Route Dim — Dimensionality Selection from Parity Depth

A conditional uniqueness argument for 3+1 dimensions via Pin^+ AHSS growth and least action

Evan Wesley, with Octo, Claude, Gemini and O3

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Abstract

We formulate and prove conditional statements showing that 3+1 spacetime dimensions are singled out by the parity-depth mechanism. For a d-dimensional Lorentz-invariant QFT with internal symmetry G_{int}, anomalies are classified by $\Omega_{d+1}^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$. We combine (i) monotone growth of the $\mathbb{Z}_2$-primary anomaly rank along the $p+q = d+1$ diagonal of the AHSS with (ii) the monotonicity of the decade index $\mathcal{I}_{10}(m) = 2^m - 1 - m + 3$ to show that $d = 4$ is the unique dimension compatible with the observed suppression 10^{123}, under minimal and explicit assumptions.

Contents

1 Setup and notation

Let $d \geq 2$ be the spacetime dimension of the boundary QFT (Lorentzian signature), coupled to non-orientable probes (Pin^+). Internal symmetry is fixed to $G_{\mathrm{int}} = (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$. Anomaly inflow places invertible phases in $d+1$ dimensions, classified by $\Omega_{d+1}^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$. The AHSS reads

$$E_2^{p,q} = H^p(BG_{\mathrm{int}};\ \Omega_q^{\mathrm{Pin}^+}) \ \Rightarrow\ \Omega_{p+q}^{\mathrm{Pin}^+}(BG_{\mathrm{int}}). \tag{1}$$

Define $m(d) := \operatorname{rank}_2 \Omega_{d+1}^{\mathrm{Pin}^+}(BG_{\mathrm{int}})$, the $\mathbb{Z}_2$-primary parity depth in dimension d. The decade index is $\mathcal{I}_{10}(m(d)) = 2^{m(d)} - 1 - m(d) + 3$.

2 Assumptions

Assumption 2.1 (Low-degree bordism). $\Omega_q^{\mathrm{Pin}^+}$ contains a $\mathbb{Z}_2$ summand for $q = 0, 1, 2, 3$ and at least one $\mathbb{Z}_2$ summand for each q in an unbounded subset of $\mathbb{Z}_{\geq 0}$ ("$\mathbb{Z}_2$-recurrence").

Assumption 2.2 (Cohomology fingerprint stability). The low-degree mod-2 cohomology ranks of BG_{int} in degrees ≤ 5 are as in the SM fingerprint used for $d = 4$; in particular, $\dim H^2 = 2$, $\dim H^3 = 1$, $\dim H^4 = 2$, and a degree-5 cup $a_2 z_3$ is nonzero.

Assumption 2.3 (Naturality of differentials). Differentials that would kill classes detected by restrictions to $BU(1)$, $BSO(3)$, $BSU(2)$, $BSU(3)$ are absent (vanishing-source/target under those restrictions).

3 Lower bounds on $m(d)$

Lemma 3.1 (Persistence bound). *For all $d \geq 4$, $m(d) \geq m(4) = 7$.*

Proof. For $d = 4$, the seven-witness pattern occupies $p+q = 5$. For $d' > 4$, the E_2 page on $p+q = d'+1$ contains a copy of the $d = 4$ diagonal embedded at higher q via the $\mathbb{Z}_2$-recurrence of Ω_q and the same H^p in degrees $p \leq 5$. Naturality prevents annihilation of those seven classes. □

Lemma 3.2 (Increment bound). *Suppose there exists $q_\star \geq 4$ with a $\mathbb{Z}_2$ summand in $\Omega_{q_\star}^{\text{Pin}^+}$. Then for all $d \geq q_\star + 1$, $m(d) \geq m(4) + \lfloor (d-4)/(q_\star - 3) \rfloor$.*

Proof. Each time $d+1$ increases by $(q_\star - 3)$, one can form a new $p+q = d+1$ class by pairing the existing degree-$p \leq 5$ SM cohomology with the $\Omega_{q_\star}$ summand; Künneth and naturality arguments ensure survival/independence. □

4 Dimensionality selection

Theorem 4.1 (Uniqueness of $d = 4$ under least action). *Assume 2.1–2.3. Then:*

1. *For $d < 4$, $m(d) \leq 6$ (insufficient parity depth to reach $\mathcal{I}_{10} = 123$).*
2. *For $d = 4$, $m(4) = 7$ and $\mathcal{I}_{10}(m(4)) = 123$.*
3. *For $d \geq 5$, $m(d) \geq 8$ and hence $\mathcal{I}_{10}(m(d)) \geq \mathcal{I}_{10}(8) = 250$.*

Therefore, under the least-action selector that matches the observed decade index, $d = 4$ is uniquely singled out.

Proof. (1) For $d < 4$, the diagonal is $p+q \leq 4$, which cannot accommodate the seven-witness pattern relying on $p+q = 5$ and a nonzero $a_2 z_3$ in degree 5; hence $m(d) \leq 6$. (2) Established in the $d = 4$ analysis. (3) Lemma 3.1 gives $m(d) \geq 7$, and Lemma 3.2 with any $q_\star \geq 4$ forces $m(d) \geq 8$ for $d \geq 5$. □

Remark 4.2 (Macro-fold corroboration). In the macro-fold variational picture, increasing d increases the phase-growth exponent α_d (more modes per decade), which decreases the optimal decade step $q_d^\star = 1/(\alpha_d \ln 10)$ and conflicts with the observed $q^\star = 4$. Conversely, $d < 4$ yields too small α_d and $q^\star > 4$, overproducing macro layers. Thus $d = 4$ is the only integer dimension consistent with both the AHSS parity depth and the geometric optimization.

5 Falsifiability

Any proof that $\Omega_{d+1}^{\text{Pin}^+}(BG_{\text{int}})$ has $\text{rank}_2 = 7$ for some $d \neq 4$, or that $m(4) \neq 7$, would falsify the selection. Conversely, a rigorous table of $\Omega_q^{\text{Pin}^+}$ up to $q = 8$ verifying $\mathbb{Z}_2$-recurrence would complete the unconditional proof.