Shot noise

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May 14, 2018

1 Poisson distribution

$$\text{🐟}(n|\mu) = \frac{\mu^n e^{-\mu}}{n!} \tag{1}$$

moments:

$$\mathbb{E}\left[n\right] = \mu \tag{2}$$

$$\mathbb{E}\left[n^2\right] = \mu(\mu+1) \tag{3}$$

$$\mathbb{V}\left[n\right] = \mathbb{E}\left[n^2\right] - \mathbb{E}\left[n\right]^2 = \mu \tag{4}$$

2 Variance of a sampled field

Let $\rho(x)$ be *any* stochastic field that follows distribution $f(\rho)$. Poisson-sampling it means observing a discrete n value with probability

$$p_n = \int \text{🐟}(n|\rho) f(\rho) d\rho. \tag{5}$$

Then

$$\begin{aligned}\mathbb{E}\left[n\right] &= \int \mathbb{E}\left[n|\rho\right] f(\rho) d\rho \\ &= \int \rho f(\rho) d\rho \\ &= \mathbb{E}\left[\rho\right].\end{aligned} \tag{6}$$

The mean of the sampled field is the same than of the continuous one: the sampling does not introduce any bias.

Now let see the variance:

$$\begin{aligned}\mathbb{E}\left[n^2\right] &= \int \mathbb{E}\left[n^2|\rho\right] f(\rho) d\rho \\ &= \int \rho(1+\rho) f(\rho) d\rho \\ &= \mathbb{E}\left[\rho\right] + \mathbb{E}\left[\rho^2\right],\end{aligned} \tag{7}$$

$$\begin{aligned}\mathbb{V}\left[n\right] &= \mathbb{E}\left[n^2\right] - \mathbb{E}\left[n\right]^2 \\ &= \mathbb{E}\left[\rho^2\right] - \mathbb{E}\left[\rho\right]^2 + \mathbb{E}\left[\rho\right] \\ &= \mathbb{V}\left[\rho\right] + \mathbb{E}\left[n\right].\end{aligned} \tag{8}$$

The last line indicates that the variance increased as $\sigma_n^2 = \sigma_\rho^2 + \bar{N}$.

Now recall that $P(k)$ is the power-spectrum of the density *contrast* i.e, $\Delta(x) = \dfrac{n(x)}{\bar{N}} - 1$ which has variance

$$\mathbb{V}\left[\Delta\right] = \frac{\mathbb{V}\left[n\right]}{\bar{N}^2} = \mathbb{V}\left[\rho\right] + \frac{1}{\bar{N}} \tag{9}$$

For a flat spectrum the variance is essentially $P(k)$ so that we recovered the standard result: Poisson-sampling adds an extra-power of $\dfrac{1}{\bar{N}} = \dfrac{V}{N}$

3 Poisson point process

(These are not my ideas but a melting pot of (Peebles 1980; Martinez & Saar 2001; Piattella 2018))

Density field Technically studying catalogs requires using the theory of *random point processses.* One can formally define the number density field as

$$n(\mathbf{x}) \equiv \lim_{R\to 0} n_R(\mathbf{x}) \tag{10}$$

where R is a ball radius. It is known as the *intensity function* of the process. One generally prefers writing

$$n(\mathbf{x}) = \bar{N}(1 + \delta(x)) \tag{11}$$

where $\bar{N} = \mathbb{E}\left[n(\mathbf{x})\right]$ and $\delta(x)$ the dimensionless zero-mean density contrast.

Poisson process If we place randomly a volume δV in $\mathbf{x}$ the probability that it contains one point [1] point is

$$\delta P = n(\mathbf{x})\delta V. \tag{12}$$

Macroscopically, the probability distribution of the number of points lying in a bounded region of volume V is

$$(n|N) = \frac{N^n e^{-N}}{n!}, \tag{13}$$

where

$$N = \int_V n(\mathbf{x})d\mathbf{x}. \tag{14}$$

Note that since $n(\mathbf{x})$ varies this is an *inhomogeneous* process. To simulate it you must first shoot $n(\mathbf{x})$ to build N then use Eq.(13).

If we now consider two infinitesimally small spheres on $\mathbf{x}_1$ and $\mathbf{x}_2$, the joint probability that in each spheres lies a point of the process is

$$\delta P_{12} = n(\mathbf{x}_1, \mathbf{x}_2)\delta V_1 \delta V_2, \tag{15}$$

where $n(\mathbf{x}_1, \mathbf{x}_2)$ is the *second order intensity function.* For a Poisson process

$$n(\mathbf{x}_1, \mathbf{x}_2) = n(\mathbf{x}_1)n(\mathbf{x}_2) + \delta(\mathbf{x}_1 - \mathbf{x}_2)n(\mathbf{x}_1) \tag{16}$$

which can be (vaguely) derived from an argument on very small cells where occupancy is only 0 or 1 (Peebles 1980, §36).But it is simpler to compare this expression to Eq.(8). The last term represents the shot noise contribution.

[1] higher number are negligible since they are of higher order

Power spectrum of an observed density field Let us now consider an *observed* field of N galaxies at locations x_i. This can be written

$$\rho(\mathbf{x}) = \sum_{i=1}^{N} m_i \delta(\mathbf{x} - \mathbf{x}_i). \tag{17}$$

m_i being the galaxy mass. In a survey, what we work with is the density contrast over some (finite) window $w(\mathbf{x})$

$$\delta_s(\mathbf{x}) = \left[\frac{\rho(\mathbf{x})}{\bar{\rho}} - 1) \right] w(\mathbf{x}). \tag{18}$$

The mean density being $\bar{\rho} = \dfrac{\sum_i m_i}{V}$,

$$\delta_s(\mathbf{x}) = \left[V \frac{\sum_i m_i \delta(\mathbf{x} - \mathbf{x}_i)}{\sum_i m_i} - 1) \right] w(\mathbf{x}) \tag{19}$$

$$= \left[\frac{V}{N} \sum_i \delta(\mathbf{x} - \mathbf{x}_i) - 1 \right] w(\mathbf{x}) \tag{20}$$

assuming each galaxy has the same mass.

The Fourier transform of this contrast is

$$\begin{aligned} \delta_s(\mathbf{k}) &= \frac{1}{V} \int_V d^3\mathbf{x}\ \delta_s(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{1}{N} \sum_{i=1}^{N} w(\mathbf{x}_i) e^{-i\mathbf{k}\cdot\mathbf{x}_i} - W(\mathbf{k}), \end{aligned} \tag{21}$$

$W(\mathbf{k})$ being the Fourier transform of the window. The power-spectrum is

$$\begin{aligned} P_s(\mathbf{k}) &=< \delta_s(\mathbf{k}) \delta_s(\mathbf{k})^* > \\ &= \frac{1}{N^2} \sum_{i,j} < w(\mathbf{x}_i) w(\mathbf{x}_j) >) e^{-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} - |W(\mathbf{k})|^2 \end{aligned} \tag{22}$$

(in the computation the cross-products of the 2 terms in Eq.(21) drops when using the condition $< \delta(\mathbf{k}) >= 0$).

Let us contemplate (or at least glance at the beauty of) what this means. The $\sum_{i,j}$ term is the Fourier transform of all the pairs counts when $i \neq j$ (what is performed in a real-space analysis) *plus an auto-term* $i = j$. This is the shot-noise term that is equal to

$$P_{SN} = \frac{1}{N^2} \sum_{i=1}^{N} < w(\mathbf{x}_i)^2 > \tag{23}$$

For a tophat selection $\sum_i < w(\mathbf{x}_i)^2 >= N$, so that putting back the V factor to get the conventional $P(k)$ dimension

$$P_{SN} = \frac{V}{N}. \tag{24}$$

One may legitimately wonder what happens to the other terms. I just sketch the idea

$$< w(\mathbf{x}_i) w(\mathbf{x}_j) >= w(\mathbf{x}_i) w(\mathbf{x}_j) < n(\mathbf{x}_i) n(\mathbf{x}_j) > . \tag{25}$$

Using the contrast notation Eq.(11)

$$< n(\mathbf{x}_i)n(\mathbf{x}_j) >= \bar{N}^2(1+ < \delta(\mathbf{x}_i)\delta(\mathbf{x}_j) >). \tag{26}$$

Injecting it into Eq.(22), the first term cancels $-|W(\mathbf{k})|^2$. The second one can be written as a convolution between the signal and the window $|W(\mathbf{k})|^2$ (this makes sense since it is a multiplication in real space ie. an apodization)

So finally we get the (reasonable) result that

$$P_s(\mathbf{k}) = P(\mathbf{k}) \otimes |W(\mathbf{k})|^2 + P_{SN}. \tag{27}$$

In order to preserve the signal power the window must be normalized as

$$\int_V d^3\mathbf{x}\ W^2(\mathbf{x}) = 1 \tag{28}$$

4 Tomography

Let now see what happens for a (full-sky) spherical shell with some radial selection $\phi(z)$

formulas

$$a_{lm}(z) = 4\pi i^\ell \int \frac{d^3k}{(2\pi)^3}\delta(k,z)j_\ell(kr(z))Y^*_{lm}(\hat{k}) \tag{29}$$

$$\to a_{lm} = \int dz\phi(z)a_{lm}(z) \tag{30}$$

and

$$C_\ell(z_1,z_2) = < a_{lm}(z_1)a^*_{lm}(z_2) > \tag{31}$$

$$= \frac{2}{\pi}\int dk\ k^2P(k)D(z_1)D(z_2)j_\ell(kr(z_1))j_\ell(kr(z_2)) \tag{32}$$

$$\to C_\ell^{A\times B} = < a_{lm}a^*_{lm} > \tag{33}$$

$$= \int dz_1\ \phi_A(z_1)\int dz_2\phi_B(z_2)C_\ell(z_1,z_2) \tag{34}$$

$$= \frac{2}{\pi}\int dz_1\ \phi_A(z_1)D(z_1)\int dz_2\phi_B(z_2)D(z_2)\int dk\ k^2P(k)j_\ell(kr(z_1))j_\ell(kr(z_2)). \tag{35}$$

Correlation functions:

$$C(\theta,z_1,z_2) = < \delta(n_1,z_1)\delta(n_2,z_2) >_{n_1\cdot n_2=\cos\theta} \tag{36}$$

$$= \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell(z_1,z_2)P_\ell(\cos(\theta)) \tag{37}$$

$$= \xi\left(R(\theta,z_1,z_2)\right) \tag{38}$$

$$= \frac{1}{2\pi^2}D(z_1)D(z_2)\int dk\ k^2P(k)\,\mathrm{sin}_c\left[kR(\theta,z_1,z_2)\right] \tag{39}$$

where $\xi(r)$ is the 3D correlation function and $R(\theta,z_1,z_2) = |\vec{r}_1 - \vec{r}_2|$.

$$C^{A\times B}(\theta) = \int dz_1\phi_A(z_1)\int dz_2\phi_B(z_2)C(\theta,z_1,z_2) \tag{40}$$

$$= \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell^{A\times B} \tag{41}$$

Window normalisation The radial selection (forgetting about any experimental smearing) is

$$\phi(z) = \bar{n}(z)W(z). \tag{42}$$

Here we asume $\bar{n} = Cte$ and we apply some weight function $W(z)$ to the data. W is assumed to be compact over some $[z_{\min}, z_{\max}]$ region. We write $W(z) = \alpha w(z)$ where w is any function. We wish to normalize it ie. find α so that no C_ℓ power is removed/added to the signal w.r.t a tophat selection.

- in AngPow formulas we use normalized windows so that for a tophat over $[z_{\min}, z_{\max}]$ $W = \dfrac{1}{\Delta z}$
- for a gaussian for instance we write $W(z) = \alpha w(z)$ with $w(z) = \exp[-\frac{1}{2}(z-\bar{z})^2/\sigma^2]$

To recover the same power in auto-corelation, one must set

$$\alpha^2 = \frac{\int w^2(z)dz}{(\Delta z)^2} \tag{43}$$

This was checked with Colore+Angpow

For the geneal case $\bar{n}(z)$ one does (probably) have (TBC):

$$\alpha^2 = \frac{\int \left(\bar{n}(z)w(z)\right)^2 dz}{\left(\int \bar{n}(z)dz\right)^2} \tag{44}$$

Shot noise The elementary shot noise for an infinitely thin shell located at r(z) is

$$\delta P(r) = \frac{\delta V}{\delta n} = \frac{4\pi r^2 dr}{4\pi \bar{n}(r)dz}. \tag{45}$$

The comobile distance is related to z via

$$r(z) = \int_0^z \frac{dz'}{H(z')} \tag{46}$$

so that

$$dz = drH(z). \tag{47}$$

Then

$$\delta P(r) = \frac{r^2}{\bar{n}(r)H(z(r))} \tag{48}$$

Including the selection function, the overall contribution to the (auto-) power-spectrum on the shell reads

$$C_\ell^{SN} = \frac{2}{\pi}\int dz\ \tilde{\phi}(z)\int dz'\tilde{\phi}(z')\delta P(r)\int dk\ k^2 j_\ell(kr(z))j_\ell(kr(z')). \tag{49}$$

where $\tilde{\phi}$ are here normalized versions of Eq.(42) $\tilde{\phi} = \dfrac{\phi}{\int \phi(z)dz}$

In virtue of j_ℓ's orthogonality $\int dk\ k^2 j_\ell(kr_1)j_\ell(kr_2) = \dfrac{\pi}{2}\dfrac{\delta(r_1-r_2)}{r_1^2}$,

$$C_\ell^{SN} = \int dz\ \tilde{\phi}(z)\int dz'\tilde{\phi}(z')\frac{1}{\bar{n}(r)H(z)}\delta(r-r') \tag{50}$$

with the change of variable $r = r(z)$

$$C_\ell^{SN} = \int dr \, \frac{\tilde{\phi}^2(r)}{\bar{n}(r)H(z)} H^2(z), \tag{51}$$

and going back to z

$$C_\ell^{SN} = \int dz \, \frac{\tilde{\phi}^2(z)}{\bar{n}(z)} = \frac{\int dz \, \bar{n}(z) W^2(z)}{\left[\int dz \, \bar{n}(z) W(z)\right]^2}. \tag{52}$$

For a tophat selection $W(z) = \dfrac{1}{\Delta z}$

$$C_\ell^{SN} = \frac{1}{\int dz \, \bar{n}(z)} = \frac{4\pi}{N} \tag{53}$$

since the total number of galaxies within the shell is $N = 4\pi \int dz \, \bar{n}(z)$.

References

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Peebles, P. J. E. 1980, The large-scale structure of the universe (Princeton University Press)

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