Copy the statement of the problem on a piece of $8\frac{1}{2} \times 11$ piece of blank computer paper, and write the solution underneath. Write neatly. Mathematics should always be written in grammatically correct English, in complete sentences.

If G is a group, H is a subgroup of G, and K is a subgroup of H, then K is a subgroup of G.

If G is a group, and H and K are subgroups of G, then their intersection $H \cap K$ is a subgroup of G.

A permutation $\alpha \in S_n$ is called *even* if it can be written as a product of an even number of transpositions; otherwise it is called *odd*. Exactly half of the permutations in S_n are even.

Set

$$A_n = \{\alpha \in S_n \mid \alpha \text{ is even}\}.$$

Then A_n is a subgroup of S_n, called the *alternating subgroup*.

Let H be a subgroup of S_n. Then either H consists of even permutations or exactly half of the permutations in H are even. Thus either $H \subset A_n$, in which case $H \cap A_n = H$, or $H \cap A_n$ is exactly half of H. We outline the proof. Suppose that H is not contained in A_n and let $K = H \cap A_n$; we want to show that $|H| = 2|K|$. Let $\alpha \in H$ be an odd permutation. Set $\alpha K = \{\alpha\kappa \mid \kappa \in K\}$. Then $K \cup \alpha K = H$, $K \cap \alpha K = \varnothing$, and $|K| = |\alpha K|$.

Let $\rho, \tau \in S_n$ be given by

$$\rho = \texttt{(1 2 ... n)} \quad \text{and} \quad \tau = \begin{cases} \texttt{(2 n)(3 n-1) ... ((n+1)/2 (n+3)/2)} & \text{if } n \text{ is odd;} \\ \texttt{(2 n)(3 n-1) ... (n/2 (n+4)/2)} & \text{if } n \text{ is even.} \end{cases}$$

Set

$$D_n = \{\epsilon, \rho, \rho^2, \dots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \dots, \tau\rho^{n-1}\} \subset S_n.$$

Then D_n is a subgroup of S_n, called the *dihedral subgroup*. The proof that this is a subgroup follows from the identity $\tau\rho = \rho^{n-1}\tau$.

Set $K_n = D_n \cap A_n$. Then K_n is a subgroup of S_n, and either $K_n = D_n$ or K_n is exactly half of D_n. This quiz examines the relationship between n and the structure of the group K_n.

Problem 1. Let $n = 4$.
(a) Compute ρ and τ in this case.
(b) Show that K_4 is a noncyclic abelian subgroup of S_4.

Problem 2. Let $n = 5$.
(a) Compute ρ and τ in this case.
(b) Show that $K_5 = D_5$.

Problem 3. Let $n = 7$.
(a) Compute ρ and τ in this case.
(b) Show that K_7 is a cyclic subgroup of S_7.

Problem 4. Try to generalize the previous problems: what can you say about K_n in the following cases?
(a) $n \equiv 0 \pmod 2$
(b) $n \equiv 1 \pmod 4$
(c) $n \equiv 2 \pmod 4$
(d) $n \equiv 3 \pmod 4$