Route Q — Quantum Applications of the Anomaly–Cohomology Framework

Geometric phases, structural topological terms, selection rules, and SPT classification (math-only)

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Abstract

This module translates the anomaly–cohomology machinery developed for the cosmological decade index into quantum-mechanical language. We give derivations and recipes for: (i) geometric phases and Berry holonomy (including time-reversal/Pin effects), (ii) structural corrections to effective Hamiltonians by topological terms, (iii) anomaly-driven selection rules in quantum transitions, and (iv) a practical AHSS/bordism pipeline for classifying symmetry-protected topological (SPT) phases in condensed-matter systems.

Contents

1 Geometric phases and Berry holonomy via cohomology/bordism **2**
1.1 Adiabatic families and line bundles . . . 2
1.2 Antiunitary symmetries and Pin structures . . . 2

2 Structural topological terms in effective actions **2**
2.1 Particle on a ring: Aharonov–Bohm energy correction . . . 2
2.2 Spin coherent states and Haldane θ term . . . 2

3 Anomaly-driven selection rules in quantum transitions **3**
3.1 Global anomalies as 1D invertible phases . . . 3

4 SPT classification recipe via AHSS (practical) **3**
4.1 General pipeline . . . 3
4.2 Example: 3D fermionic topological insulator (class AII) . . . 3
4.3 Example: 1D bosonic Haldane phase . . . 3

5 Bridge back to the decade index **4**

6 How to deploy Route Q (no code) **4**

7 Falsifiable predictions (quantum) **4**

1 Geometric phases and Berry holonomy via cohomology/bordism

1.1 Adiabatic families and line bundles

Let $\{H(\lambda)\}_{\lambda\in X}$ be a gapped family of Hamiltonians on a finite-dimensional Hilbert space, with a rank-1 ground-state vector bundle $L \to X$. The Berry connection A is a U(1) connection on L; its curvature $F = \mathrm{d}A$ has first Chern class $c_1(L) = [F/2\pi] \in H^2(X;\mathbb{Z})$. A closed loop $\gamma \subset X$ accumulates geometric phase

$$\varphi(\gamma) = \oint_\gamma A \;=\; \int_{\Sigma:\,\partial\Sigma=\gamma} F \mod 2\pi, \tag{1}$$

and the corresponding holonomy is $\exp(i\varphi(\gamma))$.

Proposition 1.1 (Quantization and stability)**.** *If $c_1(L) \neq 0$, the set of Berry phases over loops represents a nontrivial subgroup of $U(1)$ determined by $H_2(X;\mathbb{Z})$. Any continuous deformation of $H(\lambda)$ that preserves the gap keeps $c_1(L)$ invariant.* □

1.2 Antiunitary symmetries and Pin structures

Suppose an antiunitary symmetry T acts on the family and reverses time. If $T^2 = +1$ (class AI), the appropriate background structure is Pin$^+$; if $T^2 = (-1)^F$ (class AII), it is Pin$^-$. The Berry line bundle is then promoted to a real/quaternionic bundle; the mod-2 invariants are Stiefel–Whitney classes w_i.

Proposition 1.2 (Z_2 quantization under time reversal)**.** *Let X admit a T-invariant loop γ with the T-action reversing orientation. If $T^2 = (-1)^F$ (Pin$^-$ case), the Berry phase on γ is $\varphi(\gamma) \in \{0, \pi\} \bmod 2\pi$. Equivalently, the holonomy is a $\mathbb{Z}_2$ class detected by w_1 on the associated bundle.* □

Remark 1.3 (Kramers degeneracy and w_2). In class AII, a rank-1 complex bundle is obstructed globally; the relevant obstruction is w_2 of the real bundle underlying the Kramers pair. Nontrivial w_2 forces a $\mathbb{Z}_2$ holonomy (Berry phase π) on certain loops and protects degeneracies.

2 Structural topological terms in effective actions

2.1 Particle on a ring: Aharonov–Bohm energy correction

Consider a particle of moment of inertia I on S^1 in an Aharonov–Bohm flux Φ. The Euclidean action includes a topological term $i\,\frac{q\Phi}{\hbar}\,\mathrm{wind}(\theta)$. Energy levels are

$$E_n(\Phi) = \frac{\hbar^2}{2I}\Big(n - \frac{\Phi}{\Phi_0}\Big)^2, \qquad \Phi_0 = \frac{h}{q}, \quad n \in \mathbb{Z}. \tag{2}$$

This realizes $E = E_{\mathrm{std}} + \hbar\omega \cdot N_{\mathrm{top}}$ with $N_{\mathrm{top}} = n$ and $\omega = \hbar/(2I)$.

2.2 Spin coherent states and Haldane θ term

For a spin-s quantum rotor, the path integral contains the Berry (Wess–Zumino) term $S_B = s\,\Omega[\hat{\boldsymbol{n}}]$, the area on S^2. In 1D antiferromagnets, the continuum limit yields a θ-term with $\theta = 2\pi s$; integer $s \Rightarrow \theta = 0$ (gapped), half-integer $s \Rightarrow \theta = \pi$ (critical/protected).

Proposition 2.1 (Topological quantization and spectral parity)**.** *If the effective action contains $ik\int \mathrm{d}\theta$ with integer k (winding number density), then spectra and selection rules shift by k quanta: transitions changing k by an odd unit are forbidden when coupled to a Pin$^\pm$ background that detects the parity of k.* □

3 Anomaly-driven selection rules in quantum transitions

3.1 Global anomalies as 1D invertible phases

In 0+1D, a global anomaly corresponds to a nontrivial element in the deformation classes of reflection-positive invertible phases with background symmetry. Coupling a quantum system to such a background assigns a phase $\nu(\gamma) \in U(1)$ to parameter loops γ.

Proposition 3.1 (Selection rule from anomaly phase)**.** *If there exists a loop γ in parameter space such that the path integral picks a phase $\nu(\gamma) = -1$ (a $\mathbb{Z}_2$ anomaly), then any process requiring an odd number of traversals of γ has vanishing amplitude. Equivalently, the transition operator anticommutes with the anomaly character and is projected out.* □

Remark 3.2 (Pin$^\pm$ diagnostic). Placing the system on non-orientable probe loops (crosscaps in the worldline) implements time-reversal twists; the induced sign is the Arf–Brown–Kervaire invariant in simple models. Nontrivial ABK yields a robust $\mathbb{Z}_2$ selection rule.

4 SPT classification recipe via AHSS (practical)

4.1 General pipeline

Given spatial dimension d, statistics (bosonic/fermionic), and symmetry G (including crystalline), the SPT classification at low energies is computed by

$$E_2^{p,q} = H^p(BG;\ \Omega_q^{\text{structure}}) \ \Rightarrow\ \Omega_{p+q}^{\text{structure}}(BG), \tag{3}$$

where structure $=$ Spin, Pin$^\pm$, $\ldots$ and total degree is $d{+}1$. The steps are:

1. Compute low-degree bordism coefficients $\Omega_q^{\text{structure}}$ (just ranks and torsion needed).
2. Compute $H^p(BG;\cdot)$ in degrees $p \le d{+}1$ (mod torsion relevant to the problem).
3. Identify differentials d_r by naturality (restrictions to subgroups) and Steenrod operations.
4. Read off survivors at E_∞ and resolve extension problems (usually by physical arguments or subgroup data).

4.2 Example: 3D fermionic topological insulator (class AII)

Let $d = 3$, fermions with $G = U(1) \rtimes \mathbb{Z}_2^T$ and $T^2 = (-1)^F$ (Pin$^-$). The classification yields a $\mathbb{Z}_2$ strong index. The bulk topological term is the axion coupling $\theta \frac{e^2}{2\pi h} \int F \wedge F$ with $\theta = \pi$ mod 2π; surfaces carry half-quantized Hall response when gapped by symmetry breaking.

4.3 Example: 1D bosonic Haldane phase

Let $d = 1$, bosons with $G = \mathbb{Z}_2 \times \mathbb{Z}_2^T$ (spin rotations by π and time reversal). The AHSS/bordism computation gives a nontrivial $\mathbb{Z}_2$ SPT protected by G; edges host symmetry-protected Kramers doublets (degeneracy that cannot be removed without breaking G).

5 Bridge back to the decade index

The same structural ingredients appear in cosmology:

- Discrete parity bookkeeping (AHSS rank m) vs. continuous phase optimization (macro-fold depth q).
- Anomaly phases on non-orientable probes (Pin) enforce uniform reflection positivity; the minimal extensive counterterm is the cosmological constant, with decades $\mathcal{I}_{10}(m) = 2^m - 1 - m + 3$.
- In quantum systems, the same phases enforce selection rules and quantized responses.

6 How to deploy Route Q (no code)

1. **Choose a system and symmetry G.** Determine if time reversal or spatial inversion acts antiunitarily and pick Spin/Pin$^\pm$.
2. **Compute the E_2 page.** Only low degrees are needed; tabulate torsion/ranks.
3. **Use subgroup naturality.** Restrict to simple subgroups to fix differentials and prove survival.
4. **Extract invariants.** Survivors at E_∞ are the SPT indices; map them to measurable quantities (Berry $\mathbb{Z}_2$ phase, quantized pump, surface Hall response, edge degeneracy).
5. **State selection rules.** Formulate allowed/forbidden transitions based on anomaly characters.

7 Falsifiable predictions (quantum)

- **Quantized holonomy:** In class AII, a T-reversing loop in parameter space yields Berry phase 0 or π only. Any continuous deformation preserving the gap cannot change it.
- **Edge degeneracy:** Protected Kramers doublets at 1D SPT edges (Haldane-type) persist until G is broken.
- **Axion response:** For 3D AII insulators, $\theta = \pi \bmod 2\pi$ implies half-quantized surface Hall conductance on symmetry-breaking terminations.