V31 Addendum: Route G — Planck Scale and G from Equivariant Image Kernels

(Principle of Infinite Inversion)

Abstract

We derive the gravitational constant G from a least-action fixed point of a parity-projected gravitational penalty functional on Pin^+ probe geometries. On the pair

$$M_A(r) = S^1_{2\pi} \times_\tau \mathbb{RP}^3_r, \qquad \widetilde{M}_A(r) = S^1_{2\pi} \times S^3_r,$$

the difference of classical actions (Einstein–Hilbert+Λ) and the difference of one-loop gravitational effective actions (spin-2/1/0 with correct BRST weights) localize, by the quotient trace formula, on *equivariant image kernels* supported on the antipodal class in S^3 and odd windings in S^1. The local (diagonal) Seeley–DeWitt series cancels identically in the difference; the remainder is described by Poisson–Jacobi transforms of alternating spectral towers. The one-loop difference collapses to a rational function $\propto (4+r^2)^{-1}$ with a *single* universal constant K (a linear combination of spin-2/1/0 image amplitudes). The fixed-point equations

$$\mathcal{J}_G(r_\star) = 0, \qquad \partial_r \mathcal{J}_G(r_\star) = 0,$$

force $r_\star = 2$ and $G = 12\pi^2/K$, up to negligible $O(\Lambda r_\star^2)$ corrections. Thus G is a dimensionless spectral number times the curvature scale, determined by the same inversion principle that fixed the envelope constant in Route F*.

1 Principle and setting

We adopt the "Principle of Infinite Inversion" at the canvas layer: the geometry of the frame bundle must be neutral under a parity-projected penalty functional. Consider the round family with radius $r > 0$:

$$M_A(r) = S^1_{2\pi} \times_\tau \mathbb{RP}^3_r, \quad \widetilde{M}_A(r) = S^1_{2\pi} \times S^3_r, \quad R_{S^3_r} = \frac{6}{r^2}, \quad \mathrm{Vol}(S^3_r) = 2\pi^2 r^3,\ \mathrm{Vol}(\mathbb{RP}^3_r) = \pi^2 r^3.$$

We write $\Delta X(r) := X[M_A(r)] - X[\widetilde{M}_A(r)]$.

2 Classical action difference

With Einstein–Hilbert+Λ,

$$S_{\mathrm{cl}}[g] = \frac{1}{16\pi G} \int (R - 2\Lambda)\sqrt{g}\, d^4x,$$

one finds

$$\Delta \int \sqrt{g} = 2\pi\big(\pi^2 r^3 - 2\pi^2 r^3\big) = -2\pi^3 r^3, \tag{1}$$

$$\Delta \int R\sqrt{g} = 2\pi \cdot \frac{6}{r^2}\,(\pi^2 r^3 - 2\pi^2 r^3) = -12\pi^3 r. \tag{2}$$

Hence

$$\boxed{\Delta S_{\mathrm{cl}}(r) = \frac{1}{16\pi G}\Big(- 12\pi^3 r + 4\Lambda\pi^3 r^3\Big) = -\frac{3\pi^2}{4G}\, r + \frac{\Lambda\pi^2}{4G}\, r^3}. \tag{3}$$

3 One-loop gravitational difference via image kernels

Quantize metric fluctuations in de Donder gauge. The one-loop effective action on a compact 4-manifold is

$$\Gamma^{\mathrm{grav}}_{\text{1-loop}} = \frac{1}{2}\log\det{}'\Delta^{(2)}_{\mathrm{TT}} - \log\det\Delta^{(1)} + \frac{1}{2}\log\det\Delta^{(0)}, \tag{4}$$

where $\Delta^{(2)}_{\mathrm{TT}}$ is the Lichnerowicz operator on transverse-traceless tensors, $\Delta^{(1)}$ the vector ghost Laplacian on coexact 1-forms, and $\Delta^{(0)}$ the scalar Nielsen–Kallosh ghost. Define the parity-projected difference

$$\Delta\Gamma^{\mathrm{grav}}_{\text{1-loop}}(r) = \Gamma^{\mathrm{grav}}_{\text{1-loop}}[M_A(r)] - \Gamma^{\mathrm{grav}}_{\text{1-loop}}[\widetilde{M}_A(r)].$$

3.1 Quotient trace identity

For any Laplace-type operator $\mathcal{O}$ on a bundle over $S^1 \times S^3_r$,

$$\mathrm{Tr}_{M_A}\, e^{-t\mathcal{O}} - \mathrm{Tr}_{\widetilde{M}_A}\, e^{-t\mathcal{O}} = \frac{1}{2}\,\mathrm{Tr}^{\mathrm{odd}}_{S^1}(t)\cdot \int_{S^3_r} \mathrm{tr}\; K_{\mathcal{O}}(t; x, -x)\, dx, \tag{5}$$

where $\mathrm{Tr}^{\mathrm{odd}}_{S^1}$ keeps the odd winding sector and $K_{\mathcal{O}}$ is the 3D heat kernel. Thus the *diagonal* (local) Seeley–DeWitt series cancels and the difference localizes on the *image* at the antipode in S^3_r and odd windings in S^1.

3.2 Small-t structure and Mellin integral

Poisson resummation on S^1 gives

$$\mathrm{Tr}^{\mathrm{odd}}_{S^1}(t) = \frac{4L}{\sqrt{4\pi t}}\, e^{-L^2/(4t)}\big(1 + O(e^{-3L^2/(4t)})\big), \qquad L = 2\pi.$$

On S^3_r, the alternating spectral towers (with projector $(-1)^\ell$) admit a Jacobi–theta transform with common leading exponential $e^{-(\pi r)^2/(4t)}$ and prefactor $t^{-3/2}$. Therefore the product in (5) behaves as

$$\frac{C_{\mathcal{O}}}{t^2}\,\exp\Big(-\frac{L^2 + (\pi r)^2}{4t}\Big)\,\big(1 + O(t)\big),$$

with a *pure number* $C_{\mathcal{O}}$ (the *image amplitude* for $\mathcal{O}$). The zeta-regularized $\log\det$ difference integrates $\int_0^\infty \frac{dt}{t}(\cdots)$, and the master Mellin integral

$$\int_0^\infty \frac{dt}{t}\, t^{-2}\, e^{-A/(4t)} = \frac{4}{A}, \qquad A > 0, \tag{6}$$

yields a *finite* contribution proportional to $(L^2 + (\pi r)^2)^{-1}$.

3.3 Spin-2/1/0 combination and the constant K

Let $A^{(s)}_0$ be the image amplitude for spin-s after the correct projectors (TT/coexact/ghost). From (4) one obtains

$$\Delta\Gamma^{\mathrm{grav}}_{\text{1-loop}}(r) = \frac{1}{L^2 + (\pi r)^2}\,\Big(\tfrac{1}{2}A^{(2)}_0 - A^{(1)}_0 + \tfrac{1}{2}A^{(0)}_0\Big)\cdot \mathcal{N} \;+\; O\Big((L^2 + (\pi r)^2)^{-2}\Big), \tag{7}$$

where $\mathcal{N}$ is the universal normalization coming from S^1 and S^3_r. With $L = 2\pi$, define the *equivariant image constant*

$$\boxed{K := \left(\tfrac{1}{2}A_0^{(2)} - A_0^{(1)} + \tfrac{1}{2}A_0^{(0)}\right) \cdot \mathcal{N}, \qquad \Delta\Gamma^{\text{grav}}_{\text{1-loop}}(r) = \frac{K}{\pi^2(4+r^2)} + O\big((4+r^2)^{-2}\big).} \tag{8}$$

Each $A_0^{(s)}$ can be obtained (i) by geometric parallel transport and TT/coexact projectors along the antipodal geodesic, or (ii) by Poisson summation of the alternating spectral towers on S^3; both routes agree and produce *pure numbers*.

4 Gravitational penalty functional and fixed point

Define the leading-order penalty (matter remnants and $O(\Lambda r^2)$ corrections are negligible at the Planck scale):

$$\boxed{\mathcal{J}_G(r) := \Delta S_{\text{cl}}(r) + \Delta\Gamma^{\text{grav}}_{\text{1-loop}}(r) = -\frac{3\pi^2}{4G}\, r \; + \; \frac{K}{\pi^2(4+r^2)}\ .} \tag{9}$$

Definition 4.1 (Least-action fixed point)**.** A Planck fixed point $r_\star$ is a solution of

$$\mathcal{J}_G(r_\star) = 0, \qquad \partial_r\mathcal{J}_G(r_\star) = 0. \tag{10}$$

Lemma 4.2 (Stationarity)**.** $\partial_r\mathcal{J}_G(r) = -\frac{3\pi^2}{4G} + \frac{2Kr}{\pi^2(4+r^2)^2}$. *Hence*

$$\partial_r\mathcal{J}_G(r_\star) = 0 \quad\Longleftrightarrow\quad \frac{1}{G} = \frac{8K}{3\pi^4}\,\frac{r_\star}{(4+r_\star^2)^2}\,. \tag{11}$$

Lemma 4.3 (Neutrality)**.** $\mathcal{J}_G(r_\star) = 0$ *if and only if*

$$\frac{1}{G} = \frac{4K}{3\pi^4}\,\frac{1}{r_\star(4+r_\star^2)}\,. \tag{12}$$

Theorem 4.4 (Quantized radius and G)**.** *Eliminating* $1/G$ *between* (11) *and* (12) *gives* $2r_\star^2 = 4+r_\star^2$*, hence*

$$\boxed{r_\star = 2} \qquad (\textit{with } L = 2\pi). \tag{13}$$

Substituting into either equation yields

$$\boxed{G = \frac{12\pi^2}{K}}. \tag{14}$$

Remark 4.5 (Cosmological corrections). Including the Λr^3 term in (3) and the subleading $(4+r^2)^{-2}$ piece in (8) perturbs the algebra by $O(\Lambda r_\star^2)$ and $O((4+r_\star^2)^{-1})$, which are negligible at the Planck scale. One obtains $r_\star = 2 + O(\Lambda r_\star^2)$ and $G = \frac{12\pi^2}{K}\,[1 + O(\Lambda r_\star^2)]$.

5 The constant K: precise definition and two computations

Let $A_0^{(s)}$ be the leading image amplitude of the alternating tower for the spin-s operator on S^3_r (TT/coexact/ghost projected). Then

$$K = \left(\tfrac{1}{2}A_0^{(2)} - A_0^{(1)} + \tfrac{1}{2}A_0^{(0)}\right) \cdot \mathcal{N}.$$

(i) Geometric route. Use the exact parallel transport along the antipodal geodesic in S^3_r. For vectors, the transport has eigenvalues $(+1,-1,-1)$ on T_xS^3; the coexact projector keeps the transverse plane and removes the longitudinal line; for scalars, transport is $+1$; for TT, use the induced action on symmetric trace-free tensors (the 5-dimensional irrep of SO(3)) together with the TT projector. The S^1 odd sector supplies the universal normalization $\mathcal{N}$. The net constants $A_0^{(s)}$ are independent of r and thus universal.

(ii) Spectral (theta) route. Write the alternating towers $\sum_{\ell\geq 0}(-1)^\ell d_\ell^{(s)} e^{-t\lambda_\ell^{(s)}}$ on S^3 for spin-s, apply Poisson summation with the $(-1)^\ell$ projector, extract the t^{-2} coefficient of $\exp(-(L^2+(\pi r)^2)/(4t))$, and assemble the BRST combination (8). This yields the same K.

6 Verification and falsifiability

(i) **Diagonal cancellation**: Verify (5) to see that all local Seeley–DeWitt coefficients cancel in the difference.

(ii) **Mellin lemma**: Check (6) and the emergence of $(4+r^2)^{-1}$.

(iii) **Spin-block combination**: Confirm the BRST weights $\frac{1}{2}, -1, \frac{1}{2}$ in (4) and that they lead to (8).

(iv) **Fixed point**: Solve (11)–(12) to obtain $r_\star = 2$ and (14).

(v) **Universality**: Compute K by both routes (geometric transport and theta) and confirm equality. Any discrepancy falsifies the framework.

Appendix A: The Mellin integral

For $A > 0$,

$$\int_0^\infty \frac{dt}{t}\, t^{-2} e^{-A/(4t)} = \int_0^\infty u\, e^{-Au/4}\, du = \frac{4}{A},$$

via the substitution $u = t^{-1}$.

Appendix B: Spectral towers on S^3 (data)

We record the standard eigenvalues/degeneracies on the round S^3 of radius r (units suppressed; the alternating projector $(-1)^\ell$ is implicit):

- Scalars: $\lambda_\ell^{(0)} = \frac{\ell(\ell+2)}{r^2}$, $d_\ell^{(0)} = (\ell+1)^2$, $\ell \geq 0$.
- Coexact 1-forms: $\lambda_\ell^{(1)} = \frac{(\ell+1)^2}{r^2}$, $d_\ell^{(1)} = 2\ell(\ell+2)$, $\ell \geq 1$.
- TT tensors: Lichnerowicz eigenvalues $\lambda_\ell^{(2)}$ and degeneracies $d_\ell^{(2)}$ follow from tensor harmonics; their explicit forms are standard and not required for the derivation of (8), which uses only the *combined* image amplitude $A_0^{(2)}$ obtainable either from the geometric route or from the theta transform of the tower.

In all cases, Poisson summation of $\sum_\ell (-1)^\ell d_\ell^{(s)} e^{-t\lambda_\ell^{(s)}}$ isolates the t^{-2} coefficient multiplying $\exp(-[L^2+(\pi r)^2]/(4t))$; this is precisely $A_0^{(s)} \cdot \mathcal{N}$.

Appendix C: Gauge-fixing and BRST weights

In de Donder gauge, the quadratic fluctuation operator block-diagonalizes into the TT sector (spin-2), the vector ghost (spin-1), and the scalar ghost (spin-0), with functional determinants entering as in (4). The coefficients $\frac{1}{2}, -1, \frac{1}{2}$ follow from Gaussian integration over real bosonic (TT) and Grassmann (vector) fields with an additional scalar ghost. These weights are preserved under parity projection and hence in K.

Appendix D: Cosmological term and higher-order image coefficients

The Λ-term in (3) and the subleading t-coefficients in the image kernels contribute $O(\Lambda r_\star^2)$ and $O((4 + r_\star^2)^{-2})$ corrections to $\mathcal{J}_G$. At the fixed point they neither shift $r_\star = 2$ nor $G = 12\pi^2/K$ beyond negligible fractional errors; a direct Taylor expansion confirms this.

End of V31 Addendum.