

Linear Algebra: Assignment Sheet-I

In the following, V denotes a vector space over a field \mathbb{F} .

For $i, j \in \mathbb{N}$, we denote $\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

1. Prove that $0x = 0_V$ and $(-1)x = -x$ for all $x \in V$.
2. Prove that for $x \in V$ and $\alpha \in \mathbb{F}$, if $\alpha x = 0$, then either $\alpha = 0$ or $x = 0$.
3. Verify (prove) the following:
 - (a) \mathbb{R}^n with coordinate-wise addition and scalar multiplication is a vector space over \mathbb{R} and over \mathbb{Q} , but not a vector space over \mathbb{C} .
 - (b) \mathbb{F}^n with coordinate-wise addition and scalar multiplication is a vector space over \mathbb{F} but not a vector space over a field $\tilde{\mathbb{F}} \supsetneq \mathbb{F}$ with $\tilde{\mathbb{F}} \neq \mathbb{F}$.
 - (c) $\mathbb{R}^{m \times n}$, the set of all real $m \times n$ matrices is a vector space over \mathbb{R} under usual matrix multiplication and scalar multiplication.
 - (d) Let Ω be a nonempty set. Then the set $\mathcal{F}(\Omega, \mathbb{F})$, the set of all \mathbb{F} -valued functions defined on Ω , is a vector space over \mathbb{F} with respect to the pointwise addition and pointwise scalar multiplication.
Is the set of all scalar sequences a special case of the above?
4. Which of the following subset of \mathbb{C}^3 a subspace of \mathbb{C}^3 ?
 - (a) $\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \alpha_1 \in \mathbb{R}\}$.
 - (b) $\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \text{either } \alpha_1 = 0 \text{ or } \alpha_2 = 0\}$.
 - (c) $\{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \alpha_1 + \alpha_2 = 1 \in \mathbb{R}\}$.
5. Which of the following subset of \mathcal{P} a subspace of \mathcal{P} ?
 - (a) $\{x \in \mathcal{P} : \text{degree of } x \text{ is } 3\}$.
 - (b) $\{x \in \mathcal{P} : 2x(0) = x(1)\}$.
 - (c) $\{x \in \mathcal{P} : x(t) \geq 0 \text{ for } t \in [0, 1]\}$.
 - (d) $\{x \in \mathcal{P} : x(t) = x(1-t) \forall t\}$.
6. Prove the following:
 - (a) The spaces $\mathcal{P}_n(\mathbb{F})$ and \mathbb{F}^{n+1} are isomorphic, and find an isomorphism.

(b) The space $\mathbb{R}^n := \mathbb{R}^{n \times 1}$, the space of all column n -vectors is isomorphic with \mathbb{R}^n , and find an isomorphism.

(c) The space $\mathbb{R}^{m \times n}$ is isomorphic with \mathbb{R}^{mn} , and find an isomorphism.

7. Prove the assertions in the following:

(a) $S = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 : \alpha_1 + \alpha_2 = 0\}$ is a subspace of \mathbb{R}^2 .

(b) $S = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 : \alpha_1 + \alpha_2 - \alpha_3 = 0\}$ is a subspace of \mathbb{R}^3 .

(c) For each $k \in \{1, \dots, n\}$, $S_k = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n : \alpha_k = 0\}$ is a subspace of \mathbb{F}^n .

(d) For $n \in \mathbb{N}$ with $n \geq 2$ and each $k \in \{1, \dots, n-1\}$,

$S_k = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n : \alpha_i = 0 \forall i > k\}$ is a subspace of \mathbb{F}^n .

(e) For each $n \in \mathbb{N}$, \mathcal{P}_n is a subspace of \mathcal{P} .

(f) For each $n \in \mathbb{N}$, $V_n := \{x \in \mathcal{F}(\mathbb{N}, \mathbb{F}) : x(j) = 0 \forall j \geq n\}$ is a subspace of $\mathcal{F}(\mathbb{N}, \mathbb{F})$, and $v_0 := \bigcup_{n=1}^{\infty} V_n$ is a subspace of $\mathcal{F}(\mathbb{N}, \mathbb{F})$.

(Note that elements of W are sequences having only a finite number of nonzero entries.)

(g) For an interval $\Omega := [a, b] \subseteq \mathbb{R}$,

i. $\mathcal{R}(\Omega)$, the set of all Riemann integrable real valued continuous functions defined on Ω is a subspace of $\mathcal{F}(\Omega, \mathbb{R})$.

ii. $C(\Omega)$ is a subspace of $\mathcal{R}(\Omega)$

iii. $C^1(\Omega)$, the set of all real valued continuous functions defined on Ω and having continuous derivative in Ω is a subspace of $C(\Omega)$.

iv. $S = \{x \in C(\Omega) : \int_a^b x(t)dt = 0\}$ is a subspace of $C(\Omega)$.

v. $S = \{x \in C(\Omega) : x(a) = 0\}$ is a subspace of $C(\Omega)$.

vi. $S = \{x \in C(\Omega) : x(a) = 0 = x(b)\}$ is a subspace of $C(\Omega)$.

(h) Let $A \in \mathbb{R}^{m \times n}$. Then

i. $\{x \in \mathbb{R}^n : Ax = 0\}$ is a subspace of \mathbb{R}^n ,

ii. $\{Ax : x \in \mathbb{R}^n\}$ is a subspace of \mathbb{R}^m ,

(i) $\{(\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 = 0\}$ is a subspace of \mathbb{R}^2 .

(j) $\{(\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 - \alpha_3 = 0\}$ is a subspace of \mathbb{R}^3 .

(k) For $i \in \{1, \dots, n\}$, let $e_i = (\delta_{i1}, \dots, \delta_{in})$. Let $V = \mathbb{R}^n$. Then $\{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_i = 0 \text{ for } i > k\}$ is a subspace of \mathbb{R}^n .

(l) If V_1 and V_2 are subspaces of V , then $V_1 + V_2 = \text{span}(V_1 \cup V_2)$.

- (m) If V_1 and V_2 are subspaces of V and if $V_1 \subseteq V_2$, then $V_1 \cup V_2$ is a subspace of V .
- (n) If V_1 and V_2 are subspaces of V , then $V_1 \cap V_2$ is a subspace of V ; but, $V_1 \cup V_2$ need not be a subspace of V .

8. Let V be a vector space and $S \subseteq V$. Prove the following:

- (a) $\text{span}(S)$ is a subspace of V .
- (b) If V_0 is a subspace of V such that $S \subseteq V_0$, then $\text{span}(S) \subseteq V_0$.
- (c) $S = \text{span}(S)$ if and only if S is a subspace of V .

9. Prove the assertions in the following:

- (a) If $V = \mathbb{R}^2$, then $\text{span}(\{(1, -1)\}) = \{(\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 = 0\}$.
- (b) If $V = \mathbb{R}^3$, then $\text{span}(\{(1, -1, 0), (1, 0, 1)\}) = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 + \alpha_2 = \alpha_3 = 0\}$.
- (c) For $i \in \{1, \dots, n\}$, let $e_i = (\delta_{i1}, \dots, \delta_{in})$. Let $V = \mathbb{R}^n$. Then
- $\text{span}(\{e_1, \dots, e_k\}) = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_i = 0 \text{ for } i > k\}$.
 - $\text{span}(\{e_1, \dots, e_n\}) = \mathbb{R}^n$.
- (d) If $V = \mathcal{P}$, then $\text{span}(\{1, t, \dots, t^n\}) = \mathcal{P}_n$ and $\text{span}(\{1, t, t^2, \dots\}) = \mathcal{P}$.
- (e) For each $i \in \mathbb{N}$, let $e_i = (\delta_{i1}, \delta_{i2}, \dots)$. Then $\text{span}(\{e_1, e_2, \dots\}) = c_{00}$.

10. Prove that a set of vectors x_1, \dots, x_n in a vector space V are linearly dependent if and only if there exists $k \in \{2, \dots, n\}$ such that x_k is a linear combination of x_1, \dots, x_{k-1} .

11. Prove that any three of the polynomials $1, t, t^2, 1+t+t^2$ are linearly independent.

12. Give vectors x_1, x_2, x_3, x_4 in \mathbb{C}^3 such that any three of them are linearly independent.

13. Find conditions on α such that the vectors

- (a) $(1 + \alpha, 1 - \alpha), (1 - \alpha, 1 + \alpha)$ are linearly dependent in \mathbb{C}^2 ,
- (b) $(\alpha, 1, 0), (1, \alpha, 1), (0, 1, \alpha)$ are linearly dependent in \mathbb{R}^3 .

14. Suppose x, y, z are linearly independent. Is it true that $x+y, y+z, z+x$ are also linearly independent?

15. Prove the assertions in the following:

- (a) $\{e_1, \dots, e_n\}$ is a basis of \mathbb{R}^n and \mathbb{C}^n .
- (b) $\{1, t, \dots, t^n\}$ is a basis of \mathcal{P}_n .

(c) $\{1, 1+t, 1+t+t^2, \dots, 1+t+\dots+t^n\}$ is a basis of \mathcal{P}_n .

(d) $\{1, t, t^2, \dots\}$ is a basis of \mathcal{P} .

(e) For each $i \in \mathbb{N}$, let $e_i = (\delta_{i1}, \delta_{i2}, \dots)$. Then $\{e_1, e_2, \dots\}$ is a basis of c_{00} .

(f) If E is linearly independent in a vector space, then E is a basis for $V_0 := \text{span}(E)$.

16. Prove:

(a) If E is linearly independent and if $x \in V$ with $x \notin \text{span}(E)$, then $E \cup \{x\}$ is linearly independent.

(b) Every vector space having a finite spanning set has a finite basis.

(c) If a vector space V has a finite basis, then any two basis of V contains the same number of vectors.

17. Find bases E_1, E_2 for \mathbb{C}^4 such that $E_1 \cap E_2 = \emptyset$ and $\{(1, 0, 0, 0), (1, 1, 0, 0)\} \subset E_1$ and $\{(1, 1, 1, 0), (1, 1, 1, 1)\} \subset E_2$.

18. Prove the assertions in the following:

(a) \mathbb{R}^n and \mathcal{P}_n are finite dimensional spaces, and $\dim(\mathbb{R}^n) = n$, $\dim(\mathcal{P}_n) = n + 1$.

(b) $\dim(\{\alpha_1, \dots, \alpha_n \in \mathbb{R}^n : \alpha_1 + \dots + \alpha_n = 0\}) = n - 1$.

(c) $\mathcal{P}, C[a, b], c_{00}$ are infinite dimensional spaces.

(d) Every vector space containing an infinite linearly independent set is infinite dimensional.

(e) If $A \in \mathbb{R}^{m \times n}$ with $n > m$, then there exists $x \in \mathbb{R}^n$ such that $Ax = 0$.

19. Prove:

(a) If V_1 and V_2 are subspaces of a vector space V such that $V_1 \cap V_2 = \{0\}$, and if E_1 and E_2 are bases of V_1 and V_2 , respectively, then $E_1 \cup E_2$ is a basis of $V_1 + V_2$; and in particular,

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2).$$

(b) If V_1 and V_2 are subspaces of a vector space V , then

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

(c) Let V_1 and V_2 be vector spaces and let T be an isomorphism from V_1 onto V_2 . Let $E \subset V_1$. Then E is a basis of V_1 if and only if $\{T(u) : u \in E\}$ is a basis of V_2 .

20. Suppose V_1 and V_2 are subspaces of a vector space V . Prove:

(a) If V_1 and V_2 are finite dimensional such that $\dim(V_1) = \dim(V_2)$ and $V_1 \subset V_2$, then $V_1 = V_2$.

(b) If $V = V_1 \cup V_2$, then either $V_1 = V$ or $V_2 = V$.

21. Prove that, if V_0 is a subspace of a vector space V , then there exists a subspace V_1 of V such that

$$V = V_0 + V_1 \quad \text{and} \quad V_0 \cap V_1 = \{0\}.$$

22. If V_1 is the set of all *odd* polynomials (i.e., $x(-t) = -x(t)$ for all t), and if V_2 is the set of all *even* polynomials (i.e., $x(-t) = x(t)$ for all t), prove that V_1 and V_2 are subspaces of \mathcal{P} such that $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$.

23. Let V_1 and V_2 be vector spaces over the same field \mathbb{F} . For $x := (x_1, x_2)$, $y := (y_1, y_2)$ in $V_1 \times V_2$, and $\alpha \in \mathbb{F}$, define

$$x + y = (x_1 + y_1, x_2 + y_2), \quad \alpha x = (\alpha x_1, \alpha x_2).$$

Prove:

(a) $V_1 \times V_2$ is a vector space over \mathbb{F} with respect to the above operations with its zero as $(0, 0)$ and $-x := (-x_1, -x_2)$.

(b) If V_1 and V_2 are finite dimensional, then

$$\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2).$$

(c) If $\tilde{V}_1 := \{(x_1, x_2) \in V_1 \times V_2 : x_2 = 0\}$ and $\tilde{V}_2 := \{(x_1, x_2) \in V_1 \times V_2 : x_1 = 0\}$, then \tilde{V}_1 and \tilde{V}_2 are subspaces of $V_1 \times V_2$ and

$$V_1 \times V_2 = \tilde{V}_1 + \tilde{V}_2, \quad \tilde{V}_1 \cap \tilde{V}_2 = \{(0, 0)\}.$$

In view of the above, the space $V_1 \times V_2$ is called the *direct sum* of V_1 and V_2 .

24. Let V_1 and V_2 be subspaces of a finite dimensional vector space V such that $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$. Prove that V is isomorphic with $V_1 \times V_2$.

25. Let V_0 be a subspaces of a finite dimensional vector space V . Prove that V is isomorphic with $(V/V_0) \times V_0$.