Why the Divergent Sum $1 + 2 + 3 + \ldots$ Equals $-\dfrac{1}{12}$

Route $\boldsymbol{\zeta}$ — A Parity-Projection Derivation Evan Wesley

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Abstract

The famous assignment $\sum_{n=1}^{\infty} n = -\frac{1}{12}$ is often presented as a "regularisation trick." Here I derive the same value from first principles by applying the *Infinite Inversion* parity-projection used earlier to fix α, ρ_Λ and G. Working on a Möbius circle versus its orientable cover cancels all power divergences; the surviving *parity imbalance* is encoded in the Riemann zeta function at $s = -1$. The derivation is fully Lorentz and reflection positive, turning a notorious curiosity into a physical statement about vacuum energy on a non-orientable probe.

1 Why another derivation?

In string theory and the Casimir effect one regularly encounters the zeta-regularised identity

$$1 + 2 + 3 + \ldots \;=\; -\frac{1}{12}.$$

Scepticism persists because the derivation often appears as "analytic continuation prestidigitation." We show the constant arises *inevitably* when one imposes reflection positivity on a non-orientable probe, exactly the same logic that fixed $m = 7$ for the Standard Model.

2 Set-up: Möbius vs. Cover

- Take a real massless boson on a spatial circle of length $L = 2\pi$ (time direction kept continuous).
- Form the quotient by the half-period reflection $\tau : x \mapsto -x$. The resulting 1-manifold is a *Möbius circle* $M_A = S^1_{2\pi}/\tau$.
- Its orientable double cover is the ordinary $\widetilde{M}_A = S^1_{2\pi}$.

3 Parity projection of the integer tower

On the cover the mode energies are $E_n = \frac{\hbar\pi}{L}\, n \;\; (n = 1, 2, \ldots)$. Reflection splits the tower into

$$\{\text{even } n\} \quad (+), \qquad \{\text{odd } n\} \quad (-).$$

Subtracting $\widetilde{M}_A$ from M_A removes the $+$ sector:

$$\Delta E \;=\; \frac{\hbar\pi}{L} \sum_{n\,\text{odd}} n = \frac{\hbar\pi}{L} \sum_{k=1}^{\infty} (2k-1). \tag{3.1}$$

Physical meaning. The even modes cancel because they admit a consistent orientation on both manifolds; the odd modes see the orientation reversal and survive as a *global* parity imbalance.

4 Zeta regularisation and the $-1/12$

Retain the *ordering* of the series and replace termwise by the zeta regulator:

$$\sum_{k=1}^{\infty}(2k-1) := \; 2\zeta(-1) - \zeta(-1) = \zeta(-1) = -\frac{1}{12}. \tag{4.1}$$

Hence

$$\boxed{\Delta E = -\,\frac{\hbar\pi}{12L}\,.} \tag{4.2}$$

Why is this legitimate?

- Reflection positivity ensures only *differences* of spectral sums appear; power divergences cancel automatically, leaving a finite analytic continuation.
- The zeta function is the unique reflection-positive way to continue the heat-kernel trace $\sum_n e^{-tE_n}$ to negative t.

5 Casimir cross-check

Equation (**??**) equals the standard 1-D Casimir energy

$$E_{\text{Casimir}} = -\frac{\pi\hbar c}{12L},$$

confirming that the parity-imbalance derivation reproduces the well-tested physical value (e.g. string zero-point energy).

6 Interpretation in the Infinite-Inversion framework

- The **canvas** (odd modes) and **paint** (even modes) exact-cancel all local counterterms; only the global parity defect survives.
- The leftover is a pure number, $\zeta(-1) = -1/12$, fixed by the same reflection-positivity logic that forced $m = 7$, C_{env} and $r_\star = 2$.
- No "trick" is used: non-orientability + least action $\Rightarrow$ zeta-regularised parity cost.

7 Outlook

Exactly the same procedure applied to higher-dimensional integer towers gives $\zeta(-3) = -1/120$ (3-D gauge ghosts) or $\zeta(-2) = 0$ (2-D), suggesting a universal ladder of parity costs $\zeta(-d)$ ($d \in \mathbb{N}$) governed by the topology of the probe.