i) TRUE

To Show $\lambda \odot (x \oplus y) = (\lambda \odot x) \oplus (\lambda \odot y)$ [$\odot$ is left-distributive over $\oplus$]

$\lambda \odot (x \oplus y) = \lambda \odot (xy) = (xy)^\lambda = x^\lambda y^\lambda = x^\lambda \oplus y^\lambda$

$= (\lambda \odot x) \oplus (\lambda \odot y)$

Aliter i) FALSE

To show $(x \oplus y) \odot \lambda = (x \odot \lambda) \oplus (y \odot \lambda)$ [$\odot$ is right-distributive over $\oplus$]

L.H.S $(x \oplus y) \odot \lambda = (xy) \odot \lambda = (\lambda)^{xy}$

R.H.S $(x \odot \lambda) \oplus (y \odot \lambda) = \lambda^x \oplus \lambda^y = \lambda^x \cdot \lambda^y = \lambda^{x+y}$

$\Rightarrow$ L.H.S $\neq$ R.H.S

ii) FALSE

To show $\lambda \otimes (x \oplus y) = \lambda \otimes x \oplus \lambda \otimes y$ [$\otimes$ is left-distributive over $\oplus$]

L.H.S $\lambda \otimes (x \oplus y) = \lambda \otimes (xy) = \lambda^{xy}$

R.H.S $\lambda \otimes x \oplus \lambda \otimes y = \lambda^x \oplus \lambda^y = \lambda^x \lambda^y = \lambda^{x+y}$

L.H.S $\neq$ R.H.S

Aliter ii) TRUE

To show $(x \oplus y) \otimes \lambda = x \otimes \lambda \oplus y \otimes \lambda$ [$\otimes$ is right-distributive over $\oplus$]

L.H.S $(x \oplus y) \otimes \lambda = (xy) \otimes \lambda = (xy)^\lambda = x^\lambda y^\lambda$

R.H.S $x \otimes \lambda \oplus y \otimes \lambda = x^\lambda \oplus y^\lambda = x^\lambda y^\lambda$

L.H.S = R.H.S

iii) TRUE

To show $x, y \in V \Rightarrow x \otimes y \in V = R_+$

Let $x, y \in V$ Then $x \otimes y = x^y > 0$ $\because x > 0, y > 0$

$\Rightarrow x \otimes y \in R_+ = V$

iv) TRUE

Suppose $\exists\, v \in R_+ = V$ s.t. for any $x \in V = R_+$ $\exists\, \lambda \in \mathbb{R}$ s.t.

$x = \lambda \odot v = v^\lambda$

$\Rightarrow \log x = \lambda \log v \Rightarrow \lambda = \frac{\log x}{\log v}$ when $v \neq 1$

So if $v \neq \perp$ then $span\{v\} = V = \mathbb{R}+$

(b) $V = M_{3\times3}(\mathbb{R})$ &

$$W = \left\{ A = [a_{ij}] \in V \mid \sum_{i=1}^{3} a_{ij} = 0,\ j = 1,2,3 \right\}$$

it is sufficient to show that for any $\alpha, \beta \in \mathbb{R}$ & for any $A, B \in W$

$\alpha A + \beta B \in W$.

So let $A, B \in W$

Then $A = [a_{ij}]$ s.t $\sum_{i=1}^{3} a_{ij} = 0$, $j = 1,2,3$

& $B = [b_{ij}]$ s.t $\sum_{i=1}^{3} b_{ij} = 0$, $j = 1,2,3$

$\alpha A + \beta B = \alpha[a_{ij}] + \beta[b_{ij}] \in W$

Because $\alpha \sum_{i=1}^{3} a_{ij} + \beta \sum_{i=1}^{3} b_{ij} = \alpha \cdot 0 + \beta \cdot 0 = 0$, $j = 1,2,3$

Now W can be written in the form

$$W = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -a_{11}-a_{21} & -a_{12}-a_{22} & -a_{13}-a_{23} \end{bmatrix},\ a_{11}, a_{12} \cdots a_{23} \in \mathbb{R} \right\}$$

Basis for W

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

Marking Scheme:

(a)

(i) $\frac{1}{2} + \frac{1}{2} = 1$ — $\frac{1}{2}$ for true or false + $\frac{1}{2}$ for Justification

(ii) $\frac{1}{2} + \frac{1}{2} = 1$ — $\frac{1}{2}$ " + $\frac{1}{2}$ "

(iii) $\frac{1}{2} + \frac{1}{2} = 1$ — $\frac{1}{2}$ " + $\frac{1}{2}$ "

(iv) $\frac{1}{2} + \frac{1}{2} = 1$ — $\frac{1}{2}$ " + $\frac{1}{2}$ "

(b) 1 + 1 + 2 = 4

1 mark for definition of subspace

+

1 mark for complete Proof subspace

+

2 mark for basis.

Q 2) $V = \mathbb{R}^8$ $F = \mathbb{R}$

To prove: Intersection of any three subspaces, each of dimension 6, can not be a zero subspace.

Soln:- Let W_1, W_2 & W_3 be three subspaces of $\mathbb{R}^8$ st $\dim W_1 = \dim W_2 = \dim W_3 = 6$

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \quad - \frac{1}{2}$$

as $W_1 + W_2$ is a subspace of $\mathbb{R}^8$

$\therefore \dim W_1 + W_2 \leq 8.$

$\therefore \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \leq 8$

$\Rightarrow 6 + 6 - \dim(W_1 \cap W_2) \leq 8$ ($\because \dim W_1 = \dim W_2 = 6$)

$\Rightarrow \dim(W_1 \cap W_2) \geq 4$ — $\frac{1}{2}$

Also $W_1 \cap W_2 \subseteq W_1$ $\therefore \dim(W_1 \cap W_2) \leq 6.$

$$\dim((W_1 \cap W_2) \cap W_3) = \dim(W_1 \cap W_2) + \dim W_3 - \dim((W_1 \cap W_2) + W_3) \quad - 1$$

$\dim(W_1 \cap W_2) \geq 4$

$\dim((W_1 \cap W_2) + W_3) \leq 8$ — 1

$\Rightarrow -\dim((W_1 \cap W_2) + W_3) \geq -8.$ — 1

$\Rightarrow \dim((W_1 \cap W_2) \cap W_3) \geq 4 + 6 - 8 = 2$

$\Rightarrow \dim(W_1 \cap W_2 \cap W_3) \geq 2.$

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(b) (i) $N(A) \subseteq \text{Range}(I-A)$

Let $x \in N(A)$

$\Rightarrow Ax = 0$

consider

$x = Ix - A(x) \quad (\because Ax = 0)$

$\Rightarrow x = (I-A)x$

$\Rightarrow x \in \text{Range}(I-A)$ — 1

$\Rightarrow N(A) \subseteq \text{Range}(I-A)$

(ii) Is $\text{Range}(I-A) \subseteq N(A)$

counter example :- $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ — 1

$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\text{Range}(I-A) =$ col space of $(I-A) = \mathbb{R}^2$.

Hence $\text{Range}(I-A) \not\subseteq N(A)$

OR

Let $x \in \text{Range}(I-A)$

$\Rightarrow \exists\, y \in \mathbb{R}^{n \times 1}$ such that

$(I-A)y = x$

~~Apply A to~~ $\Rightarrow A(I-A)y = Ax$

$\Rightarrow (A - A^2)y = Ax$

Take $A \neq A^2$ and $y \notin N(A - A^2)$

then $Ax \neq 0$ & $x \notin N(A)$.

iii) Let $A^2 = A$

Is $\text{Range}(A) \cap \text{Range}(I-A) = \{0\}$

Let $x \in \text{Range}(A) \cap \text{Range}(I-A)$

$\Rightarrow$ $x \in \text{Range } A$ and $x \in \text{Range}(I-A)$

$x \in \text{Range } A$

$\Rightarrow \exists\, y \in \mathbb{R}^{n\times 1}$ st

$Ay = x$ — ⓐ — (1)

Also as $x \in \text{Range}(I-A)$

$\Rightarrow \exists\, z \in \mathbb{R}^{n\times 1}$ st

$(I-A)z = x$ — ⓑ

from ⓐ & ⓑ

$Ay = (I-A)z$

$\Rightarrow A \cdot Ay = A(I-A)z$

$\Rightarrow A^2 y = (A - A^2)z$ — (

$\Rightarrow Ay = (A-A)z \quad (\because A^2 = A)$

$\Rightarrow Ay = 0$

as $Ay = x \Rightarrow x = 0$

$\therefore \text{Range}(A) \cap \text{Range}(I-A) = \{0\}$

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OR

as $A = A^2$; using part (i) & (ii) we have ①

$\text{Range}(I-A) = N(A)$

(If you have done part (b) & written that if $A = A^2$ then $\text{Range}(I-A) \subseteq N(A)$)

Also $N(A) \perp R(A)$

$\Rightarrow N(A) \cap R(A) = \{0\}$

but $N(A) = \text{Range}(I-A)$ — ①

$\Rightarrow \text{Range}(I-A) \cap R(A) = \{0\}$

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4\. a) $B_1 = \{u_1, u_2\}$, $[T]_{B_1} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

So. $T(u_1) = 2u_1 + u_2$

$T(u_2) = 3u_1 + 4u_2$ $\frac{1}{2}$ mark

$B = \{v_1, v_2\}$. and $v_1 = u_1 + u_2$, $v_2 = u_1 + 2u_2$

$\therefore T(v_1) = T(u_1 + u_2) = T(u_1) + T(u_2)$

$= 2u_1 + u_2 + 3u_1 + 4u_2$

$= 5u_1 + 5u_2 = 5(u_1 + u_2)$

$= 5v_1 = 5v_1 + 0v_2$ $1\frac{1}{2}$ mark

$T(v_2) = T(u_1 + 2u_2) = T(u_1) + 2T(u_2)$

$= 2u_1 + u_2 + 6u_1 + 8u_2$

$= 8u_1 + 9u_2 = 7(u_1 + u_2) + (u_1 + 2u_2)$

$= 7v_1 + v_2$ $1\frac{1}{2}$ mark

$\therefore [T]_B = \begin{bmatrix} 5 & 7 \\ 0 & 1 \end{bmatrix}$

$\frac{1}{2}$ mark

<u>Another method</u>

Since we know $\exists$ an invertible matrix P s.t $[T]_B = P^{-1}[T]_B P$,

so here $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ as $v_1 = u_1 + u_2$ & $v_2 = u_1 + 2u_2$

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Therefore

$$[T]_B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 7 \\ 0 & 1 \end{bmatrix}$$

4. b) $T_1(x_1, x_2, x_3, x_4, \cdots) = (2x_2, 3x_3, 4x_4, \cdots)$

$\Rightarrow \ker(T_1) = \{(x_1, 0, 0, 0, \cdots) : x_1 \in \mathbb{R}\}$

$\neq \{(0, 0, 0, \cdots)\}$

$\Rightarrow T_1$ is not one-one

or

$T_1(x, x_2, x_3, x_4, \cdots) = (2x_2, 3x_3, 4x_4, \cdots)$

$\forall x \in \mathbb{R}$

$\Rightarrow T_1$ is not one-one. $\longrightarrow$ 1 mark

$T_2(x_1, x_2, x_3, x_4, \cdots) = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots\right)$

Since, $\left(x, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots\right) \in V \quad \forall x \in \mathbb{R}$

But $\left(x, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots\right) \notin \text{Image}(T_2)$

for $x \neq 0$

$\Rightarrow T_2$ is not onto.

$\longrightarrow$ 1 mark

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$T_1 \circ T_2 (x_1, x_2, x_3, x_4, \ldots)$

$= T_1 \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots\right)$

$= \left(2x_1, \frac{3x_2}{2}, \frac{4x_3}{3}, \frac{5x_4}{4}, \ldots\right)$

$\Rightarrow \ker(T_1 \circ T_2) = \{(0, 0, 0, \ldots)\}$

$\Rightarrow$ $T_1 \circ T_2$ is one-one.

If $(y_1, y_2, y_3, y_4, \ldots) \in V$ then

$T_1 \circ T_2 \left(\frac{y_1}{2}, \frac{2y_2}{3}, \frac{3y_3}{4}, \frac{4y_4}{5}, \ldots\right)$

$= (y_1, y_2, y_3, y_4, \ldots)$

$\Rightarrow T_1 \circ T_2$ is onto

$\Rightarrow T_1 \circ T_2$ is a bijection. $\longrightarrow$ 1 mark

Now, $T_2 \circ T_1 (x_1, x_2, x_3, x_4, \ldots)$

$= T_2 (2x_2, 3x_3, 4x_4, \ldots)$

$= \left(0, 2x_2, \frac{3x_3}{2}, \frac{4x_4}{3}, \ldots\right)$

$\Rightarrow T_2 \circ T_1$ is neither one-one nor onto

$\Rightarrow T_2 \circ T_1$ is not a bijection. $\longrightarrow$ 1 mark

Also one can say $\longrightarrow$

Since, T_1 is not one-one but onto, and T_2 is one-one but ~~onto~~ not onto, so $T_1 \circ T_2$ is a bijection and $T_2 \circ T_1$ is not a bijec

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5 (a) $[A|b] = \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 9 & 2 & 1 & 7 \\ 7 & 1 & 1 & 13 \end{array}\right)$

$R_2 \to R_2 - 9R_1$

$R_3 \to R_3 - 7R_1$

Then $R_3 \to R_3 - R_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & -2/5 & 38/25 \\ 0 & 0 & 0 & 11/2 \end{array}\right)$

$\text{rank}(A) = 2 < \text{rank}(A|b) = 3$

System is inconsistent -------- (2)

Least square solution : $A^T A X = A^T b$

$$\begin{pmatrix} 131 & 28 & 15 \\ 28 & 14 & 0 \\ 15 & 0 & 3 \end{pmatrix} X = \begin{pmatrix} 159 \\ 42 \\ 15 \end{pmatrix}$$ ------- (1)

After row transformations $R_2 \to \frac{R_2}{2}$, $R_3 \to \frac{R_3}{3}$ and

then $R_1 \to R_1 - 28R_2 - 15R_3$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$$

Set of least square solutions $= \{(k, 3-2k, 5-5k) : k \in \mathbb{R}\}$

------- (2)

5 (b) $u = \left(\frac{x}{\sqrt{y+z}}, \frac{y}{\sqrt{x+z}}, \frac{z}{\sqrt{x+y}}\right)$

$v = \left(\sqrt{y+z}, \sqrt{x+z}, \sqrt{x+y}\right)$ ------ (1)

$$|\langle u, v\rangle|^2 = (x+y+z)^2$$

$$\|u\|^2 \|v\|^2 = \left[\left(\frac{x^2}{y+z}\right) + \left(\frac{y^2}{x+z}\right) + \left(\frac{z^2}{x+y}\right)\right]\left[2(x+y+z)\right]$$

By C-S inequality and $x+y+z>0 \Rightarrow \frac{x+y+z}{2} \le \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}$

(2)

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