COL202: Discrete Mathematical Structures

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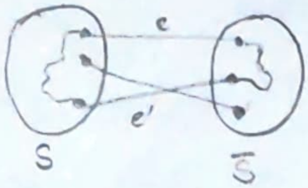
Tutorial Sheet - 8

2. **Theorem:** For any subset $S \subseteq V$, if there exist a unique edge $e \in E(S, \bar{S})$ with smallest weight among the edges in $E(S, \bar{S})$, then e must belong to every MST of graph G.

Proof: By contradiction.

Let, for some subset $S \subseteq V$ with a unique smallest weight edge $e \in E(S, \bar{S})$, $\exists$ a MST T which does not include e.

$$T = (V, E^T)$$



Now, since T is a tree $\Rightarrow$ $(V, E^T \cup \{e\})$ has a unique cycle. —— (#)

There must be an edge $e' \in E^T$ which is in this cycle and $e' \in E(S, \bar{S})$ since a cycle must cross from S to $\bar{S}$ and then back to S.

Since e is the unique minimum weight edge in $E(S, \bar{S})$,

$$\Rightarrow \quad wt(e) < wt(e')$$

Swap e and e' in T

$$T^* = (V, E^{T^*}) \quad , \quad \text{where } E^{T^*} = (E^T \setminus \{e'\}) \cup \{e\}$$

We will now show that T^* is an MST of graph G.

i) T^* is acyclic :- (V, E^T) was acyclic (T ~~was~~ is a tree) and $(V, E^T \cup \{e\})$ had a unique cycle. Now we are removing a single edge e' from this cycle to break it.

Hence $T^* = (V, E^{T^*})$ is acyclic.

ii) T^* is connected - Consider any two vertices $x, y \in V$

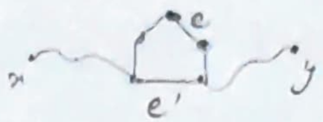
Case 1 : The x to y path in T does not contain e'.

$\Rightarrow$ x to y path is not disturbed in T^* and they are still connected.

Case 2 : The x to y path in T contains e'.

In the cycle created by adding e, bypass e' by going through the other part of cycle (containing e).

$\Rightarrow \exists$ x to y path in T^*



Hence, T^* is connected.

iii) T^* contains all vertices, since T contained all vertices and even after removing e', we have a path between the endpoints of e' (through the other part of cycle as shown in ii)). All the other vertices are undisturbed.

Hence T^* is a spanning tree of G.

Now since $wt(e) < wt(e')$

$\Rightarrow wt(T^*) < wt(T)$

i.e. we created a MST of G which has strictly smaller weight than T, which is a contradiction.

Hence proved. ■

Proof of ⊕ : Adding a new edge to a tree creates a unique cycle.

Proof by contradiction.

Let an edge (u,v) be added to a tree T and this addition results in creation of two or more cycles.

Let two such cycles be $(u, v, a_1, a_2, \dots, a_k, u)$ and $(u, v, b_1, b_2, \dots, b_j, u)$

Removing the edge (u,v) from both these cycles, we get two different paths from v to u : $(v, a_1, a_2, \dots, a_k, u)$ and $(v, b_1, b_2, \dots, b_j, u)$

But removing edge (u,v) gives back the tree T and so, T has more than one path between two nodes. But this is not possible if T is a tree. Contradiction.

Hence, addition of an edge creates a unique cycle.