Elementary Operations:-

An elementary operation on a matrix A over a field F is an operation of the following three types.

(1) Interchange of two rows of A. R_{ij} / $R_i \leftrightarrow R_j$

(2) Multiplication of a row by a non-zero scalar c in F. cR_i / $R_i \rightarrow cR_i$

(3) Addition of a scalar multiple of one row to another row. $R_j + cR_i$ / $R_j \rightarrow R_j + cR_i$

We can have similar elementary column operation. Just replace the word 'row' by the word 'column' & R_i by C_i in the above definition.

Example :-

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 7 \\ 4 & 9 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow{3R_2} \begin{pmatrix} 2 & 4 & 0 \\ 12 & 27 & 15 \\ 1 & 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow{R_2 - 4R_3} \begin{pmatrix} 2 & 4 & 0 \\ 0 & -3 & -23 \\ 1 & 3 & 7 \end{pmatrix}$$

Note that :-

(1) $$A_{m\times n} \xrightarrow{R_{ij}} B_{m\times n} \iff B = E_{ij}\, A$$

where $$I_{m\times m} \xrightarrow{R_i \leftrightarrow R_j} E_{ij}$$

(2) $$A_{m\times n} \xrightarrow{R_j + cR_i} B_{m\times n} \iff B = E_{ji}(c)\, A$$

where $$I_{m\times m} \xrightarrow{R_j + cR_i} E_{ji}(c)$$

(3) $$A_{m\times n} \xrightarrow{cR_i} B_{m\times n} \iff B = E_i(c)\, A$$

where $$I_{m\times m} \xrightarrow{cR_i} E_i(c)$$

(4) The elementary matrices E_{ij}, $E_i(c)$, $E_{ji}(c)$ are invertible and their inverses are E_{ji}, $E_i(\frac{1}{c})$, $E_{ji}(-c)$

Examples :-

$$\begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv E_{21}(-2)$$

Now,
$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 3 \end{pmatrix}$$

Row Equivalence :- Let $M_{m\times n}(F)$ be the set of all $m\times n$ matrices over a field F. A matrix $B \in M_{m\times n}(F)$ is said to be row equivalent to a matrix $A \in M_{m\times n}(F)$ if B can be obtained by successive application of a finite number of elementary row operations on A.
Similarly, A can be obtained from B by some row operations.

Finding Inverse of a Matrix :-

Suppose $A \in M_{n\times n}(F)$

Let $E_1, E_2, \cdots, E_k$ are the elementary row operations such that, $E_k E_{k-1} \cdots E_2 E_1 A = I_{n\times n}$ (If possible)

Then $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$.

If A is an invertable matrix then only you will get $I_{n\times n}$, and you can find an inverse.

If A is not invertible then you never get $I_{n\times n}$.

If you do not get $I_{n\times n}$ then A is not invertible.

Row-reduced matrix :- An $m \times n$ matrix A is said to be row-reduced if

(1) the first non-zero element in non-zero row is 1, (we call this as leading 1)

(2) in each column containing the leading 1 of some row, the leading 1 is the only non-zero elements. (other elements of that column should be zero)

Row-reduced echelon matrix :- An $m \times n$ matrix A is said to be a row-reduced echelon matrix if

(1) A is row-reduced,

(2) every zero row is below every non-zero row,

(3) if the leading 1 of i th row occurs in K_i th column then $K_1 < K_2 < \cdots < K_r$

Examples of a row-reduced echelon matrix

(1) $\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad k_1 = 1$, $k_2 = 4$, $k_3 = 5$

(2) $\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad k_1 = 2$, $k_2 = 4$

The following matrices are row-reduced but not row-reduced echelon.

(1) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad k_1 = 2$, $k_2 = 3$, $k_3 = 1$

(2) $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$ $\quad k_1 = 2$, $k_2 = 3$, $k_4 = 1$

a zero row is above a nonzero row.

Apply elementary row operations to reduce the following matrix to a row-reduced echelon matrix.

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_2]{R_3-2R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem :- A matrix A can be made row equivalent to a row echelon matrix B by elementary row operations.

Proof :-

Step 1 : Apply $R_i(c)$ & $R_{ij}(c)$ and find one row-reduced matrix.

Step 2 : Apply R_{ij} so that all zero rows bellow non zero rows.

Step 3 : Apply R_{ij} so that $K_1 < K_2 < \cdots < k_r$ for non-zero rows.