Find a basis for the row space and column space of the matrix $A = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{bmatrix}$

Solution :- $A \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 3 & 2 & 6 & 9 \\ 2 & 1 & 4 & 3 \end{bmatrix} \xrightarrow[R_2 - 3R_1]{R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & 0 & -9 \\ 0 & -1 & 0 & -9 \end{bmatrix}$

$\xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 0 & 9 \\ 0 & -1 & 0 & -9 \end{bmatrix} \xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

Then $R$ is a row-reduced echelon matrix form of $A$.

The row space of $A$ = The row space of $R$

$= \text{Span}\{(1,0,2,-3), (0,1,0,9)\}$

The basis of the row space of $A$ is $\{(1,0,2,-3), (0,1,0,9)\}$

We know the non-zero rows of row reduced echelon matrix are Linearly independent.

The Column Space of $A$ = Row Space of $A^T$

$$A^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 4 & 6 & 2 \\ 3 & 9 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 6 & 2 \\ 3 & 9 & 6 \end{bmatrix} \xrightarrow[R_4 - 3R_1]{R_2 - 2R_1,\ R_3 - 4R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow[R_4 - 3R_2]{R_1 - 2R_2,\ R_3 + 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

The basis of $B = \{(1,0,-1), (0,1,1)\}$, which is a basis of the column space of $A$.

**Problem :-** Examine linear independence of the set of vectors $\{(1,-1,2,4), (2,-1,5,7), (-1,3,1,-2)\}$ in $\mathbb{R}^4$.

Solution :- Consider a matrix $A$ whose row vectors are $(1,-1,2,4)$, $(2,-1,5,7)$, $(-1,3,1,-2)$ in $\mathbb{R}^4$.

Then $A = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 2 & -1 & 5 & 7 \\ -1 & 3 & 1 & -2 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}$

$\xrightarrow[R_3-2R_2]{R_1+R_2} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_2-R_3]{R_1-3R_3} \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Rank of the matrix $A$ is 3. Therefore, the vectors $(1,-1,2,4)$, $(2,-1,5,7)$, $(-1,3,1,-2)$ generate a vector subspace of dimension 3. Therefore, they are linearly independent.

**Linear Transformation :-** Let $V$ and $W$ be two vector spaces over the same field $F$. A map $T : V \rightarrow W$ is called a linear transformation if $T(au + bv) = aT(u) + bT(v)$ for every $u, v \in V$ and $a, b \in F$.

* Therefore, $T(0_V) = 0_W$

**Example :-**

(i) $V = W = \mathbb{R}^2$ and $F = \mathbb{R}$. Let $T(x, y) = (ax + by, cx + dy)$

Then $T(c_1(x_1, y_1) + c_2(x_2, y_2)) = T(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$

$$= \left(a(c_1x_1 + c_2x_2) + b(c_1y_1 + c_2y_2),\ c(c_1x_1 + c_2x_2) + d(c_1y_1 + c_2y_2)\right)$$

$$= c_1(ax_1 + by_1, cx_1 + dy_1) + c_2(ax_2 + by_2, cx_2 + dy_2)$$

$$= c_1 T(x_1, y_1) + c_2 T(x_2, y_2)$$

Therefore, $T$ is a linear transformation.

(ii) $T:\mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(x,y) = (x+y+1,\ x-y)$

Then $T$ is not a linear transformation as $T(0,0) \neq (0,0)$

**Null space / kernel :–** Let $T: V \to W$ be a linear transform over same field $F$. Then $\ker(T) := \{v \in V : T(v) = 0\}$ is called kernel / null space. The null space is a subspace of $V$. The dimension of the null space is called nullity of $T$.

**Image space / range space :–** Let $T: V \to W$ be a linear transformation over the same field $F$. Then $T(V) = \{T(v) : v \in V\}$ is a subspace of $W$. This subspace is called the image space or range space of $T$. The dimension of $T(V)$ is called the rank of $T$.

**Theorem :-** A linear transformation is injective if and only if its null space is the zero space.

**Proof :-** Let $T: V \to W$ be injective.
Let $v \in \ker(T)$. Then $T(v) = 0 = T(0)$. Since $T$ is injective $v = 0$. Therefore $\ker(T) = \{0\}$.

Conversely, let $\ker(T) = \{0\}$

Let $T(u) = T(v)$ for some $u, v \in V$

$$\Rightarrow T(u - v) = 0$$

$$\Rightarrow u - v \in \ker(T) = \{0\}$$

$$\Rightarrow u = v$$

Therefore, $T$ is injective.

**Rank-Nullity Theorem :-** Let $T : V \to W$ be a linear transformation. Then $\operatorname{rank}(T) + \operatorname{Nullity}(T) = \dim(V)$, where $V$ is finite dimensional.

**Proof :-** Let $\{u_1, \dots, u_m\}$ is a basis for $\ker(T)$. Then $\{u_1, u_2, \dots, u_m\}$ is a linearly independent subset of $V$. extend this to a basis $\{u_1, \dots, u_m, v_1, \dots, v_r\}$ of $V$.

Therefore, $\dim(V) = m + r$.

Note that $\operatorname{nullity}(T) = m$, we claim that $\operatorname{rank}(T) = r$

We prove that $\{T(v_1), \dots, T(v_r)\}$ is a basis of range space of $T$.

Suppose, $c_1 T(v_1) + \dots + c_r T(v_r) = 0$

$$\Rightarrow T(c_1 v_1 + \dots + c_r v_r) = 0$$

$$\Rightarrow c_1 v_1 + \dots + c_r v_r \in \ker(T) = \operatorname{span}\{u_1, \dots, u_m\}$$

$\Rightarrow c_1 v_1 + \cdots + c_r v_r = d_1 u_1 + \cdots + d_m u_m$

$\Rightarrow c_1 v_1 + \cdots + c_r v_r - d_1 u_1 - \cdots - d_m u_m = 0$

Since $\{u_1, \ldots, u_m, v_1, \ldots, v_r\}$ is linearly independent,

$$c_1 = \cdots = c_r = 0$$

Thus, $\{T(v_1), \ldots, T(v_r)\}$ is linearly independent in $T(V)$.

Now, let $w \in T(V)$. Then $\exists\ v \in V$ s.t. $T(v) = w$.

Let $v = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{r} b_i v_i$

Then $T(v) = \sum_{i=1}^{m} a_i T(u_i) + \sum_{i=1}^{r} b_i T(v_i)$

$= \sum_{i=1}^{r} b_i T(v_i) \in \text{Span}\{T(v_1), \ldots, T(v_r)\}$

Therefore, $T(V) = \text{Span}\{T(v_1), \ldots, T(v_r)\}$ and $\dim(T(V)) = r$

Therefore, $\text{rank}(T) + \text{Nullity}(T) = \dim(V)$

Problem :- Do we have a surjective linear transfor $T : \mathbb{R}^3 \to \mathbb{R}^4$?

Solution :- If $T$ is surjective then range$(T)$ $= \mathbb{R}^4$, and hence rank$(T) = 4$.

Further we have $\text{rank}(T) + \text{Nullity}(T) = \dim(V)$

$$\Rightarrow 4 + \text{Nullity}(T) = 3$$

which is not possible.

* There is no injective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ if $m > n$.
* There is no surjective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ if $m < n$.
* There is no bijecte linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ if and only if $m \neq n$.