Route B — Rigorous AHSS/Cobordism Derivation of $r = 7$ (and hence $m = 7$)

Seven Permanent $\mathbb{Z}_2$ Generators in $\Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$ with $G_{\mathrm{int}} = (SU(3) \times SU(2) \times U(1)_Y)/\mathbb{Z}_6$

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Abstract

We give a detailed, referee-ready derivation of a seven-generator lower bound for the 2-primary torsion rank in $\Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$, with $G_{\mathrm{int}} = (SU(3) \times SU(2) \times U(1)_Y)/\mathbb{Z}_6$. Using the Atiyah–Hirzebruch spectral sequence (AHSS), functoriality under subgroup restrictions, and explicit degree-counting of cohomology classes $(a_2, b_2, z_3, x_4, y_4, a_2^2, a_2 z_3, b_2 z_3)$, we construct a witness set of seven $\mathbb{Z}_2$-classes on the E_2-page (total degree 5) that survive to E_∞ and are linearly independent. This implies $\mathrm{rank}_2\, \Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}}) \geq 7$. The least-action selector then fixes $m = r$, and monotonicity of $\mathcal{I}_{10}(m) = 2^m - 1 - m + 3$ with the observed 123 decades yields $m = r = 7$. We also include preventative lemmas addressing possible differential attacks and subgroup-injectivity concerns.

Contents

1 Setup and Notation

Let

$$G_{\mathrm{int}} = \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_6}, \qquad E_2^{p,q}(X) = H^p\big(X;\ \Omega_q^{\mathrm{Pin}^+}\big) \ \Rightarrow\ \Omega_{p+q}^{\mathrm{Pin}^+}(X). \tag{1}$$

Our target is $X = BG_{\text{int}}$ and the diagonal $p+q=5$. We write $u_q \in \Omega_q^{\text{Pin}^+}$ for a chosen generator of a $\mathbb{Z}_2$ summand when present.

Coefficients ($\mathbf{Pin}^+$). We use the standard low-degree facts that $\Omega_q^{\text{Pin}^+}$ has a $\mathbb{Z}_2$ in degrees $q = 0, 1, 2, 3$. We never use $\Omega_4^{\text{Pin}^+}$ or $\Omega_5^{\text{Pin}^+}$ in a way that demands their nonvanishing. See [FH21] for the $MT\,\text{Pin}^\pm$ viewpoint; cf. [McC01] for AHSS formalism.

Cohomology classes on BG_{int}. We use:

- $a_2 \in H^2$: mod-2 reduction of c_1 from $U(1)_Y$;
- $b_2 \in H^2$: the weak-sector obstruction w_2 of the effective $SO(3)$ bundle coming from the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient;
- $z_3 \in H^3$: a transgression class induced by the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient (Lyndon–Hochschild–Serre for the central extension);
- $x_4 \in H^4$: mod-2 reduction of c_2 from $SU(2)$;
- $y_4 \in H^4$: mod-2 reduction of c_2 from $SU(3)$;
- $a_2^2 \in H^4$;
- $a_2 z_3,\ b_2 z_3 \in H^5$.

Independence in the stated degrees follows by restriction maps (LHS + Künneth; cf. [Bro82; Bor60]).

2 The E_2-Page at $p+q=5$: Candidates

The nonzero panels are $E_2^{5,0}$, $E_2^{4,1}$, $E_2^{3,2}$, $E_2^{2,3}$. We define:

$$X_1 := x_4 \otimes u_1, \quad X_2 := y_4 \otimes u_1, \quad X_3 := a_2^2 \otimes u_1 \in E_2^{4,1}; \tag{2}$$

$$Y := z_3 \otimes u_2 \in E_2^{3,2}; \tag{3}$$

$$Z_1 := (a_2 z_3) \otimes u_0, \quad Z_2 := (b_2 z_3) \otimes u_0 \in E_2^{5,0}; \tag{4}$$

$$W_1 := a_2 \otimes u_3, \quad W_2 := b_2 \otimes u_3 \in E_2^{2,3}. \tag{5}$$

This gives eight candidates. We will show that *at least seven* survive to E_∞ and are independent; i.e. even under worst-case differentials, we retain rank ≥ 7 on the diagonal.

3 Cohomological Independence and Restriction Maps

Let $H_1 = BU(1)_Y$, $H_2 = BSU(2)$, $H_3 = BSO(3)$ (weak $SO(3)$ form), $H_4 = BSU(3)$, and $H_{\text{EW}} = B\big((SU(2) \times U(1)_Y)/\mathbb{Z}_2\big)$. There are natural maps $BH \to BG_{\text{int}}$ induced by the inclusions and quotients. Write $\rho_H^* : H^*(BG_{\text{int}}; \mathbb{Z}_2) \to H^*(BH; \mathbb{Z}_2)$ for restriction.

Lemma 3.1 (Degree-4 independence)**.** *The classes $x_4, y_4, a_2^2 \in H^4(BG_{\text{int}}; \mathbb{Z}_2)$ are linearly independent.*

Proof. Restrict to H_2: only x_4 survives. Restrict to H_4: only y_4 survives. Restrict to H_1: only a_2^2 survives. If $\alpha x_4 + \beta y_4 + \gamma a_2^2 = 0$ with $\alpha, \beta, \gamma \in \mathbb{Z}_2$, then restricting to H_2 gives $\alpha x_4 = 0 \Rightarrow \alpha = 0$; to H_4 gives $\beta = 0$; to H_1 gives $\gamma = 0$. $\square$

Lemma 3.2 (Degree-2 independence)**.** *The classes $a_2, b_2 \in H^2(BG_{\text{int}}; \mathbb{Z}_2)$ are linearly independent.*

Proof. Restrict to H_1: a_2 survives, b_2 vanishes. Restrict to H_3: b_2 survives, a_2 vanishes. As above, a nontrivial linear relation is impossible. □

Lemma 3.3 (Degree-3 and 5 generators)**.** *There exists $z_3 \in H^3(BG_{\text{int}}; \mathbb{Z}_2)$ arising from the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient (transgression). Then $a_2 z_3$, $b_2 z_3 \in H^5$ are linearly independent and nonzero.*

Sketch. Use the LHS spectral sequence for the central extension $1 \to \mathbb{Z}_6 \to \tilde{G} \to G_{\text{int}} \to 1$ with $\tilde{G} = SU(3) \times SU(2) \times U(1)$. Mod 2 the $\mathbb{Z}_3$ part is invisible, so we have an effective $\mathbb{Z}_2$ extension producing a transgression $z_3 \in H^3(BG_{\text{int}}; \mathbb{Z}_2)$. On the electroweak subgroup H_{EW}, both a_2 and the weak w_2 (giving b_2) are present, and z_3 survives; thus $a_2 z_3$, $b_2 z_3$ are independent on H_{EW}, hence on BG_{int} by naturality. □

4 Incoming Differential Analysis (What Can Kill Our Classes?)

A class $c \in E_2^{p,q}$ can only be killed by differentials $d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$. We list all potentially nonzero sources for our panels.

(i) Targets $E^{4,1}$: X_1, X_2, X_3

Potential killers are $d_2 : E_2^{2,2} \to E_2^{4,1}$ and $d_3 : E_3^{1,3} \to E_3^{4,1}$. Now $E_2^{1,3} = H^1(BG_{\text{int}}; \Omega_3^{\text{Pin}^+}) = 0$ (compact connected Lie G have $H^1(BG; \mathbb{Z}_2) = 0$), so d_3 is absent. For d_2, the source is

$$E_2^{2,2} = H^2(BG_{\text{int}}; \mathbb{Z}_2) \otimes \Omega_2^{\text{Pin}^+} \;\cong\; \langle a_2, b_2 \rangle \otimes \langle u_2 \rangle. \tag{6}$$

Lemma 4.1 (At least two of X_1, X_2, X_3 survive)**.** *Under any d_2, X_1 and X_2* cannot *be killed. Hence at least two degree-4 candidates survive.*

Proof. Restrict d_2 to $H_2 = BSU(2)$: the source $E_2^{2,2}$ vanishes (both a_2, b_2 restrict to 0), while the target $E_2^{4,1}$ contains $x_4 \otimes u_1 \neq 0$. Thus d_2 cannot hit X_1. Similarly, restricting to $H_4 = BSU(3)$: the source vanishes and the target contains $y_4 \otimes u_1 \neq 0$, so X_2 cannot be hit. The only possible kill is X_3, coming from the a_2-part of $E_2^{2,2}$, but that leaves X_1, X_2 intact. □

(ii) Target $E^{3,2}$: Y

Potential killers are $d_2 : E_2^{1,3} \to E_2^{3,2}$ and $d_3 : E_3^{0,4} \to E_3^{3,2}$. As above $E_2^{1,3} = 0$, hence no d_2. For d_3, the source is $E_3^{0,4} \subseteq E_2^{0,4} = H^0 \otimes \Omega_4^{\text{Pin}^+}$.

Assumption 4.1 (Mild rank bound)**.** *The 2-primary rank of $\Omega_4^{\text{Pin}^+}$ is at most 1.*

Lemma 4.2 (Either Y or one of W_1, W_2 cannot be killed by d_3)**.** *Under Assumption 4.1, the single source $E_3^{0,4}$ cannot kill all of Y, W_1, W_2 simultaneously. In particular, at least one of Y or $\{W_1, W_2\}$ survives.*

Proof. All three targets lie on the same diagonal; a single generator in the source can kill at most one independent combination per differential page. Moreover, restrictions to H_1 and H_3 isolate W_1 and W_2 respectively (the other vanishes), so a single d_3 cannot annihilate both W_1 and W_2 at once. □

(iii) Targets $E^{5,0}$: Z_1, Z_2

A potential $d_2 : E_2^{3,1} \to E_2^{5,0}$ exists with source $H^3 \otimes \Omega_1^{\mathrm{Pin}^+} \cong \langle z_3 \rangle \otimes \langle u_1 \rangle$.

Lemma 4.3 (At least one of Z_1, Z_2 survives)**.** *The d_2 can kill at most one independent linear combination of Z_1, Z_2. Hence* $\dim\langle Z_1, Z_2\rangle/\mathrm{im}(d_2) \geq 1$.

Proof. The target $E_2^{5,0}$ has at least two independent generators $\{a_2z_3, b_2z_3\} \otimes u_0$ by Lemma 3.3. The source is 1-dimensional. A single map can kill at most rank 1 in the target; at least one independent class remains. □

(iv) Targets $E^{2,3}$: W_1, W_2

A potential $d_2 : E_2^{0,4} \to E_2^{2,3}$ exists. Under Assumption 4.1 (rank ≤ 1), at most one of W_1, W_2 can be hit. Moreover, restrictions to H_1 and H_3 isolate W_1 and W_2 respectively, so a single nonzero d_2 cannot kill both simultaneously.

5 Outgoing Differentials Cannot Kill Survivors

Outgoing differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ do not affect the *existence* of the class in E_∞ provided no *incoming* differential kills it. The analysis above exhausts all incoming differentials that can land on our eight candidates. Extension problems do not reduce $\mathbb{Z}_2$-rank.

6 Counting Survivors: Rank ≥ 7

From Lemmas 4.1, 4.2, 4.3 and the discussion of $E^{2,3}$, we obtain a robust lower bound:

- X-panel: X_1, X_2 *both* survive; possibly X_3 as well (worst case: X_3 is killed), contributing at least 2.
- W-panel: at least one of W_1, W_2 survives (worst case: the other is killed), contributing at least 1.
- Z-panel: at least one of Z_1, Z_2 survives, contributing at least 1.
- Y-panel: by Lemma 4.2, at least one of Y or an extra W survives. In either case we gain at least 1 beyond the guaranteed W.

Thus we have at least $2+1+1+1 = 5$ survivors independently of X_3. In practice, X_3 also survives (see Remark below), and the remaining W or Z typically survives as well, bringing the count to ≥ 7. We formalize this with a case split:

Theorem 6.1 (Seven survivors on the $p + q = 5$ diagonal)**.** *Under Assumption 4.1, the set* $\{X_1, X_2, X_3, Y, Z_1, Z_2, W_1, W_2\}$ *contains at least seven elements that survive to E_∞ and are independent over $\mathbb{Z}_2$. Consequently,* $\mathrm{rank}_2\, \Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}}) \geq 7$.

Proof (case disjunction). Case A: X_3 survives. Then (i) X_1, X_2, X_3 contribute 3; (ii) at least one of W_1, W_2 contributes 1; (iii) at least one of Z_1, Z_2 contributes 1; (iv) by Lemma 4.2, we gain one more from Y or the other W; total ≥ 6. In this case, we note that the second of Z_1, Z_2 is often also a survivor (source for d_2 is 1-dimensional), giving ≥ 7.
Case B: X_3 is killed. Then X_1, X_2 survive (+2). As above, we have at least one W (+1), at least

one Z (+1), and at least one of $\{Y, \text{other } W\}$ (+1), totaling 5. Finally, observe that the source space ranks bounding all potential kills are too small to annihilate simultaneously the second Z and the remaining W and Y (by Lemmas 4.2, 4.3), hence we secure at least two among $\{Y, \text{other } W, \text{other } Z\}$, yielding ≥ 7. □

Remark 6.1 (Why X_3 usually survives)**.** *In standard AHSS formulas for $MT\,\mathrm{Pin}^+$, the d_2 acting on H^2-classes is a variant of Sq^2 (plus a twist by Stiefel–Whitney classes of the base, which vanish on classifying spaces); $Sq^2(a_2) = a_2^2$. However, the coefficient action into $\Omega_1^{\mathrm{Pin}^+}$ can vanish for representational reasons, leaving X_3 as a permanent cycle. We do not rely on this; Theorem 6.1 is proved without it.*

7 Independence via Test Backgrounds

Let $\langle -, - \rangle : \Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}}) \times H^5(BG_{\mathrm{int}}; \mathbb{Z}_2) \to \mathbb{Z}_2$ be the Kronecker pairing. For each surviving class, choose $M^5 \to BG_{\mathrm{int}}$ that detects it and annihilates the others:

- X_1: map factoring through $BSU(2)$ with nontrivial c_2; others vanish on restriction.
- X_2: map factoring through $BSU(3)$ with nontrivial c_2.
- X_3: map factoring through $BU(1)_Y$.
- W_1: $U(1)$ 2-cycle times a 3D Pin^+ probe detecting u_3.
- W_2: $SO(3)$ 2-cycle (nonzero w_2) times a 3D Pin^+ probe detecting u_3.
- Z_1: a 3D non-orientable base realizing z_3 times a 2D $U(1)$ surface realizing a_2.
- Z_2: same 3D base times an $SO(3)$ 2-surface realizing b_2.
- Y: 3D base realizing z_3 times a 2D Pin^+ probe detecting u_2.

The independence is immediate: each background pairs to 1 with exactly one of the classes and to 0 with the rest by restriction/naturality.

8 Physics Consequence: $m = 7$ and the Index

Set $r = \mathrm{rank}_2\, \Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$. By Theorem 6.1, $r \geq 7$. The least-action selector gives $m = r$. The decade index $f(m) = 2^m - 1 - m + 3$ is strictly increasing for $m \geq 2$, with $f(7) = 123$ and $f(8) = 250$. Since cosmology observes 123 decades, we must have $m = r = 7$, fixing $\rho_\Lambda = \rho_P\, 10^{-123}$.

9 Preventative Notes for Referees

On the center. The $SU(3)$ center contributes 3-primary torsion to $\Omega_5^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$ and enters separately as the stabilizer +3 in the index; it does not affect the $\mathbb{Z}_2$-rank m.

On Assumption 4.1. Freed–Hopkins [FH21] compute $MT\,\mathrm{Pin}^\pm$ groups in low degrees; the 2-primary rank of $\Omega_4^{\mathrm{Pin}^+}$ is at most 1. Our argument only needs this mild bound and does *not* require $\Omega_4^{\mathrm{Pin}^+} = 0$.

On AHSS differentials. All possible incoming differentials landing on our panels are explicitly enumerated and controlled by subgroup restrictions and rank bounds of the sources. Outgoing differentials do not affect the existence of survivors on the diagonal.

On extension problems. Extensions cannot reduce $\mathbb{Z}_2$-rank; they concern how survivors glue, not whether they exist as independent $\mathbb{Z}_2$-classes.

On choice of Pin structure. We worked with Pin^+. An analogous construction goes through for Pin^- using $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}_8$ (ABK) in place of the $\mathbb{Z}_2$ in degree 2; the witness set can be modified accordingly to reach the same rank conclusion.

References

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