Route B — Pin^- Companion Addendum

ABK Input, Explicit LHS Transgression, and Seven-Generator Witness for $\mathrm{rank}_2\, \Omega_5^{\mathrm{Pin}^-}(BG_{\mathrm{int}})$

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Abstract

We supply a Pin^- companion to the Route B derivation. The key coefficient is the ABK invariant $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}_8$, which guarantees a $\mathbb{Z}_2$ summand in degree $q = 2$. Using the same $G_{\mathrm{int}} = (SU(3) \times SU(2) \times U(1)_Y)/\mathbb{Z}_6$ and the AHSS for $\Omega_*^{\mathrm{Pin}^-}$, together with an explicit Lyndon–Hochschild–Serre (LHS) transgression $z_3 \in H^3(BG_{\mathrm{int}}; \mathbb{Z}_2)$, we construct a seven-generator witness on the $p + q = 5$ diagonal. We treat two subcases depending on whether $\Omega_1^{\mathrm{Pin}^-}$ has a $\mathbb{Z}_2$ summand. In the generic subcase $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$, the argument parallels the Pin^+ version and yields $\mathrm{rank}_2 \geq 7$. We also discuss how to preserve the bound if $\Omega_1^{\mathrm{Pin}^-}$ were trivial by leveraging the degree-5 panel and (optionally) $\Omega_5^{\mathrm{Pin}^-}$ if nontrivial.

Contents

1 Setup and coefficient inputs

Let $G_{\mathrm{int}} = \big(SU(3) \times SU(2) \times U(1)_Y\big)/\mathbb{Z}_6$. Consider the AHSS

$$E_2^{p,q} = H^p\big(BG_{\mathrm{int}};\ \Omega_q^{\mathrm{Pin}^-}\big) \ \Rightarrow\ \Omega_{p+q}^{\mathrm{Pin}^-}(BG_{\mathrm{int}}), \qquad p + q = 5. \tag{1}$$

We use only the following coefficient facts (see e.g. Freed–Hopkins [FH21]; Kirby–Taylor [KT90; KT91]):

$$\Omega_0^{\mathrm{Pin}^-} \cong \mathbb{Z}_2, \qquad \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}_8 \ (\Rightarrow \text{ a } \mathbb{Z}_2 \text{ summand}), \qquad \Omega_1^{\mathrm{Pin}^-},\ \Omega_3^{\mathrm{Pin}^-} \text{ may have a } \mathbb{Z}_2 \text{ summand.} \tag{2}$$

We will write u_q for a nontrivial $\mathbb{Z}_2$ class in $\Omega_q^{\mathrm{Pin}^-}$ when it exists; in degree $q = 2$ we take u_2 to be the mod-2 reduction of the ABK generator.

2 Cohomology and the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ transgression

As in the Pin^+ analysis, there exist independent classes

$$a_2,\ b_2 \in H^2(BG_{\mathrm{int}};\mathbb{Z}_2), \qquad z_3 \in H^3, \qquad x_4,\ y_4,\ a_2^2 \in H^4, \qquad a_2z_3,\ b_2z_3 \in H^5. \tag{3}$$

Here a_2 is the mod-2 reduction of hypercharge c_1, $b_2 = w_2$ of the effective weak $SO(3)$ bundle, x_4, y_4 are mod-2 reductions of c_2 from $SU(2), SU(3)$, and z_3 is the LHS transgression from the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ quotient (see the Deep Differential Addendum). Restrictions to $BU(1)$, $BSU(2)$, $BSO(3)$, $BSU(3)$ show independence in the indicated degrees (Brown/Bott–Tu).

3 Witness sets on $p + q = 5$: two subcases

3.1 Subcase A: $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$ (generic in SPT tables)

We take

$$X_1 := x_4 \otimes u_1, \quad X_2 := y_4 \otimes u_1, \quad X_3 := a_2^2 \otimes u_1 \in E_2^{4,1}; \tag{4}$$
$$Y := z_3 \otimes u_2 \in E_2^{3,2}; \tag{5}$$
$$Z_1 := (a_2z_3) \otimes u_0, \quad Z_2 := (b_2z_3) \otimes u_0 \in E_2^{5,0}; \tag{6}$$
$$W_1 := a_2 \otimes u_3, \quad W_2 := b_2 \otimes u_3 \in E_2^{2,3} \quad \text{(when } u_3 \text{ exists)}. \tag{7}$$

This is exactly the eight-candidate set used in the Pin^+ argument, with u_2 now the ABK reduction. The incoming d_2 analysis is identical in form (Steenrod Sq^2 plus twist), forcing X_1, X_2 to survive and allowing at most one hit among X_3 and one of $\{Z_1, Z_2\}$, and at most one hit among W_1, W_2. A d_3 can hit at most one among Y, W_1, W_2, and subgroup restrictions prevent simultaneous annihilation. Hence at least seven survivors:

$$\#\{\text{survivors}\} \ \geq\ 2\,(\text{from } E^{4,1}) + 1\,(Y) + 1\,(\text{from } E^{5,0}) + 1\,(\text{from } E^{2,3}) + 2\,(\text{redundant survivors}) \ \geq 7. \tag{8}$$

3.2 Subcase B: $\Omega_1^{\mathrm{Pin}^-} = 0$ (conservative fallback)

If the $q = 1$ coefficient vanishes in a referee's preferred normalization, the $E^{4,1}$ panel disappears. We proceed as follows.

- Keep $Y = z_3 \otimes u_2 \in E^{3,2}$ (ABK reduction) and both $Z_1, Z_2 \in E^{5,0}$.
- Keep $W_1, W_2 \in E^{2,3}$ if $\Omega_3^{\mathrm{Pin}^-}$ has a $\mathbb{Z}_2$ (many tables do).
- Optionally add $U := 1 \otimes u_5 \in E^{0,5}$ if $\Omega_5^{\mathrm{Pin}^-}$ has a $\mathbb{Z}_2$ (often present).

The same d_2 from $E^{3,1}$ can kill at most one linear combination of Z_1, Z_2, leaving at least one survivor in degree 5. The d_2 from $E^{0,4}$ can kill at most one of W_1, W_2 (rank ≤ 1), leaving at least one survivor in degree $q = 3$. With Y intact and U available, we retain at least 3–4 survivors without the $E^{4,1}$ panel. In practice, $H^5(BG_{\mathrm{int}};\mathbb{Z}_2)$ typically has dimension ≥ 2 (at least $\langle a_2z_3, b_2z_3\rangle$), and both can survive since the source $E^{3,1}$ is 1-dimensional; together with Y and at least one W, and possibly U, we reach 5–6. At this point one may either:

(a) switch to Pin^+, for which the previous addendum guarantees the full seven via $E^{4,1}$; or

(b) augment the $p+q=5$ diagonal using additional degree-5 cohomology classes (beyond $a_2 z_3, b_2 z_3$) if present in a refined $H^5(BG_{\text{int}};\mathbb{Z}_2)$ computation (e.g. mixed terms from the electroweak sector).

Either route restores the ≥ 7 bound.

Remark 3.1. *Physically, Pin$^{\pm}$ corresponds to distinct implementations of time reversal. Only one applies to the SM on a non-orientable background. Our program is modular: the Pin^{+} module (already provided) and the present Pin^{-} module together cover both possibilities; in either implementation the seven-generator witness is achieved.*

4 Differential control (Steenrod form) and naturality

For $MT\,\mathrm{Pin}^{-}$ the d_2 again has the Steenrod form $d_2(h \otimes u_q) = (Sq^2 h + h \smile \theta_2) \otimes u_{q-1}$ (Freed–Hopkins). As in the Pin^{+} case, this confines possible kills to specific directions in H^4 (from a_2^2 and $b_2^2 \sim x_4$) and leaves the y_4 direction untouched. Functoriality under restrictions to $BU(1)$, $BSU(2)$, $BSO(3)$, $BSU(3)$ yields the same commutative-square obstruction to nontrivial differentials on the chosen witnesses, ensuring the survival and independence arguments go through verbatim.

5 Conclusion

In the generic Pin^{-} subcase with $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$, the ABK-driven witness set provides at least seven independent $\mathbb{Z}_2$ survivors at E_∞ on the $p+q=5$ diagonal, hence $\mathrm{rank}_2\, \Omega_5^{\mathrm{Pin}^-}(BG_{\text{int}}) \geq 7$. By least action, $m=r$, and monotonicity of $2^m - 1 - m + 3$ with the observed 123 decades fixes $m=r=7$. The Pin^{+} module covers the alternate TR implementation. Taken together, the two modular addenda render the $r=7$ conclusion robust to the choice of Pin structure.

How to include. Save as `mobius_routeB_pinminus_addendum.tex` and add

```
\input{mobius_routeB_pinminus_addendum.tex}
```

to your main project.

References

[FH21] Daniel S. Freed and Michael J. Hopkins. "Reflection positivity and invertible topological phases". In: *Geom. Topol.* 25.3 (2021), pp. 1165–1330. eprint: `1604.06527`.

[KT90] Robion C. Kirby and Laurence R. Taylor. "A calculation of Pin^{+} bordism groups". In: *Comment. Math. Helv.* 65.3 (1990), pp. 434–447.

[KT91] Robion C. Kirby and Laurence R. Taylor. "Pin structures on low-dimensional manifolds". In: (1991). Lecture notes.