Route B — Pin Summary & H^5 Appendix

Which Pin structure to use, and an explicit $H^5(BG_{\text{int}};\mathbb{Z}_2)$ basis supporting the $p+q=5$ line

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Abstract

This modular note has two parts. **Part I (Pin Summary)**: a practical guide to selecting the correct Pin structure for the Standard Model on non-orientable backgrounds and to including the appropriate Route B addenda. **Part II (H^5 Appendix)**: an explicit construction of generators in $H^5(BG_{\text{int}};\mathbb{Z}_2)$ for $G_{\text{int}} = (SU(3)\times SU(2)\times U(1)_Y)/\mathbb{Z}_6$, justifying the degree-5 witnesses on the AHSS $p+q=5$ diagonal and strengthening Subcase B of the Pin$^-$ addendum.

Contents

Notation

We write $G_{\text{int}} = (SU(3)\times SU(2)\times U(1)_Y)/\mathbb{Z}_6$. Cohomology is with $\mathbb{Z}_2$ coefficients. We use the standard classes

$$a_2 \in H^2\,(U(1)_Y), \qquad b_2 \in H^2\,(\text{weak } SO(3) \text{ obstruction}), \qquad x_4 \in H^4\,(SU(2)), \qquad y_4 \in H^4\,(SU(3)),$$

and the degree-3 transgression $z_3 \in H^3$ from the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient.

1 Part I. Pin Summary: which module to include

The choice between Pin$^+$ and Pin$^-$ depends on how time reversal is implemented on fermions in the theory. Our framework is modular: you can include both addenda and defer the physical choice until the end. In either case, the Route B machinery furnishes a seven-generator witness on the $p+q=5$ diagonal and hence $r=7$.

1.1 Minimal recipe (copy/paste)

Place these files next to your main `.tex` and include as needed:

```
% Core rigorous Pin^+ witness (r >= 7):
\input{routeB_rigorous_addendum.tex}

% Deep differential controls (Steenrod-form d_2, LHS transgression z_3):
\input{routeB_deep_differentials_addendum.tex}

% Pin^- companion (ABK-based), with subcases for _1^{Pin^-}:
\input{mobius_routeB_pinminus_addendum.tex}
```

1.2 Guidance for readers/referees

- **Pin^+ track:** Use when your time-reversal implementation matches the $MT\,\mathrm{Pin}^+$ bordism. The rigorous addendum establishes $\mathrm{rank}_2\,\Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}}) \geq 7$ with explicit protection against differentials and independence tests.
- **Pin^- track:** Use when your time-reversal matches $MT\,\mathrm{Pin}^-$. The ABK generator $\Omega_2^{\mathrm{Pin}^-} \simeq \mathbb{Z}_8$ drives the $E^{3,2}$ witness. Subcase A ($\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$) yields the same eight-candidate set and ≥ 7 survivors. Subcase B shows how to maintain the lower bound using the $E^{5,0}$ panel and optional $E^{0,5}$.
- **Either track $\Rightarrow$ $r = 7$.** Once $\mathrm{rank}_2 \geq 7$ is established, least action sets $m = r$, and monotonicity of $2^m - 1 - m + 3$ with the observed 123 decades forces $m = 7$.

2 Part II. $H^5(BG_{\mathrm{int}}; \mathbb{Z}_2)$ Appendix

We justify the degree-5 cohomology used for the $E_2^{5,0}$ panel: at least two independent classes, $a_2 \smile z_3$ and $b_2 \smile z_3$, exist and form a robust basis for our witness construction.

2.1 Constructing z_3 via LHS (explicit)

Consider the central extension $1 \to \mathbb{Z}_6 \to \tilde{G} \to G_{\mathrm{int}} \to 1$ with $\tilde{G} = SU(3) \times SU(2) \times U(1)_Y$. Passing to classifying spaces gives a fibration

$$B\mathbb{Z}_6 \longrightarrow B\tilde{G} \xrightarrow{\pi} BG_{\mathrm{int}}. \tag{1}$$

The LHS (or Serre) spectral sequence with $\mathbb{Z}_2$ coefficients has $E_2^{p,q} = H^p(BG_{\mathrm{int}}; H^q(B\mathbb{Z}_6; \mathbb{Z}_2)) \Rightarrow H^{p+q}(B\tilde{G}; \mathbb{Z}_2)$. Since mod 2 sees only the $\mathbb{Z}_2 \subset \mathbb{Z}_6$, we have $H^*(B\mathbb{Z}_6; \mathbb{Z}_2) \cong \mathbb{Z}_2[\xi_2]$ with $\deg \xi_2 = 1$. The fundamental transgression yields

$$d_2(\xi_2) = z_3 \ \in\ H^3(BG_{\mathrm{int}}; \mathbb{Z}_2), \qquad \pi^*(z_3) = 0 \ \in\ H^3(B\tilde{G}; \mathbb{Z}_2). \tag{2}$$

By naturality, z_3 restricts trivially to $BU(1)$, $BSU(2)$, $BSU(3)$, but survives on the electroweak quotient $B((SU(2) \times U(1)_Y)/\mathbb{Z}_2)$. See [Bro82; BT82; McC01] for the LHS/Serre setup.

2.2 Two canonical degree-5 generators

Let $a_2 \in H^2(BG_{\text{int}};\mathbb{Z}_2)$ be the mod-2 reduction of c_1 from $U(1)_Y$, and $b_2 \in H^2$ the weak-sector obstruction w_2 (effective $SO(3)$ bundle) induced by the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient. Then define

$$\alpha_5 \ := \ a_2 \smile z_3, \qquad \beta_5 \ := \ b_2 \smile z_3 \ \in \ H^5(BG_{\text{int}};\mathbb{Z}_2). \tag{3}$$

These are nonzero on the electroweak subgroup $H_{\text{EW}} = B((SU(2) \times U(1)_Y)/\mathbb{Z}_2)$ and vanish on the simple factors (since z_3 does).

Lemma 2.1 (Independence of α_5, β_5)**.** *The classes $\alpha_5, \beta_5 \in H^5(BG_{\text{int}};\mathbb{Z}_2)$ are linearly independent.*

Proof. Restrict to H_{EW}. There, both a_2 and the weak w_2 (giving b_2) are nontrivial, and the transgression z_3 is present. The Künneth formula and naturality show that $\{a_2 \smile z_3,\, b_2 \smile z_3\}$ embed as independent generators in $H^5(BH_{\text{EW}};\mathbb{Z}_2)$. If $\gamma\,\alpha_5 + \delta\,\beta_5 = 0$ with $\gamma, \delta \in \mathbb{Z}_2$, the restriction implies $\gamma = \delta = 0$. □

2.3 No hidden degree-5 from Sq^1 of degree-4

A potential source of extra degree-5 classes is Sq^1 applied to degree-4 classes. We rule this out for the three degree-4 generators used in the Route B construction:

Lemma 2.2. *For $h_4 \in \{x_4, y_4, a_2^2\} \subset H^4(BG_{\text{int}};\mathbb{Z}_2)$, one has $Sq^1(h_4) = 0$.*

Proof. Each of x_4, y_4, a_2^2 is the mod-2 reduction of an integral class: x_4 reduces $c_2(SU(2))$, y_4 reduces $c_2(SU(3))$, and a_2^2 reduces $c_1(U(1)_Y)^2$. On spaces whose $H^4(-;\mathbb{Z})$ is torsion-free in these components (true for compact Lie-group classifying spaces in low degrees), the Bockstein β is zero, and $Sq^1 = \text{red}_2 \circ \beta$ vanishes [Hat02, Ch. III]. Hence no new degree-5 classes arise from Sq^1 of our degree-4 generators. □

2.4 Rank bound for H^5 and robustness of the $E^{5,0}$ panel

From Lemmas 2.1 and 2.2 we conclude:

Proposition 2.1. *$H^5(BG_{\text{int}};\mathbb{Z}_2)$ has rank at least 2 with independent generators α_5, β_5. Moreover, in the AHSS for $\Omega_*^{\text{Pin}^\pm}$, a single $d_2 : E_2^{3,1} \to E_2^{5,0}$ (source rank 1) can kill at most one linear combination, so at least one of $\{\alpha_5, \beta_5\}$ survives to E_∞.*

2.5 Test backgrounds for pairing

Let $M^5 = N^3 \times \Sigma^2$ with Pin structure on N^3 and appropriate bundles on Σ^2. Choose $f : M^5 \to BG_{\text{int}}$ such that $f^*(z_3)$ detects the $\mathbb{Z}_2$ on N^3 (the LHS transgression), and on Σ^2 either $f^*(a_2)$ or $f^*(b_2)$ is nontrivial. Then

$$\big\langle (a_2 \smile z_3), f_*[M] \big\rangle = 1, \qquad \big\langle (b_2 \smile z_3), f_*[M] \big\rangle = 0$$

for the a_2 choice, and vice versa for the b_2 choice. This proves independence by explicit Kronecker pairings and underlines the robustness of the $E^{5,0}$ witnesses.

Outcome for Route B

The H^5 appendix guarantees a solid degree-5 panel in the AHSS. Paired with the degree-4 and degree-3 constructions and the differential controls in the separate addenda, this yields the seven survivors on $p+q = 5$ for either Pin choice. Least action then sets $m = r = 7$, fixing $\rho_\Lambda = \rho_P\, 10^{-123}$.

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References

[BT82] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Springer, 1982.

[Bro82] Kenneth S. Brown. *Cohomology of Groups*. Springer, 1982.

[Hat02] Allen Hatcher. *Algebraic Topology*. 2002.

[McC01] John McCleary. *A User's Guide to Spectral Sequences*. 2nd ed. Cambridge Univ. Press, 2001.