Differential Equations - Solved Problems

MTL101

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1 First Order ODEs

1.1 Problem 1

Find the family of curves which intersects the family of curves $y = cx$ at an angle of 45^o.

Sol: A slight modification to the problem of finding the orthogonal trajec tories for a curve, in this case we are asked to find the oblique trajectories instead. Nevertheless, we would best complete the first step, that is, finding the differential equation governing the given family of curves, which is,

$$\frac{dy}{dx} = \frac{y}{x}$$

Recall that the possible slopes $\tilde{m}$ of a line inclined at an angle 45^o to a given line of slope m is given by:

$$\tilde{m} = \frac{m \pm 1}{1 \mp m}$$

To find the desired family of curves $\tilde{y}(x)$, we replace $\tilde{m}$ by $\frac{d\tilde{y}}{dx}$ and m by $\frac{\tilde{y}}{x}$ (since $y = \tilde{y}$ at the point of intersection).

$$\frac{d\tilde{y}}{dx} = \frac{\tilde{y} \pm x}{x \mp \tilde{y}}$$

We choose the positive sign in the numerator (the 2nd part is left as an exercise). This is a homogeneous equation, which is separable by making the substitution $m = \frac{y}{x}$. On simplifying, we obtain,

$$x\frac{dm}{dx} = \frac{m^2 + 1}{1 - m}$$

$$\implies \int \frac{dx}{x} = \int \frac{(1 - m)dm}{1 + m^2} + C$$

On computing the integral and substituting back m, we finally obtain,

$$\boxed{\tan^{-1}\frac{y}{x} + \ln\sqrt{x^2 + y^2} = C}$$

1.2 Problem 2

Find a general solution for the following differential equation:

$$\frac{dy}{dx} = \frac{x + 2y + 1}{2x + y - 1}$$

Sol: Although this equation can be solved by converting it into an exact equation, that method is slightly tedious and not the preferred approach. Instead, we try to convert it into a separable, homogeneous form.

Consider the substitution, $x = u + a$, $y = v + b$. Clearly $\frac{dy}{dx} = \frac{dv}{du}$. Thus,

$$\frac{dv}{du} = \frac{u + 2v + a + 2b + 1}{2u + v + 2a + b - 1}$$

For this to be a homogeneous equation, we must obtain appropriate a and b by solving the system of linear equations:

$$a + 2b + 1 = 0 \quad (1)$$

$$2a + b - 1 = 0 \quad (2)$$

On solving, we get $a = 1$, $b = -1$. We will need these values when we substitute back u and v.

Coming back to our now homogeneous equation,

$$\frac{dv}{du} = \frac{u + 2v}{2u + v}$$

Solving and substituting back, as in Problem 1, we obtain the solution as,

$$\boxed{x + y = K(y - x + 2)^3}$$

1.3 Problem 3

Consider S, the set of solutions of the homogeneous, linear differential equation, $y^0 + P(x)y = 0$

Prove that S is a vector subspace of the real vector space of continuous functions. Also find $dim(S)$.

Sol: Recall that W is a subspace of V (F) if and only if $\forall\ u, v \in W$ and $a, b \in$ F, $au + bv \in W$

For any two particular solutions y_1 and y_2 of the DE, consider $ay_1 + by_2$. We have,

$(ay_1 + by_2)^0 + P(x)(ay_1 + by_2) = a(y^0{}_1 + P(x)y_1) + b(y^0{}_2 + P(x)y_2) = 0$ Thus, S is a subspace, as claimed. Also, note that any solution of the above is of the type $y = Ce^{\mathrm{R}P\,(x)dx}$, i.e. S is spanned by $e^{\mathrm{R}P\,(x)dx}$. So

$$\boxed{dim(S) = 1\ .}$$

1.4 Problem 4

Solve the following ODE, given that it admits an integrating factor which is a function of $(x + y^2)$.

$$(3y^2 - x)dx + 2y(y^2 - 3x)dy = 0$$

Sol: Comparing with $M\,dx + N\,dy = 0$, we get that $M = 3y^2 - x$ and $N = 2y(y^2 - 3x)$. Evidently, $\frac{\partial M}{\partial y} = 6y$ and $\frac{\partial N}{\partial x} = -6y$. So, the equation is not exact. We thus multiply it by an integrating factor, $F(x+y^2)$. For exactness, we must have

$$\frac{\partial(FM)}{\partial y} = \frac{\partial(F\,N)}{\partial x}$$

Simplifying, we get:

$$12yF = F^0(N - 2yM)$$

$$\Longrightarrow 3F = -(x + y^2)F^0$$

Thus, $F = \frac{1}{(x+y^2)^3}$. We now have,

$$\frac{3y^2 - x}{(x + y^2)^3}\,dx + \frac{2y(y^2 - 3x)}{(x + y^2)^3}\,dy = 0$$

We now follow the usual method. Assuming the solution is $u(x, y) = c$, write

$$\frac{\partial u}{\partial x} = \frac{3y^2 - x}{(x + y^2)^3}$$

$$\implies u = \frac{x - y^2}{(x + y^2)^2} + g(y)$$

Taking the partial derivative w.r.t. y,

$$\frac{\partial u}{\partial y} = \frac{2y(y^2 - 3x)}{(x + y^2)^3} + g^0(y) = \frac{2y(y^2 - 3x)}{(x + y^2)^3}$$

So, $g^0(y) = g(y) = 0$, and the solution $u(x, y) = c$ is thus:

$$\boxed{x - y^2 = C(x + y^2)^2}$$

1.5 Problem 5

Consider the IVP $y^0 = f(x, y)$, $y(1) = -1$, where

$$f(x, y) = \begin{cases} \frac{\sin(x+y)}{x+y} & \text{if } x + y 6= 0 \\ 1 & \text{if } x + y = 0 \end{cases}$$

Check the existence and uniqueness of the solution. 3

Sol: Consider any rectangle in the neighbourhood of a general point $(c, -c)$. We have,

$$\lim_{(x,y)\longrightarrow(c,-c)} \frac{\sin(x + y)}{x + y}$$

$$x + y = \lim_{t \longrightarrow 0} \frac{sin(t)}{t} = 1$$

Thus, $f(x, y)$ is continuous in any rectangle in R^2, and solution exists.

Now, taking the partial derivative w.r.t x, we get:

$$\frac{\partial f}{\partial x} = \frac{(x + y)cos(x + y) - sin(x + y)}{(x + y)^2}$$

Around a point $(c, -c)$, we have,

$$\lim_{(x,y) \longrightarrow (c,-c)} \frac{(x + y)cos(x + y) - sin(x + y)}{(x + y)^2} = \lim_{t \longrightarrow 0} \frac{tcos(t) - sin(t)}{t^2} = 0$$

Also, by definition, we have

$$\frac{\partial f}{\partial x}_{(c,-c)} = \frac{f(c + h, -c) - f(c, -c)}{h} = \frac{sin(h) - h}{h^2} = 0$$

So we have, $\frac{\partial f}{\partial x}$ is continuous and thus bounded in any rectangle in R^2. Hence, we must have a unique solution in the vicinity of (1,-1).

1.6 Problem 6

Consider the ODE:

$$y^0 = y(xy^3 - 1)$$

Find the general solution by reducing it to a linear form.

Sol: Rewrite the given ODE as,

$$y^0 + y = xy^4$$

This is clearly a Bernoulli equation. Dividing by y^4 on both sides of the equation,

we obtain,

$$\frac{1}{y^4} y^0 - \frac{1}{3y^3} = x \quad -1\ 3y^3$$

Now, make the substitution $z = \frac{-1}{3y^3}$. Then $z^0 = \frac{1}{y^4} y^0$, and the DE becomes

$$z^0 - 3z = x$$

The integration factor in this case is $e^{\int -3dx} = e^{-3x}$. On multiplying with the same and integrating, we then obtain

$$ze^{-3x} = \int 3 - e^{-3x}$$

$$xe^{-3x} = -xe^{-3x} - \frac{e^{-3x}}{9} + c$$

Substituting back the value of z, we finally get:

$$\boxed{y^{\frac{1}{3}} = \frac{1}{3} + x + ce^{3x}}$$

1.7 Problem 7

Find the general solution of the following ODE:

$$2xe^{2y}y' = 3x^4 + e^{2y}$$

Sol: Divide both sides of the equation by x.

$$2y'e^{2y} = 3x^3 + \frac{e^{2y}}{x}$$

Set $t = e^{2y}$. Then, $t' = 2y'e^{2y}$. And so, the DE reduces to

$$t' - \frac{1}{x}t = 3x^3$$

The integrating factor $\mu = e^{\int -\frac{dx}{x}} = \frac{1}{x}$. Thus,

$$\frac{t}{x} = \int 3x^2 dx = x^3 + C$$

Thus, on substituting back the value of t, we get

$$\boxed{e^{2y} = x^4 + Cx}$$

1.8 Problem 8

Given the IVP:

$$y' + xy = x,\ y(0) = 0$$

Use Picard's iterative method to find the solution.

Sol: Set the zeroth Picard's iterate to be $y_0 = 0$. We know that, by Picard's iteration method, the successive approximations for the solution to an ODE $y' = f(x, y)$ are given by:

$$y_n(x) = y_0 + \int_0^x f(t, y_{n-1})\,dt$$

Note that here, $f(x, y) = x(1 - y)$. The first few Picard's iterates are given by,

$$y_1(x) = 0 + \int_0^x t\,dt = \frac{x^2}{2}$$

$$y_2(x) = 0 + \int_0^x t\left(1 - \frac{t^2}{2}\right)dt = \frac{x^2}{2} - \frac{\left(\frac{x^2}{2}\right)^2}{2!}$$

$$y_3(x) = 0 + \int_0^x t\left(1 - \frac{t^2}{2} - \frac{t^4}{8}\right)dt = \frac{x^2}{2} - \frac{\left(\frac{x^2}{2}\right)^2}{2!} + \frac{\left(\frac{x^2}{2}\right)^3}{3!}$$

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Continuing this way, we see that $y_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i!}\left(\frac{x^2}{2}\right)^i$. As $n \longrightarrow \infty$, we see that

$$\boxed{y_n \longrightarrow 1 - e^{-\frac{x^2}{2}}}$$