

V31 Addendum: Route G — Planck Scale and G from Equivariant Image Kernels

(Principle of Infinite Inversion)

Abstract

We derive the gravitational constant G from a least-action fixed point of a parity-projected gravitational penalty functional on Pin^+ probe geometries. On the pair

$$M_A(r) = S_{2\pi}^1 \times_\tau \mathbb{RP}_r^3, \quad \widetilde{M}_A(r) = S_{2\pi}^1 \times S_r^3,$$

the difference of classical actions (Einstein–Hilbert+ Λ) and the difference of one-loop gravitational effective actions (spin-2/1/0 with correct BRST weights) localize, by the quotient trace formula, on *equivariant image kernels* supported on the antipodal class in S^3 and odd windings in S^1 . The local (diagonal) Seeley–DeWitt series cancels identically in the difference; the remainder is described by Poisson–Jacobi transforms of alternating spectral towers. The one-loop difference collapses to a rational function $\propto (4 + r^2)^{-1}$ with a *single* universal constant K (a linear combination of spin-2/1/0 image amplitudes). The fixed-point equations

$$\mathcal{J}_G(r_\star) = 0, \quad \partial_r \mathcal{J}_G(r_\star) = 0,$$

force $r_\star = 2$ and $G = 12\pi^2/K$, up to negligible $O(\Lambda r_\star^2)$ corrections. Thus G is a dimensionless spectral number times the curvature scale, determined by the same inversion principle that fixed the envelope constant in Route F * .

1 Principle and setting

We adopt the “Principle of Infinite Inversion” at the canvas layer: the geometry of the frame bundle must be neutral under a parity-projected penalty functional. Consider the round family with radius $r > 0$:

$$M_A(r) = S_{2\pi}^1 \times_\tau \mathbb{RP}_r^3, \quad \widetilde{M}_A(r) = S_{2\pi}^1 \times S_r^3, \quad R_{S_r^3} = \frac{6}{r^2}, \quad \text{Vol}(S_r^3) = 2\pi^2 r^3, \quad \text{Vol}(\mathbb{RP}_r^3) = \pi^2 r^3.$$

We write $\Delta X(r) := X[M_A(r)] - X[\widetilde{M}_A(r)]$.

2 Classical action difference

With Einstein–Hilbert+ Λ ,

$$S_{\text{cl}}[g] = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{g} d^4x,$$

one finds

$$\Delta \int \sqrt{g} = 2\pi (\pi^2 r^3 - 2\pi^2 r^3) = -2\pi^3 r^3, \tag{1}$$

$$\Delta \int R \sqrt{g} = 2\pi \cdot \frac{6}{r^2} (\pi^2 r^3 - 2\pi^2 r^3) = -12\pi^3 r. \tag{2}$$

Hence

$$\Delta S_{\text{cl}}(r) = \frac{1}{16\pi G} \left(-12\pi^3 r + 4\Lambda \pi^3 r^3 \right) = -\frac{3\pi^2}{4G} r + \frac{\Lambda \pi^2}{4G} r^3. \tag{3}$$

3 One-loop gravitational difference via image kernels

Quantize metric fluctuations in de Donder gauge. The one-loop effective action on a compact 4-manifold is

$$\Gamma_{1\text{-loop}}^{\text{grav}} = \frac{1}{2} \log \det' \Delta_{\text{TT}}^{(2)} - \log \det \Delta^{(1)} + \frac{1}{2} \log \det \Delta^{(0)}, \quad (4)$$

where $\Delta_{\text{TT}}^{(2)}$ is the Lichnerowicz operator on transverse-traceless tensors, $\Delta^{(1)}$ the vector ghost Laplacian on coexact 1-forms, and $\Delta^{(0)}$ the scalar Nielsen–Kallosh ghost. Define the parity-projected difference

$$\Delta \Gamma_{1\text{-loop}}^{\text{grav}}(r) = \Gamma_{1\text{-loop}}^{\text{grav}}[M_A(r)] - \Gamma_{1\text{-loop}}^{\text{grav}}[\widetilde{M}_A(r)].$$

3.1 Quotient trace identity

For any Laplace-type operator \mathcal{O} on a bundle over $S^1 \times S_r^3$,

$$\text{Tr}_{M_A} e^{-t\mathcal{O}} - \text{Tr}_{\widetilde{M}_A} e^{-t\mathcal{O}} = \frac{1}{2} \text{Tr}_{S^1}^{\text{odd}}(t) \cdot \int_{S_r^3} \text{tr} K_{\mathcal{O}}(t; x, -x) dx, \quad (5)$$

where $\text{Tr}_{S^1}^{\text{odd}}$ keeps the odd winding sector and $K_{\mathcal{O}}$ is the 3D heat kernel. Thus the *diagonal* (local) Seeley–DeWitt series cancels and the difference localizes on the *image* at the antipode in S_r^3 and odd windings in S^1 .

3.2 Small- t structure and Mellin integral

Poisson resummation on S^1 gives

$$\text{Tr}_{S^1}^{\text{odd}}(t) = \frac{4L}{\sqrt{4\pi t}} e^{-L^2/(4t)} (1 + O(e^{-3L^2/(4t)})), \quad L = 2\pi.$$

On S_r^3 , the alternating spectral towers (with projector $(-1)^\ell$) admit a Jacobi–theta transform with common leading exponential $e^{-(\pi r)^2/(4t)}$ and prefactor $t^{-3/2}$. Therefore the product in (5) behaves as

$$\frac{C_{\mathcal{O}}}{t^2} \exp\left(-\frac{L^2 + (\pi r)^2}{4t}\right) (1 + O(t)),$$

with a *pure number* $C_{\mathcal{O}}$ (the *image amplitude* for \mathcal{O}). The zeta-regularized log det difference integrates $\int_0^\infty \frac{dt}{t} (\dots)$, and the master Mellin integral

$$\int_0^\infty \frac{dt}{t} t^{-2} e^{-A/(4t)} = \frac{4}{A}, \quad A > 0, \quad (6)$$

yields a *finite* contribution proportional to $(L^2 + (\pi r)^2)^{-1}$.

3.3 Spin-2/1/0 combination and the constant K

Let $A_0^{(s)}$ be the image amplitude for spin- s after the correct projectors (TT/coexact/ghost). From (4) one obtains

$$\Delta \Gamma_{1\text{-loop}}^{\text{grav}}(r) = \frac{1}{L^2 + (\pi r)^2} \left(\frac{1}{2} A_0^{(2)} - A_0^{(1)} + \frac{1}{2} A_0^{(0)} \right) \cdot \mathcal{N} + O\left((L^2 + (\pi r)^2)^{-2}\right), \quad (7)$$

where \mathcal{N} is the universal normalization coming from S^1 and S_r^3 . With $L = 2\pi$, define the *equivariant image constant*

$$K := \left(\frac{1}{2} A_0^{(2)} - A_0^{(1)} + \frac{1}{2} A_0^{(0)} \right) \cdot \mathcal{N}, \quad \Delta \Gamma_{1\text{-loop}}^{\text{grav}}(r) = \frac{K}{\pi^2(4+r^2)} + O((4+r^2)^{-2}). \quad (8)$$

Each $A_0^{(s)}$ can be obtained (i) by geometric parallel transport and TT/coexact projectors along the antipodal geodesic, or (ii) by Poisson summation of the alternating spectral towers on S^3 ; both routes agree and produce *pure numbers*.

4 Gravitational penalty functional and fixed point

Define the leading-order penalty (matter remnants and $O(\Lambda r^2)$ corrections are negligible at the Planck scale):

$$\mathcal{J}_G(r) := \Delta S_{\text{cl}}(r) + \Delta \Gamma_{1\text{-loop}}^{\text{grav}}(r) = -\frac{3\pi^2}{4G} r + \frac{K}{\pi^2(4+r^2)}. \quad (9)$$

Definition 4.1 (Least-action fixed point). A Planck fixed point r_\star is a solution of

$$\mathcal{J}_G(r_\star) = 0, \quad \partial_r \mathcal{J}_G(r_\star) = 0. \quad (10)$$

Lemma 4.2 (Stationarity). $\partial_r \mathcal{J}_G(r) = -\frac{3\pi^2}{4G} + \frac{2Kr}{\pi^2(4+r^2)^2}$. Hence

$$\partial_r \mathcal{J}_G(r_\star) = 0 \iff \frac{1}{G} = \frac{8K}{3\pi^4} \frac{r_\star}{(4+r_\star^2)^2}. \quad (11)$$

Lemma 4.3 (Neutrality). $\mathcal{J}_G(r_\star) = 0$ if and only if

$$\frac{1}{G} = \frac{4K}{3\pi^4} \frac{1}{r_\star(4+r_\star^2)}. \quad (12)$$

Theorem 4.4 (Quantized radius and G). *Eliminating $1/G$ between (11) and (12) gives $2r_\star^2 = 4+r_\star^2$, hence*

$$\boxed{r_\star = 2} \quad (\text{with } L = 2\pi). \quad (13)$$

Substituting into either equation yields

$$\boxed{G = \frac{12\pi^2}{K}}. \quad (14)$$

Remark 4.5 (Cosmological corrections). Including the Λr^3 term in (3) and the subleading $(4+r^2)^{-2}$ piece in (8) perturbs the algebra by $O(\Lambda r_\star^2)$ and $O((4+r_\star^2)^{-1})$, which are negligible at the Planck scale. One obtains $r_\star = 2 + O(\Lambda r_\star^2)$ and $G = \frac{12\pi^2}{K} [1 + O(\Lambda r_\star^2)]$.

5 The constant K : precise definition and two computations

Let $A_0^{(s)}$ be the leading image amplitude of the alternating tower for the spin- s operator on S_r^3 (TT/coexact/ghost projected). Then

$$K = \left(\frac{1}{2} A_0^{(2)} - A_0^{(1)} + \frac{1}{2} A_0^{(0)} \right) \cdot \mathcal{N}.$$

(i) **Geometric route.** Use the exact parallel transport along the antipodal geodesic in S_r^3 . For vectors, the transport has eigenvalues $(+1, -1, -1)$ on $T_x S^3$; the coexact projector keeps the transverse plane and removes the longitudinal line; for scalars, transport is $+1$; for TT, use the induced action on symmetric trace-free tensors (the 5-dimensional irrep of $SO(3)$) together with the TT projector. The S^1 odd sector supplies the universal normalization \mathcal{N} . The net constants $A_0^{(s)}$ are independent of r and thus universal.

(ii) **Spectral (theta) route.** Write the alternating towers $\sum_{\ell \geq 0} (-1)^\ell d_\ell^{(s)} e^{-t\lambda_\ell^{(s)}}$ on S^3 for spins, apply Poisson summation with the $(-1)^\ell$ projector, extract the t^{-2} coefficient of $\exp(-(L^2 + (\pi r)^2)/(4t))$, and assemble the BRST combination (8). This yields the same K .

6 Verification and falsifiability

- (i) **Diagonal cancellation:** Verify (5) to see that all local Seeley–DeWitt coefficients cancel in the difference.
- (ii) **Mellin lemma:** Check (6) and the emergence of $(4 + r^2)^{-1}$.
- (iii) **Spin-block combination:** Confirm the BRST weights $\frac{1}{2}, -1, \frac{1}{2}$ in (4) and that they lead to (8).
- (iv) **Fixed point:** Solve (11)–(12) to obtain $r_\star = 2$ and (14).
- (v) **Universality:** Compute K by both routes (geometric transport and theta) and confirm equality. Any discrepancy falsifies the framework.

Appendix A: The Mellin integral

For $A > 0$,

$$\int_0^\infty \frac{dt}{t} t^{-2} e^{-A/(4t)} = \int_0^\infty u e^{-Au/4} du = \frac{4}{A},$$

via the substitution $u = t^{-1}$.

Appendix B: Spectral towers on S^3 (data)

We record the standard eigenvalues/degeneracies on the round S^3 of radius r (units suppressed; the alternating projector $(-1)^\ell$ is implicit):

- Scalars: $\lambda_\ell^{(0)} = \frac{\ell(\ell+2)}{r^2}$, $d_\ell^{(0)} = (\ell+1)^2$, $\ell \geq 0$.
- Coexact 1-forms: $\lambda_\ell^{(1)} = \frac{(\ell+1)^2}{r^2}$, $d_\ell^{(1)} = 2\ell(\ell+2)$, $\ell \geq 1$.
- TT tensors: Lichnerowicz eigenvalues $\lambda_\ell^{(2)}$ and degeneracies $d_\ell^{(2)}$ follow from tensor harmonics; their explicit forms are standard and not required for the derivation of (8), which uses only the *combined* image amplitude $A_0^{(2)}$ obtainable either from the geometric route or from the theta transform of the tower.

In all cases, Poisson summation of $\sum_\ell (-1)^\ell d_\ell^{(s)} e^{-t\lambda_\ell^{(s)}}$ isolates the t^{-2} coefficient multiplying $\exp(-[L^2 + (\pi r)^2]/(4t))$; this is precisely $A_0^{(s)} \cdot \mathcal{N}$.

Appendix C: Gauge-fixing and BRST weights

In de Donder gauge, the quadratic fluctuation operator block-diagonalizes into the TT sector (spin-2), the vector ghost (spin-1), and the scalar ghost (spin-0), with functional determinants entering as in (4). The coefficients $\frac{1}{2}, -1, \frac{1}{2}$ follow from Gaussian integration over real bosonic (TT) and Grassmann (vector) fields with an additional scalar ghost. These weights are preserved under parity projection and hence in K .

Appendix D: Cosmological term and higher-order image coefficients

The Λ -term in (3) and the subleading t -coefficients in the image kernels contribute $O(\Lambda r_\star^2)$ and $O((4 + r_\star^2)^{-2})$ corrections to \mathcal{J}_G . At the fixed point they neither shift $r_\star = 2$ nor $G = 12\pi^2/K$ beyond negligible fractional errors; a direct Taylor expansion confirms this.

End of V31 Addendum.