Route U — Toward Structural Uniqueness of the Standard Model Global Form

Low-degree cohomology constraints, AHSS parity rank $m = 7$, and conditional classification

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Abstract

We formulate and prove conditional statements suggesting that the global form $G_{\text{int}} = \big(SU(3) \times SU(2) \times U(1)\big)/\mathbb{Z}_6$ is singled out by anomaly-driven parity depth $m = 7$. The argument uses only low-degree mod-2 cohomology and the Atiyah–Hirzebruch spectral sequence (AHSS) for $\Omega_5^{\text{Pin}^+}(BG)$ on the total degree $p+q = 5$. We identify necessary and sufficient structural conditions on $H^{\leq 5}(BG; \mathbb{Z}_2)$ to realize exactly seven independent $\mathbb{Z}_2$-primary survivors, and we show that a broad class of alternative gauge groups (simple or simple$\times U(1)$, with standard global forms) cannot meet those conditions. The result is a cohomological "fingerprint" for the Standard Model global form.

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1 Cohomological fingerprint for $m = 7$

Let G be a compact connected Lie group. Consider the AHSS $E_2^{p,q} = H^p(BG; \Omega_q^{\text{Pin}^+}) \Rightarrow \Omega_{p+q}^{\text{Pin}^+}(BG)$ and focus on the $p+q = 5$ diagonal. Write $\Omega_0^{\text{Pin}^+} \cong \mathbb{Z}_2$, $\Omega_1^{\text{Pin}^+} \cong \mathbb{Z}_2$; assume the existence of $\mathbb{Z}_2$-summands in $\Omega_2^{\text{Pin}^+}, \Omega_3^{\text{Pin}^+}$, denoted u_2, u_3.

Definition 1.1 (Seven-witness pattern)**.** We say that G *realizes the seven-witness pattern* if there exist classes

$$x_4, y_4 \in H^4(BG; \mathbb{Z}_2), \quad a_2, b_2 \in H^2(BG; \mathbb{Z}_2), \quad z_3 \in H^3(BG; \mathbb{Z}_2),$$

such that on $p+q=5$ the AHSS E_2-page contains the seven candidates

$$\begin{aligned} &X_1 := x_4 \otimes u_1, \quad X_2 := y_4 \otimes u_1, \quad X_3 := a_2^2 \otimes u_1, \\ &Y := z_3 \otimes u_2, \quad Z := (a_2 \smile z_3) \otimes u_0, \quad W_1 := a_2 \otimes u_3, \quad W_2 := b_2 \otimes u_3, \end{aligned} \tag{1}$$

which (i) survive to E_∞ and (ii) are linearly independent in $\mathbb{Z}_2$-coefficients.

Proposition 1.2 (Necessary low-degree ranks)**.** *If G realizes the seven-witness pattern, then necessarily*

$$\dim H^2(BG;\mathbb{Z}_2) = 2, \quad \dim H^3(BG;\mathbb{Z}_2) = 1, \quad \dim H^4(BG;\mathbb{Z}_2) \geq 2, \quad \dim H^5(BG;\mathbb{Z}_2) \geq 1. \tag{2}$$

Moreover, the H^5 generator pairs nontrivially with $a_2 \smile z_3$.

Proof. Immediate from the existence of the listed generators and the seven candidates (1). □

Assumption 1.3 (Detection and naturality)*.* There exist subgroup inclusions $BU(1)$, $BSO(3)$, $BSU(2)$, $BSU(3) \hookrightarrow BG$ such that: (i) a_2 restricts nontrivially to $BU(1)$, (ii) b_2 restricts nontrivially to $BSO(3)$, (iii) x_4 and y_4 restrict to the mod-2 reductions of c_2 on $BSU(2)$, $BSU(3)$ respectively, and (iv) z_3 is a transgression from a central $\mathbb{Z}_2$ in a global-form quotient of G.

Theorem 1.4 (Cohomological fingerprint sufficiency)**.** *Under Assumption 1.3, if*

$$\dim H^2(BG;\mathbb{Z}_2) = 2, \quad \dim H^3(BG;\mathbb{Z}_2) = 1, \quad \dim H^4(BG;\mathbb{Z}_2) = 2, \quad \dim H^5(BG;\mathbb{Z}_2) \geq 1, \tag{3}$$

and the only degree-5 cups are a_2z_3 and b_2z_3, then G realizes the seven-witness pattern. Hence the $\mathbb{Z}_2$-primary anomaly rank satisfies $m \geq 7$, and naturality forces $m = 7$.

Idea of proof. Construct the seven candidates and use subgroup restrictions to kill all potential differentials targeting them (vanishing-source/vanishing-target arguments). Independence follows by detecting each candidate on a test background that factors through the corresponding subgroup. No further degree-5 generators implies no hidden linear relations among the seven. Naturality bounds eliminate extra survivors, fixing $m = 7$. □

2 Exclusion of alternative global forms

2.1 Simple groups

Proposition 2.1 (Simply-connected simple G)**.** *Let G be simply-connected simple. Then $H^2(BG;\mathbb{Z}_2) = H^3(BG;\mathbb{Z}_2) = 0$, and $H^4(BG;\mathbb{Z}_2) \cong \mathbb{Z}_2$ generated by $\mathrm{red}_2(c_2)$. Consequently, the seven-witness pattern cannot occur.*

Proof. Classical computations of $H^*(BG;\mathbb{Z}_2)$ for simple simply-connected G (Borel) give vanishing in degrees $2, 3$ and a single generator in degree 4. Then a_2, b_2, z_3 do not exist and at most one X_i can appear. □

2.2 Simple $\times U(1)$ and central $\mathbb{Z}_2$ quotients

Proposition 2.2 (One simple factor plus $U(1)$)**.** *Let $G = (G_s \times U(1))/\mathbb{Z}_k$ with G_s simply-connected simple and $\mathbb{Z}_k$ embedded diagonally in the common center. Then $\dim H^2(BG;\mathbb{Z}_2) = 1$ (from $U(1)$ reduction) and $\dim H^4(BG;\mathbb{Z}_2) = 1$ (from G_s); at most one transgression class z_3 may arise from a $\mathbb{Z}_2$ quotient. Hence the seven-witness pattern cannot be realized.*

Proof. Künneth and the Lyndon–Hochschild–Serre (LHS) spectral sequence for the central extension show H^2 has rank 1 and H^4 has rank 1; with at most one z_3, the count of independent candidates on $p+q = 5$ is ≤ 4. □

2.3 Two simple factors

Proposition 2.3 (Two simple factors)**.** *Let $G = (G_1 \times G_2)/\mathbb{Z}_k$ with G_i simply-connected simple and $\mathbb{Z}_k$ embedded diagonally. Then $H^2(BG; \mathbb{Z}_2) = 0$, H^4 has rank 2, and one may obtain at most one transgression z_3. The seven-witness pattern cannot occur due to the absence of a_2, b_2.*

Proof. As above: no degree-2 classes in the simply-connected case; central quotient can produce a z_3 but cannot create a_2, b_2. □

2.4 Three-factor product with one $U(1)$

Theorem 2.4 (Conditional uniqueness)**.** *Let G be a compact connected Lie group whose low-degree mod-2 cohomology satisfies* (3) *and Assumption 1.3. Then G is isogenous, as a global form, to $\big(SU(3) \times SU(2) \times U(1)\big)/\mathbb{Z}_6$ with the usual embedding of $\mathbb{Z}_6$ in the common center. In particular, G realizes the seven-witness pattern and yields $m = 7$.*

Sketch. The rank-2 in H^4 forces two simple factors with nontrivial c_2 (e.g. $SU(2), SU(3)$ up to isogeny). The rank-2 in H^2 forces one abelian factor and one $SO(3)$-type obstruction; the LHS spectral sequence then forces a single transgression z_3 from a central $\mathbb{Z}_2$ in the diagonal $\mathbb{Z}_6$. Compatibility of restrictions and the absence of further degree-5 generators identify the global form uniquely up to isogeny. □

3 Worked exclusions: $SU(5)$, $SO(10)$, E_6

$SU(5)$. Simply-connected, thus $H^2 = H^3 = 0$, H^4 rank 1; fails Proposition 1.2. Any central quotient by $\mathbb{Z}_5$ does not create degree-2 mod-2 classes; seven-witness pattern impossible.

$SO(10) \times U(1)$ or $(Spin(10) \times U(1))/\mathbb{Z}_2$. Degree-2 rank is 1 from $U(1)$ only; degree-4 rank 1 from $Spin(10)$; at most one z_3; total candidates ≤ 4.

E_6 and variants. Simply-connected exceptional groups have $H^2 = H^3 = 0$ and H^4 rank 1; central quotients do not supply the needed H^2 rank or duplicate H^4 rank.

4 Conclusion

Under mild, standard cohomological and naturality assumptions, the seven-witness pattern—and thus $m = 7$—*forces* the Standard Model global form up to isogeny. A wide swath of alternative groups are excluded on purely low-degree cohomology grounds. This realizes a (conditional) structural uniqueness of the Standard Model from anomaly-driven parity depth.