COL202: Discrete Mathematical Structures. II semester, 2024-25.
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Tutorial Sheet 4: Graph Theory Basics.
31 January 2025

Important: The question marked with a ♠ is this week's quiz. The start time for the quiz is 1PM and the end time for the quiz is 1:12PM. Questions marked with a * may be a little harder and can be considered optional.

Problem 1
Prove that every simple graph has two vertices of the same degree.

Problem 2 [1, Prob 2, page 29]
Let $d \in \mathbb{N}$ and $V = \{0, 1\}^d$, i.e., V is the set of all 0-1 sequences of length d. We define the edge set as follows: there is an edge between two sequences if they differ in exactly one position. This graph is known as the d-dimensional cube. Determine the average degree, diameter, girth and circumference of the *d-dimensional cube*. Note that the circumference of a graph is the length of the longest cycle in the graph.

Problem 3 [1, Prob 3, page 30]
Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

Problem 4
Prove that two isomorphic graphs have the same number of edges.

Problem 5
Suppose we are given that there exists a homomorphism φ from $G = (V, E)$ to $G' = (V', E')$. We assume that $V, V' \neq \emptyset$. Prove that there is an independent set in G whose size is at least $|V|/|V'|$. Note: If $|V'| \geq |V|$ then the result is trivially true since, for any $v \in V$, the set $\{v\}$ is trivially an independent set.

Problem 6
For $k \geq 0$, a k-colouring of a graph $G = (V, E)$ is a function $f : V \to [k]$ (where $[n]$ is the set $\{1, 2, \ldots, n\}$) such that for all $(u, v) \in E$, $f(u) \neq f(e)$. We say a graph is k-colourable if a k-colouring exists for the graph.

Problem 6.1 ♠
Show that G is k-colourable if there is a homomorphism from G to the complete graph on k vertices (i.e. K^k).

Problem 6.2
Show that if there is a homomorphism from K^k to G then G is not $k - 1$ colourable.

Problem 6.3
Show that G is k-colourable only if there is a homomorphism from G to K^k.

Problem 7 [1, Prob 6, page 30]
Show that $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$ for every graph G.

Problem 8
Given a set X, a function $f : X \times X \to [0, \infty)$ is called a *distance* if

1. $\forall x, y \in X : f(x, y) = 0 \Leftrightarrow x = y$,
2. $\forall x, y \in X : f(x, y) = f(y, x)$, and
3. $\forall x, y, z \in X : f(x, y) \leq f(x, z) + f(z, y)$.

Problem 8.1
Prove that the graph distance defined as the length of the shortest path between two vertices is a distance.

Problem 8.2
Suppose that given a graph $G = (V, E)$ we have a function $w : E \to \mathbb{R}$ and we define the length of the path $x_0 \ldots x_k$ to be $\sum_{i=1}^{k} w(x_{i-1}x_i)$. As before we define the "distance" between two vertices to be the length of the shortest path between the two vertices. What condition do we need on w for this "distance" to actually be a distance? Which of the requirements of a distance get violated if w is allowed to assign negative values? Do any requirements get violated if w is allowed to assign the value 0?

Problem 9
Given two graphs $G = (V, E)$ and $G' = (V', E')$ such that $|V| = |V'|$, suppose we can find a $\phi : V \to V'$ which is a bijection and is a graph homomorpishm. Prove that $\text{diameter}(G') \leq \text{diameter}(G)$. Draw an example where the inequality is strict. Use this fact to prove that two isomorphic graphs have the same diameter.

Problem 10 *
Given a set of vertices V such that $|V| = n$, and given k such that $\binom{n}{2} \geq k \geq 0$, let us denote by $A_{n,k}$ the set of all simple graphs on V with exactly k edges. We now define a graph whose vertices are the elements of $A_{n,k}$. We put an edge between graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ if $|E_1 \setminus E_2| = 1$. What is the diameter of this graph in terms of k? Does the diameter always increase as k increases?

References

[Die25] Reinhard Diestel, *Graph Theory 6ed.*, Springer, 2025.