Algebra II
Dr. Paul L. Bailey

Lesson 0804
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1. Sets, Elements, and Subsets

A *set* is a collection of things. The things in the set are called *elements* of the set. The things in the set are often numbers, but they don't have to be.

We can describe a set with a sentence, such as "the set of all cars in a parking lot", "the set of all cards in a deck", or "the set of integers between 2 and 7.

If a set is small enough, we can describe it by listing its elements, and surrounding the list with braces. For example, "the set of integers between 2 and 5 may be written as $\{2,3,4,5\}$. This way of specifying a set is called *roster notation*.

A set is completely determined by the elements it contains. Two sets are equal if and only if they contain exactly the same elements.

There is no notion of order in a set. The set is the same, no matter what order you list the elements. For example,

$$\{2,3,4,5\} = \{3,2,5,4\} = \{5,4,3,2\};$$

these are all the same set.

There is no notion of multiplicity in a set. The set is the same, no matter how often you list an element. For example,

$$\{2,3,4,5\} = \{2,2,3,4,4,4,5\} = \{4,2,3,4,3,3,2,2,5,2,2,2\}.$$

A thing is either in a set, or it is not; it is not in a set "multiple times".

Sets are often written uses capital letters. Let A be a set. Elements are often written using lowercase letters. Let a be a number. The notation

$$a \in A \quad \text{means} \quad \text{"}a \text{ is an element of } A\text{"}.$$

So $a \in A$ is a statement which is either true or false; it is true if a is in A, and it is false otherwise. The notation

$$a \notin A \quad \text{means} \quad \text{"}a \text{ is not an element of } A\text{"}.$$

Let $B = \{2,3,4,5\}$. Then $2 \in B$ is true, but $8 \in B$ is false. Thus $8 \notin B$.

Let A and B be sets. If all of the elements in B are also in A, we say that B is a *subset* of A. The notation

$$B \subset A \quad \text{means} \quad \text{"}B \text{ is a subset of } A\text{"}.$$

That is, $B \subset A$ is statement which is either true or false. Thus $\{1,3,5\} \subset \{1,2,3,4,5\}$ is true, but $\{3,6\} \subset \{1,2,3,4,5\}$ is false.

2. Standard Sets

A set containing no elements is called the *empty set* and is denoted $\varnothing$. Since a set is determined by its elements, there is only one empty set. Note that the empty set is a subset of any set.

The following familiar sets of numbers have standard names:

$$
\begin{aligned}
\text{Natural Numbers:} &\quad \mathbb{N} = \Big\{ 1,2,3,\dots \Big\} \\
\text{Integers:} &\quad \mathbb{Z} = \Big\{ \dots,-2,-1,0,1,2,\dots \Big\} \\
\text{Rational Numbers:} &\quad \mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p,q \in \mathbb{Z}, q \neq 0 \right\} \\
\text{Real Numbers:} &\quad \mathbb{R} = \Big\{ \text{ numbers given by decimal expansions } \Big\} \\
\text{Complex Numbers:} &\quad \mathbb{C} = \Big\{ a+ib \mid a,b \in \mathbb{R} \text{ and } i^2 = -1 \Big\}
\end{aligned}
$$

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

We invented the natural numbers in order to count. As such, the natural numbers are an ordered set. We can add and multiply natural numbers; that is, the sum or product of two natural numbers is also a natural number. We say that the set of natural numbers is *closed* under addition and multiplication. However, cannot always subtract natural numbers; for example, $3-5$ is not a natural number.

We invented the integers in order to subtract. The integers are also ordered; if a and b are natural numbers, then $-b \leq -a$ if and only if $a \leq b$. The integers are closed under addition, subtraction, and multiplication; however, they are not closed under division. The natural numbers are the positive integers.

We invented the rational numbers in order to divide. The rational numbers also are ordered; if a, b, c, d are positive integers, then $\frac{a}{b} \leq \frac{c}{d}$ if and only if $ad < bc$. For example, to see if $\frac{7}{9} < \frac{11}{14}$, check if $7 \cdot 14 < 9 \cdot 11$. Since $98 < 99$, we know that $\frac{7}{9} < \frac{11}{14}$.

We invented the real numbers in order to compute distances. If we leave home and walk 1 mile to the east, then 1 mile to the north, the distance to walk home is $\sqrt{2}$. We can show that $\sqrt{2}$ is not a rational number; thus, we need a new idea for distances.

Proposition 1. *$\sqrt{2}$ is not rational.*

Proof. Suppose that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ for some integers p and q. We may assume that p and q have no common factors, for if they did, we could cancel them. Then $q\sqrt{2} = p$, so $2q^2 = p^2$, so p^2 is even. This implies that p is even.

Since p is even, $p = 2k$ for some integer k. But then $2q^2 = (2k)^2 = 4k^2$, so $q^2 = 2k^2$, so q^2 is even, which implies that q is even.

But we assumed that p and q had no common factors, so they cannot both be even. This contradiction shows that our assumption that $\sqrt{2} = \frac{p}{q}$ is impossible; thus, $\sqrt{2}$ is not rational. $\square$

In our next lesson, we will discover that real numbers correspond to points on a line, which in turn correspond to decimal expansions. Then, we will discuss how to view rational numbers as decimal expansions, and we will discover which decimal expansions correspond to rational numbers.

Problem 1. We of the following are true? Write T or F in the blank.

(1) $-8 \in \mathbb{Z}$ ____________

(2) $0.1 \notin \mathbb{N}$ ____________

(3) $-\frac{1}{2} \notin \mathbb{Q}$ ____________

(4) $-\sqrt{3} \in \mathbb{Q}$ ____________

(5) $4 \in \mathbb{N}$ ____________