Mathematics, Physics, Programming and Methodology of Science

Under construction

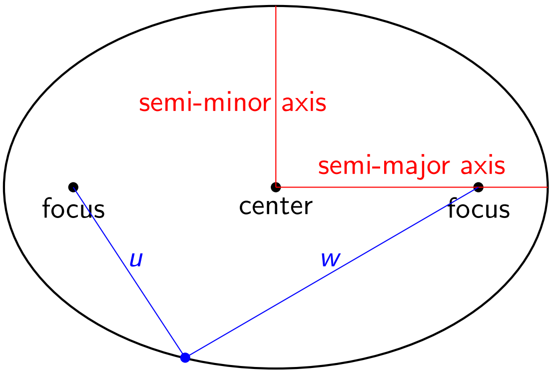
Jan Plaza

github.com/plazajan

February 2, 2025

Mathematics

Ellipse



- Draw a circle on a rubber sheet, and stretch it horizontally – you get an ellipse.
- Draw a circle on a vertically stretched rubber sheet, and let it shrink – you get an ellipse.
- $u + w =$ constant
- When the foci coincide, the ellipse is a circle.

Consider an ellipse with the foci at $(-f, 0)$ and $(f, 0)$ on a Cartesian plane, where $f \geq 0$.

Such an ellipse has a center at $(0, 0)$ and is symmetric with respect to:

- point symmetry with respect to $(0, 0)$,
- reflection with respect to the x-axis,
- reflection with respect to the y-axis.

Consider an ellipse with the foci at $(-f, 0)$ and $(f, 0)$ on a Cartesian plane, where $f \geq 0$.

left vertex, right vertex - the points where the ellipse intersects the x-axis.
lower vertex, upper vertex - the points where the ellipse intersects the y-axis.

- a - semi-major axis - the distance from the center to the right (or left) vertex.
- b - semi-minor axis - the distance from the center to the upper (or lower) vertex.
- c - linar eccentricity - the distance from the center to a focus.
- A - aphelion - the bigger of the distances from the foci to the right vertex.
- P - perihelion - the smaller of the distances from the foci to the right vertex.
- e - eccentricity $= c/a$.

$2c + 2(a - c) = 2\sqrt{b^2 + c^2}$ – from Pythagorean equation.
So, $2a = 2\sqrt{b^2 + c^2}$
So, $a = \sqrt{b^2 + c^2}$
So, $a^2 = b^2 + c^2$

$P = a - c$

$A = a + c$

$a = (A + P)/2$

$c = a - P = (A + P)/2 - P = (A - P)/2$

$b^2 = a^2 - c^2 = ((A + P)/2)^2 - ((A - P)/2)^2 = 4AP/4 = AP$
So, $b = \sqrt{AP}$

$e = c/a = \sqrt{a^2 - b^2}/a = \sqrt{1 - (b^2)/(a^2)}$
So, $c = ae = \sqrt{a^2 - b^2}$

A parametric equation of a circle with radius r and the center at $(0, 0)$:
$x = r \cos t$,
$y = r \sin t$,
where t in $[0, 2\pi)$.

A parametric equation of ellipse with semi-major axis a and semi-minor axis b and the center at $(0, 0)$:
$x = a \cos t$,
$y = b \sin t$,
where t in $[0, 2\pi)$.

Exercise

- Find a parametric equation of an ellipse with semi-major axis a and semi-minor axis b and the center at $(-C, 0)$, where C is a given number.
- Find a parametric equation of an ellipse with semi-major axis a and semi-minor axis b and the right focus in the center of the coordinate system.

Vectors and matrices in R^2

Adding vectors: ...

Length/magnitude of a vector: $|(x, y)| = \sqrt{x^2 + y^2}$

A unit vector in the direction of vector (x, y):
$v = |v| * u$, where u is the unit vector in the direction of v

2x2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
Determinant of this matrix: $\det M = ad - bc$

Area of a parallelogram spanned by two vectors $=$ the absolute value of the determinant of the matrix with such column vectors.

Physics

Newtonian mechanics, selected laws

- A body not subject to forces is at rest or moves straight at a constant speed.
- $F = ma$
- Law of gravity: $F = GMm/(r^2)$

Velocity is a vector (v_x, v_y) - may vary with time.
v_x - horizontal speed, v_y - vertical speed.
Speed is a scalar - the length/magnitude of velocity.

Acceleration is a vector (a_x, a_y)
(a_x, a_y)- horizontal and vertical acceleration.

For a body moving on a straight line, (v_x, v_y) and (a_x, a_y) are parallel.
For a body moving on a circular orbit, (v_x, v_y) and (a_x, a_y) are perpendicular.
For a body moving on a curve, (v_x, v_y) and (a_x, a_y) are not parallel.

Change of position after time dt :
$dx = v_x * dt$
$dy = v_y * dt$

Change of velocity after time dt :
$dv_x = a_x * dt$
$dv_y = a_y * dt$

Acceleration of a mass m orbiting a stationary mass M while at a distance r.
$a_x = -G * M * x/r^3$
$a_y = -G * M * y/r^3$

Centripetal force - the force that makes the body to follow a curved path:
$F = ma = mv^2/r$

$r = (x, y)$
$\hat{r} = (x/r, y/r)$ – unit vector in the direction of r

$\vec{F} = M\vec{a}$ – vector equation

$F = (GMm/r^2)$ – scalar equation
$\vec{F} = (GMm/r^2) * (-\hat{r})$ – vector equation

$m\vec{a} = -(GMm/r^2) * \hat{r}$ – vector equation
$\vec{a} = -(GM/r^2) * \hat{r}$ – vector equation
$a_x = -(GM/r^2) * x/r = -GMx/r^3$ - scalar equation
$a_y = -GMy/r^3$ – scalar equation

In our simulation:
$v_x := v_x + a_x t$ – scalar equation
$v_y := v_y + a_y t$ – scalar equation

Planetary motion (celestial mechanics)

The two-body problem parameters.

G - the gravitational constant $= 6.67430e{-}11 \;\; \mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$.

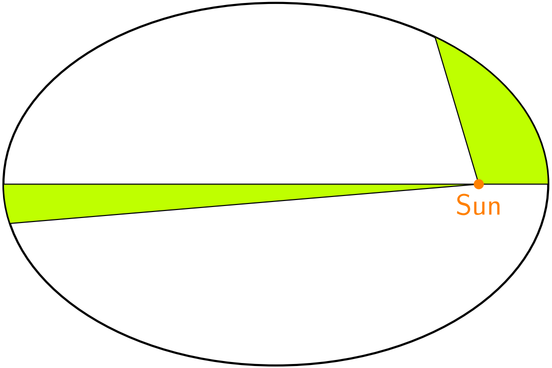
M - the bigger mass (Sun, considered stationary)
m - the smaller mass (planet, given some initial velocity, subject to M's gravity.)

$\mu = G(M + m)$ – gravitational parameter
approximately, $\mu \approx GM$ when $m \ll M$.

r - the distance between (the centers of) M and m (varies with time)
v - speed of m at its orbit around stationary M (varies with time)

T - orbital period of m in the movement around M.
a - semi-major axis of the elliptical orbit of m around M.

Kepler's laws, 1609 - 1619



Empirical laws of planetary motion
(based on Tycho's observations):

1. The orbit of a planet is an ellipse with the Sun in a focus.
2. The line joining a planet and the Sun sweeps out equal areas during equal intervals of time. (Based on observations of Mars.)
3. The square of the orbital period is proportional to the cube of the semi-major axis.

Enhancements to Kepler's laws

Kepler's 2nd law, enhanced, vis viva equation:

$$v^2 = G(M+m)\left(\frac{2}{r} - \frac{1}{a}\right)$$

Kepler's 3rd law, enhanced:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)}$$

where:
T – orbital period,
a – semi-major axis,
G – gravitational constant,
$M,\ m$ – masses,
v – speed,
r – radius i.e. distance M to m.

Keplers 1st law:

The planet moves on an elliptical orbit with the Sun in one focus.

Keplers 2nd law

The planet moves at such a speed that:

- The line connecting the Sun and the planet sweeps equal areas in equal periods of time,
- The area of the parallelogram spanned by vectors r and v is constant
- The determinant of the square matrix with columns r and v is constant.
- Vis Viva Equation: $v^2 = G(M+m)(2/r - 1/a)$

Keplers 3rd law

- $T^2/a^3 = \text{const}$
- If orbital period in years, and semi-major axis in AU (astronomical units): $P^2 = a^3$
- $T^2/a^3 = 4\pi^2/(G(M+m))$
  i.e. $T = 2\pi\sqrt{a^3/(G(M+m)}$
  (If using G in $\mathrm{m}^3/(\mathrm{kg} * \mathrm{s}^2)$, make sure to use m, s, kg everywhere.)

In the case of circular orbits, the orbital period can be easily derived from:
$F = mv^2/r$ (centripetal)
$F = GMm/(r^2)$ (gravitational)

We can verify $T^2 = a^3$ for Mars:
Mars' sami major axis = 1.524 AU, so its orbital is
$T = \sqrt{1.524^3} \approx 1.88$ years.

The maximum speed of a planet (at the perihelion)

Vis Viva Equation: $v^2 = G(M + m)(2/r - 1/a)$.

So,
$v_{\text{max}} = \sqrt{G(M + m)(2/P - 1/a)}$ where $P =$ perihelion distance.

Orbit's parameters, from mass, perihelion and maximum speed

Solve the equation above for a:
$a = PG(M + m)/(2G(M + m) - P * v_{\max}^2)$

$c = a - P$ – the linear eccentricity, i.e. center-to-focus distance
$A = a + c$ – Aphelion distance (biggest distance from the Sun)
$b = \sqrt{A * P}$ – semi-minor axis
$T = \sqrt{4\pi^2 a^3/(G(M + m))}$ – orbital period in seconds

Mass of the Sun, Earth, Mars, Jupyter

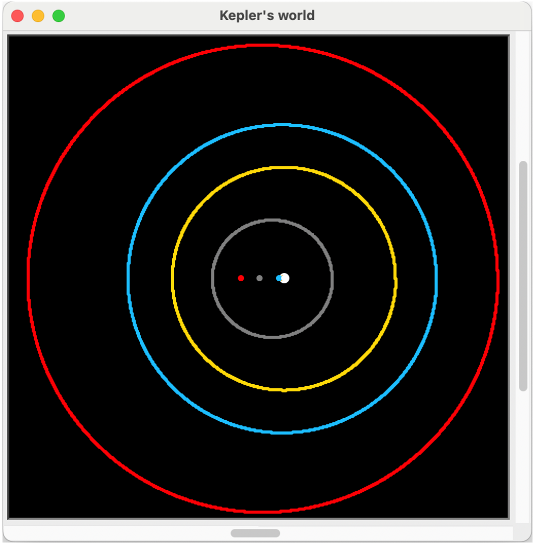
Can be calculated from the Earth-Sun distance and Earth's orbital period.
$T^2/a^3 = 4\pi^2/(GM)$ because $M \gg m$ (mass of Sun $\gg$ mass of Earth)
T - Earth's orbital period
a - Earth's semi-major axis $\approx$ distance to Sun
$T,\ a$ are observable.
From the equation, one can calculate the mass of the Sun.

Exercise

- Calculate the mass of the Earth from the distance to the Moon and the Moon's orbital period
- Calculate the mass of Mars can be calculated from the distance to its moon Phobos and Phobos' orbital period.
- Calculate the mass of Jupyter can be calculated from the distance to its moon Io and Io's orbital period.

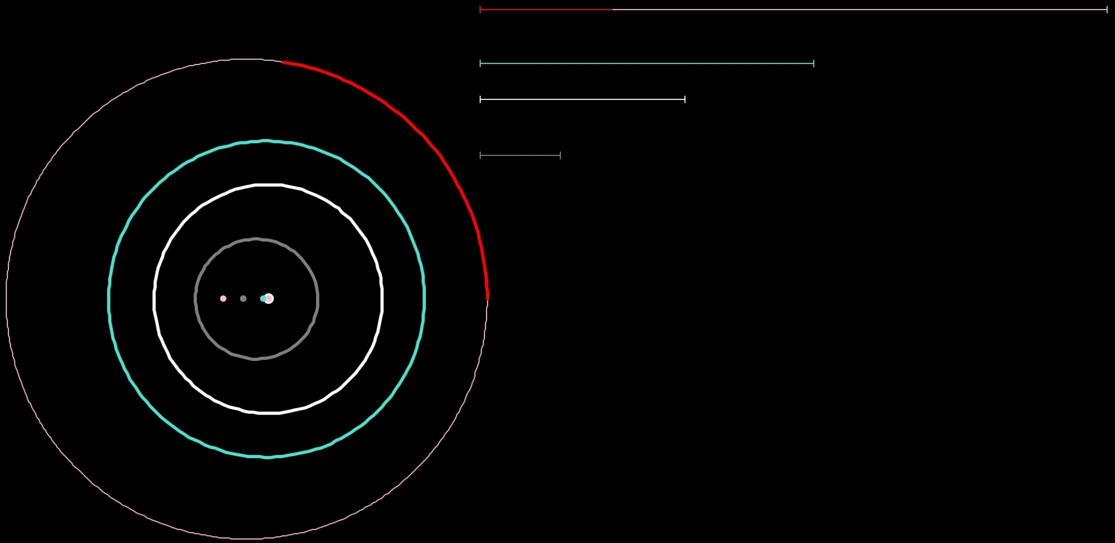
Computational Experiment

Kepler's world - a computer simulation to test Kepler's laws



- A computer simulation of orbits resulting from the continuing local effect of the Newton's law of gravity $F = G\frac{Mm}{r^2}$
- Tests if the simulated planets obey (global) Kepler's laws 1 and 3.
- The screenshot on the left:
  Mercury, Venus, Earth, Mars.
- Verified ellipses with the Sun in one focus.
- Orbital periods in days,
  simulation vs. actual:
  Mercury: 88.77 vs. .87.97 (0.91% error)
  Venus: 225.46 vs. 224.7 (0.34% error)
  Earth: 365.97 vs 365.26 (0.2% error)
  Mars: 687.85 vs 686.98 (0.13% error)
- Available at `github.com/plazajan`

Kepler's World – a computer simulation to test Kepler's laws



For every planet, Kepler's World

- draws the Sun as a white dot,
- draws in pink the ellipse predicted by Kepler's 1st law and its two foci.
- draws in pink a straight line segment, whose length is proportional to the orbital period of the planet predicted by Kepler's 3rd law; the greater the area sweep speed predicted by Vis Viva equation (from Kepler's 2nd law), the higher the line.
- Uses Newton's law of gravity to simulate the movement of the planet and marks it position (for Mars – in red),
- also graphs the value of the area sweep speed, from time 0 to the time of completion of the orbit.

Methodology of Science

Natural science

Natural science describes the natural world:

- physics (force, movement, particle, atom, radiation, ...),
  encompass astronomy and astrophysics,
- chemistry (element, molecule, reaction, ...)
- biology (cell, organism, species, evolution, ...)
- planetary science,
  encompass Earth, oceanic and atmospheric science.

Contrasted with social science:

- psychology (behavior of a person),
- anthropology, sociology (behavior of a society, culture, ...),
- political science (government, policy, ...),
- economics (supply, demand, trade, ...)

Note: Neuroscience (nervous system, brain, ...) bridges biology and psychology.

Logic, mathematics, natural science

Logic guides reasoning in mathematics and in natural science.

Mathematics consists of **theorems**. (For instance, the Pythagorean Theorem.)
A theorem is a statement that has a proof based on specific axioms/assumptions.
Theorems are universally valid – true forever and in any place in the universe.
Mathematics is a tool used in natural science.

Natural science consists of **laws of science**.
For instance, the Galilean law of addition of velocities: $v = v_1 + v_2$
– a person who walks at 2 mph on a train traveling at 20 mph is moving at 22 mph.
A law of science is our best description at the moment of an aspect of the natural world.
To be a law of science, a statement must be precise, **falsifiable** and consistent with available data. The Galilean law of addition of velocities has been falsified and is now considered only an approximate description of reality, applicable to velocities much smaller than the speed of light. Currently, Einstein's theory of relativity provides a new law of addition of velocities. This law may be falsified in the future.

The scientific method

Science is built using the **scientific method**:

- observe/experiment (obtain empirical evidence/data)
- construct a theoretical model,
- make predictions based on the model,
- test the predictions.

Methods of science:

- experimental/observational,
- theoretical/mathematical,
- computational.

Units of measurement and dimensional analysis:

International System of Units (SI), base units:
Mass: kilogram (kg) $\approx$ 2 pounds
Length: meter (m) $\approx$ 1 yard
Time: second (s),
(also ampere, kelvin, mole, candela.)

Speed – distance per time: $\mathrm{m/s} = \mathrm{m} \cdot \mathrm{s}^{-1}$.
Acceleration – speed change per second: $(\mathrm{m/s})/\mathrm{s} = \mathrm{m/s}^2 = \mathrm{m} \cdot \mathrm{s}^{-2}$.
Force – $F = ma$: Newton $= \mathrm{kg} \cdot \mathrm{m/s}^2 = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$.

Gravitational constant: G,
$F = G\frac{Mm}{r^2}$,
$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2} = x \cdot \mathrm{kg}^2\mathrm{m}^{-2}$.
So, G has dimension $x = \mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$.

Vis Viva Equation – speed vs. distance from the Sun:
$v^2 = G(M+m)(2/r - 1/a)$, where a is the length of the semi-major axis,
$(\mathrm{m} \cdot \mathrm{s}^{-1})^2 = \mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}(\mathrm{kg}+\mathrm{kg})(\mathrm{m}^{-1} - \mathrm{m}^{-1})$, i.e.
$\mathrm{m}^2 \cdot \mathrm{s}^{-4} = \mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}\mathrm{kg} \cdot \mathrm{m}^{-1}$.
This does not prove the Vis Viva Equation,
but shows there is nothing wrong with its unit dimensions.

Orbital period $T = 2\pi\sqrt{a^3/(G(M+m))}$
$\mathrm{s} = \sqrt{\mathrm{m}^3(\mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2})^{-1}(\mathrm{kg}+\mathrm{kg})^{-1}}$, i.e.
$\mathrm{s} = \sqrt{\mathrm{m}^3\mathrm{kg} \cdot \mathrm{m}^{-3}\mathrm{s}^2\mathrm{kg}^{-1}}$.
Nothing is wrong with unit dimensions.

Approximations

$$\pi = \texttt{3.14159265358979323846264433...}$$
$$\pi \approx \texttt{3.14159265}$$
$$\pi \approx \texttt{3.1415927}$$
$$\pi \approx \texttt{3.141593}$$
$$\pi \approx \texttt{3.14159}$$
$$\pi \approx \texttt{3.1416}$$
$$\pi \approx \texttt{3.142}$$
$$\pi \approx \texttt{3.14}$$
$$\pi \approx \texttt{3.1}$$
$$\pi \approx \texttt{3}$$

Absolute and relative errors

Tm – tera-meter = 10^{12} m = 1 000 000 000 000 meters.

Neptune's perihelion:
4.47 Tm, all digits significant, i.e.
4.47 ± 0.005 Tm

4 470 000 000 000 m could be an approximation of either of:

- 4 465 000 000 000 m
  absolute error = 4 470 000 000 000 - 4 465 000 000 000 = 5 000 000 000 m
  relative error = 5 000 000 000 / 4 465 000 000 000 $\approx$ 0.11%.
- 4 474 999 999 999.99 m
  absolute error = 4 474 999 999 999.99 - 4 470 000 000 000 = 4 999 999 999.99 m
  relative error = 4 999 999 999.99 / 4 474 999 999 999 $\approx$ 0.11%.

Calculations involving approximate values and the estimation of error are subjects of **numerical analysis** (which is not in the scope of this presentation).

Metric prefixes

peta	P	10^{15}	1 000 000 000 000 000
tera	T	10^{12}	1 000 000 000 000
giga	G	10^{9}	1 000 000 000
mega	M	10^{6}	1 000 000
kilo	k	10^{3}	1 000
hecto	h	10^{2}	100
deca	da	10^{1}	10
–	–	10^{0}	1
deci	d	10^{-1}	0.1
centi	c	10^{-2}	0.01
mili	m	10^{-3}	0.001
micro	μ	10^{-6}	0.000 001
nano	n	10^{-9}	0.000 000 001
pico	p	10^{-12}	0.000 000 000 001
femto	f	10^{-15}	0.000 000 000 000 001

Floating-point numbers

Floats are not like reals. Floats implement the IEEE 754-2008 standard.

- There are only finitely many floats (of any given type).
- Floats are discrete, while reals are dense and continuous.
- The farther from zero – the more sparse.
- There is a biggest numerical float. There is a smallest positive float.
- Most reals can be only approximated by floats.
- There are special float values:
  negative zero (-0), infinity (inf), negative infinity (-inf), not-a-number (NaN).
- Often, the actual result of an arithmetical operation on floats
  can be only approximated by a float.
- Associativity fails: $(1 + 1\text{e}100) + \text{-}1\text{e}100 = 0$ while $1 + (1\text{e}100 + \text{-}1\text{e}100) = 1$.

For instance, numeric values of “tiny floats” look as follows on the number line.

Sign, Exponent	00	01	10	11	Explanation
0 000	**0**	0.0625	0.125	0.1875	keep adding 1/16
0 001	0.25	0.3125	0.375	0.4375	keep adding 1/16
0 010	0.5	0.625	0.75	0.875	keep adding 1/8
0 011	**1**	1.25	1.5	1.75	keep adding 1/4
0 100	2	2.5	3	3.5	keep adding 1/2
0 101	4	5	6	7	keep adding 1
0 110	8	10	12	14	keep adding 2
0 111	inf	NaN	NaN	NaN	special values
1 000	-**0**	-0.0625	-0.125	-0.1875	
1 001	-0.25	-0.3125	-0.375	-0.4375	
1 010	-0.5	-0.625	-0.75	-0.875	
1 011	-**1**	-1.25	-1.5	-1.75	
1 100	-2	-2.5	-3	-3.5	
1 101	-4	-5	-6	-7	
1 110	-8	-10	-12	-14	
1 111	-inf	NaN	NaN	NaN	

In the previous table, 00, 01, 10, 11 in the header are values of “fraction”, aka “significand”, aka “mantissa”.

	Sign	Exponent	Fraction	Total	How many	Min pos	Max
“Tiny floats”	1 bit	3 bits	2 bits	6 bits	$55 \approx 2^6$	2^{-4}	$14 \approx 2^4$
1.4.3-minifloats	1 bit	4 bits	3 bits	8 bits	$239 \approx 2^8$	2^{-9}	$240 \approx 2^8$
Half-precision	1 bit	5 bits	10 bits	16 bits	$63487 \approx 2^{16}$	2^{-24}	$65504 \approx 2^{16}$
Single-precision	1 bit	8 bits	23 bits	32 bits	$\approx 2^{32}$	2^{-149}	$\approx 2^{128}$
Double-prec.	1 bit	11 bits	52 bits	64 bits	$\approx 2^{64}$	2^{-1074}	$\approx 2^{1024}$

Problem. For double-precision floats, calculate:

1. the exact number of numeric values other than -0, and the exact maximum numerical value;
2. the maximum relative error while approximating by a float a real number between 1 and the maximum numerical float.

Discrete vs continuous

Discrete simulation of a **continuous** process.

Recurrent/periodic process

Recurrent/periodic process - for instance planet's movement around the Sun.

Qualitative vs. quantitative statements

1. "a planet moves fastest when close to the Sun" vs. Kepler's 2nd law.
2. The line joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. $v^2 = G(M+m)\left(\frac{2}{r} - \frac{1}{a}\right)$

Local vs global

Local effects (of the law of gravity) vs. **global** properties (of orbits described by Kepler's laws).

Visualization vs. simulation.

- Could we make a simulation of planetary motion without a visualization?
- Could we make a visualization of planetary motion without an underlying simulation?

Abstraction and simplification

Abstraction - the process of removing from consideration those details which have no impact on the answer to the question.
Simplification - the process of removing from consideration those details which have minor impact on the answer to he question, or those which make the problem mathematically or computationally unmanageable. Simplification amounts to adding the problem new assumptions, which make it simpler.

While modeling the planetary motion in the Solar System:

- We abstract away the chemical composition of the planet, existence of life, the name of the planet, etc.
- We simplify the problem considering only one planet moving around the Sun (a two-body problem). The two-body problem is mathematically solvable, while a general three body system is chaotic, with an unpredictable behavior in the long term.

The Shell Theorem

Theorem (Isaac Newton, 1687)
A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.

We simplify the planetary motion problem assuming that the Sun and the planet are spherically symmetric (as they approximately are.)

Then, by the Shell Theorem we perform an abstraction – we consider point masses instead of three-dimensional bodies.

Representation

We assume that the Sun is in the center of a planar Cartesian coordinate system.

A planet is represented as a point and a velocity vector in 2 dimensions.

Ockham's razor

Given two models that give the same predictions, choose the simpler one.

For instance, even if the Ptolemaic model of the Sun and planets gave the same predictions as the Copernican/x model, we would choose the Copernican/Keplerian model because it is simpler.