$\boxed{\text{Green}}$

1. (a) $(x,y,z,w) \in W_1 \cap W_2 \iff \begin{cases} x-y+z-w=0 \\ 5x-4y+3z-2w=0 \\ x-2y+3z-4w=0 \\ x-2y+2z-w=0 \end{cases}$

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 5 & -4 & 3 & -2 \\ 1 & -2 & 3 & -4 \\ 1 & -2 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank $= 3$ $\quad \therefore$ dim (soln. space) $= 4-3 = 1$

$W_1 \cap W_2 = \{ (\lambda, 3\lambda, 3\lambda, \lambda) : \lambda \in \mathbb{R} \}$

$= \text{span} \{ (1,3,3,1) \}$

$\therefore$ $\{(1,3,3,1)\}$ is a basis for $W_1 \cap W_2$.

(b) $S = \{ (x,y,z) \in \mathbb{R}^3 : 3x+2y+z=2,\ x+y+z=1 \}$ is a straight line in $\mathbb{R}^3$ <u>not</u> passing through the origin.

So, any two points on S spans span(S).

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$S = \{ (\lambda, 1-2\lambda, \lambda) : \lambda \in \mathbb{R} \}$

Putting $\lambda = 0$ & $\lambda = 1$, we get $(0,1,0)$, $(1,-1,1) \in S$

$\therefore$ $\{ (0,1,0), (1,-1,1) \}$ is a basis for span(S).

Green

(2) $$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ \alpha & \beta & \gamma & \delta \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \alpha & \beta & \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta-\beta \end{pmatrix}$$

Rank of the matrix is 4 if $\delta-\beta \neq 0$ and 3 if $\delta-\beta=0$.

For the set to be linearly dependent the rank must be less than 4.

$\therefore$ The given set is L.D. iff. $\boxed{\beta-\delta=0}$

$\boxed{\text{Green}}$

(3). (a) $S_a = \{\sin 2x, \sin ax\}$

If $a=0$ then S_a is L.D. as $0 \in S_a$.

Assume $a \neq 0$.

S_a is L.D. $\Leftrightarrow$ $\sin ax = \lambda \sin 2x \quad \forall x \in \mathbb{R}$ for some $\lambda \in \mathbb{R}$.

Now $\sin ax = \lambda \sin 2x \quad \forall x \in \mathbb{R}$

$\Rightarrow a \cos ax = 2\lambda \cos 2x$

$\Rightarrow -a^2 \sin ax = -4\lambda \sin 2x$

$\Rightarrow a^2 \sin ax = 4\lambda \sin 2x = 4 \sin ax$

$\Rightarrow (a^2-4) \sin ax = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow a^2 - 4 = 0 \quad (\because a \neq 0)$

$\Rightarrow a = \pm 2$.

$\therefore S_a$ is L.I. if $a \notin \{0, \pm 2\}$

Also if $a = \pm 2$, $S_a = \{\sin 2x, \pm \sin 2x\}$ is L.D.

$\therefore S_a$ is L.I. iff $a \in \mathbb{R} \setminus \{0, 2, -2\}$.

(b) $\mathcal{B} = \{e^{3x}, \cos 2x, \sin 2x\}$, $W = \text{span}(\mathcal{B})$

$f(x) = e^{3x} + \sin 2x - \cos 2x$

(i) $f'(x) = 3e^{3x} + 2\cos 2x + 2\sin 2x \in \text{span}(\mathcal{B}) = W$

(ii) $[f'(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

$\boxed{\text{Green}}$

(4) (a) Take $V = \mathbb{R}^2$, $X = \{(1,0)\}$, $Y = \{(0,1)\}$

Then $\text{span}(X) = \{(x,0) : x \in \mathbb{R}\}$

$\text{span}(Y) = \{(0,y) : y \in \mathbb{R}\}$

$\text{span}(X \cup Y) = \mathbb{R}^2$

$\therefore \text{span}(X) \cup \text{span}(Y) \neq \text{span}(X \cup Y)$

Thus the statement is FALSE.

(b) FALSE: $(u - 3v) + (3v - w) + (w - u) = 0$

$\therefore \{u-3v, 3v-w, w-u\}$ is L.D.

(c) Since $A0 = 0A^t$, $0 \in W$ (0 denotes the ~~zero~~ matrix of size $n \times n$)

$\therefore W \neq \phi$.

If $X, Y \in W$ & $\lambda \in \mathbb{R}$, then

$AX = XA^t$; $AY = YA^t$

$\therefore A(X + \lambda Y) = AX + \lambda AY = XA^t + \lambda YA^t = (X + \lambda Y)A^t$

$\Rightarrow X + \lambda Y \in W$

$\therefore W$ is a subspace.