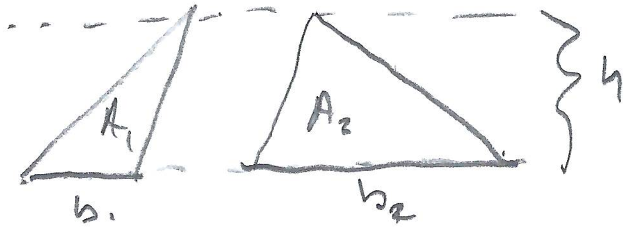
Euclid Book VI

Prop 1 Triangles which are under the same height are to one another as their bases.



Show: $\frac{A_1}{A_2} = \frac{b_1}{b_2}$

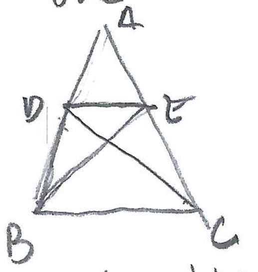
We know $A_1 = b_1 h$ and $A_2 = b_2 h$.

So, $h = \frac{A_1}{b_1}$ and $h = \frac{A_2}{b_2}$, and thus $\frac{A_1}{b_1} = \frac{A_2}{b_2}$

Multiply by A_2 and b_1 to get

$$\frac{A_1}{A_2} = \frac{b_1}{b_2}.$$

Prop 2a If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides proportionally.



$(A = \frac{1}{2}bh)$

Given $\overline{DE} \parallel \overline{BC}$.

show $BD : DA :: CE : EA$, i.e.,

$$\frac{AD}{BD} = \frac{AE}{CE}$$

By I.38, area $\triangle BDE$ = area $\triangle CDE$, so $\frac{\text{area}\,\triangle ADE}{\text{area}\,\triangle BDE} = \frac{\text{area}\,\triangle ADE}{\text{area}\,\triangle CDE}$

By VI.1, $\frac{\text{area}\,\triangle ADE}{\text{area}\,\triangle BDE} = \frac{AD}{BD}$ and $\frac{\text{area}\,\triangle ADE}{\text{area}\,\triangle CDE} = \frac{AE}{CE}$

So, by CN1, $\frac{AD}{BD} = \frac{AE}{CE}$.

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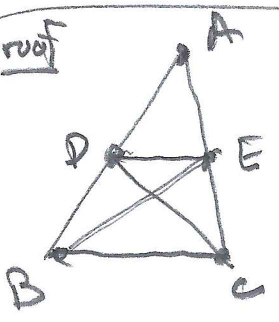
Prop 2b If the sides of a triangle be cut proportionally, the line joining the points of section will be parallel to the remaining sides of the triangle.

proof



(uses I.39)

Suppose $\frac{AD}{BD} = \frac{AE}{CE}$.

Then $\frac{\text{area } \triangle ADE}{\text{area } \triangle BDE} = \frac{\text{area } \triangle ADE}{\text{area } \triangle CDE}$.

Thus area $\triangle BDE$ = area $\triangle CDE$.

But these triangles share base DE, and triangles on the same base with equal area have the same height (they are in the same parallels).

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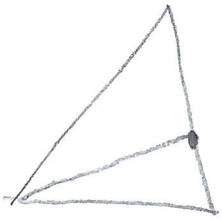
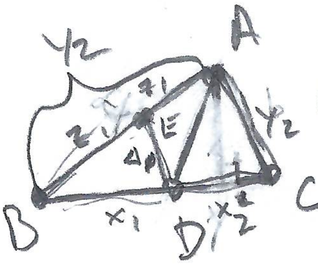
Prop 3a

If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, then the segments of the base will have the same ratio as the remaining sides of the triangle.





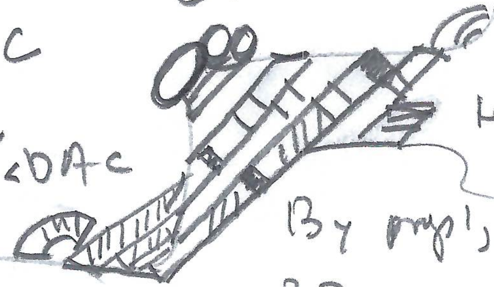
Claim: $\frac{BD}{CD} = \frac{BA}{CA}$

$\frac{x}{y} = \frac{c}{b}$

$\frac{x_1}{x_2} = \frac{y_1}{y_2}$

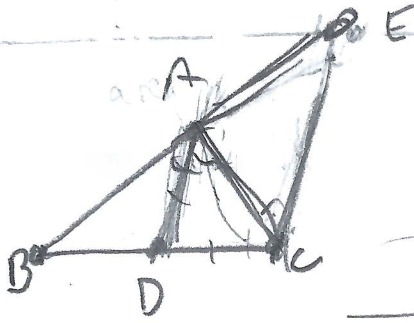
Let E = intersection of $\overleftrightarrow{AB}$ and line through C ∥ to $\overleftrightarrow{AD}$

Given: ∠BAD ≅ ∠DAC



By prop 1, $\frac{\text{area}\,\triangle ABD}{\text{area}\,\triangle ACD} = \frac{BD}{CD}$

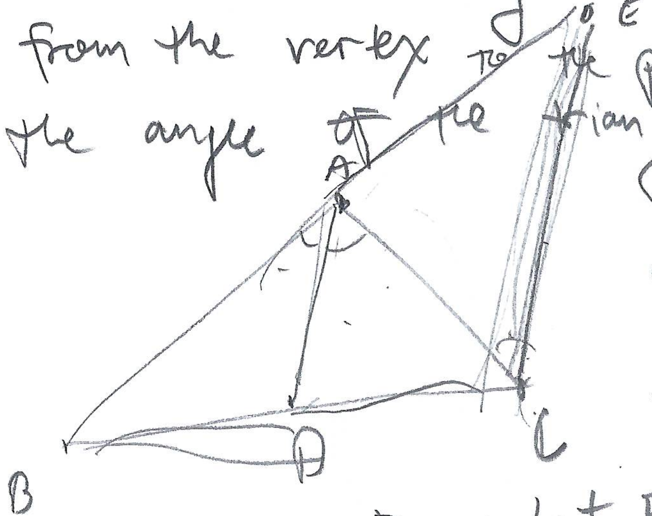
~~By~~



Claim	Reason
∠ACE ≅ ∠CAE	Alt interior angles
∠CAE ≅ ∠DAB	Given
∠ACE ≅ ∠DAB	CN1
∠DAB ≅ CEA	Corresp angles
∠ACE ≅ ∠CEA so, isosceles	CN1
$\overline{AC} \cong \overline{AE}$	Prop I.5
BD : DC :: BA : AE	Prop 2 VI
BD : DC :: BA : AC	CN1

Prop 3b Euclid Book VI

If the segments of the base of a triangle have the same ratio as the remaining sides, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.



Given $\frac{BD}{CD} = \frac{AB}{AC}$

Show: $\overline{AD}$ bisects $\angle BAC$.

Rearrange given to $\frac{AC}{CD} = \frac{AB}{AC}$

~~Draw~~ Let E be the point of intersection of $\overleftrightarrow{AB}$ and the line through C parallel to AD.

Then by prop VI.2, $\cancel{\frac{\text{area } \triangle BAD}{\text{area } \triangle BEC}} = \frac{AB}{BD} = \frac{AE}{CD} = \cancel{\frac{CD}{AC}}\ \frac{AC}{CD}$

Thus $AE = AC$, Thus $\angle E \cong \angle ECA$, so.

and $\angle ECA \cong \angle CAD$ (alt int.), so by C.N.1, $\angle E \cong \angle CAD$. But $\angle E \cong \angle BAD$ (corresponding), so $\angle CAD \cong \angle BAD$. Thus $\overline{AD}$ bisects $\angle BAC$.

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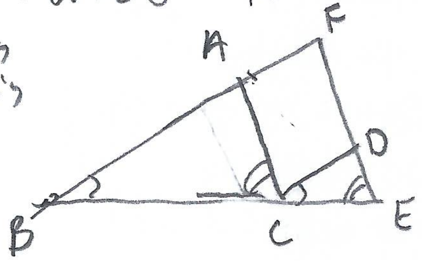
Prop 4 In equiangular triangles, the sides about equal angles are proportional.

pf: WLOG let the bases sit on a line w/ a shared endpoint: C

This is Euclid's proof.



V.16 says: $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$

V.22 says: $\frac{a}{b} = \frac{d}{e}$ and $\frac{b}{c} = \frac{e}{f} \Rightarrow \frac{a}{c} = \frac{d}{f}$.

Claim		Reason
$\angle ABC \cong \angle DCE$		Given
$\angle ACB \cong DEC$		Given
$\overline{BF} \parallel \overline{CD}$		Prop I.28 (alt interior)
$\overline{AC} \parallel \overline{FE}$		Prop I.28
AFCD is a parallelogram		Definition (?)
$\overline{AF} \cong \overline{CD}$		Prop I.34
$\overline{AC} \cong \overline{DF}$		Prop I.34
$\frac{AB}{AF} = \frac{BC}{CE}$		VI.2 since $\overline{AC} \parallel \overline{EF}$
$\frac{AB}{CD} = \frac{BC}{CE}$		Subst $AF = CD$
$\frac{AB}{BC} = \frac{CD}{CE}$	1	Prop V.16
$\frac{CE}{BC} = \frac{ED}{DF}$		VI.2 since $\overline{CD} \parallel \overline{BF}$
$\frac{CE}{BC} = \frac{ED}{AC}$		Subst $DF = AC$
$\frac{CE}{ED} = \frac{BC}{AC}$	2	Prop V.16

Claim	Reason
$\frac{AB}{AC} = \frac{CD}{ED}$	1, 2, Prop 22
That is what we wanted to show.	