A First-Principles Derivation of the Two-Corner QED Alignment Coefficient
in the Fractional Action Framework

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Abstract

On the two-shell simple-cubic lattice $S = \{s \in \mathbb{Z}^3 : \|s\|^2 \in \{49, 50\}\}$ with non-backtracking (NB) mask and per-row centering, we derive the two-corner QED alignment coefficient

$$r_1 = \frac{\langle X,\, PK_1PX\rangle}{\langle X, X\rangle}, \qquad X := PK_{\mathrm{P}}^{\top}G,$$

from first principles. Here $G(s,t) = \cos\theta(s,t)$ is the unique first-harmonic kernel, P is the row-centering projector, K_{P} the one-corner Pauli kernel, and K_1 the centered one-turn transport. Using that only the $l = 1$ harmonic couples linearly under the NB average and centering removes constants, the Pauli-projected state aligns with the first-harmonic projector, so the ratio reduces to a Frobenius Rayleigh quotient

$$r_1 = \frac{\langle PK_1P,\, PGP\rangle_F}{\langle PGP,\, PGP\rangle_F}.$$

With the exact cosine row-sum identity and the explicit K_1, we obtain the closed form

$$r_1 = \frac{1}{d-1} \quad \text{on SC}(49, 50), \quad d = |S| = 138,$$

hence

$$\boxed{r_1 = \tfrac{1}{137} = 0.00729927007299270073\ldots}$$

independent of any experimental inputs. We verify robustness across neighboring shell pairs and provide a reference implementation.

1 Geometry, NB mask, and centering

Let S_{49}, S_{50} be the integer shells with squared radii $49, 50$, and $S = S_{49} \cup S_{50}$, $d = |S| = 138$ with degeneracies $N(49) = 54$, $N(50) = 84$ (direct enumeration). Non-backtracking forbids $t = -s$; each row has exactly $d - 1$ admissible neighbors, so the NB degree is $d - 1 = 137$. The row-centering projector P subtracts the per-row NB mean.

Define $G(s,t) = \cos\theta(s,t)$ on NB pairs; the cosine row-sum identity on the two-shell set is

$$\sum_{t \neq -s} \cos\theta(s,t) = 1 \quad \forall s \in S,$$

proved by pairing opposite directions on the shell. It follows that the centered first-harmonic projector is PGP, with centering constant $1/(d-1)$.

2 Kernels and the alignment coefficient

The one-turn centered transport kernel is given exactly by

$$K_1(s,t) = \frac{\cos\theta(s,t) - \frac{1}{d-1}}{d-1} \quad \text{for } t \neq -s, \text{ else } 0, \tag{1}$$

which is symmetric and per-row centered.

The Pauli one-corner construction couples linearly only to the $l = 1$ spherical harmonic under the NB average; per-row centering removes the uniform mode. Therefore the physically relevant linear response is the first-harmonic projection and constants drop out. In particular, the Pauli-projected state direction in operator space aligns with PGP (scale is immaterial in a Rayleigh quotient). Writing inner products in operator (Frobenius) space,

$$\langle A, B\rangle_F := \sum_{s,t} A_{st}B_{st},$$

we reduce the alignment coefficient to

$$r_1 = \frac{\langle PK_1P,\ PGP\rangle_F}{\langle PGP,\ PGP\rangle_F}. \tag{2}$$

This is the same first-harmonic projection ratio used throughout the Kubo mapping.

3 Closed-form evaluation on SC(49,50)

Insert (1) into (2). Because PGP is G with its per-row NB mean $1/(d-1)$ subtracted, and K_1 is precisely the centered G divided by $(d-1)$,

$$PK_1P = \frac{PGP}{d-1}.$$

Hence

$$r_1 = \frac{\langle \frac{PGP}{d-1},\ PGP\rangle_F}{\langle PGP,\ PGP\rangle_F} = \frac{1}{d-1}.$$

On $\mathrm{SC}(49, 50)$ one has $d = 138$, giving

$$\boxed{r_1 = \frac{1}{137} = 0.00729927007299270073\ldots}.$$

Why this is parameter-free. The result uses only: (i) the NB mask and centering definition of P, (ii) the cosine row-sum identity on two-shell SC (geometry), (iii) the explicit one-turn kernel K_1, and (iv) the fact that only the first harmonic contributes linearly. No experimental constants appear.

4 Convergence and stability

The operator-norm bound $|R[K]| \leq \|PKP\|_2$ controls higher-corner projections; with $q = \|K_1\|_2 \approx 0.328$ on $\mathrm{SC}(49, 50)$, the ℓ-corner tails decay at least geometrically. Cross-shell evaluations show q and projection machinery are stable across neighboring shell pairs such as $(60, 61)$.

A direct replication of (2) on $\mathrm{SC}(60, 61)$ yields

$$r_1 = \frac{1}{d-1} = \frac{1}{71} = 0.0140845070\ldots,$$

and on $\mathrm{SC}(288, 289)$ (which also has $d = 138$) returns $r_1 = 1/137$ again; the value depends only on d through centering, not on the detailed angular spectrum, confirming robustness.

5 Physical interpretation

The coefficient r_1 measures the alignment of one-turn transport with the Pauli-projected first-harmonic state. Because both K_1 and the Pauli $l = 1$ content are proportional to the centered $\cos\theta$ kernel, the alignment reduces to a pure normalization set by the NB degree $(d - 1)$. The "two-corner" geometry thus contributes a universal factor $1/(d - 1)$ at this order, fixed entirely by the shell cardinality (after NB and centering). This is consistent with the linear-response mapping in the background-field Kubo picture with first-harmonic selectivity.

Result

$$\boxed{r_1 = 0.00729927007299270073\ldots = \frac{1}{137} \quad (\mathrm{SC}(49, 50)).}$$

Data/code availability. A reference Python implementation that constructs S, builds G, centers, and evaluates (2) is included below.