

Shot noise

S. Plaszczynski

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1 Poisson distribution

$$\text{fish} (n|\mu) = \frac{\mu^n e^{-\mu}}{n!} \quad (1)$$

moments:

$$\mathbb{E}[n] = \mu \quad (2)$$

$$\mathbb{E}[n^2] = \mu(\mu + 1) \quad (3)$$

$$\mathbb{V}[n] = \mathbb{E}[n^2] - \mathbb{E}[n]^2 = \mu \quad (4)$$

2 Variance of a sampled field

Let $\rho(x)$ be *any* stochastic field that follows distribution $f(\rho)$. Poisson-sampling it means observing a discrete n value with probability

$$p_n = \int \text{fish} (n|\rho) f(\rho) d\rho. \quad (5)$$

Then

$$\begin{aligned} \mathbb{E}[n] &= \int \mathbb{E}[n|\rho] f(\rho) d\rho \\ &= \int \rho f(\rho) d\rho \\ &= \mathbb{E}[\rho]. \end{aligned} \quad (6)$$

The mean of the sampled field is the same than of the continuous one: the sampling does not introduce any bias.

Now let see the variance:

$$\begin{aligned} \mathbb{E}[n^2] &= \int \mathbb{E}[n^2|\rho] f(\rho) d\rho \\ &= \int \rho(1 + \rho) f(\rho) d\rho \\ &= \mathbb{E}[\rho] + \mathbb{E}[\rho^2], \end{aligned} \quad (7)$$

$$\begin{aligned} \mathbb{V}[n] &= \mathbb{E}[n^2] - \mathbb{E}[n]^2 \\ &= \mathbb{E}[\rho^2] - \mathbb{E}[\rho]^2 + \mathbb{E}[\rho] \\ &= \mathbb{V}[\rho] + \mathbb{E}[\rho]. \end{aligned} \quad (8)$$

The last line indicates that the variance increased as $\sigma_n^2 = \sigma_\rho^2 + \bar{N}$.

Now recall that $P(k)$ is the power-spectrum of the density *contrast* i.e, $\Delta(x) = \frac{n(x)}{\bar{N}} - 1$ which has variance

$$\mathbb{V}[\Delta] = \frac{\mathbb{V}[n]}{\bar{N}^2} = \mathbb{V}[\rho] + \frac{1}{\bar{N}} \quad (9)$$

For a flat spectrum the variance is essentially $P(k)$ so that we recovered the standard result: Poisson-sampling adds an extra-power of $\frac{1}{\bar{N}} = \frac{V}{N}$

3 Poisson point process

(These are not my ideas but a melting pot of (Peebles 1980; Martinez & Saar 2001; Piattella 2018))

Density field Technically studying catalogs requires using the theory of *random point processes*. One can formally define the number density field as

$$n(\mathbf{x}) \equiv \lim_{R \rightarrow 0} n_R(\mathbf{x}) \quad (10)$$

where R is a ball radius. It is known as the *intensity function* of the process. One generally prefers writing

$$n(\mathbf{x}) = \bar{N}(1 + \delta(x)) \quad (11)$$

where $\bar{N} = \mathbb{E}[n(\mathbf{x})]$ and $\delta(x)$ the dimensionless zero-mean density contrast.

Poisson process If we place randomly a volume δV in \mathbf{x} the probability that it contains one point¹ point is

$$\delta P = n(\mathbf{x})\delta V. \quad (12)$$

Macroscopically, the probability distribution of the number of points lying in a bounded region of volume V is

$$\text{fish} (n|N) = \frac{N^n e^{-N}}{n!}, \quad (13)$$

where

$$N = \int_V n(\mathbf{x}) d\mathbf{x}. \quad (14)$$

Note that since $n(\mathbf{x})$ varies this is an *inhomogeneous* process. To simulate it you must first shoot $n(\mathbf{x})$ to build N then use Eq.(13).

If we now consider two infinitesimally small spheres on \mathbf{x}_1 and \mathbf{x}_2 , the joint probability that in each spheres lies a point of the process is

$$\delta P_{12} = n(\mathbf{x}_1, \mathbf{x}_2)\delta V_1\delta V_2, \quad (15)$$

where $n(\mathbf{x}_1, \mathbf{x}_2)$ is the *second order intensity function*. For a Poisson process

$$n(\mathbf{x}_1, \mathbf{x}_2) = n(\mathbf{x}_1)n(\mathbf{x}_2) + \delta(\mathbf{x}_1 - \mathbf{x}_2)n(\mathbf{x}_1) \quad (16)$$

which can be (vaguely) derived from an argument on very small cells where occupancy is only 0 or 1 (Peebles 1980, §36). But it is simpler to compare this expression to Eq.(8). The last term represents the shot noise contribution.

¹higher number are negligible since they are of higher order

Power spectrum of an observed density field Let us now consider an *observed* field of N galaxies at locations x_i . This can be written

$$\rho(\mathbf{x}) = \sum_{i=1}^N m_i \delta(\mathbf{x} - \mathbf{x}_i). \quad (17)$$

m_i being the galaxy mass. In a survey, what we work with is the density contrast over some (finite) window $w(\mathbf{x})$

$$\delta_s(\mathbf{x}) = \left[\frac{\rho(\mathbf{x})}{\bar{\rho}} - 1 \right] w(\mathbf{x}). \quad (18)$$

The mean density being $\bar{\rho} = \frac{\sum_i m_i}{V}$,

$$\delta_s(\mathbf{x}) = \left[V \frac{\sum_i m_i \delta(\mathbf{x} - \mathbf{x}_i)}{\sum_i m_i} - 1 \right] w(\mathbf{x}) \quad (19)$$

$$= \left[\frac{V}{N} \sum_i \delta(\mathbf{x} - \mathbf{x}_i) - 1 \right] w(\mathbf{x}) \quad (20)$$

assuming each galaxy has the same mass.

The Fourier transform of this contrast is

$$\begin{aligned} \delta_s(\mathbf{k}) &= \frac{1}{V} \int_V d^3\mathbf{x} \delta_s(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{1}{N} \sum_{i=1}^N w(\mathbf{x}_i) e^{-i\mathbf{k}\cdot\mathbf{x}_i} - W(\mathbf{k}), \end{aligned} \quad (21)$$

$W(\mathbf{k})$ being the Fourier transform of the window. The power-spectrum is

$$\begin{aligned} P_s(\mathbf{k}) &= \langle \delta_s(\mathbf{k}) \delta_s(\mathbf{k})^* \rangle \\ &= \frac{1}{N^2} \sum_{i,j} \langle w(\mathbf{x}_i) w(\mathbf{x}_j) \rangle e^{-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} - |W(\mathbf{k})|^2 \end{aligned} \quad (22)$$

(in the computation the cross-products of the 2 terms in Eq.(21) drops when using the condition $\langle \delta(\mathbf{k}) \rangle = 0$).

Let us contemplate (or at least glance at the beauty of) what this means. The $\sum_{i,j}$ term is the Fourier transform of all the pairs counts when $i \neq j$ (what is performed in a real-space analysis) *plus an auto-term* $i = j$. This is the shot-noise term that is equal to

$$P_{SN} = \frac{1}{N^2} \sum_{i=1}^N \langle w(\mathbf{x}_i)^2 \rangle \quad (23)$$

For a tophat selection $\sum_i \langle w(\mathbf{x}_i)^2 \rangle = N$, so that putting back the V factor to get the conventional $P(k)$ dimension

$$P_{SN} = \frac{V}{N}. \quad (24)$$

One may legitimately wonder what happens to the other terms. I just sketch the idea

$$\langle w(\mathbf{x}_i) w(\mathbf{x}_j) \rangle = w(\mathbf{x}_i) w(\mathbf{x}_j) \langle n(\mathbf{x}_i) n(\mathbf{x}_j) \rangle. \quad (25)$$

Using the contrast notation Eq.(11)

$$\langle n(\mathbf{x}_i)n(\mathbf{x}_j) \rangle = \bar{N}^2(1 + \langle \delta(\mathbf{x}_i)\delta(\mathbf{x}_j) \rangle). \quad (26)$$

Injecting it into Eq.(22), the first term cancels $-|W(\mathbf{k})|^2$. The second one can be written as a convolution between the signal and the window $|W(\mathbf{k})|^2$ (this makes sense since it is a multiplication in real space ie. an apodization)

So finally we get the (reasonable) result that

$$P_s(\mathbf{k}) = P(\mathbf{k}) \otimes |W(\mathbf{k})|^2 + P_{SN}. \quad (27)$$

In order to preserve the signal power the window must be normalized as

$$\int_V d^3\mathbf{x} W^2(\mathbf{x}) = 1 \quad (28)$$

4 Tomography

Let now see what happens for a (full-sky) spherical shell with some radial selection $\phi(z)$

formulas

$$a_{lm}(z) = 4\pi i^\ell \int \frac{d^3k}{(2\pi)^3} \delta(k, z) j_\ell(kr(z)) Y_{lm}^*(\hat{k}) \quad (29)$$

$$\rightarrow a_{lm} = \int dz \phi(z) a_{lm}(z) \quad (30)$$

and

$$C_\ell(z_1, z_2) = \langle a_{lm}(z_1) a_{lm}^*(z_2) \rangle \quad (31)$$

$$= \frac{2}{\pi} \int dk k^2 P(k) D(z_1) D(z_2) j_\ell(kr(z_1)) j_\ell(kr(z_2)) \quad (32)$$

$$\rightarrow C_\ell^{A \times B} = \langle a_{lm} a_{lm}^* \rangle \quad (33)$$

$$= \int dz_1 \phi_A(z_1) \int dz_2 \phi_B(z_2) C_\ell(z_1, z_2) \quad (34)$$

$$= \frac{2}{\pi} \int dz_1 \phi_A(z_1) D(z_1) \int dz_2 \phi_B(z_2) D(z_2) \int dk k^2 P(k) j_\ell(kr(z_1)) j_\ell(kr(z_2)). \quad (35)$$

Correlation functions:

$$C(\theta, z_1, z_2) = \langle \delta(n_1, z_1) \delta(n_2, z_2) \rangle_{n_1 \cdot n_2 = \cos \theta} \quad (36)$$

$$= \frac{1}{4\pi} \sum_\ell (2\ell + 1) C_\ell(z_1, z_2) P_\ell(\cos(\theta)) \quad (37)$$

$$= \xi(R(\theta, z_1, z_2)) \quad (38)$$

$$= \frac{1}{2\pi^2} D(z_1) D(z_2) \int dk k^2 P(k) \text{sinc}[kR(\theta, z_1, z_2)] \quad (39)$$

where $\xi(r)$ is the 3D correlation function and $R(\theta, z_1, z_2) = |\vec{r}_1 - \vec{r}_2|$.

$$C^{A \times B}(\theta) = \int dz_1 \phi_A(z_1) \int dz_2 \phi_B(z_2) C(\theta, z_1, z_2) \quad (40)$$

$$= \frac{1}{4\pi} \sum_\ell (2\ell + 1) C_\ell^{A \times B} \quad (41)$$

Shot noise The (un-normalized) radial selection function is written

$$\phi(z) = \bar{n}(z)W(z). \quad (42)$$

The elementary shot noise for an infinitely thin shell located at $r(z)$ is

$$\delta P(r) = \frac{\delta V}{\delta n} = \frac{4\pi r^2 dr}{4\pi \bar{n}(r) dz}. \quad (43)$$

The comobile distance is related to z via

$$r(z) = \int_0^z \frac{dz'}{H(z')} \quad (44)$$

so that

$$dz = dr H(z). \quad (45)$$

Then

$$\delta P(r) = \frac{r^2}{\bar{n}(r)H(z(r))} \quad (46)$$

Including the selection function, the overall contribution to the (auto-) power-spectrum on the shell reads

$$C_\ell^{SN} = \frac{2}{\pi} \int dz \tilde{\phi}(z) \int dz' \tilde{\phi}(z') \delta P(r) \int dk k^2 j_\ell(kr(z)) j_\ell(kr(z')). \quad (47)$$

where $\tilde{\phi}$ are here normalized versions of Eq.(42) $\tilde{\phi} = \frac{\phi}{\int \phi(z) dz}$

In virtue of j_ℓ 's orthogonality $\int dk k^2 j_\ell(kr_1) j_\ell(kr_2) = \frac{\pi}{2} \frac{\delta(r_1 - r_2)}{r_1^2}$,

$$C_\ell^{SN} = \int dz \tilde{\phi}(z) \int dz' \tilde{\phi}(z') \frac{1}{\bar{n}(r)H(z)} \delta(r - r') \quad (48)$$

with the change of variable $r = r(z)$

$$C_\ell^{SN} = \int dr \frac{\tilde{\phi}^2(r)}{\bar{n}(r)H(z)} H^2(z), \quad (49)$$

and going back to z

$$C_\ell^{SN} = \int dz \frac{\tilde{\phi}^2(z)}{\bar{n}(z)} = \frac{\int dz \bar{n}(z) W^2(z)}{[\int dz \bar{n}(z) W(z)]^2}. \quad (50)$$

For a tophat selection $W(z) = \frac{1}{\Delta z}$

$$C_\ell^{SN} = \frac{1}{\int dz \bar{n}(z)} = \frac{4\pi}{N} \quad (51)$$

since the total number of galaxies within the shell is $N = 4\pi \int dz \bar{n}(z)$.

Window weighting Here we assume $\bar{n} = Cte$ and we apply some weight function $W(z)$ to the data. W is assumed to be compact over some $[z_{\min}, z_{\max}]$ region. We write $W(z) = \alpha w(z)$ where w is any function. We wish to normalize it ie. find α so that no C_ℓ power is removed/added to the signal w.r.t a tophat selection.

- in AngPow formulas we use normalized windows so that for a tophat over $[z_{\min}, z_{\max}]$ $W = \frac{1}{\Delta z}$
- for a gaussian for instance we write $W(z) = \alpha w(z)$ with $w(z) = \exp[-\frac{1}{2}(z - \bar{z})^2/\sigma^2]$

To recover the same power in auto-correlation, one must set

$$\alpha^2 = \frac{\int w^2(z) dz}{(\Delta z)^2} \quad (52)$$

This was checked with Colore+Angpow

For the general case $\bar{n}(z)$ one does (probably) have (TBC):

$$\alpha^2 = \frac{\int (\bar{n}(z)w(z))^2 dz}{(\int \bar{n}(z) dz)^2} \quad (53)$$

References

- Martinez, V. & Saar, E. 2001, Statistics of the Galaxy Distribution (CRC Press)
- Peebles, P. J. E. 1980, The large-scale structure of the universe (Princeton University Press)
- Piattella, O. F. 2018, ArXiv e-prints, arXiv:1803.00070