Once released :)

Solve only those questions which are in syllabus to avoid panic

1

(a) Justify whether the uniqueness theorem is applicable for the intial value problem (IVP):

$$\frac{dy}{dx} = (x^2+1)y^{2/3}; \quad y(0) = 0.$$

(b) For the ordinary differential equation (ODE)

$$(4xy^2 + 3y)dx + (3x^2y + 2x)dy = 0,$$

find real numbers p, q such that $x^p y^q$ is an integrating factor. Then, solve this ODE by making it an exact equation.

Sol^n

1. Here, $\frac{dy}{dx} = (x^2+1)y^{2/3}$ with $y(0) = 0$.
   Thus, $f(x, y) = (x^2+1)y^{2/3}$. To see, whether the uniqueness theorem is applicable, first we check that the given function f is Lipschitz or not with respect to y. We show that f is not a Lipschitz continuous function around $(0, 0)$ with respect to y. **(1 mark)**
   Consider a rectangle $\mathcal{R}$ around $(0, 0)$. Let $f(x, y)$ be Lipschitz continuous in $\mathcal{R}$. Then, there exists a positive real constant L such that, for all $y_1, y_2 \in \mathcal{R}$,

$$\begin{aligned}
|f(x, y_1) - f(x, y_2)| &\leq L|y_1 - y_2|, \\
\Rightarrow |(x^2+1)y_1^{2/3} - (x^2+1)y_2^{2/3}| &\leq L|y_1 - y_2|, \\
\Rightarrow \frac{|(x^2+1)y_1^{2/3} - (x^2+1)y_2^{2/3}|}{|y_1 - y_2|} &\leq L,
\end{aligned}$$

   Since, the above equation is true $\forall\ y_1$ and $y_2 \in \mathcal{R}$, thus in particular it is true for $y_2 = 0$.

$$\Rightarrow (1+x^2)\frac{|y_1^{2/3}|}{|y_1|} \leq M.$$

   So, when $y_1 \to 0$, $\frac{|y_1^{2/3}|}{|y_1|} \to \infty$. But, M is a fixed finite number. Thus, f is not Lipschitz continuous. **(1 mark)**

   Remark: It is not enough to show that $\frac{\partial f}{\partial y}$ is discontinuous or unbounded, to show the non-applicability of uniqueness theorem (as if the function is Lipschitz and $\frac{\partial f}{\partial y}$ is discontinuous, then also uniqueness theorem is applicable).

2. Given ODE $(4xy^2 + 3y)dx + (3x^2y + 2x)dy = 0$, we compare this with general equation $Mdx + Ndy = 0$. Thus, we have $M = 4xy^2 + 3y$ and $N = 3x^2y + 2x$.
   Then, $\frac{\partial M}{\partial y} = 8xy + 3 \neq \frac{\partial N}{\partial x} = 6xy + 2$, hence ODE is not exact.
   Given an integrating factor $x^p y^q$ for some p, q, $x^p y^q((4xy^2 + 3y)dx + (3x^2y + 2x)dy) = 0$ is exact for some p and q.
   Consider $\bar{M} = 4x^{p+1}y^{2+q} + 3x^p y^{q+1}$ and $\bar{N} = 3x^{2+p}y^{q+1} + 2x^{p+1}y^q$.

   (1 mark)

   Then, $\frac{\partial \bar{M}}{\partial y} = (8+4q)x^{p+1}y^{q+1} + (3q+3)x^p y^q$ and $\frac{\partial \bar{N}}{\partial x} = (6+3p)x^{p+1}y^{q+1} + (2p+2)x^p y^q$.
   On equating $\frac{\partial \bar{M}}{\partial y} = \frac{\partial \bar{N}}{\partial y}$, we get, $p = 2$ and $q = 1$. Thus $x^2 y$ is an integrating factor. **(2 marks)**

$$\begin{aligned}
&\Rightarrow x^2y((4xy^2 + 3y)dx + (3x^2y + 2x)dy) = 0 \\
&\Rightarrow (4x^3y^3 + 3x^2y^2)dx + (3x^4y^2 + 2x^3y)dy = 0 \\
&\Rightarrow x^4y^3 + y^2x^3 = c
\end{aligned}$$

   (1 mark)

2

Sol^n

Question 4: [6 marks]

(a) Find the general solution of the following differential equation

$$y^{(5)} - y^{(4)} + 2y^{(3)} - 2y^{(2)} = 0.$$

Notation: Here $y^{(n)}$ denotes the n-th derivative of y.

(b) Find the general solution by using the method of undetermined coefficient of the following equation:

$$y^{(5)} - y^{(4)} + 2y^{(3)} - 2y^{(2)} = t + e^t.$$

Q 4. (a) The general sol^n of $y^{(5)} - y^{(4)} + 2y^{(3)} - 2y^{(2)} = 0$

The char eq^n is $m^5 - m^4 + 2m^3 - 2m^2 = 0$ $\frac{1}{2}$

So, $m = 0, 0, 1, i\sqrt{2}, -i\sqrt{2}$ $\frac{1}{2}$

Thus, $y_h = \underbrace{A + Bt}_{\boxed{1/2}} + \underbrace{Ce^t}_{\boxed{1/2}} + \underbrace{D\cos(\sqrt{2}t) + E\sin(\sqrt{2}t)}_{\boxed{1/2}}$ $1\frac{1}{2}$

(b) The proposed y_p for

$y^{(5)} - y^{(4)} + 2y^{(3)} - 2y^{(2)} = t + e^t$ is

$y_p = \underbrace{(P + Qt)t^2}_{\boxed{1/2}} + \underbrace{R\,te^t}_{\boxed{1/2}}$ 1

Substitute y_p in the given eq^n $\frac{1}{2}$

& compare coefficients $\frac{1}{2}$

and get $P = -\frac{1}{4}$ $\frac{1}{2}$

$Q = -\frac{1}{12}$ $\frac{1}{2}$

$R = \frac{1}{3}$ $\frac{1}{2}$

3

Question 5: [6 marks]

(a) Find the general solution of the following homogeneous ordinary differential equation (ODE):

$$t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0.$$

(b) Use variation of parameter to find a particular solution of the following non-homogeneous ODE:

$$t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = t^4.$$

Sol^n

Question 5

(a) $t^2y'' - 5ty' + 9y = 0$

This is a Cauchy-Euler equation.

Putting $y = t^m$, we get the characteristic eqn:

$m(m-1) - 5m + 9 = 0 \iff (m-3)^2 = 0$ [1 mark]

$\therefore$ $m = 3$ is a repeated root.

Hence, $y_1 = t^3$ [$\frac{1}{2}$ mark] and $y_2 = t^3\ln|t|$ [$\frac{1}{2}$ mark] are two linearly independent solutions.

$\therefore$ The general soln. is $\boxed{y = c_1t^3 + c_2t^3\ln|t|}$

(b) $t^2y'' - 5ty' + 9y = t^4$

ie. $y'' - \frac{5}{t}y' + \frac{9}{t^2}y = t^2$

Comparing with $y'' + p(t)y' + q(t)y = r(t)$,

$r(t) = t^2$ [$\frac{1}{2}$ mark]

By part (a), $y_1 = t^3$, $y_2 = t^3\ln|t|$ are two lin. indep. solns of the corresponding homogeneous ODE.

$y_p = u_1y_1 + u_2y_2$, where [1 mark for formula]

$u_1' = \frac{W_1}{W} = \frac{-y_2\,r(t)}{W}$

$u_2' = \frac{W_2}{W} = \frac{y_1\,r(t)}{W}$

$W = y_1y_2' - y_1'y_2 = t^3(t^2 + \cancel{3t^2\ln|t|}) - \cancel{3t^2\cdot t^3\ln|t|} = t^5$ [1 mark]

$\therefore$ $u_1' = -\frac{t^3\ln|t|\cdot t^2}{t^5} = -\ln|t| \Rightarrow u_1 = -t\ln|t| + t$ [1 mark]

$u_2' = \frac{t^3\cdot t^2}{t^5} = 1 \Rightarrow u_2 = t$

$\therefore$ $y_p = (-\cancel{t\ln|t|} + t)t^3 + \cancel{t\cdot t^3\ln|t|} = t^4$ is a particular soln. [$\frac{1}{2}$ mark]

Hence $\boxed{y_p = t^4}$

Remark: If $r(t)$ is taken as t^4 instead of t^2 and rest of the calculations are done correctly using that then 2.5 marks given for part (b)

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Question 6: [6 marks]
Solve the following initial value problem (IVP) using the Laplace transform:

$$y'' + 7y' + 12y = u(t-2) + \delta(t-3); \quad y(0) = 1, y'(0) = 3.$$

Here δ denotes the Dirac delta function and u denotes the Heaviside function.

Question 6:[6 marks]
Solve the following initial value problem (IVP) using the Laplace transform:

$$y'' + 7y' + 12y = u(t-2) + \delta(t-3); \qquad y(0) = 1, y'(0) = 3.$$

Here δ denotes the Dirac delta function and u denotes the Heaviside function.

Solution:

STEP 1: Taking Laplace transform of the given ODE.

$$\mathcal{L}\{y''\}(s) + 7\mathcal{L}\{y'\}(s) + 12\mathcal{L}(\{y\}(s) = \mathcal{L}\{u(t-2)\}(s) + \mathcal{L}\{\delta(-3)\}(s).$$

STEP 2: Simplifying the above equation and computing $\mathcal{L}\{y\}(s)$ using the following facts:

1) $\mathcal{L}\{y''\}(s) = s^2\mathcal{L}y(s) - sy(0) - y'(0)$
2) $\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\}(s) - y(0)$
3) $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}$
4) $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

This gives

$$\begin{aligned} s^2\mathcal{L}y(s) - sy(0) - y'(0) + 7\left(s\mathcal{L}\{y\}(s) - y(0)\right) + 12\mathcal{L}\{y\}(s) &= \frac{e^{-2s}}{s} + e^{-3s} \\ (s^2 + 7s + 12)\mathcal{L}\{y\}(s) - s - 10 &= \frac{e^{-2s}}{s} + e^{-3s} \end{aligned}$$

whence

$$\mathcal{L}\{y\}(s) = \frac{e^{-2s}}{s(s+4)(s+3)} + \frac{e^{-3s}}{(s+4)(s+3)} + \frac{s+10}{(s+4)(s+3)}.$$

STEP 3: Taking the inverse Laplace transform.

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)(s+3)}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+4)(s+3)}\right\} \\ + \mathcal{L}^{-1}\left\{\frac{s+10}{(s+4)(s+3)}\right\}. \end{aligned}$$

Using

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = u(t-a)\mathcal{L}^{-1}\left\{F(s)\right\}(t-a) = u(t-a)f(t-a)$$

we get

$$\begin{aligned} y(t) &= u(t-2)\mathcal{L}^{-1}\left\{\frac{1}{12s} + \frac{1}{4(s+4)} - \frac{1}{3(s+3)}\right\}(t-2) \\ &+ u(t-3)\mathcal{L}^{-1}\left\{\frac{1}{s+3} - \frac{1}{s+4}\right\}(t-3) + \mathcal{L}^{-1}\left\{\frac{7}{s+3} - \frac{6}{s+4}\right\} \\ &= u(t-2)\left(\frac{1}{12} + \frac{1}{4}e^{-4(t-2)} - \frac{1}{3}e^{-3(t-2)}\right) + u(t-3)\left(e^{-3(t-3)} - e^{-4(t-3)}\right) \\ &+ 7e^{-3t} - 6e^{-4t} \end{aligned}$$

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Question 8: [6 marks]
Solve the following system of linear differential equations (**without** using Laplace transform):

$$\begin{aligned} x_1' &= 7x_1 - 3x_2 + x_3 \\ x_2' &= 8x_1 - 3x_2 + 2x_3 \\ x_3' &= -x_1 + 3x_3. \end{aligned}$$

Sol^n

$\underline{Q_n\text{-}8}$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 7 & -3 & 1 \\ 8 & -3 & 2 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 7 & -3 & 1 \\ 8 & -3 & 2 \\ -1 & 0 & 3 \end{pmatrix}$$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-7 & 3 & -1 \\ -8 & \lambda+3 & -2 \\ 1 & 0 & \lambda-3 \end{vmatrix} = 0 \quad \text{——} \ 1$$

$$\Rightarrow -6 + (\lambda+3) + (\lambda-3)\left[(\lambda-7)(\lambda+3)+24\right] = 0$$

$$\Rightarrow (\lambda-3)(\lambda-7)(\lambda+3) + 25\lambda - 3\times 25 = 0$$

$$\Rightarrow (\lambda-3)\left[(\lambda-7)(\lambda+3)+25\right] = 0$$

$$\Rightarrow (\lambda-3)(\lambda-2)^2 = 0$$

$$\lambda = 2, 2, 3 \quad \text{——} \ 1$$

$\underline{\text{For } \lambda = 3}$ Let $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$4x_1 - 3x_2 + x_3 = 0$$
$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-x_1 = 0$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ 3x_2 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad \text{——} \ 1$$

$\underline{\text{For } \lambda = 2}$ let $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\left.\begin{aligned} 5x_1 - 3x_2 + x_3 &= 0 \\ 8x_1 - 5x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_3 &= x_1 \\ 6x_1 &= 3x_2 \end{aligned}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{——} \ 1$$

Now solve $(A-2I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ (*), then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ or any vector $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ satisfying (*) —— 1

The general sol^n becomes.

$$x(t) = c_1 e^{3t}\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{2t}\left[t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right] \quad \text{——} \ 1$$

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Question 9: [3 marks]
Suppose the function given by the power series $\sum_{n=0}^{\infty} a_n x^n$ is the solution of the following initial value problem (IVP):

$$y'' + xy' + x^2 y = 1 + x; \quad y(0) = 7, y'(0) = 11.$$

Find the values of a_i for $i \leq 5$.

Question 9: [3 marks]
Suppose the function given by the power series $\sum_{n=0}^{\infty} a_n x^n$ is the solution of the following initial value problem (IVP):

$$y'' + xy' + x^2 y = 1 + x; \quad y(0) = 7, y'(0) = 11.$$

Find the values of a_i for $i \leq 5$.

Solution: [Step-1: 1 mark] Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is the solution of given IVP, we have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

and

$$\begin{aligned} & (\textstyle\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}) + x(\sum_{n=1}^{\infty} n a_n x^{n-1}) + x^2(\sum_{n=0}^{\infty} a_n x^n) &= 1 + x \\ \Rightarrow \quad & (\textstyle\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}) + (\sum_{n=1}^{\infty} n a_n x^n) + (\sum_{n=0}^{\infty} a_n x^{n+2}) &= 1 + x \quad - - - (*) \end{aligned}$$

[Step-2: 1 mark] Using initial conditions we get the following:
(1) $y(0) = 7 \Rightarrow a_0 = 7$.
(2) $y'(0) = 11 \Rightarrow a_1 = 11$.
[Step-3: 1 mark] Comparing the constant term, coefficients of x, x^2, x^3 on both sides of $(*)$, we get
(1) $2a_2 = 1 \Rightarrow a - 2 = 1/2$.
(2) $6a_3 + a_1 = 1 \Rightarrow 6a_3 + 11 = 1 \Rightarrow a_3 = -10/6 = -5/3$.
(3) $12a_4 + 2a_2 + a_0 = 0 \Rightarrow 12a_4 + 1 + 7 = 0 \Rightarrow a_4 = -2/3$.
(4) $20a_5 + 3a_3 + a_1 = 0 \Rightarrow 20a_5 - 5 + 11 = 0 \Rightarrow a_5 = -6/20 = -3/10$.

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5/7

Question 7: [4+2 marks]
Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and the inverse Laplace transform.

(a) Using convolution property of the Laplace transform, find $\mathcal{L}^{-1}\left(\dfrac{4}{(s^2+4s+8)^2}\right)$.

(b) Show that $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)(s) = \dfrac{1}{s}\mathcal{L}(f)(s)$.

MTL 101, MAJOR, PROBLEM 7

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and the Laplace inverse.

(a) Using the convolution property of the Laplace transform find $\mathcal{L}^{-1}\left(\frac{4}{(s^2+4s+8)^2}\right)$

Solution. Let $\mathcal{L}(f(t) = F(s) = \frac{4}{(s^2+4s+8)^2} = \frac{4}{((s+2)^2+2^2)^2}$. Then

$$\mathcal{L}(e^{2t}f(t)) = F(s-2) = \left(\frac{2}{s^2+2^2}\right)^2$$

Thus

$$e^{2t}f(t) = \mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right)^2 = g(t)*g(t)$$

where $g(t) = \sin 2t$ and $\mathcal{L}(\sin 2t) = \frac{2}{s^2+2^2}$. Thus

$$\begin{aligned}
e^{2t}f(t) = g(t)*g(t) &= \int_0^t \sin 2\tau sin2(t-\tau)d\tau \\
&= \int_0^t 1/2\left(\cos(2\tau-2(t-\tau)) - \cos(2\tau+2(t-\tau))\right)d\tau \\
&= 1/2\int_0^t (\cos(4\tau-2t) - \cos 2t)d\tau \\
&= \frac{1}{4}\sin 2t - \frac{1}{2}t\cos 2t
\end{aligned}$$

Thus $f(t) = \frac{1}{4}e^{-2t}\sin 2t - \frac{1}{2}e^{-2t}t\cos 2t$ □

(b) Show that $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right)(s) = \frac{1}{s}\mathcal{L}(f)(s)$

Solution. Write $g(t) = \int_0^t f(\tau)d\tau$. Then $g'(t) = f(t)$ and $g(0) = 0$.

$$\begin{aligned}
\mathcal{L}(g') &= \mathcal{L}(f) \\
s\mathcal{L}(g) - g(0) &= F(s) \\
\mathcal{L}(g) &= \frac{\mathcal{L}(f)(s)}{s}
\end{aligned}$$

Remark 0.1. You can also solve the problem using integration by parts or by convolution.

□

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Question 6: [4 marks]
Consider the following ordinary differential equation (ODE):

$$\cos y \; dx + \cot x \; \sin y \; dy = 0.$$

Write $M = \cos y$ and $N = \cot x \; \sin y$.

(a) Using the fact that $(M_y - N_x)/N$ is a function of x only, find an integrating factor of the ODE so that it becomes exact.

(b) Solve the ODE with the following initial conditions: $y(\pi/4) = \pi/4$.

Sol^n

Q6 $\quad M dx + N dy = 0 \qquad y(\pi/4) = \pi/4.$

$M = \cos y, \; N = \cot x \sin y.$

$$\frac{M_y - N_x}{N} = \frac{-\sin y + \sin y \csc^2 x}{\cot x \sin y} = \frac{\sin y \cot^2 x}{\cot x \sin y}$$

$$= \cot x \qquad [\tfrac{1}{2}]$$

I.F. $\quad e^{\int \cot x \, dx} = e^{+\ln \sin x} = \sin x$ $\quad [2]$

So, $\quad \sin x \cos y \, dx + \cos x \sin y \, dy = 0$ is exact

So $\quad u(x,y) = c \quad$ is a sol^n where

$u_x = \sin x \cos y \quad \& \quad u_y = \cos x \sin y \rightarrow (*)$

Integrating

$u = -\cos x \cos y + h(y)$

so that $\quad u_y = \cos x \sin y + h'(y) \qquad [\tfrac{1}{2}]$

Comparing with $(*)$ $\quad h'(y) = 0 \Rightarrow h(y) = c$

Thus ~~sol~~ the general solution is

$u = \cos x \cos y = c \qquad [\tfrac{1}{2}]$

So, using initial conditions: $\frac{1}{2} = c$

Thus the sol^n of the IVP is

given by $\quad \cos x \cos y = \frac{1}{2} \qquad [\tfrac{1}{2}]$

or $\quad y = \cos^{-1}\left(\frac{1}{2} \sec x\right)$

Remark: part a) is of $2\frac{1}{2}$ & part b) is of $1\frac{1}{2}$.

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Question 7: [2+2+2 marks]

Justify whether the following statements are true or false.

(b) Let y_1, y_2 be two solutions of the homogeneous linear ordinary differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x), q(x)$ are continuous functions on an open interval I. Let $W(t) = W(y_1, y_2)(t)$ denote the Wronskian. Then there exist $a, b \in I$ such that $W(a) = 0$ and $W(b) \neq 0$.

(c) Consider the initial value problem (IVP): $y' + p(x)y = q(x); y(x_0) = y_0$ where $p(x)$ and $q(x)$ are continuous functions on $\mathbb{R}$. Then, this IVP satisfies the conditions of the uniqueness theorem.

Question 7(b): (FALSE)

1M

By Abel's theorem

$$W(y_1, y_2)(t) = c\, e^{\int_{x_0}^{t} p(x)\,dx}$$

where $c = W(y_1, y_2)(x_0)$

1M

If $c = 0$ then $W(t) = 0 \quad \forall t \in I$.

If $c \neq 0$ then $W(t) \neq 0 \quad \forall t \in I$

since exponential is always non-zero.

Then we cannot have $a \in I$, $b \in I$ with $W(a) = 0$ & $W(b) \neq 0$.

Question 7(c): (TRUE)

Given IVP is

$$y' = q(x) - p(x)y \;;\; y(x_0) = y_0.$$

1M

Write $y' = f(x, y)$; $y(x_0) = y_0$

where $f(x,y) = q(x) - p(x)y$.

Since $p(x), q(x)$ are continuous on $\mathbb{R}$

$f(x,y)$ is continuous on $\mathbb{R}^2$.

Then

$$\boxed{|f(x,y_1) - f(x,y_2)| = |p(x)|\,|y_1 - y_2|}$$

1M

Consider any closed rectangle containing (x_0, y_0), say $R = \{(x,y) : |x - x_0| \leq 1, |y - y_0| \leq 1\}$

Since $p(x)$ is continuous, and $[x_0 - 1, x_0 + 1]$ is closed bounded interval, it is bounded on $[x_0 - 1, x_0 + 1]$, say by $L > 0$.

Then $f(x,y)$ is Lipschitz w.r.t. y on R.

Therefore the IVP satisfies the condition of the uniqueness theorem.

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Q1) Consider the initial value problem (IVP):

$$\frac{dy}{dx} = (\sin x + x^2)\, y^{1/5}, \quad y(0) = \beta.$$

a) If $\beta > 0$, then prove that the IVP has a unique solution.

b) If $\beta = 0$, then write down two distinct solutions of the IVP.

Sb 4

The given IVP is

$$\frac{dy}{dx} = (\sin x + x^2)\, y^{1/5}, \quad y(0) = \beta.$$

Here $f(x,y) = (\sin x + x^2)\, y^{1/5}$. Notice that in any rectangle R containing $(0,\beta)$, the function f is continuous.

a) Suppose $\beta > 0$. choose a rectangle R with $(0,\beta)$ as centre but does not contain $(0,0)$. $\longrightarrow$ 1

Notice that the partial derivative

$$\frac{\partial f}{\partial y} = \frac{\sin x + x^2}{5y^{4/5}} \longrightarrow 1.5$$

is bounded in R. Thus, ~~by the~~ f is Lipschitz and hence by existence and uniqueness theorem, the given IVP has a unique solution $\longrightarrow$ (0.5)

b) One solution can be obtained by solving the given IVP, i.e, the solution

$$y(x) = \left(\frac{4}{5}\left(1 + \frac{x^3}{3} - \cos x\right)\right)^{5/4}. \longrightarrow 2$$

The other solution is $y(x) = 0$. $\longrightarrow$ 1

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Guess the Problem by looking at the solution. Good luck ;)

2 a) $y' + y\tan x = \cos^2 x \,,\quad y\left(\frac{3\pi}{4}\right) = -1$

I.F. $e^{\int \tan x\, dx} = e^{\log|\sec x|} = |\sec x| = -\sec x$ in a neighbourhood of $x = \frac{3\pi}{4}$. ($\because \sec\left(\frac{3\pi}{4}\right) = -\sqrt{2}$)

So the sol^n of the Linear ODE is (1)

$$y(-\sec x) = \int (-\sec x)\cos^2 x\, dx + c$$

$$\Rightarrow \quad y = \cos x \sin x - c\cos x \qquad (2)$$

Using initial cond^n: $-1 = -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - c\left(-\frac{1}{\sqrt{2}}\right)$

$$\Rightarrow \quad c = \sqrt{2}\left(-\frac{1}{2}\right) = -\frac{1}{\sqrt{2}} \qquad (1)$$

The sol^n is $y = \cos x \sin x + \frac{\cos x}{\sqrt{2}}$

2b) The wronskian of $y_1 = x$, $y_2 = \sin x$ is $x\cos x - \sin x$ (1)

It vanishes at $x = 0 \in (-1,1)$ & nonzero elsewhere e.g. $x = \frac{\pi}{4}$.

If the LI fns x & $\sin x$ ~~where~~ sol^{n}s of $y'' + p(x)y' + q(x)y = 0$, then the wronskian should have been nonzero on $(-1,1)$ (2)

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Q-3.

(a) Given $y_1(x) = x$ is a solution of the following homogeneous linear ODE.

$$(1+x^2)\,y'' - 2xy' + 2y = 0$$

find the general solution using the method of reduction of order.

5/4

Sol^n $\Rightarrow$ Given $y_1(x) = x$ is one solution.

Let $y_2(x) = v(x)\,y_1(x)$ be 2nd solution.

i.e., $y_2(x) = xv$

$\therefore$ $y_2' = xv' + v$

$y_2'' = xv'' + 2v'$

$\Rightarrow$ $(1+x^2)(xv''+2v') - 2x(xv'+v) + 2xv = 0.$

Let $w = v'$

then $xw' + x^3w' + 2w = 0$

$\Rightarrow$ $(x+x^3)w' + 2w = 0$

$\Rightarrow$ $w' + \dfrac{2}{x(1+x^2)}w = 0$ ——— ① marks for reduced order equation

$\Rightarrow$ $\dfrac{w'}{w} + \left[\dfrac{2}{x} - \dfrac{2x}{1+x^2}\right] = 0$

Integrating both sides

$\log w + 2\log x - \log(1+x^2) = \log c$

$\Rightarrow$ $\dfrac{wx^2}{1+x^2} = c$

$\Rightarrow$ $w = c\left(\dfrac{1+x^2}{x^2}\right)$

$\Rightarrow$ $v' = c\left(\dfrac{1}{x^2}+1\right)$

Integrating both sides

$v = c\left(-\dfrac{1}{x} + x\right) + c'$ ——— ① marks for finding v

$\Rightarrow$ $y_2 = vx$

$\Rightarrow$ $\boxed{y_2 = c(x^2-1) + c'x}$ ——— ① marks for writing general solution.

is the general solution

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(b) Using the method of variation of parameter find a particular solution of the following non-homogeneous linear ODE:

$$(1+x^2)y'' - 2xy' + 2y = x^3+x.$$

Sol^n → From part (a), we get ~~the~~ fundamental solution of the homogeneous equation as

$y_1 = x$ and $y_2 = x^2-1$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix} = x^2+1$$ — ① marks for Wronskian

Now, $(1+x^2)y'' - 2xy' + 2y = x^3+x$

$\Rightarrow \quad y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = x.$

$\Rightarrow \quad y'' + P(x)y' + Q(x)y = F(x)$

Then Particular solution $Y_p = y_1u_1 + y_2u_2$ where

$$u_1 = -\int \frac{y_2 F(x)}{W(x)} \quad \text{and} \quad u_2 = \int \frac{y_1 F(x)}{W(x)}$$

So, $u_1 = -\int \frac{(x^2-1)x}{(x^2+1)}dx$ — (½) marks for u_1

$= -\int\left(1-\frac{2}{1+x^2}\right)x\,dx = -\frac{x^2}{2} + \log(1+x^2)$

and $u_2 = \int \frac{x^2}{x^2+1}dx$ — (½) marks for u_2

$= \int\left(1-\frac{1}{1+x^2}\right)dx = x - \tan^{-1}x$

So, $Y_p = y_1u_1 + y_2u_2.$ — ① marks for particular solution

$\therefore Y_p = x\log(1+x^2) - \frac{x^3}{2} + x(x^2-1) - (x^2-1)\tan^{-1}x$

$= x\left(\log(1+x^2) - 1\right) + \frac{x^3}{2} - (x^2-1)\tan^{-1}x$

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$8\frac{3}{4}$

Question 4:

Solve the following system of differential equations (without using Laplace transform)

$$x_1' = 5x_1 + 2x_2 + x_3$$

$$x_2' = -8x_1 - 3x_2 - 2x_3$$

$$x_3' = 6x_1 + 4x_2 + 3x_3$$

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Then the above system can be rewritten as,

$X' = AX$, where

$$A = \begin{pmatrix} 5 & 2 & 1 \\ -8 & -3 & -2 \\ 6 & 4 & 3 \end{pmatrix}$$

To solve this system we have to find the eigen values and the corresponding eigenvectors of A.

The characteristic equation is given by,

$$\det(A - \lambda I) = 0$$

ie,

$$\begin{vmatrix} 5-\lambda & 2 & 1 \\ -8 & -3-\lambda & -2 \\ 6 & 4 & 3-\lambda \end{vmatrix} = 0$$

Solving, we get,

$\lambda = 1, \quad 2-i, \quad 2+i$. +2

• For $\boxed{\lambda = 1}$, solving

$$Av_1 = v_1$$

we get

$$v_1 = \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix}$$ [$\because x_3$ was the free variable]

+1

• For $\boxed{\lambda = 2-i}$, solving

$$Av_2 = (2-i)v_2$$

we get

$$v_2 = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} \\ -1-i \\ 1 \end{pmatrix}$$ +1

• For $\boxed{\lambda = 2+i}$, solving

$$Av_3 = (2+i)v_3$$

we get

$$v_3 = \begin{pmatrix} \frac{1}{2} - \frac{i}{2} \\ -1+i \\ 1 \end{pmatrix}$$ +1

Then the general solution is

$$x(t) = C_1 e^t v_1 + C_2 e^{(2+i)t} v_2 + C_3 e^{(2-i)t} v_3$$

+1

15

8/1

Let $f^{(n)}$ denote the n-th derivative of f and $\mathcal{L}$ denote the Lapace transform.

(1) Suppose f is a function such that for all integers $n \geq 0$, $f^{(n)}$ is continuous function on $\mathbb{R}$. Moreover, assume that there exists $M > 0, K > 0$ satisfying $|f^{(n)}(t)| \leq Me^{Kt}$. Using induction prove that for $n \geq 1$,

$$\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0).$$

(2) Compute the Laplace transform $\mathcal{L}(f)$ where f is given by $f(t) = t\sin(2t)$.

Solution of Part (a):
[1 mark] For $n = 1$ the statement is $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$, which we prove now. By definition of Laplace transform and doing integration by parts:

$$\begin{aligned}\mathcal{L}(f')(s) &= \textstyle\int_0^\infty e^{-st}f'(t)dt \\ &= e^{-st}f(t)|_0^\infty + \textstyle\int_0^\infty se^{-st}f(t)dt\end{aligned}$$

[1 mark] Since $|f(t)| \leq Me^{Kt}$ is given, we get $\lim_{t\to\infty} e^{-st}f(t) = 0$ for $s > K$. Then,

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0).$$

We have verified the case $n = 1$.
[1 mark] Now assume that the statement is true for $n = k$, i.e.,

$$\mathcal{L}(f^{(k)})(s) = s^k\mathcal{L}(f)(s) - s^{k-1}f(0) - s^{k-2}f^{(1)}(0) - \cdots - f^{(k-1)}(0).$$

Now we claim that the statement is true for $n = k+1$.
[1 mark] Consider

$$\begin{aligned}\mathcal{L}(f^{(k+1)})(s) &= \mathcal{L}((f^{(k)})')(s) \\ &= s(\mathcal{L}(f^{(k)})(s) - f^{(k)}(0) \qquad \text{(from the case } n = 1 \text{ since } |f^{(k)}(t)| \leq Me^{Kt}) \\ &= s\left(s^k\mathcal{L}(f)(s) - s^{k-1}f(0) - s^{k-2}f^{(1)}(0) - \cdots - f^{(k-1)}(0)\right) - f^{(k)}(0) \\ &= s^{k+1}\mathcal{L}(f)(s) - s^k f(0) - s^{k-1}f^{(1)}(0) - \cdots - f^{(k)}(0).\end{aligned}$$

Thus, the statement for $n = k+1$.
Therefore by principle of mathematical induction, the given statement is true for all $n \geq 1$.

Solution of Part (b):
[1 mark] Recall that $\mathcal{L}(tf(t))(s) = -\frac{d}{ds}\mathcal{L}(f(t)(s)$. Using this we get $\mathcal{L}(t\sin 2t)(s) = -\frac{d}{ds}\mathcal{L}(\sin 2t)(s)$.

[2 mark] Using definition and simple computation gives

$$\mathcal{L}(\sin 2t)(s) = \int_0^\infty e^{-st}t\sin(2t)dt = \cdots = \frac{2}{s^2+4}.$$

Therefore,

$$\mathcal{L}(tf(t))(s) = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}.$$

Solve the following IVP using the Laplace Transform:

$$y'' + 2y = g(t) + \delta(t - \pi/2); y(0) = 1, y'(0) = 0$$

$$g(t) = \begin{cases} 0 & \text{if } t < \pi/2 \\ \sin t & \text{if } t \geqslant \pi/2 \end{cases}$$

Solution. Apply Laplace transform to get

$$\begin{aligned}\mathcal{L}(y'') + 2\mathcal{L}(y) &= \mathcal{L}(g(t) + \mathcal{L}(\delta(t-\pi/2)) \\ s^2\mathcal{L}(y) - sy'(0) - y'(0) + 2\mathcal{L}(y) &= \mathcal{L}(g(t) + \mathcal{L}(\delta(t-\pi/2)) \\ (s^2+2)\mathcal{L}(y) &= s + \mathcal{L}(g(t)) + \mathcal{L}(\delta(t-\pi/2)) \\ \mathcal{L}(y) &= \frac{s}{s^2+2} + \frac{\mathcal{L}(g(t))}{s^2+2} + \frac{\mathcal{L}(\delta(t-\pi/2))}{s^2+2}\end{aligned}$$

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left(\frac{s}{s^2+2} + \frac{\mathcal{L}(g(t))}{s^2+2} + \frac{\mathcal{L}(\delta(t-\pi/2))}{s^2+2}\right) \\ &= \mathcal{L}^{-1}\left(\frac{s}{s^2+2}\right) + \mathcal{L}^{-1}\left(\frac{\mathcal{L}(g(t))}{s^2+2}\right) + \mathcal{L}^{-1}\left(\frac{\mathcal{L}(\delta(t-\pi/2))}{s^2+2}\right) \\ &= \cos\sqrt{2}t + \mathcal{L}^{-1}\left(\frac{\mathcal{L}(g(t))}{s^2+2}\right) + \mathcal{L}^{-1}\left(\frac{\mathcal{L}(\delta(t-\pi/2))}{s^2+2}\right) \\ &= \cos\sqrt{2}t + \mathcal{L}^{-1}\left(\frac{e^{-\pi s/2}}{s^2+2}\right) + \mathcal{L}^{-1}\left(\frac{\mathcal{L}(g(t))}{s^2+2}\right) \\ &= \cos\sqrt{2}t + \mathcal{L}^{-1}\left(e^{-\pi s/2}\frac{1}{\sqrt{2}}\mathcal{L}(\sin\sqrt{2}t)\right) + \mathcal{L}^{-1}\left(\mathcal{L}(g(t)\frac{1}{\sqrt{2}}\mathcal{L}(sin(\sqrt{2}t)\right) \\ &= \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\mathcal{L}(u(t-\pi/2)\sin\sqrt{2}(t-\pi/2))\right) + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\mathcal{L}(g(t))\mathcal{L}(sin(\sqrt{2}t)\right) \\ &= \cos\sqrt{2}t + \frac{1}{\sqrt{2}}u(t-\pi/2)\sin\sqrt{2}(t-\pi/2) + \frac{1}{\sqrt{2}}g(t) * \sin(\sqrt{2}t)\end{aligned}$$

Now computing the convolution we get

$$\begin{aligned}g(t) * \sin(\sqrt{2}t) &= \int_0^t g(\tau)\sin(\sqrt{2}(t-\tau))d\tau \\ &= \begin{cases} 0 & t < \pi/2 \\ \sqrt{2}\sin t - \sqrt{2}\cos(\sqrt{2}(t-\pi/2)) & t \geqslant \pi/2 \end{cases}\end{aligned}$$

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Question 1:[4 marks]

Consider the following initial value problem (IVP)

$$\frac{dy}{dx} = x(1+y); \quad y(0) = 1. \tag{1}$$

a) Using Picard's method compute the first three successive approximations of the above IVP.

b) Write the n-th approximation and justify it by induction (PMI).

a) By Picard's method, we have

$$y_0(x) = y(x_0), \text{ and } y_n(x) = y(x_0) + \int_{x_0}^{x} f\left(s, y_{n-1}(s)\right) ds, \quad [0.5 \text{ Marks }].$$

This gives

$$y_1(x) = y(0) + \int_0^x f\left(s, y_0(s)\right) ds = 1 + \int_0^x s(1+1)ds = 1 + x^2 \quad [0.5 \text{ Marks}].$$

$$y_2(x) = y(0) + \int_0^x f\left(s, y_1(s)\right) ds = 1 + \int_0^x s(1+1+s^2)ds = 1 + x^2 + \frac{x^4}{4} \quad [0.5 \text{ Marks}].$$

$$y_3(x) = y(0) + \int_0^x f\left(s, y_2(s)\right) ds = 1 + \int_0^x s(1+1+s^2+\frac{s^4}{4})ds = 1 + x^2 + \frac{x^4}{4} + \frac{x^6}{4.6}. \quad [0.5 \text{ Marks}].$$

b) The $n-$th Picard iteration is

$$y_n(x) = 1 + \sum_{j=1}^{n} \frac{x^{2j}}{2^{j-1} j!}. \quad [1 \text{ Marks}]$$

Now we will justify this using PMI. Clearly from part a) we have $y_1(x) = 1 + x^2$. Now suppose that it is true for $n = k$ i.e.,

$$y_k(x) = 1 + \sum_{j=1}^{k} \frac{x^{2j}}{2^{j-1} j!}, \quad [0.5 \text{ Marks}]$$

Now consider

$$\begin{aligned}
y_{k+1} &= 1 + \int_0^x s\left(1 + y_k(s)ds\right. \\
&= 1 + \int_0^x s\left(1 + 1 + \sum_{j=1}^{k} \frac{s^{2j}}{2^{j-1} j!}\right) ds \\
&= 1 + x^2 + \sum_{j=1}^{k} \frac{x^{2j+2}}{2^{j-1}(2j+2) j!} \\
&= 1 + x^2 + \sum_{j=1}^{k} \frac{x^{2j+2}}{2^{j}(j+1)!} \\
&= 1 + x^2 + \sum_{j=2}^{k+1} \frac{x^{2j}}{2^{j-1} j!} = 1 + \sum_{j=1}^{k+1} \frac{x^{2j}}{2^{j-1} j!}
\end{aligned}$$

Hence holds true for $n = k + 1$. [0.5 Marks]

Therefor, by PMI, we have

$$y_n(x) = 1 + \sum_{j=1}^{n} \frac{x^{2j}}{2^{j-1} j!}.$$

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Suppose $f(x,y) = x - y^2 + e^x$ is defined on the rectangle $R = \{(x,y) : |x| \le 1, |y| \le 1\}$. Using the existence theorem for the initial value problem

$$\frac{dy}{dx} = f(x,y); y(0) = 0$$

find an $\alpha > 0$ such that there exists a solution of the IVP on the interval $(-\alpha, \alpha)$.

MS

Marking scheme for Q2: $\frac{dy}{dx} = x - y^2 + e^x$

$|x| \le 1$, $|y| \le 1$ $\quad y(0) = 0$.

- The function $f(x,y) = x - y^2 + e^x$ is continuous [$\frac{1}{2}$]
- On the given rectangle $|f(x,y)| \le 1 + e$ [$\frac{1}{2}$]
- By existence theorem, there is a soln on $(-\alpha, \alpha)$ where $\alpha = \text{Min}\{a, \frac{b}{M}\}$ where $a = 1$, $b = 1$, $M \geq 1 + e$. [$\frac{1}{2}$]
- So $\alpha \le \frac{1}{1+e}$ (e.g. $\alpha = \frac{1}{1+e}$) [$\frac{1}{2}$]

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Happy hunting!

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Ques 3 of re-quiz

Given IVP is $\frac{dy}{dx} = f(x,y)$; $y(0) = 0$

where $f(x,y) = x\sqrt{|y|}$

(a) $f(x,y) = x\sqrt{|y|}$ is continuous on $\mathbb{R}^2$ and hence in any rectangle containing $(x_0,y_0) = (0,0)$ [1]

So, the existence theorem is applicable and it ensures the existence of at least one solution to the IVP.

(b) For $y_1 \neq y_2$ and $x \neq 0$,

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \frac{|x|\,\left|\sqrt{|y_1|} - \sqrt{|y_2|}\right|}{|y_1 - y_2|}$$

$$= \frac{|x|\,\left|\,|y_1| - |y_2|\,\right|}{|y_1 - y_2|\left(\sqrt{|y_1|} + \sqrt{|y_2|}\right)}$$

$$= \frac{|x|}{\sqrt{y_1} + \sqrt{y_2}} \quad \text{for any } y_1 > 0, y_2 > 0, y_1 \neq y_2$$

$\to \infty$ as $y_1 \to 0+, y_2 \to 0+$

Thus, $f(x,y)$ does not satisfy the Lipschitz condition on any rectangle containing $(x_0,y_0) = (0,0)$. [1]

$\therefore$ The uniqueness theorem is not applicable

(c) $\boxed{y \equiv 0}$ is clearly a solution to the IVP. [1]

To find another solution, assume $y > 0$. Then

$$\int \frac{dy}{\sqrt{y}} = \int x\,dx \Rightarrow 2\sqrt{y} = \frac{x^2}{2} + C$$

$y(0) = 0 \Rightarrow C = 0$

$$\therefore y = \frac{x^4}{16}$$

We can easily verify that $\boxed{y = \frac{x^4}{16}}$ is a solution to the IVP. [1]

Remark: $y = -\frac{x^4}{16}$ is NOT a solution because

$\frac{dy}{dx} = -\frac{x^3}{4}$ but $x\sqrt{|y|} = x\left(\frac{x^2}{4}\right) = \frac{x^3}{4}$

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Question 2: **(5 Marks)** Solve the following IVP

$$t^2x'' + 5tx' + 4x = 0, \quad x(1) = 0, \quad x'(1) = 1.$$

Sol $\cancel{7}$

Ques $t^2x'' + 5tx' + 4x = 0$, $x(1) = 0$, $x'(1) = 1$

Sol^n Making a change of (independent) variable using $t = e^s$, we get

1

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{dx}{dt}e^s = \frac{dx}{dt}t$$

$$\frac{d^2x}{ds^2} = \frac{d^2x}{dt^2}t^2 + \frac{dx}{dt}t$$

$\Rightarrow$ $t^2x'' + 5tx' + 4x = 0$ reduces to

$$\left(\frac{d^2x}{ds^2} - \frac{dx}{ds}\right) + 5\frac{dx}{ds} + 4x = 0$$

or $$\frac{d^2x}{ds^2} + 4\frac{dx}{ds} + 4x = 0$$

1 Trying the sol^n as $x = e^{ms}$, we get

$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$

two equal roots

1 General sol^n is

$$x(s) = c_1 e^{-2s} + c_2 s e^{-2s}$$

1 Using $s = \ln t$, we get the sol^n as

$x(t) = c_1 t^{-2} + c_2 t^{-2}\ln t$, $x'(t) = \frac{-2c_1}{t^3} + \frac{c_2}{t^3} - \frac{2c_2\ln t}{t^3}$

1 Using $x(1) = 0$ & $x'(1) = 1$, we get

$c_1 = 0$, $-2c_1 + c_2 = 1$ $\Rightarrow$ $c_1 = 0$, $c_2 = 1$

$\therefore$ Solution is $\boxed{x(t) = \frac{\ln t}{t^2}}$

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Question 4: (5 Marks)

(a) Discuss the Existence and Uniqueness of the solutions of the following Initial Value Problem:

$$x'(t) = t^2 + \sqrt{\tan(t + x(t))}, \quad x\left(\frac{\pi}{6}\right) = 0.$$

(b) Let $x(t)$ be a nonnegative function with

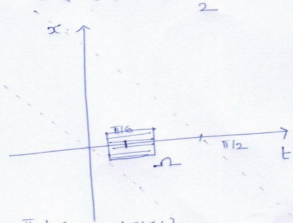
$$x(t) \le L \int_{t_0}^{t} x(s)\, ds,$$

where $t_0 \le t$ and L is a nonnegative constant. Is it true that $x(t) = 0$ for all t? Justify.

4 (a) Here $f(t, x) = t^2 + \sqrt{\tan(t+x)}$, $(t_0, x_0) = (\pi/6, 0)$

For $f = f(t, x)$ to be a well-defined continuous function, we need $\tan(\cdot) \ge 0$.

Since we need a "region" containing $(\pi/6, 0)$, consider $0 < t + x < \frac{\pi}{2}$



Choose $a > 0$, $b > 0$ sufficiently small so that the closed and bounded "rectangle"

$\Omega := \{(t, x) : |t - \frac{\pi}{6}| \le a, \ |x| \le b\}$

lies in $0 < t + x < \pi/2$.

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{\tan(t+x)}} \sec^2(t+x)$$

Note that $\frac{\partial f}{\partial x}$ is continuous on Ω, and hence bounded. Consequently, f is Lipschutz in x-variable on Ω.

By the existence and uniqueness theorem, the given IVP has a unique solution $\phi(t)$ in an interval $|t - \pi/6| \le \alpha$ for a suitable α. □

4 (b) If $x = x(t)$ is a continuous function, then given conditions imply $x \equiv 0$.

Let x be a continuous nonnegative function.

$$u(t) := \int_{t_0}^{t} x(s)\, ds.$$

By the fundamental theorem of calculus, u is differentiable (Here we use the assumption, x is continuous).

$$\frac{du}{dt} = x(t) \le L \int_{t_0}^{t} x(s)\, ds = L u(t$$

ie, $\frac{du}{dt} - L u(t) \le 0$.

This implies,

$$\frac{d}{dt}\left(u\, e^{-L(t-t_0)}\right) \le 0.$$

consequently,

$$e^{-L(t-t_0)} u(t) \le u(t_0) = 0.$$

ie $u(t) \le 0 \quad \forall \ t \ge t_0$.

This implies $u(t) = 0$ and hence $x \equiv 0$ ($\because x$ being nonnegative continuous fn)

If the additional assumption of continuity is not there, then it is not true, in general that $x \equiv 0$; give a counter example. □

Question 5: (5 Marks) Let $f_1, f_2 \in C^1(I)$ (set of all continuously differentiable functions on open interval I) be given functions such that $W(f_1, f_2)(t) = 0$ for all $t \in I$. Show that there exists an interval $I_0 \subseteq I$ such that f_1 and f_2 are linearly dependent on I_0.

$f_1, f_2 \in C^1(I)$

Given $W(f_1, f_2)(t) = 0 \quad \forall \ t \in I$

To show that there exists an interval $I_0 \subseteq I$ s.t. f_1 & f_2 are linearly dependent on I_0.

If either f_1 or f_2 is identically zero on I,

say $f_1(t) = 0 \quad \forall \ t \in I$

1 mark

then $c_1 f_1(t) + c_2 f_2(t) = 0 \quad \forall t \in I$ holds for any $c_1 \in \mathbb{R}/\{0\}$ & $c_2 = 0$.

$\Rightarrow$ f_1 & f_2 are linearly dependent on I.

4 marks

If f_1 is not identically zero on I

$\Rightarrow \exists \ t_0 \in I$ such that $f_1(t_0) \neq 0$

w.l.o.g suppose $f_1(t_0) > 0$.

Since f_1 is continuous on I

$\Rightarrow \exists$ an interval $I_0 \subseteq I$ s.t. $f_1(t) > 0 \quad \forall \ t \in I_0$

Now $W(f_1, f_2)(t) = 0 \quad \forall \ t \in I$

$$\Rightarrow W(f_1, f_2)(t) = 0 \quad \forall \ t \in I_0$$

$$f_1(t) f_2'(t) - f_2(t) f_1'(t) = 0 \quad \forall \ t \in I_0$$

Upon dividing by f_1^2 (keeping in mind $f_1(t) > 0 \ \forall t \in I_0$)

we get $$\frac{f_1(t) f_2'(t) - f_2(t) f_1'(t)}{f_1^2(t)} = 0 \quad \forall \ t \in I_0$$

$\Rightarrow \left(\frac{f_2(t)}{f(t)}\right)' = 0 \quad \forall \ t \in I_0 \quad \Rightarrow \quad f_2(t) = \underset{\in \mathbb{R}}{c} f_1(t) \quad \forall t \in I_0$

$\Rightarrow$ f_1 & f_2 are linearly independent on I_0.

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Question 6: (5 Marks) For $\alpha \in \mathbb{R}$, let

$$L(x(t)) = t^3 x''' + 3(1-\alpha)t^2 x'' + (3\alpha^2 - 3\alpha + 1)t x' - \alpha^3 x, \quad t > 0.$$

(a) Reduce the differential equation $L(x(t)) = 0$ into a differential equation with constant coefficients, say $M(x(t)) = 0$.

(b) Find the general solution of differential equation $M(x(t)) = 0$.

(c) For $\alpha = 1$, use variation of parameters method to find the general solution of $L(x(t)) = 1$.

For $\alpha \in \mathbb{R}$, let

$$L(x(t)) = t^3 x''' + 3(1-\alpha)t^2 x'' + (3\alpha^2 - 3\alpha + 1)t\alpha' - \alpha^3 x, t > 0 \tag{1}$$

Reduction of (1) *into DE with constant coefficients:* Set $t = e^s$, $\frac{dt}{ds} = e^s$, $\frac{d^2t}{ds^2} = e^s$. Now we have

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = x't$$

$$\frac{d^2x}{ds^2} = \frac{d^2x}{dt^2}\left(\frac{dt}{ds}\right)^2 + x't = x''t^2 + x't$$

The above two equations leads to $t^2x'' = \frac{d^2x}{ds^2} - \frac{dx}{ds}$. Now

$$\frac{d^3x}{ds^3} = x'''t^3 + 3x''t^2 + x't$$

$$t^3x''' = \frac{d^3x}{ds^3} - 3\left[\frac{d^2x}{ds^3} - \frac{dx}{ds}\right] - \frac{dx}{ds}$$

$$= \frac{d^3x}{ds^3} - 3\frac{d^2x}{ds^3} + 2\frac{dx}{ds}$$

Therefore $L(x(t)) = 0$ reduces to

$$\frac{d^3x}{ds^3} - 3\frac{d^2s}{dx^2} + 2\frac{dx}{ds} + 3(1-\alpha)\left[\frac{d^2x}{ds^2} - \frac{dx}{ds}\right] + (3\alpha^2 - 3\alpha + 1)\frac{dx}{ds} - \alpha^3 x = 0$$

Further simplication leads to $\mathbf{\frac{d^3x}{ds^3} - 3\alpha\frac{d^2x}{ds^2} + 3\alpha^2\frac{dx}{ds} - \alpha^3 x = 0}$ **(1 mark)**.

General solution to DE: Let $x = e^{ms}$. The characteristic polynomial is $m^3 - 3\alpha m^2 + 3\alpha^2 m - \alpha^3 = 0$ or $(m-\alpha)^3 = 0$. The roots of this equation are $m = \alpha, \alpha, \alpha$. Thus the general solution is $x(s) = c_1x_1(s) + c_2x_2(s) + c_3x_3(s)$ where $x_1(s)e^{\alpha s}, x_2(s) = se^{\alpha s}, x_3(s) = s^2e^{\alpha s}$. Thus the general solution is

$$x(s) = c_1e^{\alpha s} + c_2se^{\alpha s} + c_3s^2e^{\alpha s}$$

$$x(t) = c_1t^\alpha + c_2t^\alpha ln(t) + c_3t^\alpha(ln(t))^2 \textbf{(1 mark)}$$

General solution of $L(x(t)) = 1$ *for* $\alpha = 1$: It suffices to find the general solution of $M(x(s)) = \frac{d^3x}{ds^3} - 3\frac{d^2x}{ds^2} + 3\frac{dx}{ds} - x = 1$ (with $t = e^s$). The general solution to homogeneous DE $M(x(s)) = 0$ is $x_h(s) = c_1x_1(s) + c_2x_2(s) + c_3x_3(s)$ where $x_1(s)e^{\alpha s} = e^s, x_2(s) = se^{\alpha s} = se^s, x_3(s) = s^2e^{\alpha s} = s^2e^s$.

We try the particular solution as

$$x_p(s) = u_1(s)x_1(s) + u_2(s)x_2(s) + u_3(s)x_3(s)$$

$$x_p'(s) = u_1(s)x_1'(s) + u_2(s)x_2'(s) + u_3(s)x_3'(s) + u_1'(s)x_1(s) + u_2'(s)x_2(s) + u_3'(s)x_3(s)$$

Set

$$u_1'(s)x_1(s) + u_2'(s)x_2(s) + u_3'(s)x_3(s) = 0 \tag{2}$$

Now

$$x_p''(s) = u_1(s)x_1''(s) + u_2(s)x_2''(s) + u_3(s)x_3''(s) + u_1'(s)x_1'(s) + u_2'(s)x_2'(s) + u_3'(s)x_3'(s)$$

Set

$$u_1'(s)x_1'(s) + u_2'(s)x_2'(s) + u_3'(s)x_3'(s) = 0 \tag{3}$$

Then

$$x_p'''(s) = u_1(s)x_1'''(s) + u_2(s)x_2'''(s) + u_3(s)x_3'''(s) + u_1'(s)x_1''(s) + u_2'(s)x_2''(s) + u_3'(s)x_3''(s)$$

Upon subtituting $x_p(s)$ in $M(x(s)) = 1$ and simplifying we get

$$u_1'(s)x_1''(s) + u_2'(s)x_2''(s) + u_3'(s)x_3''(s) = 1 \tag{4}$$

Thus from (2),(3),(4)we have

$$u_1'(s)e^s + u_2'(s)se^s + u_3'(s)s^2e^s = 0$$

$$u_1'(s)e^s + u_2'(s)(s+1)e^s + u_3'(s)(2s+s^2)e^s = 0$$

$$u_1'(s)e^s + u_2'(s)(s+2)e^s + u_3'(s)(s^2+4s+2)e^s = 1$$

Using Cramer's rule we get

$$W(x_1,x_2,x_3) = \begin{vmatrix} e^s & se^s & s^2e^s \\ e^s & (s+1)e^s & (2s+s^2)e^s \\ e^s & (s+2)e^s & (s^2+4s+2)e^s \end{vmatrix} = 2e^{3s}$$

$$u_1'(s) = \frac{\begin{vmatrix} 0 & se^s & s^2e^s \\ 0 & (s+1)e^s & (s^2+2s)e^s \\ 1 & (s+2)e^s & (s^2+4s+2)e^s \end{vmatrix}}{W(x_1,x_2,x_3)} = \frac{s^2e^{2s}}{2e^{3s}} = \frac{1}{2}s^2e^{-s}$$

$$u_2'(s) = \frac{\begin{vmatrix} e^s & 0 & s^2e^s \\ e^s & 0 & (s^2+2s)e^s \\ e^s & 1 & (s^2+4s+2)e^s \end{vmatrix}}{W(x_1,x_2,x_3)} = \frac{-2se^{2s}}{2e^{3s}} = -se^{-s}$$

$$u_3'(s) = \frac{\begin{vmatrix} e^s & se^s & 0 \\ e^s & (s+1)e^s & 0 \\ e^s & (s+2)e^s & 1 \end{vmatrix}}{W(x_1,x_2,x_3)} = \frac{e^{2s}}{2e^{3s}} = \frac{1}{2}e^{-s}$$

Thus we have

$$u_1(s) = 1/2\int s^2e^{-s} = \frac{-s^2}{2}e^{-s} - se^{-s} - e^{-s}$$

$$u_2(s) = se^{-s} + e^{-s}$$

$$u_3(s) = \frac{-1}{2}e^{-s}$$

Thus the particular solution is

$$x_p(s) = u_1x_1 + u_2x_2 + u_3 + x_3$$

$$= e^s(\frac{-s^2}{2}e^{-s} - se^{-s} - e^{-s}) + (se^{-s} + e^{-s})se^s + (\frac{-1}{2}e^{-s})s^2e^s$$

$$= -\frac{-s^2}{2} - s - 1 + s^2 + s - \frac{-s^2}{2} = -1(\textbf{ 3 marks})$$

The general solution of $M(x(s)) = 1$ is $x_h(s) + x_p(s)$. The general solution of $L(x(t)) = 1$ with $\alpha = 1$ is

$$c_1t + c_2t\ln t + c_3t(\ln t)^2 - 1$$

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Question 7: (5 Marks) Find the general solution of the system of ODEs:

$$X' = AX$$

with

$$A = \begin{pmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{pmatrix}.$$

Q7

$A = \begin{bmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ $\quad \det(A - \lambda I) = \det\left(\begin{bmatrix} 8-\lambda & 12 & -2 \\ -3 & -4-\lambda & 1 \\ -1 & -2 & 2-\lambda \end{bmatrix}\right)$

$= (8-\lambda)\left[(-4-\lambda)(2-\lambda) + 2\right] - 12\left[-3(2-\lambda) + 1\right] - 2\left[6 - (4+\lambda)\right]$

$= 0 \Rightarrow -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = 0$

$\Rightarrow \lambda = 2, 2, 2$ (1)

First eigenvector $(A - 2I)\vec{u} = 0$

$\Rightarrow \begin{bmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad 6u_1 + 12u_2 - 2u_3 = 0$

$\Rightarrow -u_1 - 2u_2 = 0$

$\Rightarrow -2u_3 = 0 \Rightarrow u_3 = 0$

$u_1 = -2u_2$

take $u_2 = k = 1$

$\Rightarrow \vec{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\vec{x}^{(1)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{2t}$ (1)

$(A - 2I)\vec{v} = \vec{u} \Rightarrow \begin{bmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow v_1 = -2v_2 \Rightarrow v_3 = 1 \Rightarrow \vec{v} = \begin{bmatrix} -2k \\ k \\ 1 \end{bmatrix}$ taking $k = 1$

we get $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\vec{x}^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} t e^{2t} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} e^{2t}$ (1)

$[A - 2I](\vec{w}) = \vec{v} \Rightarrow \begin{bmatrix} 6 & 12 & -2 \\ -3 & -6 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow w_1 = -2w_2 - 1, \quad 3 + w_3 = 1 \Rightarrow w_3 = -4$

$\Rightarrow$ taking $w_2 = 1$ we get $\vec{w} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$

$\vec{x}^{(3)}(t) = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} t e^{t} + \begin{pmatrix} -3 \\ 1 \\ -4 \end{pmatrix} e^{2t}$ (1)

General Solution

$\vec{x} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} + c_3 \vec{x}^{(3)}$

$= c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} t e^{2t} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} e^{2t}\right]$

$+ c_3 \left[\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{2t} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} t e^{2t} + \begin{pmatrix} -3 \\ 1 \\ -4 \end{pmatrix} e^{2t}\right]$ (1)

No study = No Details = No Marks = No good grades = No job = Business = More money = Happy life

$\Rightarrow$ No study $\Rightarrow$ Happy life ☺

S5/4

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Question 8: (5 Marks) Find the general solution of

$$x'' + tx' + x = 0$$

using series solution methods.

$S\delta^{4}$

(8) $y'' + xy' + y = 0$

Let $y = \sum c_k x^k$ Then $y' = \sum k c_k x^{k-1}$ and

$y'' = \sum k(k-1) c_k x^{k-2}$

Substituting in the Equation we get

$$\sum \left[(k+2)(k+1) c_{k+2} + k c_k + c_k \right] x^k = 0$$

$\therefore$ $(k+2)(k+1) c_{k+2} + (k+1) c_k = 0$

$\Rightarrow$ The recurrence relation

$$c_{k+2} = -\frac{c_k}{(k+2)} \qquad k = 0, 1, 2, \ldots$$

(2 marks)

Therefore,

$c_2 = \frac{-c_0}{2}$, $c_3 = \frac{-c_1}{3}$

$c_4 = \frac{-c_2}{4} = \frac{c_0}{4 \cdot 2} = \frac{c_0}{2^2 2!}$; $c_5 = \frac{-c_3}{5} = \frac{c_1}{3 \times 5}$

$c_6 = -\frac{c_0}{6 \cdot 4 \cdot 2} = -\frac{c_0}{2^3 3!}$ (1 mark) ~~1 mark~~

Hence $c_{2m} = \frac{(-1)^m c_0}{2^m m!}$, $c_{2m+1} = \frac{(-1)^n c_1}{3 \cdot 5 \cdot 7 \cdot 2m+1}$

$\phi_1 = \sum c_{2m} x^{2m} + \sum c_{2m+1} x^{2m+1}$

is the general solution. (2 marks)

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1. Consider the following IVP [4 = 2 + 2]

$$y' = \sqrt{t} + y^2, \quad y(0) = y_0 = 0.$$

(a) Discuss the existence and uniqueness of this IVP.

(b) Find the first two Picard's iterations y_1 and y_2.

4

Q1 b) $y_{n+1} = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds$ is the ~~Pik~~ Picard's formula. Hence

$$y_1 = 0 + \int_0^t (\sqrt{s} + 0^2)\, ds = \frac{2}{3} t^{3/2}.$$

$$y_2 = 0 + \int_0^t \sqrt{s} + \left(\frac{2}{3} s^{3/2}\right)^2 ds$$

$$= \frac{2}{3} t^{3/2} + \frac{4}{9} \times \frac{t^4}{4}$$

$$= \frac{2}{3} t^{3/2} + \frac{t^4}{9}$$

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8

2. Find the general solution of [4]

$$t^2y'' - 4ty' + 6y = \sin(\ln t).$$

Major Q 2 Solution: $t^2y'' - 4ty' + 6y = \sin(\ln t)$.

This is a Cauchy-Euler equation.

Let $s = \ln t$. Then $\frac{dy}{dt} = \frac{ds}{dt}\frac{dy}{ds} = \frac{1}{t}\frac{dy}{ds}$

so that $t\frac{dy}{dt} = \frac{dy}{ds}$ & $\frac{d^2y}{dt^2} = -\frac{1}{t^2}\frac{dy}{ds} + \frac{1}{t}\frac{d}{ds}\left(\frac{dy}{ds}\right)\frac{ds}{dt}$

$\Rightarrow \frac{d^2y}{dt^2} = -\frac{1}{t^2}\cdot\frac{dy}{ds} + \frac{1}{t^2}\times\frac{d^2y}{ds^2}$

$\Rightarrow t^2\frac{d^2y}{dt^2} = \left(\frac{d^2y}{ds^2} - \frac{dy}{ds}\right)$

Substituting, $\left(\frac{d^2y}{ds^2} - \frac{dy}{ds}\right) - 4\times\frac{dy}{ds} + 6y = \sin s$

$\Rightarrow \frac{d^2y}{ds^2} - 5\frac{dy}{ds} + 6y = \sin s.$

The char. poly of the hom. part $y'' - 5y' + 6y = 0$ is $m^2 - 5m + 6 = 0$. Hence $m = 2, 3$ Thus,

$y_h(s) = c_1 e^{2s} + c_2 e^{3s}$. Then the particular solution is given by $y_p(s) = A\sin s + B\cos s$ (using method of ~~Variat~~ undetermined coefficients)

$y_p'(s) = A\cos s + B(-\sin s)$ & $y_p''(s) = -(A\sin s + B\cos s)$

Substituting y_p in the eqn we have

$-A\sin s - B\cos s - 5A\cos s + 5B\sin s + 6A\sin s + 6B\cos s = \sin s$

Comparing the coefficients of $\cos s$ & $\sin s$ we have

$-A + 5B + 6A = 1$ & $-B - 5A + 6B = 0$

$A + B = 1/5,\quad A - B = 0 \Rightarrow A = B = \frac{1}{10}.$

Thus $y_p = \frac{1}{10}(\cos s + \sin s)$. Thus the general solution is

$y(t) = c_1t^2 + c_2t^3 + \frac{1}{10}\{\cos(\ln t) + \sin(\ln t)\}.$

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3. Solve the following IVP [4]

$$y'' - 4y' + 4y = \delta(t-2) + H_1(t), \quad y(0) = y'(0) = 0.$$

Sol^n

Q3. $y'' - 4y' + 4y = \delta(t-2) + H(t)$, $y(0) = y'(0) = 0$

Taking Lapalace Transformation on both sides, with $\mathcal{L}[y] = Y$, we get

$$(s^2 - 4s + 4)\, Y = e^{-2s} + \frac{e^{-s}}{s}$$

$$\Rightarrow \quad Y = \frac{e^{-2s}}{s^2 - 4s + 4} + \frac{e^{-s}}{s(s^2 - 4s + 4)}$$

Consider $f_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 - 4s + 4}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right]$

$$= t e^{2t}$$

so $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2 - 4s + 4}\right] = H_2(t)\, f_1(t-2)$

Consider $f_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2 - 4s + 4)}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2}\right]$$

$$= \frac{1}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}t e^{2t}$$

so $\mathcal{L}^{-1}\left[\frac{e^{-s}}{s(s^2 - 4s + 4)}\right] = H_1(t)\, f_2(t-1)$

so $y = \mathcal{L}^{-1}[Y] = H_2(t)\, f_1(t-2) + H_1(t)\, f_2(t-1)$

$$= \boxed{(t-2)e^{2(t-2)} H_2(t) + \left[\frac{1}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}t e^{2t}\right] H_1(t)}$$

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4. Find the general solution of $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{pmatrix}$. [6]

Q4.

$$A = \begin{pmatrix} 8 & 12 & -2 \\ -3 & -4 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

characteristic polynomial $= \lambda^3 - 6\lambda^2 + 12\lambda - 8$

eigen values are 2, 2, 2

For X_1,

First eigen vector, $v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$(2I - A)v_1 = 0$$

$$X_1 = v_1 e^{2t}$$

For X_2, $(2I - A)v_2 = v_1$

$$v_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$X_2 = (t v_1 + v_2) e^{2t}$$

For X_3, $(2I - A)v_3 = v_2$

$$v_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$X_3 = \left(\tfrac{1}{2}t^2 v_1 + t v_2 + v_3\right) e^{2t}$$

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

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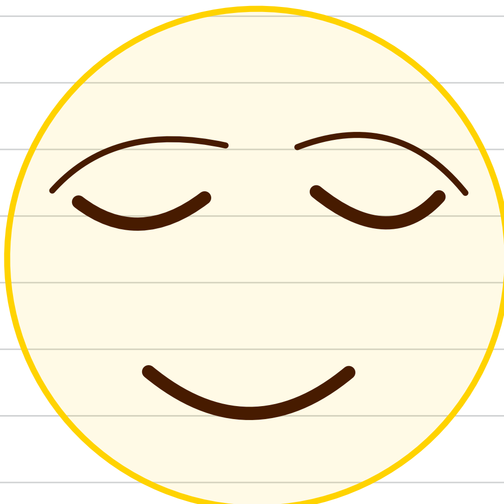
5. Find, by using the method of variation of parameters, the general solution of [4]

$$y'' + y = \tan(t).$$

नहीं मिला



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4

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6. Let

$$f(t) = \mathcal{L}^{-1}\left[\frac{e^{-\pi s/2} - e^{-3\pi s/2}}{s(s^2+2s+5)}\right].$$

Find the value of $f(2\pi)$. [4]

5

$$\frac{1}{s(s^2+2s+5)} = \frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right]$$

$$\therefore\ \mathcal{L}^{-1}\left[\frac{1}{s(s^2+2s+5)}\right] = \frac{1}{5}\left[1 - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)\right] = f_1(t)$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi}{2}s} - e^{-\frac{3\pi}{2}s}}{s(s^2+2s+5)}\right]$$

$$= H_{\frac{\pi}{2}}(t)\, f_1\left(t-\frac{\pi}{2}\right) - H_{\frac{3\pi}{2}}(t)\, f_1\left(t-\frac{3\pi}{2}\right)$$

$$f(2\pi) = f_1\left(3\frac{\pi}{2}\right) - f_1\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{5}\left[\cancel{1} - e^{-\frac{3\pi}{2}}\cos(3\pi) - \frac{1}{2}e^{-\frac{3\pi}{2}}\cancel{\sin(3\pi)}^{0}\right]$$

$$- \frac{1}{5}\left[\cancel{1} - e^{-\frac{\pi}{2}}\cos\pi - \frac{1}{2}e^{-\frac{\pi}{2}}\cancel{\sin\pi}^{0}\right]$$

$$\boxed{f(2\pi) = \frac{1}{5}\left(e^{-\frac{3\pi}{2}} - e^{-\pi/2}\right)}$$

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Find the general solution of [4]

$$x' = 2x + 6y + e^t$$
$$y' = x + 3y - e^t$$

Q6 Find the general soln of

$$x' = 2x + 6y + e^t$$
$$y' = x + 3y - e^t$$

Soln. Consider the homogeneous Problem:

$$\begin{matrix} x' = 2x + 6y \\ y' = x + 3y \end{matrix} = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ eigenvalues are $\lambda = 0, 5$

$$(A - 0I)\vec{x} = A\vec{x} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 2x_1 + 6x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = -3x_2$

So $\vec{u}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Similarly $(A - 5I) = \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 = 2x_2 \Rightarrow \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So general homogeneous soln is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{5t} c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We ~~hoo~~ need to find one non-homogeneous soln.

(a) Using method of undetermined coefficients

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t = \begin{pmatrix} 2c_1 + 6c_2 \\ c_1 + 3c_2 \end{pmatrix} e^t + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$\Rightarrow A = 2, \quad B = -\frac{1}{2} \Rightarrow$ $x(t) = 2e^t$, $y(t) = -\frac{1}{2}e^t$

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Sol 4

Discuss the existence, uniqueness and find the maximal interval of existence of the following initial value problem. [4]

$$y' = \frac{\sqrt{y}\ \sin(\sqrt{y})}{t^2-4}, \qquad y(1) = 1.$$

Q4 Ans $y' = \dfrac{\sqrt{y}\sin\sqrt{y}}{t^2-4}$ $\quad y(1) = 1.$

$f(t,y) = \dfrac{\sqrt{y}\sin\sqrt{y}}{t^2-4}.$

$\left|\dfrac{\partial f}{\partial y}\right| = \left|\dfrac{1}{t^2-4}\left(\dfrac{\cos\sqrt{y}}{2} + \dfrac{\sin\sqrt{y}}{2\sqrt{y}}\right)\right|$

$\leq \left|\dfrac{1}{t^2-4}\right| < \infty$ in any rectangle that does not contain $t=-2,2$ and $y>0$.

So f is Lipschitz in a rectangle around $(1,1)$.

By Picard's, $\exists$ unique solution around $t=1$.

Now solving the Equation

$\dfrac{y'}{\sqrt{y}\sin\sqrt{y}} = \dfrac{1}{t^2-4} = \dfrac{1}{4}\left(\dfrac{1}{t-2} - \dfrac{1}{t+2}\right)$

$\dfrac{2\,du}{\sin u} = \dfrac{1}{4}\left(\dfrac{1}{t-2} - \dfrac{1}{t+2}\right)$

Integrating

$-2\log|\operatorname{cosec} u + \cot u| = \dfrac{1}{4}\log\left|\dfrac{t-2}{t+2}\right| + \log c, \quad t \neq -2,2$

i.e., $\tan\dfrac{\sqrt{y}}{2} = c\left|\dfrac{t-2}{t+2}\right|^{1/8} \quad t \neq -2,2$

$y(1)=1 \Rightarrow \tan\frac{1}{2} = c\cdot\left(\frac{1}{3}\right)^{1/8}, \quad t \neq -2,2$

$1 \in (-2,2)$. So the maximal interval $(-2,2)$.

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Solve the following initial value problem. [3]

$$y' = -\frac{x\tan y}{1+x^2}, \qquad y(0) = \frac{\pi}{6}.$$

S_o^1

Q.5

$$y' = -\frac{x \tan y}{1+x^2}, \quad y(0) = \frac{\pi}{6}$$

Using Variable separable method

$$\frac{dy}{\tan y} = -\frac{x}{1+x^2}\,dx$$

On integration

$$\int \frac{dy}{\tan y} = -\int \frac{x\,dx}{1+x^2}$$

$$\log \sin y = -\frac{1}{2}\log(1+x^2) + \log c$$

$$\log \sin y = \log \frac{c}{\sqrt{1+x^2}}$$

$$\sin y = \frac{c}{\sqrt{1+x^2}}$$

Using initial conditions

$$\frac{1}{2} = \frac{c}{\sqrt{1+0}} = c$$

$$\Rightarrow \sin y = \frac{1}{2\sqrt{1+x^2}}$$

$$y(x) = \sin^{-1}\left(\frac{1}{2\sqrt{1+x^2}}\right), \quad \forall x \in \mathbb{R}$$

with $y(0) = \frac{\pi}{6}$

34

Suppose that a hostel houses 500 students. Following a semester break, 10 students returned to the hostel with flu and after 5 days a total of 20 students had flu. Let $y(t)$ be the number of students with flu at time t. Assume that the rate of spread of flu is proportional to

$$y(t)\left(1-\frac{y(t)}{500}\right).$$

Find the solution $y(t)$. [3]

Sol^n

Q-6 $\quad \frac{dy}{dt} = k\, y(t)\left(1-\frac{y(t)}{500}\right)$

$y(0)=10, \quad y(5)=20$

$$\frac{dy}{y\left(1-\frac{y}{500}\right)} = k\,dt$$

$$\frac{500\,dy}{y(500-y)} = k\,dt$$

$$\frac{\{(500-y)+y\}dy}{y(500-y)} = k\,dt$$

$$\int\left(\frac{1}{y}+\frac{1}{(500-y)}\right)dy = \int k\,dt$$

$$\log(y) - \ln(500-y) = kt + \ln c$$

$$\ln\left(\frac{y}{(500-y)\cdot c}\right) = kt$$

$$\frac{y}{(500-y)} = c\,e^{kt}$$

$y(0)=10$ gives

$$c = \frac{1}{49}$$

and $y(5)=20$ give

$$\frac{20}{480} = \frac{1}{49}e^{5k}$$

$$\ln\left(\frac{49}{24}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{49}{24}\right)$$

then Sol^n is given by –

$$\frac{y}{(500-y)} = \frac{1}{49}e^{kt}, \quad \text{where } k=\frac{1}{5}\ln\left(\frac{49}{24}\right)$$

35

8/

(a) Prove that $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$, where $f*g$ denotes the convolution of f and g and $\mathcal{L}(f)$ denotes the Laplace transform of f. [2]

(b) Solve the following integral equation [3]

$$y(t) = 1 - \cosh t - \int_0^t e^{\tau} y(t-\tau)d\tau.$$

3.(a) $\mathcal{L}(f*g) = \mathcal{L}(f)\,\mathcal{L}(g)$

$$f*g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$\mathcal{L}(f*g)(s) = \int_{t=0}^{\infty} \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau\; e^{-st}\, dt$$

$$= \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} f(\tau)\, g(t-\tau)\, e^{-st}\, dt\, d\tau$$

$$= \int_0^{\infty} f(\tau)\, e^{-s\tau}\, \underbrace{G(s)}_{\mathcal{L}(g)(s)}\, d\tau$$

$$= \cancel{F(s)\, G(s)}\; \mathcal{L}(f)(s)\; \mathcal{L}(g)(s).$$

(b) $y(t) = 1 - \cosh t - \int_0^t e^{\tau}\, y(t-\tau)\, d\tau$

let $g(t) = e^t$. Then $y(t) = 1 - \cosh t - y*g(t)$.

Taking Laplace transform, $Y(s) = \frac{1}{s} - \frac{s}{s^2-1} - \frac{Y(s)}{s-1}$

$$\left[1 + \frac{1}{s-1}\right] Y(s) = \frac{1}{s} - \frac{s}{s^2-1}$$

$$Y(s) = \frac{s-1}{s^2} - \frac{1}{s+1}$$

$$Y(s) = \frac{1}{s} - \frac{1}{s^2} - \left[\frac{1}{s+1}\right].$$

$$\boxed{y(t) = 1 - t - e^{-t}}$$

36

(a) Prove that $\mathcal{L}\{tf(t)\} = -F'(s)$, where $F(s)$ is the Laplace transform of $f(t)$. [2]

(b) Find the inverse Laplace transform of the function $F(s) = \tan^{-1}\left(\frac{3}{s}\right)$. [3]

Sol^n

4. (a)

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \quad F'(s) = \int_0^\infty \left(-t e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty e^{-st}\{-t f(t)\}\,dt$$

$$= \mathcal{L}\{-t f(t)\}$$

$$\Rightarrow \quad \mathcal{L}\{t f(t)\} = -F'(s)$$

(b)

$$F(s) = \tan^{-1}\left(\frac{3}{s}\right)$$

$$\Rightarrow F'(s) = \frac{1}{1+\left(\frac{3}{s}\right)^2}\cdot\left(\frac{-3}{s^2}\right) = \frac{-3}{s^2+3^2}$$

$$\Rightarrow \mathcal{L}^{-1}\{-F'(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \sin(3t)$$

$$\Rightarrow t f(t) = \sin(3t)$$

$$\Rightarrow \boxed{f(t) = \frac{\sin(3t)}{t}}$$

37

$S S^4$

Find the general solution of [4]

$$y'' + 2y' + 2y = 4e^{-x}\sec^3 x.$$

5. $y'' + 2y' + 2y = 4e^{-x}\sec^3 x$

Char. eqn. for homog. part : $\lambda^2 + 2\lambda + 2 = 0 \implies \lambda = -1 \pm i$

$y_h = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$ [1 mark]

$y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= e^{-x}\cos x \cdot e^{-x}(\cos x - \sin x) - e^{-x}\sin x \cdot e^{-x}(-\sin x - \cos x)$

$= e^{-2x}$ [1 mark]

$y_p = u_1 y_1 + u_2 y_2$

where $u_1 = -\int \frac{y_2\, r(x)}{W(y_1, y_2)}\, dx$

$= -\int \frac{\cancel{e^{-x}}\sin x \cdot 4\cancel{e^{-x}}\sec^3 x}{\cancel{e^{-2x}}}\, dx$ [1]

$= -\int 4\tan x \sec^2 x\, dx = -2\tan^2 x$

and $u_2 = \int \frac{y_1\, r(x)}{W(y_1, y_2)}\, dx = \int \frac{\cancel{e^{-x}}\cos x \cdot 4\cancel{e^{-x}}\sec^3 x\, dx}{\cancel{e^{-2x}}}$

$= 4\int \sec^2 x\, dx = 4\tan x$ [1]

$\therefore$ $\boxed{y_p = (-2\tan^2 x)\, e^{-x}\cos x + (4\tan x)\, e^{-x}\sin x}$

General soln. $\boxed{y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x + y_p}$

38

6. Solve the following using the power series method [5]

$$(1-x^2)y'' - 2xy' + 2y = 2 - x; \quad y(0) = 0, y'(0) = 1.$$

6. $(1-x^2)y'' - 2xy' + 2y = 2 - x \qquad y(0)=0, \; y'(0)=1$

(1 mark) $y(x) = \sum_{n=0}^{\infty} a_n x^n \longleftrightarrow \boxed{\begin{array}{l} y(0) = a_0 = 0 \\ y'(0) = a_1 = 1 \end{array}}$

$x y'(x) = \sum_{n=0}^{\infty} n a_n x^n$

$y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$x^2 y'' = \sum_{n=0}^{\infty} (n-1) n \, a_n x^n$

(1 mark) $\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n-1)n a_n - 2n a_n + 2a_n \right] x^n = 2 - x$

For $n=0$: $2a_2 + 2a_0 = 2$

$a_2 = 1$

(1 mark) $n=1$: $6a_3 = -1$

$a_3 = -\frac{1}{6}$

$\boxed{\text{For } n \geq 2}$ $a_{n+2} = \frac{n-1}{n+1} a_n$

$a_4 = \frac{1}{3} a_2 = \frac{1}{3}$

$a_6 = \frac{3}{5} \cdot \frac{1}{3} = \frac{1}{5}$

Assume $a_{2n} = \frac{1}{2n-1}$

Then $a_{2(n+1)} = a_{2n+2} = \frac{2n-1}{2n+1} \cdot a_{2n}$

$= \frac{1}{2(n+1)-1}$

Then induction

$a_{2n} = \frac{1}{2n-1} \quad n \geq 2$

$a_5 = \frac{2}{4} a_3 = \frac{-1}{2 \cdot 6}$

$a_7 = -\frac{4}{6} \cdot \frac{1}{2 \cdot 6} = \frac{-1}{3 \cdot 6}$

Assume $a_{2n+1} = \frac{-1}{6 \cdot n}$

Then $a_{2(n+1)+1} = a_{2n+1+2} = \frac{2n}{2n+2} a_{2n+1}$

$= -\frac{2n}{2n+2} \cdot \frac{1}{6 \cdot n}$

$= \frac{-1}{6(n+1)}$

$\therefore \boxed{y = x + \sum_{n=1}^{\infty} \frac{1}{2n-1} x^{2n} - \sum_{n=1}^{\infty} \frac{1}{6n} x^{2n+1}}$

39

7. Solve the following IVP [5]

$$y'' + 5y' + 6y = \delta\left(t - \frac{\pi}{2}\right) + u(t-\pi); \quad y(0) = 0, y'(0) = 1.$$

(Here δ and u are the Dirac's delta and Heaviside function, respectively.)

7. $y'' + 5y' + 6y = \delta(t - \frac{\pi}{2}) + u(t-\pi)$; $y(0) = 0$, $y'(0) = 1$

$\mathcal{L}(y') = sY - y(0) = sY$

$\mathcal{L}(y'') = s^2 Y - s\,y(0) - y'(0) = s^2 Y - 1$

$\mathcal{L}\{\delta(t - \frac{\pi}{2})\} = e^{-\frac{\pi}{2}s}$

$\mathcal{L}\{u(t-\pi)\} = \frac{e^{-\pi s}}{s}$

$\therefore$ Taking Laplace transform, we get

$$s^2 Y - 1 + 5sY + 6Y = e^{-\frac{\pi}{2}s} + \frac{e^{-\pi s}}{s}$$

$$\Rightarrow (s^2 + 5s + 6) Y = 1 + e^{-\frac{\pi}{2}s} + \frac{e^{-\pi s}}{s}$$

$$\Rightarrow Y = \frac{1}{(s+2)(s+3)}\left[1 + e^{-\frac{\pi}{2}s} + \frac{e^{-\pi s}}{s}\right]$$

$$= \left(\frac{1}{s+2} - \frac{1}{s+3}\right) + e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) + e^{-\pi s}\cdot\frac{1}{s(s+2)(s+3)}$$

$$\frac{1}{s(s+2)(s+3)} = \frac{1/6}{s} + \frac{-1/2}{s+2} + \frac{1/3}{s+3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) = e^{-2t} - e^{-3t}$$

$$\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right)\right\} = e^{-2(t-\frac{\pi}{2})}u(t-\tfrac{\pi}{2}) - e^{-3(t-\frac{\pi}{2})}u(t-\tfrac{\pi}{2})$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s+2)(s+3)}\right\} = \left[\frac{1}{6} - \frac{1}{2}e^{-2(t-\pi)} + \frac{1}{3}e^{-3(t-\pi)}\right]u(t-\pi)$$

So,

$$\boxed{y(t) = \left(e^{-2t} - e^{-3t}\right) + \left[e^{-2(t-\frac{\pi}{2})} - e^{-3(t-\frac{\pi}{2})}\right]u(t-\tfrac{\pi}{2}) + \left[\frac{1}{6} - \frac{1}{2}e^{-2(t-\pi)} + \frac{1}{3}e^{-3(t-\pi)}\right]u(t-\pi)}$$

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8. Solve the following system of ODEs [5]

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sec t \end{pmatrix}.$$

Sol^n

$y' = Ay + \vec{g}(t)$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\vec{g}(t) = \begin{pmatrix} 0 \\ \sec t \end{pmatrix}$

Eigen values of A are $\pm i$ with eigen vector $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

$$e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$\therefore$ Soln. of homog. $\vec{y}_h = c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

Fund. matrix $\tilde{Y} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$$\tilde{Y}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$\vec{y}_p = \tilde{Y}\vec{u}$, where $\vec{u}' = \tilde{Y}^{-1}\vec{g}(t)$

$$\vec{u}' = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 \\ \sec t \end{pmatrix} = \begin{pmatrix} -\tan t \\ 1 \end{pmatrix}$$

$\Rightarrow u_1 = -\int \tan t \, dt = \ln|\cos t|$

$u_2 = t$

$\therefore y_p = u_1\vec{y}_1 + u_2\vec{y}_2 = \ln|\cos t|\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$$\therefore \boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(c_1 + \ln|\cos t|\right)\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + (c_2 + t)\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}$$

41

9. Consider the IVP [3]

$$\frac{dy}{dx} = \frac{xy}{1 - x^2 - y^2}, \quad y(0) = \alpha.$$

Use the existence-uniqueness theorem to find the values of $\alpha \in \mathbb{R}$ for which the given IVP has a unique solution (Give proper justification that the hypotheses of the theorem are satisfied).

9. $y' = \frac{xy}{1-x^2-y^2}$ $\quad y(0) = \alpha \longrightarrow (1)$

Let $f(x,y) = \frac{xy}{1-x^2-y^2}$

The f is cts on $\mathbb{R}^2$ except the pts (x,y) such that $x^2+y^2=1$

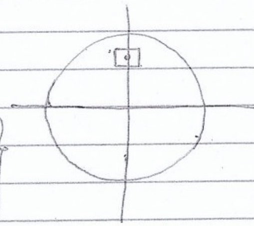
In particular, f is not cts at the pts $(0, y(0)) = (0, \alpha)$ with $\alpha^2 = 1$.

Hence we can't use existence and uniqueness theorem if $\alpha^2 = 1$.

Assume $\alpha^2 \neq 1$. Then

~~If $|\alpha| \leq 1$,~~ we can choose a small rectangle

$$R = \left\{(x,y) : \begin{array}{l} |x| \leq \left|\frac{1-|\alpha|}{4}\right| \; \& \\ |y-\alpha| \leq \left|\frac{1-|\alpha|}{4}\right| \end{array}\right\}$$



Such that f is cts on R.

It means that either R is contained in ~~subset of~~ $\{(x,y) : x^2+y^2 < 1\}$ if $|\alpha|<1$ or $R \subset \{(x,y) : x^2+y^2>1\}$ if $|\alpha| > 1$

In both cases, $f_y = \frac{x(1-x^2-y^2)+2xy^2}{(1-x^2-y^2)^2}$ is cts on R.

Summary. If $\alpha^2 \neq 1$, $\exists$ R - rectangle s.t. $(0,\alpha) \in R$

(1) f is cts on R.

(2) f_y is cts on R.

then by existence & uniqueness (1) has a unique solution.

Sol. 4

42

(1) Find the n-th Picard's iterate y_n for the following IVP. [3]

$$y' = x^2 + y, \qquad y(0) = 0.$$

Sol^n: Here $y_0 = 0$, $y_1 = 0 + \int_0^x (t^2 + 0)\,dt = \frac{x^3}{3}$

$$y_2 = 0 + \int_0^x \left(t^2 + \frac{t^3}{3}\right) dt = 2\frac{x^3}{3!} + 2\frac{x^4}{4!}$$

$$y_3 = \int_0^x \left(t^2 + 2\cdot\left(\frac{t^3}{3!} + \frac{t^4}{4!}\right)\right) dt = 2\left\{\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)$$

Induction hypothesis $y_{n-1} = 2\left\{\frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}\right\}$

$$y_n = \int_0^x t^2 + 2\cdot\left(\frac{t^3}{3!} + \cdots + \frac{t^{n+1}}{(n+1)!}\right) dt$$

$$= 2\left\{\frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^{n+1}}{(n+1)!}\right\}.$$ completed.

Thus $y(x) = \lim_{n\to\infty} y_n = 2\left\{\sum_{i=3}^{\infty} \frac{x^i}{i!}\right\} = 2\left\{\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1 - x - \frac{x^2}{2!}\right\}$

$$= 2\left(e^x - 1 - x - \frac{x^2}{2}\right)$$

43

(2) Discuss the existence and uniqueness of the following IVP in a neighborhood of $x_0 = 1$. [3]

$$\frac{dy}{dx} = f(x,y), \qquad y(1) = -1,$$

where

$$f(x,y) = \begin{cases} \dfrac{x+y}{\sin(x+y)} & \text{if } x+y \neq 0 \\ 1 & \text{if } x+y = 0 \end{cases}.$$

Sol^n

(2) Find existence & uniqueness of the following IVP on a nbhd of $x=1$.

~~$\sin(x+y)\,dy = (x+y)\,dx, \quad y(1)=-1$~~

Solution: ~~Here~~ $\frac{dy}{dx} = f(x,y) \quad y(1) = -1$

where $f(x,y) = \begin{cases} \frac{x+y}{\sin(x+y)} & \text{if } x+y \neq 0 \\ 1 & \text{if } x+y = 0. \end{cases}$

Solution:

We know, for $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$|t| < \delta \Rightarrow \left|\frac{t}{\sin t} - 1\right| < \varepsilon$

Now for any $a \in \mathbb{R}$,

$|x-a| < \delta/2$ & $|y+a| < \delta/2 \Rightarrow |x+y| = |(x-a)+(y+a)|$
$\leq |x-a| + |y+a|$
$< \delta$

Thus, $|x-a| < \delta/2$ & $|y+a| < \delta/2 \Rightarrow \left|\frac{x+y}{\sin(x+y)} - 1\right| < \varepsilon$.

Therefore $f(x,y)$ is continuous at $(a,-a)$ $\forall a \in \mathbb{R}$.

Since $x+y$ & $\sin(x+y)$ are continuous & the denominator does not vanish for (a,b) $(b \neq -a)$, $f(x,y)$ is continuous at (a,b) $(b \neq -a)$.

For $x+y \neq 0$ $\quad f_y = \dfrac{\sin(x+y) - (x+y)\cos(x+y)}{\sin^2(x+y)}$

and for $f_y(a,-a) = \lim\limits_{h\to 0} \dfrac{\left\{\frac{(a+h)+(-a+h)}{\sin(a+h+(-a)+h)} - 1\right\}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{2h}{\sin 2h} - 1}{h} = 0$

Then observe, $\lim\limits_{\substack{x\to a \\ y \to -a}} f_y(x,y) = 0$

Since f is continuous on a closed rectangle R around $(1,-1)$ and f_y exists & continuous on R, by existence & uniqueness theorem, the IVP has a unique solution on a nbhd of $x=1$.

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4. Consider the IVP:

$$y' = xy^2 + 1, \; y(0) = 0.$$

Find the first three iterates of the Picard's method of successive approximations. [3]

4. $y' = f(x,y) \;, \; y(0) = 0$

where $f(x,y) = xy^2 + 1 \;; \; x_0 = 0 \;, \; y_0 = 0$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$

$$= 0 + \int_0^x 1\,dt = x$$

So, $\boxed{y_1(x) = x}$

$\therefore \; f(t, y_1(t)) = t(t)^2 + 1 = t^3 + 1$

$$\therefore \; y_2(x) = 0 + \int_0^x (t^3+1)\,dt = \boxed{\frac{x^4}{4} + x = y_2(x)}$$

$$f(t, y_2(t)) = t\left(\frac{t^4}{4} + t\right)^2 + 1 = \frac{t^9}{16} + \frac{t^6}{2} + t^3 + 1$$

$$\therefore \; y_3(x) = 0 + \int_0^x \left(\frac{t^9}{16} + \frac{t^6}{2} + t^3 + 1\right) dt$$

$$\boxed{y_3(x) = \frac{x^{10}}{160} + \frac{x^7}{14} + \frac{x^4}{4} + x}$$

X

45

Consider the ODE

$$(xy-1)dx + (x^2 - xy)dy = 0$$

(a) Find a general solution to the given ODE.

(b) Find a particular solution in *explicit form* to the given ODE satisying the initial condition $y(1) = 2$.

(c) Using the existence-uniqueness theorem justify that the solution obtained in (b) is unique in a small enough interval $(1-\alpha, 1+\alpha)$. $[2+2+2=6]$

Sol^n

5.(a) The eqn. is of the form $M\,dx + N\,dy = 0$

where $M = xy - 1$ & $N = x^2 - xy$

$M_y = x$, $N_x = 2x - y$

Since $M_y \neq N_x$, the eqn. is not exact.

But, $\dfrac{M_y - N_x}{N} = \dfrac{-x+y}{x(x-y)} = -\dfrac{1}{x}$ is a fn. of x only.

$\therefore$ $\mu(x) = \exp\left(\int \dfrac{M_y - N_x}{N}\,dx\right)$

$= \exp\left(\int -\frac{1}{x}\,dx\right) = e^{-\ln x} = \dfrac{1}{x}$

is an integ. factor.

Thus, $\left(y - \frac{1}{x}\right)dx + (x-y)dy = 0$ is exact.

$\therefore$ We can find $u(x,y)$ s.t. $\dfrac{\partial u}{\partial x} = y - \frac{1}{x}$ & $\dfrac{\partial u}{\partial y} = x - y$

$\dfrac{\partial u}{\partial x} = y - \frac{1}{x} \Rightarrow u = xy - \ln|x| + h(y)$

$\Rightarrow \dfrac{\partial u}{\partial y} = x + h'(y)$

$\therefore$ $h'(y) = -y$, so $h(y) = -\dfrac{y^2}{2}$ works.

$\therefore$ $\boxed{xy - \ln|x| - \frac{y^2}{2} = c}$ is a general soln.

(b) Using $y(1) = 2$, $c = 2 - 0 - \frac{2^2}{2} = 0$

$\therefore$ $xy - \ln x - \frac{y^2}{2} = 0$, $x > 0$

$\Rightarrow y^2 - 2xy + 2\ln x = 0$

$\Rightarrow y = \dfrac{2x \pm \sqrt{4x^2 - 8\ln x}}{2} = x \pm \sqrt{x^2 - 2\ln x}$

Since $y(1) = 2$, $\boxed{y = x + \sqrt{x^2 - 2\ln x}, \quad x > 0}$

is a soln. to the given IVP.

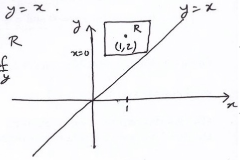
5. (c) Writing the eqn. $(xy-1)dx + (x^2 - xy)dy = 0$ in the form $\dfrac{dy}{dx} = f(x,y)$, we have

$f(x,y) = \dfrac{xy-1}{x(y-x)}$, $x \neq 0$, $y \neq x$

So, $f(x,y)$ & $\dfrac{\partial f}{\partial y}$ are continuous everywhere except the lines $x = 0$ & $y = x$.

Clearly, we can find a rectangle R about $(1,2)$ such that both f & $\frac{\partial f}{\partial y}$ are continuous on R.

(as shown in the figure)



Hence the uniqueness theorem guarantees a unique soln. to the IVP in a small enough interval $(1-\alpha, 1+\alpha)$.

Since in part (b) we have found a soln. to the IVP defined on an interval containing 1, it must be the unique soln. in $(1-\alpha, 1+\alpha)$.

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Find the general solution of the following ODE: [4]

$$(3xy + 2y^2 + 4y)dx + (x^2 + 2xy + 2x)dy = 0.$$

The eqn. is of the form $M\,dx + N\,dy = 0$

with $M = 3xy + 2y^2 + 4y$

$N = x^2 + 2xy + 2x$

Here $M_y = 3x + 4y + 4$ & $N_x = 2x + 2y + 2$

Since $M_y \neq N_x$ the eqn. is not exact. (1 mark)

But, $\dfrac{M_y - N_x}{N} = \dfrac{x + 2y + 2}{x^2 + 2xy + 2x} = \dfrac{1}{x}$

$\therefore$ There is an integrating factor

$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) = \exp\left(\int \frac{1}{x} dx\right) = x$ [1 mark]

Multiplying the given eqn. by x, we get

$(3x^2y + 2xy^2 + 4xy)dx + (x^3 + 2x^2y + 2x^2)dy = 0$

which is an exact eqn.

We find $u(x,y) = x^3y + x^2y^2 + 2x^2y + h(y)$ [1 mark]

and $x^3 + 2x^2\cancel{y} + 2x^2 = u_y = x^3 + 2x^2y + 2x^2 + h'(y)$

$\Rightarrow h'(y) = 0$ So, we can take $h(y) = 0$.

$\therefore u(x,y) = x^3y + x^2y^2 + 2x^2y$

The general solution is

$$\boxed{x^3y + x^2y^2 + 2x^2y = c}$$

[1 mark]

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Sol

Consider the IVP: [4 = 2 + 2]

$$(x^2-1)y' = 4y, \; y(x_0) = y_0.$$

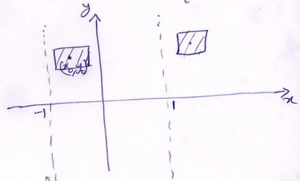
(a) Find the values of (x_0, y_0) for which a unique solution is guaranteed by the existence-uniqueness theorem.

(b) Show that if $(x_0, y_0) = (1, 0)$, then the IVP has infinitely many solutions.

5. (a) The given ODE can be written as

$$y' = f(x,y) = \frac{4y}{x^2-1} \quad \text{for } x \neq \pm 1.$$

$f(x,y)$ & $\frac{\partial f}{\partial y} = \frac{4}{x^2-1}$ are both continuous at all points in the xy-plane except on the lines $x=-1$ & $x=1$. [1 mark]



If $x_0 \neq -1$ & $x_0 \neq 1$, then for any $y_0 \in \mathbb{R}$, we can find a small enough closed rectangle R centered at (x_0, y_0) so that $f(x,y)$ & $\frac{\partial f}{\partial y}(x,y)$ are both cont. on R. [1 mark]

Thus the existence-uniqueness thm. guarantees a unique soln. for the IVP if $(x_0,y_0) \in \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R}\setminus\{\pm 1\}, y \in \mathbb{R}\}$

If $x_0 = -1$ or $x_0 = 1$, the theorem is not applicable.

(b) Solving the ODE $(x^2-1)\frac{dy}{dx} = 4y$ using the method of separation of variables, we get [1 mark]

$$y = c\left(\frac{x-1}{x+1}\right)^2 \; ; \; c \in \mathbb{R}.$$

are solutions. if $x^2-1 \neq 0, y \neq 0$

But the solution is defined for $x \neq -1$, and by differentiating, we can check that

$$y = \frac{c(x-1)^2}{(x+1)^2}, \; x \neq -1$$

is a soln.

Since this satisfies $y(1) = 0$ for every $c \in \mathbb{R}$, we get infinitely many solns. [1 mark]

$$\boxed{y = c\left(\frac{x-1}{x+1}\right)^2}, \; x \in (-1, \infty)$$

to the IVP with $y(1) = 0$.

QWA link.

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Good Luck!