Schrödinger Equations — Fraction Physics (Worked Examples Edition)

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Abstract

A Fraction Physics DLC pack for the Schrödinger equation with exact rational locks, minimal-description-length (MDL) accounting, and fully worked examples so new contributors can reproduce every step. Made for Teaching!

Contents

Conventions and MDL

- Natural units unless noted: $\hbar = 1$, and often $m = 1$. Transcendentals (e.g. π) stay explicit and carry no MDL charge. Only rationals p/q are scored by $L = \lceil \log_2 p \rceil + \lceil \log_2 q \rceil$.
- We prefer dimensionless forms with unit kinetic coefficient; all model info is isolated in a dimensionless potential U.

1 Core Modules (recap)

1.1 M–SCH–01: Time-Dependent Schrödinger Equation (TDSE)

$\mathrm{i}\,\partial_t \psi = (-\nabla^2/2m + V)\psi$. Lock: $\frac{1}{2}$ (bit-cost 1 when $m = 1$). Dimensionless with $E_0 = 1/(2mL^2)$: $\mathrm{i}\partial_\tau \psi = (-\partial_\xi^2 + U)\psi$ (unit coefficient).

1.2 M–SCH–02: Time-Independent Schrödinger Equation (TISE)

$(-\nabla^2/2m + V)\phi = E\phi$. Dimensionless eigenproblem: $(-\partial_\xi^2 + U)\phi = \epsilon\phi$ with $\epsilon = E/E_0$.

1.3 M–SCH–03: Continuity and Current

$\rho = |\psi|^2$, $\mathbf{j} = \frac{1}{m}\,\mathrm{Im}(\psi^* \nabla\psi)$, $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$. In $m = 1$ units, current prefactor is 1.

1.4 M–SCH–04..08 (Models)

Free particle: $E = p^2/2m$. Infinite well: $E_n = n^2\pi^2/(2mL^2)$. HO: $E_n = (n + \frac{1}{2})\omega$. Hydrogenic: $E_n = -\mu\alpha^2/(2n^2)$. Delta: $E_0 = -g^2/2$ (in $\hbar = 2m = 1$).

1.5 M–SCH–09..10 (Principles)

Variational: $E_0 = \min\, \langle\psi|H|\psi\rangle \,/\, \langle\psi|\psi\rangle$. Time evolution: $U(t) = \mathrm{e}^{-\mathrm{i}Ht}$.

2 Unitization Recipe (do this first)

Pick a length scale L and set

$$\xi = \frac{x}{L}, \qquad E_0 = \frac{1}{2mL^2}, \qquad \tau = E_0 t, \qquad U(\xi) = \frac{V(x)}{E_0}. \tag{1}$$

Then TDSE becomes

$$\mathrm{i}\,\partial_\tau \psi(\xi,\tau) = \left[-\partial_\xi^2 + U(\xi,\tau)\right]\psi(\xi,\tau), \tag{2}$$

so all kinetic structure is locked to coefficient 1. **Teaching tip:** students solve the *same* PDE each time; only U changes.

3 Worked Example 1: Infinite Square Well

Setup. $V = 0$ on $x \in (0, L)$, $V = \infty$ outside. With $\phi(0) = \phi(L) = 0$ the TISE is $-\phi'' = k^2\phi$.

Solution. $\phi_n = A\sin(n\pi x/L)$, $k_n = n\pi/L$, $n \in \mathbb{N}$. Normalize: $A = \sqrt{2/L}$.

Energies. $E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2}{2mL^2}$; **lock:** $E_n/E_1 = n^2$ (integer).

Expectations. $\langle x\rangle = L/2$; $\langle x^2\rangle = L^2\big(\frac{1}{3} - \frac{1}{2n^2\pi^2}\big)$. Momentum: $\langle p^2\rangle = 2mE_n$; virial holds with $\langle T\rangle = E_n$, $\langle V\rangle = 0$.

MDL. Rational locks: 1/2 (kinetic), integer n^2.

4 Worked Example 2: Harmonic Oscillator (Ladders)

Define. $a = \frac{1}{\sqrt{2}}(\xi + \partial_\xi)$, $a^\dagger = \frac{1}{\sqrt{2}}(\xi - \partial_\xi)$ with $[a, a^\dagger] = 1$ and $H/\omega = \frac{1}{2}(aa^\dagger + a^\dagger a)$.

Ground state. $a\phi_0 = 0 \Rightarrow \phi_0(\xi) = \pi^{-1/4}\mathrm{e}^{-\xi^2/2}$, $E_0 = \frac{1}{2}\omega$.

Spectrum. $\phi_n = (a^\dagger)^n\phi_0/\sqrt{n!}$, $E_n = (n+\frac{1}{2})\omega$ (**lock:** 1/2). Orthogonality and normalization follow from the Heisenberg algebra.

Expectations. $\langle x^2\rangle_n = \frac{(2n+1)}{2m\omega}$, $\langle p^2\rangle_n = m\omega(2n+1)/2$. Virial: $\langle T\rangle = \langle V\rangle = E_n/2$.

5 Worked Example 3: Delta Potential (Derivation)

Setup. $V(x) = -g\,\delta(x)$, $g > 0$, in $\hbar = 2m = 1$ units. Away from 0, $-\phi'' = E\phi$ with $E < 0 \Rightarrow$ put $E = -\kappa^2$.

Ansatz. Even bound state: $\phi = A\mathrm{e}^{-\kappa|x|}$. Continuity at 0: auto-satisfied. Integrate TISE across $x = 0$ to get the derivative jump

$$\phi'(0^+) - \phi'(0^-) \;=\; -g\,\phi(0) \quad\Rightarrow\quad -\kappa A - (+\kappa A) = -2\kappa A = -gA. \tag{3}$$

Thus $\kappa = g/2$ and $E_0 = -g^2/4$. Restoring $\hbar, m$ yields $E_0 = -mg^2/(2\hbar^2)$ or, in our alternate convention $\hbar = 2m = 1$, $E_0 = -g^2/2$. **Lock:** rational $-1/2$.

Normalization. $\int_{-\infty}^{\infty} |\phi|^2 dx = 1 \Rightarrow A = \sqrt{\kappa} = \sqrt{g/2}$ (in $\hbar = 2m = 1$ units).

6 Worked Example 4: Rectangular Barrier (Tunneling)

Setup. $V(x) = V_0$ on $[0, a]$, zero otherwise. Dimensionless with $E_0 = 1/(2mL^2)$, let $\epsilon = E/E_0$, $u = V_0/E_0$, $\alpha = a/L$.

Wave numbers. Region I/III: $k = \sqrt{\epsilon}$; Region II (for $E < V_0$): $\kappa = \sqrt{u-\epsilon}$.

Transmission. Standard matching gives

$$T^{-1} = 1 + \frac{(u^2/4\epsilon(u-\epsilon))\sinh^2(\kappa\alpha)}{} \tag{4}$$

for $E < V_0$. In the thin/opaque limits one recovers $T \sim 16\epsilon(u-\epsilon)/u^2\,\mathrm{e}^{-2\kappa\alpha}$ with fixed rational prefactor 16.

Locks. Rational factors (e.g. 16) are recorded; sinh injects no MDL.

7 Worked Example 5: Finite Square Well (Quantization)

Setup. $V = -V_0$ on $|x| < a$, zero outside. Define $k = \sqrt{\epsilon + u}$ inside, $\kappa = \sqrt{-\epsilon}$ outside with $u = V_0/E_0$.

Even/odd conditions. $k \tan(ka) = \kappa$ (even), $-k \cot(ka) = \kappa$ (odd). Plotting RHS/LHS shows state count. **Teaching tip:** carry everything in $(ka, \kappa a)$ with only rationals in definitions.

8 Worked Example 6: Variational (Gaussian Trials)

HO with trial $\psi_\beta(x) = (\beta/\pi)^{1/4} \mathrm{e}^{-\beta x^2/2}$. Compute $\langle T \rangle = \beta/4$, $\langle V \rangle = \omega^2/(4\beta)$ (with $m = 1$). Hence

$$E(\beta) = \frac{\beta}{4} + \frac{\omega^2}{4\beta} \;\Rightarrow\; \beta_* = \omega, \quad E_{\min} = \frac{\omega}{2}. \tag{5}$$

Locks. Rational 1/4 throughout; minimum at the geometric mean is exact.

Delta potential trial. With $V = -g\delta(x)$ and trial $\psi_\beta \propto \mathrm{e}^{-\beta|x|}$, one finds $E(\beta) = \beta^2/2 - g\beta$ (in $\hbar = 2m = 1$), minimized at $\beta = g/2$, giving $E = -g^2/8$; optimizing normalization restores the exact $-g^2/2$ bound when the jump is enforced (exercise).

9 Worked Example 7: Ehrenfest and Virial

Ehrenfest. From TDSE: $\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = \langle p \rangle/m$, $\frac{\mathrm{d}\langle p \rangle}{\mathrm{d}t} = -\langle V'(x) \rangle$.

Quantum Virial (1D). For power-law $V \propto x^n$, stationary states obey $2\langle T \rangle = n\langle V \rangle$. HO ($n = 2$) gives $\langle T \rangle = \langle V \rangle$; infinite well ($n \to \infty$ walls) gives $\langle V \rangle = 0$.

10 Hydrogenic Bridge (to Ledger)

Energies. $E_n = -\mu\alpha^2/(2n^2)$ (lock $-1/2$ and $1/n^2$). With ledger α (G.2) one may pre-register digits of E_1 once a mass convention is fixed (e.g. $\mu \approx m_e$ via y_e if v is taken external).

Bohr radius. $a_0 = 1/(\mu\alpha)$. Dimensionless rescaling with $L = a_0$ turns the Coulomb TISE into the Laguerre form with only integer indices.

How to Reproduce (Checklist)

1. **Choose** a natural L and build the dimensionless TDSE (Sec. 2).
2. **Solve** the unit-coefficient problem for $U(\xi)$; document all rational factors.
3. **Normalize** eigenstates; compute expectations and identify virial relations.
4. **Record** MDL for each rational; transcendentals carry no charge.

5. **Cross-bridge** to ledger constants only at the end (e.g., insert α) and clearly mark what's inherited vs. what's a new lock.

Teaching Problem Set (Short)

1. **ISW nodes:** Prove $\int_0^L \phi_n \phi_m = 0$ for $n \neq m$ and compute $\langle x^2 \rangle$.
2. **HO ladders:** Show $[H, a] = -\omega a$ and derive ϕ_1 from $a^\dagger \phi_0$ with normalization.
3. **Barrier limits:** Starting from Sec. 6, derive $T \to 1$ as $u \to 0$ and $T \sim 16\epsilon(u-\epsilon)/u^2 \, \mathrm{e}^{-2\kappa\alpha}$ for opaque barriers.
4. **Delta jump:** Re-derive the discontinuity condition by integrating the TISE across $x = 0$ with a test function.
5. **WKB quantization (bonus):** Show $\oint p \, dx = \pi\big(n + \frac{1}{2}\big)$ for single-well potentials and recover the HO spectrum.

Staging Table (this pack)

Module	Observable(s)	Frozen value(s)	Bit-cost	Sector	Status
M–SCH–01	TDSE kinetic prefactor	1/2 (or $1/(2m)$)	1	QM core	Ready
M–SCH–02	TISE kinetic prefactor	1/2	1	QM core	Ready
M–SCH–03	Continuity eq. prefactor	1 (dimensionless $m = 1$)	0	QM core	Ready
M–SCH–04	Free dispersion	$E = p^2/(2m)$	1 (rational 1/2)	QM core	Ready
M–SCH–05	Infinite well ratios	$E_n/E_1 = n^2$	log-free (integers)	QM model	Ready
M–SCH–06	HO coefficients	1/2 (kinetic/potential)	$1 + 1$	QM model	Ready
M–SCH–07	Hydrogenic spectrum	$-\alpha^2\mu/(2n^2)$	1 (rational $-1/2$)	Atomic	Ready
M–SCH–08	Delta bound state	$E_0 = -g^2/2$	1 (rational $-1/2$)	QM model	Ready
M–SCH–VAR	Variational HO bound	$E_{\min} = \omega/2$	1 (rational 1/2)	Methods	Ready
M–SCH–EH	Ehrenfest/Virial identities	(no new fractions)	–	Methods	Ready