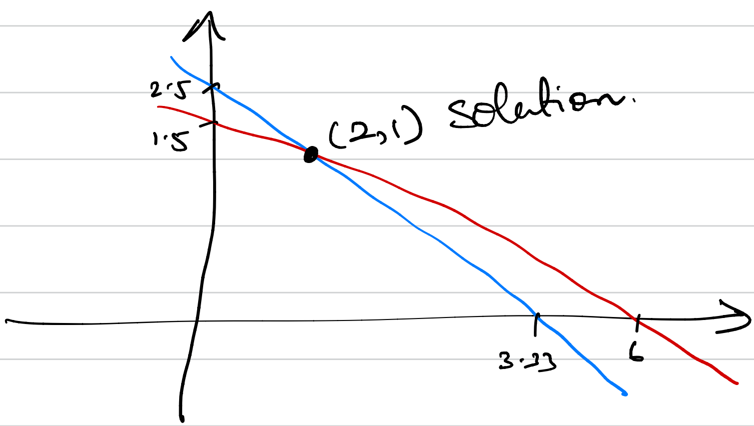
Lecture 2

§ System of linear equations.

Ex: $3x + 4y = 10$
$2x + 8y = 12$ } linear system of equations

Solution to a linear system is a ordered set (a, b)
s.t $3a + 4b = 10$ and $2a + 8b = 12$



Def: A system of linear equations is a collection of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad (*)$$

a_{ij} $\quad 1 \le i \le m$, $1 \le j \le n$ $\quad b_1, \dots, b_m$ are all constants (real or complex)

and $x_1, \dots, x_n$ are variables (unknowns)

- A solution $(s_1, \dots, s_n)$ is an ordered set of n numbers such that if we set $x_i = s_i$ $\; 1 \le i \le n$ then the above system is satisfied.

i.e., $a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i \quad 1 \le i \le m$

Ex1:

$$3x + 4y = 10 \quad (1)$$
$$2x + 8y = 12 \quad (2)$$

(1) $\times(-2)$ $\quad -6x - 8y = -20$

(2) $\quad 2x + 8y = 12$

$$-4x = -8$$
$$\boxed{x = 2}$$

Sub $x = 2$ in (1) (or (2)) to get $3(2) + 4(y) = 10$

$$\Rightarrow \boxed{y = 1}$$

Ex2:

$$6x - 10y = 0 \quad (1)$$
$$-3x + 5y = 8 \quad (2)$$

(1) $\quad 6x - 10y = 0.$

(2) $\times 2$ $\quad -6x + 10y = 16$

$$0 = 16$$ ← makes no sense!

No solutions

Ex 3:

$$4x + 10y = 14 \quad (1)$$
$$-6x - 15y = -21 \quad (2).$$

$(1) \times 3/2$ $\quad 6x + 15y = 21$

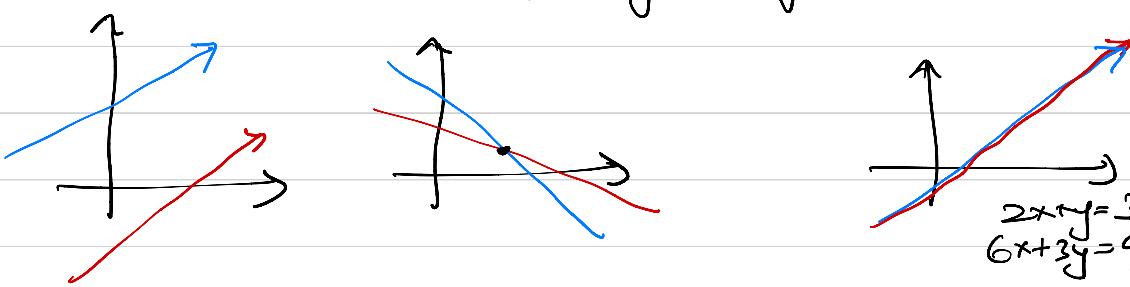
(2) $\quad -6x - 15y = -21$

$$0 = 0$$

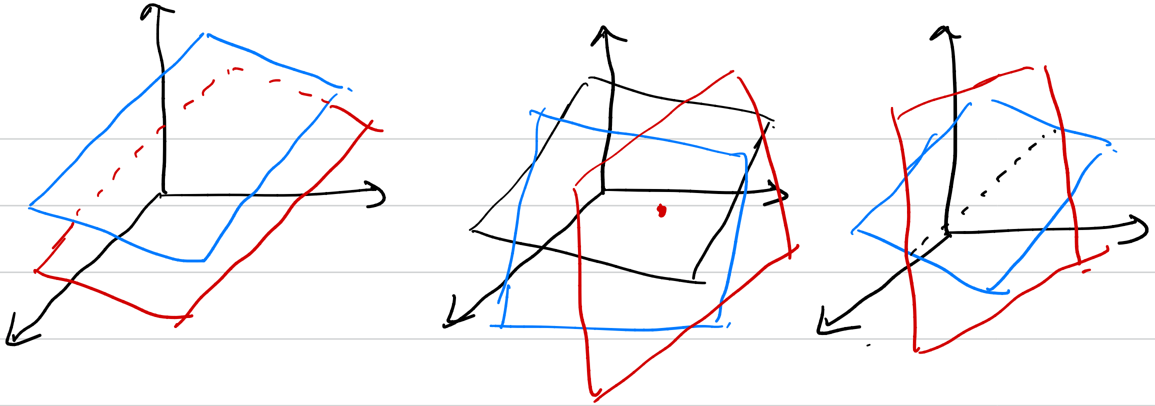
$$x = \frac{7}{2} - \frac{5}{2}y.$$

$$x = 7/2 - 5/2\, s$$
$$y = s. \leftarrow \text{free parameter.}$$

Theorem: A system of linear equations has
(1) no solutions
(2) unique solution
(3) infinitely many solutions.



$2x+y=3$
$6x+3y=9$



§ Solving a System of linear equation.

§ Elementary Operators.

There are 3 elementary operators

1. Interchange the position of two equation
2. Multiply an equation by a nonzero constant
3. Add a multiple of one equation to another.

⇒ These elementary operations do not change the solution set.

$\boxed{\text{I}}$ Write (*) in the matrix form

$$A\,\underline{x} = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times n$ $\quad$ $n \times 1$ $\quad$ $m \times 1$

$$A\underline{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

$\boxed{\text{II}}$ $$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$ is an Augmented matrix.

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- If $b_1 = b_2 = \cdots = b_m = 0$ then we say that the system is Homogeneous
- If the linear system is not homogeneous its called a non-homogeneous system.

Ex:

$$\begin{aligned} 2x_1 - 3x_2 + 10x_3 &= -2 \\ x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + 3x_2 + x_3 &= 4. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 10 & -2 \\ 1 & -2 & 3 & -2 \\ -1 & 3 & 1 & 4 \end{array}\right]$$

Augmented form.

Interchange rows:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 2 & -3 & 10 & -2 \\ -1 & 3 & 1 & 4 \end{array}\right]$$

Add (-2) times Row 1 to Row 2

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ -1 & 3 & 1 & 4 \end{array}\right]$$

Add Row 1 to Row 3

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \end{array}\right]$$

Add (-1) times Row 2 to Row 3

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$$x_1 - 2x_2 + 3x_3 = -2$$
$$x_2 + 4x_3 = 2$$

$x_3 = s$ ← free parameter

$x_2 = 2 - 4s$

$x_1 = 2 - 11s$.

Substitute these solutions into the original system

$2(2-11s) - 3(2-4s) + 10s = 4 - 22s - 6 + 12s + 10s = -2$

$(2-11s) - 2(2-4s) + 3s = 2 - 11s - 4 + 8s + 3s = -2$

$-(2-11s) + 3(2-4s) + s = -2 + 11s + 6 - 12s + s = 4$

Def: Elementary Row operations

1. Interchange two rows
2. Multiply a row by a nonzero constant c
3. Replace a row with a sum of that row and the scalar multiple of another row.

• Equivalent Matrices: Two matrices are equivalent if one can be obtained from the other through a sequence of elementary row operations.

→ Hence equivalent augmented matrices correspond to equivalent systems.

Theorem: Two equivalent system of linear equations have the same solution.

- A matrix is in echelon form if
  1. Every leading term is in a column to the left of the leading term of the row below it
  2. Any zero rows are at the bottom of the matrix.

- For a matrix in echelon form, the pivot positions are those that contain a leading term.
- pivot columns are the columns that contain pivot positions.

Ex:

$$2x_1 - 2x_2 - 6x_3 + x_4 = 3$$
$$-x_1 + x_2 + 3x_3 - x_4 = -3$$
$$x_1 - 2x_2 - x_3 + x_4 = 2$$

Augmented matrix
$$\left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array}\right]$$

$R_1 \longleftrightarrow R_3$
$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{array}\right]$$

$R_2 \to R_2 + R_1 \qquad R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 2 & -4 & -1 & -1 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right]$$

$$x_1 - 2x_2 - x_3 + x_4 = 2$$
$$-x_2 + 2x_3 = -1$$
$$-x_4 = -3$$

$x_4 = 3 \qquad x_3 = s \qquad x_2 = 1 + 2s \qquad x_1 = 1 + 5s$