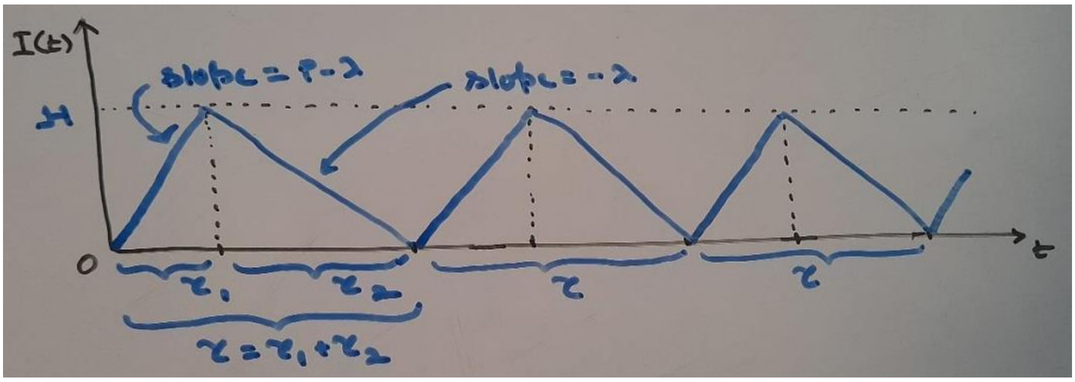
EPQ model

If we look at applicability of the EOQ model, it applies to retailers facing a deterministic and steady demand and a constant lead time. If we consider the case of distributor, its demand is generated by the retailers it supplies to. As the EOQ model suggests, retailers would order in bulk, which makes the distributor's demand unsteady in short time-windows. For example, consider a retailer facing a steady demand of 50 units every day and it orders 350 units once in a week. Then the distributor's demand is unsteady at the level of day, but it is steady at the level of week. If we model this demand by a steady rate, the error will be insignificant in the long-run. Thus, if retailer's orders are steady in terms of quantity and gap between successive orders, then the distributor's demand can be modelled as deterministic and steady. If its lead time is constant, then EOQ model applies to the distributor as well.

When it comes to the manufacturer, if distributor's orders are steady in terms of quantity and gap between successive orders, then the manufacturer's demand can be modelled as steady and deterministic. Lead time for manufacturer is its production throughput time, which can be regarded as constant. This setting is suitable for the EOQ model, but there is one distinction from the case of retailer/distributor – production happens in a steady manner, unlike supply that happens in bulk. Let P denote the rate of production. Then we need $P \geq \lambda$, the demand rate, in order to meet the demand. Again, we have a cyclic pattern involving production up-time and down-time. During the up-time, production takes place, which is consumed during the whole cycle. Since $P \geq \lambda$, during the up-time, denoted by r_1, inventory builds up at the rate $P - \lambda$. So, the maximum inventory level $H = (P - \lambda)r_1$. During the down-time, denoted by r_2, this build-up inventory is consumed at the rate of λ, i.e., $H = \lambda r_2$, and the next cycle begins with zero stock. This is depicted in the diagram below.



Let $r = r_1 + r_2$ denote the cycle time and Q denote the production/consumption quantity in a cycle. Then $Q = Pr_1 = \lambda r$ and $H = (P - \lambda)r_1 = (P - \lambda)(Q/P) = Q(1 - \lambda/P)$. Since the average inventory level during a cycle is $H/2$, holding cost rate is $hH/2 = hQ(1 - \lambda/P)/2 = h'Q/2$, where $h' = h(1 - \lambda/P)$. The production cost rate is similar to the ordering cost rate of the EOQ model, i.e., $K/Q + c\lambda$, of which $c\lambda$, the variable part of the production cost, does not depend on Q. So, we need to minimize $G(Q) = K/Q + h'Q/2$. This is similar to the EOQ model, and thus, the optimal solution is: $Q^* = \sqrt{2\lambda K/h'}$. Then the optimal policy is to produce Q^* units or run production for $r_1 = Q^*/P$ time whenever the stock-level reaches zero. This is known as the economic production quantity (EPQ) model.

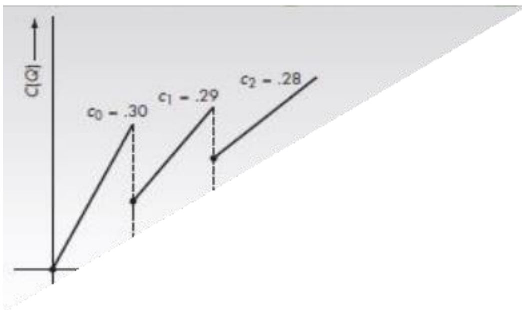
All-unit discount scheme

In the EOQ model, cost of ordering quantity Q is $K + cQ$ for all $Q > 0$. This assumes that the cost of an item c is independent of the order quantity. This may not be true, particularly in the case of fast-moving consumer goods. There are discounting schemes that attempts to induce greater consumption by decreasing cost of an item for higher order quantities. The following example illustrates one such discounting scheme known as all-unit discount.

Variable part of the ordering cost

$$C(Q) = \begin{cases} 0.30Q \text{ for } Q < 500 \\ 0.29Q \text{ for } 500 \leq Q < 1000 \\ 0.28Q \text{ for } Q \geq 1000 \end{cases}$$

$C(Q)$ is shown in the diagram to the right. Note that $C(500) < C(500^-)$ and $C(1000) < C(1000^-)$. It may appear as irrational!

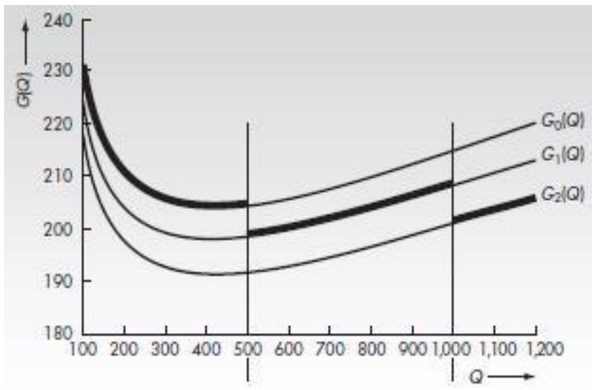


Let $c(Q) = C(Q)/Q$ denote the variable cost of ordering an item. $c(Q) = 0.30$ for $Q < 500$, 0.29 for $500 \leq Q < 1000$, and 0.28 for $Q \geq 1000$. Let the fixed cost of ordering $K = 8$ and demand rate $\lambda = 600$ per year. Since the item cost $c(Q)$ changes with Q, it is appropriate to consider the holding cost as: $h(Q) = I \times c(Q)$, where I denotes the annual interest capturing the holding cost per item per year. Let us take $I = 20\%$. Then the total cost rate

$$G(Q) = \frac{\lambda K}{Q} + \lambda c(Q) + h(Q)\frac{Q}{2}$$

$$= \begin{cases} G_0(Q) \text{ for } Q < 500, \\ G_1(Q) \text{ for } 500 \leq Q < 1000, \\ G_2(Q) \text{ for } Q \geq 1000. \end{cases}$$

Plugging in the values,
$G_0(Q) = 600 \cdot 8/Q + 180 + 0.060\, Q/2$,
$G_1(Q) = 600 \cdot 8/Q + 174 + 0.058\, Q/2$,
$G_2(Q) = 600 \cdot 8/Q + 168 + 0.056\, Q/2$.



Due to the similarity of the above cost rates with that of the EOQ model, $Q^* = \sqrt{2\lambda K/h}$ is the *unconstraint optimal* solution in each case. Let Q'_0, Q'_1, Q'_2 denote these.

$$Q'_0 = \sqrt{2 \times 600 \times 8/0.060} = 400 \sim \text{in } (0,500)$$
$$Q'_1 = \sqrt{2 \times 600 \times 8/0.058} = 406 \sim \text{not in } [500,1000)$$
$$Q'_2 = \sqrt{2 \times 600 \times 8/0.056} = 414 \sim \text{not in } [1000,\infty)$$

Let Q^*_0, Q^*_1, Q^*_2 denote the *constrained optimal* solutions of $G_0(Q), G_1(Q), G_2(Q)$. Since Q'_0 is in its range, $Q^*_0 = Q'_0 = 400$. Since Q'_1, Q'_2 are out of their ranges and $G_1(Q), G_2(Q)$ are convex, the nearest in-range point is the constrained optima. Therefore, $Q^*_1 = 500$ and $Q^*_2 = 1000$. Now, we compare the cost rates at the constrained optimal solutions.

$$G_0(Q^*_0) = 600 \cdot 8/400 + 180 + 0.060 \cdot 400/2 = 204,$$
$$G_1(Q^*_1) = 600 \cdot 8/500 + 174 + 0.058 \cdot 500/2 = 198.1,$$
$$G_2(Q^*_2) = 600 \cdot 8/1000 + 168 + 0.056 \cdot 1000/2 = 200.8.$$

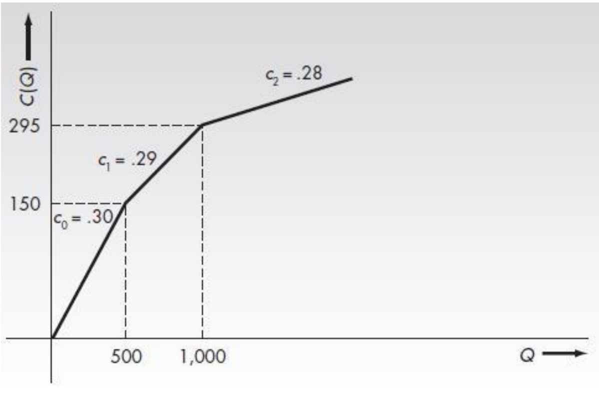
Thus, the overall optimal order quantity $Q^* = 500$. We can apply the above procedure in any case of all-unit quantity discount scheme and obtain the optimal solution.

Incremental discount scheme

Let us consider another discounting scheme known as incremental discount.

$$C(Q) = \begin{cases} 0.30Q \text{ for } Q < 500 \\ 0.3 \times 500 + 0.29(Q - 500) = 5 + 0.29Q \text{ for } 500 \le Q < 1000 \\ (5 + 0.29 \times 1000) + 0.28(Q - 1000) = 15 + 0.28Q \text{ for } Q \ge 1000 \end{cases}$$

$C(Q)$ is shown in the figure to the right. Note that it is continuous, unlike the case of all-unit discounting scheme. Here, the variable cost of ordering an item has a complex form, as shown next.



$$c(Q) = C(Q)/Q$$
$$= \begin{cases} 0.30 \text{ for } Q < 500 \\ 0.29 + 5/Q \text{ for } 500 \le Q < 1000 \\ 0.28 + 15/Q \text{ for } Q \ge 1000 \end{cases}$$

Let us consider same values for the other cost parameters, i.e., $K = 8$, $\lambda = 600$ per year, and $h(Q) = I \times c(Q)$ with $I = 20\%$ per year. Then the total cost rate

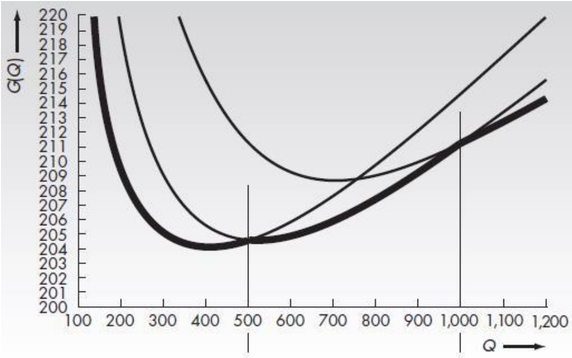
$$G(Q) = \frac{\lambda K}{Q} + \lambda c(Q) + h(Q)\frac{Q}{2} = \begin{cases} G_0(Q) \text{ for } Q < 500, \\ G_1(Q) \text{ for } 500 \le Q < 1000, \\ G_2(Q) \text{ for } Q \ge 1000. \end{cases}$$

Plugging in the values,

$G_0(Q) = 600 \cdot 8/Q + 180.0 + 0.060\, Q/2,$

$G_1(Q) = 600 \cdot 13/Q + 174.5 + 0.058\, Q/2,$

$G_2(Q) = 600 \cdot 23/Q + 169.5 + 0.056\, Q/2.$



These are shown in the diagram to the right. Note that $G(Q)$, which is a combination of $G_0(Q), G_1(Q), G_2(Q)$ is continuous, unlikethe case of all-unit discount.

EOQ approximation

So far, we have considered steady deterministic demand, which was modelled in a continuous fashion. Demand during interval $(t_1, t_2]$ was captured by $\lambda(t_2 - t_1)$ for $0 \leq t_1 < t_2$, where λ denotes the steady demand rate. Here, we consider situations where the demand rate changes with time. Some such cases are listed below.

(1) Items having seasonal demand patterns.
(2) Items in the start or end of their product life cycle.
(3) Production or supply to fulfill contractual obligation that require certain quantities to be delivered to the customer on specific dates such that demand is not steady.

In (3), demand is deterministic, but in (1) and (2), some amount of randomness is expected. If the amount of variation with respect to the mean is 'limited', then a deterministic model is reasonable. Let us consider demand to be time-varying but deterministic. One way to model this kind of demand is to consider a time-varying demand rate $\lambda(t)$ for $t \geq 0$. Then demand during interval $(t_1, t_2]$ is $\int_{t_1}^{t_2} \lambda(t)dt$. The major problem with this approach is to find out $\lambda(t)$ that fits the demand pattern and is simple enough for the optimization. It may work for rare cases, but it's impossible to develop a general model like EOQ with this approach. Thus, *we discretize time*, which is a major departure from the EOQ setting.

With discretized time, demand is modeled as: d_t in period-t where $t = 1,2,3, ...$ One time- period can be of any length, e.g., a month. This allows modelling all kinds of time-varying demand. Once such pattern, which we will use for illustrations, is shown below.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
t	1	2	3	4	5	6	7	8	9	10	11	12
d_t	10	62	12	130	154	129	88	52	124	160	238	41

Consider the demand of 10 units in January. Discrete modelling approach does not tell how these 10 units are realized in the month, which continuous modelling approach does. We can make a reasonable assumption that demand within a period is steady. Another assumption, which would appear unrealistic, can be that *a period's demand fully realizes in the beginning of the period*. The second approach greatly simplifies the optimization procedure without any significant compromise on the solution quality (to be explained later). So, we follow the later approach. With this, *holding/back-order cost is applicable on the end-of-period stock level*. With the other approach, holding/back-order cost would depend on both starting and ending stock level of a period. For now, *we will not allow any shortage, which will be relaxed later*. We also assume zero lead time, which can be easily relaxed to include the case of constant lead time, as we have seen in the case of EOQ model. We continue to consider the same kind of ordering, holding, and back-order costs.

An important implication of the above considerations is as follows: *For a planning horizon of T periods, in an optimal ordering decision, order quantity in period-t, for* $t = 1,2, \dots, T$, *mustbe* 0 *or* d_t *or* $d_t + d_{t+1}$ *or* $d_t + d_{t+1} + d_{t+2}$ *or* $\cdots$ *or* $d_t + d_{t+1} + \cdots + d_T$. Suppose we donot follow this. Then there exists $t \in \{1,2, \dots, T\}$ such that the order quantities up to period-t fulfill demands up to period t' for some $t' \geq t$ (as shortages are not allowed) and still carry some left-over stock, say $\delta > 0$, which is insufficient for fulfilling demand of period $t' + 1$ in case $t' < T$. If $t' = T$, then we could have ordered δ less and fulfill demand of all periods, thereby saving inventory holding cost. In case $t' < T$, an order of at least $d_{t^F+1} - \delta$ is placed in period $t' + 1$, as shortages are not allowed. If we increase this order by δ and reduce past orders accordingly, then no additional cost is incurred, and we save inventory holding cost while fulfilling all demands. Thus, the claim is true.

The above observation greatly simplifies the task of finding optimal order quantities. First, we approach through the EOQ route. We ignore the period-to-period variation of demand and consider a steady demand of $\lambda = \sum^T{}_{t=1} d_t/T$ per period, which is 100 per month in the above example. Let fixed cost of ordering $K = 54$ and inventory holding cost $h = 0.4$ per item per month. Then the EOQ model would suggest $Q^* = \sqrt{2\lambda K/h} = 164.3$. Given our observation on optimal order quantities, we shall adjust Q^* to match with the nearest cumulative demand. This process, when repeated as shown below, gives an approximate solution.

Month	1	2	3	4	5	6	7	8	9	10	11	12	Total
Demand	10	62	12	130	154	129	88	52	124	160	238	41	1200
Order	214	–	–	–	154	129	140	–	124	160	238	41	1200
Inventory	204	142	130	0	0	0	52	0	0	0	0	0	528

Cost of this solution is: $54 \times 8 + 0.4 \times 528 = 643.2$.

Wagner-Whitin algorithm

Here, we discuss a dynamic programming procedure developed by Wagner-Whitin that finds optimal ordering decision. We illustrate the method with the earlier example. Consider Q_1. It must be d_1 or $d_1 + d_2$ or $d_1 + d_2 + d_3$ or $\cdots$ or $d_1 + d_2 + \cdots + d_T$. Costs associated with these options are: $K, K + d_2h, K + d_2h + 2d_3h, \cdots\cdots, K + d_2h + 2d_3h + \cdots + (n-1)d_nh$. These are shown in the first row of the table below. Now, consider Q_2. It must be 0 or d_2 or $d_2 + d_3$ or $d_2 + d_3 + d_4$ or $\cdots\cdots$ or $d_2 + d_3 + \cdots + d_T$. Costs associated with these options are: $C^*_1, C^*_1 + K, C^*_1 + K + d_3h, C^*_1 + K + d_3h + 2d_4h, \cdots\cdots, C^*_1 + K + d_3h + 2d_4h + \cdots + (n-2)d_nh$, where C^*_1 is the minimum cost of fulfilling demand of the first period. C^*_1 is shown in bold in the first column and the rest are shown in the second row of the table below.

Consider Q_3. It must be 0 or d_3 or $d_3 + d_4$ or $d_3 + d_4 + d_5$ or $\cdots\cdots$ or $d_3 + d_4 + \cdots + d_T$. Costs associated with these options are: C^*_2, $C^*_2 + K$, $C^*_2 + K + d_4h$, $C^*_2 + K + d_4h + 2d_5h$, $\cdots\cdots$, $C^*_2 + K + d_4h + 2d_5h + \cdots + (n-3)d_nh$, where C^*_2 is the minimum cost of fulfilling demand of the first two periods. C^*_2 is shown in bold in the second column and the rest are shown in the third row of the table below. This way, all options are shown.

Month	1	2	3	4	5	6	7	8	9	10	11	12
1	**54**	**79**	**88**	244	491	749	960	1106	1502	2078	3030	3211
2		108	113	217	402	608	784	909	1256	1768	2625	2789
3			133	185	308	463	604	708	1005	1453	2215	2362
4				**142**	204	307	413	496	744	1128	1794	1926
5					**196**	**248**	318	381	579	899	1470	1585
6						250	**286**	327	476	732	1208	1306
7							302	**323**	422	614	995	1077
8								340	389	517	803	868
9									**377**	441	631	680
10										**431**	526	559
11											**485**	**501**
12												539
Order	84	--	--	130	283	--	140	--	124	160	279	--
Demand	10	62	12	130	154	129	88	52	124	160	238	41
Inventory	74	12	0	0	129	0	52	0	0	0	41	0

The minimum entry in the last column, that is C^*_T gives us the minimum cost of fulfilling all demands. We can trace backward from the last column and figure out how that least cost has been arrived at, which gives us optimal ordering decision. In the example, $C^*_{12} = 501$ and we trace backward along the underlined cells and figure out that the order quantities mentioned in the bottom of the table is optimal. Cost associated with the optimal decision is $54 \times 7 + 0.4 \times 308 = 501.2$, which is about 22% less than the EOQ approach.

Wagner-Whitin's approach did not become as popular as the EOQ model, primarily because of the finite planning horizon. With larger T, our solution improves, but demand forecasting error increases. Thus, we choose the largest possible T such that forecasting error does not exceed a threshold. Assume that $T = 12$ months is such a length for the example problem. In the 1st month, we obtain the optimal solution and implement $Q^*_1 = 84$. When we go to the 2nd month, we have 74 units of stock and demand forecast of month-13, which was unavailable earlier. With the new data, $Q^*_1 = 84$, which has already been implemented, may become sub- optimal. This is one major problem associated with Wagner-Whitin's approach. It essentially makes the Wagner-Whitin's optimal solution for the planning horizon $\{1,2, \ldots, T\}$ merely a heuristic solution when time periods beyond T are considered.

Lot sizing - Heuristics

We have discussed Wagner-Whitin's optimization approach for the lot sizing problem. Here, we develop heuristics for the same and consider some relaxations in the model.

Heuristic approach

Rolling plan, that is, changing planning horizon with passage of time, is very common in real life, makes the Wagner-Whitin's approach a heuristic. Here, we discuss some more heuristics with less computation requirements than Wagner-Whitin's approach.

Least per period cost: In this approach, we calculate cost per period for each option of Q_1. If $Q_1 = d_1 + d_2 + \cdots + d_t$, then cost per period is $\{K + d_2h + 2d_3h + \cdots + (t-1)d_th\}/t$ for $t \in \{1,2, \dots , T\}$. This quantity initially decreases with t and then increases. Let the decrease occurs till period t' and then it increases. Then we choose $Q_1 = d_1 + d_2 + \cdots + d_{t^F}$ and $Q_t = 0$ for $t = 2,3, \dots, t'$. The same process is repeated from period $t' + 1$. The following table shows these calculations and the ordering decision. It is same as that of the Wagner-Whitin's method, and thus, cost is also the same.

Month	1	2	3	4	5	6	7	8	9	10	11	12
1	54.0	39.4	**29.5**	61.1								
4				**54.0**	57.8							
5					54.0	**52.8**	61.6					
7							54.0	**37.4**	68.3			
9									**54.0**	59.0		
10										**54.0**	74.6	
11											54	**35.2**
Order	84	--	--	130	283	--	140	--	124	160	279	--
Demand	10	62	12	130	154	129	88	52	124	160	238	41
Inventory	74	12	0	0	129	0	52	0	0	0	41	0

Least per unit cost: This one is very similar to the least per unit cost approach, except that we calculate cost per unit for each option of the order quantity. If $Q_1 = d_1 + d_2 + \cdots + d_t$, then cost per unit is $\{K + d_2h + 2d_3h + \cdots + (t-1)d_th\}/\sum_{i=1}^{t} d_i$. The table below shows these calculations and the ordering decision. Cost of this decision is 558.8.

Month	1	2	3	4	5	6	7	8	9	10	11	12
1	5.40	1.09	**1.05**	1.14								
4				0.42	**0.41**	0.53						
6						0.42	**0.41**	0.49				
8								1.04	**0.59**	0.69		
10										**0.34**	0.37	

11											**0.23**	0.25
12												**1.32**
Order	84	--	--	284	--	217	--	176	--	160	238	41
Demand	10	62	12	130	154	129	88	52	124	160	238	41
Inventory	74	12	0	154	0	88	0	124	0	0	0	0

Part-period balancing: In this approach, we choose order quantities such that the fixed cost of ordering and the total holding cost for that order are closest to one another. Recall that in the EOQ model, ordering and holding costs are equal at the optimal decision. Here, we chose $Q_1 = d_1 + d_2 + \cdots + d_t$ such that K and $d_2h + 2d_3h + \cdots + (t-1)d_th$ are closest. This is repeated to provide the complete decision. The following table shows these calculations and the ordering decision. Cost of this decision is 600.0.

Month	1	2	3	4	5	6	7	8	9	10	11	12
1	0.0	24.8	**34.4**	190								
4				0.0	**61.6**							
6						0.0	**35.2**	76.8				
8								0.0	**49.6**	177		
10										0.0	**95.2**	
12												**0.0**
Order	84	--	--	284	--	217	--	176	--	398	--	41
Demand	10	62	12	130	154	129	88	52	124	160	238	41
Inventory	74	12	0	154	0	88	0	124	0	238	0	0

Wagner-Whitin's approach, as a heuristic, may be better than the above three heuristics, but with rolling plan, Wagner-Whitin's solution may change with passage of time. This does not happen with the three heuristics, except for the last order quantity. From a practical point of view, this is an advantage. In real-life applications, EOQ approximation is preferred due to its relatively steady order quantities, despite its inferior solution quality.

Practice problems

Book-1: Production and Operations Analysis by Nahmias and Olsen [7th Edition]

EPQ model
Book-1, Chapter-4, Problem No. 17, 20

Discount schemes
Book-1, Chapter-4, Problem No. 21, 24, 25

Book-2: Inventory and Production Management in Supply Chains by Silver, Pyke, andThomas [4th Edition]

Lot sizing problem
Book-2, Chapter-5, Problem No. 4, 10, 15, 18