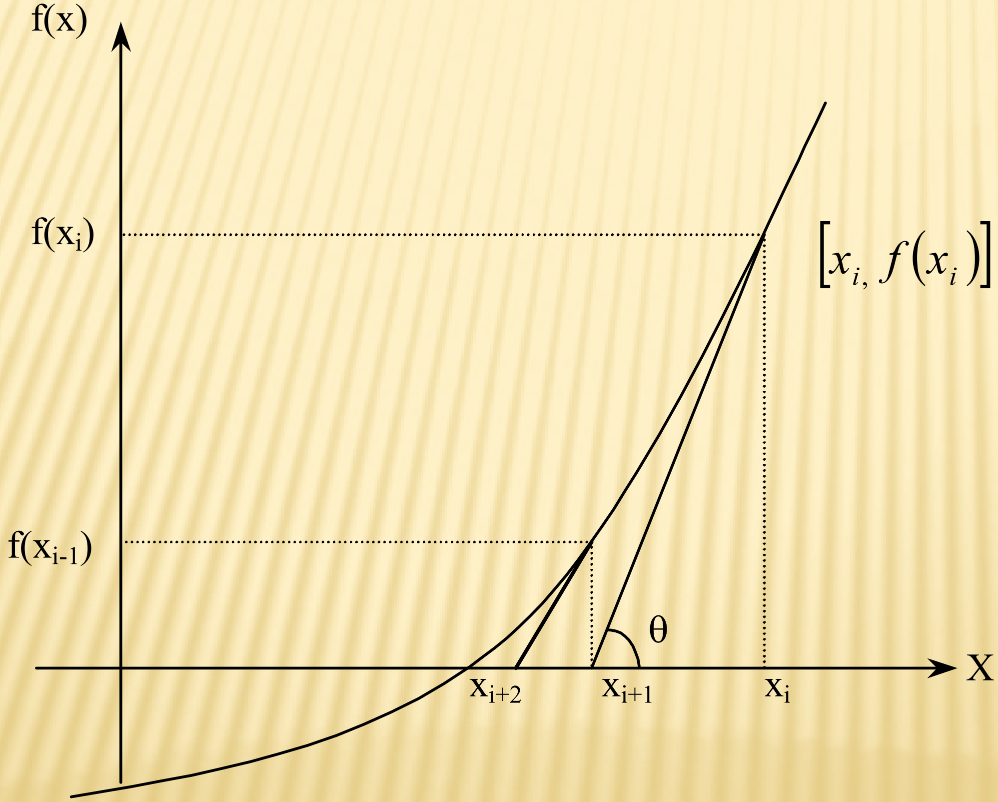
NEWTON-RAPHSON METHOD

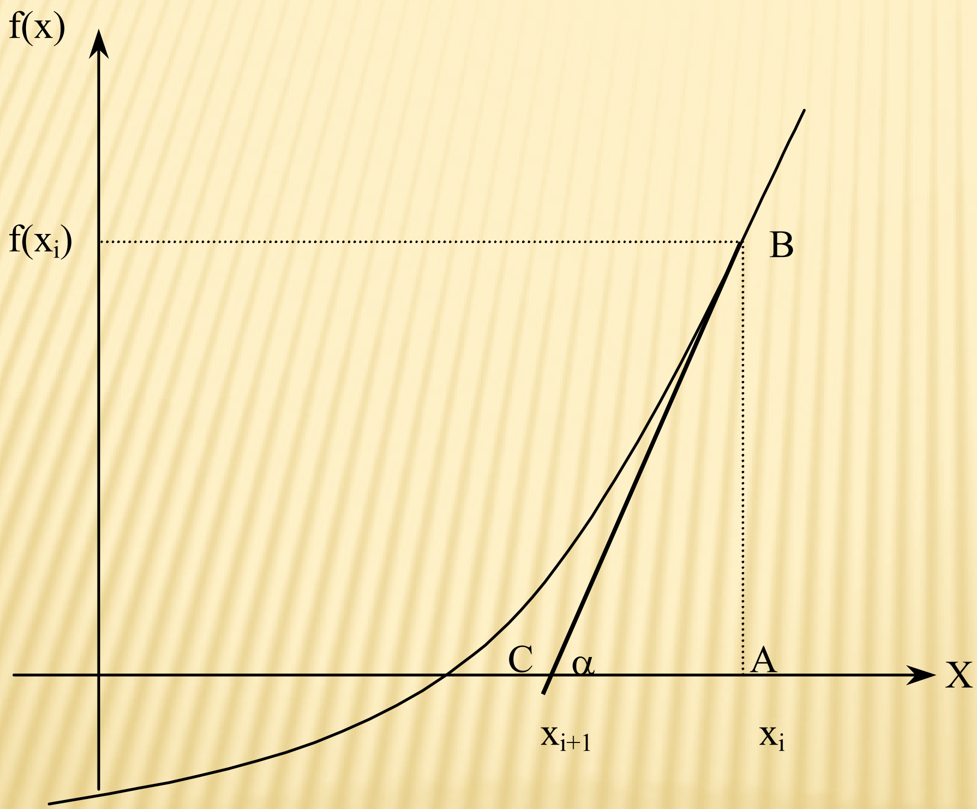
NEWTON-RAPHSON METHOD



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 1 Geometrical illustration of the Newton-Raphson method.

DERIVATION



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

NEWTON-RAPHSON METHOD

Let us consider x_i be the approximation of exact root 'α' of equation, f(x)=0

So we assume that, $\alpha = x_i + \Delta x_i$

Hence, $f(\alpha) = 0$ i.e. $f(x_i + \Delta x_i) = 0$

According to Taylor's Series,

$$f(x_i + \Delta x_i) = f(x_i) + \Delta x_i f'(x_i) + \frac{(\Delta x_i)^2}{2!} f''(x_i) + \cdots\cdots$$

Since x_i is very close to the exact α, Δx_i is small and $(\Delta x_i)^2$ and higher powers of Δx_i can be neglected.

$$f(x_i + \Delta x_i) \cong f(x_i) + \Delta x_i f'(x_i) = 0$$

$$\Delta x_i = -\frac{f(x_i)}{f'(x_i)}$$

NEWTON-RAPHSON METHOD

Using the above expression of Δx_i , a better approximation of x_0 can be obtained as,

$$x_1 = x_0 + \Delta x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are x_2, x_3, ·····, x_{n+1} which can be obtained accordingly, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

ALGORITHM FOR NEWTON-RAPHSON METHOD

STEP 1

Evaluate $f'(x)$ symbolically.

STEP 2

Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

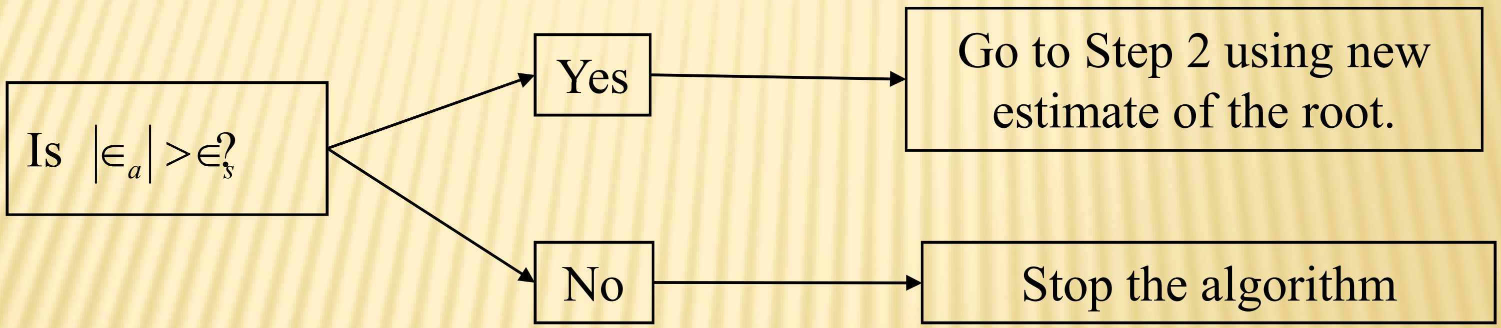
STEP 3

Find the absolute relative approximate error $\left|\in_a\right|$ as

$$\left|\in_a\right| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

STEP 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$.



Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

EXAMPLE 1 CONT.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\begin{aligned}\left|\in_a\right| &= \left|\frac{x_2 - x_1}{x_2}\right| \times 100 \\ &= \left|\frac{0.06238 - 0.06242}{0.06238}\right| \times 100 \\ &= 0.0716\ \%\end{aligned}$$

The maximum value of m for which $\left|\in_a\right| \leq 0.5 \times 10^{2-m}$ is 2.844. Hence, the number of significant digits at least correct in the answer is 2.

ADVANTAGES AND DRAWBACKS OF NEWTON RAPHSON METHOD

ADVANTAGES

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

DRAWBACKS

1. Divergence at inflection points
   Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function $f(x)$ may start diverging away from the root in ther Newton-Raphson method.

For example, to find the root of the equation $f(x) = (x-1)^3 + 0.512 = 0$

The Newton-Raphson method reduces to $x_{i+1} = x_i - \frac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$

Table 1 shows the iterated values of the root of the equation.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$.

Eventually after 12 more iterations the root converges to the exact value of $x = 0.2$.

DRAWBACKS – INFLECTION POINTS

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000

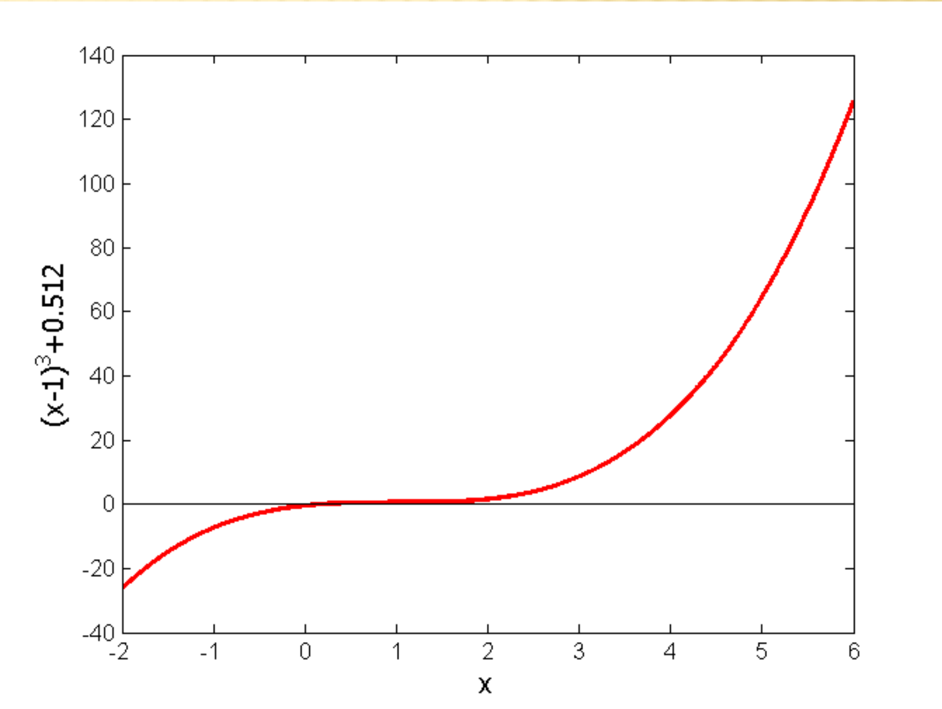


Figure 8 Divergence at inflection point for

$$f(x) = (x-1)^3 + 0.512 = 0$$

DRAWBACKS – DIVISION BY ZERO

2. Division by zero

   For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4\times10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4\times10^{-6}}{3x_i^2 - 0.06x_i}$$

For $x_0 = 0$ or $x_0 = 0.02$, the denominator will equal zero.

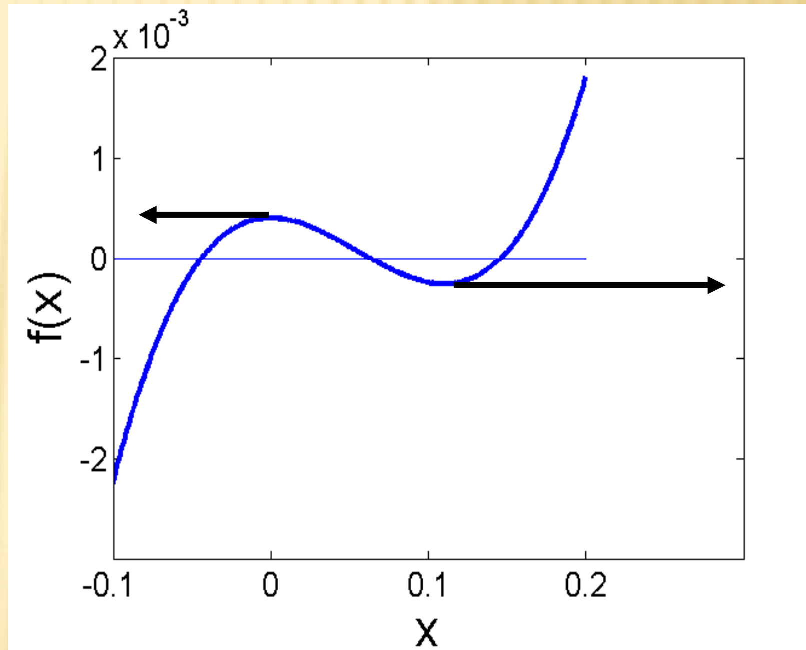


Figure 9 Pitfall of division by zero or near a zero number

DRAWBACKS – OSCILLATIONS NEAR LOCAL MAXIMUM AND MINIMUM

3. Oscillations near local maximum and minimum

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

Eventually, it may lead to division by a number close to zero and may diverge.

For example for $f(x) = x^2 + 2 = 0$ the equation has no real roots.

DRAWBACKS – OSCILLATIONS NEAR LOCAL MAXIMUM AND MINIMUM

Table 3 Oscillations near local maxima and mimima in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	–1.0000	3.00	
1	0.5	2.25	300.00
2	–1.75	5.063	128.571
3	–0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	–0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96

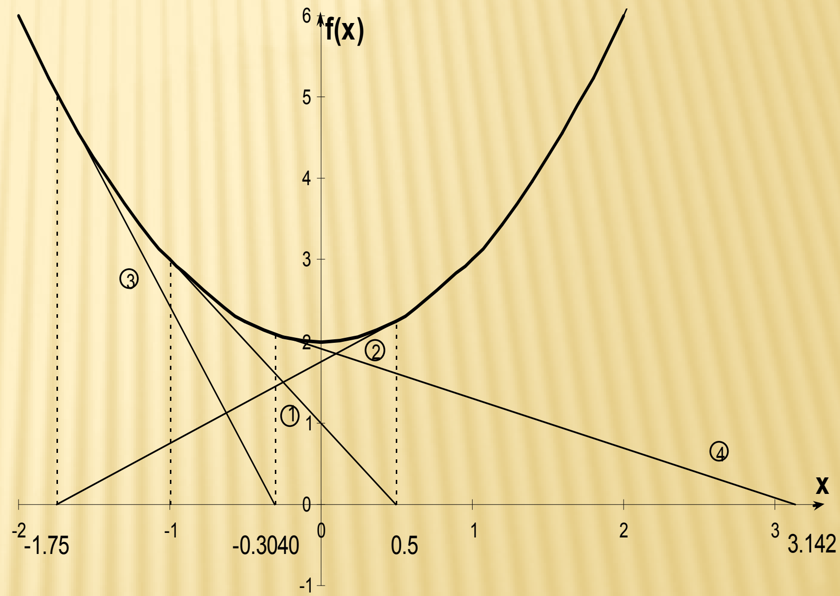


Figure 10 Oscillations around local minima for $f(x)=x^2+.2$