1. Characterization of Rational Numbers

We have defined the rational numbers as fractions, where the numerator and denominator are integers. We have defined real numbers as decimal expansions, which correspond to points on a line. Now we ask whether are not it makes sense to view rational numbers as real numbers; do they have a position on the number line?

The answer is yes; using decimal long division, we can find the decimal expansion of any rational number. Indeed, we have seen that rational numbers have decimal expansions that either terminate, or are repeating, with repeating part of length less than the denominator. This of course is under the assumption that the fraction is reduced; that is, in "lowest form", so that any common prime factors in the numerator and denominator have been cancelled.

If a decimal expansion terminates, we may view it as having a repeating zero at the end; for example, $\frac{1}{8} = 0.125 = 0.125\overline{0}$. With this understood, it is true and convenient to say that *all* rational numbers have a repeating decimal expansion. For brevity, let us call any number with a repeating decimal expansion a *repeater*.

The next question is whether or not this is a *complete characterization* of the rational numbers. Is every real number with a repeating decimal expansion a rational number?

The answer again is yes; we will be convince of this if we know how to convert a repeating decimal expansion into a fraction. There is a trick for doing this.

Example 1. Write $0.\overline{3}$ as a fraction.

Solution. Let $x = 0.\overline{3}$. We may write x as $x = 0.3\overline{3}$. Then $10x = 3.\overline{3}$. If we subtract x from $10x$, the repeating parts cancel:

$$\begin{aligned} 10x &= 3.\overline{3} \\ -\,x &= 0.\overline{3} \\ \hline 9x &= 3 \end{aligned}$$

Divide both sides by 9 to see that $x = \frac{3}{9} = \frac{1}{3}$. □

Example 2. Write $1.23\overline{4}$ as a fraction.

Solution. Let $x = 1.23\overline{4}$. Then $10x = 12.34\overline{4}$. Subtract these equations, and the repeating parts cancel:

$$\begin{aligned} 10x &= 12.34\overline{4} \\ -\,x &= 1.23\overline{4} \\ \hline 9x &= 11.11 \end{aligned}$$

Divide both sides by 9 to see that $x = \frac{11.11}{9} = \frac{1111}{900}$. □

Example 3. Write $9.8\overline{765}$ as a fraction.

Solution. Let $x = 9.8\overline{765}$. Then $1000x = 9876.5\overline{765}$. Subtract these equations, and the repeating parts cancel:

$$\begin{aligned} 1000x &= 9876.5\overline{765} \\ -\ x &= 9.8\overline{765} \\ \hline 999x &= 9866.7 \end{aligned}$$

Divide both sides by 999 to see that $x = \dfrac{9866.7}{999} = \dfrac{98667}{9990} = \dfrac{10963}{1110}$. $\square$

Here are the steps we took.

- We are given a number. Call it x. Let n be the length of the repeating part.
- Multiply x by 10^n to shift the decimal point right by n places. So for example if $n = 4$, multiply by 10000.
- Subtract x from $10^n x$. You now have $(10^n - 1)x =$ a decimal number with no repeating part .
- Divide both sides by $(10^n - 1)$ to get $x =$ a fraction whose top might have a decimal .
- If the top has a decimal part, multiply the top and bottom by a power of ten to make the top an integer.

We now know how to convert any reduced fraction into a decimal expansion, and that when we do so, the expansion will either terminate, or repeat such that the length of the repeating part is less than the denominator. Moreover, we also know how to take any repeating decimal expansion, and find a fraction which represents the same number. We are therefore convinced of the following proposition.

Proposition 1. *A real number is rational if and only if it has a terminating or repeating decimal expansion.*