Certainly! To represent the steady-state solution for the inhibitory-stabilized network, we need to set the time derivatives to zero and rewrite the system of equations in the form:

[

$$\frac{dV}{dt} = AV + X$$

]

where:

- (

$$V = \begin{pmatrix} V_E \\ V_I \end{pmatrix}$$

) is the vector of membrane potentials for the excitatory and inhibitory neurons,
- (A) is a (2 \times 2) matrix representing the interactions between neurons,
- (X) is a vector representing the contributions from the resting potential and external inputs.

Step 1: Start with the Original Equations

From our previous equations:

1. **Equation for the excitatory neuron (E):**

[

$$\frac{dV_E}{dt} = -(V_E - V_0) + \beta\left[w_{EE}(V_E - V_0) - w_{EI}(V_I - V_0)\right] + u_E$$

]

2. **Equation for the inhibitory neuron (I):**

[

$$\frac{dV_I}{dt} = -(V_I - V_0) + \beta\left[w_{IE}(V_E - V_0) - w_{II}(V_I - V_0)\right] + u_I$$

]

Step 2: Expand and Rearrange the Equations

Let's expand the terms and collect like terms for each equation.

For the excitatory neuron (E):

Expand the terms:

1. **Passive decay and external input:**

[

$$-(V_E - V_0) + u_E = -V_E + V_0 + u_E$$

]

2. **Synaptic input terms:**

[

$$\beta\left[w_{EE}(V_E - V_0) - w_{EI}(V_I - V_0)\right] = \beta w_{EE}(V_E - V_0) - \beta w_{EI}(V_I - V_0)$$

]

Expand further:

[

$$\beta w_{EE} V_E - \beta w_{EE} V_0 - \beta w_{EI} V_I + \beta w_{EI} V_0$$

]

Combine all terms:

[

$$\frac{dV_E}{dt} = (-V_E + \beta w_{EE} V_E) + (-\beta w_{EI} V_I) + (V_0 - \beta w_{EE} V_0 + \beta w_{EI} V_0 + u_E)$$

]

Collect coefficients:

1. **Coefficient of (V_E):**

[

$$(-1 + \beta w_{EE}) V_E$$

]

2. **Coefficient of (V_I):**
   [
   $$

- \beta w_{EI} V_I

\]

3. **Constant terms:**

[

$$[V_0 - \beta w_{EE} V_0 + \beta w_{EI} V_0 + u_E]$$

]

Factor out (V_0):

[

$$[(1 - \beta w_{EE} + \beta w_{EI}) V_0 + u_E]$$

]

For the inhibitory neuron (I):

Expand the terms:

1. **Passive decay and external input:**

[

$$-(V_I - V_0) + u_I = -V_I + V_0 + u_I$$

]

2. **Synaptic input terms:**

[

$$\beta \left[w_{IE}(V_E - V_0) - w_{II}(V_I - V_0) \right] = \beta w_{IE}(V_E - V_0) - \beta w_{II}(V_I - V_0)$$

]

Expand further:

[

$$\beta w_{IE}V_E - \beta w_{IE}V_0 - \beta w_{II}V_I + \beta w_{II}V_0$$

]

Combine all terms:

[

$$\frac{dV_I}{dt} = (-V_I - \beta w_{II}V_I) + (\beta w_{IE}V_E) + (V_0 - \beta w_{IE}V_0 + \beta w_{II}V_0 + u_I)$$

]

Collect coefficients:

1. **Coefficient of (V_E):**

[

$$\beta w_{IE}V_E$$

]

2. **Coefficient of (V_I):**

[

$$(-1 - \beta w_{II})V_I$$

]

3. **Constant terms:**

[

$$[V_0 - \beta w_{IE}V_0 + \beta w_{II}V_0 + u_I]$$

]

Factor out (V_0):

[

$$[(1 - \beta w_{IE} + \beta w_{II})V_0 + u_I]$$

]

Step 3: Formulate the Matrix Equation

We can now write the system of equations in matrix form:

[

$$\frac{dV}{dt} = AV + X$$

]

where:

- (V = \begin{pmatrix} V_E \ V_I \end{pmatrix})
- (A) is the interaction matrix,
- (X) is the external input vector.

Components of Matrix (A):

1. **(a_{EE}):**

[

$$a_{EE} = -1 + \beta w_{EE}$$

]

2. **(a_{EI}):**

$$a_{EI} = -\beta w_{EI}$$

3. **(a_{IE}):**

$$a_{IE} = \beta w_{IE}$$

4. **(a_{II}):**

[

$$a_{II} = -1 - \beta w_{II}$$

]

Components of Vector (X):

1. **(X_E):**

[

$$X_E = (1 - \beta w_{EE} + \beta w_{EI})V_0 + u_E$$

]

2. **(X_I):**

[

$$X_I = (1 - \beta w_{IE} + \beta w_{II})V_0 + u_I$$

]

Final Matrix Equation:

[

$$\begin{pmatrix} \frac{dV_E}{dt} \\ \frac{dV_I}{dt} \end{pmatrix} = \begin{pmatrix} a_{EE} & a_{EI} \\ a_{IE} & a_{II} \end{pmatrix} \begin{pmatrix} V_E \\ V_I \end{pmatrix} + \begin{pmatrix} X_E \\ X_I \end{pmatrix}$$

Substituting the expressions:

$$\begin{pmatrix} \frac{dV_E}{dt} \\ \frac{dV_I}{dt} \end{pmatrix} = \begin{pmatrix} -1 + \beta w_{EE} & -\beta w_{EI} \\ \beta w_{IE} & -1 - \beta w_{II} \end{pmatrix} \begin{pmatrix} V_E \\ V_I \end{pmatrix} + \begin{pmatrix} (1 - \beta w_{EE} + \beta w_{EI})V_0 + u_E \\ (1 - \beta w_{IE} + \beta w_{II})V_0 + u_I \end{pmatrix}$$

]

Interpretation:

- The matrix (A) encapsulates the synaptic interactions and intrinsic decay of the neurons.
- Diagonal terms ((a*{EE}, a*{II})) represent self-influence and decay.
- Off-diagonal terms ((a*{EI}, a*{IE})) represent synaptic coupling between neurons.
- The vector (X) includes contributions from the resting potential and external inputs.
- This formulation allows us to analyze the stability and steady-state behavior by examining the eigenvalues and eigenvectors of matrix (A).

Steady-State Solution:

To find the steady-state membrane potentials V_E^* and V_I^*, set $\frac{dV}{dt} = 0$:

$$0 = AV^* + X$$

Solving for V^*:

$$V^* = -A^{-1}X$$

Provided that (A) is invertible, this equation gives the steady-state membrane potentials in terms of the network parameters and external inputs.

Summary:

We have successfully rewritten the original system of equations in the form ($\frac{dV}{dt} = AV + X$), where (V) is the vector of membrane potentials, (A) is the interaction matrix, and (X) is the input vector. This form is conducive to analytical and numerical methods for studying the dynamics and steady-state solutions of the inhibitory-stabilized network.

Here is your message with the requested replacements:

To derive the expression for the steady-state membrane potential of the inhibitory neuron (

$$V_I^*$$

) in terms of the network parameters, we'll start from the steady-state condition and solve the system of equations we previously formulated.

Step 1: Recall the Matrix Equation at Steady State

At steady state, the time derivatives are zero:

$$\frac{dV}{dt} = 0 = AV^* + X$$

where:

- $$V^* = (V_E^* V_I^*) \$\$ is the vector of steady-state membrane potentials.$$
- $$A$$
  is the interaction matrix:
  $$A = \begin{pmatrix} -1 + \beta w_{EE} - \beta w_{EI} \\ \beta w_{IE} - 1 - \beta w_{II} \end{pmatrix}$$
- $$X$$
  is the vector of constants:
  $$X = \begin{pmatrix} (1 - \beta w_{EE} + \beta w_{EI})V_0 + u_E \\ (1 - \beta w_{IE} + \beta w_{II})V_0 + u_I \end{pmatrix}$$
  $$= \begin{pmatrix} X_E \\ X_I \end{pmatrix}$$

Step 2: Write Down the System of Equations

The matrix equation

$$0 = AV^* + X$$

corresponds to the following system:

1. **Equation for the excitatory neuron (E):**
   $$a_{EE}V_E^* + a_{EI}V_I^* + X_E = 0$$
2. **Equation for the inhibitory neuron (I):**
   $$a_{IE}V_E^* + a_{II}V_I^* + X_I = 0$$

where:

- $$a_{EE} = -1 + \beta w_{EE}$$
- $$a_{EI} = -\beta w_{EI}$$

- $$a_{IE} = \beta w_{IE}$$
- $$a_{II} = -1 - \beta w_{II}$$

Step 3: Solve for

$$V_I^*$$

in Terms of Network Parameters

Our goal is to find an expression for

$$V_I^*$$

solely in terms of the network parameters and external inputs.

Step 3a: Solve the First Equation for

$$V_E^*$$

From the first equation:

$$a_{EE}V_E^* + a_{EI}V_I^* + X_E = 0$$

Rewriting:

$$V_E^* = -\frac{a_{EI}V_I^* + X_E}{a_{EE}}$$

Step 3b: Substitute

$$V_E^*$$

into the Second Equation

Insert

$$V_E^*$$

into the second equation:

$$a_{IE}\left(-\frac{a_{EI}V_I^* + X_E}{a_{EE}}\right) + a_{II}V_I^* + X_I = 0$$

Simplify:

$$-\frac{a_{IE}a_{EI}}{a_{EE}}V_I^* - \frac{a_{IE}X_E}{a_{EE}} + a_{II}V_I^* + X_I = 0$$

Step 3c: Collect Like Terms

Group the terms with

$$V_I^*$$

and constants separately:

1. **Terms with**

   $$V_I^*$$

   :

   $$\left(-\frac{a_{IE}a_{EI}}{a_{EE}} + a_{II}\right)V_I^*$$

2. **Constant terms:**

$$-\frac{a_{IE}X_E}{a_{EE}} + X_I$$

Step 3d: Solve for

$$V_I^*$$

Rewriting the equation:

$$\left(-\frac{a_{IE}a_{EI}}{a_{EE}} + a_{II}\right)V_I^* + \left(-\frac{a_{IE}X_E}{a_{EE}} + X_I\right) = 0$$

Move the constants to the other side:

$$\left(-\frac{a_{IE}a_{EI}}{a_{EE}} + a_{II}\right)V_I^* = \frac{a_{IE}X_E}{a_{EE}} - X_I$$

Divide both sides by the coefficient of

$$V_I^*$$

:

$$V_I^* = \frac{\frac{a_{IE}X_E}{a_{EE}} - X_I}{-\frac{a_{IE}a_{EI}}{a_{EE}} + a_{II}}$$

Simplify numerator and denominator:

Numerator:

$$\frac{a_{IE}X_E}{a_{EE}} - X_I = \frac{a_{IE}X_E - a_{EE}X_I}{a_{EE}}$$

Denominator:

$$-\frac{a_{IE}a_{EI}}{a_{EE}} + a_{II} = \frac{-a_{IE}a_{EI} + a_{EE}a_{II}}{a_{EE}}$$

Putting it Together:

$$V_I^* = \frac{a_{IE}X_E - a_{EE}X_I}{-a_{IE}a_{EI} + a_{EE}a_{II}}$$

This can be rewritten as:

$$V_I^* = \frac{a_{IE}X_E - a_{EE}X_I}{\det(A)}$$

where

$$\det(A)$$

is the determinant of matrix

$$A$$

:

$$\det(A) = a_{EE}a_{II} - a_{IE}a_{EI}$$

Step 4: Express

$$V_I^*$$

in Terms of Network Parameters

Now, substitute back the expressions for

$$a_{ij}$$

and

$$X_i$$

:

Compute the Numerator

$$a_{IE}X_E$$

:

$$a_{IE}X_E = \beta w_{IE}\left[(1 - \beta w_{EE} + \beta w_{EI})V_0 + u_E\right]$$

$$a_{EE}X_I$$

:

$$a_{EE}X_I = (-1 + \beta w_{EE})\left[(1 - \beta w_{IE} + \beta w_{II})V_0 + u_I\right]$$

Numerator:

$$\text{Numerator} = a_{IE}X_E - a_{EE}X_I$$

Compute the Denominator

$$a_{EE}a_{II}$$

:

$$a_{EE}a_{II} = (-1 + \beta w_{EE})\,(-1 - \beta w_{II})$$

$$a_{IE}a_{EI}$$

:

$$a_{IE}a_{EI} = (\beta w_{IE})\,(-\beta w_{EI}) = -\beta^2 w_{IE}w_{EI}$$

Denominator (

$$\det(A)$$

):

$$\det(A) = a_{EE}a_{II} - a_{IE}a_{EI} = (-1 + \beta w_{EE})\,(-1 - \beta w_{II}) + \beta^2 w_{IE}w_{EI}$$

$$\det(A) = [1 + \beta w_{II} - \beta w_{EE} - \beta^2 w_{EE}w_{II}] + \beta^2 w_{IE}w_{EI}$$

Simplify the determinant:

$$\det(A) = 1 + \beta(w_{II} - w_{EE}) + \beta^2(w_{IE}w_{EI} - w_{EE}w_{II})$$

Final Expression for

$$V_I^*$$

Substitute the expressions back into

$$V_I^*$$

:

$$V_I^* = \frac{\beta w_{IE}\left[(1 - \beta w_{EE} + \beta w_{EI})V_0 + u_E\right] - (-1 + \beta w_{EE})\left[(1 - \beta w_{IE} + \beta w_{II})V_0 + u_I\right]}{1 + \beta(w_{II} - w_{EE}) + \beta^2(w_{IE}w_{EI} - w_{EE}w_{II})}$$

Simplify the Numerator

Expand numerator terms:

1. **First term

:**

$$\beta w_{IE}\,(1 - \beta w_{EE} + \beta w_{EI})V_0 + \beta w_{IE}u_E$$

2. **Second term:**

$$- (-1 + \beta w_{EE})\,(1 - \beta w_{IE} + \beta w_{II})V_0 - (-1 + \beta w_{EE})u_I$$

Combine Terms

Collect all terms involving

$$V_0$$

and

$$u_E, u_I$$

:

- $$V_0$$

  terms:

Combine all coefficients of

$$V_0$$

.

- $$u$$

  terms:

Combine

$$\beta w_{IE} u_E$$

and

$$(-1 + \beta w_{EE})u_I$$

.

Note: Simplifying further would involve messy algebra, but the expression can be left in terms of known parameters.

Summary

The steady-state membrane potential of the inhibitory neuron

$$V_I^*$$

in terms of the network parameters is:

$$V_I^* = \frac{\beta w_{IE}\left[(1 - \beta w_{EE} + \beta w_{EI})V_0 + u_E\right] - (-1 + \beta w_{EE})\left[(1 - \beta w_{IE} + \beta w_{II})V_0 + u_I\right]}{1 + \beta(w_{II} - w_{EE}) + \beta^2(w_{IE}w_{EI} - w_{EE}w_{II})}$$

This expression provides

$$V_I^*$$

as a function of:

- Synaptic weights

  $$w_{EE}, w_{EI}, w_{IE}, w_{II}$$

- Gain

  $$\beta$$

- Resting potential

  $$V_0$$

- External inputs

  $$u_E, u_I$$

Note:

- The determinant

$$\det(A)$$

  in the denominator ensures the invertibility of matrix

$$A$$

  and affects the stability of the system.
- This expression can be further evaluated numerically if the specific values of the parameters are known.

Interpretation

- **Numerator:** Represents the combined effect of excitatory input to the inhibitory neuron and the inhibitory influence from the excitatory neuron, adjusted by the network connectivity and external inputs.
- **Denominator:** Captures how the network's feedback loops and synaptic connections influence the overall response of the inhibitory neuron.

By analyzing

$$V_I^*$$

, one can understand how changes in synaptic weights or external inputs impact the inhibitory neuron's steady-state activity within the network.