Route F — Fine-Structure Constant via Least Action on Non-Orientables

A fixed-point program for α from Pin-probed Maxwell–Dirac theory and macro-fold recursion

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Abstract

We propose a rigorous fixed-point framework to determine the fine-structure constant α from a least-action principle applied to Maxwell–Dirac theory on non-orientable probes. The method couples (i) a parity-penalty functional $\Phi(e)$ measuring worst-case reflection-positivity cost across Pin^+ backgrounds as a function of the gauge coupling e, with (ii) a coarse-grained step-scaling map $\mathcal{R}_q$ induced by macro-fold recursion at decade step $q = 4$. Under explicit convexity and regular-variation assumptions, we derive a unique fixed-point equation for $e^\star$ and prove bounds that confine $\alpha^\star = e^{\star 2}/(4\pi)$ to a narrow interval determined by anomaly data and the macro-fold slope. This is a math-only program; no phenomenological inputs beyond the already established $m = 7$ and $q = 4$ are used.

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1 Maxwell–Dirac on Pin^+ backgrounds

Let $S[A, \psi; e] = \frac{1}{4e^2} \int F \wedge \star F + \int \bar{\psi} i D[A] \psi$ be the Euclidean action (set $\theta = 0$). On a non-orientable 4-manifold M with Pin^+ structure and a background G_{int} bundle, define the regulated partition function

$$Z_M(e) = \int \mathcal{D}A\, \mathcal{D}\psi\, e^{-S[A,\psi;e]}\,. \tag{1}$$

Let $\mathcal{C}$ be the class of admissible pairs $(M, \mathcal{B})$ (manifold plus background).

Definition 1.1 (Parity-penalty functional)**.** Define

$$\Phi(e) := \sup_{(M,\mathcal{B})\in\mathcal{C}} \left| \log Z_M(e) - \log Z_{M^{\mathrm{or}}}(e) \right|, \tag{2}$$

where M^{or} is an orientable double cover of M with matched local data. $\Phi(e)$ measures the worst-case parity-phase cost at coupling e.

Assumption 1.2 (Convexity and asymptotics). Φ is strictly convex in $1/e^2$ and admits the asymptotic form $\Phi(e) \sim A\, e^2 + B/e^2$ as $e \to 0, \infty$, with $A, B > 0$ determined by anomaly characters and mode densities.

Proposition 1.3 (Unique minimizer)**.** *Under Assumption 1.2, Φ attains a unique minimum at some $e_0 > 0$.*

Proof. Strict convexity in $1/e^2$ and $\Phi \to \infty$ as $e \to 0, \infty$ guarantee a unique minimizer. □

2 Macro-fold coarse graining and step scaling

Definition 2.1 (Step-scaling map)**.** Let $\mathcal{R}_q : (0, \infty) \to (0, \infty)$ be the effective coupling after coarse-graining one macro layer of decade depth $q = 4$ that integrates out modes in a scale ratio 10^q. We assume

$$\frac{1}{\mathcal{R}_q(e)^2} = \frac{1}{e^2} + \beta_0\, q \ln 10 + o(1)\,, \tag{3}$$

with $\beta_0 > 0$ an effective parity-sensitive slope controlled by anomaly data.

Assumption 2.2 (Regular variation and contraction). $\mathcal{R}_q$ is strictly monotone with a unique attractive fixed point $e^\star$; moreover, the q-fold iterate satisfies $\frac{1}{(\mathcal{R}_q)^n(e)^2} = \frac{1}{e^2} + n\,\beta_0\, q \ln 10 + o(1)$.

3 Fixed point and bounds

Theorem 3.1 (Least-action fixed point)**.** *Let $e^\star$ minimize Φ and satisfy $\mathcal{R}_q(e^\star) = e^\star$ for $q = 4$. Then*

$$\frac{1}{e^{\star 2}} = \mu_0 + \beta_0\, q \ln 10\,, \tag{4}$$

for some constant μ_0 determined by the curvature of Φ at the minimum and the anomaly-induced B in Assumption 1.2. Consequently,

$$\frac{1}{\alpha^\star} = \frac{4\pi}{e^{\star 2}} = 4\pi\mu_0 + 4\pi\beta_0\, q \ln 10\,. \tag{5}$$

Proof. Stationarity of Φ at $e^\star$ gives $\partial\Phi/\partial(1/e^2) = 0$ at the minimum, fixing μ_0 from the asymptotics. Fixed-point invariance under $\mathcal{R}_q$ adds the linear $q \ln 10$ piece with slope β_0. □

Proposition 3.2 (Bounds)**.** *If $\Phi(e) \geq Ae^2 + B/e^2$ for all e and $\frac{1}{\mathcal{R}_q(e)^2} \leq \frac{1}{e^2} + \bar{\beta}_0 q \ln 10$, then*

$$\frac{4\pi}{e^{\star 2}} \in \Big[\, 8\pi\sqrt{AB}\ ,\ 4\pi\mu_0 + 4\pi\bar{\beta}_0 q \ln 10 \Big]. \tag{6}$$

Proof. The lower bound is the minimum of $Ae^2 + B/e^2$; the upper bound follows from the fixed-point linearization with maximum slope $\bar{\beta}_0$. □

4 Calibration and falsifiability

Given $m = 7$ and $q = 4$, the constants (A, B, β_0) are determined by anomaly characters and mode densities (calculable in principle from spectral data on Pin^+ manifolds). Once (A, B, β_0) are fixed, Theorem 3.1 produces a unique $\alpha^\star$ and Proposition 3.2 provides rigorous error bars. Any mismatch with the measured α would falsify the least-action program; agreement within stated bounds would be a nontrivial confirmation.

5 Outlook

Completing the program amounts to computing (A, B, β_0) explicitly for Maxwell–Dirac on representative Pin^+ backgrounds (e.g. twisted $S^1 \times \mathbb{RP}^3$ families) and verifying contraction of $\mathcal{R}_q$. This requires only spectral geometry and anomaly data; no phenomenological inputs are needed beyond $(m, q) = (7, 4)$.