EOQ model

Costs

1. *Holding cost*: Inventory holding/carrying cost is the sum of all costs that are proportional to the on-hand inventory level at any point in time. It consists of
    1. a) Cost of capital tied up with the stock
    2. b) Cost of providing physical space occupied
    3. c) Breakage, spoilage, deterioration, and obsolescence
    4. d) Insurance, taxes, etc.

    We generally consider holding cost to be linear with inventory level and duration of inventory holding. Let h denote holding cost per unit quantity and unit time. Let $I(t)$

    denote the inventory level at time t. Then the holding cost during $[t_1, t_2]$ is h $\int^{t_2} I(t)dt$ = h$I_a(t_2 - t_1)$, where I_a = $\int^{t_2} I(t)dt/(t_2 - t_1)$ is the average inventory level.

2. *Ordering cost*: Holding cost includes all those costs that are proportional to the inventory on-hand, whereas the ordering cost captures all those costs that depend on the order size or the production quantity. Typically, it has a fixed component and a variable component. Let K denote the fixed component and c denote the variable component. Then ordering cost for size x is $K + cx$ if $x > 0$ and zero if $x = 0$. K consists of fixed cost of production or procurement, material handling, bookkeeping, etc. c consists of per-unit production or purchasing cost, per-unit material handling cost, etc.

3. *Shortage cost*: Shortage/stock-out cost/penalty is the cost of not having sufficient stock to satisfy a demand when it occurs. This cost has a different interpretation depending upon whether excess demand is back-ordered (i.e., unfulfilled orders are held on the books until the next shipment arrives) or lost (known as lost sales). In the back-order case, shortage cost includes bookkeeping cost and late fulfillment penalty (as agreed between the parties involved). In the lost sales case, shortage cost includes the lost profit (that could not be made due to the shortage). In either case, there is a loss of goodwill.

    Depending on the situation, shortage cost can be measured on a per unit quantity basis, particularly in the case of lost sales, or on a per unit quantity and unit time basis (like the holding cost). Sometimes, shortage cost is not considered explicitly and is replaced by a minimum service level or fill rate requirement (to be explained later).

With increased procurement or production quantity, total ordering cost in a planning horizon decreases (due to the presence of the fixed component). It decreases shortage cost too. On the other hand, it increases the holding cost. In inventory management, we design and optimize procurement and production policies that balances these conflicting costs. Depending on the features of the inventory system, different kinds of models are developed.

Features

1. Stationary vs. dynamic demand
2. Deterministic vs. stochastic demand
3. Nature of lead time (constant vs. random)
4. Back-order vs. lost sales vs. service level/fill rate
5. Continuous vs. periodic review system

Other factors such as perishability, price, etc.

Once we complete our study of inventory control for each stage in the production system, we will take a holistic view and try to streamline the whole system, which is the focus of supply chain management. There are other kind of flows in the supply chain, e.g., money, data, etc., which we will not consider in this course. In service operations management, movement of material takes the backstage. There we will study quite different topics.

EOQ model

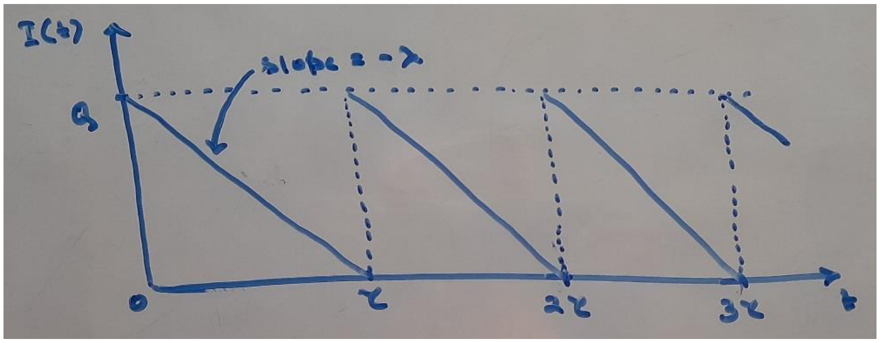
Economic order quantity (EOQ) model is the simplest and the most fundamental inventory model. It applies to situations when demand is stationary as well as deterministic, lead time is constant, and shortages are back-ordered. Consumable items without seasonality effect shows these characteristics. Note that we did not mention about the nature of the review system. It will be evident later. To begin with, we impose some additional restrictions.

1. 1) Demand can be viewed as 'continuous', and due to its stationarity and deterministic nature, demand during time t can be represented as λt where λ is the demand rate.
2. 2) Lead time is zero (to be relaxed later)
3. 3) Shortages are not permitted (to be relaxed later)
4. 4) Cost components are fixed cost of ordering K per order, variable cost of ordering c per

   item ordered, inventory holding cost h per unit quantity and unit time.

Note that (1) and (4) are not really restrictions, these are modelling approaches. With these, let Q denote the optimal order quantity at time zero. It is implicitly assumed that the stock level at t = 0 is zero, else it makes sense to place the first order at t = $I(0)/\lambda$ to avoid unnecessary holding cost, and in that case, we can regard $I(0)/\lambda$ as the 'beginning of time'. Since shortage is not permitted, the second order must be placed by t = Q/λ. Again, for avoiding unnecessary holding cost, it makes sense to place the second order at t = Q/λ. Observe that the state of the system at t = 0 and t = Q/λ are identical. Therefore, the second order must be same as Q, the optimal first order quantity. The same must be the case with all subsequent orders. This identical ordering makes the gap between two successive orders Q/λ identical too. Thus, we can operate this inventory system both as continuous and as periodic review system. In the continuous review, we place an order of size Q whenever stock level reaches zero, and in the periodic review, we place the order after every Q/λ units of time. Let us determine the optimal value of Q in terms of cost parameters.

With Q as the order quantity, the system returns to the same state after every $\tau = Q/\lambda$ units of time. So, there is a cyclic behavior, as shown in the inventory build-up diagram below.



There is one order in every cycle. So, the per cycle ordering cost is: $K + cQ$. In every cycle, inventory level starts at Q and linearly reduces to 0 in the end. So, the average inventory level in a cycle $I_a = Q/2$, and the per cycle holding cost is: $\mathrm{h}(Q/2)\tau$. There is no shortage cost. So, the total cost per cycle is: $K + cQ + \mathrm{h}Q\tau/2$. Now, cycle cost is not a good measure of system cost (*why*?). We need to decide a planning horizon first, and then minimize total cost in that horizon. This brings the problem of boundary effect, i.e., location of the end of the horizon with respect to the above cycles. To avoid this complication, *we assume infinite horizon*. In the infinite horizon, system cost is infinite too. So, we choose the long-run cost rate as the performance measure. Observe that the long-run cost rate is same as the cost rate in a cycle, i.e., $(K + cQ + \mathrm{h}Q\tau/2)/\tau = \lambda K/Q + c\lambda + \mathrm{h}Q/2$.

Since $c\lambda$ does not depend on Q, it is appropriate to minimize $G(Q) = \lambda K/Q + \mathrm{h}Q/2$.

$G'(Q) = -\lambda K/Q^2 + \mathrm{h}/2 \Rightarrow G'(Q) = 0$ gives $Q^* = \sqrt{2\lambda K/\mathrm{h}}$ as the stationary point. $G''(Q) = 2\lambda K/Q^3 > 0$ for $Q > 0 \Rightarrow G(Q)$ is convex and Q^* is the optimal order quantity. It shall be noted that at the optimal order quantity, i.e., $Q = Q^*$, the fixed part of ordering cost rate, i.e., $\lambda K/Q$, and the holding cost rate, i.e., $\mathrm{h}Q/2$, are the same.

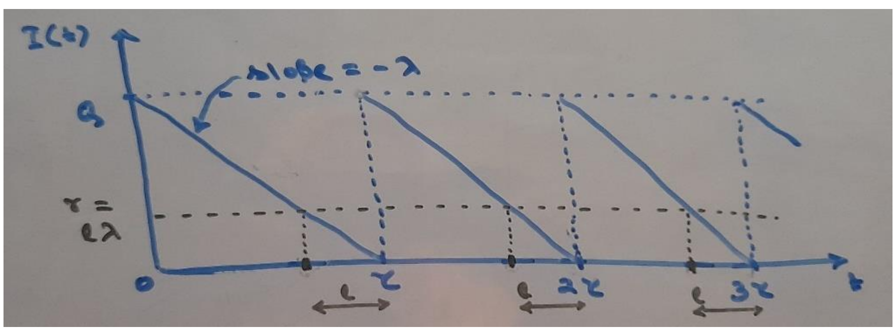
Positive lead time

Consider a setting consistent with the EOQ model, with λ = 15 per week, K = 150, c = 13, and h is 20% of c per year, i.e., h = (20% of 13)/52 = 1/20 per item-week. The optimal

order quantity $Q^* = \sqrt{2\lambda K/\text{h}} = \sqrt{2 \times 15 \times 150 \times 20} = 300$. So, an order of size 300 is placed whenever the stock level becomes zero, equivalently, an order of size 300 is placed after 300/15 = 20 weeks elapse since the last ordering.

Let us consider a constant lead time of 4 weeks. If we place order as above, then there will be stock-out for 4 weeks after every order. The simplest way of overcoming this is to place an order 4 weeks before the stock level becomes zero, which is same as placing the order whenever stock level reaches 4 × 15 = 60. Then the stock level becomes zero by the time the supply arrives. So, *shortage is avoided with no additional cost*.

In general, if l is the lead time, an order of size $Q^* = \sqrt{2\lambda K/\text{h}}$ shall be placed whenever the stock level reaches $r = l\lambda$. Here, r is known as the reorder point. In a periodic review system, an order of size Q^* is placed after $\tau = Q^*/\lambda$ time elapse since the last ordering – this is same as $l = 0$ case, except that we need to ensure that the stock level is $l\lambda$ at the time of placing the first order. The following diagram depicts these.



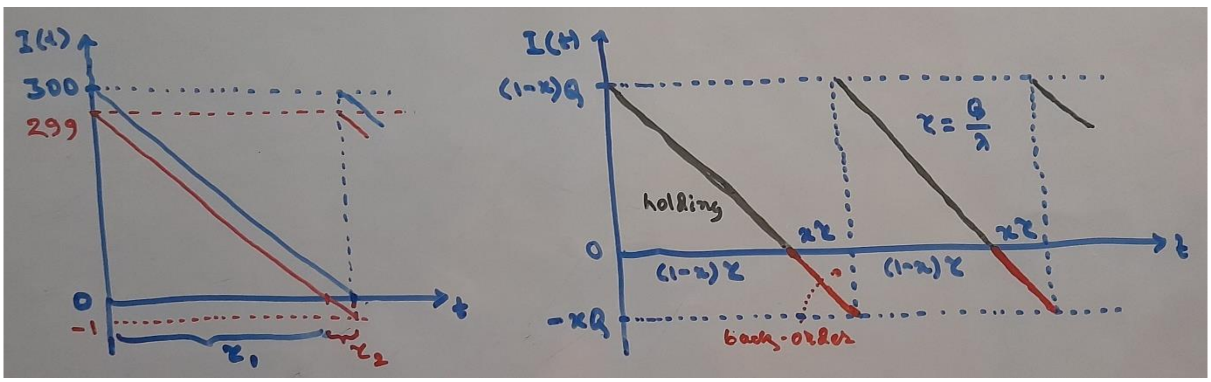
Planned back-order

Due to the deterministic setting in the EOQ model, it's possible to avoid shortage completely. Then from a practical point of view, it makes sense to avoid shortage. However, if the system permits back-ordering, then it may make economic sense to induce some back-ordering in a planned manner, as illustrated below.

Consider the above example where λ = 15, K = 150, c = 13, h = 0.05, and shortage is not permitted. We determined Q^* = 300, and then the minimum cost rate $G(Q^*) = \lambda K/Q^* + \text{h}Q^*/2$ = 7.5 + 7.5. These two terms are due to the ordering and holding costs respectively. These are same at Q^*, as noted earlier. Now, let us consider a back-ordering cost of b = 0.1 per item-week (same unit as that of h). If we place the order little late and allow a back-order of 1 unit, then the next cycle begins with 299 units of stock, as shown below.

This reduces holding cost as the average inventory level reduces, and it brings back-ordering cost. Note that the order quantity Q = 300 and the cycle length $\tau = Q/\lambda$ = 20 remains the same. The stock level is positive for $\tau_1 = 299/15$ units of time, and it is negative during the remaining $\tau_2 = \tau - \tau_1 = 1/15$ units of time. The new average (positive) stock level during the entire cycle is $(0.5 \times 299 \times \tau_1)/\tau = 149.002$, and the corresponding holding cost rate is $0.05 \times 149.002 = 7.45$, which has reduced. The average back-order (i.e., negative stock) during the entire cycle is $(0.5 \times 1 \times \tau_2)/\tau = 0.0017$, and the corresponding back-ordering cost rate is $b \times 0.0017 = 0.00017$, which was absent earlier. Overall, there is a reduction in the total cost rate. So, planned back-order can be economically beneficial.

Let us determine the optimal level of planned back-order quantity. Let Q denote the order quantity and xQ denote the maximum level of back-order, where $x \in [0,1]$ denotes the back-order level. Then $(1 - x)Q$ is the maximum (positive) inventory level, which is exhausted in $\tau_1 = (1-x)Q/\lambda = (1-x)\tau$ amount of time. During the remaining $\tau_2 = \tau - \tau_1 = x\tau$ time, there is shortage. These are shown in the above diagram. First, we obtain the optimal value of x for a given Q, and then find the optimal Q.



Average inventory level during the entire cycle $I_a = (0.5 \times (1 - x)Q \times \tau_1)/\tau = (1 - x)^2 \cdot Q/2$. Similarly, average back-order level during the whole cycle $B_a = (0.5 \times xQ \times \tau_2)/\tau = x^2 \cdot Q/2$. Then the total cost rate, consisting of ordering, holding and back-ordering costs, is

$$G(x|Q) = \frac{\lambda K}{Q} + (1 - x)^2 \frac{hQ}{2} + x^2 \frac{bQ}{2}$$; note that ordering cost rate does not depend on x

$G'(x|Q) = -(1 - x)hQ + xbQ \Rightarrow x^* = h/(h + b)$ is the only stationary point

$G''(x|Q) = (h + b)Q > 0$ for $Q > 0 \Rightarrow G(x|Q)$ is convex and x^* is optimal