Mathematical Appendix: EB/TB Angle Estimation without Code

Exact rotation of power spectra, unbiased estimators, variance/weights, self-calibration bias, and cross-spectrum disentangling

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Abstract

This appendix gives a *code-free* derivation of the EB/TB-based angle estimation formalism used in the external validation plan. We (i) derive the *exact* transformation laws of TT, EE, BB, TE, EB, TB under a uniform polarization rotation α, (ii) construct *unbiased*, small-angle estimators $\hat{\alpha}_{EB}, \hat{\alpha}_{TB}$, (iii) compute their *variances* and *optimal weights*, (iv) prove the *self-calibration bias* (why enforcing EB=0 absorbs a real cosmic rotation), and (v) show how *cross-spectra* between two experiments disentangle sky rotation from instrument angles in a linear algebraic way. We also state the pseudo-C_ℓ mixing-matrix correction needed for publication-grade analyses.

Contents

1 Preliminaries and notation

Let Q, U be Stokes fields and $P := Q + iU$. A uniform rotation by α acts as

$$P' = e^{2i\alpha} P, \qquad \begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix}. \tag{1}$$

Denote by E and B the scalar/pseudoscalar polarization fields obtained from Q, U via the standard spin-2 harmonics. A uniform rotation acts as

$$\begin{pmatrix} E' \\ B' \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}. \tag{2}$$

We use isotropic power spectra C_ℓ^{XY} for $X, Y \in \{T, E, B\}$ and write $f_{\rm sky}$ for the effective sky fraction.

2 Exact rotation of power spectra

Theorem 2.1 (Exact transformation laws)**.** *Under a uniform rotation by α, the power spectra transform as*

$$\begin{aligned}
C_\ell^{EE'} &= C_\ell^{EE}\cos^2 2\alpha + C_\ell^{BB}\sin^2 2\alpha - C_\ell^{EB}\sin 4\alpha, &(3)\\
C_\ell^{BB'} &= C_\ell^{EE}\sin^2 2\alpha + C_\ell^{BB}\cos^2 2\alpha + C_\ell^{EB}\sin 4\alpha, &(4)\\
C_\ell^{EB'} &= \tfrac{1}{2}\,(C_\ell^{BB} - C_\ell^{EE})\sin 4\alpha + C_\ell^{EB}\cos 4\alpha, &(5)\\
C_\ell^{TE'} &= C_\ell^{TE}\cos 2\alpha - C_\ell^{TB}\sin 2\alpha, &(6)\\
C_\ell^{TB'} &= C_\ell^{TE}\sin 2\alpha + C_\ell^{TB}\cos 2\alpha. &(7)
\end{aligned}$$

Proof. Apply (2) to harmonic coefficients $(E_{\ell m}, B_{\ell m})$ and average over m, using bilinearity and the definitions $C_\ell^{XY} = \frac{1}{2\ell+1}\sum_m X_{\ell m} Y^*_{\ell m}$. The T field is unchanged by polarization rotation, hence (6)–(7) follow from rotating only E, B. □

Remark 2.2 (Small-angle limits). If the fiducial cosmology has $C_\ell^{EB} = C_\ell^{TB} = 0$ and $C_\ell^{BB} \ll C_\ell^{EE}$, then for $|\alpha| \ll 1$ (radians)

$$C_\ell^{EB'} \simeq 2\alpha\, C_\ell^{EE}, \qquad C_\ell^{TB'} \simeq 2\alpha\, C_\ell^{TE}, \tag{8}$$

to first order in α. Sign conventions are fixed by (2).

3 EB/TB angle estimators: unbiasedness and optimal weights

Define weighted estimators

$$\hat\alpha_{EB} = \frac{\sum_\ell w_\ell\, C_\ell^{EB'}}{2\sum_\ell w_\ell\, C_\ell^{EE}}, \qquad \hat\alpha_{TB} = \frac{\sum_\ell w_\ell\, C_\ell^{TB'}}{2\sum_\ell w_\ell\, C_\ell^{TE}}. \tag{9}$$

Proposition 3.1 (Unbiased to first order)**.** *Assume the fiducial (unrotated) spectra satisfy $C_\ell^{EB} = C_\ell^{TB} = 0$. Then, for $|\alpha| \ll 1$, $\mathbb{E}[\hat\alpha_{EB}] = \mathbb{E}[\hat\alpha_{TB}] = \alpha + O(\alpha^3)$.*

Proof. Insert the small-angle limits of Remark 2.2 into (9); numerator and denominator share the same weights w_ℓ, hence the factors of C_ℓ^{EE} (or C_ℓ^{TE}) cancel. □

Proposition 3.2 (Variance and optimal weights)**.** *Under the Gaussian approximation, the variances of binned spectra satisfy*

$$\mathrm{Var}(C_\ell^{EB'}) \simeq \frac{(C_\ell^{EE'} + N_\ell^{EE})(C_\ell^{BB'} + N_\ell^{BB}) + (C_\ell^{EB'})^2}{(2\ell+1) f_{\rm sky}}, \tag{10}$$

$$\mathrm{Var}(C_\ell^{TB'}) \simeq \frac{(C_\ell^{TT} + N_\ell^{TT})(C_\ell^{BB'} + N_\ell^{BB}) + (C_\ell^{TB'})^2}{(2\ell+1) f_{\rm sky}}. \tag{11}$$

Then

$$\mathrm{Var}(\hat\alpha_{EB}) = \frac{\sum_\ell w_\ell^2\, \mathrm{Var}(C_\ell^{EB'})}{4\left(\sum_\ell w_\ell C_\ell^{EE}\right)^2}, \qquad \mathrm{Var}(\hat\alpha_{TB}) = \frac{\sum_\ell w_\ell^2\, \mathrm{Var}(C_\ell^{TB'})}{4\left(\sum_\ell w_\ell C_\ell^{TE}\right)^2}. \tag{12}$$

The variance-minimizing *weights are*

$$w_\ell^{(EB)} \propto \frac{C_\ell^{EE}}{\mathrm{Var}(C_\ell^{EB'})}, \qquad w_\ell^{(TB)} \propto \frac{C_\ell^{TE}}{\mathrm{Var}(C_\ell^{TB'})}, \tag{13}$$

up to an overall normalization.

Proof. Equation (12) follows by linear error propagation and independence of ℓ-modes in the Gaussian approximation. Minimization of a Rayleigh quotient yields (13). □

4 Instrument angle degeneracy and the self-calibration bias

Let the instrument polarization angle be offset by δ. The measured rotation is $\alpha_{\mathrm{meas}} = \alpha + \delta$.

Proposition 4.1 (Self-calibration absorbs cosmic rotation)**.** *If one* defines *the instrument angle by enforcing $C_\ell^{EB'} \approx 0$ on the sky, then the calibration solution is $\hat{\delta} \simeq -\alpha$ (to first order), i.e. the procedure* absorbs *any real cosmic birefringence.*

Proof. With an unknown δ, small-angle EB is $C_\ell^{EB'} \simeq 2(\alpha+\delta)\, C_\ell^{EE}$. Setting this to zero forces $\delta = -\alpha$. □

Remark 4.2. Therefore, absolute-angle calibration must come from *external* calibrators or cross-experiment comparisons; one must not impose EB/TB nulls when searching for cosmic rotation.

5 Two-experiment cross-spectra: disentangling α from δ_A, δ_B

Let experiments A, B have instrument offsets δ_A, δ_B. Consider cross-spectra built from (T_A, E_A, B_A) and (T_B, E_B, B_B).

Lemma 5.1 (Linearized cross-spectrum system)**.** *To first order in small angles and neglecting primordial EB, TB,*

$$C_{\ell,AB}^{EB'} \simeq (\alpha + \delta_A + \alpha + \delta_B)\, C_\ell^{EE} \;=\; (2\alpha + \delta_A + \delta_B)\, C_\ell^{EE}, \tag{14}$$

$$C_{\ell,AB}^{TB'} \simeq (\alpha + \delta_B)\, 2\, C_\ell^{TE}, \qquad C_{\ell,BA}^{TB'} \simeq (\alpha + \delta_A)\, 2\, C_\ell^{TE}. \tag{15}$$

Proof. Each map's E, B is rotated by its own net angle $(\alpha + \delta_{A/B})$; the cross EB uses E_A with B_B, hence it depends on the sum. TB depends on the rotation of the B-map only. □

Proposition 5.2 (Solving for $\alpha, \delta_A, \delta_B$)**.** *Given measurements of $C_{\ell,AB}^{EB'}, C_{\ell,AB}^{TB'}, C_{\ell,BA}^{TB'}$ (appropriately weighted over ℓ), the linear system in Lemma 5.1 solves uniquely for $\alpha, \delta_A, \delta_B$. In particular,*

$$\hat{\alpha} = \tfrac{1}{2}\left(\frac{C_{AB}^{EB'}}{C^{EE}} - \frac{C_{AB}^{TB'} + C_{BA}^{TB'}}{2\, C^{TE}}\right), \qquad \hat{\delta}_A = \frac{C_{BA}^{TB'}}{2\, C^{TE}} - \hat{\alpha}, \qquad \hat{\delta}_B = \frac{C_{AB}^{TB'}}{2\, C^{TE}} - \hat{\alpha}, \tag{16}$$

where we omit the ℓ-weighting symbols for clarity.

Remark 5.3. With more than two experiments, the system is overdetermined and can be solved by least squares, providing robustness to residual systematics.

6 Pseudo-C_ℓ mixing and publication-grade correction

On a masked sky, pseudo-spectra $\tilde{C}_\ell^{XY}$ are related to true spectra by a coupling matrix M:

$$\tilde{C}_\ell^{XY} = \sum_{\ell'} M_{\ell\ell'}^{XY} C_{\ell'}^{XY} \quad (+\text{ leakage terms between } E \leftrightarrow B \text{ if the mask breaks parity}). \tag{17}$$

The MASTER procedure inverts this mixing (with binning and regularization) to yield unbiased estimates. For the EB/TB estimators, one must replace $C_\ell^{EB'}, C_\ell^{EE}, C_\ell^{TB'}, C_\ell^{TE}$ in (9) by their MASTER-corrected counterparts. In the *quick-look* regime, generous masks and identical binning for numerators/denominators minimize residual bias, but publication results should use the full coupling correction.

7 Summary

The above derivations provide a *self-contained*, mathematical treatment of EB/TB angle estimation. It establishes exact rotation laws, unbiased estimators, optimal weighting, the self-calibration pitfall, and cross-spectrum strategies for disentangling sky rotation from instrument angle offsets.