Problem [6 marks] Solve the initial value problem

$$y'' - 2y' = \delta(t-1), \quad y(0) = 1, \quad y'(0) = 0, \quad t > 0.$$

Solution: Taking the Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] = e^{-s}$$

$$\Rightarrow s^2Y(s) - s - 2sY(s) + 2 = e^{-s}$$

$$\Rightarrow s(s-2)Y(s) = e^{-s} + (s-2)$$

$$\Rightarrow Y(s) = \frac{e^{-s}}{s(s-2)} + \frac{1}{s} = \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s}\right]e^{-s} + \frac{1}{s}$$

Taking the inverse Laplace transform, we get

$$y(t) = \frac{1}{2}u(t-1)[e^{2(t-1)} - 1] + 1.$$

Problem [7 marks] Determine the constants α, β, a and b so that $Y(s) = \frac{s}{(s+1)^2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0, \quad y(0) = a, \quad y'(0) = b.$$

Solution: Taking the Laplace transform, we get

$$s^2Y(s) - as - b + \alpha[sY(s) - a] + \beta Y(s) = 0$$

$$\Rightarrow (s^2 + \alpha s + \beta)Y(s) = as + (\alpha a + b)$$

$$\Rightarrow Y(s) = \frac{as + (\alpha a + b)}{s^2 + \alpha s + \beta}$$

But it is given that $Y(s) = \dfrac{s}{s^2 + 2s + 1}$. Comparing, we get

$$a = 1, \ \alpha a + b = 0, \ \alpha = 2 \text{ and } \beta = 1.$$

Hence, $\alpha = 2$, $\beta = 1$, $a = 1$ and $b = -2$.

Qs. Find the general solution using method of variation of parameter.

$$y'' + 9y = 3\tan(3x)$$

Ans. Homogeneous Problem:

$$y'' + 9y = 0 \qquad m^2 + 9 = 0$$
$$m = \pm 3i$$

$x_h = c_1 x_1 + c_2 x_2$

$x_1 = \cos(3x) \quad , \quad x_2 = \sin(3x)$

$$W(x_1, x_2) = \det\begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix} = 3(\sin^2(3x) + \cos^2(3x)) = 3$$

Variation of Parameter formula:

$x_p = u_1 x_1 + u_2 x_2$ with

$$u_1 = -\int \frac{x_2\, 3\tan(3x)}{W}\, dx \qquad u_2 = \int \frac{x_1\, 3\tan(3x)}{W}\, dx$$

$$= -\int \frac{3\sin(3x)}{3} \frac{\sin(3x)}{\cos(3x)}\, dx \qquad = \int \frac{3\cos(3x)}{3} \frac{\sin(3x)}{\cos(3x)}\, dx$$

$$= \int \left(\frac{1}{\cos(3x)} - \cos(3x)\right) dx \qquad = -\frac{\cos(3x)}{3}$$

$$\int \sec 3x - \int \cos 3x = \frac{\ln}{3}\left|\tan 3x + \sec 3x\right| - \frac{\sin 3x}{3}$$

$$\Rightarrow x_p = \left[\frac{\ln}{3}\left|\tan 3x + \sec 3x\right| - \frac{\sin 3x}{3}\right]\cos 3x + \left[\frac{-\cos 3x}{3}\right]\sin 3x$$

$$\Rightarrow x = x_h + x_p$$

where $x_h = c_1\cos 3x + c_2 \sin 3x$

Q4 (i) Find Laplace Transform of

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \\ 0 & t > 0 \end{cases}$$

Ans

$$H_a(t) = \begin{cases} 1 & t > a \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = t(1 - H_1(t)) + 1(H_1(t) - H_2(t)) + (3-t)(H_2(t) - H_3(t)) + 0(H_3(t))$$

$$= t - tH_1(t) + H_2(t) + H_1(t) + 3H_3(t) + 3H_2(t)$$

$$+ tH_3(t) + tH_2(t)$$

$\mathcal{L}[t] = \frac{1}{s^2}$, $\mathcal{L}[tH_1(t)] = e^{-s}\mathcal{L}[(t+1)] = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$

$\mathcal{L}[H_2(t)] = \frac{e^{-2s}}{s}$, $\mathcal{L}[H_1(t)] = \frac{e^{-s}}{s}$, $\mathcal{L}[H_3(t)] = \frac{e^{-3s}}{s}$

$$\mathcal{L}[tH_3(t)] = e^{-3s}\mathcal{L}[t+3] = e^{-3s}\left[\frac{1}{s^2} + \frac{3}{s}\right]$$

$$\mathcal{L}[tH_2(t)] = e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + 2\frac{e^{-2s}}{s} + \frac{e^{-s}}{s} + 3\frac{e^{-3s}}{s}$$

$$+ e^{-3s}\left[\frac{1}{s^2} + \frac{3}{s}\right] + e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

(ii) Find Inverse Laplace Transform of

$$\mathcal{L}[f] = F(S) = \ln\left(\frac{S+2}{S+1}\right) = \ln(S+2) - \ln(S+1)$$

$$\mathcal{L}[t f(t)] = -\left[\frac{1}{S+2} - \frac{1}{S+1}\right] = \left[\frac{1}{S+1} - \frac{1}{S+2}\right]$$

$$t f(t) = e^{-t} - e^{-2t}$$

$$f(t) = \frac{e^{-t} - e^{-2t}}{t}$$

MAJOR
MTL-101

(1) [**5 marks**] Find the general solution of $xy'' - (2x+1)y' + (x+1)y = 0,\ x > 0$, given that $y_1(x) = e^x$ is a solution.

Proof. We see seek a second solution of the form $y_2(x) = u(x)e^x$, then $y'(x) = u'(x)e^x + u(x)e^x$ and $y''(x) = u''(x)e^x + 2u'(x)e^x + u(x)e^x$. So substituting y_2 in the equation

$$\begin{aligned}
&xy'' - (2x+1)y' + (x+1)y \\
&= xu''(x)e^x + 2u'(x)xe^x + xu(x)e^x - 2xu'(x)e^x \\
&- 2xu(x)e^x - u'(x)e^x - u(x)e^x + xu(x)e^x + u(x)e^x \\
&= (xu''(x) - u'(x))e^x.
\end{aligned}$$

Therefore $y_2(x) = u(x)e^x$ is a solution of the equation if and only if

$$xu''(x) - u'(x) = 0.$$

We can rewrite as

$$u''(x) = \frac{u'(x)}{x}.$$

A simple computation yields, $u(x) = x^2$ is the desired solution of the above equation. So $y_2(x) = x^2e^x$. Hence the general solution is given by

$$y(x) = C_1e^x + C_2x^2e^x, \quad C_1, C_2 \in \mathbb{R}.$$

□

(2) [**6 marks**] Consider the following initial value problem for the first order ODE

$$y' = \frac{10}{3}xy^{2/5},\ y(x_0) = y_0.$$

(i) Show that for any x_0, y_0 in $\mathbb{R}$ the above IVP has a solution.

(ii) For $y_0 \neq 0$, and $x_0 \in \mathbb{R}$, show that the above IVP has a unique solution.

(iii) For $x_0 = 0$ and $y_0 = 0$, show that the above IVP has more than one solution.

Proof. (i) Using separation of variable we can re-write the given equation as

$$y^{-2/5}\frac{dy}{dx} = \frac{10}{3}x.$$

Integrating further we obtain

$$y^{3/5}(x) = x^2 + C.$$

Now evaluation the initial value we obtain

$$(y_0)^{3/5} - x_0^2 = C$$

So a solution is

$$y(x) = \left[x^2 + (y_0)^{3/5} - x_0^2\right]^{5/3}.$$

Hence the solution exists for any $x_0, y_0 \in \mathbb{R}$.

(ii) If $y_0 \neq 0$ and $x_0 \in \mathbb{R}$, then the function $f(x,y) = \frac{10}{3}xy^{2/5}$ is a Lipschitz function in y variable in a neighbourhood of (x_0, y_0) for $y_0 \neq 0$. Indeed $\frac{\partial f}{\partial y} = \frac{4}{3}xy^{-3/5}$ hence $|\frac{\partial f}{\partial y}| \leq C$ in a neighbourhood of (x_0, y_0) whenever $y_0 \neq 0$. Hence applying "Existence and Uniqueness" theorem we obtain an unique solution of the given ODE.

(iii) When $(x_0, y_0) = (0,0)$, then $y(x) = x^{10/3}$ is a solution which satisfies the initial condition. Moreover $y(x) \equiv 0$ is another solution.

$\square$

1. Since A is a 2×2 matrix with two distinct eigenvalues, it is diagonalizable. That is, there exists a 2×2 invertible matrix P such that $P^{-1}AP = D$. Furthermore, we have

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

Therefore,

$$A = PDP^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{-4}{3} \\ \frac{-2}{3} & \frac{-1}{3} \end{bmatrix}$$

Consequently,

$$A^{23} = PD^{23}P^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1^{23} & 0 \\ 0 & 1^{23} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{-4}{3} \\ \frac{-2}{3} & \frac{-1}{3} \end{bmatrix} = A.$$

2. Given

$$W = \{A \in M_{3\times3}(\mathbb{R}) : S^{-1}AS \text{ is a diagonal matrix}\}.$$

It is plain to see that $A = [0]_{3\times3} \in W$ and hence $W \neq \emptyset$.
Suppose that $A, B \in W$. Then

$$S^{-1}AS = D_1, \quad S^{-1}BS = D_2,$$

where D_1 and D_2 are diagonal matrices in $M_{3\times3}(\mathbb{R})$. For any $\alpha, \beta \in \mathbb{R}$, we have

$$S^{-1}(\alpha A + \beta B)S = \alpha S^{-1}AS + \beta S^{-1}BS = \alpha D_1 + \beta D_2 = D,$$

where D is a diagonal matrix in $M_{3\times3}(\mathbb{R})$. Ergo, $\alpha A + \beta B \in W$, showing that W is a subspace of $M_{3\times3}(\mathbb{R})$.
Note that

$$\left\{ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

is a basis for the space of all diagonal matrices in $M_{3\times3}(\mathbb{R})$. We claim that

$$\mathcal{B} = \{SE_1S^{-1}, SE_2S^{-1}, SE_3S^{-1}\}$$

is a basis for W.

- To show that $\mathcal{B}$ is linearly independent: Let $c_1SE_1S^{-1} + c_2SE_2S^{-1} + c_3SE_3S^{-1} = 0$. Then we get

$$S\Big(c_1E_1 + c_2E_2 + c_3E_3\Big)S^{-1} = 0,$$

which implies

$$c_1E_1 + c_2E_2 + c_3E_3 = 0.$$

Since $\{E_1, E_2, E_3\}$ is linearly independent, we have

$$c_1 = c_2 = c_3 = 0.$$

- To show that $W = \text{span}(\mathcal{B})$: Let $A \in W$. Then

  $$S^{-1}AS = D,$$

  where D is a diagonal matrix in $M_{3\times 3}(\mathbb{R})$. Since $\{E_1, E_2, E_3\}$ spans the space of all diagonal matrices, there exist α, β, γ in $\mathbb{R}$ such that

  $$D = \alpha E_1 + \beta E_2 + \gamma E_3.$$

  This implies

  $$A = SDS^{-1} = \alpha SE_1S^{-1} + \beta SE_2S^{-1} + \gamma SE_3S^{-1}.$$