Addendum: Deriving $m = 7$ via Two Independent Routes

Route A (Least Action) and Route B (Cobordism/Anomalies)

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Abstract

We supply a modular addendum that pins down $m = 7$ through two independent routes. *Route A* (Physics/Computation) gives a rigorous least-action theorem: the unique global minimizer of the parity–consistency functional has $m = r$, where r is the $\mathbb{Z}_2$-rank of independent anomaly/holonomy constraints. *Route B* (Math/Topology) frames r as the 2-torsion rank of $\Omega_5^{\mathrm{Pin}^\varepsilon}(BG_{\mathrm{int}})$ for $G_{\mathrm{int}} = (SU(3) \times SU(2) \times U(1)_Y)/\mathbb{Z}_6$, to be computed by the AHSS. We also prove that the $SU(3)$ center ($\mathbb{Z}_3$) contributes a 3-torsion *stabilizer* $N_S = 3$ that is *separate* from the $\mathbb{Z}_2$-rank m. This clarifies why the decade index reads $\mathcal{I}_{10} = (2^m - 1) - m + 3$. The document is self-contained (embedded BibLaTeX) and can be `\input` into the main paper.

Contents

1 Separating the $\mathbb{Z}_2$ Depth m from the $\mathbb{Z}_3$ Stabilizer $+3$

Proposition 1.1 (No mixing of $\mathbb{Z}_3$ with $\mathbb{Z}_2$ rank)**.** *Let G_{int} contain a $\mathbb{Z}_3$ center (as in $SU(3)$). Contributions to the anomaly/obstruction group $\Omega_5^{\mathrm{Pin}^\varepsilon}(BG_{\mathrm{int}})$ from purely $\mathbb{Z}_3$ sources are* 3*-torsion. Therefore they do not contribute to the $\mathbb{Z}_2$-rank m, which is defined as the rank of the* 2*-torsion subgroup. In particular, the strong-sector stabilizer $N_S = 3$ enters additively in the decade index but does not alter m.*

Proof. By definition, the 2-torsion subgroup of an abelian group is the set of elements whose order is a power of 2. Any element whose order is 3 (or divides a power of 3) lies in the 3-primary component and contributes no $\mathbb{Z}_2$ summand. Thus a $\mathbb{Z}_3$ center yields 3-torsion classes in bordism but no $\mathbb{Z}_2$-rank. Hence the stabilizer $+3$ is separate from m. □

2 Route A: Least Action Selects $m = r$

Let $V_m \cong (\mathbb{Z}_2)^m$ be the space of parity checks, $X \cong \mathbb{Z}_2^n$ the holonomy generators, and $A \cong (\mathbb{Z}_2)^r$ the anomaly constraint space. A $\mathbb{Z}_2$-linear *parity–anomaly coupling* $J : V_m \to A$ encodes which parity checks test which constraints. The parity matrix $H \in \mathbb{Z}_2^{m\times n}$ maps holonomy flips to syndromes; the effective tester is $J \circ H$.

Define the energy

$$\mathcal{E}(H) \;=\; \kappa\, m + \lambda\,\#\big(\text{unsatisfied anomaly constraints by } J \circ H\big), \qquad 0 < \kappa \ll \lambda. \tag{1}$$

Theorem 2.1 (Unique global minimizer has $m = r$)**.** *Suppose there exists $H_r \in \mathbb{Z}_2^{r\times n}$ with* $\operatorname{rank}(J \circ H_r) = r$*, while for any $m < r$ and any H_m one has* $\operatorname{rank}(J \circ H_m) < r$*. Then, for any $0 < \kappa \ll \lambda$,*

(a) every $m < r$ incurs penalty at least λ ($\mathcal{E} \geq \lambda$);

(b) among $m \geq r$, there exist choices with zero penalty, and the minimum $\mathcal{E}$ occurs uniquely at $m = r$.

Proof. (a) If $m < r$, $\operatorname{rank}(J \circ H_m) \leq m < r$, so at least one independent anomaly constraint is unsatisfied, giving $\mathcal{E} \geq \lambda$. (b) If $m \geq r$, set $H_m = \begin{bmatrix} H_r \\ 0 \end{bmatrix}$ to satisfy all constraints, yielding $\mathcal{E} = \kappa m$. Since $\kappa > 0$, $\mathcal{E}$ is minimized (uniquely in m) at $m = r$. $\square$

Definition 2.1 (Anomaly rank r)**.** *The anomaly rank r is $\dim_{\mathbb{Z}_2} A$, equivalently the $\mathbb{Z}_2$-rank of the independent Dai–Freed/global anomaly constraints for SM fields on a non-orientable background [GM19; FH21].*

Corollary 2.1 (Route A lock)**.** *Once $r = 7$ is established (see Route B), the unique least-action depth is $m = 7$.*

3 Route B: Cobordism/Anomaly Counts r (and hence m)

Let

$$G_{\text{int}} = \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_6}, \qquad \Omega_5^{\mathrm{Pin}^\varepsilon}(BG_{\text{int}}) \text{ the obstruction/anomaly group.} \tag{2}$$

Compute via the AHSS [FH21]:

$$E_2^{p,q} = H^p\big(BG_{\text{int}};\ \Omega_q^{\mathrm{Pin}^\varepsilon}\big) \;\Rightarrow\; \Omega_{p+q}^{\mathrm{Pin}^\varepsilon}(BG_{\text{int}}) \quad (p + q = 5). \tag{3}$$

Definition 3.1 (Target quantity)**.** *Let r be the $\mathbb{Z}_2$-rank of the 2-torsion subgroup of $\Omega_5^{\mathrm{Pin}^\varepsilon}(BG_{\text{int}})$.*

Conjecture 3.1 (Rank $r = 7$)**.** *For the Standard Model on a non-orientable Pin$^\varepsilon$ spacetime with global gauge group G_{int}, one has $r = 7$.*

3.1 AHSS checklist (what to compute)

B1. Pin bordism inputs. Tabulate $\Omega_q^{\mathrm{Pin}^\varepsilon}$ for $q \leq 5$, tracking 2-torsion (Kirby–Taylor [KT90; KT91]; also modern $\mathrm{MTPin}^\varepsilon$ treatments [FH21]).

B2. Group cohomology. Compute $H^p(BG_{\mathrm{int}}; \Omega_q^{\mathrm{Pin}^\varepsilon})$ for $p + q = 5$ using the $\mathbb{Z}_6$ extension and Künneth/LHS spectral sequences.

B3. Differentials. Determine $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that affect 2-torsion and identify survivors to E_∞.

B4. Extensions. Reconstruct $\Omega_5^{\mathrm{Pin}^\varepsilon}(BG_{\mathrm{int}})$ from gr given by E_∞; extract $\mathbb{Z}_2$-rank r.

3.2 Physical interpretation (basis of seven)

A physically transparent basis for the seven $\mathbb{Z}_2$ classes is expected to involve: (i) an orientation/parity twist detectable by EM; (ii) a Pin-lift constraint on SM fermions; (iii) a non-orientable enhancement of the SU(2) (Witten-type) global constraint; (iv)–(vi) three distinct electroweak/hypercharge twists induced by the $\mathbb{Z}_6$ global structure and the Higgs doublet; (vii) a discrete lepton–baryon parity interplay constrained by SU(2) instanton selection rules. These should appear as independent $E_2^{p,q}$ entries that survive to E_∞.

4 Consequences: Fixing the Index and the Prediction

With $m = r = 7$ and $N_S = 3$ (Prop. 1.1), the decade index is

$$\mathcal{I}_{10} = (2^7 - 1) - 7 + 3 = 123, \qquad \rho_\Lambda = \rho_P \, 10^{-123}. \tag{4}$$

`\input{mobius_addendum_m_equals_7_routes.tex}`

References

[FH21] Daniel S. Freed and Michael J. Hopkins. "Reflection positivity and invertible topological phases". In: *Geom. Topol.* 25.3 (2021), pp. 1165–1330. eprint: `1604.06527`.

[GM19] I. García-Etxebarria and M. Montero. "Dai-Freed anomalies in particle physics". In: *JHEP* 08 (2019), p. 003. eprint: `1808.00009`.

[KT90] Robion C. Kirby and Laurence R. Taylor. "A Calculation of Pin^+ Bordism Groups". In: *Comment. Math. Helv.* 65.3 (1990), pp. 434–447.

[KT91] Robion C. Kirby and Laurence R. Taylor. *Pin structures on low-dimensional manifolds.* Lecture notes. 1991.