Stochastic models

We have not considered any kind of uncertainty yet, which is our agenda in this lecture.

Implications of uncertainty

So far, we have considered deterministic demand (static as well as time-varying) and constant lead time. We did not really assume everything to be deterministic. Rather, we considered the amount of variation to be insignificant compared to the mean level. For example, imagine a normally distributed annual demand with mean 10000 and standard deviation 50. With the Chebyshev's Inequality, this demand is in $\pm 1\%$, $\pm 2\%$, $\pm 5\%$, and $\pm 10\%$ around its mean with probabilities at least 75%, 93.75%, 99%, and 99.75%. If the mean demand is 1000 (instead of 10000), then the corresponding probability bounds are 0%, 0%, 0%, and 75%. Given the robustness of the EOQ model, considering a steady and deterministic demand rate of $\lambda = 10000$ per year in the first case is unlikely to induce any significant error. We cannot say the same for the second case. Here, we study the implications of different uncertainties in the inventory models, and design good replenishment systems.

<u>Continuous vs. periodic review</u>: Let us consider an EOQ model with $Q^* = 120$ and $T^* = 12$ days. Then the (mean) daily demand $\lambda = 10$. Let us consider small variation in the demand – let us assume that the daily demand can take three values $9,10,11$ with probabilities such that the mean is 10. Then the cycle demand can take values $108,109, \ldots ,132$ with probabilities such that the mean is 120. Let us consider a constant lead time $l = 2$ days.

In the EOQ model, continuous and periodic review policies result in the same outcome. In the continuous review system, we place an order of size $Q^* = 120$ every time the inventory level reaches the reorder point $r = l\lambda = 20$. If we ignore demand variation, then these 20 units are just enough to meet the demand during lead time. Then the shipment of 120 units arrive, and after 10 days, the stock level reaches the reorder point. So, an order is placed every $2 + 10 = 12 = T^*$ days, which is essentially the periodic review system.

Now, imagine a situation where the daily demands of 12 days since the placement of an order are $10,11,11, \ldots ,11$. Then the inventory level would reach the reorder point on the 11th day (instead of $T^* = 12$ days). If the daily demands of the next 13 days are $11,10,9,9, \ldots ,9$, then the inventory level would reach the reorder point on the 13th day (instead of $T^* = 12$ days). Demand uncertainty in the continuous review system results in variation in the time between successive orders, which does not happen in the periodic review system.

Let us consider the periodic review system. Here, we place orders at constant gaps of $T^* = 12$ days. Since the shipments arrive after $l = 2$ days, at the time of the first order inventory level should have been $l\lambda = 20$. These 20 units plus the order quantity shall be just enough to meet the requirements of next $l + T^* = 14$ days, because the second order is placed on 12th day and it arrives on 14th day. So, the first order size should be $(l + T^*)\lambda - l\lambda = 120$. This pattern repeats every time we place an order. So, a constant order $Q^* = 120$ is placed every time the inventory level reaches $l\lambda = 20$, just like the continuous review system.

We refer to $S = (l + T^*)\lambda$ as the order-up-to level. Then the order size in periodic review system is $S - (\text{inventory level at the time of orderimng})$. Due to steady and deterministic demand, inventory level at the time of ordering (at a constant gap of T^*) is always the same, which made the order quantities identical. With demand uncertainty, the order quantities are no more identical, just like the time between successive orders in continuous review system isno more a constant (due to the same demand uncertainty).

<u>Inventory policies</u>: Since the continuous and periodic review policies are no more identical, we need to study them separately. In the table below, we list down the above and some more replenishment policies with their operating mechanisms and features.

Policy	Review type	Mechanism	Feature
(r, Q)	Continuous	Order Q whenever inventory level reaches r	Fixed order quantity Variable cycle length
(T, S)	Periodic	Order to reach level S every T units of time	Variable order quantity Fixed cycle length
(r, S)	Continuous	Order to reach level S whenever inventory level reaches or goes below r	Variable order quantity Variable cycle length
(T, r, S)	Periodic	Every T units of time, order to reach level S if inventory level is r or below, else do nothing	Variable order quantity Variable but regular (as in multiples of T) cycle length

(r, S) policy is designed to react to the situation when inventory level can fall from $r + \epsilon$ to $r - \delta$ instantaneously for some $\epsilon > 0$ and $\delta \geq 0$. It would place an order of size $(S - r + \delta)$. Depending on δ in different cycles, the orders can be different. (r, Q) policy always places an order of size Q ($\equiv S - r$). If customer orders are of unit size, then $\epsilon = 1$ and $\delta = 0$, and then (r, S) policy places constant orders of size $S - r$, just like the (r, Q) policy.

(T, r, S) policy is a generalization of (T, S) as well as (r, S) policies. Consider $r = S^-$ in the (T, r, S) policy. Then it would place order to reach level S every T units of time, just like the (T, S) policy. Similarly, $T = 0^+$ in the (T, r, S) policy will force continuous review and order placement to reach level S whenever inventory level reaches or goes below r.

Choice of the policy depends on the item being considered and the supply environment. A continuous review policy is preferred for high value, fast moving, and critical items. For low value, slow moving, and non-critical items, a periodic review policy is preferred. In each of these policies, demand uncertainty is translated into variability of order quantities or cycle lengths or both, which is faced by the supplier. It may prefer one kind of uncertainty over the other, which also influences the policy choice. Given a policy, our objective here is to decide the policy variables so that cost is minimized.

The idea of safety stock

Let us consider the example mentioned in the beginning of the previous section. Consider the (r, Q) policy with $Q = 120$ and $r = \lambda l = 10 \times 2 = 20$. The daily demand can take values 9,10,11 with 10 as the mean. Then the mean cycle length $T = Q/\lambda = 12$ days. Let us break-down this cycle into two parts – time between arrival of a shipment and the next ordering, which is 10 days on an average, and the time between an ordering and its arrival, which is fixed 2 days. Assume customer orders to be of unit size. Let D_L denote demand during the second part, which is popularly known as the lead time demand.

Assume customer orders to be of unit size. Then the inventory level at the time of ordering is always $r = 20$. Inventory level just before the arrival of a shipment is $r - D_L = 20 - D_L$, which can be negative, signifying unplanned shortage. This does not happen in the other part of the cycle. With other inventory policies too, there is always some part in the cycle that is prone to unplanned shortage if we set policy variables using mean demand. For example, the entire cycle in the (T, S) policy is prone to unplanned shortage.

A simple strategy to avoid this unplanned shortage is to carry some additional stock, known as the safety stock. In the above example, inventory level just before the arrival of a shipment $r - D_L = 20 - D_L \geq -2$. So, if we carry safety stock of $s = 2$, we can avoid the unplanned shortage completely. This increases average inventory level, and thus, increases holding cost. We need to choose it considering the gain from avoiding unplanned shortages and increased holding cost. We do not consider safety stock as a policy variable, because we can indirectly capture it through the other variables. For example, $s = 2$ can be captured by changing the reorder point as: $r = \lambda l + s = 22$.

Since safety stock is not a policy variable, we need to define it precisely. Along with this, let us define some more terminologies to avoid confusion.

a) *On-hand stock*: This is the physical stock that can be used to serve customers. Inventory holding cost is incurred on the on-hand stock. It cannot be negative.
b) *Back-order*: This is the total requirements of the waiting customers when on-hand stock is zero. Back-ordering cost is incurred on the back-order level. This cannot be negative like the on-hand stock; however, both cannot be positive at the same time.
c) *Net-stock*: It is On-hand stock $-$ Back-order. It can be positive (implying holding of physical stock) as well as negative (implying waiting customers).
d) *Transit stock*: It is the total outstanding orders, i.e., orders placed but shipments yet to arrive. It is non-negative, and it does not arise when lead time is zero.
e) *Inventory position*: It is Net-stock $+$ Transit stock. It is used for initiating ordering. So, in the table of policies, 'inventory level' is essentially inventory position.

Note: If we use on-hand stock or net-stock to initiate ordering, then we may end up placing extra order, as these does not keep track of outstanding orders.

f) *Safety stock*: It is defined as the average level of net-stock just before shipments arrive. Note that all other terms are defined over time, i.e., they change with time, while safety stock does not change with time. It is an implicit policy variable.
Note: In the EOQ model without shortage, safety stock is zero. In the case of planned back-order, it is negative. In the stochastic models, it is generally positive.

Stock-out penalty

We have noticed that uncertainty induces unplanned shortages in the inventory models. Here, we discuss about different approaches for measuring stock-out penalty. Next, we capture this penalty for different approaches for the (r, Q) policy.

Stock-out penalty

Since demand uncertainty induces unplanned shortages, it would be part of every stochastic model and influence the policy variables. Cost of shortage is not easy to measure unlike the ordering and holding costs. Here, we list down some of the measures.

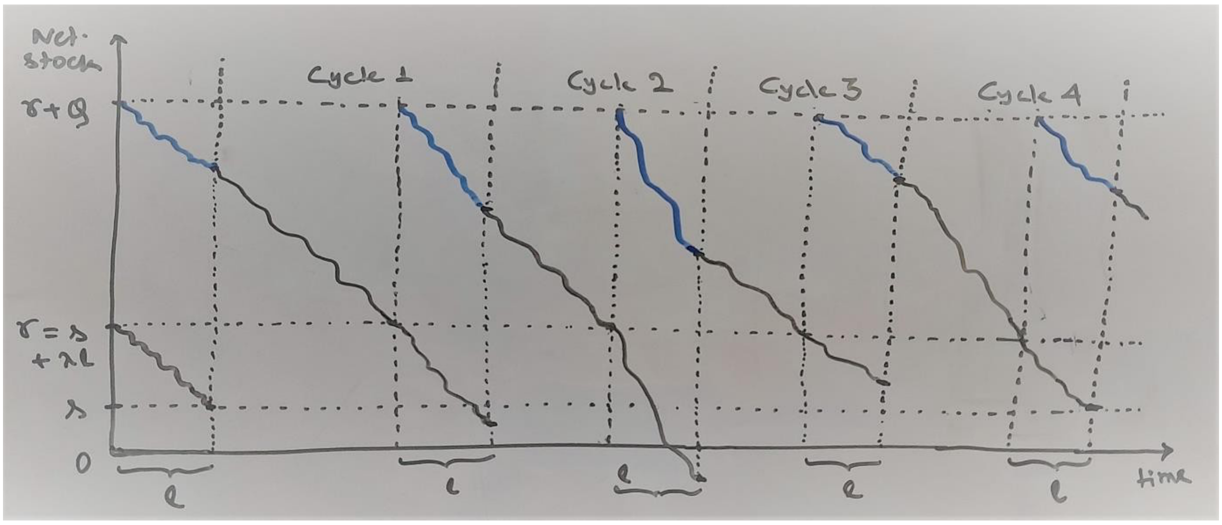
1) *Cost per stock-out occasion*: In this measure, a fixed cost (b_1) is charged every time a shortage happens, irrespective of the level and duration of the shortage. In the lost-sales case, once the on-hand stock becomes zero, it is generally difficult to know the amount of unmet demand. Then this approach works well.
2) *Cost per unit short*: In this measure, a cost per unit short (b_2) is charged, irrespective of the duration of the shortage. In the lost sales case, if the level of unmet demand can be measured, then this measure works well. In the back-ordering case too, this works if the waiting customer's activities are unaffected by the shortage duration.
3) *Cost per unit short per unit time*: In this measure, a cost per unit short per unit time (b_3) is charged. In the back-ordering case, if the waiting customer's activities are affected until its requirements are fulfilled, then this measure works well.
4) *Service level requirement*: In this requirement, the probability of no stock-out, i.e., the long-run fraction of cycles without stock-out, must be at least a specified level known as the service level (α). Note that this requirement is not concerned with the level and duration of the shortage, like the cost per stock-out occasion.
5) *Fill rate requirement*: In this requirement, the fraction of demand fulfilled from on-hand stock must be at least a specified level known as the fill rate (β). Unlike service level, this requirement is concerned with the shortage level.

If cost of shortage (b_1, b_2, b_3, whichever is applicable) can be measured, then we minimize total cost consisting of ordering, holding, and back-ordering/lost sales costs to determine the policy variables. However, cost of shortage is not always easy to measure. Then we minimize ordering and holding with a service level or fill rate constraint (whichever is applicable). This approach has greater use in practice due to the ease in its implementation. Also, shortages are viewed negatively, even if it offers economic benefit. The service level/fill rate constraint can control the occurrence/level of shortage better than the cost-based approach.

(r, Q) Policy: Features

In the (r, Q) policy, *inventory position* is monitored continuously and whenever it reaches the reorder point r, then an order of size Q is placed, which arrives after a constant lead time l.

The following diagram shows the inventory build-up diagram of a typical (r, Q) policy. Here, l is small enough to ensure at most one outstanding order at any point in time. Then net-stock (shown in black) and inventory position (shown in blue) are different only between placing an order and receiving it. Shortage can happen only during this time. Cycle 2 in the diagram shows shortage, which has been back-ordered. If it is a lost-sales case, then the net-stock line would have run along the horizontal axis instead of going below it.



Let D_l denote the lead time demand. Its mean $E[D_l] = \lambda l$, where λ denotes the mean demand rate. *Let us consider the reorder point* $r = \lambda l + s$. In the EOQ model, $s = 0$. Let us assume unit customer order size. Then the inventory position just before placing an order is always r, which is same as the net-stock unless l is large. Just before receiving the shipment, net-stock is $r - D_l$, which on an average is $(\lambda l + s) - \lambda l = s$. So, this s in $r = \lambda l + s$ is essentially the safety stock. We will determine s, and r will be automatically decided.

Let us capture cost rates of different kind. Mean cycle length is still $r = Q/\lambda$. So, the average ordering cost rate is: $K/r = \lambda K/Q$. Just before receiving a shipment, average net-stock is s, and immediately after receiving the shipment, it is $s + Q$. Both are positive, and therefore, are same as the average on-hand stock levels at the end and start of a cycle. Then the average on-hand stock is $s + Q/2$. In rare cases, when back-ordering cost is very small, we may choose a negative s. Then $s + Q/2$ is a good approximation. Thus, $h(s + Q/2)$ is the average holding cost rate. Higher safety stock decreases stock-outs but it increases holding cost too.

Now, we capture stock-out penalty through the approaches mentioned earlier. Let B denote the back-order level just before receiving a shipment. Note that B is the highest level of back-order in a cycle. For the lost-sales case, B represents the total amount of unmet demand in a cycle. Net-stock just before receiving a shipment is $\lambda l + s - D_l$. If this quantity is positive or zero, then $B = 0$, else $B = D_l - r$. Thus, $B = \max\{0, D_l - \lambda l - s\}$. It plays a central role in capturing stock-out penalty. In order to capture behavior of B, we would need distribution and density (or mass) functions of lead time demand D_l. Let $F_{D_l}(\cdot)$ denote the distribution function and $f_{D_l}(\cdot)$ denote the density function. Observe that we did not need distribution of demand during other parts of the cycle to capture anything.

Case-1: If a cost per stock-out occasion (b_1) is applicable, then the average shortage cost rate is $b_1 p_u/r = \lambda b_1 p_u/Q$, where p_u denotes the probability of stock-out in a cycle. Since a positive B is same as stock-out, $p_u = P(B > 0)$, that is,

$$p_u = P(\max\{0, D_l - \lambda l - s\} > 0) = P(D_l > \lambda l + s) = 1 - F_{D_l}(\lambda l + s) \quad \cdots (1a)$$

Let D_l follows normal distribution, which is most commonly observed. Note that $E[D_l] = \lambda l$, irrespective of the distribution. Let σ_l^2 denote variance of D_l. *Let us express the safety stock as:* $s = z\sigma_l$. Then determining z would leads to the value of s, which leads to the value of r. We can express p_u using the standard normal distribution function $\Phi(\cdot)$ and z as

$$p_u = 1 - F_{N(\lambda l, \sigma_l^2)}(\lambda l + z\sigma_l) = 1 - \Phi(\mathrm{z}) \quad \cdots (1b)$$

Case-2: If a cost per unit short (b_2) is applicable, then the average shortage cost rate would be $b_2 E[B]/r = \lambda b_2 E[B]/Q$, where expected quantity short is obtained as

$$E[B] = E[\max\{0, D_l - \lambda l - s\}] = \int_{\lambda l + s}^{\infty} (x - \lambda l - s) f_{D_l}(x) dx \quad \cdots (2a)$$

If $D_l \sim N(\lambda l, \sigma_l^2)$ and $s = z\sigma_l$, then using the normal density function

$$E[B] = \int_{\lambda l + z\sigma_l}^{\infty} (x - \lambda l - z\sigma_l) \frac{1}{\sqrt{2\pi\sigma_l^2}} e^{-\frac{(x-\lambda l)^2}{2\sigma_l^2}} dx\,; \text{ replacing } \frac{x - \lambda l}{\sigma_l} = v$$

$$= \int_{z}^{\infty} (v\sigma_l - z\sigma_l) \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = \sigma_l \int_{z}^{\infty} (v - z)\phi(v) dv$$

where $\phi(\cdot)$ is the standard normal density function. Let us define $\Psi(z) = \int_z^{\infty} (v - z)\phi(v) dv$. It is known as the loss function. It can be calculated numerically just like the standard normal distribution $\Phi(\cdot)$. There are tables for $\Psi(z)$ for $z \in \mathbb{R}$. $\Psi(z)$ is decreasing in z, unlike $\Phi(z)$, which is increasing in z. We can express $E[B]$ using $\Psi(\cdot)$ and z as

$$E[B] = \sigma_l \int_{z}^{\infty} (v - z)\phi(v) dv = \sigma_l \Psi(z) \quad \cdots (2b)$$

Case-3: If a cost per unit short per unit time (b_3) is applicable, then the mean shortage cost rate is $b_3 B_a$, where B_a denotes the average quantity short in a cycle. Shortage, if happens, starts at some point between the point of ordering and the point of shipment, and it ends at the point of shipment with $E[B]$ as the average quantity short. With λ as the mean demand rate, the average duration of shortage is $E[B]/\lambda$. Then B_a can be calculated as

$$B_a = \frac{(1/2) \cdot E[B] \cdot (E[B]/\lambda)}{r = Q/\lambda} = \frac{E^2[B]}{2Q} \quad \cdots (3a)$$

If $D_l \sim N(\lambda l, \sigma_l^2)$ and $s = z\sigma_l$, then we can express B_a using $\Psi(\cdot)$ and z as

$$B_a = \frac{\sigma_l^2 \Psi^2(z)}{2Q} \quad \cdots (3b)$$

Case-4: If there is a service level requirement of $\alpha \in (0,1)$, then there is no shortage cost. We just need to ensure that the probability of no stock-out is at least α. Probability of stock-out p_u has already been obtained in (1a). Then we want the probability of no stock-out, i.e., $1 - p_u$ to be at least α. So, the requirement is

$$1 - p_u \geq \alpha \equiv F_{D_l}(\lambda l + s) \geq \alpha \quad \cdots (4a)$$

If $D_l \sim N(\lambda l, \sigma_l^2)$ and $s = z\sigma_l$, then (4a) can be expressed using $\Phi(\cdot)$ and z as

$$F_{N(\lambda l, \sigma_l^2)}(\lambda l + s) \geq \alpha \equiv \Phi(\mathrm{z}) \geq \alpha \quad \cdots (4b)$$

Case-5: If there is a fill rate requirement of $\beta \in (0,1)$, then shortage cost is not applicable. We just need to ensure that the fraction of demand fulfilled from on-hand stock is at least β. Average cycle demand is $\lambda r = Q$ and the average quantity short in a cycle is $E[B]$. So, the average fraction of demand fulfilled from on-hand stock is $(Q - E[B])/Q = 1 - E[B]/Q$. We want it to be at least β. Using (2a) for $E[B]$, the requirement is

$$1 - \frac{E[B]}{Q} \geq \beta \equiv \int_{\lambda l + s}^{\infty} (x - \lambda l - s) f_{D_l}(x)dx \leq (1 - \beta)Q \quad \cdots (5a)$$

If $D_l \sim N(\lambda l, \sigma_l^2)$ and $s = z\sigma_l$, then using (2b) for $E[B]$, the fill rate requirement of β can be expressed in terms of $\Psi(\cdot)$ and z as

$$1 - \frac{E[B]}{Q} \geq \beta \equiv \Psi(z) \leq (1 - \beta)\frac{Q}{\sigma_l} \quad \cdots (5b)$$

Policy optimization

We have discussed different aspects of (r, Q) policy. Here, we optimize the policy variables.

(r, Q) Policy: Optimization

In the $(r,\ Q)$ replenishment system, average ordering and holding cost rates are $\lambda K/Q$ and $h(s + Q/2)$, where $r = \lambda l + s$. Shortage penalty has five different forms that are expressed using distribution function of lead time demand D_l. Each distribution leads to a different kind of optimal solution for Q and s. Here, we consider the normal distribution $D_l \sim N(\lambda l, \sigma_l^2)$. We also write $s = z\sigma_l$, and obtain optimal solution for Q and z.

Case-1: A cost per stock-out occasion (b_1) is applicable. Then the average shortage cost rate is $\lambda b_1 p_u/Q$, where $p_u = 1 - \Phi(z)$. The optimization problem is:

$$\underset{Q,z}{\text{Minimize}}\ g(Q,z) = \frac{\lambda K}{Q} + h\left(\frac{Q}{2} + z\sigma_l\right) + \frac{\lambda b_1}{Q}\{1 - \Phi(z)\}$$

$g(Q,z)$ is a convex function. Hence, finding its stationary point is sufficient.

$$\frac{\partial g}{\partial Q} = -\frac{\lambda K + \lambda b_1\{1 - \Phi(z)\}}{Q^2} + \frac{h}{2};\ \Rightarrow Q = \sqrt{\frac{2\lambda}{h}[K + b_1\{1 - \Phi(z)\}]} \quad \cdots (1a)$$

$$\frac{\partial g}{\partial z} = h\sigma_l - \frac{\lambda b_1}{Q}\phi(z);\ \Rightarrow \phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} = \frac{h\sigma_l Q}{\lambda b_1} \Rightarrow z = \sqrt{2\ln\left(\frac{\lambda b_1}{\sqrt{2\pi}h\sigma_l Q}\right)} \quad \cdots (1b)$$

Plugging-in one expression into the other does not lead to a closed-form solution. Numerical technique is the way out. We start with the EOQ optimal solution, i.e., $Q_0 = \sqrt{2\lambda K/h}$, as our candidate for Q. It leads to a value of z, say z_0. Then we use z_0 to obtain a new value of Q, say Q_1. Next, Q_1 leads to a new value of z, say z_1. This process is repeated until successive values are close enough. During the iterations, it is possible that the input to the $\ln(\cdot)$ function in (1b) is less than 1. Then z is the square-root of a negative number, which is absurd. In such cases, we set z at its lowest possible value (specified by the management).

A heuristic solution freezes Q and then obtains z. Since $0 \le 1 - \Phi(z) \le 1$, $\sqrt{2\lambda K/h} \le Q \le \sqrt{2\lambda(K + b_1)/h} \equiv Q_0 \le Q \le Q_0\sqrt{1 + b_1/K}$, where $Q_0 = \sqrt{2\lambda K/h}$ is the EOQ solution. We can settle at some value between these two bounds and obtain the corresponding z.

Case-2: A cost per unit short (b_2) is applicable. The average shortage cost rate is $\lambda b_2 E[B]/Q$, where $E[B] = \sigma_l \Psi(z)$ with $\Psi(z) = \int_z^\infty (v - z)\phi(v)dv$ denoting the loss function. Using the Leibniz integral rule, $\Psi'(z) = \int_z^\infty -\phi(v)dv = \Phi(z) - 1$. The optimization problem is:

$$\underset{Q,z}{\text{Minimize}}\, g(Q,z) = \frac{\lambda K}{Q} + h\left(\frac{Q}{2} + z\sigma_l\right) + \frac{\lambda b_2 \sigma_l \Psi(z)}{Q}$$

$g(Q,z)$ is a convex function. Hence, finding its stationary point is sufficient.

$$\frac{\partial g}{\partial Q} = -\frac{\lambda K + \lambda b_2 \sigma_l \Psi(z)}{Q^2} + \frac{h}{2};\ \Rightarrow Q = \sqrt{\frac{2\lambda}{h}\{K + b_2\sigma_l\Psi(z)\}} \quad \cdots (2a)$$

$$\frac{\partial g}{\partial z} = h\sigma_l + \frac{\lambda b_2 \sigma_l}{Q}\{\Phi(z) - 1\};\ \Rightarrow \Phi(z) = 1 - \frac{hQ}{\lambda b_2} \Rightarrow z = \Phi^{-1}\left(1 - \frac{hQ}{\lambda b_2}\right) \quad \cdots (2b)$$

Again, plugging-in one expression into the other does not lead to a closed-form solution. We can solve the equation numerically in an iterative manner, starting with the EOQ solution $Q_0 = \sqrt{2\lambda K/h}$. During the iterations, it is possible that the input to the $\Phi^{-1}(\cdot)$ function in (2b) is zero or less. Then $\Phi(z)$ is non-positive, which is absurd. In such cases, we set z at its lowest possible value (specified by the management).

During the iterations, we would need to obtain $\Psi(z)$. There are tables for it, but MS Excel does not have a dedicated function for it. The following expression of $\Psi(z)$ is implemented in excel. Then we can perform the iterations easily.

$$\Psi(z) = \int_z^\infty (v - z)\phi(v)dv = \int_z^\infty v\phi(v)dv - z\int_z^\infty \phi(v)dv$$

$$= \int_z^\infty -\phi'(v)dv - z\{1 - \Phi(z)\}, \text{as } \phi'(v) = \frac{d}{dv}\left(\frac{e^{-v^2/2}}{\sqrt{2\pi}}\right) = \phi(v)(-v)$$

$$= -\phi(v)|_z^\infty - z\{1 - \Phi(z)\} = \phi(z) - z\{1 - \Phi(z)\}$$

A heuristic solution freezes Q and then obtains z. Since $\Psi(z) \geq 0, Q \geq \sqrt{2\lambda K/h} = Q_0$, the EOQ solution. We do not have an upper bound, as $\Psi(z) \to \infty$ as $z \to -\infty$. However, with a reasonably small value of z, we can obtain an upper bound. Then we can settle at some point between the bounds and obtain the corresponding z.

<u>Case-3</u>: A cost per unit short per unit time (b_3) is applicable. Then the average shortage cost rate is $b_3 B_a$, where $B_a = \sigma_l^2\Psi^2(z)/2Q$. The optimization problem is:

$$\underset{Q,z}{\text{Minimize}}\, g(Q,z) = \frac{\lambda K}{Q} + h\left(\frac{Q}{2} + z\sigma_l\right) + \frac{b_3\sigma_l^2\Psi^2(z)}{2Q}$$

$g(Q,z)$ is a convex function. Hence, finding its stationary point is sufficient.

$$\frac{\partial g}{\partial Q} = -\frac{2\lambda K + b_3\sigma_l^2\Psi^2(z)}{2Q^2} + \frac{h}{2};\ \Rightarrow Q = \sqrt{\frac{2\lambda K}{h} + \frac{b_3\sigma_l^2\Psi^2(z)}{h}} \quad \cdots (3a)$$

$$\frac{\partial g}{\partial z} = h\sigma_l + \frac{b_3\sigma_l^2}{Q}\Psi(z)\{\Phi(z) - 1\};\ \Rightarrow \Psi(z)(1 - \Phi(z)) = \frac{hQ}{b_3\sigma_l} \quad \cdots (3b)$$

Obtaining z from (3b) is no easy task. There is a connection between the above optimization problem and the one in Case-5. If we consider a fill rate requirement of $\beta = b_3/(b_3 + h)$ in Case-5, then the optimal solution of Case-5 will be optimal for the above problem.

Case-4: There is a service level requirement of $\alpha \in (0,1)$, which is same as $\Phi(\mathrm{z}) \geq \alpha$. There is no explicit shortage cost. The (constrained) optimization problem is:

$$\underset{Q,z}{\text{Minimize}}\, g(Q,z) = \frac{\lambda K}{Q} + h\left(\frac{Q}{2} + z\sigma_l\right) \text{ s.t. } \Phi(\mathrm{z}) \geq \alpha$$

Dependence of $g(Q, z)$ on Q and z is additive with $g(Q, z)$ being convex in Q and increasing in z. Thus, it makes sense to select the smallest z satisfying the constraint, i.e., $z^* = \Phi^{-1}(\alpha)$. With z decided, the optimal order quantity $Q^* = \sqrt{2\lambda K/h} = Q_0$, the EOQ solution.

Case-5: There is a fill rate requirement of $\beta \in (0,1)$, which is same as $\Psi(z) \leq (1 - \beta)\, Q/\sigma_l$. There is no explicit shortage cost. The (constrained) optimization problem is:

$$\underset{Q,z}{\text{Minimize}}\, g(Q,z) = \frac{\lambda K}{Q} + h\left(\frac{Q}{2} + z\sigma_l\right) \text{ s.t. } \Psi(z) \leq (1-\beta)\frac{Q}{\sigma_l}$$

Dependence of $g(Q, z)$ on Q and z is additive with $g(Q, z)$ being convex in Q and increasing in z. However, the constraint connects z and Q, which was not the case earlier. Given a Q, we shall select the smallest z satisfying the constraint. Since $\Psi(z)$ is decreasing in z, $z^*(Q) = \Psi^{-1}((1-\beta)Q/\sigma_l)$ is optimal. Then the optimization problem becomes

$$\underset{Q}{\text{Minimize}}\, g(Q) = \left(\frac{\lambda K}{Q} + \frac{hQ}{2}\right) + h\sigma_l\, \Psi^{-1}\left((1-\beta)\frac{Q}{\sigma_l}\right)$$

Even though the above optimization problem is single-variable and unconstrained, we won't be able to solve it using the standard approach due to the presence of $\Psi^{-1}(\cdot)$. We can solve it numerically. The 1st part of $g(Q)$ is the EOQ cost function and it is convex in Q with $Q_0 = \sqrt{2\lambda K/h}$ being its minima. Now, $\Psi(\cdot)$ is a decreasing function that makes $\Psi^{-1}(\cdot)$ decreasing too. Then the 2nd part of $g(Q)$, i.e., $h\sigma_l\Psi^{-1}((1 - \beta)Q/\sigma_l)$, is decreasing in Q. Combining these two observations, we see that $g(Q)$ decreases in Q in the range $(0, Q_0]$, and beyond Q_0, 1st part increases and the 2nd part decreases. Then the balance is achieved at some $Q > Q_0$. Starting at Q_0, we can increase Q gradually as long as $g(Q)$ decreases. A heuristic approach freezes Q at some point above Q_0 and then obtains the corresponding z.

Stochastic lead time

We have analyzed and optimized the $(r,\ Q)$ policy. Here, we discuss some additional aspects of the (r,Q) policy, notably the randomness of lead time.

Lost-sales in (r,Q) policy

We did not differentiate between back-ordering and lost-sales while capturing the stock-out penalty, as both cases are governed by the quantity short $B = \max\{0,\ D_l - \lambda l - s\}$. However, the calculation of average on-hand stock that attracts holding cost changes slightly. In Lecture 9, we implicitly assumed back-ordering and argued that the net-stock varies from $\lambda l + s - D_l + Q$ to $\lambda l + s - D_l$ during a cycle. Their expected values are $s + Q$ and s, which we assumed positive, and obtained the average on-hand stock during the cycle to be $s + Q/2$. In the case of lost sales, net-stock cannot be negative, and hence, it varies from $\max\{0, \lambda l + s - D_l\} + Q$ to $\max\{0, \lambda l + s - D_l\}$ during a cycle. Let us define

$$I_e = E[\max\{0, \lambda l + s - D_l\}] = \int_0^{\lambda l+s} (\lambda l + s - x) f_{D_l}(x) dx$$

to be the expected end-of-cycle on-hand stock. If $D_l \sim N(\lambda l, \sigma_l^2)$ and $s = z\sigma_l$, then

$$\begin{aligned}
I_e &= \int_0^{\lambda l+z\sigma_l} (\lambda l + z\sigma_l - x) f_{N(\lambda l,\sigma_l^2)}(x) dx \approx \int_{-\infty}^{\lambda l+z\sigma_l} (\lambda l + z\sigma_l - x) f_{N(\lambda l,\sigma_l^2)}(x) dx \\
&= \int_{-\infty}^{\infty} (\lambda l + z\sigma_l - x) f_{N(\lambda l,\sigma_l^2)}(x) dx + \int_{\lambda l+z\sigma_l}^{\infty} (x - \lambda l - z\sigma_l) f_{N(\lambda l,\sigma_l^2)}(x) dx \\
&= \{\lambda l + z\sigma_l - E[N(\lambda l, \sigma_l^2)]\} + \sigma_l \Psi(z) = \sigma_l \{z + \Psi(z)\}
\end{aligned}$$

Then the expect beginning-of-cycle on-hand stock is $I_e + Q$, and the average on-hand stock during the cycle is $I_e + Q/2$, which is slightly more than $s + Q/2$. This changes the holding cost rate, and the policy optimization becomes complicated. A heuristic approach would be to use the policy variables of the back-ordering case.

Stochastic lead time: Implications

We have assumed lead time to be a constant. This, too, may be uncertain like the demand. Let L denote the random lead time. In our study of (r,Q) policy, lead time featured in two places: (i) in deciding the reorder point r and (ii) in deciding the safety stock s. We have obtained the reorder point as: $r = \lambda l + s$ where l denotes the constant lead time. In the random lead time case, it makes sense to consider $r = \lambda E[L] + s$. We have obtained s using the distribution of shortage quantity $B = \max\{0,\ D_l - \lambda l - s\}$ and the model for capturing stock-out penalty. In the random lead time case, shortage quantity becomes: $B = \max\{0, D_L - \lambda E[L] - s\}$ where D_L denotes the demand during the random lead time. If we know the distribution of D_L, then we can obtain the safety stock using our previous approach.

Randomness in lead time increases variability of lead time demand. To better understand this, consider L to be integer valued and let $D_1, D_2, \ldots$ denote random demand in periods 1,2, ... Then $D_L = D_1 + D_2 + \cdots + D_L$. Observe the random index in the sum. Let $D_1, D_2, \ldots$ be *iid* random variables with mean λ and variance σ^2. Consider $D_1, D_2, \ldots$ to be independent of N. Let l denote the expected value of L. Then

$$E[D_L] = E[D_1 + D_2 + \cdots + D_L] = E_L[E[D_1 + D_2 + \cdots + D_L | L = l]]$$
$$= \sum_l E[D_1 + D_2 + \cdots + D_l]P(L = l) = \sum_l \lambda l P(L = l) = \lambda E[L] = \lambda l$$

In the constant lead time case, expected lead time demand is $E[D_l] = E[D_1 + D_2 + \cdots + D_l] = \lambda l$, which is same as the above. So, randomness in lead time does not change the expected lead time demand. Let us check variance of lead time demand. In the constant lead time case, $Var(D_l) = Var(D_1 + D_2 + \cdots + D_l) = l\sigma^2$. In the random lead time case,

$$Var(D_L) = Var(D_1 + D_2 + \cdots + D_L) = E[(D_1 + D_2 + \cdots + D_L)^2] - E^2[D_1 + D_2 + \cdots + D_L]$$
$$= E_L[E[(D_1 + D_2 + \cdots + D_L)^2 | L = l]] - (\lambda l)^2$$
$$= E_L[Var[Y|L = l] + E^2[Y|L = l]] - \lambda^2 l^2, \text{where } Y = D_1 + D_2 + \cdots + D_L$$
$$= E_L[Var(D_1 + D_2 + \cdots + D_l) + E^2[D_1 + D_2 + \cdots + D_L]] - \lambda^2 l^2$$
$$= \sum_l (l\sigma^2 + \lambda^2 l^2)P(L = l) - \lambda^2 E^2[L] = \sigma^2 E[L] + \lambda^2(E[L^2] - E^2[L])$$
$$= l\sigma^2 + \lambda^2 Var(L) = Var(D_l) + \lambda^2 Var(L)$$

It's evident that the randomness in lead time increases variability of lead time demand. This, in turn, increases safety stock in every model of stock-out penalty. To understand this better, consider $Z_{D_L} = (D_L - E[D_L])/\sqrt{Var(D_L)}$. It's the standardized version of random variable D_L. Note hat $E[Z_{D_L}] = 0$ and $Var(Z_{D_L}) = 1$. Z_{D_L} captures essence of the randomness in D_L, while keeping mean at 0 and variance at 1. Consider any normal random variable and the standard normal random variable for comparison. With Z_{D_L} defined, we can express $D_L = E[D_L] + Z_L\sqrt{Var(D_L)} = \lambda l + \sigma_L Z_L$ where $Var(D_L) = \sigma^2{}_L$. Note that $\sigma_L = \sqrt{l\sigma^2 + \lambda^2 Var(L)}$, which reduces to $\sigma_L = \sqrt{l\sigma^2} = \sqrt{l}\sigma$ in the constant lead time case.

Let $F_{Z_{D_L}}(\cdot)$ and $f_{Z_{D_L}}(\cdot)$ denote the distribution and density functions of Z_{D_L}. Then probability of stock-out in a cycle, which is used in 1st and 4th model can be calculated as:

$$p_u = P(\max\{0, D_L - \lambda E[L] - s\} > 0) = P(\lambda l + \sigma_L Z_L > \lambda l + s) = P(Z_L > s/\sigma_L)$$
$$= 1 - F_Z\left(s/\sqrt{l\sigma^2 + \lambda^2 Var(L)}\right)$$

Observe that the randomness in lead time increases p_u. The purpose of safety stock s in the 1st and 4th model is to protect against the possibility of stock-out. Since p_u has increased, s must be increased accordingly to compensate.

Now, we calculate expected quantity short that is used in 2nd, 3rd, and 4th model.

$$E[B] = E[\max\{0, D_L - \lambda E[L] - s\}] = E[\max\{0, (\lambda l + \sigma_L Z_L) - \lambda l - s\}]$$
$$= E[\sigma_L \max\{0, Z_L - s/\sigma_L\}] = \sigma_L \int_{s/\sigma_L}^{\infty} (z - s/\sigma_L) f_Z(z) dz$$

As σ_L increases due to randomness in lead time, the integrand, the range of integration, and the multiplier outside the integration, all increases, and thus, $E[B]$ increases. The purpose of safety stock s in the 2nd, 3rd, and 4th model is to protect against the magnitude of stock-out. Since $E[B]$ has increased, s must be increased accordingly to compensate.

Stochastic lead time: Optimization

While the above arguments talk about qualitative change in the policy variables as a result of the randomness in lead time, obtaining optimal value of s becomes lot more challenging, as the distribution of lead time demand D_L becomes complex. Normal distribution typically fits well to demand data during a fixed duration. So, it's reasonable to consider $D_1, D_2, \dots$ to be independent $N(\lambda, \sigma^2)$ random variables. In the case of constant lead time, the demand during lead time $D_l = D_1 + D_2 + \cdots + D_l \sim N(l\lambda, l\sigma^2) = N(\lambda l, \sigma_l^2)$, where $\sigma_l^2 = l\sigma^2$, is normally distributed. This, however, changes with random lead time.

Consider L takes values 1,2,3, ... with probabilities $p_1, p_2, p_3, \dots$. Let $E[L] = l$. Then $E[D_L] = \lambda l$ and $Var(D_L) = l\sigma^2 + \lambda^2 Var(L) = \sigma_L^2$, say. We may expect D_L to be normally distributed with mean λl and variance σ_L^2, but that's not the case.

$$F_{D_L}(x) = P(D_L \le x) = \sum_k P(D_L \le x | L = k) P(L = k) = \sum_k p_k P(D_1 + D_2 + \cdots + D_k \le x)$$
$$= \sum_k p_k P(N(k\lambda,\ k\sigma^2) \le x) = \sum_k p_k \Phi\left(\frac{x - k\lambda}{\sqrt{k}\sigma}\right) \text{ for all } x \in \mathbb{R}$$
$$\Rightarrow f_{D_L}(x) = \sum_k \frac{p_k}{\sqrt{k}\sigma} \phi\left(\frac{x - k\lambda}{\sqrt{k}\sigma}\right) \text{ for all } x \in \mathbb{R}$$

Distribution of the above kind is called mixture distribution. Let us obtain the probability of stock-out p_u and expected quantity short $E[B]$ in a cycle in terms of safety stock s.

$$p_u = P(\max\{0, D_L - \lambda E[L] - s\} > 0) = P(D_L > \lambda l + s) = 1 - F_{D_L}(\lambda l + s)$$
$$= 1 - \sum_k p_k \Phi\left(\frac{\lambda l + s - k\lambda}{\sqrt{k}\sigma}\right) = 1 - \sum_k p_k \Phi\left(\frac{(l-k)\lambda + s}{\sqrt{k}\sigma}\right)$$

$$E[B] = E[\max\{0, D_l - \lambda E[L] - s\}] = \int_{\lambda l + s}^{\infty} (x - \lambda l - s) f_{D_L}(x) dx$$
$$= \int_{\lambda l + s}^{\infty} (x - \lambda l - s) \sum_k \frac{p_k}{\sqrt{k}\sigma} \phi\left(\frac{x - k\lambda}{\sqrt{k}\sigma}\right) dx =$$
$$= \sum_k p_k \int_{\lambda l + s}^{\infty} (x - \lambda l - s) \phi\left(\frac{x - k\lambda}{\sqrt{k}\sigma}\right) \frac{dx}{\sqrt{k}\sigma}; \text{ replacing } \frac{x - k\lambda}{\sqrt{k}\sigma} = v$$

$$= \sum_k p_k \int_{\frac{(l-k)\lambda+s}{\sqrt{k}\sigma}}^{\infty} \{v\sqrt{k}\sigma - (l-k)\lambda - s\}\phi(v)dv$$

$$= \sum_k p_k\sqrt{k}\sigma \int_{\frac{(l-k)\lambda+s}{\sqrt{k}\sigma}}^{\infty} \{v - \frac{(l-k)\lambda+s}{\sqrt{k}\sigma}\} \phi(v)dv$$

$$= \sigma \sum_k p_k\sqrt{k}\Psi\left(\frac{(l-k)\lambda+s}{\sqrt{k}\sigma}\right), \text{where } \Psi(\cdot) \text{ is the loss function}$$

Unlike the constant lead time case, p_u and $E[B]$ are not simple functions of s (or $z = s/\sigma_l$). Therefore, we cannot use the above expressions in the optimization models. One way out is to freeze s and obtain the values of p_u and $E[B]$ from the above expressions, which then can be used in the optimization models to determine the optimal order quantity $Q^*(s)$ for that s. Let s_0 denote the optimal value of s in the constant lead time model with $l = E[L]$ as the lead time. Since the optimal s in the stochastic lead time model must be at least s_0, we can obtain $Q^*(s)$ for some $s \geq s_0$ and pick an appropriate s and the corresponding $Q^*(s)$.

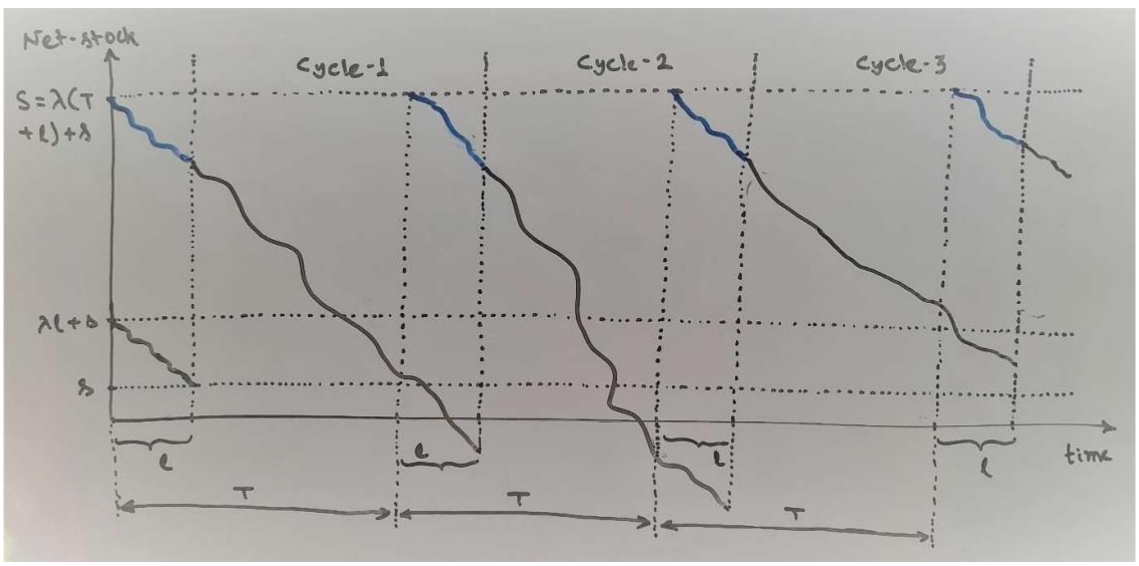
Our discussion so far has been from the point of view of an operations researcher. We took the randomness as given and worked out the best decision in that situation. The operations manager, however, can work towards reducing the randomness. For example, entering into a contract with supplier and developing long-term relation, that require commitment from both sides, reduces lead time uncertainty. Similarly, initiatives like membership, pre-booking, etc., which requires some discounts, can reduce demand uncertainty. We can perform cost-benefit analysis of such initiatives, and implement the ones have a positive net benefit. Then we can apply our inventory models to deal with the residual uncertainty.

(T, S) Policy

We have studied the (r, Q) policy in details. Now, we discuss the (T, S) policy.

Policy analysis

In the (T, S) replenishment system, inventory position is checked every T units of time and an order is placed such that the inventory position reaches the level S. The shipment arrives after a constant lead time l. The following diagram shows the inventory build-up diagram of a typical (T, S) policy. Here, l is small enough to ensure at most one outstanding order at any point in time. Then net-stock (shown in black) and inventory position (shown in blue) are different only between placing an order and receiving it. Shortage can happen anywhere in a cycle. Cycles 1 and 2 in the diagram shows shortages, which has been back-ordered. Cycle-1 shortage occurs between placement of an order and receiving the shipment, whereas Cycle-2 shortage happens between receiving a shipment and placing the next order. In the lost-sales case, the net-stock line would have run along the horizontal axis instead of going below it.



Let I_b and I_e denote the net-stock at the beginning and end of a cycle, which are just after and before receiving a shipment. Let D_{T+l} denote the demand during a cycle and the preceding lead time. Then $I_e = S - D_{T+l}$. Since $E[I_e] = s$, the safety stock, then it makes sense to set $S = \lambda(T + l) + s$, where λ is the mean demand rate. Then $E[I_e] = S - E[D_{T+l}] = \{\lambda(T + l) + s\} - \lambda(T + l) = s$, as required. We need to determine s and T that minimizes cost.

Let us capture cost rates of different kind. Here, cycle length is fixed at T. Then the ordering cost rate is K/T. In case lead time is stochastic (with l as the mean), then cycle length is no more fixed, but its expected value is still T. Then K/T is the average ordering cost rate. Mean net-stock at the end of a cycle $E[I_e] = s$, which we assume to be positive. Net-stock at the beginning of a cycle $I_b = S - D_l$, where D_l denotes the lead time demand. Then $E[I_b] = S -$

$E[D_l] = \{\lambda(T + l) + s\} - \lambda l = \lambda T + s$. Then the average on-hand stock is $(E[I_b] + E[I_s])/2 = \lambda T/2 + s$. In rare cases, when back-ordering cost is very small, we may choose a negative s. Then $\lambda T/2 + s$ is a good approximation. Thus, $h(\lambda T/2 + s)$ is the average holding cost rate. This, however, increases for the lost-sales case.

Now, we capture stock-out penalty through the approaches mentioned earlier. Let B denote the quantity short just before receiving a shipment. B is the highest level of back-order in a cycle. In the lost-sales case, B represents the total amount of unmet demand in a cycle. Net-stock just before receiving a shipment is $S - D_{T+l}$. If this quantity is positive or zero, then $B = 0$, else $B = D_{T+l} - S$. Thus, $B = \max\{0, D_{T+l} - S\} = \max\{0, D_{T+l} - \lambda(T + l) - s\}$. It plays a central role in capturing stock-out penalty.

Let $F_{D_{T+l}}(\cdot)$ and $f_{D_{T+l}}(\cdot)$ denote the distribution and density () functions of D_{T+l}. These functions are influenced by T, which is a policy variable. The probability of stock-out p_u and the expected quantity short $E[B]$ in a cycle can be calculated as:

$$p_u = P(B > 0) = P(\max\{0, D_{T+l} - \lambda(T + l) - s\} > 0) = P(D_{T+l} > \lambda(T + l) + s)$$
$$= 1 - F_{D_{T+l}}(\lambda(T + l) + s) \quad \cdots (1)$$

$$E[B] = E[\max\{0, D_{T+l} - \lambda(T + l) - s\}] = \int_{\lambda(T+l)+s}^{\infty} (x - \lambda(T + l) - s) f_{D_{T+l}}(x) d \quad \cdots (2)$$

With these expressions, we can express stock-out penalty as shown in the table below.

Approach	Cost rate/Constraint
Cost per stock-out occasion (b_1)	$b_1 p_u/T$
Cost per unit short (b_2)	$b_2 E[B]/T$
Cost per unit short per unit time (b_3)	$b_3 E^2[B]/2\lambda T$
Service level requirement (α)	$1 - p_u \geq \alpha$
Fill rate requirement (β)	$1 - E[B]/\lambda T \geq \beta$

Policy optimization

Shortage penalty depends on the distribution D_{T+l}, which is influenced by the policy variable T. It is very difficult to express $F_{D_{T+l}}$ and $f_{D_{T+l}}$ in terms of T, except for the case of normally distributed demand. Let $D_1, D_2, \ldots$ denote demands of periods 1,2, ... Consider these to be *iid* $N(\lambda, \sigma^2)$ random variables. Then $D_{T+l} \sim N(\lambda(T + l), (T + l)\sigma^2)$. This does not hold if lead time is stochastic (as observed in the last class). Let us express the safety stock as: $s = z\sigma_{T+l}$, where $\sigma^2_{T+l} = (T + l)\sigma^2$. It makes the holding cost rate $\lambda T/2 + z\sigma_{T+l} = \lambda T/2 + z\sqrt{T + l}\sigma$. Now, determining z would leads to the value of s. We can express p_u and $E[B]$ using the standard normal distribution and loss functions and z as follows.

$$p_u = 1 - F_{N(\lambda(T+l),\sigma^2_{T+l})}(\lambda(T + l) + z\sigma_{T+l}) = 1 - \Phi(z) \quad \cdots (1a)$$

$$E[B] = \int_{\lambda(T+l)+z\sigma_{T+l}}^{\infty} (x - \lambda(T+l) - z\sigma_{T+l}) \frac{1}{\sqrt{2\pi\sigma_{T+l}^2}} e^{-\frac{(x-\lambda(T+l))^2}{2\sigma_{T+l}^2}} dx$$

$$= \int_{z}^{\infty} (v\sigma_{T+l} - z\sigma_{T+l}) \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv \text{, by replacing } \frac{x - \lambda(T+l)}{\sigma_{T+l}} = v$$

$$= \sigma_{T+l} \int_{z}^{\infty} (v - z)\phi(v)dv = \sqrt{T+l}\sigma\Psi(z) \quad \cdots (2a)$$

With the above expressions, our optimization problems are:

Case 1: $\underset{T,z}{\text{Minimize}}\ g(T,z) = \frac{K}{T} + h\left(\frac{\lambda T}{2} + z\sqrt{T+l}\sigma\right) + \frac{b_1}{T}\{1 - \Phi(z)\}$

Case 2: $\underset{T,z}{\text{Minimize}}\ g(T,z) = \frac{K}{T} + h\left(\frac{\lambda T}{2} + z\sqrt{T+l}\sigma\right) + \frac{b_2}{T}\sqrt{T+l}\sigma\Psi(z)$

Case 3: $\underset{T,z}{\text{Minimize}}\ g(T,z) = \frac{K}{T} + h\left(\frac{\lambda T}{2} + z\sqrt{T+l}\sigma\right) + \frac{b_3}{2\lambda}\left(1 + \frac{l}{T}\right)\sigma^2\Psi^2(z)$

Case 4: $\underset{T,z}{\text{Minimize}}\ g(T,z) = \frac{K}{T} + h\left(\frac{\lambda T}{2} + z\sqrt{T+l}\sigma\right)$ s.t. $\Phi(z) \geq \alpha$

Case 5: $\underset{T,z}{\text{Minimize}}\ g(T,z) = \frac{K}{T} + h\left(\frac{\lambda T}{2} + z\sqrt{T+l}\sigma\right)$ s.t. $\Psi(z) \leq \frac{(1-\beta)\lambda T}{\sqrt{T+l}\sigma}$

Each of the above problem is difficult to solve. So, we follow a heuristic approach where T is decided first. The EOQ solution, i.e., $T_0 = \sqrt{2K/\lambda h}$, is a good choice. Also, external factors influence T. A periodic review policy is typically followed for low-value and slow-moving items, and many such items are ordered together. Then a common review period T is decided first, and then individual order-up-to levels are decided.

Once the review period T is decided, then we have a single-variable optimization problem in terms of z, which is manageable. Tn the first three cases (as above), one part of the objective function is linear in z and the other part decreases with z. We can find the optimal z numerically. In the fourth case, $z = \Phi^{-1}(\alpha)$ is the optimal solution, and in the fifth case, $z = \Psi^{-1}(\gamma)$, where $\gamma = (1-\beta)\lambda T/(\sqrt{T+l}\sigma)$. This approach works for non-normal demands as well. Once T is decided, we can obtain $F_{D_{T+l}}$ and $f_{D_{T+l}}$ and then solve for s.

Newsvendor model

We have studied (r, Q) and (T, S) policies in detail. Here, we briefly discuss other inventory policies, and then study a perishable inventory model - the newsvendor model.

Other policies

(r, S) policy: It is a continuous-review policy like the (r, Q) policy, except that the order size is not fixed any more. Inventory position is monitored continuously and whenever it reaches or goes below the reorder point r, an order is placed such that the inventory position becomes the order-up-to level S. Let I_o denote the inventory position just before placing an order. In the (r, Q) system, we assume $I_o = r = \lambda l + s$, which is true when customer orders are of unit size. Then in the (r, S) system, too, $I_o = r$ and orders are of fixed size $Q = S - r$. When customer order sizes follow a general distribution, then $I_0 \leq r$, and these policies differ.

Let $U = r - I_0$ denote the amount of undershoot (with respect to r) of the customer order that triggers ordering. Let V denote the random variable that describes customer order size. Then $V > U$ and it strongly influences the distribution of U. Other than V, the gap between S and r influences U, though it can be ignored if the gap is much larger than the order size V, which usually is the case. Then $E[U] \approx E[V^2]/2E[V] = E[V]/2 + Var(V)/2E[V]$. If may seem that $E[U]$ should be $E[V]/2$, but it is more than that due to the *inspection paradox*. One can easily estimate mean and variance of customer order size, and thus, estimate $E[U]$.

With the expected undershoot obtained, it makes sense to set $r = E[U] + \lambda l + s$ so that $E[I_o] = r - E[U] = \lambda l + s$. Now, we consider a (r, Q) system with the above r. Since $E[I_o] = \lambda l + s$, as is assumed in (r, Q) system, optimal policy variables of the modified (r, Q) system is same as that of a typical system. We can use these optimal values, i.e., Q^* and s^*, to obtain a good solution for the (r, S) system as: $r = E[U] + \lambda l + s^*$ and $S = r + Q^*$. Better solution can be obtained by considering distribution of the undershoot.

(T, r, S) policy: In the (T, r, S) replenishment system, inventory position is checked every T units of time, and if it touches or goes below the reorder point r, then an order is placed such that the inventory position reaches the order-up-to level S. If inventory position at the point of review is above the reorder point, then no order is placed. It is a generalization of (r, S) and (T, S) policies, as explained earlier, and it performs better than the other policies. Analysis and optimization of (T, r, S) policy is quite complicated, and we skip it here.

Stochastic dynamic demand: Demand can be stochastic as well as time varying. It's the most general form of demand, and the most difficult to manage. A simplistic solution approach is to apply the methods of the lot-sizing problem considering the expected demands of different periods, and then add appropriate safety stock considering the demand variability.

Newsvendor model

Consider the problem faced by a newspaper vendor in a railway station. At the end of a day, it must decide how many newspapers to order for the next day. If he orders too many, there will be unsold stock at the end of the day, which does not have much value on the next day. On the other hand, if he orders too little, then there will be lost sales and associated penalties. The ordering decision must balance these two costs. This simple inventory model has many applications, particularly in inventory management of perishable and seasonal items. Some of the assumptions of this model are listed below.

1) The planning horizon consists of a single selling season of the product. Multiple selling seasons are treated as independent of one another.
2) There is only one opportunity for procurement/production for a season, which must be completed before the random demand is realized.
3) Unmet demand at the end of the selling season, if any, are lost and attracts penalty.
4) Excess stock at the end of the selling season, if any, are either sold at a discounted price (lower than the purchase price) or disposed-off.
5) Holding cost is incurred on the end-of-season stock (if any), and fixed cost of ordering is not too high to make ordering uneconomical.

With the above assumptions, there are two relevant costs: under-stocking cost of c_u per unit short and over-stocking cost of c_o per unit excess. Observe that holding cost is part of c_o and c_u includes missed profit (due to lost sales) as well as goodwill loss. Let X denote the random demand with known distribution and density functions $F(\cdot)$ and $f(\cdot)$. Let Q denote the order quantity. Then $\max\{0, X - Q\}$ is the quantity short and $\max\{0, Q - X\}$ is the excess stock. Total cost associated with these shortage and excess is:

$$C(Q,X) = c_o \max\{0, Q - X\} + c_u \max\{0, X - Q\}$$

$$\Rightarrow E[C(Q)] = c_o \int_0^Q (Q - x) f(x)dx + c_u \int_Q^\infty (x - Q) f(x)dx$$

$$\Rightarrow \frac{dE[C(Q)]}{dQ} = c_o \int_0^Q f(x)dx - c_u \int_Q^\infty f(x)dx \text{, by Leibniz integral rule}$$

$$= c_o F(Q) - c_u\{1 - F(Q)\}$$

$$\Rightarrow \frac{d^2E[C(Q)]}{dQ^2} = c_o f(Q) + c_u f(Q) > 0 \Rightarrow E[C(Q)] \text{ is convex}$$

Optimal solution: $c_o F(Q^*) - c_u\{1 - F(Q^*)\} = 0 \Rightarrow F(Q^*) = \dfrac{c_u}{c_o + c_u}$

We have a remarkably simple-looking expression for the optimal solution. The fraction $\xi = c_u/(c_o + c_u)$ is known as the critical fractile. The optimal order quantity increases with the critical fractile, which is decreasing in c_o and increasing in c_u. If the demand is normally distributed with mean μ and variance σ^2, i.e., $X \sim N(\mu. \sigma^2)$, then $F(Q^*) = \Phi((Q^* - \mu)/\sigma)$

$= \xi \Rightarrow (Q^* - \mu)/\sigma = \Phi^{-1}(\xi) \Rightarrow Q^* = \mu + \sigma\Phi^{-1}(\xi)$. In case the demand is modelled as a discrete random variable, then the equation $F(Q) = \xi$ may not be satisfied by any Q. Then the optimal order quantity is: $Q^* = \min\{Q: F(Q) \geq \xi\}$.