A function is called Periodic function and T- Periodic if $f(t+T) = f(t)$

Theorem :- If f is T- Periodic then

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(t) e^{-st} \, dt$$

Proof :-

$$L(f(t)) = \int_0^\infty f(t) e^{-st} \, dt$$

$$= \int_0^T f(t) e^{-st} \, dt + \int_T^{2T} f(t) e^{-st} \, dt + \cdots$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t) e^{-st} \, dt$$

By taking,

$z = t - nT$

$$= \sum_{n=0}^{\infty} \int_0^T f(z+nT) \, e^{-sz} \cdot e^{-nTs} \, dz$$

$$= \left(\int_0^T f(z+nT)\, e^{-sz}\, dz \right) \left(\sum_{n=0}^{\infty} e^{-nTs} \right)$$

$$= \left(\int_0^T f(z)\, e^{-sz}\, dz \right) \left(1 + e^{-Ts} + \left(e^{-Ts}\right)^2 + \cdots \right)$$

$$= \int_0^T f(t)\, e^{-st}\, dt \cdot \frac{1}{1 - e^{-Ts}}$$

$$\therefore \quad L(f(t)) = \frac{1}{1 - e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt$$

Problem:– Determine the Laplace transformation of

$$f(t) = \begin{cases} 1 & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} \le t \le T \end{cases} \quad \& \quad f(t+T) = f(t).$$

Sol^n :- Here f is T- Periodic.

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T f(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^{\frac{T}{2}} e^{-st}\, dt$$

$$= \frac{1}{1-e^{-sT}} \left[\frac{e^{-st}}{-s}\right]_0^{\frac{T}{2}}$$

$$= \frac{1}{s(e^{-sT}-1)} \cdot \left(e^{-\frac{sT}{2}} - 1\right)$$

$$= \frac{e^{-\frac{sT}{2}}-1}{s\left(e^{-\frac{sT}{2}}+1\right)\left(e^{-\frac{sT}{2}}-1\right)} = \frac{1}{s\left(1+e^{-\frac{sT}{2}}\right)}, \; s>0$$

Solve $y'' - 2y' = \delta(t-1)$, $y(0) = 1$, $y'(0) = 0$, $0 \le t \le 2$

Solution :- Taking Laplace on both sides gives

$$L(y'') - 2L(y') = L(\delta(t-1))$$

$$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) - 2s\,L(y) + 2y(0) = e^{-s}$$

$$\Rightarrow (s^2 - 2s)\,L(y) - s + 2 = e^{-s}$$

$$\Rightarrow s(s-2)\,L(y) = e^{-s} + (s-2)$$

$$\Rightarrow L(y) = \frac{e^{-s}}{s(s-2)} + \frac{1}{s}$$

$$= \frac{1}{s} + \frac{e^{-s}}{2}\left[\frac{1}{s-2} - \frac{1}{s}\right]$$

$$= \frac{1}{s} + \frac{e^{-s}}{2(s-2)} - \frac{e^{-s}}{2s}$$

$$y(t) = L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2} L^{-1}\left(\frac{e^{-s}}{s-2}\right) - \frac{1}{2} L^{-1}\left(\frac{e^{-s}}{s}\right)$$

NOTE:- $L\left(u(t-a) f(t-a)\right) = e^{-as} F(s)$

Here $a = 1$, $F(s) = \frac{1}{s-2}$ & $\frac{1}{s}$

i.e., $f(t) = e^{2t}$ & 1 respectively.

$$y(t) = 1 + \frac{1}{2} e^{2(t-1)} \cdot u(t-1) - \frac{1}{2} u(t-1)$$

$$y(t) = \begin{cases} 1 & , \; 0 \leq t < 1 \\ \frac{1}{2} + \frac{1}{2} e^{2(t-1)} & , \; 1 < t \leq 2 \end{cases}$$

Solve the initial value Problem

$$y'' = \delta(t-1) - \delta(t-3), \quad y(0)=0, \quad y'(0)=0, \quad 0 \le t \le 6.$$

Solution: $s^2 L(y) - s\,y(0) - y'(0) = e^{-s} - e^{-3s}$

$$\Rightarrow s^2 L(y) = e^{-s} - e^{-3s}$$

$$\Rightarrow L(y) = \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2}$$

$$\therefore y = L^{-1}\left(\frac{e^{-s}}{s^2}\right) - L^{-1}\left(\frac{e^{-3s}}{s^2}\right)$$

Note that $L(t) = \frac{1}{s^2}$ & $L\big(u(t-a) f(t-a)\big) = F(s)$

$$\Rightarrow y(t) = (t-1)\,u(t-1) - (t-3)\,u(t-3)$$

$$= \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 < t < 3 \\ 2, & 3 < t \le 6 \end{cases}$$

Shifted data Problem :-

Solve $y'' + y = 2t$, $y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$, $y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$

Solution :- Let $T = t - \frac{\pi}{4}$

Then $y(t) = y\left(T + \frac{\pi}{4}\right) = \tilde{y}(T)$

$y'(t) = \tilde{y}'(T)$

$y''(t) = \tilde{y}''(T)$

Then $\tilde{y}''(T) + \tilde{y}(T) = 2\left(T + \frac{\pi}{4}\right)$

$\tilde{y}(0) = \frac{\pi}{2}$, $\tilde{y}'(0) = 2 - \sqrt{2}$

Taking Laplace on both side we get,

$$s^2 L(\tilde{y}) - s\,\tilde{y}(0) - \tilde{y}'(0) + L(\tilde{y}) = \frac{2}{s^2} + \frac{\pi}{2s}$$

$$\Rightarrow (s^2+1)\,L(\tilde{y}) - \frac{s\pi}{2} - (2-\sqrt{2}) = \frac{2}{s^2} + \frac{\pi}{2s}$$

$$\Rightarrow (s^2+1)\,L(\tilde{y}) = \frac{2}{s^2} + \frac{\pi}{2s} + \frac{s\pi}{2} + (2-\sqrt{2})$$

$$= \left(\frac{2}{s^2} + 2\right) + \frac{\pi}{2}\left(\frac{1}{s} + s\right) - \sqrt{2}$$

$$= \frac{2(1+s^2)}{s^2} + \frac{\pi}{2}\,\frac{1+s^2}{s} - \sqrt{2}$$

$$\Rightarrow L(\tilde{y}) = \frac{2}{s^2} + \frac{\pi}{2}\cdot\frac{1}{s} - \frac{\sqrt{2}}{1+s^2}$$

$$\Rightarrow \tilde{y}(T) = 2T + \frac{\pi}{2} - \sqrt{2}\,\sin(T)$$

Then $y(t) = 2\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2} - \sqrt{2}\,\sin\left(t - \frac{\pi}{4}\right)$

$$= 2t - \sqrt{2}\,\sin\left(t - \frac{\pi}{4}\right), \quad t \geq \frac{\pi}{4}$$

Solve $ty'' - ty' + y = 0$ using Laplace transformation.

$y(0) = 0, \; y'(0) = 2$

Solution :- Note that $L(t\,f(t)) = -F'(s)$

$$L(y'') = s^2 L(y) - s\,y(0) - y'(0)$$

$$L(y') = s\,L(y) - y(0)$$

Then $-\frac{d}{ds}\left(s^2 L(y) - s\,y(0) - y'(0)\right) + \frac{d}{ds}\left(s\,L(y) - y(0)\right)$

$$+ L(y) = 0$$

$$\Rightarrow -s^2 \frac{d}{ds}(L(y)) - L(y)(2s) + s\frac{d}{ds}(L(y)) + L(y) + L(y) = 0$$

$$\Rightarrow (s - s^2)\frac{d}{ds} L(y) + (2 - 2s) L(y) = 0$$

$$\Rightarrow s\frac{d}{ds} L(y) + 2L(y) = 0 \quad [\text{dividing } 1-s] \quad \text{Take } s > 1$$

$$\Rightarrow s^2 \frac{d}{ds} L(y) + 2s\, L(y) = 0 \quad \left[\text{Multiplying by } s \right]$$

$$\Rightarrow \frac{d}{ds}\left(s^2 L(y) \right) = 0$$

$$\Rightarrow s^2 L(y) = c$$

$$\Rightarrow L(y) = \frac{c}{s^2}$$

$$\therefore \quad y = c\, L^{-1}\left(\frac{1}{s^2} \right)$$

$$y(t) = ct$$

Now, $\quad y'(t) = c$

$$y'(0) = 2 \Rightarrow c = 2$$

Therefore, $\quad y(t) = 2t. \quad t \geq 0$