Recall, If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Use the Laplace transformation to solve the initial value problem.

$$y' = \begin{bmatrix} 5 & -4 \\ 5 & -4 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution :- Taking Laplace of both sides and using the initial condition we find

$$s\,L(y) - y(0) = \begin{bmatrix} 5 & -4 \\ 5 & -4 \end{bmatrix} L(y) + \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 5 & -4 \\ 5 & -4 \end{bmatrix} \right) L(y) = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s-5 & 4 \\ -5 & s+4 \end{bmatrix} L(y) = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow L(y) = \frac{1}{(s-5)(s+4)+20} \begin{bmatrix} s+4 & -4 \\ 5 & s-5 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix}$$

$$= \frac{1}{s^2 - s - 20 + 20} \begin{bmatrix} -\frac{4}{s} \\ \frac{s-5}{s} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{s^2(s-1)} \\ \frac{s-5}{s^2(s-1)} \end{bmatrix}$$

$$\frac{1}{s^2(s-1)} = \frac{s-(s-1)}{s^2(s-1)} = \frac{1}{s(s-1)} - \frac{1}{s^2}$$

$$= \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}$$

$$\frac{s-5}{s^2(s-1)} = \frac{1}{s(s-1)} - \frac{5}{s^2(s-1)}$$

$$= \frac{1}{s-1} - \frac{1}{s} + \frac{5}{s^2} + \frac{5}{s} - \frac{5}{s-1}$$

$$= \frac{5}{s^2} + \frac{4}{s} - \frac{4}{s-1}$$

$$L(y) = \begin{bmatrix} \frac{4}{s^2} + \frac{4}{s} - \frac{4}{s-1} \\ \frac{5}{s^2} + \frac{4}{s} - \frac{4}{s-1} \end{bmatrix} \Rightarrow y(t) = \begin{bmatrix} 4t + 4 - 4e^t \\ 5t + 4 - 4e^t \end{bmatrix}$$

Use Laplace transformation to solve the initial value problem

$$y'' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} y + \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad y'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution :- Taking Laplace of both sides and using the initial condition we get,

$$s^2 L(y) - s y(0) - y'(0) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} L(y) + \begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right) L(y) = s \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s^2 - 1 & 1 \\ -1 & s^2 + 1 \end{bmatrix} L(y) = \begin{bmatrix} \frac{2}{s} \\ s + \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \quad L(y) = \begin{bmatrix} s^2-1 & 1 \\ -1 & s^2+1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{s} \\ \frac{1+s^2}{s} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(s^2-1)(s^2+1)+1} \begin{bmatrix} s^2+1 & -1 \\ 1 & s^2-1 \end{bmatrix} \begin{bmatrix} \frac{2}{s} \\ \frac{1+s^2}{s} \end{bmatrix}$$

$$= \frac{1}{s^4 - \cancel{1} + \cancel{1}} \begin{bmatrix} \frac{2(s^2+1)}{s} - \frac{(1+s^2)}{s} \\ \frac{2}{s} + \frac{s^4-1}{s} \end{bmatrix}$$

$$= \frac{1}{s^4} \begin{bmatrix} \frac{1+s^2}{s} \\ \frac{s^4+1}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{s^5} + \frac{1}{s^3} \\ \frac{1}{s} + \frac{1}{s^5} \end{bmatrix}$$

$$\Rightarrow y(t) = \begin{bmatrix} \frac{t^2}{2} + \frac{t^4}{\lfloor 4} \\ 1 + \frac{t^4}{\lfloor 4} \end{bmatrix} \qquad L(t^n) = \frac{\lfloor n}{s^{n+1}}$$

Problem :- The Laplace transformation was applied to the initial value problem $y' = Ay$, $y(0) = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ and A is a 2×2 constant matrix.

If $L(y(t)) = \frac{1}{s^2 - 9s + 18} \begin{bmatrix} s-2 & -1 \\ 4 & s-7 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix}$, then find A.

Solution :- Taking the Laplace transformation on both sides of $y' = Ay$ with $y(0) = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ we have,

$$s\,L(y) - y(0) = A\,L(y)$$

$$\Rightarrow (sI - A)\,L(y) = y(0)$$

$$\Rightarrow L(y) = (sI - A)^{-1}\begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\therefore\ L(y)(s) = (sI - A)^{-1}\begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

given, $L(y)(s) = \dfrac{1}{s^2 - 9s + 18}\begin{bmatrix} s-2 & -1 \\ 4 & s-7 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix}$

By Putting $s=0$, we have

$$(-A)^{-1} = \frac{1}{18}\begin{bmatrix} -2 & -1 \\ 4 & -7 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{2}{18} & \frac{1}{18} \\ -\frac{4}{18} & \frac{7}{18} \end{bmatrix} \Rightarrow A = \frac{1}{\frac{14+4}{(18)^2}}\begin{bmatrix} \frac{7}{18} & -\frac{1}{18} \\ \frac{4}{18} & \frac{2}{18} \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix}$$

Problem :- Let $f(t) = t^2 e^{-t}$

(i) Is f continuous on $(0, \infty)$, discontinuous but piecewise continuous on $(0, \infty)$ or neither?

(ii) Are there any fixed numbers a & M such that $|f(t)| \leq M e^{at}$ for $0 \leq t < \infty$?

Solution :- (i) Since t^2 and e^{-t} are continuous on $[0, \infty)$, $f(t)$ is also continuous on $[0, \infty)$

(ii) By L'Hospital rule, $\lim\limits_{t \to \infty} \dfrac{t^2}{e^{t}} = 0$.

Since $f(0) = 0$, $f(t)$ is bounded.

Now, $f'(t) = e^{-t} \cdot 2t + t^2 (-e^{-t}) = (2t - t^2) e^{-t}$.

$f(t)$ has maximum when $t = 2$. $f(2) = 4e^{-2}$ is the maximum value, $M = 4e^{-2}$ & $a = 0$.

Find the Laplace transformation of the periodic function $f(t) = \begin{cases} 0, & 0 \le t \le 1 \\ t-1, & 1 \le t < 2 \end{cases}$ $\quad f(t+2) = f(t)$

Solution :- Here f is 2- periodic, i.e., $T = 2$.

$$L(f(t)) = \frac{1}{1-e^{-2s}} \int_0^2 f(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2t}} \int_1^2 (t-1)\, e^{-st}\, dt.$$

Now, $$\int_1^2 (t-1)\, e^{-st} = \int_1^2 t\, e^{-st}\, dt - \int_1^2 e^{-st}\, dt$$

$$= \left[t \frac{e^{-st}}{-s} \right]_1^2 - \int_1^2 \frac{e^{-st}}{-s}\, dt - \int_1^2 e^{-st}\, dt$$

$$= \left[\frac{t}{-s} e^{-st} + \frac{e^{-st}}{-s^2} - \frac{e^{-st}}{-s} \right]_1^2$$

$$= -\frac{2}{s}e^{-2s} + \cancel{\frac{1}{s}e^{-s}} - \frac{1}{s^2}e^{-2s} + \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-2s} - \cancel{\frac{1}{s}e^{-s}}$$

$$= -\frac{1}{s}e^{-2s} - \frac{1}{s^2}e^{-2s} + \frac{1}{s^2}e^{-s}$$

$$= -\frac{1}{s^2}\left(e^{-2s} + s\,e^{-2s} - e^{-s}\right)$$

$$= \frac{e^{-s}}{s^2}\left(1 - (1+s)e^{-s}\right).$$

$$\therefore \quad L\left(f(t)\right) = \frac{e^{-s}}{s^2\left(1-e^{-2s}\right)}\left[1 - (1+s)e^{-s}\right]$$