COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 7

QUIZ 1

JAN 17, 2023 | ROHIT VAISH

PROBLEM 1

Given two collections of numbers

$$a_1 \le a_2 \le \cdots \le a_n \quad \text{and} \quad b_1 \le b_2 \le \cdots \le b_n$$

Suppose we want to make n disjoint pairs of numbers, consisting of one a and one b.

Prove that the sum of products of the members of each pair is maximized when, for each $i \in \{1, 2, \cdots, n\}$, a_i is paired with b_i.

PROBLEM 1

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$P(n)$: Statement of the theorem.

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Base case: $P(2)$ is TRUE because

$$a_1 \cdot b_1 + a_2 \cdot b_2 \geq a_1 \cdot b_2 + a_2 \cdot b_1$$

$$\iff (a_2 - a_1)(b_2 - b_1) \geq 0$$

which holds because $a_2 \geq a_1$ and $b_2 \geq b_1$.

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$P(n)$: Statement of the theorem.

Induction Step: $\forall n \geq 2 \quad P(n) \Rightarrow P(n+1)$.

Consider any pairing of $a_1 \leq \cdots \leq a_{n+1}$ and $b_1 \leq \cdots \leq b_{n+1}$.

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Consider any pairing of $a_1 \leq \cdots \leq a_{n+1}$ and $b_1 \leq \cdots \leq b_{n+1}$.

If a_1 is paired with b_1, then sum of products is maximized if $(a_2, b_2), (a_3, b_3), \ldots, (a_{n+1}, b_{n+1})$ are paired.

Using $P(n)$

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So, $P(n+1)$ holds in this case.

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Consider any pairing of $a_1 \leq \cdots \leq a_{n+1}$ and $b_1 \leq \cdots \leq b_{n+1}$.

If a_1 is **not** paired with b_1, then it must be paired with b_k for some $k \neq 1$.

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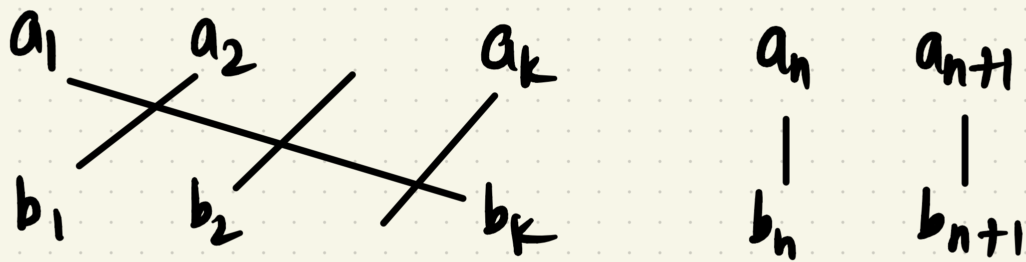
Using $P(n)$, the remaining numbers should be paired as:



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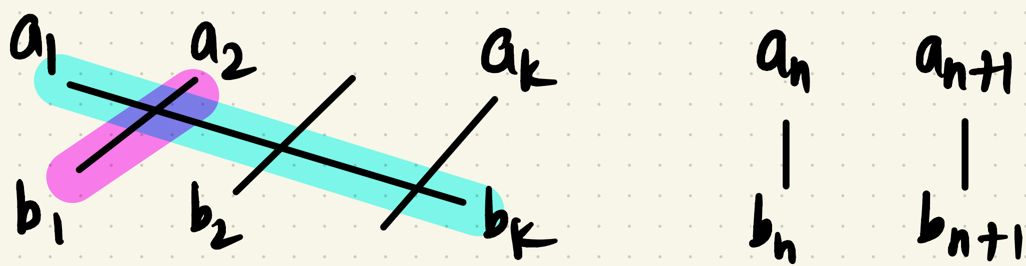

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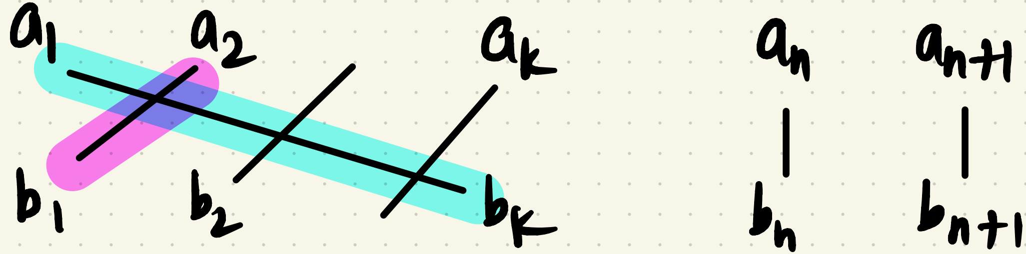

From earlier argument, $\{(a_1, b_1), (a_2, b_k)\}$ improves upon $\{(a_1, b_k), (a_2, b_1)\}$.

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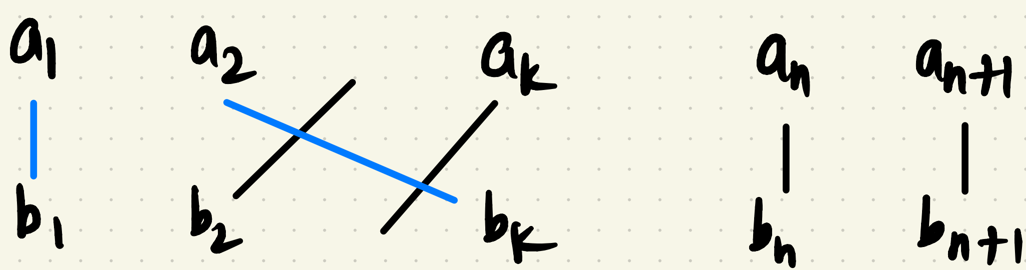

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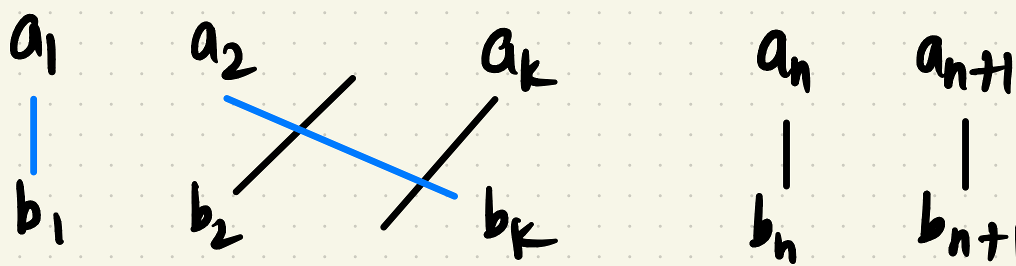

Now apply $P(n)$ to $a_2 \cdots a_{n+1}$ and $b_2 \cdots b_{n+1}$. $\square$

PROBLEM 1

Total = 20 points

Identifying proof by induction [2 pts]

Writing P(n) [2 pts]

Base case [6 pts]

Inductive step → Identifying the use of base case-like argument [7 pts]

→ Applying P(n) [3 pts]

PROBLEM 2

Imagine a board with the numbers $1, 2, \cdots, 2n$ written on it, where n is an odd positive integer.

Erase any pair of numbers i, j from the board and write $|i-j|$ instead.

Continue until only one integer is left on the board.

Show that this integer must be odd.

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This invariant certainly holds at the beginning, since the numbers $1, 3, 5, \ldots, 2n-1$ are odd, and there are n such numbers (we are given that n is odd).

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At any step, there are three cases for the numbers erased.

* Both numbers are even
* Both numbers are odd
* One number is even, other is odd

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At any step, there are three cases for the numbers erased.

* Both numbers are even
  ↳ Their difference is even. So, no. of odd numbers is unchanged.
* Both numbers are odd
  ↳ Their difference is even. So, no. of odd numbers decreases by 2.
* One number is even, other is odd
  ↳ Their difference is odd. So, no. of odd numbers is unchanged.

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After each step, the number of odd numbers on the board either remains the same or decreases by 2.

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Lemma: For any $k < 2n$, after k steps, there is an odd number of odd numbers on the board.

Proof of Lemma: Using invariant and repeated application of modus ponens (since n is finite).

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Proof:

Theorem follows by invoking Lemma for $k = 2n-1$.

$\blacksquare$

Lemma: For any $k < 2n$, after k steps, there is an odd number of odd numbers on the board.

Proof of Lemma: Using invariant and repeated application of modus ponens (since n is finite).

PROBLEM 2

Total = 15 points

Identifying proof by invariant [2 pts]

Identifying the correct invariant [6 pts]

Proving invariant for ONE step [2 pts]

Proving invariant for K steps [5 pts]