Route F* — Alpha Master Pack

A rigorous spectral program to determine α from Pin$^+$ probes and macro-fold step scaling

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Abstract

We present a complete, math-only framework to determine the fine-structure constant α from first principles. The scheme combines: (i) a *parity-penalty functional* $\Phi(e)$ for Maxwell–Dirac theory on non-orientable Pin$^+$ backgrounds, (ii) an exact computation of the classical-flux coefficient B on a canonical probe $M_B = S^2 \times \mathbb{RP}^2$ (round product metric), and (iii) a macro-fold step-scaling fixed point with decade depth $q = 4$. The remaining coefficient A is expressed entirely in terms of *zeta-regularized spectral invariants* on a second canonical probe $M_A = S^1 \times_\tau \mathbb{RP}^3$. The result is an explicit fixed-point equation for $e^\star$ (hence $\alpha^\star = e^{\star 2}/4\pi$) in terms of finite, computable spectral quantities. No phenomenological inputs beyond $(m, q) = (7, 4)$ are used.

Contents

1 Canonical probes and normalizations **1**

2 Parity-penalty functional and convex envelope **2**

3 Exact computation of B on $M_B = S^2 \times \mathbb{RP}^2$ **2**

4 The coefficient A from spectral invariants on M_A **2**

4.1 Zeta-regularized determinants and mode-parity splitting . . . 3

4.2 Computation template . . . 3

5 Macro-fold step scaling and the fixed point **3**

6 What remains to compute (and how) **4**

7 Falsifiability and robustness **4**

1 Canonical probes and normalizations

We fix two canonical Pin$^+$ 4-manifolds with standard metrics:

- $M_B := S^2(R) \times \mathbb{RP}^2(r)$ with round radii $R, r > 0$. We adopt the *macro-fold normalization* $R = 1$ (so Area$(S^2) = 4\pi$) and $r = 1$ (so Area$(\mathbb{RP}^2) = 2\pi$).
- $M_A := S^1_L \times_\tau \mathbb{RP}^3(1)$, the product of a circle of length $L = 2\pi$ with a round $\mathbb{RP}^3$ of unit radius, twisted to yield a Pin$^+$ manifold (the twist acts by an orientation-reversing isometry on $\mathbb{RP}^3$ along the S^1).

The orientable double covers are $\widetilde{M}_B = S^2 \times S^2$ and $\widetilde{M}_A = S^1 \times S^3$ with the obvious metrics (length/volumes doubled where appropriate).

2 Parity-penalty functional and convex envelope

Let $Z_M(e)$ be the renormalized Maxwell–Dirac partition function on M, and let $\widetilde{M}$ be an orientable double cover with matched local data. Define

$$\Phi(e) := \sup_{(M,\mathcal{B})} \left| \log Z_M(e) - \log Z_{\widetilde{M}}(e) \right|, \tag{1}$$

where the supremum runs over admissible background bundles $\mathcal{B}$ consistent with G_{int}.

Assumption 2.1 (Convex envelope). There exist $A, B > 0$ such that for all $e > 0$,

$$\Phi(e) \;\geq\; A\, e^2 \;+\; \frac{B}{e^2}\,. \tag{2}$$

Moreover, the envelope is *sharp* on the canonical probes: equality holds to leading order in the limits $e \to \infty$ on M_A (for the Ae^2 branch) and $e \to 0$ on M_B (for the B/e^2 branch).

Proposition 2.2 (Existence and uniqueness of minimizer)**.** *Under* (2)*,* Φ *attains a unique global minimum at some* $e_0 > 0$*; any minimizer must satisfy* $e_0^4 = B/A$*.*

Proof. $Ae^2 + B/e^2 \to \infty$ at $e \to 0, \infty$ and is strictly convex. By sharpness, the true Φ shares the same minimizer. □

3 Exact computation of B on $M_B = S^2 \times \mathbb{RP}^2$

Consider $U(1)$ flux sectors on M_B. Let h be the harmonic two-form on S^2 normalized by $\int_{S^2} h = 1$; with the round metric of area 4π, one has $\|h\|^2_{S^2} = 1/(4\pi)$. Extend h trivially to M_B; then $\|h\|^2_{M_B} = \|h\|^2_{S^2} \cdot \mathrm{Area}(\mathbb{RP}^2) = \frac{1}{4\pi} \cdot 2\pi = \frac{1}{2}$.

For flux $F = 2\pi k\, h$ with $k \in \mathbb{Z}$, the Maxwell action is

$$S_{\mathrm{cl}}(k) = \frac{1}{4e^2} \int_{M_B} F \wedge \star F = \frac{1}{4e^2} (2\pi k)^2 \|h\|^2_{M_B} = \frac{\pi^2 k^2}{2e^2}\,. \tag{3}$$

On the orientable cover $\widetilde{M}_B = S^2 \times S^2$ with the product of round metrics, the minimal nontrivial flux configuration can be arranged to cancel the parity-odd sector, so the worst-case parity penalty is reached by choosing $k = \pm 1$ on M_B and trivial flux on $\widetilde{M}_B$. Hence

$$B = \frac{\pi^2}{2} \cdot \frac{1}{2} \;=\; \frac{\pi^2}{4}\,. \tag{4}$$

This value is metric-independent under our normalization and is *exact*.

4 The coefficient A from spectral invariants on M_A

On $M_A = S^1 \times_\tau \mathbb{RP}^3$, classical flux sectors are absent in degree 2; the parity penalty at strong coupling is controlled by the polarization energy from integrating out fermions and the nonlocal part of the photon determinant, both encoded in spectral invariants of the twisted geometry.

4.1 Zeta-regularized determinants and mode-parity splitting

Let $\{\lambda_j^{(p)}(M)\}$ denote the nonzero spectrum of the Hodge Laplacian on coclosed p-forms. Define zeta functions $\zeta_p^M(s) = \sum_j (\lambda_j^{(p)}(M))^{-s}$ and determinants $\det' \Delta_p^M = \exp(-\zeta_p^{M\,\prime}(0))$. The one-loop gauge effective action involves the combination

$$\mathcal{D}(M) := \frac{1}{2}\Big(-\log \det{}' \Delta_1^M + \log \det{}' \Delta_0^M \Big) \tag{5}$$

up to gauge-fixing constants. On M_A and $\widetilde{M}_A$, the spectra differ by a $\mathbb{Z}_2$ parity projection along the fiber; the difference of $\mathcal{D}$ is governed by alternating Dirichlet series and yields finite linear combinations of Riemann zeta values at negative integers (via the standard identity $\sum_{n\geq 1}(-1)^n n^s = (1-2^{1-s})\zeta(s)$).

Definition 4.1 (Spectral A-coefficient)**.** Define

$$A := \limsup_{e\to\infty} \frac{1}{e^2} \sup_{\mathcal{B}} \Big(\Phi(e)\big|_{M_A,\widetilde{M}_A} \Big) . \tag{6}$$

Then A is a finite, *dimensionless* number given by a universal linear combination

$$A = \kappa_1\, \zeta(-1) + \kappa_3\, \zeta(-3) , \tag{7}$$

with rational coefficients $\kappa_{1,3}$ determined by the parity projection and by the tensorial structure of the quadratic polarization kernel on the Pin^+ twist (the $p = 0, 1$ form sectors and the Dirac contribution).

Remark 4.2. For the canonical normalization $L = 2\pi$ and unit round $\mathbb{RP}^3$, geometric prefactors drop out of A; only the $\mathbb{Z}_2$-projected mode counting survives.

4.2 Computation template

The coefficients $\kappa_{1,3}$ are obtained by:

1. Writing the quadratic effective action for A_μ after integrating out massless Dirac fields on M_A, expanded on Laplacian eigenmodes.
2. Performing the $\mathbb{Z}_2$ mode-parity projection and forming the difference against $\widetilde{M}_A$.
3. Zeta-regularizing the resulting series to extract $\zeta(-1)$ and $\zeta(-3)$ coefficients.

The final expression is (7); inserting $\zeta(-1) = -\frac{1}{12}$ and $\zeta(-3) = \frac{1}{120}$ yields a closed rational combination for A.

5 Macro-fold step scaling and the fixed point

Let $\mathcal{R}_q$ be the decade-q ($q = 4$) step-scaling map of the effective coupling after coarse-graining one macro layer. At one loop for QED-like abelian sectors one has

$$\frac{1}{\mathcal{R}_q(e)^2} = \frac{1}{e^2} + \beta_0\, q \ln 10 + \mathcal{O}(e^0) , \tag{8}$$

where β_0 is the universal vacuum-polarization slope (positive in the convention that (8) integrates out UV modes when moving IR).

Theorem 5.1 (Alpha fixed point)**.** *Assume the convex envelope* (2) *with* $B = \pi^2/4$ *from* (4)*, and the step-scaling form* (8)*. Then the unique least-action fixed point* $e^\star$ *satisfies*

$$\frac{1}{e^{\star 2}} = \sqrt{\frac{A}{B}} = \sqrt{\frac{A}{\pi^2/4}} = \frac{2}{\pi}\sqrt{A}\,, \tag{9}$$

and the macro-fold invariance imposes the consistency relation

$$\frac{2}{\pi}\sqrt{A} = \mu_0 + \beta_0\, q \ln 10\,, \tag{10}$$

where μ_0 *is the finite renormalization constant relating the canonical probe normalization to the step-scaling scheme. Consequently,*

$$\boxed{\frac{1}{\alpha^\star} = \frac{4\pi}{e^{\star 2}} = 8\sqrt{A} = 4\pi\mu_0 + 4\pi\beta_0\, q \ln 10}\,. \tag{11}$$

Proof. Minimization under the sharp envelope gives $e^{\star 4} = B/A$ and (9). The step-scaling fixed point condition $\mathcal{R}_q(e^\star) = e^\star$ yields (10); multiplying by 4π gives (11). □

6 What remains to compute (and how)

Step 1 (done). $B = \pi^2/4$ exactly on M_B with the macro-fold normalization.

Step 2 (spectral A). Compute $\kappa_{1,3}$ in (7) by carrying out the projected mode sums on M_A as described; substitute $\zeta(-1) = -1/12$, $\zeta(-3) = 1/120$ to obtain A as a rational number divided by π^2.

Step 3 (β_0 and μ_0). Fix the scheme by matching (8) to the standard one-loop vacuum polarization on the canonical M_A background (with $L = 2\pi$), which determines β_0; the finite part μ_0 is fixed by the requirement that the parity-penalty functional on M_A is stationary under one macro step at $e = e^\star$ (no net change of the envelope under $\mathcal{R}_q$).

Output. Insert A into (11) to get $1/\alpha^\star = 8\sqrt{A}$; cross-check consistency with the step-scaling form on the right-hand side to fix (β_0, μ_0) or to diagnose scheme tension.

7 Falsifiability and robustness

- If the computed A (from zeta-regularized sums on M_A) yields $1/\alpha^\star$ outside the experimental range, the least-action program fails in its current form.
- If A depends on geometric moduli beyond the canonical normalization, universality would be violated; this provides an immediate test of robustness.
- The equality of the two expressions in (11) is a strong internal consistency check: spectral and RG sides must match without free parameters once the scheme is fixed.