COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 29

MINOR-II DISCUSSION & RECURRENCES

MAR 28, 2023 | ROHIT VAISH

PROBLEM 1

For any tree $T = (V, E)$ with n vertices and $|E| \geq 1$,

Show that

$$y_1 = 2 + \sum_{k=3}^{n-1} (k-2)\, y_k$$

where y_k = number of vertices with degree k.

deg = 1   deg = 2   deg = k

$$\text{No. of edges} = \frac{1}{2}\left(1\,y_1 + 2\cdot y_2 + \cdots + k\cdot y_k + \cdots + (n-1)\,y_{n-1}\right)$$

$|E|$

deg = 1 $\qquad$ deg = 2 $\qquad \cdots \qquad$ deg = k

$$\underset{|E|}{\text{No. of edges}} = \frac{1}{2}\left(1\,y_1 + 2\cdot y_2 + \cdots + k\cdot y_k + \cdots + (n-1)\,y_{n-1}\right)$$

$$\Rightarrow \quad 2|E| - 2y_2 - \sum_{k=3}^{n-1} k\cdot y_k = y_1 \qquad \text{———} \quad (1)$$

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$$\Rightarrow \quad 2y_1 + 2y_2 + 2\sum_{k=3}^{n-1} y_k = 2|E| + 2 \quad \text{---} \; (2)$$

Recall from (1) :

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$$(1) + (2) \Rightarrow \quad y_1 = 2 + \sum_{k=3}^{n-1} (k-2)\, y_k .$$

$\boxed{\text{//}}$

PROBLEM 1 GRADING SCHEME

Correct expression for no. of edges (identifying double counting) [5 pts]

Second expression using the property [4 pts]

Adding the two to get correct answer [1 pt]

PROBLEM 2

(a) What's wrong with the following proof for "A connected graph with $|V| = |E| + 1$ is a tree."

We are given that the graph is connected, so all we need to prove is that it has no cycle.

We proceed by induction on the number of vertices. For $|V| = 1$, there is a single vertex and no edge, and the statement holds.

So assume the implication holds for any graph $G = (V, E)$ on n vertices. We want to prove it also for a graph $G' = (V', E')$ arising from G by adding a new vertex. In order that the assumption $|V'| = |E'| + 1$ holds for G', we must also add one new edge, and because we assume G' is connected, this new edge must connect the new vertex to some vertex in V. Hence the new vertex has degree 1 and so it cannot be contained in a cycle. Because G has no cycle (by the inductive hypothesis), we get that neither does G' have a cycle, as desired.

The induction step should consider an arbitrary graph G' with $|V'| = |E'| + 1$, not just the ones arising from another graph G with $|V| = |E| + 1$ by adding a deg 1 vertex.

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PROBLEM 2(a) GRADING SCHEME

Identifying the problematic part of the proof [3 pts]

Correct explanation of the erroneous step identified above [4 pts]

PROBLEM 2

(b) Prove that a connected graph $G = (V, E)$ that satisfies $|V| = |E| + 1$ is a tree.

Thm: A connected graph $G=(V,E)$ with $|V|=|E|+1$ is a tree.

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Proof: By contradiction.

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Observation: Above step maintains connectivity of G.

Why?

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Why? For any pair of vertices u, v, there is a walk from u to v (and therefore a path from u to v).

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However, by edge removal, $|V'| \geq |E'| + 1$.

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$\Rightarrow$ G' must be a tree $\Rightarrow$ $|V'| = |E'| + 1$.

However, by edge removal, $|V'| \geq |E'| + 1$.

Contradiction! Original graph G must be a tree. $\blacksquare$

PROBLEM 2(b) GRADING SCHEME

Identifying proof by contradiction. [1 pt]

Mentioning the "edge removal" procedure [3 pts]

Observing that edge removal preserves connectedness [2 pts]

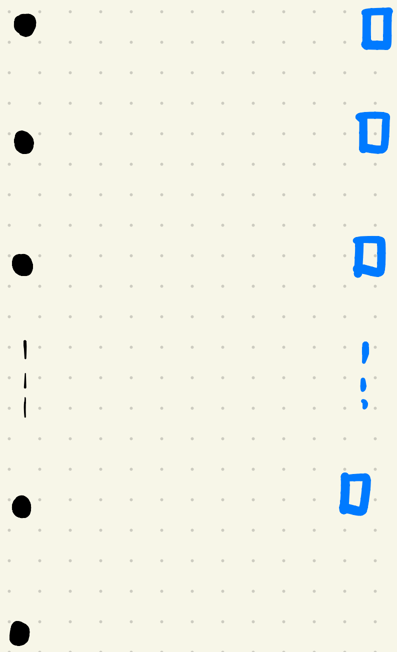
Deriving contradiction for graph G' returned by edge removal procedure [2 pts]

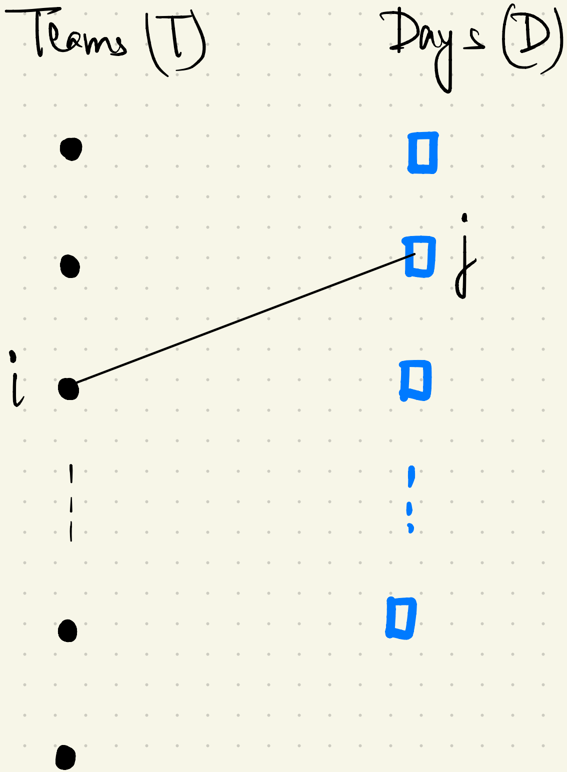
PROBLEM 3

$2n$ teams play a round-robin tournament that runs for $2n-1$ days.

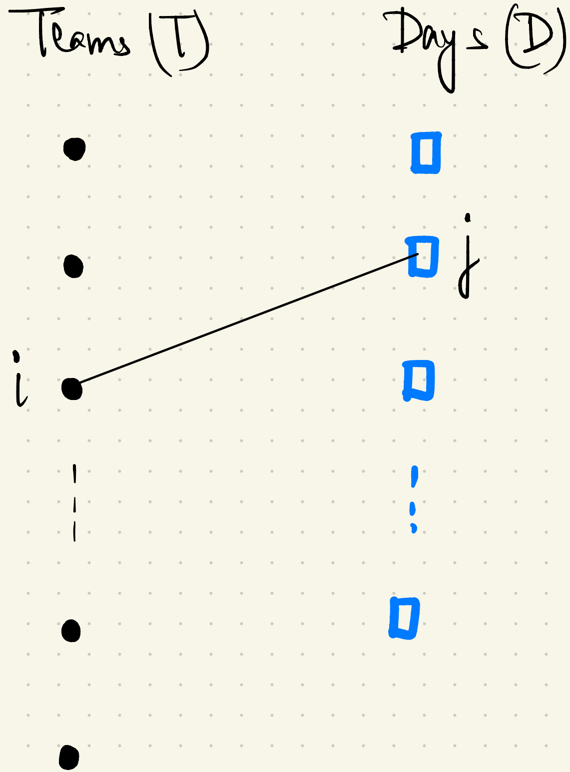
Show that for each day a unique winning team can be selected.

Teams (T) Days (D)





Edge between team i and day j if team i wins on day j.



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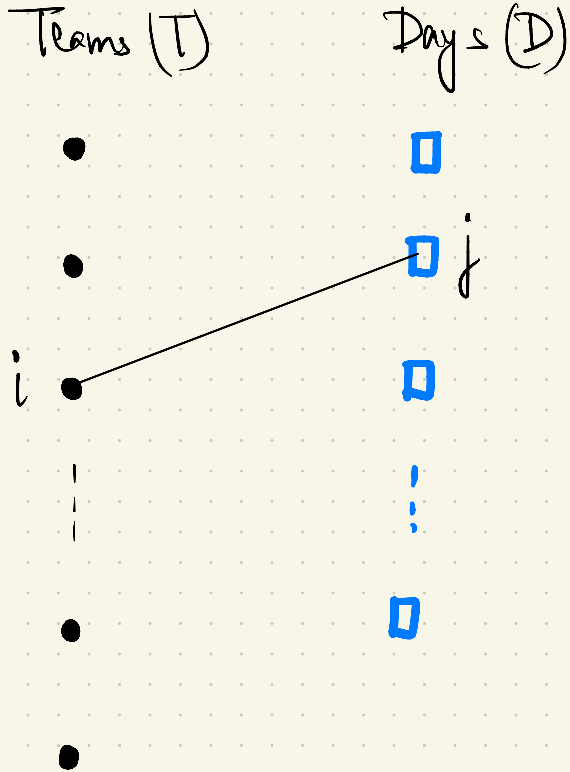
Want: A right-perfect matching.



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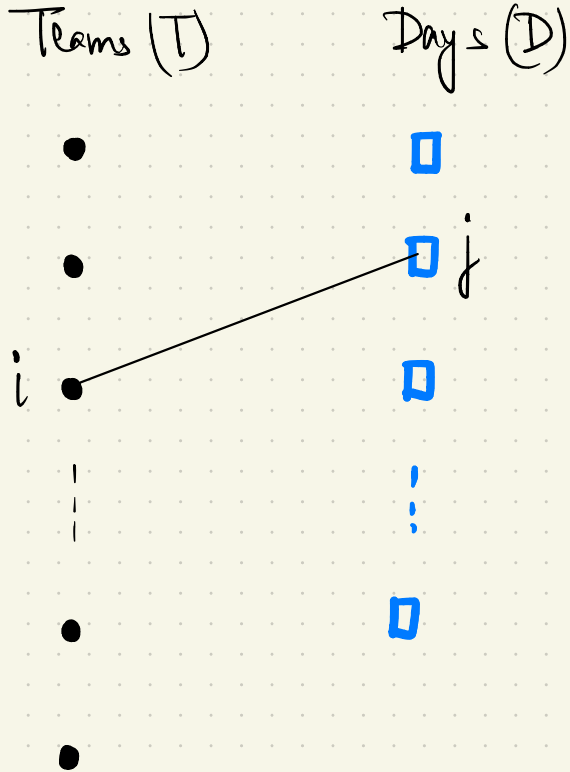

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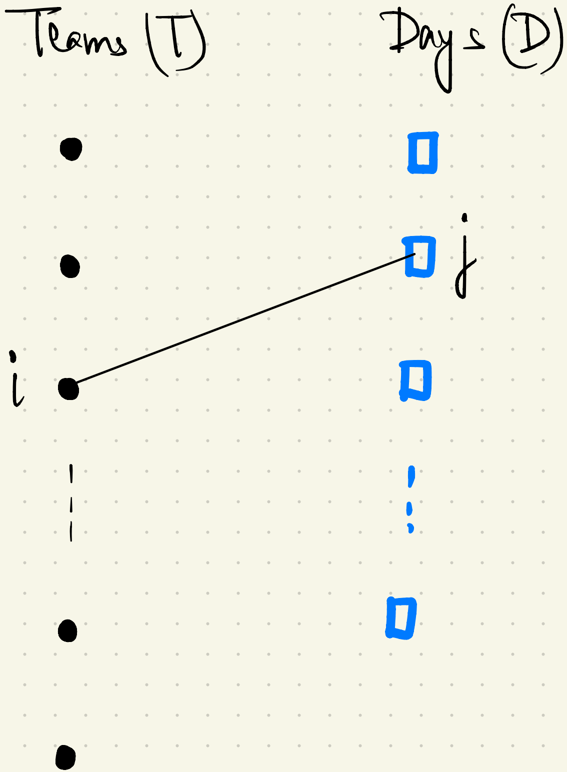
Hall's theorem $\Longrightarrow \exists\, S \subseteq D$ such that

$$|S| > |N(S)|$$



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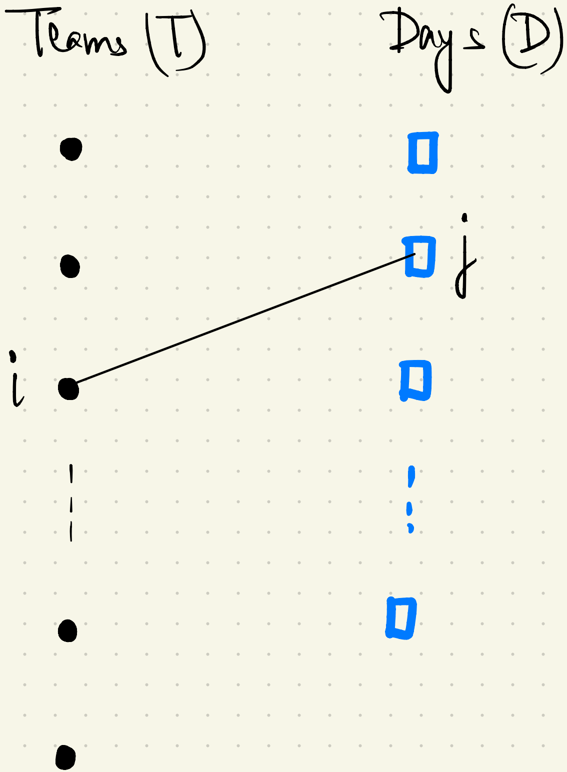

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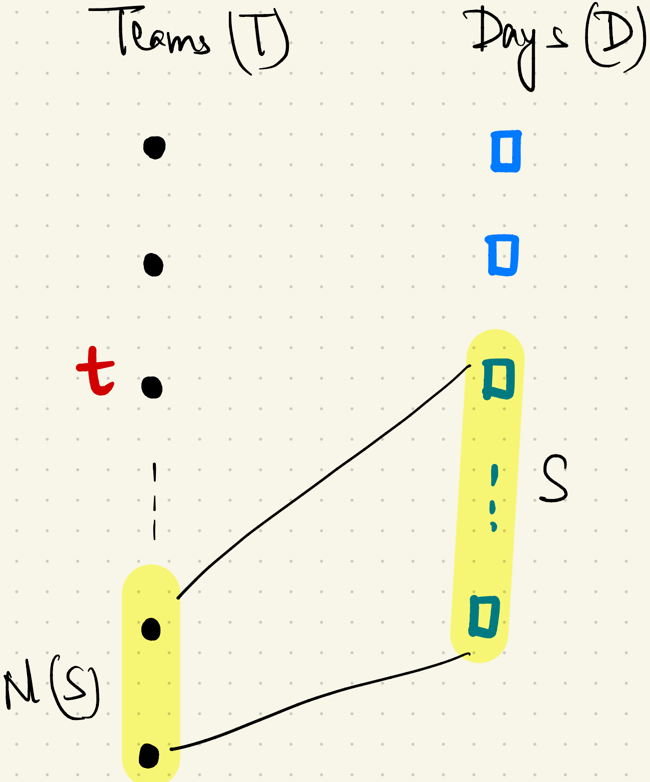

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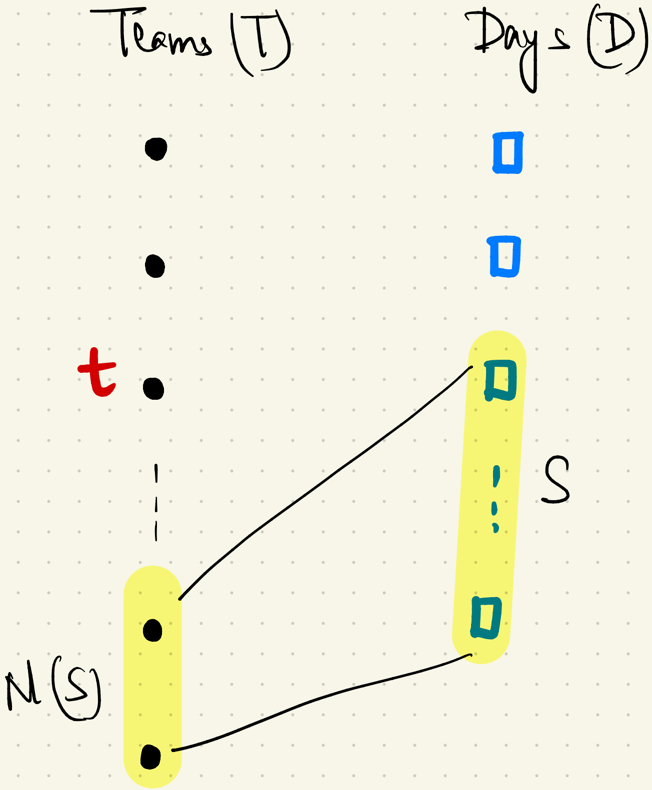
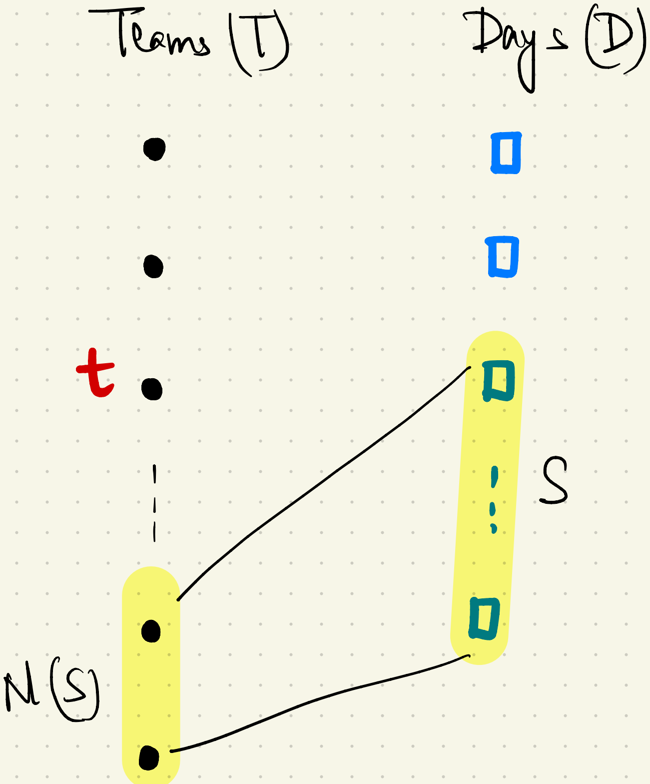


Because $t \notin N(S)$, t must lose all its games on days in S.



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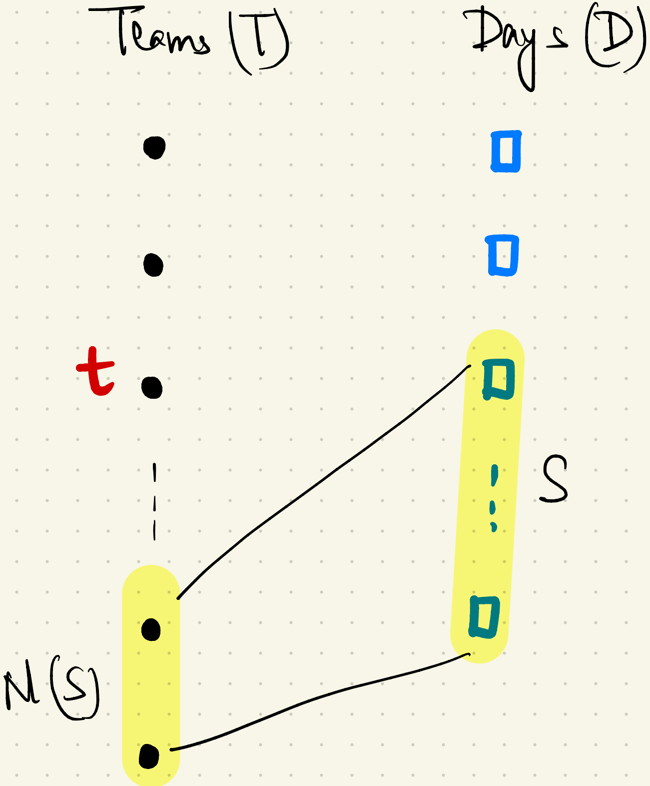
Round robin $\Longrightarrow$ t loses to a different team on each day in S.



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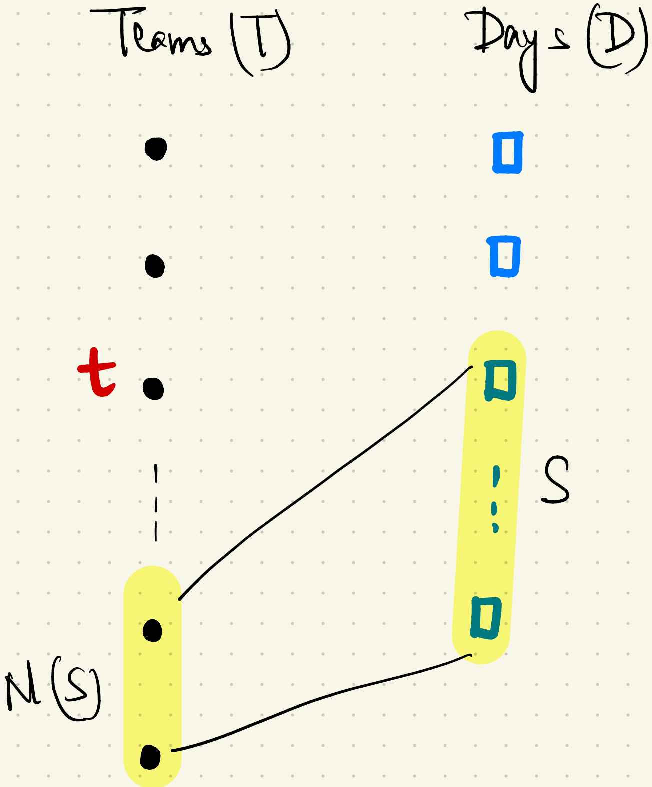
Round robin $\Rightarrow$ t loses to a different team on each day in S.

$\Rightarrow$ at least $|S|$ distinct winning teams for days in S



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$\Rightarrow$ at least $|S|$ distinct winning teams for days in S

$\Rightarrow$ $|N(S)| \geq |S|$ Contradiction.

$\boxed{\cancel{}}$

PROBLEM 3 GRADING SCHEME

Identifying proof by contradiction.	[2 pt]
Modeling as bipartite matching problem	[5 pts]
Recognizing that perfect matching is required	[2 pts]
Applying Hall's theorem.	[2 pts]
Observing that round-robin-ness gives contradiction	[5 pts]