Indian Institute of Technology Delhi
MTL101 (Major Test)
November 2015

Max Time: 2 hours | Max Marks: 40

Note: No marks will be awarded without appropriate arguments.

1. Let $\lambda \in \mathbb{R}$ (an arbitrary scalar). The space of 3×3 matrices with real entries is denoted by $M_3(\mathbb{R})$. Define $\psi_\lambda : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ by $\psi_\lambda(A) = A + \lambda A^t$, where A^t is the transpose of A. [6 = 2 × 3]

   (a) Show that ψ_λ is a linear transformation for every λ.

   (b) Show that ψ_λ is one-to-one if and only if $\lambda \neq \pm 1$.

   (c) Show that the range space of ψ_1 is the same as the null space of ψ_{-1}.

2. Let $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & s & t \end{pmatrix}$ where $s, t \in \mathbb{R}$. Find s and t, if the characteristic polynomial of A is $x^3 - 4x - 1$. [2]

3. (a) Prove that $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$, where $f * g$ denotes the convolution of f and g and $\mathcal{L}(f)$ denotes the Laplace transform of f. [2]

   (b) Solve the following integral equation [3]
$$y(t) = 1 - \cosh t - \int_0^t e^\tau y(t-\tau)\,d\tau.$$

4. (a) Prove that $\mathcal{L}\{tf(t)\} = -F'(s)$, where $F(s)$ is the Laplace transform of $f(t)$. [2]

   (b) Find the inverse Laplace transform of the function $F(s) = \tan^{-1}\left(\frac{3}{s}\right)$. [3]

5. Find the general solution of [4]
$$y'' + 2y' + 2y = 4e^{-x}\sec^3 x.$$

6. Solve the following using the power series method [5]
$$(1-x^2)y'' - 2xy' + 2y = 2 - x; \quad y(0) = 0, y'(0) = 1.$$

7. Solve the following IVP [5]
$$y'' + 5y' + 6y = \delta\left(t - \frac{\pi}{2}\right) + u(t-\pi); \quad y(0) = 0, y'(0) = 1.$$
(Here δ and u are the Dirac's delta and Heaviside function, respectively.)

8. Solve the following system of ODEs [5]
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sec t \end{pmatrix}.$$

9. Consider the IVP [3]
$$\frac{dy}{dx} = \frac{xy}{1 - x^2 - y^2}, \quad y(0) = \alpha.$$
Use the existence-uniqueness theorem to find the values of $\alpha \in \mathbb{R}$ for which the given IVP has a unique solution (Give proper justification that the hypotheses of the theorem are satisfied).

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