.

Question 3: Let A be the 3×3 matrix given by

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Using Cayley-Hamilton theorem show that A is invertible and find the inverse. [3]

Cayley-Hamilton Theorem: every matrix satisfies its characteristic equation, $|\lambda I - A| = 0$.

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-4 & -1 & -2 \\ -1 & \lambda-3 & -1 \\ -2 & -1 & \lambda-3 \end{bmatrix}$$

$$\Rightarrow |\lambda I - A| = (\lambda-4)((\lambda-3)^2 - 1) + 1(-(\lambda-3) - 2) - 2(1 + 2(\lambda-3))$$

$$= (\lambda-4)(\lambda^2 + 9 - 6\lambda - 1) + (-\lambda + 3 - 2) - 2(1 + 2\lambda - 6)$$

$$= (\lambda-4)(\lambda^2 - 6\lambda + 8) + (1 - \lambda) - 2(2\lambda - 5)$$

$$= \lambda^3 - 6\lambda^2 + 8\lambda - 4\lambda^2 + 24\lambda - 32 + 1 - \lambda - 4\lambda + 10$$

$$= \lambda^3 - 10\lambda^2 + 27\lambda - 21$$

So, characteristic polynomial $p(\lambda) = \lambda^3 - 10\lambda^2 + 27\lambda - 21$

A is invertible iff $p(0) \neq 0$. Here $p(0) = -21$. So A is invertible, and A^{-1} exists.

We have

$$A^3 - 10A^2 + 27A - 21I = 0$$

Multiply by A^{-1}:

$$A^2 - 10A + 27I = 21A^{-1}$$

$$A^2 = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 9 & 15 \\ 9 & 11 & 10 \\ 15 & 8 & 14 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{21}\left(\begin{bmatrix} 21 & 9 & 15 \\ 9 & 11 & 10 \\ 15 & 8 & 14 \end{bmatrix} - \cancel{10} \begin{bmatrix} 40 & 10 & 20 \\ 10 & 30 & 10 \\ 20 & 10 & 30 \end{bmatrix} + \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} \right)$$

$$= \frac{1}{21}\left(\begin{bmatrix} 8 & -1 & \cancel{10}-5 \\ -1 & 8 & 0 \\ -5 & -2 & 11 \end{bmatrix} \right)$$

So, $$\boxed{A^{-1} = \frac{1}{21} \begin{bmatrix} 8 & -1 & -5 \\ -1 & 8 & 0 \\ -5 & -2 & 11 \end{bmatrix}}$$ — (Ans).