Route B — Unified Addendum

+3 from the Color Center 1-Form Anomaly, a Sharp $\boldsymbol{r = 7}$ Bound from AHSS, and a Verification Appendix

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Abstract

We combine three ingredients in a single modular addendum. **(I) Center Stabilizer:** the $\mathbb{Z}_3$ one-form symmetry of color produces a *3-primary* anomaly inflow in 5d Pin bordism, yielding an *additive* $\underline{+3}$ in the decade index that is independent of the 2-primary depth m. **(II) Upper Bound:** an explicit E_2-diagonal rank count for the AHSS computing $\Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$ shows $\mathrm{rank}_2 \le 7$; with the constructive lower bound $\mathrm{rank}_2 \ge 7$ from the witness modules, this pins $\boldsymbol{r = 7}$. **(III) Verification Appendix:** a concise checklist for referees to verify the input ranks and a belt-and-suspenders argument (via LHS/Serre and global-form considerations) that $\dim H^4(BG_{\mathrm{int}}; \mathbb{Z}_2) = 2$.

Contents

1 Part I. Center Stabilizer: $\mathbb{Z}_3$ 1-Form Anomaly and the $\underline{+3}$ Constant

1.1 Color center as a 1-form symmetry

The central extension $1 \to \mathbb{Z}_3 \to SU(3) \to PSU(3) \to 1$ gives a classifying fibration

$$B\mathbb{Z}_3 \longrightarrow BSU(3) \xrightarrow{\pi} BPSU(3). \tag{1}$$

This equips any $PSU(3)$-bundle with a universal 2-form background $B \in H^2(-; \mathbb{Z}_3)$ measuring the obstruction to lifting it to an $SU(3)$-bundle (Postnikov k-invariant $\kappa \in H^3(BPSU(3); \mathbb{Z}_3)$).

1.2 Classifying map and Pin inflow

Let $G_{\text{int}} = (SU(3) \times SU(2) \times U(1)_Y)/\mathbb{Z}_6$. Projecting to the color factor and composing with the universal map $BPSU(3) \to B^2\mathbb{Z}_3$ yields

$$\Phi : \; BG_{\text{int}} \;\longrightarrow\; BPSU(3) \;\longrightarrow\; B^2\mathbb{Z}_3. \tag{2}$$

Pin inflow for a 2-form $\mathbb{Z}_3$ background B is measured by the 3-primary part of $\Omega_5^{\text{Pin}}(B^2\mathbb{Z}_3)$. One has (bosonic version $H^5(B^2\mathbb{Z}_3; U(1)) \cong \mathbb{Z}_3$; the Pin refinement agrees on the 3-primary):

$$\Omega_5^{\text{Pin}}(B^2\mathbb{Z}_3) \;\cong\; \mathbb{Z}_3. \tag{3}$$

Composing with Φ gives a natural homomorphism

$$\Phi_* : \; \Omega_5^{\text{Pin}}(BG_{\text{int}}) \;\longrightarrow\; \Omega_5^{\text{Pin}}(B^2\mathbb{Z}_3) \;\cong\; \mathbb{Z}_3. \tag{4}$$

Proposition 1.1 (Surjectivity on the 3-primary part)**.** *The map Φ_* is surjective on 3-primary torsion.*

Proof sketch. Take $M^5 = N^3 \times \Sigma^2$ with Pin structure on N^3 and choose a $PSU(3)$-bundle whose obstruction class realizes any prescribed element of $H^2(\Sigma^2; \mathbb{Z}_3) \cong \mathbb{Z}_3$. The classifying map $M^5 \to BG_{\text{int}} \xrightarrow{\Phi} B^2\mathbb{Z}_3$ then pairs to the chosen element in $\Omega_5^{\text{Pin}}(B^2\mathbb{Z}_3) \cong \mathbb{Z}_3$. Naturality gives surjectivity. □

Corollary 1.1 (Decoupling and the $\underline{+3}$ in the index)**.** *The 3-primary anomaly in $\Omega_5^{\text{Pin}}(BG_{\text{int}})$ is an order-3 obstruction independent of the 2-primary parity-depth m. Neutralizing it is a three-way choice contributing an additive $\underline{+3}$ to the decade index $2^m - 1 - m + \underline{3}$ while leaving m unchanged.*

Remark 1.1. *This formalizes, in anomaly-inflow language, the heuristic "center-flux neutralization": it is a $\mathbb{Z}_3$ choice stemming from a 1-form symmetry anomaly, not a modification of the $\mathbb{Z}_2$-rank count.*

2 Part II. Sharp Upper Bound: $r \le 7$ (hence $r = 7$) from the E_2 Diagonal

We compute the $\mathbb{Z}_2$-rank on the AHSS E_2-page for $\Omega_5^{\text{Pin}^+}(BG_{\text{int}})$, at total degree $p+q = 5$. Two inputs are required.

2.1 Input A: Pin$^+$ low-degree coefficients

We use only the $\mathbb{Z}_2$-ranks (see e.g. [FH21]):

$$\operatorname{rank}_2 \Omega_0^{\text{Pin}^+} = 1, \quad \operatorname{rank}_2 \Omega_1^{\text{Pin}^+} = 1, \quad \operatorname{rank}_2 \Omega_2^{\text{Pin}^+} = 1, \quad \operatorname{rank}_2 \Omega_3^{\text{Pin}^+} = 1. \tag{5}$$

2.2 Input B: Mod-2 cohomology ranks of BG_{int}

Using the LHS spectral sequence for the central $\mathbb{Z}_6$ quotient and restrictions to $BU(1)$, $BSU(2)$, $BSU(3)$, $BSO(3)$ one checks:

$$\dim H^2 = 2 \;(\langle a_2, b_2 \rangle), \qquad \dim H^3 = 1 \;(\langle z_3 \rangle), \qquad \dim H^4 = 2 \;(\langle x_4, y_4 \rangle), \qquad \dim H^5 = 2 \;(\langle a_2 z_3, b_2 z_3 \rangle). \tag{6}$$

(See Appendix 3 for a verification checklist and Appendix 4 for a belt-and-suspenders argument that $\dim H^4 = 2$.)

2.3 E_2 diagonal count and conclusion

On the $p+q = 5$ diagonal we have panels $E_2^{5,0}, E_2^{4,1}, E_2^{3,2}, E_2^{2,3}$, with ranks given by products of (5) and (6):

$$\mathrm{rank}_2\, E_2^{5,0} = 2, \qquad \mathrm{rank}_2\, E_2^{4,1} = 2, \qquad \mathrm{rank}_2\, E_2^{3,2} = 1, \qquad \mathrm{rank}_2\, E_2^{2,3} = 2.$$

Hence

$$\sum_{p+q=5} \mathrm{rank}_2\, E_2^{p,q} \;=\; 2+2+1+2 \;=\; 7. \tag{7}$$

Since spectral sequence differentials can only *decrease* the rank on the diagonal, we obtain

$$\mathrm{rank}_2\, \Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}}) \;\leq\; 7. \tag{8}$$

Combined with the constructive lower bound $\mathrm{rank}_2 \geq 7$ from the witness modules, we conclude:

Theorem 2.1 (Equality purely from topology)**.** *$\mathrm{rank}_2\, \Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}}) = 7$.*

3 Part III. Verification Appendix: a Referee Checklist

This checklist lets a referee confirm inputs (5)–(6) independently.

(1) Pin^+ coefficients (low degrees)

Consult a standard reference such as [FH21] for $MT\,\mathrm{Pin}^{\pm}$ in low degrees; only the $\mathbb{Z}_2$-ranks of $\Omega_q^{\mathrm{Pin}^+}$ for $q = 0,1,2,3$ are required, and each is 1. No information about $\Omega_4^{\mathrm{Pin}^+}$ or $\Omega_5^{\mathrm{Pin}^+}$ is needed.

(2) $\dim H^2 = 2$ (a_2, b_2)

Here a_2 is the mod-2 reduction of $c_1(U(1)_Y)$. The class b_2 is the weak-sector obstruction w_2 of the effective $SO(3)$ bundle induced by the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient. Restricting to $BU(1)$ and $BSO(3)$ isolates a_2 and b_2 respectively, proving independence (Brown/Bott–Tu).

(3) $\dim H^3 = 1$ (z_3)

Use the LHS/Serre sequence for the central extension $1 \to \mathbb{Z}_6 \to \tilde{G} \to G_{\mathrm{int}} \to 1$. Mod 2, only the $\mathbb{Z}_2$ factor contributes. With $H^*(B\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2[\xi_2]$ and $\deg \xi_2 = 1$, the fundamental transgression gives $d_2(\xi_2) = z_3 \in H^3(BG_{\mathrm{int}};\mathbb{Z}_2)$. By naturality, z_3 vanishes on the simple subgroups and survives on the electroweak quotient.

(4) $\dim H^5 = 2$ $(a_2 z_3, b_2 z_3)$

Form the two cup products $\alpha_5 = a_2 \smile z_3$ and $\beta_5 = b_2 \smile z_3$. On the electroweak subgroup, both are nonzero and independent; on simple subgroups they vanish (since z_3 restricts to 0). Hence $\dim H^5 \geq 2$. No further degree-5 generators arise from Sq^1 of x_4, y_4, a_2^2 because these reduce integral classes (Bockstein $= 0$ on them), so $Sq^1 = 0$ [Hat02, Ch. III]. Thus $\dim H^5 = 2$.

(5) $\dim H^4 = 2$ (x_4, y_4)

See the next section for a belt-and-suspenders argument. The short version: before the $\mathbb{Z}_6$ quotient there are three natural degree-4 classes $\{x_4, y_4, a_2^2\}$. The central quotient yields *exactly one* independent relation among them (mod 2), leaving rank 2.

4 Part IV. Belt-and-Suspenders for $\dim H^4 = 2$

We give two complementary arguments.

4.1 Route A: LHS/Serre perspective (cohomology of a central quotient)

Consider the fibration $B\mathbb{Z}_6 \to B\tilde{G} \xrightarrow{\pi} BG_{\text{int}}$ with $\tilde{G} = SU(3) \times SU(2) \times U(1)_Y$. The Serre spectral sequence has

$$E_2^{p,q} = H^p\big(BG_{\text{int}};\ H^q(B\mathbb{Z}_6; \mathbb{Z}_2)\big) \Rightarrow H^{p+q}(B\tilde{G}; \mathbb{Z}_2). \tag{9}$$

Since mod 2 only the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ is visible, $H^*(B\mathbb{Z}_6; \mathbb{Z}_2) = \mathbb{Z}_2[\xi_2]$ with $|\xi_2| = 1$. The first nontrivial differential is $d_2(\xi_2) = z_3 \in H^3(BG_{\text{int}}; \mathbb{Z}_2)$. The next potentially relevant differential hitting degree 4 on the base is $d_4(\xi_2^3) : E_4^{0,3} \to E_4^{4,0}$. Naturality with respect to the inclusions $BSU(2)$, $BSU(3)$, and $BU(1)$ shows that $d_4(\xi_2^3)$ must be a *single* linear combination of $\{x_4, y_4, a_2^2\}$ which restricts to 0 on each simple subgroup (since ξ_2 pulls back trivially there). This forces exactly *one* independent relation among the three degree-4 classes, hence $\dim H^4(BG_{\text{int}}; \mathbb{Z}_2) = 2$. (A diagram chase in the spectral sequence makes this precise; see [Bro82; BT82; Hat02].)

4.2 Route B: Global-form constraint

Before quotienting by $\mathbb{Z}_6$ the mod-2 degree-4 ring is generated by $\{x_4, y_4, a_2^2\}$ and has rank 3. The global-form identification by $\mathbb{Z}_6$ imposes:

- The $\mathbb{Z}_3$ center of $SU(3)$ is invisible mod 2 (no effect).
- The $\mathbb{Z}_2$ identification between $(-1) \in SU(2)$ and $e^{i\pi} \in U(1)_Y$ creates an $SO(3)$ weak bundle with obstruction $b_2 = w_2$, and relates the degree-4 characteristic classes by *one* mod-2 constraint. Concretely, the square $b_2^2 = w_4$ aligns with the pullback of a unique linear combination of $\{x_4, a_2^2\}$ on the electroweak quotient, producing exactly one relation among $\{x_4, y_4, a_2^2\}$ on BG_{int}.

Hence $\dim H^4 = 2$.

Remark 4.1. *Either route shows there is a* unique *mod-2 relation among the three natural degree-4 classes after the $\mathbb{Z}_6$ quotient. Combined with subgroup restrictions (which prove no further relations), we obtain* $\dim H^4 = 2$.

Synthesis

Part I promotes the $\underline{+3}$ in the index to a universal 1-form anomaly statement (independent of m). Part II pins $r = 7$ purely topologically by combining a sharp E_2 diagonal count with the constructive lower bound. Part III provides a checklist and belt-and-suspenders verifications for the few input ranks.

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References

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