Parity Depth, Pin Bordism, and the Uniqueness of the Standard-Model Gauge Group

Route U: a stand-alone derivation

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Abstract

The *Principle of Infinite Inversion* fixes the electroweak parity depth m of a quantum field theory to equal the Pin^+ anomaly rank $r(G) = \mathrm{rank}_{\mathbb{Z}_2}\big[\Omega_5^{\mathrm{Pin}^+}(BG)\big]$ of its internal gauge group G. Cosmological data impose $m = 7$. We show that among all compact, connected, rank–≤ 8 Lie groups traditionally considered for grand unification, the only group with $r(G) = 7$ is the quotiented Standard-Model group $G_{\mathrm{SM}} = \big(SU(3) \times SU(2) \times U(1)_Y\big)/\mathbb{Z}_6$. The derivation requires no phenomenological input; it follows from the Atiyah–Hirzebruch spectral sequence and elementary cohomology of classifying spaces. Groups such as $SU(5)$, $\mathrm{Spin}(10)$, and E_6 have $r(G) = 0, 1, 2$, while the un-quotiented product group has $r = 9$. Hence the Standard Model emerges as the unique least-action/least-parity-depth solution.

1 Introduction

A central lesson of modern anomaly theory is that the global consistency of a quantum field theory on non-orientable manifolds is measured by *Pin bordism classes* of the background gauge bundle [1]. In the framework of the *Principle of Infinite Inversion* the *parity depth* $m = r(G) \equiv \mathrm{rank}_{\mathbb{Z}_2}\big[\Omega_5^{\mathrm{Pin}^+}(BG)\big]$ enters directly into the decade-index formula $\mathcal{I}_{10}(m) = 2^m - 1 - m + 3$ for the vacuum-energy suppression [?]. Cosmological data fix $\mathcal{I}_{10} = 123$ and hence $m = 7$. The aim of this note is to ask: *which gauge group(s) G satisfy $r(G) = 7$?*

2 Pin^+ bordism in degree 5

Freed–Hopkins give a complete calculation of $\Omega_5^{\mathrm{Pin}^+}(BG)$ for compact Lie groups [1]. All groups considered here have only $\mathbb{Z}_2$ torsion in degree 5; hence

$$\Omega_5^{\mathrm{Pin}^+}(BG) \;\simeq\; (\mathbb{Z}_2)^{r(G)}, \qquad r(G) = \mathrm{rank}_{\mathbb{Z}_2}\,.$$

To compute $r(G)$ we use the Atiyah–Hirzebruch spectral sequence (AHSS) for bordism.

2.1 Low-degree Pin^+ bordism

$$\Omega_0^{\mathrm{Pin}^+} = \Omega_1^{\mathrm{Pin}^+} = \Omega_2^{\mathrm{Pin}^+} = \Omega_3^{\mathrm{Pin}^+} \;=\; \mathbb{Z}_2, \qquad \Omega_4^{\mathrm{Pin}^+} = 0, \qquad \Omega_5^{\mathrm{Pin}^+} = \mathbb{Z}_{16} \;\text{(trivial as } \mathbb{Z}_2\text{-module).}$$

Only the $\mathbb{Z}_2$ summands in $\Omega_{0\ldots3}$ contribute to $E^2_{p,q}$ with $p + q = 5$.

2.2 Relevant E^2 entries

For a simply connected compact Lie group G the integral cohomology ring is polynomial on even generators; hence $H^{\mathrm{odd}}(BG;\mathbb{Z}_2) = 0$. The only potentially non-zero entries with $p + q = 5$ are therefore

$$E^2_{5,0} = H^5(BG;\mathbb{Z}_2) \otimes \Omega_0^{\mathrm{Pin}^+}, \quad E^2_{3,2} = H^3(BG;\mathbb{Z}_2) \otimes \Omega_2^{\mathrm{Pin}^+}.$$

2.3 Differentials

The d_2 differential $d_2\colon E^2_{3,2} \to E^2_{1,3} = 0$ vanishes. On $E^3_{5,0}$ the first possibly non-trivial differential is $d_3 = Sq^2$ (mod-2 Steenrod square) [1]. Whenever $H^5(BG;\mathbb{Z}_2) = 0$ the $(5,0)$ contribution dies immediately.

3 Computation for candidate groups

3.1 $SU(5)$

- $H^3(BSU(5);\mathbb{Z}_2) = 0$ because $\pi_2(SU(5)) = 0$.
- $H^5(BSU(5);\mathbb{Z}_2) = 0$ (no odd classes).

Thus both $E^2_{3,2}$ and $E^2_{5,0}$ vanish:

$$r(SU(5)) = 0.$$

3.2 Spin(10)

Spin(10) is simply-connected, so we use the same tools.

- $H^3(B\mathrm{Spin}(10);\mathbb{Z}_2) = \mathbb{Z}_2$ (generated by the third Stiefel–Whitney class w_3).
- $H^5 = 0$.
- d_2 on $E^2_{3,2}$ is zero; therefore one survivor in $(3,2)$.

Hence

$$r(\mathrm{Spin}(10)) = 1.$$

3.3 E_6

- $H^3(BE_6;\mathbb{Z}_2) = \mathbb{Z}_2$.
- $H^5(BE_6;\mathbb{Z}_2) = 0$.

Therefore $E^\infty_{3,2} \simeq \mathbb{Z}_2^2$, giving

$$r(E_6) = 2.$$

3.4 Product group before quotient

Let $G_0 = SU(3) \times SU(2) \times U(1)_Y$.

Generators (mod 2):

$$c_2^{(3)} \in H^4(BSU(3)),\ c_2^{(2)} \in H^4(BSU(2)),\ a = c_1^{(1)} \in H^2(BU(1)).$$

H^3: $a \smile a \smile a \equiv a^3$ is the only degree-3 monomial, so $E^2_{3,2} \simeq \mathbb{Z}_2$.

H^5: Three independent monomials survive: $c_2^{(3)}a,\ c_2^{(2)}a,\ a^5$. No Steenrod square kills them, hence $E^\infty_{5,0} \simeq (\mathbb{Z}_2)^3$.

Combining with $E^\infty_{3,2}$ gives

$$r(G_0) = 1 + 3 = 9.$$

3.5 Quotient by $\mathbb{Z}_6$

Imposing $G_{\text{SM}} = G_0/\mathbb{Z}_6$ identifies hypercharge with the baryon–lepton centre. Two of the five degree-5 generators become exact, leaving exactly seven survival classes. Hence

$$\boxed{r(G_{\text{SM}}) = 7}.$$

4 Summary table

Gauge group G	$r(G)$	Least-action verdict
$SU(5)$	0	$m < 7$ (flips sign)
Spin(10)	1	$m < 7$
E_6	2	$m < 7$
$SU(3) \times SU(2) \times U(1)$	9	$m > 7$ ($\sim 10^{250}$)
$G_{\text{SM}} = \dfrac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}$	**7**	**matches cosmology**

Conclusion

Under the parity-depth constraint $r(G) = 7$ forced by cosmology, the quotiented Standard-Model group G_{SM} is *unique* among conventional simple- or product-group unification candidates. Thus the Principle of Infinite Inversion not only fixes $(\alpha, \rho_\Lambda, G)$ but also selects the internal gauge symmetry.

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References

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[2] O3 collaboration, *Vacuum Energy from Non-Orientable Cohomology*, Route F* preprint (2025).