(1) $\boxed{\text{White}}$

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \mu & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\alpha, \beta \in \mathbb{R}$ $\qquad$ $\mu \in \mathbb{R}$

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(5) $\boxed{\text{Green}}$

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{pmatrix}, \quad \begin{pmatrix} 1 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\alpha, \beta \in \mathbb{R}$ $\qquad$ $\mu \in \mathbb{R}$

$\boxed{\text{White I}}$

2. (a) Since $AOB = O = BOA$, $O \in W$
( $O$ denotes the zero matrix of size $n \times n$)

$\therefore$ $W \neq \phi$

If $X, Y \in W$ and $\lambda \in \mathbb{R}$, then

$AXB = BXA$ and $AYB = BYA$

$$\therefore \quad A(X + \lambda Y)B = AXB + \lambda AYB$$
$$= BXA + \lambda BYA = B(X + \lambda Y)A$$

$\Rightarrow$ $X + \lambda Y \in W$

Hence, $W$ is a subspace.

(b) <u>FALSE</u>: Take $V = \mathbb{R}$, $A = \{1\}$, $B = \{2\}$

Then $\text{span}(A) = \mathbb{R}$, $\text{span}(B) = \mathbb{R}$

But $\text{span}(A \cap B) = \{0\}$.

(c) <u>FALSE</u>: $(u - 2v) + (2v - w) + (w - u) = 0$

$\Rightarrow$ $\{u - 2v, 2v - w, w - u\}$ is linearly dependent.

$\boxed{\text{White}}$

3. (a) $(x, y, z, w) \in W_1 \cap W_2 \iff \begin{cases} x+y+z+w=0 \\ 2x+3y+4z+5w=0 \\ 4x+3y+2z+w=0 \\ x+2y+2z+w=0 \end{cases}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} \underset{R_4 \to R_4 - R_1}{\overset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1}}{\sim}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\overset{\substack{R_1 \to R_1 - R_4 \\ R_3 \to R_3 + R_2}}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R_3 \to R_3 - R_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_2 \to R_2 - R_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 3 \Rightarrow \dim(\text{soln. space}) = 4 - 3 = 1$

$W_1 \cap W_2 = \{(-\lambda, 3\lambda, -3\lambda, \lambda) : \lambda \in \mathbb{R}\}$

$= \text{span}\{(-1, 3, -3, 1)\}$

$\therefore$ A Basis for $W_1 \cap W_2$ is $\{(-1, 3, -3, 1)\}$

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3. (b) $S = \{(x, y, z) \in \mathbb{R}^3 : x+y+z=1,\ x+2y+3z=2\}$ is a straight line in $\mathbb{R}^3$ <u>not</u> passing through $(0,0,0)$.

So, any two points on $S$ spans $\text{span}(S)$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$S = \{(\lambda, 1-2\lambda, \lambda) : \lambda \in \mathbb{R}\}$

Putting $\lambda = 0$ & $\lambda = 1$, we get $(0,1,0), (1,-1,1) \in S$.

$\therefore \{(0,1,0), (1,-1,1)\}$ is a basis for $\text{span}(S)$.

White

(4) $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ a & b & c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ a & b & c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix} \overset{R_4 \to R_4 - aR_1 - bR_2 - cR_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d-b \end{pmatrix}$$

Rank of the matrix is 4 if $d \neq b$ and 3 if $d = b$.

For linearly dependent the rank must be less than 4.

$\therefore$ The given set is L.D. iff $\boxed{b - d = 0}$

$\boxed{\text{White}}$

5. (a) $S_a = \{\sin 3x, \sin ax\}$

If $a=0$, $S_a$ is L.D. because $0 \in S_a$.

Assume $a \neq 0$.

$S_a$ is L.D. $\Leftrightarrow$ $\sin ax = \lambda \sin 3x \quad \forall x \in \mathbb{R}$ for some $\lambda \in \mathbb{R}$.

Now $\sin ax = \lambda \sin 3x \Rightarrow a \cos ax = 3\lambda \cos 3x$

$\Rightarrow -a^2 \sin ax = -9\lambda \sin 3x$

$\Rightarrow a^2 \sin ax = 9\lambda \sin 3x$

$= 9 \sin ax$

$\Rightarrow (a^2-9)\sin ax = 0 \quad \forall x \in \mathbb{R}.$

$\Rightarrow a^2 = 9 \quad (\because a \neq 0)$

Clearly if $a = \pm 3$, $S_a$ is L.D.

$\therefore$ $S_a$ is L.I. iff $a \in \mathbb{R} \setminus \{0, 3, -3\}$.

(b) $\mathcal{B} = \{e^{2x}, \sin 3x, \cos 3x\}$, $W = \text{span}(\mathcal{B})$

$f(x) = e^{2x} + \sin 3x - \cos 3x$

(i) $f'(x) = 2e^{2x} + 3\cos 3x + 3\sin 3x \in \text{span}(\mathcal{B}) = W$

(ii) $[f'(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$