Q Suppose $a, b \in \mathbb{R}$. Consider the following system of eqns:

$$2x_1 + x_2 + ax_3 = 1, \quad x_1 + x_3 = 2, \quad x_1 + x_2 - x_3 = b$$

a) Apply suitable elementary row operations on the augmented matrix to find an equivalent system such that its coefficient matrix is RRE for every a & b.

b) Using part (a) find all possible $a, b \in \mathbb{R}$ s.t. the system has i) no solutions ii) a unique solution, iii) infinitely many solutions.

c) Write all the solutions when the system is consistent.

Solution: The augment matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & a & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & b \end{array}\right) \overset{R_1 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 1 & a & 1 \\ 1 & 1 & -1 & b \end{array}\right) \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & a-2 & -3 \\ 0 & 1 & -2 & b-2 \end{array}\right)$$

$$\overset{R_3 \to R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & a-2 & -3 \\ 0 & 0 & -a & b+1 \end{array}\right).$$ If $a=0$, the coefficient matrix is RRE.

Suppose $a \neq 0$. Apply $R_3 \to -a^{-1}R_3$, $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & a-2 & -3 \\ 0 & 0 & 1 & -\frac{b+1}{a} \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 + \frac{b+1}{a} \\ 0 & 1 & 0 & -3 + \frac{(b+1)(a-2)}{a} \\ 0 & 0 & 1 & -\frac{b+1}{a} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2a+b+1}{a} \\ 0 & 1 & 0 & \frac{ab-2a-2b-2}{a} \\ 0 & 0 & 1 & -\frac{b+1}{a} \end{array}\right).$$ where the coefficient matrix is RRE.

Thus if $a \neq 0$, the system has a unique solution & it is

$$\left(\frac{2a+b+1}{a}, \frac{ab-2(a+b+1)}{a}, -\frac{b+1}{a}\right).$$

If $a=0$ but $b+1 \neq 0$ i.e. $b \neq -1$, then the system is inconsistent.

If $a=0$ & $b=-1$, the system is $x_1 + x_3 = 2$, $x_2 - 2x_3 = -3$ where x_3 is the free unknown. Thus the (infinite number of) solutions are (by putting $x_3 = \lambda$) $(2-\lambda, -3+2\lambda, \lambda)$ for every $\lambda \in \mathbb{R}$.

Q Suppose u, v, w are vectors in an arbitrary vector spa
V over a field $\mathbb{F}$. Show that $\{u, v, w\}$ is linearly indepe
if & only if $\{u+v, v+w, w+u\}$ is linearly independent

<u>Solution:</u> "if" Suppose $\{u+v, v+w, w+u\}$ is linearly in
We will show that $\{u, v, w\}$ is linearly independent.
Let $w' = u+v$, $u' = v+w$, $v' = w+u$. Then $\{u', v', w'\}$ is lin
independent & $x w' + y u' + z v' = (x+z)u + (x+y)v + (y+z)w$.

Suppose $a, b \in \mathbb{R}$ ~~Then~~ such that $au + bv + cw = 0$
We need to show that $a = b = c = 0$.
First we find x, y, z such that

$$x+z = a, \quad x+y = b, \quad y+z = c \longrightarrow (*)$$

Augmented matrix is

$$\begin{pmatrix} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & 1 & c \end{pmatrix} \overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 1 & 1 & c \end{pmatrix}$$

$$\overset{R_3 \to R_3 - R_2}{\sim} \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 2 & c-b+a \end{pmatrix} \overset{R_3 \to \frac{1}{2}R_3}{\sim} \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 1 & \frac{c-b+a}{2} \end{pmatrix}$$

$$\overset{\substack{R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a - \frac{c-b+a}{2} \\ 0 & 1 & 0 & b-a+\frac{c-b+a}{2} \\ 0 & 0 & 1 & \frac{c-b+a}{2} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{a+b-c}{2} \\ 0 & 1 & 0 & \frac{-a+b+c}{2} \\ 0 & 0 & 1 & \frac{a-b+c}{2} \end{array}\right)$$

Thus $x = \frac{a+b-c}{2}$, $y = \frac{-a+b+c}{2}$, $z = \frac{a-b+c}{2}$

If we make these choices of x, y & z then

$(x+z)u + (x+y)v + (y+z)w = 0$

~~$a w' + bu$~~
$\Rightarrow x w' + y u' + z v' = 0 \Rightarrow x = y = z = 0$ ($\because \{u', v', w'\}$ is linearly independe

Then $a = 0, b = 0, c = 0$ (since x, y, z satisfy $(*)$)
"Only if" part is easier & left to the students.

Problem: Suppose W_1 and W_2 are subspaces of a vector space V over a field $\mathbb{F}$. Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Solution: ($\Leftarrow$) $W_1 \subset W_2 \Rightarrow W_1 \cup W_2 = W_2$, which is a subspace of V

$W_2 \subset W_1 \Rightarrow W_1 \cup W_2 = W_1$, which is a subspace of V.

So, in either case $W_1 \cup W_2$ is a subspace of V.

($\Rightarrow$) Given: $W_1 \cup W_2$ is a subspace of V.

To show: either $W_1 \subset W_2$ or $W_2 \subset W_1$.

We'll prove this by contradiction.

Suppose that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$.

Then there exist vectors w_1 and w_2 such that

$$\left.\begin{array}{l} w_1 \in W_1, \quad w_1 \notin W_2 \\ \& \quad w_2 \in W_2, \quad w_2 \notin W_1 \end{array}\right\} (*)$$

So, both w_1 and w_2 belong to $W_1 \cup W_2$.

Claim: $w_1 + w_2 \notin W_1 \cup W_2$

$w_1 + w_2 \in W_1 \Rightarrow w_2 = (w_1+w_2) - w_1 \in W_1$ ($\because W_1$ is a subspace)

Illy, $w_1 + w_2 \in W_2 \Rightarrow w_1 = (w_1+w_2) - w_2 \in W_2$ ($\because W_2$ is a subspace)

So, we arrive at a contradiction to $(*)$ if we assume $w_1 + w_2 \in W_1 \cup W_2$.

Hence, $w_1 + w_2 \notin W_1 \cup W_2$ which implies that $W_1 \cup W_2$ is not a subspace as $w_1, w_2 \in W_1 \cup W_2$.

This completes the proof of the "only if" part.

WARNING: Just providing an example where $W_1 \not\subset W_2$, $W_2 \not\subset W_1$ and $W_1 \cup W_2$ is not a subspace, is NOT a correct proof.

Problem: Let $V = M_n(\mathbb{R})$, $W_1 = \{A \in V : A^t = A\}$

and $W_2 = \{A \in V : A^t = -A\}$

Prove that $V = W_1 \oplus W_2$.

Soln: Note that W_1 and W_2 are subspaces of V (Prove it!)

Further let $A \in M_n(\mathbb{R})$.

Write $B = \frac{1}{2}(A + A^t)$ and $C = \frac{1}{2}(A - A^t)$

Then $B^t = B$ and $C^t = -C$

So, $B \in W_1$ & $C \in W_2$

Also, $A = \frac{1}{2}(A+A^t) + \frac{1}{2}(A-A^t)$

$= B + C \in W_1 + W_2$.

This proves $V = W_1 + W_2$.

To show $V = W_1 \oplus W_2$, we need $W_1 \cap W_2 = \{0\}$.

Suppose $A \in W_1 \cap W_2$

$\Rightarrow$ $A^t = A$ and $A^t = -A$

$\Rightarrow$ $A = -A$ $\Rightarrow$ $2A = 0$ $\Rightarrow$ $A = 0$ (the zero matrix)

$\therefore$ $W_1 \cap W_2 = \{0\}$ and $V = W_1 \oplus W_2$.

5Q Show that if ~~if~~ $u_1 = (x_1, x_2, x_3)$, $u_2 = (y_1, y_2, y_3)$, $u_3 = (z_1, z_2, z_3)$ ~~are~~ such that $B = \{u_1, u_2, u_3\}$ is linearly independent subset of $\mathbb{R}^3$, then ~~any vector $v = (\alpha, \beta, \gamma)$~~, B is a basis of $\mathbb{R}^3$.

Solution: Since B is given to be linearly independent, we have to show that every vector $v = (\alpha, \beta, \gamma)$ can be written as a linear combination of vectors in B (i.e. in $\mathrm{Span}(B)$), that is to find scalars $a, b, c \in \mathbb{R}$ s.t.

$$(\alpha, \beta, \gamma) = a(x_1, x_2, x_3) + b(y_1, y_2, y_3) + c(z_1, z_2, z_3)$$

$$\Leftrightarrow (\alpha, \beta, \gamma) = (ax_1 + by_1 + cz_1, ax_2 + by_2 + cz_2, ax_3 + by_3 + cz_3)$$

$$\Leftrightarrow ax_1 + by_1 + cz_1 = \alpha, \quad ax_2 + by_2 + cz_2 = \beta, \quad ax_3 + by_3 + cz_3 = \gamma$$

$$\Leftrightarrow \begin{aligned} x_1 a + y_1 b + z_1 c &= \alpha \\ x_2 a + y_2 b + z_2 c &= \beta \\ x_3 a + y_3 b + z_3 c &= \gamma \end{aligned}$$

Remark: Finding the coeffints a, b, c boils down to solving a system of non homogeneous linear equations.

Which has a solution if and only if

$$\mathrm{Rank}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} x_1 & y_1 & z_1 & \alpha \\ x_2 & y_2 & z_2 & \beta \\ x_3 & y_3 & z_3 & \gamma \end{pmatrix} \quad \cdots\cdots (*)$$

Since B is linearly independent, $\mathrm{Rank}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 3$ & Rank of the augmented matrix is at least the rank of the coeficin't matrix and at most 3, ~~the equati~~ the relation $(*)$ always holds. Therefore there exist $a, b, c \in \mathbb{R}$ s.t. $v = au_1 + bu_2 + cu_3$.

6Q. Prove the following ~~statemt~~ statement:
A map $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear tranformation if and only if $T(x,y) = (ax+by, cx+dy)$ for some $a, b \in \mathbb{R}$.

Solution: Suppose $T(x,y) = (ax+by, cx+dy)$. We will show that T is a linear trawsformation, that is,

$$T(p(x_1,y_1) + q(x_2,y_2)) = p\,T(x_1,y_1) + q\,T(x_2,y_2)$$

for any $p, q \in \mathbb{R}$ & $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$.

$$\begin{aligned} LHS &= T(px_1+qx_2, py_1+qy_2) = \big(a(px_1+qx_2)+b(py_1+qy_2), c(px_1+qx_2)+d(py_1+qy_2)\big) \\ &= \big(p(ax_1+by_1)+q(ax_2+by_2), p(cx_1+dy_1)+q(cx_2+dy_2)\big) \\ &= p(ax_1+by_1, cx_1+dy_1) + q(ax_2+by_2, cx_2+dy_2) \\ &= RHS. \end{aligned}$$

Next, Suppose $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation

Then $T(x,y) = T(xe_1+ye_2)$ (where $e_1=(1,0), e_2=(0,1)$)
$= xT(e_1) + yT(e_2)$

Let $T(e_1) = (\alpha, \beta)$, $T(e_2) = (\gamma, \delta) \in \mathbb{R}^2$

Then $T(x,y) = x(\alpha,\beta) + y(\gamma,\delta) = (\alpha x + \gamma y, \beta x + \delta y)$.

which we wanted to prove.

7Q. "The map $T: P_n \rightarrow P_{n+1}$ defined by $T(p)(x) = xp(x) + p(1)$ is a linear transformation." Prove this statement.

(Here $P_n := \{p \in \mathbb{F}[x] : \deg(p) < n\}$.

Solution: ~~T(a~~ We have to show $T(ap+bq) = aT(p)+bT(q)$ ~~where~~ for $p, q \in$ ~~$\mathbb{F}$~~ P_n & $a, b \in \mathbb{F}$. By definition $(ap+bq)(x) = ap(x) + bq(x)$. ~~So~~ & ~~$T(ap+bq)(x) = x(ap+bq)(x)+$~~

$$\begin{aligned} T(ap+bq)(x) &= x(ap+bq)(x) + (ap+bq)(1) = x(ap(x)+bq(x)) + ap(1)+bq(1) \\ &= a\{xp(x)+p(1)\} + b\{xq(x)+q(1)\} = aT(p)(x) + bT(q)(x). \end{aligned}$$

Thus $T(ap+bq)(x) = \{aT(p)+bT(q)\}(x)$.

8Q. Suppose $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are linear transformations. Show that $T_2 \circ T_1 : V_1 \to V_3$ is a linear transformation.

Solution: We have to ~~verify~~ show for $a, b \in \mathbb{F}$, $u, v \in V_1$,

$$(T_2 \circ T_1)(au + bv) = a\,(T_2 \circ T_1)(u) + b\,(T_2 \circ T_1)(v)$$

that is,

$$T_2\left(T_1(au+bv)\right) = a\,T_2(T_1(u)) + b\,T_2(T_1(v)) \text{ ------ } (*)$$

$$\cancel{\Leftrightarrow T_2\left(aT_1(u) + bT_1(v)\right)}$$

$$LHS = T_2\left(aT_1(u) + bT_1(v)\right) = a\,T_2(T_1(u)) + b\,T_2(T_1(v))$$

(since T_2 is a linear transformation)

$$= RHS.$$

Thus $(*)$ is verified.

9Q. Show that if $T : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation, ~~show that~~ then $T = a_1 P_1 + a_2 P_2 + \cdots + a_n P_n$ for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and P_i is the map which maps $(x_1, x_2, \ldots, x_n)$ to x_i.

Solution: $T(x_1, x_2, \ldots, x_n) = T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$ (where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$).

$$= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \quad (\because T \text{ is linear})$$

$T(e_1), T(e_2), \ldots, T(e_n) \in \mathbb{R}$, Suppose $T(e_i) = b_i$.

$$\text{Then } T(x_1, x_2, \ldots, x_n) = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n$$

$$= b_1 x_1 + b_2 x_2 + \cdots + b_n x_n$$

$$= b_1 P_1(x_1, \ldots, x_n) + b_2 P_2(x_1, x_2, \ldots, x_n) + \cdots + b_n P_n(x_1, x_2, \ldots, x_n).$$

$$= (b_1 P_1 + b_2 P_2 + \cdots + b_n P_n)(x_1, x_2, \ldots, x_n)$$

Thus $T = b_1 P_1 + b_2 P_2 + \cdots + b_n P_n$ with $b_1, b_2, \ldots, b_n \in \mathbb{R}$.

10Q. Suppose $S: \mathbb{R}^m \to \mathbb{R}^n$ is a map and $p_i: \mathbb{R}^n \to \mathbb{R}$ is the ith projection (i.e. $p_i(x_1, x_2, \dots x_n) = x_i$ for each $i = 1, 2, \dots, n$). Show that S is a linear transformation if and only if $p_i \circ S$ is linear for each $i = 1, 2, \dots, n$.

Solution: First we show that $p_i: \mathbb{R}^n \to \mathbb{R}$ is a linear transformation for each $1 \le i \le n$, that is,

$$p_i(a(x_1, x_2, \dots, x_n) + b(y_1, y_2, \dots, y_n)) = a\, p_i(x_1, x_2, \dots, x_n) + b\, p_i(y_1, y_2, \dots y_n).$$

for any $a, b \in \mathbb{R}$ & $(x_1, \dots, x_n); (y_1, \dots, y_n) \in \mathbb{R}^n$.

$$LHS = p_i(ax_1 + by_1, ax_2 + by_2, \dots, ax_n + by_n)$$

$$= \cancel{p}\ ax_i + by_i \quad (\text{by definition of } p_i)$$

$$= a\, p_i(x_1, x_2, \dots, x_n) + b\, p_i(y_1, y_2, \dots, y_n).$$

One can show that composition of two linear transformations is a linear transformation (see QWA7, Q8). Therefore, if S is linear, so is $p_i \circ S$ for each i (since, ~~each~~ p_i is a linear transformation for each i).

Next, suppose ~~each~~ $p_i \circ S$ is a linear transformation for each i.

By definition of $p_1, p_2, \dots, p_n$, for any vector $v \in \mathbb{R}^n$

$$v = (p_1(v), p_2(v), \dots, p_n(v))$$

~~Since~~ Let $a, b \in \mathbb{R}$, $u, w \in \mathbb{R}^m$. Then $S(au + bw) \in \mathbb{R}^n$.

Hence
$$S(au + bw) = (p_1(S(au + bw)), p_2(S(au + bw)), \dots, p_n(S(au + bw)))$$
$$= ((p_1 \circ S)(au + bw), (p_2 \circ S)(au + bw), \dots, (p_n \circ S)(au + bw))$$

Observe, $\because$ $p_i \circ S(au + bw) = a(p_i \circ S)(u) + b(p_i \circ S)(w)$,

$$S(au + bw) = (a(p_1 \circ S)(u), a(p_2 \circ S)(u), \dots, a(p_n \circ S)(u)) + (b(p_1 \circ S)(w), b(p_2 \circ S)(w), \dots, b(p_n \circ S)(w))$$
$$= a(p_1(S(u)), p_2(S(u)), \dots, p_n(S(u))) + b(p_1(S(w)), p_2(S(w)), \dots, p_n(S(w)))$$
$$= a\, S(u) + b\, S(w)$$ which we wanted to show.

11Q. Write the following space as the solution space of a system of homogeneous linear equations:

$$V = \cancel{\{}\ \text{span}\{(1,2,3,4),\ (1,1,2,2)\}$$

as a subspace of $\mathbb{R}^4$.

<u>Solution:</u> Consider the following matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

which is obtained by writing the tuples as rows. Now,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix} \overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

$$\underset{R_1 \to R_2 - 2R_1}{\overset{R_2 \to R_2 \times (-1)}{\sim}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

which is row reduced echelon. Then

$$\text{Span}\{(1,2,3,4),(1,1,2,2)\}$$

$$= \text{Span}\{(1,0,1,0),\ (0,1,1,2)\}$$

$$= \cancel{\text{Span}\{a}$$

$$= \{a(1,0,1,0) + b(0,1,1,2) : a, b \in \mathbb{R}\}$$

$$= \{(a,\ b,\ a+b,\ 2b) : a, b \in \mathbb{R}\}$$

$$= \{(x,y,z,w) \in \mathbb{R}^4 : z = x+y,\ w = 2y\}$$

$$= \left\{ (x,y,z,w) \in \mathbb{R}^4 : \begin{matrix} x+y-z=0 \\ 2y-w=0 \end{matrix} \right\}.$$

12Q. Let $W_1 = \{(x,y,z,w) : 2x+y=0,\ x+y+z=0\}$

$W_2 = \{(x,y,z,w) : x+y+z+w=0,\ x+2y+3z+w=0\}$

Find a basis of $W_1 \cap W_2$, a basis of $W_1 + W_2$ and write W_1+W_2 as the solution space of a system of homogeneous linear equations.

<u>Solution:</u> $W_1 \cap W_2 = \{(x,y,z,w) : 2x+y=0,\ x+y+z=0,\ x+y+z+w=0,\ x+2y+3z+w=0\}$.

Consider the coefficient matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \underset{\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}}{\overset{R_1 \leftrightarrow R_2}{\sim}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \overset{R_2 \to R_2 - 2R_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$$\underset{R_2 \to (-1)R_2}{\overset{\substack{R_1 \to R_1 + R_2 \\ R_4 \to R_4 + R_2}}{\sim}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ which is RRE. Hence,

$W_1 \cap W_2 = \{(x,y,z) : x - z = 0,\ y+2z=0,\ w=0\}$

$= \{(\lambda, -2\lambda, \lambda, 0) : \lambda \in \mathbb{R}\}$ (since z is the only free unknown is the above system set $z = \lambda$)

Thus $\{(1,-2,1,0)\}$ is a basis of $W_1 \cap W_2$.

Rank of $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 2$ & rank of $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} = 2$

so $\dim W_1 = 4-2 = 2$. & $\dim W_2 = 4-2 = 2$.

To get a basis of W_i by extending $\{(1,-2,1,0)\}$ we need one more vector for each $i = 1,2$.

$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ &

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$.

~~(1,-2,1,0)~~

$e_4 \in W_1$ & $\{(1,-2,1,0), e_4\}$ is linearly ind.

$(-1,0,0,1) \in W_2$ & $\{(1,-2,1,0), (-1,0,0,1)\}$ is linearly ind.

$\{(1,-2,1,0), (0,0,0,1), (-1,0,0,1)\}$ is a basis of $W_1 + W_2$

Solution of 12b continued:

We have seen that $\{(1,-2,1,0), (0,0,0,1), (-1,0,0,1)\}$ is a basis of W_1+W_2. To write W_1+W_2 as the solution space of a hom system of linear eqns

Consider
$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$
which is row equivalent to
$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
by $R_3 \to (-1)R_3$, $R_3 \leftrightarrow R_1$, $R_1 \leftrightarrow R_2$

$$\underset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{R_2 \to R_2(-\frac{1}{2})}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $W_1+W_2 = \text{Span of } \{(1,0,0,-1), (0,1,-\frac{1}{2},-\frac{1}{2}), e_4\}$

$= \left\{\left(a, b, -\frac{b}{2}, -a-\frac{b}{2}+c\right) : a, b, c \in \mathbb{R}\right\}$

$= \left\{(x,y,z,w) : z = -\frac{y}{2}\right\}$ $\cancel{w = x - \frac{y}{2}}$ (the fourth component can be any real number)

$= \{(x,y,z,w) : y + 2z = 0\}$

$= \{(\lambda, -2\mu, \mu, \delta) : \lambda, \mu, \delta \in \mathbb{R}\}$ (note that x, z, w are free unknowns)

Thus $\{(1,0,0,0), (0,-2,1,0), (0,0,0,1)\}$ is also a basis → not asked.

Q13. Write the following space as the solution space of a system of homogeneous linear equations.

$$W = \text{Span}\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \right\}$$ as a subspace of $M_{2\times 2}(\mathbb{R})$.

<u>Solution:</u> Consider the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

whose first row is obtained from $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ & the second from $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, (remember how you got)

Now,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \overset{R_2 \to R_2 - 2R_1}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \end{pmatrix}$$

$$\overset{R_2 \to -\frac{1}{3} R_2}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \end{pmatrix} \overset{R_1 \to R_1 - 2R_2}{\sim} \begin{pmatrix} 1 & 0 & \frac{5}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \end{pmatrix}.$$

~~Thus $W = \{a(1, 0, \frac{5}{3}, \frac{2}{3}) + b(0, 1, \frac{2}{3}, \frac{5}{3}$~~

Thus $W = \text{Span}\left\{ \begin{pmatrix} 1 & 0 \\ \frac{5}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{5}{3} \end{pmatrix} \right\}$ (see how we got these matrices from the rows)

$$= \left\{ a\begin{pmatrix} 1 & 0 \\ \frac{5}{3} & \frac{2}{3} \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ \frac{2}{3} & \frac{5}{3} \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ \frac{5a+2b}{3} & \frac{2a+5b}{3} \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) ; z = \frac{5x+2y}{3}, \; w = \frac{2x+5y}{3} \right\}$$

$$= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : 5x + 2y - 3z = 0, \; 2x + 5y - 3w = 0 \right\}$$

Q14 Show that the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & \frac{9}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{6}{5} \end{pmatrix}$ is diagonalizable.

Find an invertible matrix P s.t. $P^{-1}AP$ is diagonal.

<u>Soln</u>: The characteristic polynomial is

$\det \begin{pmatrix} x-1 & 0 & 0 \\ \frac{2}{5} & x-\frac{9}{5} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} & x-\frac{6}{5} \end{pmatrix} = (x-1)^2(x-2)$. The eigen values are 1 & 2.

To find the eigenspace corresponding to 1:

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ -\frac{2}{5}x+\frac{9}{5}y+\frac{2}{5}z \\ -\frac{x}{5}+\frac{2}{5}y+\frac{6}{5}z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{array}{l} x=x \\ -2x+4y+2z=0, \; -x+2y+z=0 \end{array}$

$\Leftrightarrow -x+2y+z=0$. The coefft matrix of the system $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The solution space is ~~one~~ two dimensional (since the rank of the coefft matrix is one). &

$(2,1,0)$ (putting $z=0$, $y=1$) &

$(1,0,1)$ (putting $y=0$ $z=1$)

are linearly independent eigen vectors. Thus $\{(2,1,0),(1,0,1)\}$ is a basis of the eigenspace corresponding to 1.

To find the eigen space corresponding to 2:

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{array}{l} x=2x \\ -2x-y+2z=0 \\ -x+2y-4z=0 \end{array} \Leftrightarrow \left.\begin{array}{l} x=0 \\ x-2y-4z=0 \\ 2x+y-2z=0. \end{array}\right\} \Rightarrow \begin{array}{l} x=0 \\ y-2z=0 \end{array}$

The coefft matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & -4 \\ 2 & +1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ which is of rank 2.

The dimension of the soln space is one & $(0,2,1)$ is an eigen vector. Thus $\{(0,2,1)\}$ is a basis of the eigen space corresponding to 2.

If $P = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ then $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(See how to get P from the bases of the eigen spaces.

Q.15 Let ~~V =~~ $P_3 = \{a+bx+cx^2 : a,b,c \in \mathbb{R}\}$.

Define $\langle f | g \rangle = \int_0^1 f(x)\,g(x)\,dx$.

Find an orthogonal basis of P_3.

Solution: ~~Obser~~ Recall $\{1, x, x^2\}$ is a basis of P_3.

Use Gram-Schmidt process:

$$v_1 = 1, \quad v_2 = x - \frac{\langle x | 1 \rangle}{\|1\|^2} 1. = x - \int_0^1 t\,dt \Big/ \left(\int_0^1 1\,dt\right)^2 = x - \frac{1}{2}$$

$$v_3 = x^2 - \frac{\langle x^2 | 1 \rangle}{\|1\|^2} 1. - \frac{\langle x^2 | x-\frac{1}{2} \rangle}{\|x-\frac{1}{2}\|^2}\left(x-\frac{1}{2}\right)$$

$$= x^2 - \int_0^1 t^2\,dt - \frac{\int_0^1 t^2(t-\frac{1}{2})\,dt}{\int_0^1 (t-\frac{1}{2})^2\,dt} \cdot \left(x-\frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - \frac{\left(\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{3}\right)}{\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right)}\left(x-\frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - \frac{3-2}{12} \times \frac{12}{(4-6+3)}\left(x-\frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - \cancel{1}\left(x-\frac{1}{2}\right)$$

$$= x^2 - x + \frac{1}{6}.$$ Thus $\{1, x-\frac{1}{2}, x^2-x+\frac{1}{6}\}$ is an orthogonal basis of P_3

Not a part of the solution.

Checks: $\langle 1 | x-\frac{1}{2} \rangle = \int_0^1 (t-\frac{1}{2})\,dt = \frac{t^2}{2}\Big|_0^1 - \frac{1}{2} = 0$

- $\langle 1 | x^2-x+\frac{1}{6} \rangle = \int_0^1 t^2 - t - \frac{1}{6} = \frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{6}\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$

- $\langle (x-\frac{1}{2}) | (x^2-x+\frac{1}{6}) \rangle = \int_0^1 (t-\frac{1}{2})(t^2-t+\frac{1}{6})\,dt = \int_0^1 t^3 - t^2 + \frac{t}{6} - \frac{t^2}{2} + \frac{t}{2} - \frac{1}{12}\,dt$

$$= \int_0^1 t^3 - \frac{3}{2}t^2 + \frac{2}{3}t - \frac{1}{12}\,dt = \frac{t^4}{4} - \frac{3}{2}\times\frac{1}{3}t^3 + \frac{2}{3}\cdot\frac{1}{2}t^2 - \frac{1}{12}t\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{2} + \frac{1}{3} - \frac{1}{12} = \frac{3-6+4-1}{12} = 0.$$

Q.16 Show that the Lipschitz condn is satisfied by the function $f = |\sin y| + x$ at every point on the xy-plane though its partial derivative with respect to y does not exist on the line $y=0$.

Soln: $\lim_{t\to 0} \frac{|\sin t| - |\sin 0|}{t} = \lim_{t\to 0} \frac{|\sin t|}{t}$ does not exist, so $\frac{d(|\sin y|)}{dy}$ does not exist at $y=0$.

$$\frac{|f(y_2, x) - f(y_1, x)|}{|y_2 - y_1|} = \frac{|(|\sin y_2| + x) - (|\sin y_1| + x)|}{|y_2 - y_1|}$$

$$= \frac{||\sin y_2| - |\sin y_1||}{|y_2 - y_1|} \leq \frac{|\sin y_2 - \sin y_1|}{|y_2 - y_1|}$$

$$= \frac{2\left|\sin \frac{y_2 - y_1}{2}\right| \left|\cos \frac{y_2 - y_1}{2}\right|}{|y_2 - y_1|}$$

Using $\left|\frac{\sin(x)}{x}\right| < 1$ we have

$$\frac{|f(y_2, x) - f(y_1, x)|}{|y_2 - y_1|} < \left|\cos\left(\frac{y_2 - y_1}{2}\right)\right| \leq 1.$$

Thus $|f(y_2, x) - f(y_1, x)| \leq |y_2 - y_1|$

Q.17 Consider the IVP $y\,dy = x\,dx$, $y(0)=\beta$. Find all possible $\beta \in \mathbb{R}$ for which the IVP has (a) a unique sol^n (b) More than one solution (c) no solutions.

<u>Sol^n:</u> By solving the ODE, $y^2 = x^2 + C$. Using the initial condition $\beta^2 = C$. Hence the sol^{n}s are given by $y^2 = \beta^2 + x^2$. Thus $y = \pm\sqrt{\beta^2+x^2}$.

If $\beta = 0$, $y = x$ as well as $y = -x$ are solutions of the given IVP. If $\beta \neq 0$, then $y = \sqrt{\beta^2+x^2}$ when $\beta > 0$ (Observe that $y = -\sqrt{\beta^2+x^2}$ is not a sol^n).

If $\beta < 0$, $y = -\sqrt{\beta^2+x^2}$ is a solution, & $y = \sqrt{\beta^2+x^2}$ is not a solution.

Q.18 Find all the curves in the XY-plane whose tangents pass through (a,b).

<u>Solution:</u> The equation of the angent to the curve $y = y(x)$, which passes (a,b) is

$$\frac{y-b}{x-a} = \frac{dy}{dx}.$$ Solving $\frac{dy}{y-b} = \frac{dx}{x-a} \Leftrightarrow \ln(y-b) = \ln(x-a) + \ln\lambda$

$$\Rightarrow \underline{y-b = \lambda(x-a)}$$ which is a straight line.

Q. Solve the following system.

$$y_1' = 3y_1 - 4y_2$$
$$y_2' = y_1 - y_2$$

Soln: This can be written as

$$y' = Ay, \text{ with } A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Eigenvalues of A:

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda - 3 & 4 \\ -1 & \lambda + 1 \end{vmatrix} = 0$$
$$\Rightarrow (\lambda-3)(\lambda+1) + 4 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$
$$\Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1.$$

- Eigenvectors for $\lambda = 1$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -u_1 + 2u_2 = 0 \Leftrightarrow u_1 = 2u_2$$

So, $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector & eigenspace is one dimensional.

$\therefore \; y^{(1)} = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a soln.

To find a second soln. L.I. to this we need to solve ~~$(\lambda I - A)$~~ $(A - \lambda I)v = u$

i.e. $\begin{pmatrix} +2 & -4 \\ +1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Leftrightarrow v_1 - 2v_2 = 1$

Choosing $v_2 = 0$, we get $v = \begin{pmatrix} +1 \\ 0 \end{pmatrix}$

So, $y^{(2)} = te^{\lambda t}u + e^{\lambda t}v$ with $\lambda = 1$

$$= te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} +1 \\ 0 \end{pmatrix}$$

$\therefore$ The general soln. is $y = c_1 y^{(1)} + c_2 y^{(2)}$

i.e. $\boxed{y = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^t \left[t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} +1 \\ 0 \end{pmatrix} \right]}$

Q. Solve the following system.

$$y' = \begin{pmatrix} -1 & -4 & 2 \\ 2 & 5 & -1 \\ 2 & 2 & 2 \end{pmatrix} y \quad , \text{ where } y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Soln: The eigenvalues of A are $0, 3, 3$

Corresponding to $\lambda = 0$, we get an eigenvector $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

For $\lambda = 3$, the eigenspace is two dimensional and $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ forms a basis for the eigenspace.

(The calculations are left to the students).

In this example, we see that we get a complete set of eigenvectors (ie. the matrix is diagonalizable). This is the easier case when eigenvalues are repeated.

The general soln. is given by

$$y = c_1 e^{0t} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

ie $$\boxed{y = c_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$$

Q. Use Frobenius method to find a basis of solutions.

$$(x^2+x)y'' + (4x+2)y' + 2y = 0$$

Soln. Letting $y = x^r \sum_{m=0}^{\infty} a_m x^m$ and substituting y, y', y'' in the ODE, we get

$$(x^2+x)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} \qquad -(*)$$

$$+ (4x+2)\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + 2\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

The lowest power is x^{r-1} and equating the coeff. of x^{r-1} to zero gives the indicial eqn.

$$r(r-1)a_0 + 2r a_0 = 0$$

$$\Rightarrow \quad r^2 + r = 0 \quad \Rightarrow \quad r = 0, -1$$

The roots differ by an integer.

We have $r_1 = 0$ & $r_2 = -1$ (Recall r_1 is the larger root)

First soln: From $(*)$ with $r = r_1 = 0$ we have

$$m(m-1)a_m + (m+1)m a_{m+1} + 4m a_m + 2(m+1)a_{m+1} + 2a_m = 0,$$
$$m = 0, 1, 2, \ldots$$

This gives $a_{m+1} = -a_m$, $m = 0, 1, 2, \ldots$

Choosing $a_0 = 1$, $a_m = (-1)^m$

and $\boxed{y_1 = \sum_{m=0}^{\infty}(-1)^m x^m = \frac{1}{1+x}}$, $|x| < 1$

Second soln: Apply reduction of order method, $y_2 = u y_1$

After substituting & simplifying, we get

$$x u'' + 2u' = 0 \quad \Rightarrow \quad \frac{u''}{u'} = -\frac{2}{x}$$

This gives $u' = \frac{1}{x^2}$, so $u = -\frac{1}{x}$

$$\therefore \quad y_2 = \frac{-1}{x(x+1)} = -\frac{1}{x} + \frac{1}{x+1}$$

Since $\frac{1}{x+1} = y_1$ is a soln., we may take $\boxed{y_2 = \frac{1}{x}}$ as the 2nd soln. for the basis.

Note that y_1 & y_2 are clearly linearly independent.

Q. Find the eigenvalues & eigenfunctions of the following Sturm-Liouville problem.

$$y'' + \lambda y = 0 \quad , \quad y(0) = y(2\pi) \ , \ y'(0) = y'(2\pi) \ .$$

Soln: For $\lambda < 0$, let $\lambda = -\mu^2$, $\mu \neq 0$.

Then $y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$y(0) = y(2\pi) \Rightarrow c_1 + c_2 = c_1 e^{2\pi\mu} + c_2 e^{-2\pi\mu}$

$y'(0) = y'(2\pi) \Rightarrow \mu c_1 - \mu c_2 = \mu c_1 e^{2\pi\mu} - \mu c_2 e^{-2\pi\mu}$

The only soln. is $c_1 = c_2 = 0$.

So, $y \equiv 0$. Thus there are $\boxed{\text{no negative eigenvalues.}}$

For $\lambda = 0$, the general soln. is

$$y = c_1 + c_2 x$$

$y(0) = y(2\pi) \Rightarrow c_2 = 0$.

$y = c_1$ satisfies the 2nd cond. $y'(0) = y'(2\pi)$ also.

Thus $\boxed{\lambda = 0 \text{ is an eigenvalue with } y_0 = 1 \text{ as an eigenfn.}}$

For $\lambda > 0$, let $\lambda = \mu^2$, $\mu \neq 0$

The general soln. is $y = c_1 \cos \mu x + c_2 \sin \mu x$

and $y' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x$

The boundary cond. $\Rightarrow \begin{cases} (1 - \cos 2\pi\mu) c_1 - (\sin 2\pi\mu) c_2 = 0 \\ (\sin 2\pi\mu) c_1 + (1 - \cos 2\pi\mu) c_2 = 0 \end{cases}$

For non-trivial soln., $\begin{vmatrix} 1 - \cos 2\pi\mu & -\sin 2\pi\mu \\ \sin 2\pi\mu & 1 - \cos 2\pi\mu \end{vmatrix} = 0$

i.e. $(1 - \cos 2\pi\mu)^2 + \sin^2 2\pi\mu = 0$

i.e. $\cos 2\pi\mu = 1$ i.e. $\mu = \pm n$, $n \in \mathbb{N}$.

$$\boxed{\begin{aligned} &\text{So, } \lambda_n = n^2 \text{ is an eigenvalue for each } n \in \mathbb{N} \\ &\text{and } y_n = \cos nx \ ; \ \sin nx \text{ are eigenfns. for } n \in \mathbb{N} \end{aligned}}$$

Q Solve IVP

$$y'' - 2y' - 3y = 0$$

$$y(1) = -3 \quad , \quad y'(1) = -17$$

Define $t = \tilde{t} + 1$ so. $y(t) = y(\tilde{t}+1) = \tilde{y}(\tilde{t})$

then we get

$$\tilde{y}'' - 2\tilde{y}' - 3\tilde{y} = 0$$

$$\tilde{y}(0) = -3 \quad , \quad \tilde{y}'(0) = -17$$

Taking L.T. $\tilde{Y} = \mathcal{L}(\tilde{y})$

$\cancel{s^2 - 3}$ $\quad s^2 \cancel{Y + 3s + 17 - 2sY + 3 + 3Y}$

$$(s^2 - 2s - 3)\tilde{Y} = -3s - 17 + 6$$

$$\Rightarrow \tilde{Y} = \frac{2}{s+1} - \frac{5}{s-3}$$

$\tilde{y}(\tilde{t}) = 2e^{-\tilde{t}} - 5e^{3\tilde{t}}$ putting $\tilde{t} = t - 1$

$$\boxed{y(t) = 2e^{-(t-1)} - 5e^{3(t-1)}}$$

Q
Solve $y'' + 3y' + 2y = r(t)$

where
$$r(t) = \begin{cases} 1 & 0<t<1 \\ 0 & t>1 \end{cases}$$

with Initial conditions.

$y(0) = 0$, $y'(0) = 0$

$$(s^2 + 3s + 2)Y = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y = \left(\frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1}\right)(1 - e^{-s})$$

$$y(t) = \mathcal{L}^{-1}(Y) =$$

$$\left(\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}\right) - \left(\frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)}\right)u(t-1)$$

$$= \begin{cases} \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} & 0<t<1 \\ \frac{1}{2}e^{-2t}(1-e^{2}) - e^{-t}(1-e) & \text{if } t>1 \end{cases}$$

Q Solve $y(t) + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau = t+e^t$

This can be written as

$y(t) + y(t) * \cosh(t) = t + e^t$

Take L.T. we get

$Y(s) + Y(s)\left(\frac{s}{s^2-1}\right) Y(s) = \frac{1}{s^2} + \frac{1}{s-1}$

So $Y(s) = \frac{1}{s^2} + \frac{1}{s}$

$y(t) = \mathcal{L}^{-1}(Y(s)) = t+1.$

Q Solve

$$y_1' = 2y_1 + y_2$$

$$y_2' = 4y_1 + 2y_2 + 64t\,u(t-1)$$

$$y_1(0) = 2$$

$$y_2(0) = 0$$

Taking L.T.

$$sY_1 = 2 + 2Y_1 + Y_2$$

$$sY_2 = 4Y_1 + 2Y_2 + 64e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$Y_1(s) = 2\left[\frac{1}{s-4} - \frac{2}{s(s-4)} + \frac{32e^{-s}}{s-4}\left(\frac{1}{s^2} + \frac{1}{s^3}\right)\right]$$

Now

$$\frac{1}{s-4} - \frac{2}{s(s-4)} = \frac{1}{2(s-4)} + \frac{1}{2s}$$

$$\frac{64}{\cancel{s}}\left(\frac{1}{s^2(s-4)} + \frac{1}{s^3(s-4)}\right) = -\frac{5}{s} - \frac{20}{s^2} - \frac{16}{s^3} + \frac{5}{s-4}$$

So

$$\mathcal{L}^{-1}\left(e^{-s}\left(\frac{-5}{s} - \frac{20}{s^2} - \frac{16}{s^3} + \frac{5}{s-4}\right)\right)$$

$$= \left(-5 - 20(t-1) - 8(t-1)^2 + 5e^{4(t-1)}\right) u(t-1)$$

Hence $y_1(t) = 1 + e^{4t} + u(t-1)\left[-8t^2 - 4t + 7 + 5e^{4(t-1)}\right]$

Similarly

$$y_2(t) = -2 + 2e^{4t} + u(t-1)\left[16t^2 - 8t - 18 + 10e^{4(t-1)}\right].$$

QWA

1. Discuss the existence and uniqueness of solutions and find the maximal interval of existence for the IVP: $ty' = t + |y|, y(-1) = 1$.

   Answer: This is not a linear or separable equation. The non-linearity $f(t, y) = \frac{t+|y|}{t}$ is locally Lipschitz in any closed rectangle that does not intersect the line $t = 0$. So By Picard's theorem the solution exists and unique in an interval around -1.

   To find the maximal interval, We notice that the nonlinearity is Lipschitz in domain $(-\infty, -\delta] \times I\!R$ for any $\delta > 0$. Indeed,

$$|f(t, x) - f(t, y)| = \frac{1}{|t|}||x| - |y|| \le \frac{1}{\delta}|x - y|.$$

   Therefore, By global existence theorem, solution uniquely exits in $(-\infty, -\delta]$ for any $\delta > 0$. So the maximal interval of existence is the union of all these intervals which is $(-\infty, 0)$.

2. Discuss the existence and uniqueness of solutions and find the maximal interval of existence guaranteed by Picard's theorem for the IVP: $y' = t + y^2, y(0) = 0$.

   Answer: The nonlinearity $f(t, y) = t + y^2$ is locally Lipschitz in the rectangle

$$R_{ab} = \{(t, y) : |t| \le a, |y| \le b\}$$

   so by Picard's theorem, there exists solution in the interval $[-h, h]$ where $h = \min(a, b/M)$, $M = a + b^2$. So taking the function $h(x) = \frac{x}{(a+x^2)}$, we see that maximum of h is at $\frac{1}{2\sqrt{a}}$. Therefore, $h = \min\{a, \frac{1}{2\sqrt{a}}\}$. We have to find largest a such that we get h maximum. i.e., $h = \max_a \min\{a, \frac{1}{2\sqrt{a}}\} = \sqrt[3]{\frac{1}{4}}$.

3. Discuss the existence and uniqueness of the problem and then find the solution.

$$y' = 1 + y^2, \ y(0) = 1$$

   Answer: The function $f(t, y) = 1+y^2$ is locally Lipschitz in any bounded rectangle $R = \{(t, y) : |t| \le a, |y - 1| \le b\}$ around $(0, 1)$.

$$|\frac{\partial f}{\partial y}| \leq |2y| \leq 2(b+1).$$

Hence $f(t,x)$ is locally Liptschitz in R. So by Picard's theorem, there exists a unique solution in $[-h,h]$ where $h = \min(a, \frac{b}{M})$, $M = \max|f(t,x)|$. We can find the exact solution as the given equation is separable.

$$\frac{y'}{1+y^2} = 1$$

On integration, we get $\tan^{-1} y = t + c$ and $y = \tan(t+c)$ for $t + c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the one-parameter solution. substitute $t = 0$, we get $1 = \tan c$. So, the solution is $y = \tan(t + \frac{\pi}{4})$.

4. Solve the IVP

$$\frac{dy}{dt} = e^{(2t+y)}, \qquad y(0) = 0.$$

Answer: Note that $f(t,y)$ is continuous function and Lipschitz continuous w.r.t. y is any bounded interval containing $(0,0)$. So, we will have a unique solution in a neighborhood of $(0,0)$. By separating the variables, we get $e^{-y}dy = e^{2t}dt$. Integrating and simplifying results in $e^{-y} = -\frac{1}{2}e^{2t} - c_1$ for some constant c_1. Using $y(0) = 0$ we get $c_1 = -3/2$. Taking logarithm both sides we get,

$$y = -\ln\left(-\frac{1}{2}e^{2t} + 3/2\right)$$

Now domain of ln is $(0,\infty)$, so solution is valid in a domain where $e^{2t} < 3$ i.e. $t < \ln(3)/2 \approx 0.5493$.

5. The wood in a Egyptian burial case is found to contain 79% of carbon ^{14}C. What is the age of burial case. Half life of ^{14}C is 5730 years.

Answer: Decay of ^{14}C is modeled by $y' = ky$ and has solution, $y = y_0 e^{kt}$ Now using $y(0) = 1$ and $y(5730) = 0.5$ we get,

$$k = \frac{\ln(0.5)}{5730} = -\frac{\ln(2)}{5730}.$$

Now solving $0.79 = e^{kt}$ for t we get $t = 1948.63$ years.