LU Decomposition

LU Decomposition

Method

For most non-singular matrix $[A]$ that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

Method: [A] Decomposes to [L] and [U]

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

[U] is the same as the coefficient matrix at the end of the forward elimination step.

[L] is obtained using the *multipliers* that were used in the forward elimination process

On multiplying $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ *and* $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$, we get,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$$u_{11}=a_{11}; \quad u_{12}=a_{12}; \quad u_{13}=a_{13}$$

$$l_{21}u_{11}=a_{21} \;\Rightarrow l_{21}=\frac{a_{21}}{u_{11}} \;; \quad l_{31}u_{11}=a_{31} \;\Rightarrow l_{31}=\frac{a_{31}}{u_{11}}$$

$$l_{21}u_{12}+u_{22}=a_{22} \;\Rightarrow u_{22}=a_{22}-l_{21}u_{12};$$

$$l_{21}u_{13}+u_{23}=a_{23} \;\Rightarrow u_{23}=a_{23}-l_{21}u_{13};$$

simililarly,

$$l_{31}u_{12}+l_{32}u_{22}=a_{32}, \quad l_{31}u_{13}+l_{32}u_{23}+u_{33}=a_{33} \;\; gives \;\; l_{32} \;\; and \;\; u_{33}$$

Solve the following system of equations by LU decomposition.

2x+3y+z=9

x+2y+3z=6

3x+y+2z=8.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$u_{11} = 2; \quad u_{12} = 3; \quad u_{13} = 1$$

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{1}{2} \; ; \quad l_{31} = \frac{a_{31}}{u_{11}} = \frac{3}{2}$$

$$u_{22} = a_{22} - l_{21}u_{12} = 2 - \frac{1}{2} \times 3 = \frac{1}{2};$$

$$u_{23} = a_{23} - l_{21}u_{13} = 3 - \frac{1}{2} \times 1 = \frac{5}{2};$$

$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{1 - \frac{3}{2} \times 3}{\frac{1}{2}} = -7 \quad and$$

$$u_{33} = u_{33} = a_{33} - (l_{31}u_{13} + l_{32}u_{23}) = 2 - \left(\frac{3}{2} \times 1 + (-7) \times \frac{5}{2}\right) = 2 - \left(\frac{3}{2} - \frac{35}{2}\right) = 18$$

How does LU Decomposition work?

If solving a set of linear equations	$[A][X] = [C]$
If $[A] = [L][U]$ then	$[L][U][X] = [C]$
Multiply by	$[L]^{-1}$
Which gives	$[L]^{-1}[L][U][X] = [L]^{-1}[C]$
Remember $[L]^{-1}[L] = [I]$ which leads to	$[I][U][X] = [L]^{-1}[C]$
Now, if $[I][U] = [U]$ then	$[U][X] = [L]^{-1}[C]$
Now, let	$[L]^{-1}[C]=[Z]$
Which ends with	$[L][Z] = [C]$ (1)
and	$[U][X] = [Z]$ (2)

LU Decomposition

How can this be used?

Given $[A][X] = [C]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [C]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

Finding the $[U]$ matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

Matrix after Step 1: $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Step 2: $\dfrac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \qquad \ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = \ ?$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the $[L]$ and $[U]$ matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 10$$

$$2.56 z_1 + z_2 = 177.2$$

$$5.76 z_1 + 3.5 z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for [Z]

$$
\begin{aligned}
z_1 &= 106.8 \\
z_2 &= 177.2 - 2.56 z_1 \\
&= 177.2 - 2.56(106.8) \\
&= -96.2 \\
z_3 &= 279.2 - 5.76 z_1 - 3.5 z_2 \\
&= 279.2 - 5.76(106.8) - 3.5(-96.21) \\
&= 0.735
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}
$$

Example

Set $[U][X] = [Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$0.7a_3 = 0.735$$

Example

From the 3rd equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example

Substituting in a_3 and a_2 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$= 0.2900$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Finding the inverse of a square matrix

The inverse [B] of a square matrix $[A]$ is defined as

$$[A][B] = [I] = [B][A]$$

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of $[B]$ to be $[b_{11}\ \ b_{12}\ \ \ldots\ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of $[B]$

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of $[B]$

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in $[B]$ can be found in the same manner

Example: Inverse of a Matrix

Find the inverse of a square matrix [A]

$$[A]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the [L] and [U] matrices are found to be

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving for the each column of $[B]$ requires two steps

1) Solve $[L]\ [Z] = [C]$ for $[Z]$

2) Solve $[U]\ [X] = [Z]$ for $[X]$

Step 1: $$[L][Z]=[C]\rightarrow\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

This generates the equations:

$$z_1 = 1$$

$$2.56z_1 + z_2 = 0$$

$$5.76z_1 + 3.5z_2 + z_3 = 0$$

Example: Inverse of a Matrix

Solving for [Z]

$$
\begin{aligned}
z_1 &= 1 \\
z_2 &= 0 - 2.56z_1 \\
&= 0 - 2.56(1) \\
&= -2.56 \\
z_3 &= 0 - 5.76z_1 - 3.5z_2 \\
&= 0 - 5.76(1) - 3.5(-2.56) \\
&= 3.2
\end{aligned}
$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving $[U][X] = [Z]$ for $[X]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$
$$-4.8b_{21} - 1.56b_{31} = -2.56$$
$$0.7b_{31} = 3.2$$

Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of $[A]$ is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix

The inverse of $[A]$ is

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Inverse of a Matrix by Gauss-Jordan method

- Gauss-Jordan method is almost 1.5 times the total number of divisions and multiplications required for Gauss elimination.
- The most important application of this method is to find the inverse of a nonsingular matrix.

Inverse of a Matrix by Gauss-Jordan method

Find the inverse of the following matrix A by Gauss-Jordan method.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}.$$

Inverse of a Matrix by Gauss-Jordan method

Find the inverse of the following matrix A by Gauss-Jordan method.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

Inverse of a Matrix by Gauss-Jordan method

Find the inverse of the following matrix A by Gauss-Jordan method.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \text{by } R_2 \to R_2 - 4R_1 \\ \text{by } R_3 \to R_3 - 3R_1 \end{array}$$

Inverse of a Matrix by Gauss-Jordan method

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \text{by } R_2 \to R_2 - 4R_1 \\ \text{by } R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \text{by } R_2 \to -R_2$$

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$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \begin{array}{l} \text{by } R_1 \to R_1 - R_2 \\ \text{by } R_3 \to R_3 - 2R_2 \end{array}$$

Inverse of a Matrix by Gauss-Jordan method

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$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] \text{by } R_3 \to -\frac{1}{10} R_3$$

Inverse of a Matrix by Gauss-Jordan method

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$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] \begin{array}{l} \text{by } R_1 \to R_1 + 4R_3 \\ \text{by } R_2 \to R_2 - 5R_1 \end{array}$$

Inverse of a Matrix by Gauss-Jordan method

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] \begin{array}{l} \text{by } R_1 \to R_1 + 4R_3 \\ \text{by } R_2 \to R_2 - 5R_1 \end{array}$$

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}.$$

Matrix Inversion by Gauss Elimination

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix Inversion by Gauss Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly the above equation is equivalent to the three equations

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Matrix Inversion by Gauss Elimination

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Clearly the above equation is equivalent to the three equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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Matrix Inversion by Gauss Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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We can therefore solve each of these systems using Gaussian elimination method and the result in each case will be the corresponding column of $X = A^{-1}$.

Matrix Inversion by Gauss Elimination

Using Gaussian elimination, find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

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In the first stage, A is converted into an upper triangular form, using Gaussian elimination method

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$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & \frac{7}{2} & \frac{17}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{by } R_2 \to R_2 - \frac{3}{2}R_1 \\ \text{by } R_3 \to R_3 - \frac{1}{2}R_1 \end{array}$$

Matrix Inversion by Gauss Elimination

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$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \text{by } R_3 \to R_3 - 7R_{21}$$

Matrix Inversion by Gauss Elimination

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right]$$

The above is equivalent to the following three systems:

Matrix Inversion by Gauss Elimination

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Matrix Inversion by Gauss Elimination

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Matrix Inversion by Gauss Elimination

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Matrix Inversion by Gauss Elimination

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which on back substitution gives $x_{31} = -5, \quad x_{21} = 12, \quad x_{11} = -3.$

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$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & -\frac{1}{2} \end{bmatrix}.$$

Jacobi's Iteration method

Consider a linear system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}\right\} \quad \ldots (1)$$

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\end{aligned}\right\} \quad \ldots (1)$$

in which the diagonal elements a_{ii} do not vanish.

$$\left.\begin{aligned}
x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \cdots - \frac{a_{1n}}{a_{11}}x_n \\
x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \cdots - \frac{a_{2n}}{a_{22}}x_n \\
x_3 &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \cdots - \frac{a_{2n}}{a_{33}}x_n \\
&\vdots \\
x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \cdots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}
\end{aligned}\right\} \quad \ldots (2)$$

Jacobi's Iteration method

$x_1^{(0)},\ x_2^{(0)},\ \ldots, x_n^{(0)}$ as initial values to the variables $x_1,\ x_2,\ \ldots, x_n$.

Step 1: Determination of first approximation $x_1^{(1)},\ x_2^{(1)},\ \ldots, x_n^{(1)}$ using $x_1^{(0)},\ x_2^{(0)},\ \ldots, x_n^{(0)}$.

$$\left.\begin{aligned}
x_1^{(1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(0)} \\
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&\ \ \vdots \\
x_n^{(1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(0)} - \frac{a_{n2}}{a_{nn}} x_2^{(0)} - \cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(0)}
\end{aligned}\right\} \quad \ldots (3)$$

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x_2^{(1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(0)} - \frac{a_{23}}{a_{22}}x_3^{(0)} - \cdots - \frac{a_{2n}}{a_{22}}x_n^{(0)} \\
x_3^{(1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1^{(0)} - \frac{a_{32}}{a_{33}}x_2^{(0)} - \cdots - \frac{a_{2n}}{a_{33}}x_n^{(0)} \\
&\ \vdots \\
x_n^{(1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(0)} - \frac{a_{n2}}{a_{nn}}x_2^{(0)} - \cdots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(0)}
\end{aligned}\right\} \quad \dots (3)$$

Step 2: Similarly, $x_1^{(2)},\ x_2^{(2)},\ \dots, x_n^{(2)}$ are evaluated by just replacing $x_r^{(0)}$ in the right hand sides equations in (3) by $x_r^{(1)}$.

Jacobi's Iteration method

Step $n+1$: In general, if $x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}$ are a system of n th approximations, then the next approximation is given by the formula

$$\left.\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(n)} - \frac{a_{13}}{a_{11}}x_3^{(n)} - \;\cdots\; - \frac{a_{1n}}{a_{11}}x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(n)} - \frac{a_{23}}{a_{22}}x_3^{(n)} - \;\cdots\; - \frac{a_{2n}}{a_{22}}x_n^{(n)} \\
x_3^{(n+1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1^{(n)} - \frac{a_{32}}{a_{33}}x_2^{(n)} - \;\cdots\; - \frac{a_{2n}}{a_{22}}x_n^{(n)} \\
&\;\;\vdots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(n)} - \frac{a_{n2}}{a_{nn}}x_2^{(n)} - \;\cdots\; - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(n)}
\end{aligned}\right\} \qquad \dots\ (4)$$

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&\vdots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \frac{a_{n2}}{a_{nn}} x_2^{(n)} - \quad \cdots \quad - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(n)}
\end{aligned}\right\} \qquad \ldots (4)$$

The system in (4) can also be briefly described as follows:

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)} \qquad (r = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots, n)$$

Gauss Seidel Iteration method

A simple modification to Jacobi's iteration method is given by *Gauss-Seidel* method.

Step 1 (*Gauss-Seidel method*): Determination of first approximation $x_1^{(1)},\ x_2^{(1)},\ \ldots,x_n^{(1)}$ using $x_1^{(0)},\ x_2^{(0)},\ \ldots,x_n^{(0)}$.

$$\left.\begin{aligned}
x_1^{(1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(0)} - \frac{a_{13}}{a_{11}}x_3^{(0)} - \quad \cdots \quad - \frac{a_{1n}}{a_{11}}x_n^{(0)} \\
x_2^{(1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(1)} - \frac{a_{23}}{a_{22}}x_3^{(0)} - \quad \cdots \quad - \frac{a_{2n}}{a_{22}}x_n^{(0)} \\
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&\vdots \\
x_n^{(1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(1)} - \frac{a_{n2}}{a_{nn}}x_2^{(1)} - \quad \cdots \quad - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(1)}
\end{aligned}\right\} \qquad \ldots\ (5)$$

Gauss Seidel Iteration method

Step $n+1$: In general, if $x_1^{(n)},\ x_2^{(n)},\ \ldots,x_n^{(n)}$ are a system of n th approximations, then the next approximation is given by the formula

$$\left.\begin{array}{l}
x_1^{(n+1)} = \dfrac{b_1}{a_{11}} - \dfrac{a_{12}}{a_{11}} x_2^{(n)} - \dfrac{a_{13}}{a_{11}} x_3^{(n)} - \quad \cdots \quad - \dfrac{a_{1n}}{a_{11}} x_n^{(n)} \\[2ex]
x_2^{(n+1)} = \dfrac{b_2}{a_{22}} - \dfrac{a_{21}}{a_{22}} x_1^{(n+1)} - \dfrac{a_{23}}{a_{22}} x_3^{(n)} - \quad \cdots \quad - \dfrac{a_{2n}}{a_{22}} x_n^{(n)} \\[2ex]
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\qquad\qquad\qquad \vdots \\[2ex]
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\end{array}\right\} \qquad \ldots\ (6)$$

Gauss Seidel Iteration method

Step $n+1$: In general, if $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ are a system of n th approximations, then the next approximation is given by the formula

$$\left.\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - \quad \cdots \quad - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n+1)} - \frac{a_{23}}{a_{22}} x_3^{(n)} - \quad \cdots \quad - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
x_3^{(n+1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(n+1)} - \frac{a_{32}}{a_{33}} x_2^{(n+1)} - \quad \cdots \quad - \frac{a_{2n}}{a_{33}} x_n^{(n)} \\
&\vdots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n+1)} - \frac{a_{n2}}{a_{nn}} x_2^{(n+1)} - \quad \cdots \quad - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(n+1)}
\end{aligned}\right\} \quad \ldots \ (6)$$

(6) can be briefly described as follows:

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(r+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)} \quad (r = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots, n).$$

Solve the following system of equations using

(a) Jacobi's iteration method

(b) Gauss-Seidel iteration method.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9.$$

Jacobi's Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	-0.9

Jacobi's Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	−0.9
2	0.78	1.74	2.7	−0.18

Jacobi's Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	−0.9
2	0.78	1.74	2.7	−0.18
3	0.9	1.908	2.916	−0.108

Jacobi's Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	−0.9
2	0.78	1.74	2.7	−0.18
3	0.9	1.908	2.916	−0.108
4	0.9624	1.9608	2.9592	−0.036
5	0.9845	1.9848	2.9851	−0.0158
6	0.9939	1.9938	2.9938	−0.006
7	0.9975	1.9975	2.9976	−0.0025
8	0.9990	1.9990	2.9990	−0.0010
9	0.9996	1.9996	2.9996	−0.0004
10	0.9998	1.9998	2.9998	−0.0002
11	0.9999	1.9999	2.9999	−0.0001
12	1.0	2.0	3.0	0.0

Gauss Seidel Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368

Gauss Seidel Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248

Gauss Seidel Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248
3	0.9836	1.9899	2.9924	−0.0042

Gauss Seidel Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248
3	0.9836	1.9899	2.9924	−0.0042
4	0.9968	1.9982	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001

Gauss Seidel Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248
3	0.9836	1.9899	2.9924	−0.0042
4	0.9968	1.9982	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001
6	0.9999	1.9999	3.0	0.0
7	1.0	2.0	3.0	0.0

Gauss Seidel Iteration method

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
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4	0.9968	1.9982	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001
6	0.9999	1.9999	3.0	0.0
7	1.0	2.0	3.0	0.0

Comparison

- Gauss-Seidel iteration converges more rapidly than the Jacobi iteration since it uses the latest updates
- But there are some cases that *Jacobi* iteration *does converge* but *Gauss-Seidel* does *not*
- To accelerate the Gauss-Seidel method even further, successive over relaxation method can be used

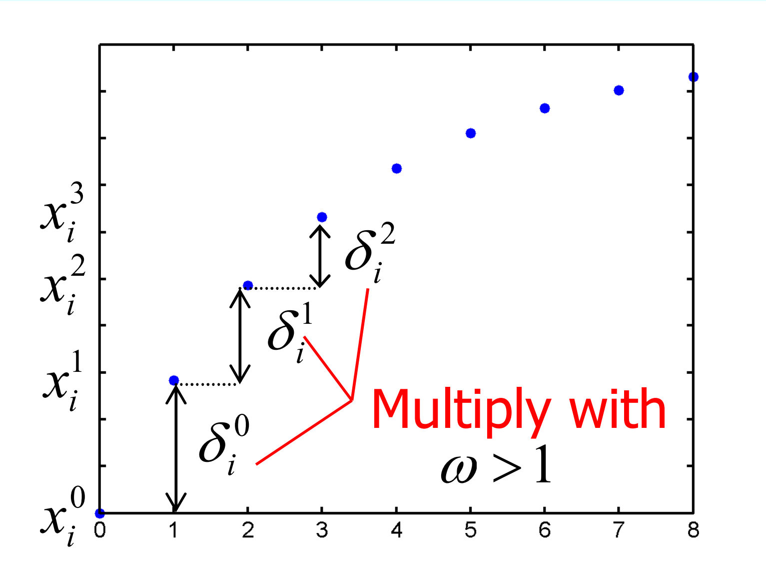
Successive Over Relaxation Method

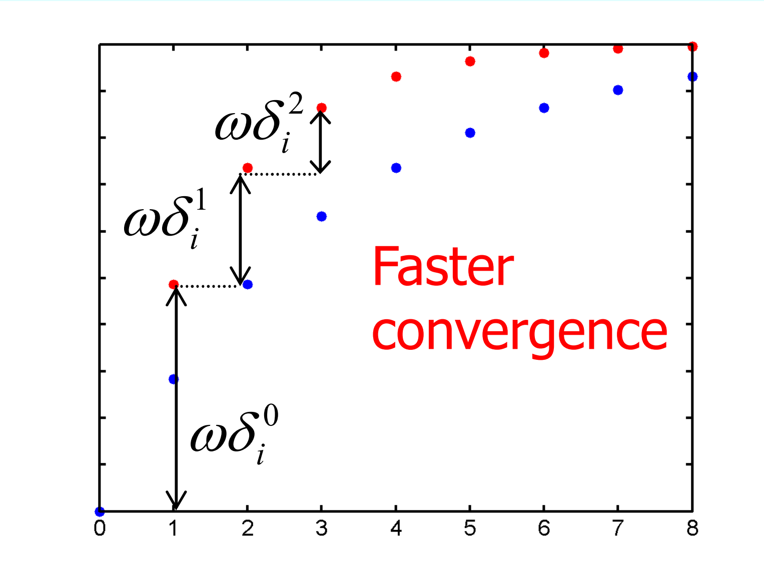
- GS iteration can be also written as follows

$$x_i^{k+1} = x_i^k + \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i}^{n} a_{ij}x_j^k\right]$$

$$x_i^{k+1} = x_i^k + \delta_i^k$$

Correction term





SOR

$$x_i^{k+1} = x_i^k + \omega \delta_i^k$$

$$x_i^{k+1} = x_i^k + \omega \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i}^{n} a_{ij} x_j^k \right]$$

$$x_i^{k+1} = (1-\omega) x_i^k + \omega \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k \right]$$

$1<\omega<2$ over relaxation (faster convergence)
$0<\omega<1$ under relaxation (slower convergence)
There is an optimum value for ω
Find it by trial and error (usually around 1.6)

Jacobi	$x_i^{(k+1)} = x_i^{(k)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)}\right)$
Gauss-Seidel	$x_i^{(k+1)} = x_i^{(k)} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)}\right)$
SOR	$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)}\right)$

THE END