

① Find the inverse of A , where $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}$.

Solution :- Note that, if $E_k E_{k-1} \cdots E_2 E_1 A = I_{n \times n}$ then

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1 I_{n \times n}$$

Thus, we form 3×6 matrix $(A | I_{3 \times 3})$ and perform elementary row operations to reduce A to a row-reduced echelon matrix.

$$(A | I_{3 \times 3}) = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}} \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{pmatrix}$$

$$R_1 - 2R_3 \rightarrow \left(\begin{array}{cccccc} 1 & 0 & 0 & 8 & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right) = \left(I_{3 \times 3} \mid A^{-1} \right)$$

Therefore, $A^{-1} = \begin{pmatrix} 8 & -\frac{1}{2} & -2 \\ -1 & \frac{1}{2} & 0 \\ -3 & 0 & 1 \end{pmatrix}$

Rank of a Matrix :- Let A be a non-zero matrix of order $m \times n$. The rank of A is defined to be the greatest positive integer γ such that A has at least one non-zero minor of order γ .

- The rank of a zero matrix is defined to be 0.
- If $\text{rank}(A) = \gamma$ then every minor of order m is zero for $m > \gamma$.

Theorem :- If a row-reduced echelon matrix A has τ non-zero rows, then $\text{rank}(A) = \tau$.

Proof :- Since A has τ non-zero rows, every square sub-matrix of order $\tau+1$ contains a zero row. So each minor of A of order $\tau+1$ is zero. Thus $\text{rank}(A) < \tau+1$.

If we take the sub-matrix formed from $1, 2, \dots, \tau$ -th rows and k_1, k_2, \dots, k_τ -th columns, then it is $I_{\tau \times \tau}$, and has determinant 1.

Thus A has one non-zero minor of order τ . Thus $\text{rank}(A) = \tau$

Theorem :- If A is row equivalent to B then $\text{rank}(A) = \text{rank}(B)$.

Let a system of linear equations have m number of equations (row) and n number of variables (column).

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

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$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Then the system can be expressed as $A X = B$

where,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix $(A | B)_{m \times (n+1)}$ is called the augmented matrix.

Note that $\text{rank}(A|B) \geq \text{rank}(A)$ and $\text{rank}(A) \leq n$.

- * If $\text{rank}(A|B) = \text{rank}(A) = n$ then the system of linear equations has a unique solution.
- * If $\text{rank}(A|B) = \text{rank}(A) < n$ then the system of linear equations has an infinite number of solutions.
- * If $\text{rank}(A|B) > \text{rank}(A)$ then the system of linear equations has no solutions.
- * If $B=0$, i.e., for a system of homogeneous linear equations ($x_1=0, \dots, x_n=0$) is always a solution.

Problem :- Solve the system of linear equation
(over the field \mathbb{R}).

$$x + 3y + 2z = 7$$

$$2x + y - z = 5$$

$$-x + 2y + 3z = 4$$

Solution :- The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 2 & 1 & -1 & 5 \\ -1 & 2 & 3 & 4 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 0 & -5 & -5 & -9 \\ 0 & 5 & 5 & 11 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 0 & -5 & -5 & -9 \\ 0 & 0 & 0 & 2 \end{array} \right)$$

$$\begin{array}{l}
 \xrightarrow{-\frac{1}{5}R_2} \left(\begin{array}{cccc|c} 1 & 3 & 2 & 7 \\ 0 & 1 & 1 & \frac{2}{5} \\ 0 & 0 & 0 & 1 \end{array} \right) \quad \xrightarrow{R_1 - 3R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 8 \\ 0 & 1 & 1 & \frac{2}{5} \\ 0 & 0 & 0 & 1 \end{array} \right)
 \end{array}$$

$$\begin{array}{l}
 \xrightarrow{R_1 - \frac{8}{5}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \\
 \xrightarrow{R_2 - \frac{2}{5}R_3}
 \end{array}$$

$$\text{Rank}(A|B) = 3 \quad \text{and} \quad \text{Rank}(A) = 2$$

Since $\text{Rank}(A|B) > \text{Rank}(A)$, the system of linear equations has no solution.

Solve the system of linear equations (over the field \mathbb{R})

$$x_1 - 2x_2 + 2x_3 - x_4 = -14$$

$$3x_1 + 2x_2 - x_3 + 2x_4 = 17$$

$$2x_1 + 3x_2 - x_3 - x_4 = 18$$

$$-2x_1 + 5x_2 - 3x_3 - 3x_4 = 26$$

Solution :- The Augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 3 & 2 & -1 & 2 & 17 \\ 2 & 3 & -1 & -1 & 18 \\ -2 & 5 & -3 & -3 & 26 \end{array} \right)$$

$$\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 + 2R_1 \end{matrix} \rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 0 & 8 & -7 & 5 & 59 \\ 0 & 7 & -5 & 1 & 46 \\ 0 & 1 & 1 & -5 & -2 \end{array} \right)$$

$$\begin{array}{l}
 \xrightarrow[R_3 - 7R_4]{R_2 - 8R_4} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 0 & 0 & -15 & 45 & 75 \\ 0 & 0 & -12 & 36 & 60 \\ 0 & 1 & 1 & -5 & -2 \end{array} \right) \xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 0 & 1 & 1 & -5 & -2 \\ 0 & 0 & -12 & 36 & 60 \\ 0 & 0 & -15 & 45 & 75 \end{array} \right) \\
 \\
 \xrightarrow{-\frac{1}{12}R_3} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 0 & 1 & 1 & -5 & -2 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & -3 & -5 \end{array} \right) \xrightarrow{R_4 - R_3} \left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & -14 \\ 0 & 1 & 1 & -5 & -2 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \\
 \\
 \xrightarrow[R_1 - 2R_3]{R_2 - R_3} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 5 & -4 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)
 \end{array}$$

Since $\text{rank}(A|B) = \text{rank}(A) = 3 < 4 = \text{number of variable}$
 The system of linear equations has an infinite number of
 solutions. Let $x_4 = c \in \mathbb{R}$ arbitrary. Then
 $x_1 = -c + 2, x_2 = 2c + 3$
 $x_3 = 3c - 5$

Cramer's Rule for $AX = B$ (if $\det(A) \neq 0$).

$$\text{Let } \det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\text{Then } x_i \det(A) = \begin{vmatrix} x_1 a_{11} & a_{12} & \cdots & a_{1n} \\ x_1 a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ x_1 a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\text{i-th column} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\text{Thus, } x_i = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}}{\det(A)}$$