**Name:** ________________ **Group No.**_________ **Entry No.**_______________

# DEPARTMENT OF MATHEMATICS

INDIAN INSTIUTUTE OF TECHNOLOGY DELHI

MID SEMESTER EXAMINATION 2023-24 SECOND SEMESTER

MTL101 (LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS)

**Time: 2 hours** **Max. Marks : 30**

Write your Name and Group number and Entry number at the places specified above. Attempt all questions. All notations are standard. All parts of a question must be answered at one place. Exhibit clearly all the steps. Use of any electronic gadget including calculator is NOT allowed. Attach question paper with the answer book. Do not do any rough work on the question paper. No query will be entertained.

**Q1:** Use elementary row operations to find the inverse of the following complex matrix (4)

$$\begin{bmatrix} i & -i & i & 0 \\ 0 & i & -i & i \\ i & 0 & i & -i \\ i & i & 0 & i \end{bmatrix}.$$

**Q2:** Find the values of $\lambda$ and $k$ for which the following system

$$x + y + z = 3$$

$$x + 2y + \lambda z = 4$$

$$2x + 3y + 2\lambda z = k$$

has (i) no solution (ii) unique solution (iii) infinite number of solutions (3)

**Q3:** If $W_1$ and $W_2$ are subspaces of a finite dimensional vector space $V(F)$, then prove (4)

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$

**Q4:** Let $T: V(F) \to V(F)$ be a linear transformation such that $T^2 = T$. Show that (4)

$$V = Null(T) \oplus Range(T).$$

**Q5:** Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be linear transformation such that $T^k \neq 0$ for $k = 1,2,3$ and $T^4 = 0$. Prove or disprove that there exists $x \in \mathbb{R}^4$, $x \neq 0$ such that $\{x, T(x), T^2(x), T^3(x)\}$ is linearly independent. Here, $T^n$ denotes the composition of $T$ with itself $n$ times. (4)

**Q6:** Consider the vector space $P_2(\mathbb{R})$ of real polynomials of degree less than or equal to two with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

Starting with the basis $\{1, x, x^2\}$ obtain an orthonormal basis of $P_2(\mathbb{R})$. (4)

**Q7a:** Use Cauchy-Schwarz inequality show that if $a_1, a_2, \ldots \ldots . a_n$ are positive reals, then (2)

$$(a_1 + a_2 + \ \ldots \ldots . a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + . \ldots . . + \frac{1}{a_n}\right) \geq n^2.$$

**Q7b:** Show that any orthogonal set of nonzero vectors in an inner product space is linearly independent. (2)

**Q8:** Let $\mathbb{R}^{2\times 2}$ be the vector space of all $2 \times 2$ matrices over the field of reals and $A \in \mathbb{R}^{2\times 2}$ be a fixed matrix. Define $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by (4)

$$T(B) = AB - BA$$

Show that $T$ is linear. Find the matrix of $T$ with respect to the ordered basis

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$