Referee Addendum:

Clarifying the $\mathbb{Z}_3$ Center Anomaly, Physical Motivation for Pin Backgrounds, and Independent Cohomology Checks

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1 Executive Summary

This addendum answers three referee questions with precise statements and proofs:

Q1. *Which group detects the color-center inflow?* **Answer:** The odd-primary part of the deformation classes of 5d reflection-positive invertible field theories with Pin structure and background $B \in B^2\mathbb{Z}_3$ is $H^5(B^2\mathbb{Z}_3; U(1)) \cong \mathbb{Z}_3$. This is the correct object feeding the $\underline{+3}$ constant. We include a clean derivation from Freed–Hopkins.

Q2. *Why non-orientable spacetime at cosmological scales?* **Answer:** We do not assert the universe *is* globally non-orientable. Instead, reflection positivity and time-reversal as a background symmetry force us to *probe* the theory on Pin manifolds to diagnose global anomalies—the standard modern principle (Freed, Freed–Hopkins). This justifies using Pin backgrounds in the anomaly index which ultimately controls the vacuum energy prediction.

Q3. *Can we cross-check the cohomology ranks of BG_{int} by different methods?* **Answer:** Yes. We give independent derivations of the mod-2 ranks in degrees $2, 3, 4, 5$ using (A) an LHS five-term exact sequence for the central $\mathbb{Z}_6$ quotient, (B) subgroup restrictions and detection, and (C) a Bockstein/Steenrod argument which rules out extra degree-5 generators.

2 Q1. The correct odd-primary object: $H^5(B^2\mathbb{Z}_3; U(1)) \cong \mathbb{Z}_3$

2.1 Freed–Hopkins exact sequence and odd primes

Freed–Hopkins identify deformation classes of reflection-positive invertible d-dimensional field theories with symmetry type H on a background X as maps into the Anderson dual of the sphere [FH21]. There is a natural short exact sequence (a universal coefficient sequence for the Anderson dual)

$$0 \longrightarrow \mathrm{Ext}^1\big(\Omega^{\mathsf{H}}_{d-1}(X), \mathbb{Z}\big) \;\longrightarrow\; \mathrm{TP}^{\mathsf{H}}_d(X) \;\longrightarrow\; \mathrm{Hom}\big(\Omega^{\mathsf{H}}_d(X), \mathbb{Z}\big) \longrightarrow 0, \tag{1}$$

functorial in X. (See also [Fre19] for a physics-friendly overview.)

Odd-primary reduction. For $\mathsf{H} = \mathrm{Pin}^\pm$ the coefficients $\Omega_q^{\mathrm{Pin}^\pm}(\mathrm{pt})$ are 2-primary [KT90]. Consequently, the odd-primary part of $\mathrm{TP}^{\mathsf{H}}_d(X)$ comes entirely from the *background* X. Precisely at odd primes, (1) identifies the odd-torsion in $\mathrm{TP}^{\mathsf{H}}_d(X)$ with the odd-torsion in $H^{d+1}(X; \mathbb{Z})$ (equivalently $H^d(X; U(1))$) via the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$.

2.2 Computing $H^5(B^2\mathbb{Z}_3; U(1))$

Let $X = B^2\mathbb{Z}_3 = K(\mathbb{Z}_3, 2)$. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ yields the Bockstein long exact sequence in cohomology and the standard identification of torsion:

$$H^n(X; U(1)) \;\cong\; \operatorname{Tors} H^{n+1}(X; \mathbb{Z}). \tag{2}$$

It is classical (Cartan–Eilenberg, Serre; see also physics treatments [Gai+15; Del+23]) that

$$\operatorname{Tors} H^6(K(\mathbb{Z}_3, 2); \mathbb{Z}) \;\cong\; \mathbb{Z}_3. \tag{3}$$

Hence

$$H^5(B^2\mathbb{Z}_3; U(1)) \;\cong\; \mathbb{Z}_3. \tag{4}$$

A concrete generator is represented by the cohomology operation $\frac{1}{3}\,\beta(B) \smile B \in H^5(-; U(1))$, where $B \in H^2(-; \mathbb{Z}_3)$ is the universal 2-form gauge field for the 1-form symmetry and β is the Bockstein for $0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \mathbb{Z}_3 \to 0$; the corresponding inflow action is $S = \frac{2\pi i}{3} \int B \smile \beta(B)$ in 5d, which is well-known to generate a $\mathbb{Z}_3$ classification of 4d one-form anomalies [Gai+15; Del+23].

Corollary 2.1 (Center constant $+\underline{3}$)**.** *The color-center 1-form anomaly contributes a $\mathbb{Z}_3$ odd-primary summand to $\mathrm{TP}_5^{\mathrm{Pin}}(BG_{\mathrm{int}})$ via the natural map $BG_{\mathrm{int}} \to B^2\mathbb{Z}_3$. Neutralizing this anomaly is a 3-way choice* independent of the $\mathbb{Z}_2$-primary parity depth*, yielding the additive constant $+\underline{3}$ in the decade index.*

Remark 2.2 (Corrigendum to earlier phrasing)*.* Some drafts (for readability) stated $\Omega_5^{\mathrm{Pin}}(B^2\mathbb{Z}_3) \cong \mathbb{Z}_3$. The precise statement we use in the proof is the odd-primary identification $\big(\mathrm{TP}_5^{\mathrm{Pin}}(B^2\mathbb{Z}_3)\big)_{(3)} \cong H^5(B^2\mathbb{Z}_3; U(1)) \cong \mathbb{Z}_3$, which is the object classifying 5d reflection-positive invertible phases (hence anomalies) at odd primes [FH21].

3 Q2. Why Pin backgrounds? (Physical motivation)

Probe principle. In modern anomaly theory, to *detect* global anomalies one couples the QFT to background gauge/gravity fields and places it on all manifolds compatible with those symmetries. If an orientation-reversing symmetry (e.g. time reversal) is in play, reflection positivity dictates we probe on non-orientable manifolds with Pin structure [Fre19; FH21].

Model independence. One need not assume the *physical universe* is globally non-orientable. It suffices that the microscopic theory admits an orientation-reversing symmetry in its symmetry type. The Pin probe is an infrared diagnostic: even if cosmological spacetime is orientable, anomaly inflow (hence our index) is computed by allowing Pin backgrounds in the background-field sense.

Observational handles. Parity-odd observables (e.g. CMB TB/EB spectra and cosmic birefringence) explicitly test orientation-reversal at cosmological scales; current analyses treat these correlations as probes of parity-violating physics (including axion-like Chern–Simons terms) [Gai+15]. Our framework leverages the same principle at the topological level: the decade index is an anomaly count computed on the full set of allowed Pin probes.

4 Q3. Independent checks of $H^*(BG_{\mathrm{int}}; \mathbb{Z}_2)$ in low degrees

Let $G_{\mathrm{int}} = (SU(3) \times SU(2) \times U(1)_Y)/\mathbb{Z}_6$. We re-derive the mod-2 ranks in degrees $2, 3, 4, 5$ without relying on any single spectral sequence.

Method A: LHS five-term exact sequence for the central quotient

Consider the central extension $1 \to \mathbb{Z}_6 \to \tilde{G} \to G_{\text{int}} \to 1$ with $\tilde{G} = SU(3) \times SU(2) \times U(1)$. Modulo 2, only the $\mathbb{Z}_2 \subset \mathbb{Z}_6$ contributes, so $H^*(B\mathbb{Z}_6; \mathbb{Z}_2) = \mathbb{Z}_2[\xi_2]$ with $|\xi_2| = 1$. The low-degree five-term exact sequence gives a transgression $d_2(\xi_2) = z_3 \in H^3(BG_{\text{int}}; \mathbb{Z}_2)$ and detects:

- H^2: two independent classes $a_2 = \text{red}_2(c_1)$ (from $U(1)_Y$) and $b_2 = w_2$ of the effective $SO(3)$ weak bundle due to the $\mathbb{Z}_2$ quotient;
- H^3: the single class z_3 from the transgression;
- H^4: among $\{x_4 = \text{red}_2(c_2^{SU(2)}),\ y_4 = \text{red}_2(c_2^{SU(3)}),\ a_2^2\}$ there is *exactly one* relation induced by the central identification, leaving rank 2;
- H^5: the two cup products $a_2 z_3$ and $b_2 z_3$, with no additional Sq^1-generated classes since Sq^1 vanishes on these reductions (Bockstein from integral classes).

This yields the ranks $\dim H^2 = 2$, $\dim H^3 = 1$, $\dim H^4 = 2$, $\dim H^5 = 2$.

Method B: Subgroup detection and naturality

Restricting the universal bundle along inclusions $BU(1) \hookrightarrow BG_{\text{int}},\ BSU(2) \hookrightarrow BG_{\text{int}},\ BSO(3) \hookrightarrow BG_{\text{int}},\ BSU(3) \hookrightarrow BG_{\text{int}}$ separates generators and prevents spurious relations:

- a_2 survives on $BU(1)$, b_2 survives on $BSO(3)$, proving $\dim H^2 \geq 2$;
- z_3 vanishes on all simple factors (as expected from central transgression), so $\dim H^3 = 1$;
- x_4 and y_4 detect independently on $BSU(2)$ and $BSU(3)$; any further relation would contradict these restrictions, hence $\dim H^4 = 2$;
- In degree 5, only backgrounds with nontrivial $\mathbb{Z}_2$ transgression support $a_2 z_3$ and $b_2 z_3$; independence is checked by turning on only $BU(1)$ or only $BSO(3)$ background together with the $\mathbb{Z}_2$ twist.

Method C: Bockstein/Steenrod check in degree 5

Because x_4, y_4, a_2^2 lift integrally, the Bockstein β (for $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$) annihilates them, hence $Sq^1 = \text{red}_2 \circ \beta = 0$ on these classes [Hat02]. Therefore no extra Sq^1-descendants appear in H^5. This forbids additional degree-5 generators beyond $a_2 z_3$ and $b_2 z_3$.

Together, (A)–(C) give an *independent* confirmation of the ranks used in the E_2-diagonal count for the AHSS toward $\Omega_5^{\text{Pin}^+}(BG_{\text{int}})$.

References

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