TWO LINES AND A CIRCLE

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Abstract. In this document, we will examine situations which arise when two lines intersect a circle. There are five ways this can happen:

(1) The lines intersect on the circle, creating an inscribed angle.
(2) The lines intersect inside the circle.
(3) The lines intersect outside the circle, and both are tangent.
(4) The lines intersect outside the circle, one is tangent and one is secant.
(5) The lines intersect outside the circle, and both are secant.

In cases 2 through 5, we have computationally significant results regarding both segments and angles. We will address each of these cases separately.

1. Inscribed Angles

We summarize the main result and call it the Central Angle Theorem.

Proposition 1. Central Angle Theorem
The measure of an angle inscribed in a circle is half the measure of the arc it subtends.

The angle is right if and only if its base is a diameter (Thales's Theorem).

The angle is acute if and only if it subtends a minor arc.

The angle is obtuse if and only if it subtends a major arc.

Proof. Euclid Proposition III.20 proves the Central Angle Theorem in the case that the angle subtends a minor arc. Thales' Theorem supplies the case that the angle subtends a diameter, and is included in Euclid Proposition III.31. We provide the case that the angle subtends a major arc.

Let $\widehat{AC}$ be a minor arc on $\odot O$, and let B be a point on the arc and let Z be a point on its dual. We claim that $m\angle ABC = \frac{1}{2} m\widehat{AZC}$.

Let $\alpha = m\angle AOB$ and $\beta = m\angle OCB$. We know that $OA = OB = OC$, since they are radii of $\odot O$. Thus, $\triangle AOB$ and $\triangle BOC$ are isosceles. By the Base Angle Theorem, we see that $m\angle ABO = \alpha$ and $m\angle BOC = \beta$.

Since the sum of the measures of the angles in a triangle is $180°$, we compute that $m\angle AOB = 180 - 2\alpha$, and $m\angle BOC = 180 - 2\beta$. Thus, $m\widehat{AC} = m\angle AOC = m\angle AOB + m\angle BOC = 360 - 2(\alpha + \beta)$. This shows that the measure of the dual arc, which is the included arc of the angle, is $360° - (360° + 2(\alpha + \beta)) = 2(\alpha + \beta)$.

Therefore,

$$m\angle ABC = m\angle ABO + m\angle BOC = \alpha + \beta = \frac{1}{2} m\widehat{AZC}.$$

$\square$

Date: February 28, 2021.

2. Intersecting Chords

2.1. **Similar Triangles.** We wish to find a simpler proof of Euclid Proposition III.35. Our proof will use the notion of similar triangles, which we will study in Euclid Book VI.

Definition 1. We say that two triangles are *similar* if there is a correspondence between the vertices such that corresponding angles are congruent.

We write $\triangle ABC \sim \triangle DEF$ to mean that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$.

Proposition 2. Angle-Angle Similarity (AA)
If two triangles admit a correspondence between their vertices such that two pairs of corresponding angles are congruent, then the third pair is also congruent, and the triangles are similar.

Proof. Suppose $\triangle ABC$ and $\triangle DEF$ have $\angle A \cong \angle D$ and $\angle B \cong \angle EF$. Then

$$m\angle C = 180^\circ - (m\angle A + m\angle B) = 180^\circ - (m\angle D + m\angle E) = m\angle F.$$

Thus $m\angle C = m\angle F$, that is, $\angle C \cong \angle F$. $\square$

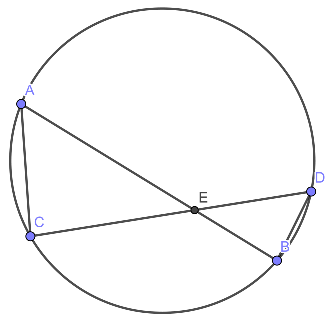
The following theorem is a consequence of Euclid Propositions VI.4 and VI.5, which we will prove later.

Proposition 3. Similarity Ratio Theorem
Two triangles are similar if and only if there exists a correspondence between the vertices such that the ratios of corresponding sides are equal. That is,

$$\triangle ABC \sim \triangle DEF \quad \Leftrightarrow \quad \frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC}.$$

2.2. **Intersecting Chord Segments.** First we prove that we have similar triangles, a fact that will make the proof of Euclid Proposition III.35 much easier.

Proposition 4. *Consider a circle and two chords, $\overline{AB}$ and $\overline{CD}$ which intersect at a point E.*



Then

$$\boxed{\triangle AEC \sim \triangle DEB.}$$

Proof. Recall that to prove that two triangles are similar, it suffices to show that two pairs of corresponding angles are congruent.

Join $\overline{AC}$ and $\overline{BD}$.

Claim	Reason
$m\angle AEC = m\angle DEB$	Vertical Angles (Prop I.15)
$m\angle CAE = \frac{1}{2} m\overset{\frown}{CB}$	Central Angle Theorem
$m\angle BDE = \frac{1}{2} m\overset{\frown}{CB}$	Central Angle Theorem
$m\angle CAE = m\angle BDE$	CN1
$\triangle AEC \sim \triangle DEB$	Angle-Angle Similarity

□

Proposition 5. Euclid Proposition III.35
Consider a circle and two chords, $\overline{AB}$ and $\overline{CD}$ which intersect at a point E. Then

$$\boxed{(AE)(BE) = (CE)(DE).}$$

Proof. Join $\overline{AC}$ and $\overline{BD}$ to form two triangles.

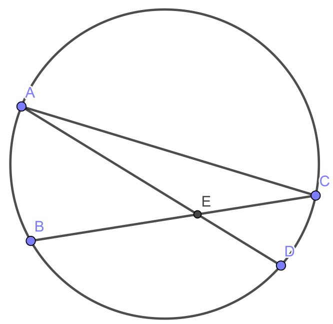
By Proposition 4, $\triangle AEC \sim \triangle DEB$. By the Similarity Ratio Theorem,

$$\frac{AE}{DE} = \frac{CE}{BE}.$$

Multiply both sides by BE and DE to get $(AE)(BE) = (CE)(DE)$. □

2.3. **Intersecting Chord Angles.**

Proposition 6. *Consider a circle and two chords, $\overline{AD}$ and $\overline{BC}$ which intersect at a point E.*



Then

$$\boxed{m\angle AEC = \frac{1}{2}(m\widehat{AB} + m\widehat{CD}).}$$

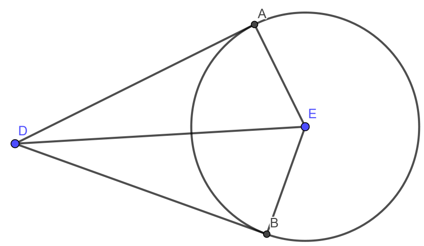
Proof. Join $\overline{AC}$.

Claim	Reason
$m\angle BCA = \frac{1}{2} m\widehat{AB}$	Central Angle Theorem
$m\angle CAD = \frac{1}{2} m\widehat{CD}$	Central Angle Theorem
$m\angle AEB = m\angle BCA + m\angle CAD$	Exterior Angle Theorem
$m\angle AEC = \dfrac{1}{2}(m\widehat{AB} + m\widehat{CD})$	CN1

$\square$

3. Tangent-Tangent

Problem 1. Consider a circle and a point D outside of it. Let $\overline{DA}$ and $\overline{DB}$ be tangent to the circle at points A and B.



Then

$$\boxed{DA = DB.}$$

Proof. Let E denote the center of the circle. Join $\overline{EA}$, $\overline{EB}$, and $\overline{DE}$.

Claim	Reason
$\overline{DA}$ and $\overline{DB}$ are tangent	Given
$\angle DAE$ and $\angle DBE$ are right	Euclid Proposition III.18
$\overline{DE} \cong \overline{DE}$	Reflexive Property of Congruence
$\overline{EA} \cong \overline{EB}$	Radii of $\odot EA$
$\triangle DAE \cong \triangle DBE$	HL Congruence Theorem
$\overline{DA} \cong \overline{DB}$	CPCTC

□

Problem 2. Consider a circle and a point D outside of it. Let $\overline{DA}$ and $\overline{DB}$ be tangent to the circle at points A and B. Then

$$\boxed{m\angle D + m\widehat{AB} = 180°.}$$

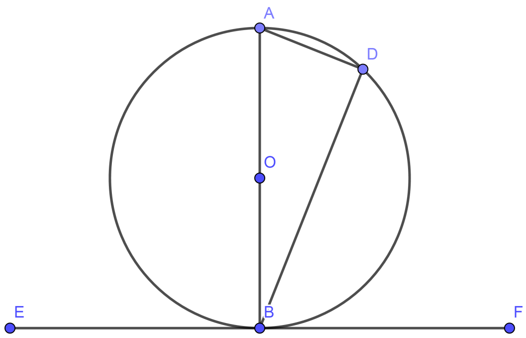
Proof. Let E denote the center of the circle. Join $\overline{EA}$ and $\overline{EB}$.

Consider the quadrilateral $DAEB$. The sum of the angles in any quadrilateral is 360°. However, $\angle DAE$ and $\angle DBE$ are right angles, by Euclid Proposition III.18. Thus, the sum of the other two angles in the quadrilateral is 180°. That is, $m\angle D + m\angle AEB = 180°$. But the measure of the arc $\widehat{AB}$ is, by definition, the measure of the subtending central angle $\angle AEB$. Thus, by substitution, $m\angle D + m\widehat{AB} = 180°$. □

4. Tangent-Secant

4.1. **Tangent Arc Relation.** The following includes part of Euclid Proposition III.32, and a consequence.

Proposition 7. *Let $\overleftrightarrow{EF}$ be tangent to circle $\odot OB$ at the point B. Let $\overline{AB}$ be a diameter of $\odot OB$. Let D be a point on $\odot OB$. Suppose $\angle FBD$ is acute.*



Then

$$\boxed{m\angle FBD = \frac{1}{2} m\widehat{BD}.}$$

Proof. Since $\overline{EF}$ is tangent to $\odot OB$ at B, $\overline{BF} \perp \overline{AB}$, so $\angle ABF$ is right. Thus $\angle FBD$ and $\angle ABD$ form a right angle.

Moreover, $\angle ADB$ is right, by Thales Theorem, so $\angle ABD$ and $\angle BAD$ together form a right angle.

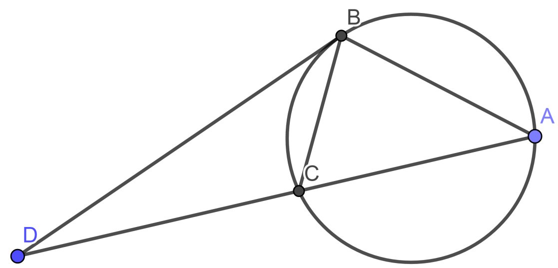
By Common Notion 3, $\angle FBD \cong \angle BAD$.

By the Central Angle Theorem, $m\angle BAD = \frac{1}{2} m\widehat{BD}$.

By Common Notion 1, $m\angle FBD = \frac{1}{2} m\widehat{BD}$. □

4.2. **Tangent-Secant Segments.** Next, we prove a lemma for our proof of Euclid Proposition III.36.

Proposition 8. *Consider a circle and a point D outside of it. Let $\overline{DB}$ touch the circle at B, and let $\overline{DA}$ cut the circle at C. Join $\overline{BC}$ and $\overline{BA}$.*



Then

$$\boxed{\triangle DBC \sim \triangle DAB.}$$

Proof. Recall that to prove that two triangles are similar, it suffices to show that two pairs of corresponding angles are congruent.

Claim	Reason
$m\angle BDC = m\angle ADB$	Reflexive Property
$m\angle DAB = \frac{1}{2}m\widehat{BC}$	Central Angle Theorem
$m\angle DBC = \frac{1}{2}m\widehat{BC}$	Proposition 7
$m\angle DBC = m\angle DAB$	CN1
$\triangle DBC \sim \triangle DAB$	Angle-Angle Similarity

□

Using the previous proposition, we supply an alternate proof of Euclid Proposition III.36.

Proposition 9. *Consider a circle and a point D outside of it. Let $\overline{DB}$ touch to the circle at B, and let $\overline{DA}$ cut the circle at C. Then*

$$\boxed{(DB)^2 = (DA)(DC).}$$

Proof. Join $\overline{AB}$ and $\overline{BC}$.

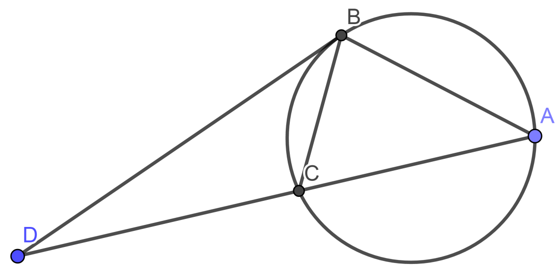
By Proposition 8, $\triangle DBC \sim \triangle DAB$. Therefore

$$\frac{DB}{DA} = \frac{DC}{DB}.$$

Multiply both sides by DA and DB to arrive at $(DB)^2 = (DA)(DC)$. □

4.3. **Tangent-Secant Angles.**

Proposition 10. *Consider a circle and a point D outside of it. Let $\overline{DB}$ touch to the circle at B, and let $\overline{DA}$ cut the circle at C. Join $\overline{BC}$ and $\overline{BA}$.*



Then

$$\boxed{m\angle D = \frac{1}{2}(m\widehat{AB} - m\widehat{BC}).}$$

Proof. Join $\overline{AB}$ and $\overline{BC}$.

By Proposition 7, $m\angle DBC = \dfrac{1}{2}m\widehat{BC}$.

By the Central Angle Theorem, $m\angle ACB = \frac{1}{2}m\widehat{AB}$.

However, by the Exterior Angle Theorem, $m\angle ACB = m\angle D + m\angle DBC$.

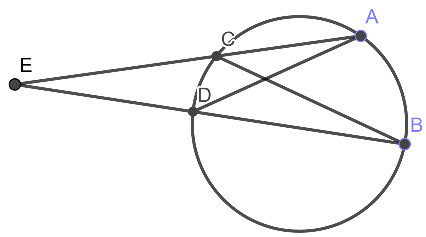
Thus

$$m\angle D = m\angle ACB - m\angle DBC = \frac{1}{2}m\widehat{AB} - \frac{1}{2}m\widehat{BC};$$

that is, $m\angle D = \dfrac{1}{2}(m\widehat{AB} - m\widehat{BC})$. □

5. Secant-Secant

Proposition 11. *Consider a circle and a point E outside of it. Let A, B be points on the opposite side of the circle, and let $\overline{EA}$ cut the circle at C and $\overline{EB}$ cut the circle at D.*



Then

$$\boxed{\triangle EAD \sim \triangle EBC.}$$

Proof. Join $\overline{AD}$ and $\overline{BC}$.

Claim	Reason
$m\angle AEC = m\angle BEC$	Reflexive Property
$m\angle A = \frac{1}{2}m\widehat{CD}$	Central Angle Theorem
$m\angle B = \frac{1}{2}m\widehat{CD}$	Central Angle Theorem
$m\angle A = m\angle B$	Common Notion 1
$\triangle EAD \sim \triangle EBC$	Angle-Angle Similarity

□

Proposition 12. *Consider a circle and a point E outside of it. Let A, B be points on the opposite side of the circle, and let $\overline{EA}$ cut the circle at C and $\overline{EB}$ cut the circle at D. Then*

$$\boxed{(EA)(EC) = (EB)(ED).}$$

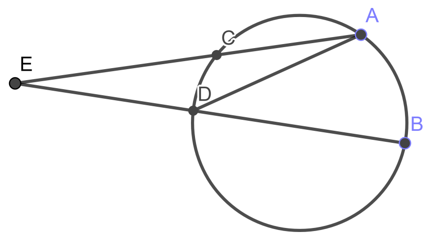
Proof. Join $\overline{AD}$ and $\overline{BC}$.

By Proposition 11, $\triangle EAD \sim \triangle EBC$. Therefore, ratios of corresponding sides are equal, so

$$\frac{EA}{EB} = \frac{ED}{EC}.$$

Multiply by EB and EC to get $(EA)(EC) = (EB)(ED)$. □

Proposition 13. *Consider a circle and a point E outside of it. Let A, B be points on the opposite side of the circle, and let $\overline{EA}$ cut the circle at C and $\overline{EB}$ cut the circle at D.*



Then

$$\boxed{m\angle E = \frac{1}{2}(m\overset{\frown}{AB} - m\overset{\frown}{CD}).}$$

Proof. Join $\overline{AD}$.

Claim	Reason
$m\angle A = \frac{1}{2}m\overset{\frown}{CD}$	Central Angle Theorem
$m\angle ADB = \frac{1}{2}m\overset{\frown}{AB}$	Central Angle Theorem
$m\angle ADB = m\angle E + m\angle A$	Exterior Angle Theorem
$m\angle ADB = m\angle E + \frac{1}{2}m\overset{\frown}{CD}$	Substitution
$m\angle E + \frac{1}{2}m\overset{\frown}{CD} = \frac{1}{2}m\overset{\frown}{AB}$	Common Notion 1
$m\angle E = \frac{1}{2}(m\overset{\frown}{AB} - m\overset{\frown}{CD})$	Algebra

□

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