Vector Space :- A Vector space over a field F is a non-empty set V together with two binary operations that satisfy the eight axioms listed below.

(i) **addition :-** For every pair $u, v \in V$, $u+v \in V$.

(ii) **Scalar multiplication :-** For every $v \in V$ and $c \in F$, $cv \in V$.

(iii) **Commutativity :-** $u+v = v+u$ for all $u, v \in V$

(iv) **Associativity :-** $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$

$(ab)v = a(bv)$ $a, b \in F$

(v) **Addative identity :-** There exists an element $0 \in V$ such that $v+0 = v$ for all $v \in V$.

(vi) Addative inverse :- For every $v \in V$, there exists $w \in V$ such that $v + w = 0$

(vii) Multiplicative identity :- $1v = v$ for all $v \in V$.

(viii) Distributive Properties :- $a(u+v) = au + av$ for all $a, b \in F$
$(a+b)u = au + bv$ $u, v \in V$.

If V is a vector space over a field F then the elements of V are called 'vectors' or 'points'.

* A vector space over the field $\mathbb{R}$ is called a real vector space.
* A vector space over the field $\mathbb{C}$ is called a complex vector space.

Examples :-

(i) The simplest vector space contains only one point namely '0'. Thus $\{0\}$ is a vector space over any field.

(ii) $\mathbb{R}$ is a vector space over itself. In fact, every field is a vector space over itself.

(iii) $\mathbb{C}$ is a vector space over $\mathbb{R}$.

(iv) $\mathbb{R}$ is a vector space over the field $\mathbb{Q}$ - rational numbers.

(v) For any field F, $F^n := F \times F \times \cdots \times F$ (n times) is a vector space over F.

(vi) For any field F, $F[x] :=$ the set of polynomials $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_i \in F$ & $n \in \mathbb{N} \cup \{0\}$ is a vector space over F.

(vii) For any field F, $F^\infty = \{(x_1, x_2, \cdots) : x_i \in F\}$, the set of all sequences of F is a vector space over F.

(viii) For a set S, F^S denotes the set of functions from S to F.

F^S is a vector space over F.

Note that F^n & F^∞ are special cases of F^S as F^n can be taken as $F^{\{1,2,\cdots,n\}}$ & F^∞ can be taken as $F^{\{1,2,3,\cdots\}}$.

(ix) $M_{m\times n}(F)$ is vector space over F.

Sub space :- Let V be a vector space over the field F. Let $U \subseteq V$. If U is also a vector space using same addition and scalar multiplicati[on] of on V then U is called a sub space of V.

Example :-

(i) $\{(x_1, x_2, 0) : x_1, x_2 \in F\}$ is a sub space of F^3.

(ii) The set of all continuous real-valued function on the interval $[0,1]$ (is a vector space over $\mathbb{R}$) is a sub space of $\mathbb{R}^{[0,1]}$.

Theorem :- A subset U of V is a subspace if and only if U satisfies the following three conditions.
(i) $0 \in U$ (ii) $u, v \in U \Rightarrow u+v \in U$ (iii) $a \in F, u \in U \Rightarrow au \in U$.

Proof :- Easy to verify.

Sum of Subsets :- Let $V_1, V_2, \cdots, V_n$ be subsets. Then the sum of subsets $V_1 + V_2 + \cdots + V_n$ is the set of all possible sums of elements of $V_1, \cdots, V_n$.

More precisely,

$$V_1 + \cdots + V_n = \{ v_1 + v_2 + \cdots + v_n : v_1 \in V_1, \cdots, v_n \in V_n \}$$

Examples :- $V = F^3$, $U = \{(x,0,0) : x \in F\}$, $W = \{(0,y,0) : y \in F\}$
Then $U + W = \{(x,y,0) : x, y \in F\}$.

Theorem: Let $V_1, V_2, \cdots, V_n$ be subspaces. Then $V_1 + V_2 + \cdots + V_n$ is the smallest subspace containing $V_1, V_2, \cdots, V_n$.

Proof:- Since $0 \in V_1 + \cdots + V_n$ and $V_1 + \cdots + V_n$ is closed under addition & scalar multiplication, $V_1 + \cdots + V_n$ is a subspace.

Clearly, $V_i \subseteq V_1 + \cdots + V_n$ as $v_i \in V_i$ can be written as $v_i = 0 + \cdots + 0 + v_i + 0 \cdots + 0$, where $0 \in V_1, \cdots, V_{i-1}, V_{i+1}, \cdots, V_n$

Let W be a subspace containing $V_1, \cdots, V_n$. Then for any $v_1 + \cdots + v_n \in V_1 + \cdots + V_n$ we have $v_i \in V_i \subseteq W$. Thus $v_1 + \cdots + v_n \in W$

Therefore, $V_1 + \cdots + V_n \subseteq W$

Direct sum :- The sum $V_1 + \cdots + V_n$ is called a direct sum and is denoted by $V_1 \oplus \cdots \oplus V_n$ if each element of $V_1 + \cdots + V_n$ can be written as a unique way $v_1 + \cdots + v_n$.

Example :- Let $V = F^3$ be a vector space over F.

$$V_1 = \{(x,0,0) : x \in F\}$$

$$V_2 = \{(0,y,0) : y \in F\}$$

$$V_3 = \{(x,y,0) : x,y \in F\}$$

Then $V_3 = V_1 \oplus V_2$

$V_1 + V_2 + V_3$ is same as V_3. But we can not write $V_1 \oplus V_2 \oplus V_3$ as elements can not be uniquely expressed.