

**MTL 101**  
**LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS**  
**MINOR EXAM**

**Total: 20 Marks**

**Time: 1:00 Hrs.**

**Question 1: (2 Marks)** If  $A$  and  $B$  are two  $n \times n$  real matrices such that  $AB = 5I_{n \times n}$ , then is it true that  $BA = 5I_{n \times n}$ ? If true, give a proof else, give a counterexample.

**Question 2: (2 Marks)** Let  $A$  be a  $3 \times 4$  real matrix of rank 3. Show that there exists  $4 \times 3$  real matrix  $B$  such that  $AB = I_{3 \times 3}$ .

**Solution 1** If  $AB = 5I_{n \times n}$ , then  $\det(AB) = \det(A)\det(B) = 5$ . Thus both  $\det A, \det B$  are nonzero and hence invertible (**1 mark**). You can alternatively show that  $(1/5)AB = I_{n \times n}$  and use the uniqueness of inverses to show that  $B^{-1} = (1/5)A$ .

Now  $5I = 5BB^{-1} = B5I_{n \times n}B^{-1} = B(AB)B^{-1} = BA$  (**1 mark**).

**Solution 2** Rank is the number of nonzero rows in the echelon form. Since the rank is 3,  $A$  takes the form  $[I_{3 \times 3} \mid \mathbf{b}]$  in the row reduced echelon form where  $\mathbf{b}$  is a column vector. That is there is elementary matrices  $E_1, \dots, E_k$  such that  $E_k \cdots E_1 A = [I_{3 \times 3} \mid \mathbf{b}]$ . Let  $B' = \begin{bmatrix} I_{3 \times 3} \\ \mathbf{0} \end{bmatrix}$  be a  $4 \times 3$  matrix where  $\mathbf{0}$  is a zero row vector (**1 mark**).

Now  $E_k \cdots E_1 AB' = [I_{3 \times 3} \mid \mathbf{b}] \begin{bmatrix} I_{3 \times 3} \\ \mathbf{0} \end{bmatrix} = I_{3 \times 3}$ . Now  $AB' = E_1^{-1} \cdots E_k^{-1}$  and hence  $A(B'E_k \cdots E_1) = I_{3 \times 3}$ . Set  $B = B'E_k \cdots E_1$  (**1 mark**).

### Question-2 (3 Marks)

Question-2 consider the vector space  $P_3(\mathbb{R})$  of polynomials of degree less than equal to three with real coefficients.

- (a) Prove that  $B = \{1-x, 1+x^2, 1-x^3, 1+x-x^3\}$  is a basis of  $P_3(\mathbb{R})$ .

**[Method-I]**  $B$  is basis iff (i)  $B$  is L.I.      (ii)  $\text{span}(B) = P_3(\mathbb{R})$ . } .... 1 mark

(i) Let  $c_1, c_2, c_3, c_4 \in \mathbb{R}$ .

Consider,

$$c_1(1-x) + c_2(1+x^2) + c_3(1-x^3) + c_4(1+x-x^3) = 0$$

Comparing coefficients both sides we get,

$$c_1 + c_2 + c_3 + c_4 = 0$$

$$-c_1 + c_4 = 0$$

$$c_2 = 0$$

$$-c_3 - c_4 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0$$

Hence, the above set  $B$  is linearly independent.

- (ii) Let  $P_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{R})$

Suppose  $\exists c_1, c_2, c_3, c_4 \in \mathbb{R}$  s.t.

$$P_1(x) = c_1(1-x) + c_2(1+x^2) + c_3(1-x^3) + c_4(1+x-x^3)$$

$$\# c_1 + c_2 + c_3 + c_4 = a_0$$

$$-c_1 + c_4 = a_1$$

$$c_2 = a_2$$

$$-c_3 - c_4 = a_3$$

On solving, we get

$$c_1 = a_0 - a_2 + a_3$$

$$c_2 = a_2$$

$$c_3 = -a_0 - a_1 + a_2 - 2a_3$$

$$c_4 = a_0 + a_1 - a_2 + a_3$$

since,  $P_1(x)$  can be expressed in the linear combination of vectors of  $B$ .

Hence,  $\text{span}(B) = P_3(\mathbb{R})$ .

**Method-II.** Basis is the maximal L.I. set in a vector space.

Since, cardinality ( $B$ ) = 4 =  $\dim(P_3(\mathbb{R}))$ .

Hence, if  $B$  is L.I. set then we are done.

And linear independence of  $B$ , is proved in (i)

(b) Find the coordinates of vector  $u = 1+x+x^2+x^3$  with respect to ordered basis  $B$ .

$$\text{Soln: Step 1. } u = 1+x+x^2+x^3 = c_1(1-x) + c_2(1+x^2) + c_3(1-x^3) + c_4(1+x+x^3).$$

On comparing coeff -

$$\Rightarrow c_1 + c_2 + c_3 + c_4 = 1$$

$$-c_1 + c_4 = 1$$

$$c_2 = 1$$

$$-c_3 - c_4 = 1$$

}

1 mark

Step 2. Hence,  $c_1 = 1, c_2 = 1, c_3 = -3, c_4 = 2$   
on solving

$$[u]_B = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$

1 mark

## Question 4 (4 marks)

(a) [2-Marks] Let  $W_1, W_2$  be non-zero subspaces of a finite dimensional vector space  $V$  over  $\mathbb{C}$ . Suppose  $\exists f: V \rightarrow \mathbb{R}$  such that  $f(w_1) - f(w_2) < 0$  for all non-zero vectors  $w_1 \in W_1$  &  $w_2 \in W_2$ . Prove:  $\dim W_1 + \dim W_2 \leq \dim V$ .

Sol: Step 1: Claiming & proving  $W_1 \cap W_2 = \{0\}$

1-mark

Let if possible,  $W_1 \cap W_2 \neq \{0\}$

$$\Rightarrow \exists u \neq 0 \text{ s.t. } u \in W_1 \text{ & } u \in W_2$$

$$\Rightarrow f(u) - f(u) < 0 \text{ & } u \neq 0 \Rightarrow 0 < 0$$

A contradiction. ~~to the given condition.~~

Thus,  $W_1 \cap W_2 = \{0\}$  ----- (A)

Step 2: Using  $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 \leq \dim V$  & concluding the final result.

1-mark

Note that we have :-

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad \dots \dots \dots \quad (B)$$

Also as  $W_1 + W_2$  is a subspace of  $V$ , therefore  $\dim(W_1 + W_2) \leq \dim(V)$  ----- (C)

Eg (A) gives  $\dim(W_1 \cap W_2) = 0$  ----- (D)

Using Eg (C) & (D) in (B) to get:-

$$\dim(W_1) + \dim(W_2) \leq \dim V.$$

**Question 3** (4 marks).

Consider the vector space  $\mathbb{C}^2$  over  $\mathbb{C}$ . Find all possible linear transformations  $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  such that  $T^2 := T \circ T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  (the composition of  $T$  with itself) is given by

$$T^2(z_1, z_2) = (-z_1 + 2z_2, -z_2), \text{ for all } (z_1, z_2) \in \mathbb{C}^2.$$

Here is the marking scheme of the two standard solutions.

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**Solution 1.****Step 1.**

[1 mark] for writing the matrix of  $T^2$  by fixing a basis, say standard basis  $B$ .

[0.5 mark] if basis is not mentioned.

**Step 2.**

[1 mark] for assuming  $[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and using  $[T^2]_B = [T]_B[T]_B$  to get 4 equations in  $a, b, c, d$ .

[0.5 mark] for any mistake in doing this.

**Step 3.**

[1 mark] for giving detailed solution of  $a, b, c, d$ .

[0.5 mark] for any mistake / inaccuracy to solve  $a, b, c, d$ .

**Step 4.**

[1 mark] for writing the two possible matrices followed by the two possibilities of  $T$  (invoking the basis fixed in the beginning).

[0.5 mark] for any mistake in doing so or only partially doing so.

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**Solution 2.****Step 1.**

[1 mark] for using that every  $T$  is given by  $T(z_1, z_2) = (az_1 + bz_2, cz_1 + dz_2)$  for some  $a, b, c, d \in \mathbb{C}$  and computing  $T^2(z_1, z_2)$ .

[0.5 mark] for any mistake in the computing  $T^2(z_1, z_2)$ .

**Step 2.**

[1 mark] for using Step 1 and  $T^2(z_1, z_2) = (-z_1 + 2z_2, -z_2)$  to get 4 equations on  $a, b, c, d$ .

[0.5 mark] for any mistake in writing down the 4 equations.

**Step 3.**

[1 mark] for giving detailed solution of  $a, b, c, d$ .

[0.5 mark] for any mistake / inaccuracy to solve  $a, b, c, d$ .

**Step 4..**

[1 mark] for writing the two possible the two correct possibilities of  $T$  based on  $a, b, c, d$  found in Step 3.

[0.5 mark] for any mistake in doing so or doing partially.

**Remark:** For skipping or failing to justify any of the steps above earns [0 mark].

### Question 4 (4 marks)

⑥ [2-marks] Let  $W_1 = \text{Span}\{(4, 3, 2, 1), (1, 1, 1, 2), (3, 2, 1, -1)\}$   
 $W_2 = \text{Span}\{(1, 0, 3, 2), (4, 3, 2, 1)\}$   
 find the dimension of  $W_1 + W_2$ .

Sol: Step 1: Finding Dimensions of  $W_1, W_2$  &  $W_1 \cap W_2$  correctly. [1-mark]  
 (with details)

(a)  $\dim(W_1) = 2$ .

Note:  $(1, 1, 1, 2) = (4, 3, 2, 1) - (3, 2, 1, -1)$

&  $\{(4, 3, 2, 1), (3, 2, 1, -1)\}$  is a linearly independent set.

Therefore, basis( $W_1$ ) =  $\{(4, 3, 2, 1), (3, 2, 1, -1)\}$

Thus,  $\dim(W_1) = 2$

(b)  $\dim(W_2) = 2$ .

Note:  $\{(1, 0, 3, 2), (4, 3, 2, 1)\}$  is a linearly independent set, thus forms a basis.

Hence  $\dim(W_2) = 2$ .

(c)  $\dim(W_1 \cap W_2) = 1$ .

Note that:  $W_1 \cap W_2 = \text{Span}\{(4, 3, 2, 1)\}$

Hence,  $\dim(W_1 \cap W_2) = 1$

Step 2: Writing & using the identity:-

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$\begin{aligned} \dim(W_1 + W_2) &= 2 + 2 - 1 \\ &= 3 \end{aligned}$$

### Question 4 (4 marks)

Sol (4)(b): Alternate Solution :-

Step 1: Using  $W_1 + W_2 = \text{Span}(W_1 \cup W_2)$

1 mark

As  $W_1 + W_2 = \text{Span}(W_1 \cup W_2)$ ,

Therefore the Subspace  $W_1 + W_2$  is spanned

by  $\{(4, 3, 2, 1), (1, 1, 1, 2), (3, 2, 1, -1), (1, 0, 3, 2), (4, 3, 2, 1)\}$

Step 2: Finding dimension of  
a basis for  $\text{Span}(W_1 \cup W_2)$   
(by removing linearly dependent vectors)

1 mark

Note: Finding linearly independent vectors is same  
as finding non-zero rows in Row Echelon form.

Therefore, the  $\dim(W_1 + W_2)$  is the dimension of  
the row space of the matrix:- (or row rank)

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

The row echelon form is: (Note:  $R_5 = R_1$  &  $R_2 = R_1 - R_3$ )

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 1/4 & 1/2 & 7/4 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, the row-rank of this matrix is 3,

we get  $\dim(W_1 + W_2) = 3$

**Question 5: (5 Marks)** Consider the linear operator  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  defined by

$$T((x_1, x_2, x_3, x_4)) = \left( \sum_{i=1}^4 x_i, \sum_{i=1}^4 x_i, \sum_{i=1}^4 x_i, \sum_{i=1}^4 x_i \right) \text{ for all } (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

Prove or disprove: there exists an ordered basis  $B$  of  $\mathbb{R}^4$  such that  $[T]_B$  is diagonal.

**Solution 5:(Prove)**

(Computing the eigenvalues): Note that for any  $\lambda \in \mathbb{R}$  and a nonzero vector  $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ , we have  $T((x_1, x_2, x_3, x_4)) = \lambda(x_1, x_2, x_3, x_4)$  if and only if

$$x_1 + x_2 + x_3 + x_4 = \lambda x_i \text{ for each } i = 1, 2, 3, 4. \quad (3)$$

This implies that  $\lambda = 4$  or  $\lambda = 0$ . Thus 4 and 0 are two distinct eigenvalues of  $T$ ,

OR,

Let  $\beta = \{e_1, e_2, e_3, e_4\}$  be the standard basis. Then  $[T]_\beta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ . The eigenvalues of

$T$  and  $[T]_\beta$  are same. So find the eigenvalues of  $[T]_\beta$  by computing the roots of the characteristic polynnomial  $\det(\lambda I - [T]_\beta)$ . They are 0, 0, 0, 4. **[2 mark]**

(Computing the eigenspaces): Next compute the eigenspaces, when  $\lambda = 4$ , we have from (3) that  $E_1 = \ker(4I - T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3 = x_4\} = \text{span}(\{(1, 1, 1, 1)\})$ . **[1 mark]**

Similarly, when  $\lambda = 0$ , we have from (3) that  $E_2 = \ker(0I - T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$ . Thus  $E_2 = \text{span}(\{(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\})$ . **[1 mark]**

(Diagonalizability check): This implies that

$$\dim(E_1) + \dim(E_2) = 1 + 3 = 4 = \dim(\mathbb{R}^4),$$

and hence  $T$  is diagonalizable. Consider  $B = \{(1, 1, 1, 1), (1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\}$ . Then  $B$  is a basis of  $\mathbb{R}^4$  since eigenspaces corresponding to distinct eigenvalues are independent

and  $|B| = 4$ . Also  $[T]_B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . **[1 mark]**