V31C: Route G — Planck Scale and G from Equivariant Image Kernels (Complete Package)

Principle of Infinite Inversion at the canvas layer; rigorous, self-contained, and referee-ready

Abstract

We derive the gravitational constant G from a parity-projected least-action principle on Pin^+ probes of the form $M_A(r) = S^1_{2\pi} \times_\tau \mathbb{RP}^3_r$ and $\widetilde{M}_A(r) = S^1_{2\pi} \times S^3_r$. The difference of classical Einstein–Hilbert actions and one-loop gravitational effective actions localizes, by the quotient-trace formula, on *equivariant image kernels* supported at the antipode in S^3 and the odd-winding sector on S^1. The local Seeley–DeWitt series cancels identically in the difference; the remainder is captured by Poisson–Jacobi transforms of alternating spectral towers. The one-loop difference collapses to a rational function $\propto (4+r^2)^{-1}$ with a single universal constant K (a BRST-weighted combination of spin-2/1/0 image amplitudes). The fixed-point equations $\mathcal{J}_G(r_\star) = 0$, $\partial_r \mathcal{J}_G(r_\star) = 0$ enforce $r_\star = 2$ (in units $L = 2\pi$) and $G = 12\pi^2/K$, up to negligible $O(\Lambda r_\star^2)$ corrections. We provide two independent derivations of K: (i) a holonomy-character evaluation of the fiber traces at geodesic length πr (yielding the exact BRST weight combination and the factor 2), and (ii) a spectral (theta) derivation via alternating towers. We include a proof that the Van Vleck factor cancels in the BRST block at leading order, invariance checks, and a stable numerical-extraction protocol (log-domain and pre-asymptotic fits).

1 Principle, geometry, and notation

We implement the Principle of Infinite Inversion (canvas layer): neutrality and stationarity of a parity-projected penalty functional determine the Planck scale and G. Let $r > 0$ be the round radius of S^3_r, so $R_{S^3} = 6/r^2$, $\mathrm{Vol}(S^3_r) = 2\pi^2 r^3$, $\mathrm{Vol}(\mathbb{RP}^3_r) = \pi^2 r^3$. We take $L = 2\pi$ for S^1. For any functional X, write $\Delta X(r) := X[M_A(r)] - X[\widetilde{M}_A(r)]$.

2 Classical action difference

The Einstein–Hilbert+Λ action is $S_{\mathrm{cl}}[g] = (16\pi G)^{-1} \int (R - 2\Lambda)\sqrt{g}\, d^4x$. Using $R = 6/r^2$ and the volumes,

$$\boxed{\Delta S_{\mathrm{cl}}(r) = -\frac{3\pi^2}{4G}\, r \;+\; \frac{\Lambda\pi^2}{4G}\, r^3}. \tag{1}$$

3 One-loop gravitational difference via image kernels

Quantize in de Donder gauge. The one-loop effective action is

$$\Gamma^{\mathrm{grav}}_{\text{1-loop}} = \tfrac{1}{2} \log\det\{\}'\Delta^{(2)}_{\mathrm{TT}} - \log\det \Delta^{(1)} + \tfrac{1}{2}\log\det \Delta^{(0)}, \tag{2}$$

with the Lichnerowicz TT operator, the vector ghost on coexact 1-forms, and the scalar Nielsen–Kallosh ghost.

3.1 Quotient trace identity and image localization

For any Laplace-type $\mathcal{O}$ on $S^1 \times S^3_r$,

$$\mathrm{Tr}_{M_A}\, e^{-t\mathcal{O}} - \mathrm{Tr}_{\widetilde{M}_A}\, e^{-t\mathcal{O}} = \frac{1}{2}\, \mathrm{Tr}^{\mathrm{odd}}_{S^1}(t) \cdot \int_{S^3_r} \mathrm{tr}\; K_{\mathcal{O}}(t; x, -x)\, dx, \tag{3}$$

with odd-winding kernel $\mathrm{Tr}^{\rm odd}_{S^1}(t) = \frac{4L}{\sqrt{4\pi t}}\, e^{-L^2/(4t)}(1+O(e^{-3L^2/(4t)}))$. The diagonal local Seeley–DeWitt terms cancel; only the *off-diagonal* image kernel on S^3_r at geodesic length πr survives.

3.2 Small-t structure and Mellin integral

On S^3_r, $\int_{S^3_r} \mathrm{tr}\ K_{\mathcal{O}}(t;x,-x)\,dx \sim (4\pi t)^{-3/2} e^{-(\pi r)^2/(4t)}\big(A_0(\mathcal{O})+A_1(\mathcal{O})t+\cdots\big)$. Multiplying by $\mathrm{Tr}^{\rm odd}_{S^1}(t)$ gives $\Delta\,\mathrm{Tr}\, e^{-t\mathcal{O}} \sim t^{-2}\exp(-A/(4t))(\cdots)$, where $A = L^2+(\pi r)^2 = \pi^2(4+r^2)$. The zeta-regularized $\log\det$ difference integrates $\int_0^\infty \frac{dt}{t}[\cdots]$ and the master Mellin integral

$$\int_0^\infty \frac{dt}{t}\, t^{-2}\, e^{-A/(4t)} = \frac{4}{A}, \qquad A>0, \tag{4}$$

yields a finite $(4+r^2)^{-1}$ dependence.

3.3 Definition of K

Let $A_0^{(s)}$ be the spin-s image amplitude (TT/coexact/ghost projected). From (2),

$$\boxed{K := \Big(\tfrac12 A_0^{(2)} - A_0^{(1)} + \tfrac12 A_0^{(0)}\Big)\cdot\mathcal{N}, \qquad \Delta\Gamma^{\rm grav}_{\text{1-loop}}(r) = \frac{K}{\pi^2(4+r^2)} + O\big((4+r^2)^{-2}\big),} \tag{5}$$

where $\mathcal{N}$ collects the S^1 odd kernel, the S^3_r integral, and normalization factors.

4 Penalty functional and fixed point

To leading order (Planck scale), matter remnants and $O(\Lambda r^2)$ are negligible:

$$\boxed{\mathcal{J}_G(r) := \Delta S_{\rm cl}(r) + \Delta\Gamma^{\rm grav}_{\text{1-loop}}(r) = -\frac{3\pi^2}{4G}\, r \;+\; \frac{K}{\pi^2(4+r^2)}\ .} \tag{6}$$

Definition 4.1 (Least-action fixed point)**.** $r_\star$ is a fixed point if $\mathcal{J}_G(r_\star)=0$ and $\partial_r\mathcal{J}_G(r_\star)=0$.

Lemma 4.2 (Stationarity)**.** $\partial_r\mathcal{J}_G(r) = -\frac{3\pi^2}{4G} + \frac{2Kr}{\pi^2(4+r^2)^2}$. *Thus* $\partial_r\mathcal{J}_G(r_\star)=0 \iff \frac{1}{G} = \frac{8K}{3\pi^4}\frac{r_\star}{(4+r_\star^2)^2}$.

Lemma 4.3 (Neutrality)**.** $\mathcal{J}_G(r_\star)=0 \iff \frac{1}{G} = \frac{4K}{3\pi^4}\frac{1}{r_\star(4+r_\star^2)}$.

Theorem 4.4 (Quantized radius and G)**.** *Eliminating* $1/G$ *between Lemmas 4.2–4.3 yields* $2r_\star^2 = 4+r_\star^2$*, hence*

$$\boxed{r_\star = 2} \qquad (L=2\pi), \tag{7}$$

and

$$\boxed{G = \frac{12\pi^2}{K}}. \tag{8}$$

Remark 4.5 (Cosmological corrections). Including Λr^3 (in (1)) and subleading $(4+r^2)^{-2}$ terms (in (5)) gives $r_\star = 2+O(\Lambda r_\star^2)$ and $G = \frac{12\pi^2}{K}[1+O(\Lambda r_\star^2)]$, negligible at the Planck scale.

5 Analytic structure of K: holonomy characters and fiber traces

Let $x \in S^3_r$ and γ the minimizing geodesic to $-x$ (length πr). Denote by $P^{(s)}(\pi)$ the parallel transport in the relevant bundle along γ. On the tangent space $T_x S^3$, in an adapted orthonormal frame, $P^{(1)}(\pi)$ has eigenvalues $+1, -1, -1$ (fixes the tangent, flips the normals). The induced action on $ST^2(T_x S^3)$ (TT fiber) is the 5D irrep of SO(3). Equivalently, using $S^3 \simeq SU(2)$, the holonomy corresponds to the group element of angle $\theta = \pi$; the character of the spin-j irrep is $\chi_j(\theta) = \frac{\sin((2j+1)\theta/2)}{\sin(\theta/2)}$. Thus

$$\alpha_0 = \chi_{j=0}(\pi) = +1, \qquad \alpha_1 = \chi_{j=1}(\pi) = -1, \qquad \alpha_2 = \chi_{j=2}(\pi) = +1. \tag{9}$$

Proposition 5.1 (BRST combination at the fiber level). *The BRST-weighted combination of fiber traces is* $\frac{1}{2}\alpha_2 - \alpha_1 + \frac{1}{2}\alpha_0 = 2$.

6 Normalization and Van Vleck cancellation

The off-diagonal parametrix yields, for each spin block, $\int_{S^3_r} \mathrm{tr}\, K^{(s)}(t; x, -x)\, dx \sim (4\pi t)^{-3/2}\, e^{-(\pi r)^2/(4t)}\, \Delta^{1/2}(\pi r)\ \mathrm{tr}\, P$ $\cdots$, where $\Delta(\pi r)$ is the Van Vleck–Morette determinant, depending only on geodesic length in a constant-curvature space.

Lemma 6.1 (Common Van Vleck factor). *For S^3_r, $\Delta^{1/2}(\pi r)$ is a scalar depending only on r. It is identical across spin blocks and hence factors out of the BRST combination at order t^0.*

Proposition 6.2 (Structure of K). *Let $\mathcal{N}$ denote the universal normalization from the S^1 odd kernel, the S^3_r image integral (including $\Delta^{1/2}$), and the Mellin factor. Then*

$$\boxed{K = \left(\tfrac{1}{2}A_0^{(2)} - A_0^{(1)} + \tfrac{1}{2}A_0^{(0)}\right) \cdot \mathcal{N} = \left(\tfrac{1}{2}\alpha_2 - \alpha_1 + \tfrac{1}{2}\alpha_0\right) \cdot \mathcal{N} = 2\,\mathcal{N}\ .} \tag{10}$$

Remark 6.3. Higher t-coefficients contain curvature tensors and projector effects; they enter only at order $(4 + r^2)^{-2}$ in $\Delta\Gamma^{\mathrm{grav}}_{\text{1-loop}}(r)$ and do not affect $r_\star = 2$ nor $G = 12\pi^2/K$ at leading order.

7 Invariance checks and falsifiability

(i) **Scaling in L.** Repeating the construction with $L \neq 2\pi$ yields $A = L^2 + (\pi r)^2$, but K is independent of L. Only $(L^2 + (\pi r)^2)^{-1}$ changes; the fixed point still gives $r_\star/L = 1/\pi$, hence $r_\star = 2$ when $L = 2\pi$.

(ii) **Independence in r.** K is a pure number (no r-dependence) because the leading image coefficient is evaluated at fixed geodesic length πr, and the r-dependence cancels between the S^1 factor, the S^3 measure, and the Mellin transform in $\mathcal{N}$.

(iii) **Two-route consistency.** Computing K via holonomy characters (fiber method) and via Poisson-summed alternating towers (spectral method) must agree. Any mismatch falsifies the framework.

8 Stable numerical extraction (optional, no code needed)

Define for each spin s: $Y_s(t) := t^2 e^{A/(4t)}\,\Delta\,\mathrm{Tr}^{(s)}_{4D}(t)$. Then $Y_s(t) = C_s + c_s t + O(t^2)$ as $t \to 0^+$.

(a) **Log-domain.** Compute $\log \Delta\,\mathrm{Tr}^{(s)}_{4D}(t)$, then add $+A/(4t) + 2\log t$, and exponentiate at the end; this avoids overflow.

(b) **Pre-asymptotic fit.** Use $t \in [0.03, 0.2]$, fit $Y_s(t) = C_s + c_s t$, take the intercept C_s.

(c) **Theta route.** Apply Poisson summation to the alternating towers; the leading coefficient is read off directly as C_s without exponentials.

Finally $K = \frac{1}{2}C_2 - C_1 + \frac{1}{2}C_0$, and $G = 12\pi^2/K$.

9 Synthesis: Infinite Inversion at two layers

Route $\mathrm{F}^\star$ fixed matter constants by balancing nonlocal parity image kernels against envelope normalization, producing C_{env}. Route G balances classical curvature against the same nonlocal parity geometry, producing K. The fixed-point equations quantize the Planck radius and reduce G to a single spectral constant. Thus (α, ρ_Λ) and G are dual outputs of one principle viewed through internal vs. external lenses.

Appendix A: Mellin integral

For $A > 0$, substituting $u = t^{-1}$ gives $\int_0^\infty \frac{dt}{t}\, t^{-2} e^{-A/(4t)} = \int_0^\infty u\, e^{-Au/4}\, du = \frac{4}{A}$.

Appendix B: Spectral towers on S^3 (data)

Scalars: $\lambda^{(0)}_\ell = \ell(\ell+2)/r^2$, $d^{(0)}_\ell = (\ell+1)^2$, $\ell \geq 0$. Coexact 1-forms: $\lambda^{(1)}_\ell = (\ell+1)^2/r^2$, $d^{(1)}_\ell = 2\ell(\ell+2)$, $\ell \geq 1$. TT tensors: Lichnerowicz eigenvalues/degeneracies are standard; they are not needed for the leading image coefficient when the holonomy-character route is used, but can be inserted into the theta method to reproduce C_2.

Appendix C: Holonomy characters on $S^3 \simeq SU(2)$

For a geodesic rotation by angle $\theta = \pi$, the SU(2) character is $\chi_j(\theta) = \frac{\sin((2j+1)\theta/2)}{\sin(\theta/2)}$. Thus $\chi_0(\pi) = 1$, $\chi_1(\pi) = -1$, $\chi_2(\pi) = 1$, yielding (9).

Appendix D: Van Vleck factor at antipode

In a constant-curvature space, the Van Vleck–Morette determinant $\Delta(x, y)$ depends only on the geodesic distance $d(x, y)$. At $d = \pi r$ it is a scalar common to all spin blocks; hence at order t^0 the BRST combination factors it out, leading to (10).

Appendix E: BRST weights and leading-order cancellation

Gaussian integration over bosonic (TT, scalar) and Grassmann (vector ghost) fields yields the coefficients $\frac{1}{2}, -1, \frac{1}{2}$ in (2). These weights commute with parity projection and survive unchanged in the image term; hence they appear in K exactly as in (5).

Appendix F: Error control in numerical extraction

If the pre-asymptotic fit is used, the residual $R_s(t) = Y_s(t) - (C_s + c_s t)$ obeys $|R_s(t)| \leq Ct^2$ on a small interval; choose the largest t in the fit so that the relative error is below tolerance. This gives certified error bars on C_s, and hence on K and on G.

End of V31C.