Routes S + C Unified Pack

A Discrete Compiler for Parity Rank and a Design Calculus for Vacuum-Energy Stability

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August 5, 2025

Abstract

We present a unified framework consisting of (i) a finite-cell "discrete compiler" that reproduces the seven-witness anomaly pattern on Pin^+ backgrounds (parity rank $m = 7$) and (ii) a design calculus that determines how m and the decade index $\mathcal{I}_{10}(m) = 2^m - 1 - m + 3$ change under controlled modifications of the internal symmetry. The compiler supplies a minimal CW model whose cohomology and cup products realize the anomaly fingerprint; the calculus provides rigorous change rules for adding factors, taking central quotients, and related moves. Together, they establish robustness of $m = 7$ and explain the instability of $\mathcal{I}_{10}$ under most extensions, thereby supporting the observed value $\mathcal{I}_{10} = 123$.

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Executive Summary

Goal. Exhibit a compact, verifiable foundation for parity rank $m = 7$ and derive model-independent rules governing how m changes under symmetry modifications.

Main statements.

- **Discrete compiler (Route S).** There exists a minimal finite CW complex X with nonzero cohomology only in degrees $2, 3, 4, 5$ and with prescribed cup products such that, upon coupling to the Standard Model global form $G_{\text{int}} = (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$, the $p+q = 5$ diagonal of the Pin^+ AHSS contains seven linearly independent survivors (the "seven witnesses") that match the anomaly fingerprint yielding $m = 7$. Survival is proved by subgroup detection and naturality; independence by test backgrounds that pair uniquely with each witness.

- **Design calculus (Route C).** For any symmetry move that injects new mod-2 classes into degrees $2, 3, 4$ of $H^*(BG; \mathbb{Z}_2)$, the $p+q = 5$ diagonal gains new candidates that survive by naturality, implying $\Delta m \geq 1$ (or ≥ 2 for diagonal central $\mathbb{Z}_2$ quotients introducing transgressions). At $m = 7$, this generically shifts $\mathcal{I}_{10}$ by at least 126 decades, far from observation.

Key assumptions (made explicit).

1. Low-degree bordism coefficients $\Omega_q^{\mathrm{Pin}^+}$ contain $\mathbb{Z}_2$ summands for $q = 0, 1, 2, 3$ (standard).
2. Subgroup naturality: background restrictions to $BU(1)$, $BSO(3)$, $BSU(2)$, $BSU(3)$ detect the relevant low-degree classes.
3. No contrived differentials: differentials that would kill detected classes are forbidden by vanishing-source/vanishing-target arguments on subgroup restrictions.

Consequences.

- *Rigidity:* Preserving $m = 7$ essentially requires preserving low-degree mod-2 cohomology of the classifying space.
- *Instability of $\mathcal{I}_{10}$:* Any $\Delta m \geq 1$ at $m = 7$ induces $\Delta\mathcal{I}_{10} \geq 126$, incompatible with the measured $\mathcal{I}_{10} \approx 123$.
- *Falsifiability:* A verified extension that injects new degree-2/3/4 mod-2 classes yet maintains $\mathcal{I}_{10} \approx 123$ would refute the mechanism.

Referee verification checklist.

1. Check that X in Part I has H^2 rank 2, H^3 rank 1, H^4 rank 2, H^5 rank 1 and cup products $a_2^2 \neq 0$, $a_2 \smile z_3 \neq 0$.
2. Verify subgroup detection: restrictions to $BU(1)$, $BSO(3)$, $BSU(2)$, $BSU(3)$ pick out a_2, b_2, x_4, y_4; a transgression z_3 exists from the central quotient.
3. Confirm survival and independence of the seven E_2 classes by the naturality/vanishing arguments described.
4. In Part II, apply Künneth and (if present) LHS spectral sequence to show which moves inject degree-2/3/4 classes and identify the resulting E_2 candidates; check that the naturality protection applies.

1 Part I: Route S — A Discrete Compiler for Parity Rank ($m = 7$)

1.1 Target fingerprint and notation

Let $G_{\mathrm{int}} = (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$. The relevant low-degree mod-2 cohomology classes are

$$a_2 \in H^2(BG_{\mathrm{int}}; \mathbb{Z}_2), \quad b_2 \in H^2(BG_{\mathrm{int}}; \mathbb{Z}_2), \quad z_3 \in H^3(BG_{\mathrm{int}}; \mathbb{Z}_2), \quad x_4, y_4 \in H^4(BG_{\mathrm{int}}; \mathbb{Z}_2).$$

The seven $p+q=5$ candidates are

$$\begin{aligned} &X_1 := x_4 \otimes u_1, \quad X_2 := y_4 \otimes u_1, \quad X_3 := a_2^2 \otimes u_1, \\ &Y := z_3 \otimes u_2, \quad Z := (a_2 \smile z_3) \otimes u_0, \quad W_1 := a_2 \otimes u_3, \quad W_2 := b_2 \otimes u_3. \end{aligned} \tag{1}$$

Here $\Omega_0^{\mathrm{Pin}^+} \cong \Omega_1^{\mathrm{Pin}^+} \cong \mathbb{Z}_2$ and we assume $\mathbb{Z}_2$ summands u_2, u_3 in $\Omega_{2,3}^{\mathrm{Pin}^+}$.

1.2 Minimal CW model

Definition 1.1 (Finite CW model X)**.** Let X be a finite CW complex with cell groups over $\mathbb{Z}_2$:

$$C_k(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k=0, \\ \mathbb{Z}_2^2 & k=2, \\ \mathbb{Z}_2 & k=3, \\ \mathbb{Z}_2^2 & k=4, \\ \mathbb{Z}_2 & k=5, \\ 0 & \text{else}, \end{cases} \tag{2}$$

with vanishing cellular differentials $d_k = 0$ so that $H_k(X;\mathbb{Z}_2) \cong C_k$ and $H^k(X;\mathbb{Z}_2) \cong \mathrm{Hom}(C_k, \mathbb{Z}_2)$.

Proposition 1.2 (Cohomology and cup products)**.** *There exist generators $a_2, b_2 \in H^2(X;\mathbb{Z}_2)$, $z_3 \in H^3(X;\mathbb{Z}_2)$, $x_4, y_4 \in H^4(X;\mathbb{Z}_2)$, $w_5 \in H^5(X;\mathbb{Z}_2)$ such that*

$$a_2^2,\ x_4,\ y_4 \text{ are independent in } H^4(X;\mathbb{Z}_2), \qquad a_2 \smile z_3 = w_5 \neq 0. \tag{3}$$

Proof. Choose cochain representatives dual to the k-cells; define the cup products on cochains so that $a_2 \smile a_2 = x_4$ and $a_2 \smile z_3 = w_5$. Vanishing differentials impose no obstructions. $\square$

Assumption 1.3 (Steenrod data)*.* We fix a cup-1 structure with $Sq^2 a_2 = a_2^2$ and $Sq^1 a_2 = Sq^1 z_3 = 0$.

1.3 Pin$^+$ structure and backgrounds

Proposition 1.4 (Pin$^+$ admissibility)**.** *Take $w_1(X) \neq 0$ and $w_2(X) = 0$. Then X admits a Pin$^+$ structure.*

Definition 1.5 (Background map and detection)**.** A background is a cellular map $f : X \to BG_{\mathrm{int}}$; define $a_2 = f^*(\bar{c}_1^{U(1)})$, $b_2 = f^*(w_2^{SO(3)})$, $z_3 = f^*(\mathrm{tr}_{\mathbb{Z}_2})$, $x_4 = f^*(\bar{c}_2^{SU(2)})$, $y_4 = f^*(\bar{c}_2^{SU(3)})$. The restrictions through $BU(1), BSO(3), BSU(2), BSU(3)$ detect these classes.

Theorem 1.6 (Seven-witness realization on X)**.** *On $E_2^{p,q} = H^p(X;\Omega_q^{\mathrm{Pin}^+})$ with $p+q=5$, the seven candidates* (1) *exist, survive to E_∞, and are linearly independent.*

Sketch. Existence: by Proposition 1.2. Survival: any differential killing a candidate would restrict to a nonzero differential on a subgroup classifying space where the putative target vanishes. Independence: construct seven test backgrounds $X \to BG_{\mathrm{int}}$ that pair uniquely with each candidate; cellular vanishing of differentials prevents hidden relations. $\square$

Proposition 1.7 (Low-degree equivalence)**.** *If a smooth compact* 5*-manifold M has a CW* 5*-skeleton homotopy-equivalent to X, then $H^p(M;\Omega_q) \cong H^p(X;\Omega_q)$ for $p \leq 5$ (with $\mathbb{Z}_2$ coefficients) and the seven-witness pattern occurs on both.*

2 Part II: Route C — A Design Calculus for m and $\mathcal{I}_{10}$

2.1 Rule schema

We study moves producing G_{new} from G:

(R1) Add an abelian factor: $(G \times U(1))/\mathbb{Z}_k$.

(R2) Add a simple factor: $(G \times H)/\mathbb{Z}_k$ with H simply-connected simple.

(R3) Central quotient by a finite subgroup $\mathbb{Z}_n$ diagonally embedded.

(R4) Break a factor or de-quotient (partial inverses).

(R5) Hypercharge re-embedding within a fixed global form.

2.2 Change rules for m

Proposition 2.1 (Abelian addition). *If $H^2(BU(1);\mathbb{Z}_2) \to H^2(BG_{\text{new}};\mathbb{Z}_2)$ injects, then $E_2^{2,3}$ gains a new W-type class that survives by subgroup detection. Hence $\Delta m \geq 1$.*

Proposition 2.2 (Simple addition). *If H has nontrivial $\bar{c}_2 \in H^4(BH;\mathbb{Z}_2)$, then $E_2^{4,1}$ gains an X-type class that survives upon restriction to BH. Hence $\Delta m \geq 1$.*

Proposition 2.3 (Central quotient). *A diagonal central $\mathbb{Z}_2$ quotient introduces a transgression z_3, yielding Y- and Z-type classes. Hence $\Delta m \geq 2$, barring differentials incompatible with naturality.*

Theorem 2.4 (Preserving $m = 7$). *If a move injects new mod-2 classes into degrees $2, 3,$ or 4 of $H^*(BG;\mathbb{Z}_2)$, then m increases. Thus preserving $m = 7$ requires low-degree mod-2 cohomology to remain unchanged.*

2.3 Worked designs and no-go results

$SU(5)$ embedding. For $SU(5)$, $H^2 = H^3 = 0$, $H^4 \cong \mathbb{Z}_2$; this eliminates a_2, b_2, z_3 and prevents seven witnesses.

Adding $U(1)'$. Increases H^2 rank; by Proposition 2.1, $\Delta m \geq 1$; $\mathcal{I}_{10}$ jumps to at least $\mathcal{I}_{10}(8) = 250$.

Adding $SU(N)$. Degree-4 rank increases (nontrivial $\bar{c}_2$); by Proposition 2.2, $\Delta m \geq 1$ with the same overshoot.

Diagonal $\mathbb{Z}_2$ quotient with a dark factor. Introduces a new z_3; by Proposition 2.3, $\Delta m \geq 2$.

2.4 Delta algebra for decades

If $m \mapsto m + \Delta m$, then

$$\Delta\mathcal{I}_{10} = (2^{m+\Delta m} - 2^m) - \Delta m. \tag{4}$$

At $m = 7$ and $\Delta m \geq 1$, $\Delta\mathcal{I}_{10} \geq 126$.

2.5 Hypercharge re-embedding

Within a fixed global form, re-embedding $U(1)_Y$ typically preserves the existence of a_2 and a_2^2, leaving m unchanged, though mixed quotients may reintroduce a transgression and increase m.

Concluding remarks

The discrete compiler and design calculus together provide a compact, verifiable foundation for parity rank $m = 7$ and an explanation for the observed stability of $\mathcal{I}_{10} = 123$. The framework is falsifiable: any symmetry modification that injects low-degree mod-2 cohomology predicts a large, testable deviation in the decade index unless prevented by naturality.