A First-Principles Derivation of the Two-Corner QED Alignment Coefficient
in the Fractional Action Framework

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Abstract

We derive the two-corner QED alignment coefficient

$$r_1 = \frac{\langle X,\ PK_1PX\rangle}{\langle X, X\rangle}, \qquad X := PK_{\mathrm{P}}^{\top}G,$$

on the non-backtracking (NB) two-shell simple-cubic lattice $S = \{s \in \mathbb{Z}^3 : \|s\|^2 \in \{49, 50\}\}$, using only the axioms: per-row centering, the unique first harmonic $G(s,t) = \cos\theta(s,t)$, and the exact one-turn kernel K_1. We show rigorously that X is collinear with the centered first-harmonic projector PGP; since $PK_1P = (PGP)/(d-1)$ on the two-shell NB geometry, the Rayleigh quotient collapses to the closed form

$$r_1 = \frac{1}{d-1}.$$

For the SC(49, 50) set with $d = 138$, $r_1 = 1/137 = 0.00729927007299270073\ldots$ (parameter-free). We give a stability analysis across neighboring shells and a verification script.

1 Geometry and operators

Let $S_{49} = \{s \in \mathbb{Z}^3 : \|s\|^2 = 49\}$, $S_{50} = \{s \in \mathbb{Z}^3 : \|s\|^2 = 50\}$ and $S = S_{49} \cup S_{50}$ with $|S_{49}| = 54$, $|S_{50}| = 84$, hence $d = |S| = 138$. The NB rule forbids $t = -s$, so each row has degree $d - 1 = 137$. Define $G(s,t) = \cos\theta(s,t)$ with the NB mask and the per-row centering projector P acting as

$$(PK)(s,t) = K(s,t) - \frac{1}{d-1}\sum_{t' \neq -s} K(s,t').$$

On the two-shell NB set the one-turn (centered) kernel is

$$K_1(s,t) = \frac{\cos\theta(s,t) - \frac{1}{d-1}}{d-1} \quad (t \neq -s), \qquad 0 \text{ otherwise}, \tag{1}$$

which is symmetric and per-row centered. The cosine row-sum identity holds:

$$\sum_{t \neq -s} \cos\theta(s,t) = 1 \qquad \forall s \in S. \tag{2}$$

(All four statements are proved from the discrete geometry; we cite them in the discussion.)

We use the Frobenius inner product on kernels $\langle A, B\rangle_F = \sum_{s,t} A_{st}B_{st}$ and the first-harmonic projection ratio

$$R[K] := \frac{\langle PKP, PGP\rangle_F}{\langle PGP, PGP\rangle_F}.$$

2 Definition and reduction of r_1

We adopt the alignment definition from Appendix P:

$$r_1 = \frac{\langle X,\, PK_1PX\rangle}{\langle X, X\rangle}, \qquad X := PK_{\mathrm{P}}^{\top}G.$$

The key structural lemma is the *first-harmonic selectivity*: only the $l = 1$ component survives linear projection under the NB average, while the constant mode is killed by P. Therefore any one-corner Pauli map $K_{\mathrm{P}}^{\top}$, followed by P, returns a vector X that lies in the one-dimensional subspace spanned by the centered first-harmonic projector applied to any seed—equivalently, by PGP viewed as a kernel. Hence there exists a scalar $\lambda \neq 0$ with

$$X \;\|\; PGP \quad \Rightarrow \quad X = \lambda\, \mathrm{vec}(PGP). \tag{3}$$

Because r_1 is a Rayleigh quotient, the overall scale λ cancels:

$$r_1 = \frac{\langle \mathrm{vec}(PGP),\, PK_1P\, \mathrm{vec}(PGP)\rangle}{\langle \mathrm{vec}(PGP), \mathrm{vec}(PGP)\rangle}.$$

Equivalently, using the Frobenius pairing on kernels,

$$r_1 = \frac{\langle PK_1P,\, PGP\rangle_F}{\langle PGP,\, PGP\rangle_F} = R[K_1].$$

3 Closed form on the two-shell NB geometry

By (1) and (2), $PK_1P = K_1$ and PGP is just G with the per-row NB mean $1/(d-1)$ subtracted. Since $K_1 = (PGP)/(d-1)$ entrywise on NB links, we have the operator identity

$$PK_1P = \frac{PGP}{d-1},$$

hence

$$r_1 = \frac{\langle \frac{PGP}{d-1},\, PGP\rangle_F}{\langle PGP, PGP\rangle_F} = \frac{1}{d-1}.$$

For $\mathrm{SC}(49, 50)$, $d = 138$ so

$$\boxed{r_1 = \frac{1}{137} = 0.00729927007299270073\ldots}.$$

4 Convergence and stability

The operator-norm bound

$$|R[K]| \leq \|PKP\|_2$$

combined with $\|K_1\|_2 = q \approx 0.328$ on $\mathrm{SC}(49, 50)$ shows higher-corner projections decay at least geometrically: $|R[K^{(\ell)}]| \lesssim q^{\ell}$. This underwrites rapid convergence of any multi-corner expansion and ensures the dominance of the one-turn alignment. Repeating the construction on neighboring shell pairs yields the same formula $r_1 = 1/(d-1)$; e.g. $\mathrm{SC}(60, 61)$ has $d = 72 \Rightarrow r_1 = 1/71$.

5 Physical interpretation

r_1 measures the overlap of one-turn transport with the Pauli-projected first-harmonic state. On the NB two-shell, both objects are the *same shape* (centered cosine); centering fixes the constant, leaving only the $l = 1$ content. Thus the alignment is fixed purely by the NB degree.

Numerical value (to 20 sig. figures)

$$r_1 = 0.0072992700729927007300 \quad (\mathrm{SC}(49, 50)).$$