Mechanism Design - II
Optimal Auctions

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Overview

Overview

Introduction to Auctions

- When does an auction happen?
  It happens when there are one or more agents vying for an item that is ready to be sold.
- What does designing an auction mean?
  Deciding who should be allocated the item(s) and how much they pay. Mathematically, it is the design of two functions: the allocation function q and the payment function t. They are functions of the bid profiles of the agents.
- The allocation and payment functions are designed based on the objective of the auctioneer.

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Efficient and Optimal Auctions

- Two broad classes of auctions in the literature: *Efficient* and *Optimal* auctions.
- Assume z_i to be the valuation of agent i. Then, an *efficient auction* maximizes $\sum_{i=1}^{n} z_i q_i(z_i, z_{-i})$.
  - Satisfies the property of allocative efficiency.
  - Objective is to maximize social welfare.
  - Vickrey auctions and VCG auctions are efficient.
- An *optimal auction* maximizes $\sum_{i=1}^{n} \mathbb{E}_z t_i(z_i, z_{-i})$.
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1 Introduction to Optimal Auctions

2 One-item Setting

- Simplifying the Optimization Problem
- The Solution

3 Two-item Setting

4 Our Work

5 Summary

Auction Setup: One-item Setting

- Consider the problem of auctioning a single indivisible object to n players.
- Their valuations $z_i \in [d_i^-, d_i^+]$, $z_i \sim f_i$. cdf is F_i. They bid $\hat{z}_i$.
- z_i is private information, but f_i is common knowledge.
- Based on the $\hat{z}_i$ values, the auctioneer decides the allocation probability vector $q : \times_{i=1}^n [d_i^-, d_i^+] \to \mathcal{P}^n$, and the payment vector $t : \times_{i=1}^n [d_i^-, d_i^+] \to \mathbb{R}_+^n$.
- Consider a quasi-linear mechanism, where the utility function of the player i is given by $u_i(z_i, \hat{z}_i, \hat{z}_{-i}) = z_i q_i(\hat{z}_i, \hat{z}_{-i}) - t_i(\hat{z}_i, \hat{z}_{-i})$.
- The expected utility w.r.t. z_{-i} is given by $U_i(z_i, \hat{z}_i) = \mathbb{E}_{z_{-i}} u_i(z_i, \hat{z}_i, z_{-i}) = z_i Q_i(\hat{z}_i) - T_i(\hat{z}_i)$.

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Optimization problem

- In an optimal auction, we need to maximize the expected revenue generated, subject to the constraints that (i) the bidders prefer truthful bidding, and (ii) the bidders are asked to pay at most their bid.
- The auctioneer thus solves the following optimization problem:

Maximize the expected revenue ($\max_{Q(\cdot),T(\cdot)} \sum_{i=1}^{n} \mathbb{E}_{z_i \sim f_i} T_i(z_i)$)

subject to (1) **BIC**: $U_i(z_i, z_i) \geq U_i(z_i, \hat{z}_i)\ \forall z_i, \forall i.$
(2) **IIR**: $U_i(z_i) \geq 0, \forall z_i, \forall i.$

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- Recall that Myerson received the Nobel prize for economics in 2007, for his contributions to the mechanism design theory.

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Simplifying the constraints

- The BIC constraint can be simplified by the following theorem.

Theorem

The mechanism is Bayesian Incentive Compatible iff (i) $Q_i(\cdot)$ is increasing for all i, and (ii) $U_i(z_i, z_i) = U_i(d_i^-, d_i^-) + \int_{d_i^-}^{z_i} Q_i(r_i)\, dr_i$.

- Let the mechanism satisfy BIC. Then we have

$$\begin{aligned} U_i(z_i, z_i) &\geq U_i(z_i, s_i) \\ \Leftrightarrow z_i Q_i(z_i) - T_i(z_i) &\geq z_i Q_i(s_i) - T_i(s_i) \\ \Leftrightarrow z_i Q_i(z_i) - T_i(z_i) &\geq s_i Q_i(s_i) - T_i(s_i) + (z_i - s_i) Q_i(s_i) \\ \Leftrightarrow U_i(z_i, z_i) &\geq U_i(s_i, s_i) + (z_i - s_i) Q_i(s_i) \\ \Leftrightarrow U_i(s_i, s_i) &\geq U_i(z_i, z_i) + (s_i - z_i) Q_i(z_i). \end{aligned}$$

- $(z_i - s_i) Q_i(s_i) \leq U_i(z_i, z_i) - U_i(s_i, s_i) \leq (z_i - s_i) Q_i(z_i)$.
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- Choose $z_i - s_i = \delta > 0$. We then have

$$Q_i(s_i)\delta \leq U_i(s_i + \delta) - U_i(s_i) \leq Q_i(s_i + \delta)\delta$$

$$Q_i(s_i) \leq \lim_{\delta \to 0} \frac{U_i(s_i + \delta) - U_i(s_i)}{\delta} \leq Q_i(s_i)$$

$$U_i'(s_i) = Q_i(s_i) \Rightarrow U_i(s_i) = U_i(d_i^-) + \int_{d_i^-}^{s_i} Q_i(r_i)\, dr_i.$$

- Let Q and U satisfy the constraints specified in the theorem. Then we have

$$U_i(z_i) = U_i(s_i) + \int_{s_i}^{z_i} Q_i(r_i)\, dr_i \geq U_i(s_i) + \int_{s_i}^{z_i} Q_i(s_i)\, dr_i$$
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- Recall that we must maximize $\sum_{i=1}^{n} \mathbb{E}_{z_i} T_i(z_i)$. We choose $U_i(d_i^-) = 0$ to maximize $T_i(z_i)$.
- Auctioneer now has the following optimization problem to solve:

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Simplifying the objective function

$$\int_{d_i^-}^{d_i^+}\int_{d_i^-}^{z_i} Q_i(r_i)f_i(z_i)\,dr_i\,dz_i = \int_{d_i^-}^{d_i^+}\int_{z_i}^{d_i^+} f_i(z_i)Q_i(r_i)\,dz_i dr_i$$
$$= \int_{d_i^-}^{d_i^+}[1 - F_i(z_i)]Q_i(z_i)\,dz_i$$

Now the objective function becomes

$$\int_{d_i^-}^{d_i^+}\left[z_iQ_i(z_i) - \int_{d_i^-}^{z_i} Q_i(r_i)\,dr_i\right] f_i(z_i)\,dz_i$$
$$= \int_{d_i^-}^{d_i^+}\left[z_i - \frac{1-F_i(z_i)}{f_i(z_i)}\right] Q_i(z_i)f_i(z_i)\,dz_i = \int_{d_i^-}^{d_i^+}\phi_i(z_i)Q_i(z_i)f_i(z_i)\,dz_i,$$

where $\phi_i(z_i) = z_i - \frac{1-F_i(z_i)}{f_i(z_i)}$.

Overview

The solution

- We now need to maximize

$$\sum_{i=1}^{n} \int_{d_i^-}^{d_i^+} \phi_i(z_i) Q_i(z_i) f_i(z_i)\, dz_i,$$

$$\Rightarrow \sum_{i=1}^{n} \int_{d_1^-}^{d_1^+} \cdots \int_{d_n^-}^{d_n^+} \phi_i(z_i) q_i(z_i) f(z)\, dz.$$

- Let $\phi_i(\cdot)$ be increasing. Then the expression can be maximized point-wise: $q_j(z_j) = 1$ if $\{\phi_j(z_j) = \max_i \phi_i(z_i),\ \phi_j(z_j) \geq 0\}$.
- In other words, the item is allocated to the bidder with the highest positive ϕ. The item is not allocated to anyone if the highest ϕ is negative.
- Winner pays $t_i(z) = \phi_i^{-1}([\phi_{i'}(z_{i'})]_+)$, where i' corresponds to the user having the second highest $\phi_j(z_j)$.

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- Let $\phi_i(\cdot)$ be increasing. Then the expression can be maximized point-wise: $q_j(z_j) = 1$ if $\{\phi_j(z_j) = \max_i \phi_i(z_i),\ \phi_j(z_j) \geq 0\}$.
- In other words, the item is allocated to the bidder with the highest positive ϕ. The item is not allocated to anyone if the highest ϕ is negative.
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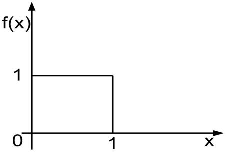

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- So, if no agent bids over $1/2$, the object is not allocated.
- If there is at least one agent bidding above $1/2$, he pays either the second highest price, or $1/2$, whichever is the maximum.
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- Notice that whenever the valuations are identically distributed, the revenue optimal auction is actually a second price auction with a reserve price.

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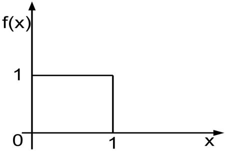

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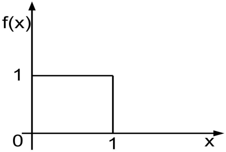

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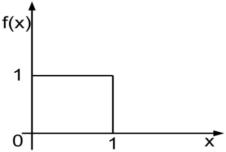

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- $F_i(z_i) = 1 - \exp -(\lambda_i z_i)$, $f_i(v_i) = \lambda_i \exp -(\lambda_i) z_i$, and thus $\phi_i(z_i) = z_i - 1/\lambda_i$.
- If $\lambda_1 = \lambda_2 = \lambda$, then it is second price auction with a reserve price of $1/\lambda$.
- If not, there are three cases based on the number of agents having $\phi_i(z_i) > 0$:
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  - **One :** The item is allocated to the agent having $\phi_i(z_i) > 0$, and he pays $1/\lambda_i$.
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- In general, Myerson's optimal auction is a second price auction conducted using $n + 1$ bids: 0, and the virtual valuations of n players. If 0 wins, the auctioneer keeps the item with himself.
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- In the one-buyer setting, we must maximize $\int_{d^-}^{d^+} (zf(z) - (1 - F(z))q(z) \, dz$.
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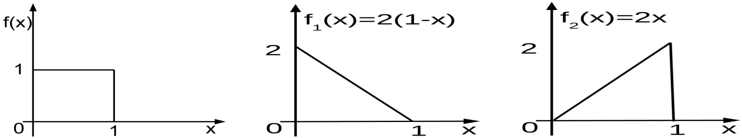

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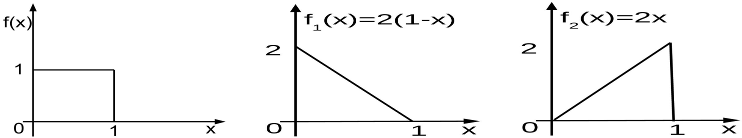

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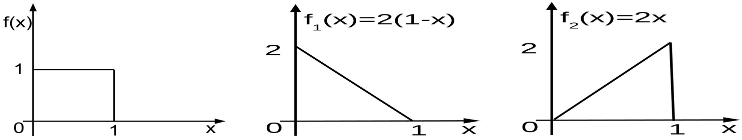

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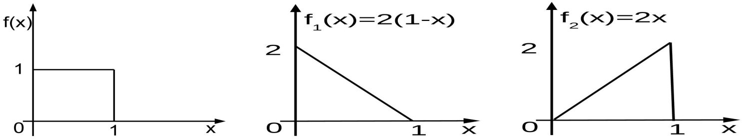

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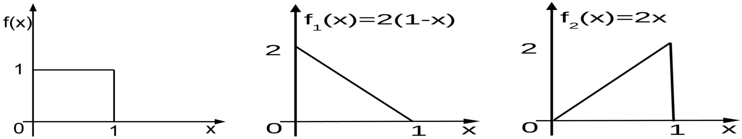

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- Optimal auction is an auction mechanism that maximizes the expected revenue to the auctioneer, subject to the constraints that the players are incentivized for truthful bidding, and that they are not asked to pay more than what they bid.
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Two-item Optimal Auction

- How do we maximize revenue when there are two items for selling to a single buyer? Can we perform Myerson's auction separately for each item?
- Consider this example: $z_1, z_2 \sim \text{Unif}\{1, 2\}$ i.i.d.
  - In Myerson auction, the reserve price$= 1$.
  - So, average revenue$= 1 + 1 = 2$.
- Instead, let us consider the following auction.

- This generates an average revenue of $3 \times (3/4) = 2.25$.

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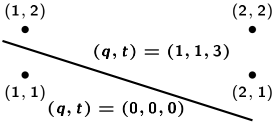

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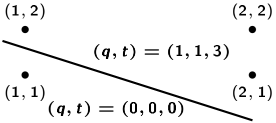
- Instead, let us consider the following auction.



- This generates an average revenue of $3 \times (3/4) = 2.25$.

Two-Item Optimal Auction (contd...)

- How about bundling the items together, and now perform Myerson's auction on the bundle?
- Consider $z_1 \sim \text{Unif}\{1, 2\}$ and $z_2 \sim \text{Unif}\{1, 3\}$. The optimal auction turns out to be as follows.

- The allocations are no more deterministic. The solutions turn out to be more and more complex.
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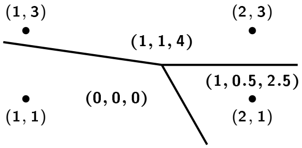

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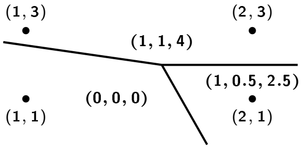

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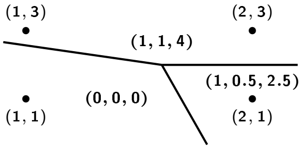

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- Solutions for certain distributions are known, however. When $z \sim \text{Unif}[0, b_1] \times [0, b_2]$, (Daskalakis et al. [2013]).

- So the buyer gets only item 1 if $\mathbf{z_1 > 2b_1/3}$ (and z_2 is small), only item 2 if $\mathbf{z_2 > 2b_2/3}$ (and z_1 is small), and both items if $\mathbf{z_1 + z_2 > (2b_1 + 2b_2 - \sqrt{2b_1 b_2})/3}$.
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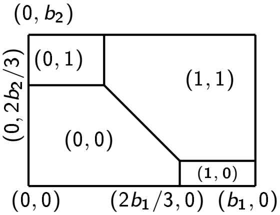

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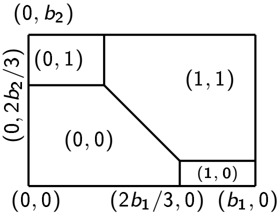

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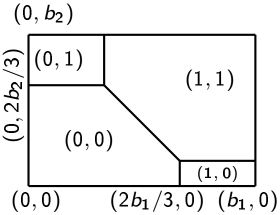

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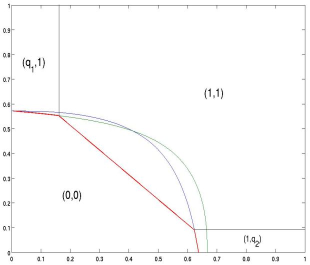
Available solutions

Consider $f_1(z_1) = \frac{1}{30} z_1^2 (1 - z_1)^2$, and $f_2(z_2) = \frac{1}{60} z_2^2 (1 - z_2)^3$, for $z \in [0, 1]^2$. Daskalakis et al. [2013] provides the following solution:



- $a_1(z_1) = -s_1'(z_1)$, $a_2(z_2) = -1/s_2'(z_2)$.
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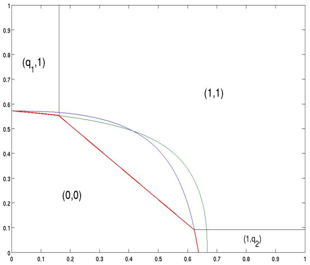

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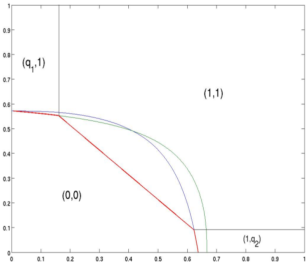

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What other distributions?

- Daskalakis et al. [2013,2014] computed the optimal mechanisms for distributions that give rise to a "well-formed canonical partition".
- Cases such as $f = \text{Unif}[0, b_1] \times [0, b_2]$, $f = \exp(\lambda_1) \times \exp(\lambda_2)$, and $f = \text{Beta}(m_1, n_1) \times \text{Beta}(m_2, n_2)$ belong to this category.
- The procedure crucially uses the fact that the support sets D are of the form $[0, b_1] \times [0, b_2]$.
- Solutions available in the literature are largely for distributions that are bordered by the coordinate axes.
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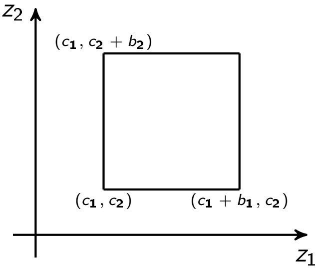
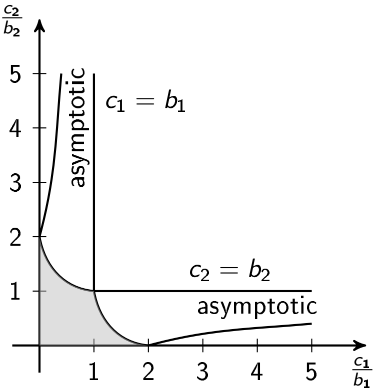
Overview

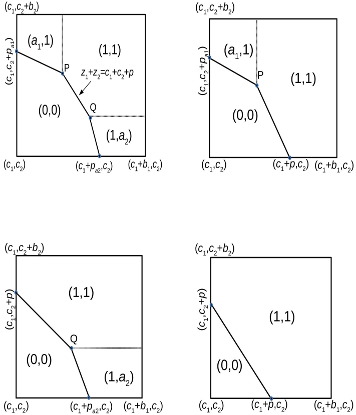
Uniform distribution on arbitrary rectangles

We solve the two-item single-buyer optimal mechanism design problem when $z \sim \text{Unif}[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$, for arbitrary nonnegative values of (c_1, c_2, b_1, b_2). We prove that the structure of the optimal solution falls within a class of eight simple structures.



Our results





Interplay optimizes revenue when (c_1, c_2) is very low.

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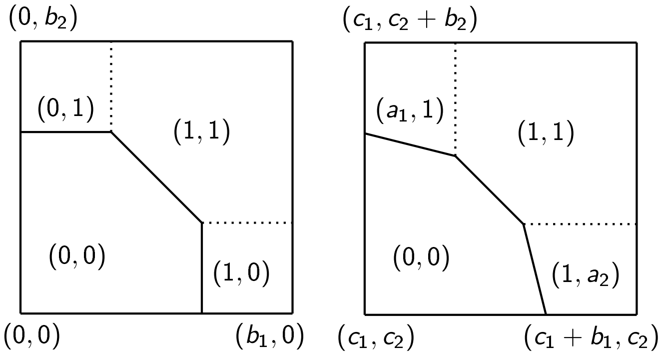
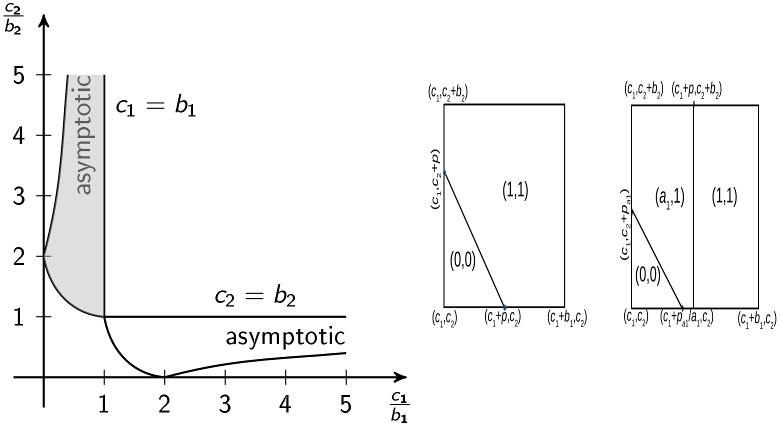


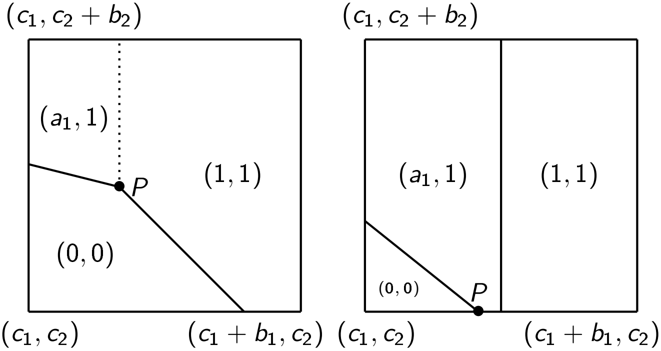
When $(c_1, c_2) > 0$, the seller finds it optimal to offer item i at a small probability a_i, when the buyer's valuation profile is such that z_i is low and z_{-i} is high.

Our results



When c_1 is low, but c_2 is high.

Our results



When c_2 goes higher, the point P drops to the bottom boundary of the support set.

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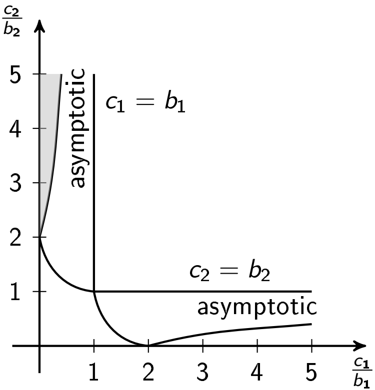
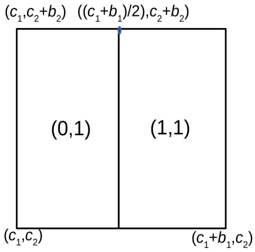
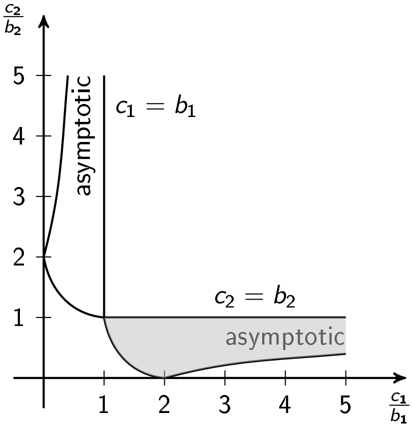
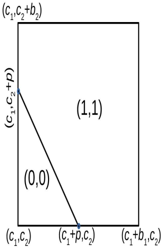


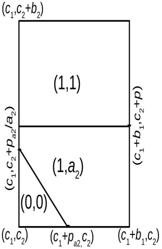


When c_1 is low and c_2 is very high, item 2 is offered for c_2, but conducts a Myerson's auction is sold item 1. The revenue is optimized by individual sale.

Our results







When c_2 is low, but c_1 is high.

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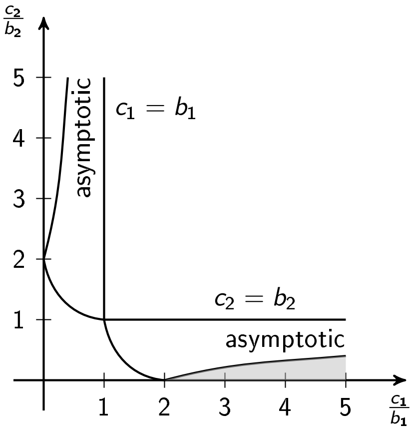

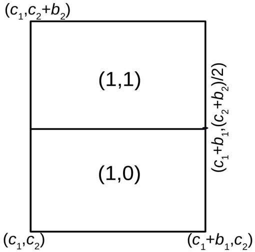

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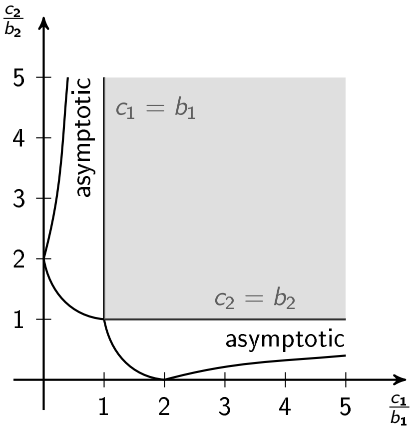
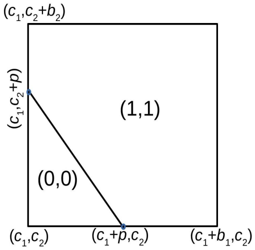
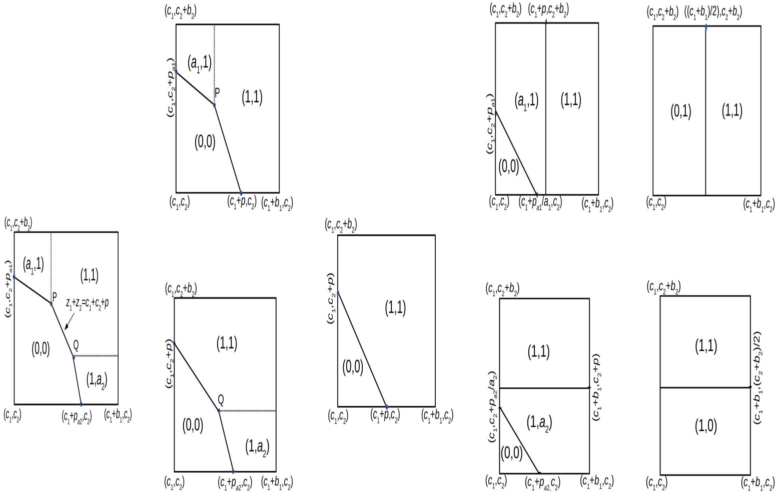




When both c_1 and c_2 are high, bundling optimizes the revenue.
The seller sets a threshold for the bundle close to $c_1 + c_2$.

The general structure



Interpreting the results

- Why so many structures?
- Observe the optimal mechanisms for (a) $[0,1]^2$, (b) $[0,1] \times [100,101]$, (c) $[100,101] \times [0,1]$, and (d) $[100,101]^2$.
  - For $[100,101]^2$, the revenue is optimized by bundle sale.
  - For $[0,1] \times [100,101]$ and the vice-versa, the revenue is optimized by individual sale.
  - For $[0,1]^2$, the revenue is optimized by having an interplay of both bundle sale and individual sale.
- The values of (c_1, c_2) represent the guarantee amount from the buyer. These values thus offer a huge information to the seller.

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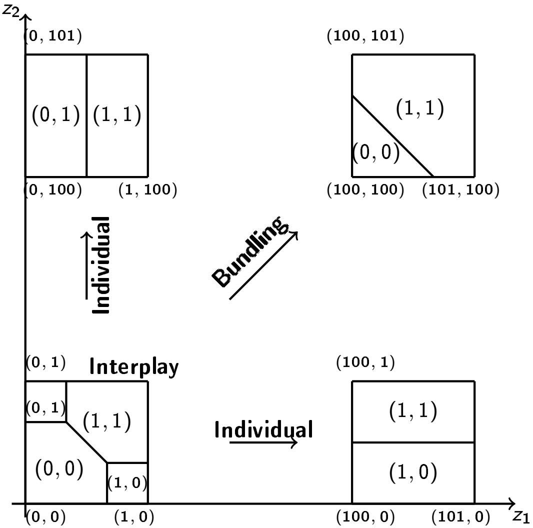

Overview

Summary

- Optimal auctions maximize the expected revenue of the seller.
- The optimal auction for a single item is a second price auction with 0 and the virtual valuations as the bids. In a single bidder setting, it is a take-it-or-leave-it auction for a reserve price.
- The two-item case is a notoriously hard problem to solve. The solutions could be non-deterministic, could have a continuum of lotteries, and could be highly complex. The literature had solutions largely for distributions with support sets $[0, b_1] \times [0, b_2]$.
- In our work, we prove that the optimal auction when $z \sim \mathrm{Unif}[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$ is to sell the items according to one of eight simple structures.