COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 10

Announced on: Mar 27 (Wed)

1. Based on Problem 22.1 in [LLM17].

   The running time of an algorithm A is described by the recurrence $T(n) = 7 \cdot T(n/2) + n^2$. A competing algorithm B has a running time of $T(n) = a \cdot T(n/4) + n^2$. For what values of a is A asymptotically faster than B? How does your answer change if the running time of B is modified to $T(n) = aT(n/4) + n^4$?

2. In the Tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.

   a) Find a recurrence relation for the number of moves required to solve the puzzle for n disks with this added restriction.

   b) Solve the recurrence derived in part (a).

3. Based on Problem 22.2 in [LLM17].

   Use the Akra-Bazzi formula to find $\Theta(\cdot)$ asymptotic bounds for the following divide and-conquer recurrence:

   $$T(n) = 3T(\lfloor n/3 \rfloor) + n,$$

   where $T(0) = 0$.

4. Based on Problem 16.15 in [LLM17].

   Solve the linear recurrence given below:

   $$T(n) = 2T(n-1) + 2T(n-2) \text{ for } n > 1,$$

   where $T(1) = 1$ and $T(0) = 0$.

5. Based on Problem 15.3, 15.6, and 15.33 in [LLM17].

   a) How many functions are there in total from set A to set B if $|A| = 3$ and $|B| = 7$?

   b) How many of the billion numbers in the integer interval $[1, 10^9]$ contain the digit 1 in their decimal representation?

   c) There is a robot that steps between integer positions in 3-dimensional space. Each step of the robot increments one coordinate and leaves the other two unchanged. How many paths can the robot follow going from the origin $(0, 0, 0)$ to $(3, 4, 5)$?

6. **[Bonus problem: Not to be included in quiz.]** Show that for any $n \in \mathbb{N}$, any sequence of $n^2 + 1$ distinct real numbers contains a subsequence of $n + 1$ numbers that is either strictly increasing or strictly decreasing.

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.