Euler Method

Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

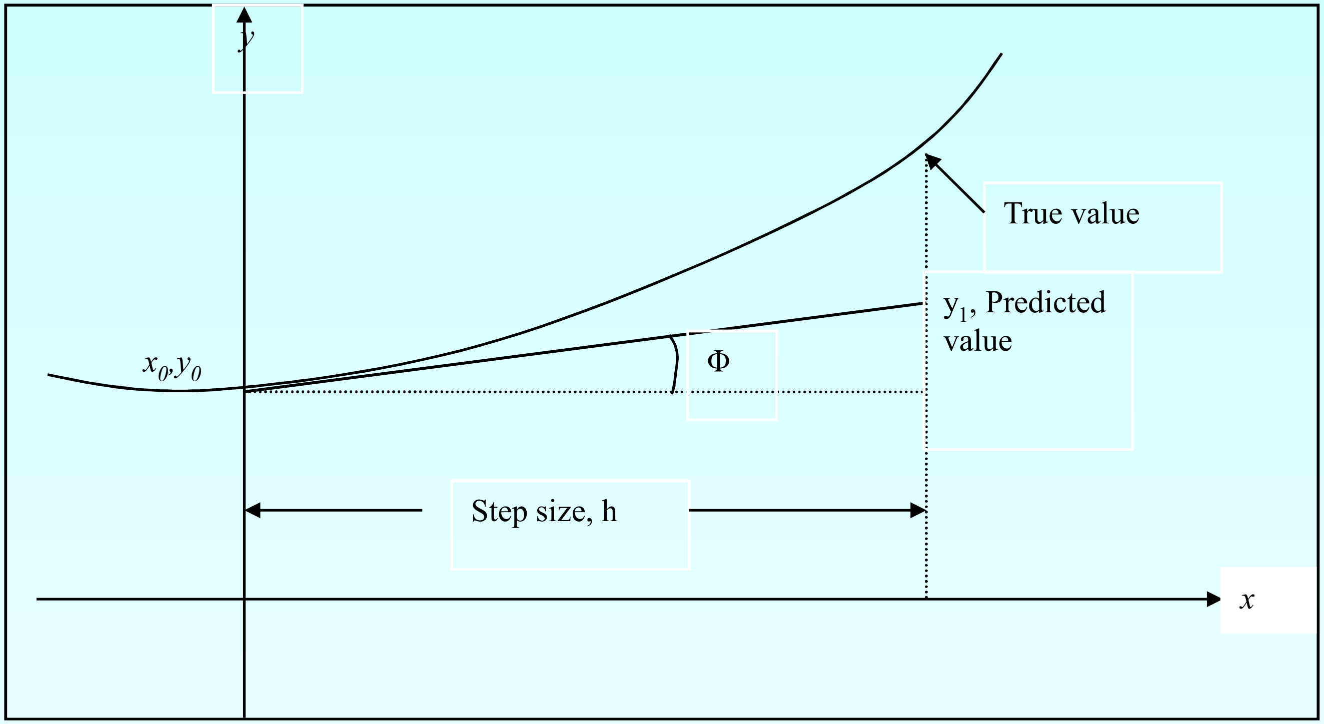

Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

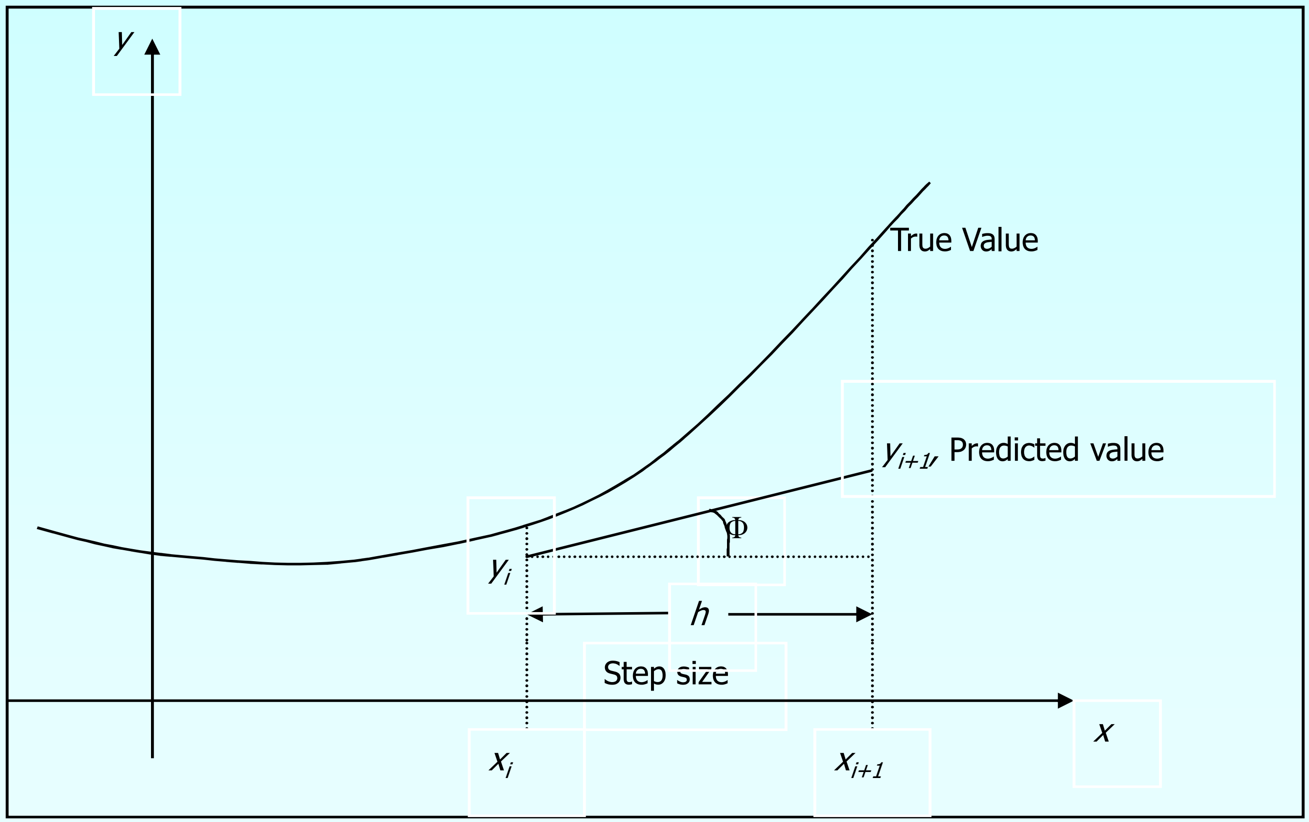

Figure 2. General graphical interpretation of Euler's method

Euler's Method

Use Euler's method with h = 0.1 to solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$ in the range $0 \le x \le 0.5$.

Euler's Method

Use Euler's method with h = 0.1 to solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$ in the range $0 \le x \le 0.5$.

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

Euler's Method

Use Euler's method with h = 0.1 to solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$ in the range $0 \le x \le 0.5$.

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

$x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.2$, $x_3 = x_2 + h = 0.3$, $x_4 = x_3 + h = 0.4$, $x_5 = x_4 + h = 0.5$.

Euler's Method

Use Euler's method with h = 0.1 to solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$ in the range $0 \le x \le 0.5$.

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

$x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.2$, $x_3 = x_2 + h = 0.3$, $x_4 = x_3 + h = 0.4$, $x_5 = x_4 + h = 0.5$.

$$y_{i+1} = y_i + f(x_i, y_i)h$$

Euler's Method

Use Euler's method with h = 0.1 to solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$ in the range $0 \le x \le 0.5$.

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

$x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.2$, $x_3 = x_2 + h = 0.3$, $x_4 = x_3 + h = 0.4$, $x_5 = x_4 + h = 0.5$.

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$y_1 = y_0 + 0.1(x_0^2 + y_0^2) = 0 + 0.1(0 + 0) = 0.$$

Euler's Method

Use Euler's method with h = 0.1 to solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$ in the range $0 \le x \le 0.5$.

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

$x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.2$, $x_3 = x_2 + h = 0.3$, $x_4 = x_3 + h = 0.4$, $x_5 = x_4 + h = 0.5$.

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$y_1 = y_0 + 0.1(x_0^2 + y_0^2) = 0 + 0.1(0 + 0) = 0.$$

$$y_2 = y_1 + 0.1(x_1^2 + y_1^2) = 0 + 0.1\left[(0.1)^2 + 0^2\right] = 0.001.$$

Euler's Method

Use Euler's method with h = 0.1 to solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$ in the range $0 \le x \le 0.5$.

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

$x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.2$, $x_3 = x_2 + h = 0.3$, $x_4 = x_3 + h = 0.4$, $x_5 = x_4 + h = 0.5$.

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$y_1 = y_0 + 0.1(x_0^2 + y_0^2) = 0 + 0.1(0+0) = 0.$$

$$y_2 = y_1 + 0.1(x_1^2 + y_1^2) = 0 + 0.1\left[(0.1)^2 + 0^2\right] = 0.001.$$

$$y_3 = y_2 + 0.1(x_2^2 + y_2^2) = 0.001 + 0.1\left[(0.2)^2 + (0.001)^2\right] = 0.005.$$

$$y_4 = y_3 + 0.1(x_3^2 + y_3^2) = 0.005 + 0.1\left[(0.3)^2 + (0.005)^2\right] = 0.014.$$

$$y_5 = y_4 + 0.1(x_4^2 + y_4^2) = 0.014 + 0.1\left[(0.4)^2 + (0.014)^2\right] = 0.0300196.$$

Modified Euler's Method

$$\frac{dy}{dx} = f(x, y)$$

$$dy = f(x, y)dx$$

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)dx$$

$$y_1 - y_0 = \int_{x_0}^{x_1} f(x, y)dx$$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)dx$$

Trapezoidal rule

$$\int_{x_0}^{x_1} f(x, y)dx = \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

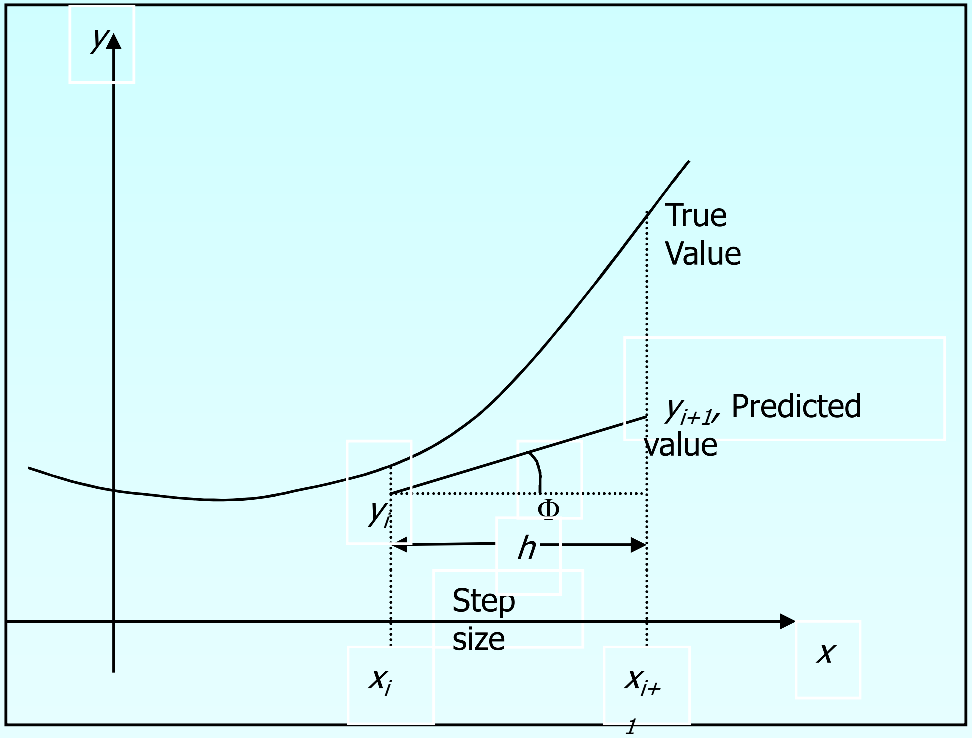

Figure 2. General graphical interpretation of Euler's method

Modified Euler's Method

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

Modified Euler method is given by the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})], \quad n = 0, 1, 2, \cdots$$

Modified Euler's Method

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

Modified Euler method is given by the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})], \quad n = 0, 1, 2, \cdots$$

where $y_1^{(n)}$ is the nth approximation to y_1.

Modified Euler's Method

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

Modified Euler method is given by the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[\, f(x_0, y_0) + f(x_1, y_1^{(n)})], \quad n = 0, 1, 2, \cdots$$

where $y_1^{(n)}$ is the nth approximation to y_1.

The iteration formula can be started by choosing $y_1^{(0)}$ from Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0).$$

Modified Euler's Method

Using modified Euler's method, determine the value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + \sqrt{y}; \; y(0) = 1. \quad (\text{Take } h = 0.2)$$

Modified Euler's Method

Using modified Euler's method, determine the value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + \sqrt{y};\ y(0) = 1. \quad \text{(Take } h = 0.2\text{)}$$

Here $f(x, y) = x + \sqrt{y};\ x_0 = 0,\ y_0 = 1.$

Modified Euler's Method

Using modified Euler's method, determine the value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + \sqrt{y};\ y(0) = 1. \quad \text{(Take } h = 0.2\text{)}$$

Here $f(x, y) = x + \sqrt{y};\ x_0 = 0,\ \ y_0 = 1.$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.2(0+1) = 1.2$$

Modified Euler's Method

Using modified Euler's method, determine the value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + \sqrt{y};\ y(0) = 1. \quad (\text{Take } h = 0.2)$$

Here $f(x, y) = x + \sqrt{y};\ x_0 = 0,\ \ y_0 = 1.$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.2(0+1) = 1.2$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2}] = 1.2295.$$

Modified Euler's Method

Using modified Euler's method, determine the value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + \sqrt{y};\ y(0) = 1. \quad (\text{Take } h = 0.2)$$

Here $f(x, y) = x + \sqrt{y};\ x_0 = 0,\ y_0 = 1.$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.2(0+1) = 1.2$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2}] = 1.2295.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2295}] = 1.2309.$$

Modified Euler's Method

Using modified Euler's method, determine the value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + \sqrt{y}; \ y(0) = 1. \quad (\text{Take } h = 0.2)$$

Here $f(x, y) = x + \sqrt{y}; \ x_0 = 0, \ \ y_0 = 1.$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.2(0+1) = 1.2$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2}] = 1.2295.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2295}] = 1.2309.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.2}{2}[1 + (0.2 + \sqrt{1.2309}] = 1.2309.$$

Runge-Kutta 2nd Order Method

Runge-Kutta 2nd Order Method

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

According to Euler method

$$y_1 = y_0 + hf\left(x_0, y_0\right)$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, (y_0 + hf(x_0, y_0)\right]$$

Let

$$f\left(x_0, y_0\right) = f_0$$

$$f(x_1, (y_0 + hf(x_0, y_0)) = f((x_0 + h), (y_0 + hf_0)) = f((x_0 + h), (y_0 + k_0))$$

$$y_1 = y_0 + \frac{1}{2}\left[hf_0 + hf((x_0 + h), (y_0 + k_0)\right] = y_0 + \frac{1}{2}(k_0 + l_0)$$

Runge-Kutta 2nd Order Method

$$y_1 = y_0 + \frac{1}{2}\left[hf_0 + hf((x_0+h),(y_0+k_0)\right] = y_0 + \frac{1}{2}(k_0 + l_0)$$

Use second order Runge-Kutta method with $h = 0.1$ to find $y(0.2)$, given

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0) = 0.$$

Runge-Kutta 2nd Order Method

$$y_1 = y_0 + \frac{1}{2}\left[hf_0 + hf((x_0 + h), (y_0 + k_0))\right] = y_0 + \frac{1}{2}(k_0 + l_0)$$

Use second order Runge-Kutta method with $h = 0.1$ to find $y(0.2)$, given

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0) = 0.$$

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

Runge-Kutta 2nd Order Method

$$y_1 = y_0 + \frac{1}{2}\left[hf_0 + hf((x_0 + h),(y_0 + k_0)\right] = y_0 + \frac{1}{2}(k_0 + l_0)$$

Use second order Runge-Kutta method with $h = 0.1$ to find $y(0.2)$, given

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0) = 0.$$

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

$$k_0 = 0.2(x_0^2 + y_0^2) = 0.1(0^2 + 0^2) = 0.$$

$$l_0 = 0.2(x_1^2 + (y_0 + k_0)^2) = 0.1\left[(0.1)^2 + (0+0)^2)\right] = 0.001$$

Runge-Kutta 2nd Order Method

$$y_1 = y_0 + \frac{1}{2}\left[hf_0 + hf((x_0+h),(y_0+k_0))\right] = y_0 + \frac{1}{2}(k_0 + l_0)$$

Use second order Runge-Kutta method with $h = 0.1$ to find $y(0.2)$, given

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0) = 0.$$

Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$.

$$k_0 = 0.2(x_0^2 + y_0^2) = 0.1(0^2 + 0^2) = 0.$$

$$l_0 = 0.2(x_1^2 + (y_0 + k_0)^2) = 0.1\left[(0.1)^2 + (0+0)^2)\right] = 0.001$$

$$y_1 = y_0 + \frac{1}{2}(k_0 + k_0) = 0 + \frac{1}{2}(0 + 0.001) = 0.0005.$$

Runge-Kutta 2nd Order Method

$$k_1 = 0.2(x_1^2 + y_1^2) = 0.1\left[(0.1)^2 + (0.0005)^2\right] = 0.001$$

Runge-Kutta 2nd Order Method

$$k_1 = 0.2(x_1^2 + y_1^2) = 0.1\left[(0.1)^2 + (0.0005)^2\right] = 0.001$$

$$l_1 = 0.2(x_2^2 + (y_1 + k_1)^2) = 0.1\left[(0.2)^2 + (0.0015)^2)\right] = 0.004$$

Runge-Kutta 2nd Order Method

$$k_1 = 0.2(x_1^2 + y_1^2) = 0.1\left[(0.1)^2 + (0.0005)^2\right] = 0.001$$

$$l_1 = 0.2(x_2^2 + (y_1 + k_1)^2) = 0.1\left[(0.2)^2 + (0.0015)^2)\right] = 0.004$$

$$y_2 = y_1 + \frac{1}{2}(k_1 + l_1) = 0.0005 + \frac{1}{2}(0.001 + 0.004) = 0.003$$

Hence $y(0.1) = 0.0005,$ $\qquad y(0.2) = 0.003.$

Example

$$\frac{dy}{dx} + xy = 0\,,\, x = 0,\, y(0) = 1$$

Solve the differential equation from x=0 to x=0.25 using Runge-Kutta and Euler's method
Take h=0.05

x	0	0.05	0.1	0.15	0.2	0.25	
Runge	1	0.999	0.994	0.988	0.979	0.968	
Euler	1	1	0.997	0.992	0.985	0.975	

Solution

Step 1:

$$\frac{dy}{dx} = f(x, y) = -xy, h = 0.05$$

$$y_1 = y_0 + f(x_0, y_0)h$$
$$= 1 + 0.05(-0.1)$$
$$= 1$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$
$$= 1 + 0.5(0.05(-0.1) + 0.05(-0.05.1))$$
$$= 0.99875 \approx 0.999$$

$$y_2 = y_1 + f(x_1, y_1)h$$
$$= 1 + 0.05(-0.05.1)$$
$$= 0.9975$$

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$
$$= 0.9988 + 0.5(0.05(-0.05 \times 0.9988) + 0.05(-0.15 \times 0.9963))$$
$$= 0.9938 \approx 0.994$$