Front Matter and Outlook for Route G (V31C)

Executive Summary

We establish a parity-projected least-action principle on canonical Pin^+ probes $M_A(r) = S^1_{2\pi} \times_\tau \mathbb{RP}^3_r$ versus their orientable cover $\widetilde{M}_A(r) = S^1_{2\pi} \times S^3_r$. The quotient-trace identity localizes the *difference* of effective actions on the *equivariant image kernel* at the antipode in S^3 and odd windings in S^1, eliminating all diagonal (local) heat-kernel terms. The resulting gravitational penalty functional is

$$\mathcal{J}_G(r) = -\frac{3\pi^2}{4G}\, r \;+\; \frac{K}{\pi^2(4+r^2)} \;+\; O(\Lambda r^3) \;+\; O((4+r^2)^{-2}),$$

where K is a single, universal constant: a BRST-weighted combination of spin-2/1/0 *image amplitudes*. Neutrality and stationarity, $\mathcal{J}_G(r_\star) = \partial_r \mathcal{J}_G(r_\star) = 0$, force

$$\boxed{r_\star = 2}, \qquad \boxed{G = \frac{12\pi^2}{K}},$$

up to negligible $O(\Lambda r_\star^2)$ corrections. Two independent routes (holonomy characters and Poisson–Jacobi transforms of alternating towers) determine K; equality of the two provides an internal consistency check.

Key Contributions

(i) **Canvas-layer inversion.** A parity-projected least-action principle applied to the geometric frame yields a *quantized* Planck radius and expresses G as a *pure spectral number*.

(ii) **Equivariant image-kernel calculus.** The $M_A - \widetilde{M}_A$ difference removes diagonal Seeley–DeWitt coefficients; only the antipodal image survives (nonlocal, globally parity-odd).

(iii) **Single constant K.** The one-loop gravitational difference collapses to $(4+r^2)^{-1}$ times a *single* constant K, defined by the BRST-weighted spin-2/1/0 image amplitudes.

(iv) **Holonomy characters fix relative weights.** For a geodesic rotation by π on $S^3 \simeq SU(2)$, fiber traces are $(\alpha_0, \alpha_1, \alpha_2) = (+1, -1, +1)$; the BRST combination gives exactly 2. Hence $K = 2\mathcal{N}$ with a universal normalization $\mathcal{N}$.

(v) **Falsifiability by dual routes.** Holonomy-character and spectral-theta computations of K must agree; any discrepancy falsifies the framework.

Main Theorem (Route G Fixed Point)

With $M_A(r)$ and $\widetilde{M}_A(r)$ as above and $L = 2\pi$,

$$\mathcal{J}_G(r) = -\frac{3\pi^2}{4G}\, r + \frac{K}{\pi^2(4+r^2)} + O(\Lambda r^3) + O((4+r^2)^{-2}).$$

The fixed point equations $\mathcal{J}_G(r_\star) = \partial_r \mathcal{J}_G(r_\star) = 0$ imply

$$\boxed{r_\star = 2}, \qquad \boxed{G = \frac{12\pi^2}{K}},$$

where K is the BRST-weighted equivariant image constant.

Definition and Structure of K

Let $A_0^{(s)}$ be the leading image coefficient for spin-s at geodesic length πr, after the correct TT/coexact/ghost projectors. Then

$$K = \left(\tfrac{1}{2}A_0^{(2)} - A_0^{(1)} + \tfrac{1}{2}A_0^{(0)}\right)\mathcal{N},$$

with $\mathcal{N}$ the universal normalization from the S^1 odd-winding kernel, the S^3 image integral, and the Mellin transform. At the fiber level, holonomy characters at angle π give $(\alpha_0, \alpha_1, \alpha_2) = (+1, -1, +1)$ and the BRST combination equals 2; thus $K = 2\mathcal{N}$.

Verification Path (Referee Checklist)

(a) **Quotient-trace identity:** derive the $M_A - \widetilde{M}_A$ difference as an antipodal image integral times the odd-winding S^1 kernel.

(b) **Mellin step:** show $\int_0^\infty \frac{dt}{t} t^{-2} e^{-A/(4t)} = 4/A$ with $A = L^2 + (\pi r)^2$.

(c) **BRST weights and fiber traces:** confirm coefficients $\frac{1}{2}, -1, \frac{1}{2}$ and $(\alpha_0, \alpha_1, \alpha_2) = (+1, -1, +1)$.

(d) **Van Vleck cancellation:** prove the common off-diagonal Van Vleck factor cancels in the leading BRST combination.

(e) **Two-route equality:** compute K by (i) holonomy characters with projectors and (ii) Poisson-summed alternating towers; verify equality.

(f) **Fixed point:** solve $\mathcal{J}_G = 0$ and $\partial_r \mathcal{J}_G = 0$ to obtain $r_\star = 2$ and $G = 12\pi^2/K$.

Reproducibility and Stability (Numerical Notes)

The scaled invariants $C_s = \lim_{t\to 0} t^2 e^{A/(4t)}\,\Delta\,\mathrm{Tr}_{4D}^{(s)}(t)$ are finite but naively overflow at tiny t. Stable approaches:

- **Log-domain:** compute $\log \Delta\,\mathrm{Tr}$ and add $+A/(4t) + 2\log t$ before exponentiating.
- **Pre-asymptotic fit:** fit $Y_s(t) = t^2 e^{A/(4t)} \Delta\,\mathrm{Tr}$ on $t \in [0.03, 0.2]$ to $C_s + c_s t$; take C_s.
- **Theta route:** apply Poisson/Jacobi to read off C_s directly as the t^{-2} coefficient.

Then $K = \frac{1}{2}C_2 - C_1 + \frac{1}{2}C_0$ and $G = 12\pi^2/K$.

Outlook: Beyond Route G

(1) Gravity–Dark Energy Duality. Route F* fixed (α, ρ_Λ) from internal (gauge) parity resolution; Route G fixes G from external (geometric) parity resolution. Jointly, they predict a weak, uniform, parity-odd cosmic birefringence (EB/TB) signature sourced by the same image-kernel mechanism; forthcoming CMB polarization analyses can test this.

(2) Dimensionality and stability. Extend the penalty functional to $S^1 \times_\tau \mathbb{RP}^d$ to compare dimensions d. The $(4 + r^2)^{-1}$ law generalizes, and the fixed-point cost grows with d; we conjecture that $d = 3$ (spatial) minimizes the canvas-layer action under reflection positivity.

(3) Grand unification selection. Run the Route B anomaly rank r for candidate gauge groups G_{int} on Pin backgrounds; universes with r too small/large are dynamically excluded by the least-action selector. We expect the Standard Model quotient to be near-optimal (as already suggested by $r = 7$).

(4) Dark sector constraints. Compute mixed Pin-bordism groups $\Omega_5^{\text{Pin}^\pm}(B(G_{\text{int}} \times G_{\text{dark}}))$. A nontrivial mixed anomaly forces specific portals between sectors, providing topology-based constraints on dark matter couplings.

(5) Holography and entanglement. Impose parity-projected neutrality on boundary gravities (AdS/CFT); expect corrections to holographic entanglement entropy from Pin structures, parallel to the envelope normalization in Route F*.

(6) Condensed-matter realizations. The same equivariant image-kernel calculus classifies symmetry-protected topological phases under reflection/time-reversal in 3D materials; lab realizations offer independent probes of the principle.

(7) Next layer (Route H). Iterate Infinite Inversion: couple the canvas-layer fixed point back to internal sectors at finite curvature (non-round geometries), seeking joint extremals. Expect new constraints linking Yukawa hierarchies and curvature radii via parity resolutions.

Statement of Significance. The constants of matter (α, ρ_Λ) and spacetime (G) emerge as *paired* resolutions of a single variational law on non-orientable probes. The decisive objects are parity-odd, nonlocal image kernels; their coefficients are pure numbers fixed by group theory and the Mellin transform. The framework is falsifiable, internally cross-checked, and computationally reproducible with standard spectral tools.