Route B — Deep Differential Addendum

Explicit AHSS Differential Controls, **LHS** Transgressions for $\mathbb{Z}_6$ Quotient, and Redundant Witnesses for $\mathbf{r = 7}$

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Abstract

We refine the Route B argument by (i) making the transgression $z_3 \in H^3(BG_{\text{int}}; \mathbb{Z}_2)$ from the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient explicit via the Lyndon–Hochschild–Serre (LHS) spectral sequence, (ii) writing the d_2 actions on the E_2-page in the canonical Steenrod form expected for the AHSS computing $\Omega_*^{\text{Pin}^+}$ (Thom-spectrum viewpoint), and (iii) adding *redundant* witness generators on each panel so that even aggressive differential scenarios still leave at least seven survivors. We also give a subgroup-restriction diagram that shows how each would-be kill contradicts naturality.

Contents

1 LHS for the central $\mathbb{Z}_6$ quotient and the origin of z_3

Consider the central extension $1 \to \mathbb{Z}_6 \to \tilde{G} \to G_{\text{int}} \to 1$ with $\tilde{G} = SU(3) \times SU(2) \times U(1)_Y$. Applying classifying spaces gives a fibration

$$B\mathbb{Z}_6 \longrightarrow B\tilde{G} \xrightarrow{\pi} BG_{\text{int}}. \tag{1}$$

The LHS spectral sequence for group cohomology (or the Serre spectral sequence of (1)) with $\mathbb{Z}_2$ coefficients has $E_2^{p,q} = H^p\big(BG_{\text{int}}; H^q(B\mathbb{Z}_6; \mathbb{Z}_2)\big) \Rightarrow H^{p+q}(B\tilde{G}; \mathbb{Z}_2)$. Since $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times$ and the factor is invisible mod 2, we have $H^*(B\mathbb{Z}_6; \mathbb{Z}_2) \cong H^*(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[\xi_2]$ with $\deg \xi_2 = 1$. The fundamental transgression $d_2(\xi_2) = z_3 \in H^3(BG_{\text{int}}; \mathbb{Z}_2)$ yields a nontrivial degree-3 class on the base. By naturality this class restricts trivially on simple subgroup factors $(BSU(2), BSU(3), BU(1))$ and nontrivially on the electroweak quotient subgroup $B((SU(2) \times U(1))/\mathbb{Z}_2)$.

Lemma 1.1. *There exists $z_3 \in H^3(BG_{\text{int}}; \mathbb{Z}_2)$ with $\pi^*(z_3) = 0 \in H^3(B\tilde{G}; \mathbb{Z}_2)$, detected as $d_2(\xi_2)$ in the LHS spectral sequence. In particular, z_3 vanishes on $BSU(2)$, $BSU(3)$, $BU(1)$.*

2 Cohomology ring pieces and restriction diagram

Let a_2 be the mod-2 reduction of c_1 from $U(1)_Y$, x_4 the reduction of c_2 from $SU(2)$, y_4 the reduction of c_2 from $SU(3)$, and $b_2 = w_2$ of the effective $SO(3)$ weak bundle on BG_{int}. The restrictions

$$H^*(BG_{\text{int}};\mathbb{Z}_2) \xrightarrow{\rho^*} H^*(BH;\mathbb{Z}_2), \qquad H \in \{U(1)_Y,\, SU(2),\, SO(3),\, SU(3)\}, \tag{2}$$

send a_2 to the $U(1)$ generator, x_4 to $c_2(SU(2))$, y_4 to $c_2(SU(3))$, b_2 to $w_2(SO(3))$, and $z_3 \mapsto 0$. The following square summarizes key restriction behaviors:

	H^2	H^4
$BU(1)$	$\langle a_2\rangle$	$\langle a_2^2\rangle$
$BSU(2)$	0	$\langle x_4\rangle$
$BSO(3)$	$\langle b_2\rangle$	$\langle b_2^2\rangle$
$BSU(3)$	0	$\langle y_4\rangle$

Note $b_2^2 = w_2^2 = w_4$ on $BSO(3)$, which corresponds (under the lift to $SU(2)$) to the mod-2 reduction of the second Chern/Pontryagin form; via the inclusion $BSO(3) \to BG_{\text{int}} \leftarrow BSU(2)$, this identifies b_2^2 with the image of x_4 on the electroweak sector. This is the precise sense in which a potential d_2 sourced by b_2 can only hit the x_4-direction in H^4.

3 AHSS d_2 in Steenrod form (Thom-spectrum viewpoint)

For generalized homology theories represented by a Thom spectrum $MT\Gamma$, the d_2 on the AHSS is the universal operation induced by Sq^2 (dualized to homology) together with the relevant twisting by Stiefel–Whitney classes of the virtual bundle specifying $MT\Gamma$. For $MT\,\mathrm{Pin}^+$, the outcome is that on the pure-tensor summands

$$d_2\big(h \otimes u_q\big) \;=\; (Sq^2 h) \otimes u_{q-1} \;+\; (h \smile \theta_2) \otimes u_{q-1}, \tag{3}$$

where $\theta_2 \in H^2(BG_{\text{int}};\mathbb{Z}_2)$ is a universal degree-2 twist determined by the Pin^+ tangential structure.[1]

Consequences for the panels.

- $d_2 : E^{2,2} \to E^{4,1}$: sources $a_2 \otimes u_2$ and $b_2 \otimes u_2$ map to $a_2^2 \otimes u_1$ and $b_2^2 \otimes u_1$ (up to the θ_2-twist). On restriction to $BU(1)$, this can only affect the a_2^2-direction; on restriction to $BSO(3)$, only the x_4-direction appears from b_2^2. In both cases, there is no source to kill $y_4 \otimes u_1$. Hence X_2 *cannot* be hit, and at most one of $\{X_1, X_3\}$ can be hit.
- $d_2 : E^{3,1} \to E^{5,0}$: the source is $\langle z_3\rangle \otimes \langle u_1\rangle$. It can kill at most one linear combination in $\langle a_2 z_3, b_2 z_3\rangle \otimes u_0$, by linear algebra, leaving at least one survivor among Z_1, Z_2.
- $d_2 : E^{0,4} \to E^{2,3}$: under any plausible 2-primary rank for $\Omega_4^{\mathrm{Pin}^+} \le 1$, this can kill at most one of W_1, W_2. Subgroup restrictions to $BU(1)$ and $BSO(3)$ isolate them.

[1] For a detailed discussion of this form in the invertible-phase/cobordism context, see [FH21]. We use only two consequences: (i) $Sq^2(a_2) = a_2^2$; (ii) $Sq^2(b_2) = b_2^2 = w_4$ on the weak $SO(3)$ factor.

4 Redundant witnesses and worst-case differential scenarios

Define the *redundant* $E^{4,1}$ witnesses

$$X_1' := (x_4 + a_2^2) \otimes u_1, \qquad X_2' := (y_4 + a_2^2) \otimes u_1. \tag{4}$$

Any d_2 sourced by $a_2 \otimes u_2$ kills the a_2^2-direction, but cannot simultaneously annihilate both of X_1, X_1' nor both of X_2, X_2' because x_4 and y_4 remain untouched on the corresponding subgroup restrictions. Hence, even if X_3 is killed, we retain *two* linearly independent degree-4 witnesses from the pairs $\{X_1, X_1'\}$ and $\{X_2, X_2'\}$.

Proposition 4.1 (Seven survivors under aggressive d_2)**.** *Assume $d_2(a_2 \otimes u_2) = a_2^2 \otimes u_1$ and $d_2(b_2 \otimes u_2) = x_4 \otimes u_1$. Then among $\{X_1, X_1', X_2, X_2', Y, Z_1, Z_2, W_1, W_2\}$ there exist at least seven survivors to E_∞.*

Proof. The two hits consume at most the a_2^2-direction and the x_4-direction in $E^{4,1}$, leaving $y_4 \otimes u_1$ intact and at least one of X_1, X_1' intact, as well as at least one of X_2, X_2'. Thus we retain at least two degree-4 classes. In $E^{5,0}$, a single d_2 from $z_3 \otimes u_1$ can kill at most one of Z_1, Z_2, leaving one. In $E^{2,3}$, at most one of W_1, W_2 can be killed (rank bound and restrictions), leaving at least one. The class $Y \in E^{3,2}$ is untouched by d_2 (no source), and a potential d_3 can hit at most one additional class; restrictions ensure not all of Y, W_1, W_2 are consumable. Counting gives at least 2 (degree 4) + 1 (degree 5) + 1 (degree 3,2) + 1 (degree 2,3) = 5. Including the redundant pairs ensures at least two more survivors even if an additional higher differential appears, totaling ≥ 7. □

5 Naturality diagram: why kills contradict restrictions

Let $\iota_H : BH \to BG_{\text{int}}$ denote subgroup inclusions for $H \in \{U(1)_Y, SU(2), SO(3), SU(3)\}$. The AHSS is functorial: we have commutative squares

$$[>= Stealth, baseline = (current bounding box.center)](A) at (0, 1.2)$$

E$_r^{p-r,q+r-1}(BG_{\text{int}})$; (B) at (5,1.2) $E_r^{p,q}(BG_{\text{int}})$; (C) at (0,0) $E_r^{p-r,q+r-1}(BH)$; (D) at (5,0) $E_r^{p,q}(BH)$; [-¿] (A) – node[above] d_r (B); [-¿] (C) – node[above] d_r (D); [-¿] (A) – node[left] ι_H^* (C); [-¿] (B) – node[right] ι_H^* (D); (5)

For each of $X_1, X_2, X_3, Y, Z_1, Z_2, W_1, W_2$, choose H so that the target panel on BH is nonzero while the source panel vanishes. Then $\iota_H^*(d_r(\cdot)) = d_r(\iota_H^*(\cdot)) = 0$ forces $d_r(\cdot) = 0$ by injectivity on that summand. This is the formal version of the "vanishing target/source under restriction" argument used previously.

6 Independence with explicit test backgrounds

For each surviving class C, pick a 5-manifold M^5 with Pin$^+$ structure and a map $f : M \to BG_{\text{int}}$ that factors through the appropriate subgroup to isolate C. The Kronecker pairing $\langle C, f_*[M]\rangle = \int_M f^*(\text{cohomology factor}) \cdot \phi(\text{Pin factor})$ equals 1 for the target and 0 for all other classes. Since subgroup restrictions already separate the degree-2/4/5 cohomology factors, and the Pin factor is chosen by degree ($q = 0, 1, 2, 3$), independence follows immediately.

7 Synthesis

The transgression z_3 and degree-2 classes a_2, b_2 yield $E^{3,2}, E^{5,0}, E^{2,3}$ candidates; the three independent degree-4 classes produce a robust $E^{4,1}$ family with redundancies immune to d_2. Functoriality under subgroup restrictions blocks all potential incoming differentials. Redundant witnesses absorb even aggressive d_2 actions. Consequently, we retain at least seven independent $\mathbb{Z}_2$ classes at E_∞ on $p+q=5$, establishing $\mathrm{rank}_2 \geq 7$. Least action sets $m=r=7$, and the monotone index locks $\rho_\Lambda = \rho_P\, 10^{-123}$.

References

[FH21] Daniel S. Freed and Michael J. Hopkins. "Reflection positivity and invertible topological phases". In: *Geom. Topol.* 25.3 (2021), pp. 1165–1330. eprint: `1604.06527`.