Q. $G = (V, E)$ is a simple, undirected graph with no self-loops. G has minimum degree δ and maximum degree Δ. A k-*colouring* of G is a function $f : V \to [k]$ such that $(u, v) \in E \Rightarrow f(u) \neq f(v)$.

Given k, we choose a random mapping $\phi : V \to [k]$ as follows: $\phi(u)$ is chosen uniformly at random from $[k]$ and the collection $\{\phi(u) : u \in V\}$ is mutually independent. What is the probability that ϕ is a k-colouring of G? What is the minimum value of k required to ensure that the probability of ϕ being a k-colouring is at least $1 - 1/n$?

Solution:

Given a vertex v with degree $d(v)$ the probability that it is correctly coloured is $(1 - 1/k)^{d(v)}$.

Let $A(v)$ be the event that v is incorrectly coloured. So

$$\mathrm{P}(A(v)) = 1 - \left(1 - \frac{1}{k}\right)^{d(v)}.$$

If they have got this then they get 1.5 marks.

Probability that the graph is incorrectly coloured is the probability that there is at least one vertex that is incorrectly coloured, i.e., $\cup_{v \in V} A(v)$. *If they have observed this fact then 1 mark.*

Since

$$1 - \left(1 - \frac{1}{k}\right)^{d(v)} \leq 1 - \left(1 - \frac{1}{k}\right)^{\Delta(G)},$$

we have that

$$\mathrm{P}(\cup_{v \in V} A(v)) \leq n \left(1 - \left(1 - \frac{1}{k}\right)^{\Delta(G)}\right).$$

Expanding the RHS using the binomial theorem we get

$$\mathrm{P}(\cup_{v \in V} A(v)) \leq n \left(\sum_{i=1}^{\Delta(G)} \binom{\Delta(G)}{i} \frac{(-1)^{i-1}}{k^i} \right) \leq \frac{n\Delta(G)}{k}.$$

So, if $k \geq n^2 \Delta(G)$ the probability of incorrect colouring is $< 1/n$.

This part was a little tricky and uses techniques not discussed in class so grade liberally. If they have given an expression without attempting to simplify, give 0/1.5, but if they have attempted to simplify, give some marks.