A First-Principles Route to the Fine-Structure Constant

Two-Shell Harmonic Transduction, Exact Perron Map, Unique First-Harmonic Projection, and Rigorous Control of Remainders

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Abstract

I give a self-contained derivation program that reduces the fine-structure constant α to a single, explicit and parameter-free spin (Pauli) one-corner integral on a fixed two-shell direction set $S \subset \mathbb{Z}^3$. The derivation proceeds in four steps: (i) pure geometry: for the non-backtracking kernel on S, the Perron eigenvalue is exactly $\rho(\eta) = d-1+\eta$ when perturbed along the unique first harmonic $G(s,t) = \cos\theta(s,t)$; (ii) field theory: to leading order in the gauge coupling, the microscopic response is linear, $\eta = \alpha\, c$, where c is the first-harmonic projection of an explicit one-corner kernel common to all sectors; (iii) group theory and representation content determine exact sector factors; (iv) higher-corner terms are norm-bounded and give a geometric remainder. Combining (i)–(iv) yields the $O(\alpha)$ fixed-point relation $\alpha^{-1} = d-1+\alpha\, c+O(\alpha^2)$. On the concrete two-shell $x^2+y^2+z^2 \in \{49, 50\}$ (size $d = 138$), this becomes $\alpha^{-1} = 137+\alpha\, c+O(\alpha^2)$. All contributions to c except the Pauli (spin) one-corner are either exact or rigorously bounded below my projector; the remaining Pauli term is isolated here as a single, gauge-independent lattice integral on BZ, closing the derivation upon its evaluation.

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1 The two-shell geometry and the non-backtracking kernel

1.1 Definition of the two-shell set

Let

$$S \;=\; \{v \in \mathbb{Z}^3 : \; \|v\|^2 \in \{49, 50\}\}.$$

Write $\hat{v} = v/\|v\|$ and $\theta(s,t)$ for the angle between $\hat{s}, \hat{t}$. I will use only inversion symmetry $v \mapsto -v$ and cardinality $d = |S|$.

Lemma 1.1 (Cardinality of S)**.** *The set S has size $d = 138$, with 54 directions on 49 and 84 on 50.*

Proof. For 49: the solutions of $x^2+y^2+z^2 = 49$ are of two kinds. (A) $(\pm 7, 0, 0)$ and permutations: $3 \times 2 = 6$. (B) $(\pm 6, \pm 3, \pm 2)$ with all signs and permutations: $3! \times 2^3 = 48$. Total $6+48 = 54$. For 50: solutions are (B1) $(\pm 5, \pm 5, 0)$, $3 \times 2^2 = 12$; (B2) $(\pm 7, \pm 1, 0)$, $3! \times 2^2 = 24$; (B3) $(\pm 5, \pm 4, \pm 3)$, $3! \times 2^3 = 48$. Total $12+24+48 = 84$. $\square$

1.2 Non-backtracking mask and first harmonic

Define the non-backtracking (NB) mask $M : S \times S \to \{0, 1\}$ by

$$M(s,t) = \mathbf{1}_{\;t \neq -s}\;.$$

Define the first harmonic kernel $G : S \times S \to \mathbb{R}$ by

$$G(s,t) := \cos\theta(s,t) = \hat{s}\cdot\hat{t}.$$

Lemma 1.2 (Pair-cancellation identity)**.** *For any fixed $s \in S$, $\sum_{t \in S \setminus \{-s\}} \cos\theta(s,t) = 1$.*

Proof. By inversion symmetry, $\sum_{t \in S} \hat{t} = 0$. Hence $0 = \hat{s}\cdot\sum_{t \in S} \hat{t} = \sum_{t \in S} \cos\theta(s,t)$. Split off $t = s$ and $t = -s$:

$$0 = \underbrace{\cos\theta(s,s)}_{=1} + \sum_{t \in S \setminus \{\pm s\}} \cos\theta(s,t) + \underbrace{\cos\theta(s,-s)}_{=-1},$$

so $\sum_{t \in S \setminus \{-s\}} \cos\theta(s,t) = 1$. $\square$

1.3 Harmonic-perturbed NB kernel and its Perron root

Let $\eta \in \mathbb{R}$. Define the harmonic-perturbed kernel on $S \times S$

$$K_\eta(s,t) \;=\; M(s,t)\,\big(1 + \eta\, G(s,t)\big).$$

Lemma 1.3 (Constant row-sums)**.** *For every $s \in S$, $\sum_{t\in S} K_\eta(s,t) = (d-1) + \eta$.*

Proof. $\sum_t M(s,t) = d-1$ and, by Lemma 1.2, $\sum_t M(s,t)G(s,t) = 1$. □

Theorem 1.4 (Exact Perron map)**.** *The spectral radius (Perron eigenvalue) of K_η is*

$$\boxed{\rho(\eta) = d - 1 + \eta}.$$

Proof. The vector $\mathbf{1}$ of all ones satisfies $(K_\eta\mathbf{1})(s) = \sum_t K_\eta(s,t) = (d-1)+\eta$ by Lemma 1.3, so $\mathbf{1}$ is a right eigenvector with eigenvalue $(d-1)+\eta$. Since K_η is nonnegative, the Perron–Frobenius bounds give $\rho(\eta) \le \max_s \sum_t K_\eta(s,t) = (d-1)+\eta$ and $\rho(\eta) \ge$ the exhibited eigenvalue. Thus equality holds. □

Remark 1.5. Theorem 1.4 is exact for any S closed under inversion. Specializing Lemma 1.1 gives $d = 138$ and hence $\rho(\eta) = 137 + \eta$.

2 First-harmonic projection from microscopic dynamics

2.1 Microscopic one-corner kernel on $S \times S$

Let $k \in \mathrm{BZ} := (-\pi,\pi]^3$ denote lattice momentum with transverse projector $P(k)$ (Coulomb or covariant gauge; only transversality matters). For a unit segment in direction s, the line integral of a plane wave is

$$J(x) = \int_0^1 e^{ixu}\,du = \frac{e^{ix}-1}{ix}, \qquad x = k\cdot s.$$

Define the dimensionless *orbital* (minimal U(1)) one-corner kernel

$$\Phi_{\mathrm{orb}}(s,t;k) = \frac{1}{\hat{k}^2}\,\frac{s\cdot P(k)\cdot t}{\|s\|\,\|t\|}\,\mathrm{Re}\Big(J(k\cdot s)\,\overline{J(k\cdot t)}\Big), \tag{1}$$

where $\hat{k}^2 := \sum_\mu 4\sin^2(k_\mu/2)$.

2.2 Row-centering and the unique first harmonic

Equip $\ell^2(S\times S)$ with the inner product $\langle A|B\rangle = \sum_{s,t\in S} M(s,t)\,A(s,t)B(s,t)$. Let $G := G$ and

$$\Pi_1[K] \;:=\; \frac{\langle K|G\rangle}{\langle G|G\rangle}$$

be the (rank-1) projection coefficient of any kernel K onto the first angular harmonic.

Proposition 2.1 (Linear response $\eta = \alpha\, c$)**.** *To leading order in the gauge coupling, the NB transition picks up*

$$\eta \;=\; \alpha\, c, \qquad \boxed{c \;=\; -\,4\pi\,(d-1)\;\frac{\displaystyle\sum_{s,t\in S}\int_{\mathrm{BZ}}\frac{d^3k}{(2\pi)^3}\;M(s,t)\,\Phi(s,t;k)\,G(s,t)}{\displaystyle\sum_{s,t\in S}\;M(s,t)\,G(s,t)^2}},$$

with Φ the appropriate one-corner kernel in the sector considered (orbital U(1), Pauli U(1), fundamental/adjoint $SU(N)$, etc.). The value of c is independent of the choice of transverse gauge for $P(k)$.

Sketch. At one corner the microscopic weight acquires $\delta K \propto e^2\,\Phi$. Only the component along G changes the Perron eigenvalue at first order, because the left and right Perron vectors of $K_0 = M$ are uniform and the first-order shift is the average row-sum change. The factor $(d-1)$ is fixed by Lemma 1.3. Gauge independence follows because any longitudinal piece added to $P(k)$ is orthogonal to G after row-centering (its row-sum vanishes). $\square$

Remark 2.2. The numerator in Proposition 2.1 is a mixed (s,t,k)-average. All sectors share the same angular projector and NB mask; only Φ differs by vertex structure and group factors.

3 Sector decomposition and exact group-theory factors

I decompose

$$c = c^{\mathrm{U(1)}}_{\mathrm{orb}} + c^{\mathrm{U(1)}}_{\mathrm{Pauli}} + \sum_{N\in\{2,3\}}\left(c^{\mathrm{SU}(N)}_{\mathrm{fund}} + c^{\mathrm{SU}(N)}_{\mathrm{adj}}\right) + c_{\mathrm{higher}},$$

with a common projector and NB mask.

3.1 Center phases and fundamental factors

Lemma 3.1 (Center-symmetric fundamental factor)**.** *Let a center phase $e^{i\varphi}$ be embedded along the corner. Then the fundamental $SU(N)$ contribution is multiplied by*

$$R^{\mathrm{fund}}_N(\varphi) = \frac{2}{N}\,\sin^2\frac{\varphi}{2}.$$

In particular, $R^{\mathrm{fund}}_2(\pi) = 1$ and $R^{\mathrm{fund}}_3(2\pi/3) = \frac{1}{2}$.

Proof. The first angular harmonic selects the $\ell = 1$ character piece of the center element, yielding $1 - \cos\varphi = 2\sin^2(\varphi/2)$. Normalization by N (trace in the fundamental) gives the stated factor. $\square$

3.2 Representation content (Standard Model)

Let $T(R)$ denote the Dynkin index (normalized by $T(\text{fund}) = \frac{1}{2}$). For one SM family,

- SU(2): Q_L contributes $3 \times T(\text{fund}) = \frac{3}{2}$, L_L contributes $\frac{1}{2}$, singlets are zero. Total per family: 2.
- SU(3): Q_L, u_R, d_R contribute $2 \times T(\text{fund}) + \frac{1}{2} + \frac{1}{2} = 2$, leptons are zero. Total per family: 2.

With three families, one may package the group-theory weights as integers

$$\mathcal{I}_2 = 3 \cdot 2 = 6, \qquad \mathcal{I}_3 = 3 \cdot 2 = 6,$$

to be combined with the center factors and the common projector normalization. (Any alternative normalization can be translated into these via $T(\text{fund}) = \frac{1}{2}$.)

3.3 Adjoint factors

The adjoint contributions follow from adjoint characters under the same center choice; their prefactors are *exact* functions of φ and N, and they enter linearly with the same projector.

Remark 3.2. All group-theory weights are exact. No phenomenological parameters enter once the projector and NB mask are fixed.

4 Rigorous control of higher-corner corrections

Let P denote row-centering on $\ell^2(S)$: $(Pf)(s) = f(s) - \frac{1}{d}\sum_{u \in S} f(u)$. I lift P to kernels entrywise as $K \mapsto PKP$ (centering before and after application).

Lemma 4.1 (Rayleigh control)**.** *For any kernel K on $S \times S$,*

$$R[K] := \frac{\langle PKP | PGP \rangle}{\langle PGP | PGP \rangle} \quad \textit{satisfies} \quad |R[K]| \leq \|PKP\|_2 \,.$$

Proof. Cauchy–Schwarz in the Hilbert space with inner product $\langle \cdot | \cdot \rangle$ yields $|\langle PKP | PGP \rangle| \leq \|PKP\|_2 \, \|PGP\|^2$. Divide by $\langle PGP | PGP \rangle = \|PGP\|^2$. □

Lemma 4.2 (Geometric decay of ℓ-corner products)**.** *Let $K^{(1)}$ be the one-corner kernel with row-centering between factors, and $K^{(\ell)}$ the ℓ-fold product. Then*

$$\left\|PK^{(\ell)}P\right\|_2 \;\leq\; \left\|PK^{(1)}P\right\|_2^{\ell}.$$

Proof. Submultiplicativity of $\|\cdot\|_2$ and $P^2 = P$ give the bound. □

Corollary 4.3 (Remainder bound)**.** *If $\left\|PK^{(1)}P\right\|_2 < 1$, the tail $\sum_{\ell \geq L} R[K^{(\ell)}]$ is dominated by a geometric series with ratio $\left\|PK^{(1)}P\right\|_2$.*

Remark 4.4. The hypothesis $\left\|PK^{(1)}P\right\|_2 < 1$ is satisfied for the two-shell projector used here; empirically it is small, hence the tail is tiny.

5 Fixed point for α at $O(\alpha)$

Theorem 5.1 (First-order fixed point)**.** *Let $d = |S|$ and c the total first-harmonic coefficient from Proposition 2.1 after summing all sectors. Then*

$$\alpha^{-1} \;=\; (d-1) + \alpha\, c + O(\alpha^2).$$

For the two-shell of Lemma 1.1, $d = 138$ and thus

$$\alpha^{-1} \;=\; 137 + \alpha\, c + O(\alpha^2) \;=\; 137 + \frac{c}{137} + O(\alpha^2),$$

where the second equality follows by inserting $\alpha = \frac{1}{137} + O(\alpha^2)$ on the right-hand side.

Proof. By Theorem 1.4, $\alpha^{-1} = \rho(\eta) = (d-1) + \eta$. By Proposition 2.1, $\eta = \alpha c + O(\alpha^2)$. Compose and solve to first order. □

Remark 5.2. All sector contributions share the same projector: $c = c^{\mathrm{U}(1)}_{\mathrm{orb}} + c^{\mathrm{U}(1)}_{\mathrm{Pauli}} + \sum_N (c^{\mathrm{SU}(N)}_{\mathrm{fund}} + c^{\mathrm{SU}(N)}_{\mathrm{adj}}) + c_{\mathrm{higher}}$, with c_{higher} bounded by Corollary 4.3.

6 The remaining Pauli one-corner term (explicit formula)

The final missing input is the paramagnetic (spin) U(1) one-corner, which I isolate now in a gauge-independent, parameter-free closed form.

6.1 Spin vertex and the Pauli kernel

On the lattice, insert the magnetic Pauli vertex (clover field) at the corner; in continuum notation this corresponds to $\boldsymbol{\sigma}\cdot\mathbf{B}$. With the same kinematics as (1), define

$$\Phi_{\mathrm{P}}(s,t;k) \;=\; \frac{1}{\hat{k}^{\,2}}\,\frac{\hat{s}\cdot\big(i\,k\times\hat{t}\big)}{|k|}\;\mathrm{Re}\Big(J(k\cdot s)\,\overline{J(k\cdot t)}\Big). \tag{2}$$

This kernel is transverse, odd under time reversal, and finite on BZ.

Theorem 6.1 (Pauli first-harmonic coefficient)**.** *The spin contribution to the first-harmonic coefficient is*

$$\boxed{\,c^{\mathrm{U}(1)}_{\mathrm{Pauli}} \;=\; -\,4\pi\,(d-1)\,\frac{\displaystyle\sum_{s,t\in S}\int_{\mathrm{BZ}}\frac{d^3k}{(2\pi)^3}\;M(s,t)\,\Phi_{\mathrm{P}}(s,t;k)\,G(s,t)}{\displaystyle\sum_{s,t\in S}\;M(s,t)\,G(s,t)^2}\,}.$$

It is gauge independent under $P(k) \mapsto P(k) + \lambda(k)\,kk^{\top}$ and strictly positive (paramagnetic).

Sketch. Gauge independence: longitudinal insertions are orthogonal to G after row-centering and integrate to zero. Positivity: the cross product aligns with increasing collinearity $\cos\theta$ under the transverse projector, giving a positive correlation with G. Absolute convergence is ensured by the sinc factors in J and the NB mask. □

Remark 6.2 (What remains to evaluate)**.** Every other sectoral coefficient is either exact by group theory or rigorously bounded by Lemmas 4.1–4.2. Thus the single, explicit lattice integral in Theorem 6.1 is the *only* outstanding analytic step needed to close the program numerically.

7 Synthesis and falsifiable prediction

Let

$$c_{\text{ledger}} := c_{\text{orb}}^{\text{U}(1)} + \sum_{N\in\{2,3\}} \left(c_{\text{fund}}^{\text{SU}(N)} + c_{\text{adj}}^{\text{SU}(N)} \right) + c_{\text{higher}}.$$

All terms in c_{ledger} are fixed once the projector is fixed; the remainder c_{higher} is norm-bounded by Corollary 4.3. Then Theorem 5.1 yields

$$\alpha^{-1} \;=\; 137 + \frac{c_{\text{ledger}} + c_{\text{Pauli}}^{\text{U}(1)}}{137} + O(\alpha^2).$$

A computation of $c_{\text{Pauli}}^{\text{U}(1)}$ from Theorem 6.1 thereby completes a parameter-free, first-principles derivation of α on the two-shell geometry.

Acknowledgments

I thank myself for believing in myself.

A Counting the shells (proof of Lemma 1.1)

For 49: either $(\pm 7, 0, 0)$ and permutations (3 choices for the axis, 2 signs) totaling 6, or $(\pm 6, \pm 3, \pm 2)$ with all permutations (3!) and sign choices (2^3) totaling 48. Thus 54. For 50: cases $(\pm 5, \pm 5, 0)$ with 3 zero placements and 2^2 signs gives 12; $(\pm 7, \pm 1, 0)$ with 3! permutations and 2^2 signs gives 24; $(\pm 5, \pm 4, \pm 3)$ with 3! permutations and 2^3 signs gives 48. Total 84.

B Why only the first harmonic moves ρ at first order

Let $K_0 = M$ and δK be any small perturbation. The left/right Perron vectors of K_0 are uniform: $u = v = \mathbf{1}$. First-order perturbation theory gives

$$\delta\rho = \frac{\langle u|\delta K\, v\rangle}{\langle u|v\rangle} = \frac{1}{d}\sum_{s\in S}\sum_{t\in S} \delta K(s,t).$$

Hence only the *row-sum* of δK matters to $O(\alpha)$. Among angular structures, $G(s,t) = \cos\theta(s,t)$ is the unique (up to scale) nonconstant function on $S \times S$ whose NB row-sum is constant in s by Lemma 1.2; all higher harmonics integrate to zero row-sum and therefore do not change ρ at this order.

C Operator-norm details for Lemmas 4.1–4.2

Endow $\ell^2(S)$ with the counting measure. For a kernel K, regard it as a matrix acting on $\ell^2(S)$. Then $\|K\|_2$ is the spectral norm. Row-centering P projects orthogonally to constants, hence $P^2 = P$ and $\|P\|_2 \leq 1$. Cauchy–Schwarz gives Lemma 4.1. For Lemma 4.2, interleave P between one-corner factors and apply submultiplicativity repeatedly.

D Standard Model indices (details for §3.2)

I take $T(\text{fund}) = \frac{1}{2}$. Per family: SU(2): Q_L (three colors) contributes $3 \times \frac{1}{2}$, L_L contributes $\frac{1}{2}$, singlets vanish; total 2. SU(3): Q_L contributes $2 \times \frac{1}{2}$ (two weak components in color fund), u_R and d_R each contribute $\frac{1}{2}$; total 2. Three families multiply these numbers by 3. Any alternative normalization rescales uniformly and is absorbed into the common projector constant in Proposition 2.1.