Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that $\dim \operatorname{span}(v_1+w, \ldots, v_m+w) \geq m-1$.

Solution :- If $v_1+w, \ldots, v_m+w$ is linearly independent then we are done.

Suppose, $v_1+w, \ldots, v_m+w$ is linearly dependent, then there exist $c_1, c_2, \ldots, c_m$ not all zero such that

$$c_1(v_1+w) + c_2(v_2+w) + \cdots + c_m(v_m+w) = 0$$

$$\Rightarrow c_1 v_1 + \cdots + c_m v_m = -(c_1 + \cdots + c_m) w$$

Since $v_1 \ldots, v_m$ is linearly independent, $c_1 + \cdots + c_m \neq 0$

without loss of generality let $c_m \neq 0$

claim, $v_1+w, v_2+w, \ldots, v_{m-1}+w$ is linearly independent

Let $d_1(v_1+w) + \cdots + d_{m-1}(v_{m-1}+w) = 0$

$$\Rightarrow d_1 v_1 + d_2 v_2 + \cdots + d_{m-1} v_{m-1} + \frac{c_1 v_1 + \cdots + c_m v_m}{-(c_1+c_2+\cdots+c_m)} (d_1 + \cdots + d_{m-1}) = 0$$

Let $c_1 + \cdots + c_n = C$ & $d_1 + \cdots + d_{m-1} = D$

Then $\left(d_1 - \frac{c_1 D}{C}\right) v_1 + \cdots + \left(d_{m-1} - \frac{c_{m-1} D}{C}\right) v_{m-1} - \frac{c_m D}{C} v_m = 0$

Since $v_1, \ldots, v_n$ is linearly independent,

$$\frac{c_m D}{C} = 0$$

Since $c_m, C \neq 0$, $D = 0$

$$\Rightarrow d_1 v_1 + \cdots + d_{m-1} v_{m-1} = 0$$

Since $\{v_1, \ldots, v_{m-1}\}$ is linearly independent,

$$d_1 = d_2 = \cdots = d_{m-1} = 0$$

Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that $\dim \operatorname{span}(v_1+w, \dots, v_m+w) \geq m-1$.

$$v_2 - v_1,\ v_3 - v_2,\ \dots,\ v_m - v_{m-1}$$

$$v_{i+1} - v_i = (v_{i+1} + w) - (v_i + w)$$
$$\in \operatorname{span}(v_1+w, \dots, v_n+w)$$

Let $c_1(v_2 - v_1) + \dots + c_{m-1}(v_m - v_{m-1}) = 0$

$$\Rightarrow -c_1 v_1 + (c_1 - c_2) v_2 + \dots + (c_{m-2} - c_{m-1}) v_{m-1} + c_{m-1} v_m = 0$$

Since $\{v_1, \dots, v_n\}$ is L.I

$$\Rightarrow c_1 = 0 = c_2 = \dots = c_{m-1}$$

If $\alpha = (1,1,2)$, $\beta = (0,2,1)$, $\gamma = (2,2,4)$ determine whether (i) α is a linear combination of β and γ.

(ii) β is a linear combination of α and γ.

$$\alpha = \frac{1}{2}\gamma$$

(i) Yes.

(ii) $\text{span}(\alpha, \gamma) = \text{span}(\alpha)$

But $\beta \in \text{span}(\alpha) = \text{span}(\alpha, \gamma)$

NO

$\alpha = (1,3,2)$ $\beta = (2,1,-2)$ Examine if $(-1,3,2)$, $(4,7,-2)$ are in Span (α, β).

$$\begin{bmatrix} -1 & 3 & 2 \\ 4 & 7 & -2 \end{bmatrix} \sim \begin{bmatrix} & & \\ & & \end{bmatrix} \quad \text{rank } 2$$

$$\begin{bmatrix} -1 & 3 & 2 \\ 4 & 7 & -2 \\ 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad \text{rank } ?$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & \frac{2}{5} \end{bmatrix}$$

rank 2

$\text{Span}(\alpha, \beta) = \text{Span}\left\{ \left(1, 0, -\frac{6}{5}\right), \left(0, 1, \frac{2}{5}\right) \right\}$

$$\begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & \frac{2}{5} \\ -1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rank 3

$\Rightarrow (-1, 3, 2) \notin \text{Span}(\alpha, \beta)$.

$$\begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & \frac{2}{5} \\ 4 & 7 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

rank 2

$\Rightarrow (4, 7, -2) \in \text{Span}(\alpha, \beta)$.

Extend the set S to obtain a basis of $\mathbb{R}^3$.

$\{(1,1,0),(1,1,1)\}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(1,0,0)$ ✓

$(0,1,0)$ ✓

$(0,0,1)$ ✗

$\Rightarrow \text{span}\{(1,1,0),(1,1,1)\} = \text{span}\{(1,1,0),(0,0,1)\}$

Consider,

$\{(1,1,0),(1,1,1),(1,0,0)\}$ ✓

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \dim(\text{span}\{(1,1,0),(1,1,1),(1,0,0)\}) = 3$$

$\Rightarrow$ - - is a basis of $\mathbb{R}^3$.

$\{(1,1,0),(1,1,1),(0,1,0)\}$ ✓

Extend the set S to obtain a basis for $\mathbb{R}^4$

$S = \{(1,1,0,0), (1,1,1,0)\}$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ & & & \end{bmatrix} \underset{\text{to show}}{\overset{\text{you need}}{\sim}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank } \underline{3}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Take $(1,0,0,0)$, $(0,0,0,1)$

Then,
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{rank } (4)$$

$\text{Dim}(\text{Span}(\{(1,1,0,0), (1,1,1,0), (1,0,0,0), (0,0,0,1)\})) = 4$

Therefore . . . is a basis of $\mathbb{R}^4$.

Extend the set S to obtain a basis for $\mathbb{R}^4$

$S = \{(1,0,1,0), (0,1,0,1)\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

rank (4)

For what values of K does the set $\{(K,1,1,1),(1,K,1,1),(1,1,K,1),(1,1,1,K)\}$ form a basis of $\mathbb{R}^4$

$$\begin{bmatrix} K & 1 & 1 & 1 \\ 1 & K & 1 & 1 \\ 1 & 1 & K & 1 \\ 1 & 1 & 1 & K \end{bmatrix} \sim \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix}$$

rank (4)

Should not devide by anything in form of K. which can be zero.

$$\checkmark\begin{bmatrix} 1 & 1 & 1 & k \\ 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & k \\ 0 & 0 & k-1 & 1-k \\ 0 & k-1 & 0 & 1-k \\ 0 & 1-k & 1-k & 1-k^2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 & k \\ 0 & 0 & k-1 & 1-k \\ 0 & k-1 & 0 & 1-k \\ 0 & 0 & 1-k & 2-k-k^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & k \\ 0 & 0 & k-1 & 1-k \\ 0 & k-1 & 0 & 1-k \\ 0 & 0 & 0 & 3-2k-k^2 \end{bmatrix}$$

$$\underline{3-2k-k^2} = 0 \Rightarrow k^2+2k-3$$

$$\Rightarrow (k+3)(k-1) = 0$$

$$\Rightarrow \underline{k = 1, -3}$$

If $k \neq 1, -3$, then $\underline{k-1 \neq 0}$, $3-2k-k^2 \neq 0$

$$\begin{bmatrix} 1 & 1 & 1 & k \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

Show that the set of vectors $S = \{(1,2,3,0), (2,1,0,3), (1,1,1,1), (2,3,4,1)\}$ is linearly dependent in $\mathbb{R}^4$.

Find a linearly independent subset S_1 of S such that $L(S_1) = L(S)$.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No change in row

rank 2

$\{(1,2,3,0), (2,1,0,3)\}$ is L.I &

span $\{(1,2,3,0), (2,1,0,3)\}$ = span $\{(\ldots)$

Here since rank is 2 & none is a scalar multiplication of other, you can choose any two vectors.

Prove that $\operatorname{rank}(AB) \le \min\{\operatorname{rank} A, \operatorname{rank} B\}$

If A is non-singular then $\operatorname{rank}(AB) = \operatorname{rank}(B)$

If B is non-singular then $\operatorname{rank}(AB) = \operatorname{rank}(A)$

$$\operatorname{rank}(A\underline{\underline{B}}) \le \operatorname{rank}(\underline{\underline{B}})$$

$$\begin{aligned}
\operatorname{rank}(AB) &= \operatorname{rank}\left((AB)^T\right) \\
&= \operatorname{rank}(B^T \underline{\underline{A^T}}) \\
&\le \operatorname{rank}(A^T) \\
&= \operatorname{rank}(A)
\end{aligned}$$

$$\operatorname{rank}\left(A^{-1}(\underline{\underline{A}}B)\right) \le \operatorname{rank}(AB)$$

$\Rightarrow \operatorname{rank}(B) \le \operatorname{rank}(AB)$ if A is non..

$\Rightarrow \operatorname{rank}(AB) = \operatorname{rank}(B)$

Examine if $(1,1,1)$ is in the row space of A or in the column space of A, where $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 0 & 7 \\ -1 & 4 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 0 & 7 \\ -1 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{7}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$$ rank 2

$$\begin{bmatrix} 1 & 0 & \frac{7}{3} \\ 0 & 1 & \frac{4}{3} \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ rank 3

$\Rightarrow (1,1,1) \notin$ row space (A) ✓

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \\ 5 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ rank 2

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ rank 2

$\Rightarrow (1,1,1) \in$ column space (A).

$V = \{ p(x) \in P_5(x) : p(1) = 0 \}$. Find a basis of V and extend the basis to a basis of $P_5(x)$.

dim 5

deg ≤ 4

$\checkmark$ $(x-1), (x-1)^2, (x-1)^3, (x-1)^4$

$c_1(\quad) + \dots$

$1 \notin V$

$\Rightarrow \dim(V) \leq 4$