

# MTL 101 Major Brief Solutions

4(a).  $\{x_1 - x, \dots, x_n - x\}$  is L.D. if there exist scalars  $d_1, \dots, d_n$  not all zero such that  $\sum_{i=1}^n d_i (x_i - x) = 0$   
 i.e.  $\sum_{i=1}^n d_i x_i - \left(\sum_{i=1}^n d_i\right) x = 0 \rightarrow \textcircled{1}$

Claim:  $\sum_{i=1}^n d_i \neq 0$

If  $\sum_{i=1}^n d_i = 0$ , then from  $\textcircled{1}$ ,  $d_i = 0, \forall i$ , since  $x_1, \dots, x_n$  are L.I.

So we discard this case.

If  $\sum_{i=1}^n d_i \neq 0$ , then from  $\textcircled{1}$ ,  $x = \frac{\sum_{i=1}^n d_i x_i}{\sum_{i=1}^n d_i}$   
 $\Rightarrow \frac{d_i}{\sum_{i=1}^n d_i} = \beta_i$   
 $\Rightarrow \sum_{i=1}^n \beta_i = 1$

1(b).

$$T\left(\sum_{i=0}^3 a_i x^i\right) = \sum_{i=0}^3 a_i (x+1)^i$$

$$\text{Let } B = \{v_1 = 1, v_2 = 1+x, v_3 = 1+x^2, v_4 = 1+x^3\}$$

$$T(v_1) = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4$$

$$T(v_2) = 1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4$$

$$T(v_3) = -1 \cdot v_1 + 2 \cdot v_2 + 1 \cdot v_3 + 0 \cdot v_4$$

$$T(v_4) = -5 \cdot v_1 + 3 \cdot v_2 + 3 \cdot v_3 + 1 \cdot v_4$$

$$[T]_B = \begin{bmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2(a).

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Tx = 0 \Rightarrow Ax = 0 \Rightarrow \begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{matrix} \therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ spans Null space of } T$$

$$\text{Since } \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is L.I.}$$

Therefore this set forms a basis of Null space of  $T$ .

$$2(b) \left( \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & k \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & k-10 \end{array} \right)$$

For unique solution:  $P(A) = 3 = P(A|b)$ . This is possible if  $\lambda-3 \neq 0$  or  $\lambda \neq 3$ .

For infinitely many solutions.  $P(A) = P(A|b) < 3$   
This is possible if  $\lambda = 3$  &  $k = 10$ .

For No solution:  $P(A) \neq P(A|b)$ . This is possible if  $\lambda = 3$  and  $k \neq 10$ .

$$3. A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \text{ Eigenvalues of } A \text{ are: } \lambda = -3, -1$$

An eigenvector corresponding to  $\lambda = -3$  is  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

An eigenvector corresponding to  $\lambda = -1$  is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\text{Let } X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow X^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Define  $Z = X^{-1}Y$  or  $Y = XZ$ . Then

$$Y' = AX + b \Rightarrow Z' = DZ + h, \text{ where}$$

$$D = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, h = X^{-1}b = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2e^{-2} \\ 3x \end{pmatrix}$$

$$Z_j' = \lambda_j Z_j + h_j, \quad j=1, 2$$

$$\therefore Z_j = c_j e^{\lambda_j x} + e^{\lambda_j x} \int e^{-\lambda_j x} h_j(x) dx, \quad j=1, 2$$

$$Z_1 = c_1 e^{-3x} + \frac{1}{2} e^{-x} - \frac{1}{2}x + \frac{1}{6}$$

$$Z_2 = c_2 e^{-x} + x e^{-x} + \frac{3}{2}x - \frac{3}{2}$$

$$\therefore Y = XZ$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 + Z_2 \\ -Z_1 + Z_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-3x} + c_2 e^{-x} + x e^{-x} + \frac{1}{2} e^{-x} + x - \frac{4}{3} \\ -c_1 e^{-3x} + c_2 e^{-x} + x e^{-x} - \frac{1}{2} e^{-x} + 2x - \frac{5}{3} \end{pmatrix}$$

4(a).

$$xy'' + y' + y = 0 \Leftrightarrow x^2 y'' + xy' + xy = 0$$

Let  $y_1(x) = x^{\lambda} \sum_{m=0}^{\infty} C_m x^m$ ,  $C_0 \neq 0$ . Substitute  $y_1(x)$  and its derivatives in the given D.E. This gives

$$\sum_{m=0}^{\infty} \left\{ [(m+\lambda)(m+\lambda-1)C_m + (m+\lambda)C_m] x^{m+\lambda} + C_m x^{m+\lambda+1} \right\} = 0$$

$$\Rightarrow (m+\lambda)^2 C_m + C_{m-1} = 0 \quad (\text{Coeff. of } x^{m+\lambda}, C_{-1} = 0)$$

$$m=0 \Rightarrow \lambda^2 = 0 \quad (\text{indicial equation}) \Rightarrow \lambda = 0 \text{ is double root.}$$

$$\Rightarrow C_m = -\frac{1}{m^2} C_{m-1}$$

$$\Rightarrow y_m = C_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2}, \quad (\text{with } C_0 = 1).$$

Since it is the case of double root, we have

$$y_2(x) = (\log x) y_1 + x^0 \sum_{m=1}^{\infty} b_m x^m.$$

Substitute  $y_2$  and its derivatives in the given D.E. This leads to

$$0 = 2xy_1' + \sum_{m=1}^{\infty} [b_m x^{m+1} + m b_m x^m + m(m-1)b_m x^m]$$

$$\Rightarrow \frac{2(-1)^m m}{(m)^2} + b_{m-1} + m^2 b_m = 0, \quad m = 1, 3, \dots$$

$$\text{Where } b_0 = 0 \Rightarrow b_1 = 2$$

$$\Rightarrow b_m = -\frac{b_{m-1}}{m^2} - \frac{2(-1)^m}{m(m)^2}$$

$$\Rightarrow b_m = -\frac{2(-1)^m}{(m)^2} h_m, \quad \text{where } h_m = 1 + \frac{1}{2} + \dots + \frac{1}{m}.$$

$$\therefore y_2(x) = (\log x) \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m h_m x^m}{(m)^2}.$$

Hence the Complete solution is given by

$$y(x) = C_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} + C_1 \left[ \log x \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(m)^2} - 2 \sum_{m=1}^{\infty} \frac{(-1)^m h_m x^m}{(m)^2} \right].$$

4(b).

$$(i) \frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x). \quad (ii) \frac{d}{dx} [\bar{x}^n J_n(x)] = -\bar{x}^n J_{n+1}(x).$$

Let  $\alpha, \beta$  be two consecutive positive zeros of  $J_{n+1}(x)$ . Using  $\frac{d}{dx} [x^{n+1} J_{n+1}(x)] = x^{n+1} J_n(x)$  (part (i) with  $n = n+1$ ) By Rolle's theorem, there is a ' $r$ ' such that  $\alpha < r < \beta$  and  $J_n(r) = 0$ . If there were more than one zero, say  $\alpha < r < \delta < \beta$  using  $\frac{d}{dx} [\bar{x}^n J_n(x)] = -\bar{x}^n J_{n+1}(x)$  (part (ii)) Rolle's theorem would imply a zero of  $J_{n+1}(x)$  in between  $r$  and  $\delta$ , which would contradict consecutiveness of  $\alpha$  and  $\beta$ . This completes the proof.

5(a).  $((1+x^2)y')' + \lambda(x^2+1)^{-1}y = 0$ , where prime denotes differentiation with respect to  $x$ .

$$\left( (1+x^2) \frac{dy/dt}{dx/dt} \right)' + \lambda(x^2+1)^{-1}y = 0 \rightarrow$$

Substituting  $\frac{dx}{dt} = x^2+1$

$$\int \frac{dx}{x^2+1} = dt \Rightarrow t = \arctan x \text{ or } x = \tan t$$

Substituting in  $(*)$ ,  $\Rightarrow \frac{d^2y}{dt^2} + \lambda y = 0$ , whose general solution is

$$y = C_1 \cos \sqrt{\lambda} t + C_2 \sin \sqrt{\lambda} t$$

$$0 = y(x=0) = y(t=0) \Rightarrow C_1 = 0, \text{ therefore } y = C_2 \sin \sqrt{\lambda} t$$

$$0 = y(x=1) = y(t=\frac{\pi}{4}) \Rightarrow \sin \sqrt{\lambda} \frac{\pi}{4} = 0 = \sin n\pi \quad (\text{for nontrivial solution } C_2 \neq 0)$$

Therefore, the eigenvalues are  $\lambda_n = 16n^2$

and the corresponding eigenfunctions are  $y_n = \sin(4nt)$   
 $= \sin(4n \arctan x)$   
 $n = 1, 2, 3, \dots$

5(b). Let  $\phi_i(x) = \cos \frac{i\pi x}{L}$ ,  $\phi_j(x) = \cos \frac{j\pi x}{L}$

$$\text{For } i \neq j, \text{ Consider } \langle \phi_i, \phi_j \rangle = \int_{-L}^L \phi_i(x) \phi_j(x) dx = \int_{-L}^L \cos \frac{i\pi x}{L} \cos \frac{j\pi x}{L} dx = 0$$

This shows that the given set of functions are orthogonal

$$\text{Now, the norm of } \phi_i = \left\| \cos \frac{i\pi x}{L} \right\| = \langle \phi_i, \phi_i \rangle^{1/2} = \left( \int_{-L}^L \cos^2 \frac{i\pi x}{L} dx \right)^{1/2} = L^{1/2} = \sqrt{L}$$

$$\therefore \text{Orthonormal set is } \frac{\phi_i}{\|\phi_i\|} = \left\{ \frac{1}{\sqrt{L}} \cos \frac{i\pi x}{L} \right\}_{i=1}^{\infty}$$

$$\text{For } i=0, \|\phi_0\| = \langle \phi_0, \phi_0 \rangle^{1/2} = \left( \int_{-L}^L dx \right)^{1/2} = \sqrt{2L}$$

$$\therefore \text{Orthonormal set is } \left\{ \frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}} \cos \frac{i\pi x}{L} \right\}_{i=1}^{\infty}$$

6(a). Prove existence of Laplace transform

$$6(b). \mathcal{L}\{t^{\frac{1}{2}}\} = \frac{\Gamma(\frac{3}{2})}{s^{\frac{3}{2}}} = \sqrt{\frac{\pi}{s}}; \mathcal{L}\{e^{-t} t^{\frac{1}{2}}\} = \sqrt{\frac{\pi}{s+1}}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \frac{1}{\sqrt{\pi}} \int_0^t e^{-\tau} \tau^{-\frac{1}{2}} d\tau = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} x^{-1} 2x dx = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx = \operatorname{erf}(\sqrt{t}).$$

( $\because \frac{F(s)}{s} = \mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\}$ )