COL202: Discrete Mathematical Structures. I semester, 2022-23.
Amitabha Bagchi
Tutorial Sheet 8: Infinite sets.
20 October 2022

**Important:** The question marked with a ♠ is to be written on a sheet of paper and submitted to your TA within the first 10 minutes of the beginning of your tutorial session. Questions marked with a * are optional challenge problems and are not to be discussed in the tutorial.

**Problem 1**
In class we discussed Cantor's predicament when a fleet of buses indexed by $\mathbb{N}$ appeared at his hotel, each with seats indexed by $\mathbb{N}$. We suggested that Cantor can accommodate the guests as follows

- Currently resident guest in room $i$ can shift to room $2^i$.
- The $j$ passenger of bus $k$ can move into room $p_{k+1}^j$ where $p_k$ denotes the $k$th prime number, i.e., $p_1 = 2$.

**Problem 1.1**
Extend the solution discussed in class to the case where the fleet of buses is indexed by $\mathbb{N}^2$.

**Problem 1.2**
Now let us consider a situation where the government declares that only rooms whose number is a multiple of 6 can be used. What should Cantor do to accomodate the guests now? In this case assume the buses are indexed by $\mathbb{N}$.

**Problem 1.3**
Use the solution of Problem 1.2 to give a simple way of accomodating a fleet of buses indexed by $\mathbb{N}^k$ for any $k \geq 1$.

**Problem 2**
Which of the following is true?

1. $A$ strict $\mathbb{N} \Leftrightarrow A$ is finite.
2. $A$ strict $\mathbb{N} \Leftrightarrow \mathbb{N}$ surj$A$.
3. $A$ strict $\mathbb{N} \Leftrightarrow \exists n \in \mathbb{N} : |A| < n$.

**Problem 3 ♠ [1]**
Prove that the set $\{0, 1\}^*$ of all finite strings on 0 and 1 is countable.

**Problem 4**
Prove that the set of positive rationals is countable. Then extend this proof to show that the set of rationals is countable.

**Problem 5 [1]**
Prove that the set $\mathbb{N}^*$ of all finite sequences of natural numbers is countable.

**Problem 6 [1]**
Suppose we have an infinite sequence $\{f_i\}_{i\geq 1}$ of functions from $\mathbb{N}$ to $\mathbb{R}_+$ (positive reals). A function $h : \mathbb{N} \to \mathbb{R}_+$ is said to *majorize* the sequence if for each $k \in \mathbb{N}$ there is some $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 : f_k(n) \leq h(n)$.

**Problem 6.1**
Before going to the main question do the following

1. Show that for any finite $A \subset \mathbb{R}_+$, $\sup A \in \mathbb{R}_+$.
2. Show that there exist infinite sets of the form $A \subset \mathbb{R}_+$ such that $\sup A \notin \mathbb{R}_+$ (i.e. the supremum is $\infty$).

**Problem 6.2**
Give an explicit construction for $h$ using Problem 6.1 as a hint. Is there some way of doing it without using this hint?

**Problem 6.3**
Also show that there is an $h$ such that $f_k(n)$ is $o(h(n))$ for every $k \in \mathbb{N}$.

**Problem 7**
In class we proved the Schroder-Bernstein theorem using Tarski's Fixed Point Theorem. However the better known proof proceeds more explicitly. First try to work out a proof for the finite case. Then go and look at the structure of the proof for the general case provided in Problem 8.14 of [1]. Work out the proof according to the directions provided there. (It's very long so I am not copying it all out here).

**Problem 8 ***
Suppose we are given a graph $G = (V, E)$ where $V$ is an infinite set. We say that such a graph is connected if there is a finite length path between any two vertices. Prove that every connected graph has a spanning tree. (Hint: Consider the poset of the trees contained in $G$ ordered by the subgraph relation and see if you can apply Zorn's Lemma to prove the result.)

**Problem 9**
Prove that $\mathbb{R}$ is uncountable.

**Problem 10 [1]**
An infinite binary string is called OK if the 1s are only allowed to appear in perfect square positions, i.e., at positions $1, 4, 9, \ldots$. Note that not all the perfect square positions must be 1, but all non-perfect square positions must be 0. Prove that the set of OK strings is uncountable.

# References


[1] E. Lehman, F. T. Leighton, and A. R. Meyer. Mathematics for Computer Science June 2018, MIT Open Courseware.