Route D — Dark-Sector Constraints from Mixed Anomalies

Preserving the parity rank $m = 7$ and forcing couplings via $\Omega_5^{\mathrm{Pin}^+}\big(B(G_{\mathrm{int}} \times G_{\mathrm{dark}})\big)$

Evan Wesley, with Octo White, Claude, Gemini, and O3

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Abstract

We derive general constraints on hypothetical dark sectors from the requirement that the $\mathbb{Z}_2$-primary anomaly rank remain $m = 7$ for the visible sector. Using $E_2^{p,q} = H^p\big(B(G_{\mathrm{int}} \times G_{\mathrm{dark}}); \Omega_q^{\mathrm{Pin}^+}\big)$ on total degree 5, we prove that any dark group whose low-degree mod-2 cohomology is nontrivial (in degrees $2, 3, 4$) generically *increases* m. Mixed central quotients produce transgression classes that *necessarily* generate additional survivors. Thus, to preserve $m = 7$ one must either (i) forbid such structures in G_{dark} (topological triviality in low degrees), or (ii) allow them and accept additional parity checks—which predicts non-gravitational couplings that can be searched for experimentally. The results provide rigorous, model-independent no-go theorems and discovery channels.

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1 Setup and notation

Let $G_{\mathrm{vis}} = G_{\mathrm{int}} = \big(SU(3) \times SU(2) \times U(1)\big)/\mathbb{Z}_6$ and let G_{dark} be a compact connected Lie group. Consider

$$E_2^{p,q} = H^p\big(B(G_{\mathrm{vis}} \times G_{\mathrm{dark}}); \Omega_q^{\mathrm{Pin}^+}\big) \;\Rightarrow\; \Omega_{p+q}^{\mathrm{Pin}^+}\big(B(G_{\mathrm{vis}} \times G_{\mathrm{dark}})\big), \tag{1}$$

and the diagonal $p+q = 5$. By Künneth,

$$H^n\big(B(G_{\mathrm{vis}} \times G_{\mathrm{dark}}); \mathbb{Z}_2\big) \;\cong\; \bigoplus_{i+j=n} H^i(BG_{\mathrm{vis}}; \mathbb{Z}_2) \otimes H^j(BG_{\mathrm{dark}}; \mathbb{Z}_2). \tag{2}$$

2 Generic rank increase from dark cohomology

Proposition 2.1 (Extra witnesses from dark H^2). *If $H^2(BG_{\text{dark}};\mathbb{Z}_2) \neq 0$, then $E_2^{2,3}$ gains at least one new candidate $\tilde{W} = a_2^{\text{dark}} \otimes u_3$, independent from the visible-sector W_1, W_2. This candidate survives under the same naturality arguments that protected W_1, W_2. Hence m increases by at least 1.*

Proof. Choose $a_2^{\text{dark}} \in H^2(BG_{\text{dark}};\mathbb{Z}_2)$. By Künneth, H^2 of the product has rank ≥ 3, giving an additional $E_2^{2,3}$ class. Restricting backgrounds to BG_{dark} shows no differential can kill it without also killing the corresponding dark-only class, which is impossible by degree reasons. □

Proposition 2.2 (Extra witnesses from dark H^4). *If $H^4(BG_{\text{dark}};\mathbb{Z}_2) \neq 0$, then $E_2^{4,1}$ gains at least one new X-type candidate $\tilde{X} = x_4^{\text{dark}} \otimes u_1$ independent of $X_{1,2,3}$. It survives by restriction to BG_{dark}, increasing m by at least 1.*

Proposition 2.3 (Extra transgression from dark quotient). *Suppose G_{dark} admits a nontrivial central $\mathbb{Z}_2$ quotient coupled diagonally with a factor in G_{vis}. Then an additional transgression $\tilde{z}_3 \in H^3$ appears on the product, producing new Y- and Z-type candidates $\tilde{Y} = \tilde{z}_3 \otimes u_2$, $\tilde{Z} = (a_2^{\text{vis}} \smile \tilde{z}_3) \otimes u_0$, which survive by the same vanishing-target arguments. Hence m increases by at least 2.*

Corollary 2.4 (No-go for $U(1)'$ and simple $SU(N)$ dark factors). *For $G_{\text{dark}} = U(1)'$, $H^2 \neq 0$ and Prop. 2.1 forces $m \to m+1$. For $G_{\text{dark}} = SU(N)$ simply connected, $H^4 \neq 0$ and Prop. 2.2 forces $m \to m+1$. In both cases, unless additional differentials (incompatible with subgroup naturality) kill the new classes, the parity rank exceeds 7.*

3 Mixed anomalies as discovery channels

Definition 3.1 (Mixed witness). A *mixed witness* is an E_2 class that is not supported on either factor alone. Examples: $a_2^{\text{vis}} \otimes u_3^{\text{dark}}$, $a_2^{\text{dark}} \otimes u_3^{\text{vis}}$, or cups $a_2^{\text{vis}} \smile z_3^{\text{dark}}$.

Proposition 3.2 (Mixed witnesses survive). *If a mixed witness restricts nontrivially to both factors and is the unique such class detected by a given pair of subgroup restrictions, then no AHSS differential can annihilate it without contradiction. Hence mixed anomalies furnish stable, testable signatures of dark-visible coupling.*

Proof. Use naturality under the two independent subgroup restrictions to BG_{vis} and BG_{dark}; any differential hitting the mixed class would have to be nontrivial on at least one restriction where the target group vanishes. □

4 Conditions to preserve $m = 7$

Theorem 4.1 (Topological triviality conditions). *If $H^2(BG_{\text{dark}};\mathbb{Z}_2) = H^3(BG_{\text{dark}};\mathbb{Z}_2) = H^4(BG_{\text{dark}};\mathbb{Z}_2) = 0$ and no central quotient of G_{dark} couples diagonally with G_{vis}, then the $p+q = 5$ AHSS diagonal for the product contains exactly the seven visible-sector survivors, and $m = 7$ is preserved.*

Proof. Künneth forbids new E_2 entries in degrees $2, 3, 4$ from the dark factor; absence of mixed quotients forbids new z_3; thus the seven visible-sector classes persist uniquely. □

Remark 4.2 (Implication). Any dark sector with nontrivial low-degree cohomology or a nontrivial central quotient *must* interact (topologically) with the visible sector, either changing m or producing mixed witnesses. Both cases are experimentally meaningful: the former shifts ρ_Λ (ruled out), the latter predicts non-gravitational portals.

5 Conclusion

Mixed-anomaly logic yields sharp, model-independent constraints: to keep $m = 7$ intact, a dark sector must be topologically trivial in low degrees and avoid central identifications with G_{vis}. Conversely, any deviation predicts either a change in parity depth (excluded by ρ_Λ) or the existence of specific mixed couplings—providing discovery targets independent of direct detection.