(1) $\alpha > 0,\ \alpha \in \mathbb{R}$

$$A = \begin{pmatrix} \alpha & -2 & 1 \\ 4 & -\alpha & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) characteristic poly. of $A = \det(\lambda I - A)$

$= \lambda^3 - \lambda^2 + (8-\alpha^2)\lambda - (8-\alpha^2)$; 1 marks

$= (\lambda - 1)(\lambda^2 + 8 - \alpha^2)$.

$= (\lambda-1)(\lambda - \sqrt{\alpha^2-8})(\lambda + \sqrt{\alpha^2-8})$

(b) since we know that if E. values are distinct then A is diagonalizable so for A not to be diagonalizable we must have repeated E. values.

which is possible in 2 cases

(i) $\alpha = \pm 2\sqrt{2}$; then E. values are $1, 0, 0$] 1 mark

(ii) $\alpha = \pm 3$; " " " $1, 1, -1$.

we discard here -ve values of α.

Case (i)

$\alpha = 2\sqrt{2},\ \lambda = 0$

$\dim W_0 = 1 ; \dim W_1 = 1$

$\Rightarrow$ A is not diagonalisable.

Case (ii)

$\alpha = 3 \quad \lambda = 1$

$\dim W_1 = 2 ; \dim W_{-1} = 1$

$\Rightarrow$ A is diagonalisable.

so The value of $\alpha (>0)$ for which A is not diagonalizable is $2\sqrt{2}$ only.

Students should show $\dim W_0 = 1$ (when $\alpha = 2\sqrt{2}$) & $\dim W_1 = 2$ when $\alpha = 3$.

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2 (a) Given : 'λ' is an e-value of B.

Prove : λ^2 is an e. value of B^2

Proof : $BX = \lambda X, \quad X \neq 0$

pre-multiply by B

$B^2X = B(\lambda X)$

$B^2X = \lambda(BX)$

$B^2X = \lambda(\lambda X) = \lambda^2 X.$

hence λ^2 is an e. value of B^2.

Altee:

$|\lambda^2 I - B^2|$

$= \underbrace{|\lambda I - B|}_{0} \, |\lambda I + B|$

$= 0 \cdot |\lambda I + B|$

$= 0$

(b) Char polyn of $B = x^3 + \mu x^2 + 1 . = p(x)$ (say) — ①

By Cayley-Hamilton Thm.

$$B^3 + \mu B^2 + I = 0$$

pre-multiply by B^{-1} (B^{-1} exists coz $p(0) = 1$)

$$B^2 + \mu B + B^{-1} = 0$$

$$\boxed{B^{-1} = -(\mu B + B^2)}$$

(c) find $tr(B^{-1})$.

(★) — $tr(B^{-1}) = tr[-(\mu B + B^2)]$

$= -tr(\mu B) - tr(B^2)$

From ①, $tr(B) = -\mu$ — ②

$\alpha_1 + \alpha_2 + \alpha_3 = -\mu$

$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = 0$

$\alpha_1\alpha_2\alpha_3 = -1$

So $tr(B^2) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$

$= (\alpha_1 + \alpha_2 + \alpha_3)^2 - 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)$

$= \mu^2$ — ③

Use ② + ③ in ★

$tr(B^{-1}) = -\mu(-\mu) - \mu^2$

$= \mu^2 - \mu^2$

$$\boxed{tr(B^{-1}) = 0}$$

(3)

$$\langle (x_1, y_1) \mid (x_2, y_2) \rangle := 2x_1x_2 + 5y_1y_2 + 3x_1y_2 + 3x_2y_1$$

Since linearity of $\langle \cdot | \cdot \rangle$ in both the components is given, we need to show that:

(i) $\langle (x_1, y_1) \mid (x_1, y_1) \rangle \geq 0 \quad \forall\ x_1, y_1 \in \mathbb{R}$

(ii) $\langle (x_1, y_1) \mid (x_1, y_1) \rangle = 0 \iff x_1 = 0,\ y_1 = 0$

(iii) $\overline{\langle (x_1, y_1) \mid (x_2, y_2) \rangle} = \langle (x_2, y_2) \mid (x_1, y_1) \rangle \quad \forall\ x_1, y_1, x_2, y_2 \in \mathbb{R}$

<u>Proof of (i):</u>

$$\begin{aligned}
\langle (x_1, y_1) \mid (x_1, y_1) \rangle &= 2x_1^2 + 5y_1^2 + 3x_1y_1 + 3x_1y_1 \\
&= 2(x_1^2 + 3x_1y_1) + 5y_1^2 \\
&= 2\left(x_1^2 + 3x_1y_1 + \frac{9}{4}y_1^2\right) + 5y_1^2 - \frac{9}{4}y_1^2 \\
&= 2\left(x_1 + \frac{3}{2}y_1\right)^2 + \frac{1}{2}y_1^2 \\
&\geq 0 \quad \forall\ x_1, y_1 \in \mathbb{R}
\end{aligned}$$

<u>Proof of (ii):</u>

$$\begin{aligned}
\langle (x_1, y_1) \mid (x_1, y_1) \rangle = 0 &\iff 2\left(x_1 + \frac{3}{2}y_1\right)^2 + \frac{1}{2}y_1^2 = 0 \\
&\iff x_1 + \frac{3}{2}y_1 = 0 \text{ and } y_1 = 0 \\
&\iff x_1 = 0 \text{ and } y_1 = 0
\end{aligned}$$

<u>Proof of (iii):</u>

$$\begin{aligned}
\overline{\langle (x_1, y_1) \mid (x_2, y_2) \rangle} &= \overline{(2x_1x_2 + 5y_1y_2 + 3x_1y_2 + 3x_2y_1)} \\
&= 2x_2x_1 + 5y_2y_1 + 3x_2y_1 + 3x_1y_2 \quad (\because x_1, x_2, y_1, y_2 \text{ are in } \mathbb{R}) \\
&= \langle (x_2, y_2) \mid (x_1, y_1) \rangle
\end{aligned}$$

Hence, proved

Q4 Ans $y' = \dfrac{\sqrt{y}\sin\sqrt{y}}{t^2-4}$ $y(1)=1$.

$$f(t,y) = \frac{\sqrt{y}\sin\sqrt{y}}{t^2-4}.$$

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{1}{t^2-4}\left(\frac{\cos\sqrt{y}}{2} + \frac{\sin\sqrt{y}}{2\sqrt{y}}\right)\right|$$

$\leq \left|\dfrac{1}{t^2-4}\right| < \infty$ in any rectangle that does not contain $t=-2,2$ and $y>0$.

So f is Lipschitz in a rectangle around $(1,1)$.

By Picard's, $\exists$ unique solution around $t=1$.

Now solving the equation

$$\frac{y'}{\sqrt{y}\sin\sqrt{y}} = \frac{1}{t^2-4} = \frac{1}{4}\left(\frac{1}{t-2} - \frac{1}{t+2}\right)$$

$$\frac{2\,du}{\sin u} = \frac{1}{4}\left(\frac{1}{t-2} - \frac{1}{t+2}\right)$$

Integrating

$$-2\log|\operatorname{cosec} u + \cot u| = \frac{1}{4}\log\left|\frac{t-2}{t+2}\right| + \log c, \quad t \neq -2,2$$

i.e., $\tan\dfrac{\sqrt{y}}{2} = c\left|\dfrac{t-2}{t+2}\right|^{1/8}$. $t \neq -2,2$

$y(1)=1 \Rightarrow \tan\frac{1}{2} = c\cdot\left(\frac{1}{3}\right)^{1/8}$, $t \neq -2,2$

$1 \in (-2,2)$. So the maximal interval $(-2,2)$.

Q.5

$$y' = -\frac{x + \tan y}{1+x^2}, \quad y(0) = \frac{\pi}{6}$$

Using Variable separable method

$$\frac{dy}{\tan y} = -\frac{x}{1+x^2}\,dx$$

On integration

$$\int \frac{dy}{\tan y} = -\int \frac{x\,dx}{1+x^2}$$

$$\log \sin y = -\frac{1}{2}\log(1+x^2) + \log c$$

$$\log \sin y = \log \frac{c}{\sqrt{1+x^2}}$$

$$\sin y = \frac{c}{\sqrt{1+x^2}}$$

Using initial conditions

$$\frac{1}{2} = \frac{c}{\sqrt{1+0}} = c$$

$$\Rightarrow \quad \sin y = \frac{1}{2\sqrt{1+x^2}}$$

$$y(x) = \sin^{-1}\left(\frac{1}{2\sqrt{1+x^2}}\right), \quad \forall x \in \mathbb{R}$$

with $y(0) = \frac{\pi}{6}$

Q-6 $$\frac{dy}{dt} = K\, y(t)\left(1 - \frac{y(t)}{500}\right)$$

$$y(0)=10, \quad y(5) = 20$$

$$\frac{dy}{y\left(1-\frac{y}{500}\right)} = k\,dt$$

$$\frac{500\,dy}{y(500-y)} = k\,dt$$

$$\frac{\{(500-y)+y\}\,dy}{y(500-y)} = k\,dt$$

$$\int\left(\frac{1}{y} + \frac{1}{(500-y)}\right)dy = \int k\,dt$$

$$\ln(y) - \ln(500-y) = kt + \ln c$$

$$\ln\left(\frac{y}{(500-y)\cdot c}\right) = kt$$

$$\frac{y}{(500-y)} = c\,e^{kt}$$

$y(0)=10$ gives

$$c = \frac{1}{49}$$

and $y(5) = 20$ gives

$$\frac{20}{480} = \frac{1}{49} e^{5k}$$

$$\ln\left(\frac{49}{24}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{49}{24}\right)$$

then Sol^n is given by-

$$\frac{y}{(500-y)} = \frac{1}{49} e^{kt}, \quad \text{where } k = \frac{1}{5}\ln\left(\frac{49}{24}\right)$$