Route H — Holography, Replica Non-Orientables, and Entanglement

Parity-anomaly contributions to Rényi entropies and reflection-positivity bounds

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Abstract

We give a mathematical framework for parity-sensitive corrections to entanglement measures in conformal field theories using non-orientable replica geometries. The analysis identifies a $\mathbb{Z}_2$ anomaly character whose inflow contribution produces a universal constant shift in Rényi entropies on crosscap replicas, reflecting the same parity-depth mechanism that fixes the cosmological decade index. We formalize the replica construction on Pin manifolds, state and prove anomaly-inflow identities, and derive reflection-positivity inequalities that saturate under the least-action selector.

Contents

1 Replica geometries with crosscaps

1.1 Replica construction

Let $(\mathcal{H}, \rho)$ be a CFT state on a spatial manifold Σ. For an entangling region $A \subset \Sigma$, the n-th Rényi entropy is computed by a path integral on an n-sheeted branched cover $\mathcal{M}_n$ of the Euclidean spacetime. We introduce a *non-orientable replica* $\widehat{\mathcal{M}}_n$ by inserting a crosscap across each sheet before gluing, so that $\widehat{\mathcal{M}}_n$ carries a Pin structure compatible with antiunitary symmetries.

Definition 1.1 (Parity-twisted Rényi)**.** Define the parity-twisted Rényi entropy by

$$S_n^{(\mathrm{P})}(A) := \frac{1}{1-n} \log \frac{Z[\widehat{\mathcal{M}}_n]}{(Z[\widehat{\mathcal{M}}_1])^n}, \tag{1}$$

where $Z[\cdot]$ is the reflection-positive, regulated CFT partition function with appropriate background gauge fields.

1.2 Anomaly inflow on non-orientables

Let $\nu \in \mathrm{Hom}(\Omega_{d+1}^{\mathrm{Pin}^+}(BG), U(1))$ be the invertible phase character. Inflow assigns to a bounding $(d+2)$-manifold W with $\partial W = \widehat{\mathcal{M}}_n$ a phase $\nu(W)$ such that

$$Z[\widehat{\mathcal{M}}_n] \;=\; \nu(W)\, Z_{\mathrm{bulk}}[\widehat{\mathcal{M}}_n]\,, \qquad |\nu(W)| = 1. \tag{2}$$

Proposition 1.2 (Z$_2$ anomaly character on crosscaps). *If the $\mathbb{Z}_2$-primary parity depth is m, then the restriction of ν to crosscap replicas takes values in $\{\pm 1\}$ and can be written as $(-1)^{\chi_n}$ for a $\mathbb{Z}_2$-valued characteristic χ_n detecting the n-dependent Pin cycle.*

Proof. $\mathbb{Z}_2$-primary classes evaluate to ± 1; invariance under deformations and sheet permutations implies dependence only on the Pin cycle class, yielding the stated form. □

2 Parity contribution to Rényi entropies

Theorem 2.1 (Universal constant shift). *Let d be the boundary spacetime dimension. The parity-twisted Rényi entropy satisfies*

$$S_n^{(\mathrm{P})}(A) = S_n^{(\mathrm{or})}(A) + \frac{i\pi}{1-n}\,\chi_n \pmod{2\pi}, \tag{3}$$

where $S_n^{(\mathrm{or})}$ is the usual orientable Rényi entropy and $\chi_n \in \{0,1\}$ is the anomaly character on $\widehat{\mathcal{M}}_n$.

Proof. Take the logarithm of (2); the magnitude of $\nu(W)$ is one, so only its phase contributes. For $\mathbb{Z}_2$-primary inflow, $\nu(W) = \pm 1 = e^{i\pi\chi_n}$, yielding the constant term. □

Corollary 2.2 (Reflection positivity bound). *Reflection positivity requires $S_n^{(\mathrm{P})}(A) \in \mathbb{R}$; hence χ_n must vanish or the imaginary contribution must cancel across replicas. The least-action selector enforces the* minimal *nonvanishing choice of χ_n consistent with parity depth, reproducing the same minimizing pattern as for the cosmological index.*

3 Holographic interpretation

Proposition 3.1 (Bulk dual constraint). *In AdS_{d+1}/CFT_d, the invertible $\mathbb{Z}_2$ character corresponds to a bulk topological term on a Pin$(d{+}1)$ extension of $\widehat{\mathcal{M}}_n$. The minimal phase consistent with reflection positivity in the bulk reproduces the boundary constant shift and fixes the sign of the bulk counterterm universally.*

Proof. By anomaly inflow, the boundary character is the boundary variation of a bulk invertible phase. On a Pin extension, the topological term reduces to a $\mathbb{Z}_2$ phase when evaluated on the filling of the crosscap cycle. Minimality is the bulk realization of the least-action selector. □

4 Consequences and falsifiability

A nontrivial χ_n implies a universal, geometry-independent constant contribution to parity-twisted Rényi entropies. Any CFT with the SM global symmetry content should exhibit the same sign pattern across n, providing a boundary probe of the same parity-depth mechanism. A proof that $\chi_n = 0$ for all n in a theory matching the SM symmetry would falsify the universality claim.