1. Linear Quadratic Expressions

A *linear expression* is an expression of the form $mx + n$, where m and n are numbers, and x is a variable.

A *quadratic expression* is an expression of the form $ax^2 + bx + c$, where $a \neq 0$. The numbers a, b, and c are called *coefficients*. We call a the *leading coefficient*, b the *coefficient of* x, and c the *constant coefficient*.

The product of two linear expressions is a quadratic expression. Moreover, quadratic expression factors into the product of binomials like this:

$$ax^2 + bx + c = a(x - r)(x - s).$$

The numbers r and s are called the *roots* of the expression. To understand this better, we will start with the case where $a = 1$.

2. Factoring Quadratics

Factoring is the reverse of multiplying. We multiply 2 times 3 to get 6. We factor 6 to get 2 times 3. We can do this with expressions as well as numbers. The product of two linear expressions is a quadratic expression.

Let r and s be numbers, and let x be a variable. Notice that

$$(x + r)(x + s) = x^2 + xs + rx + rs = x^2 + (r + s)x + rs.$$

This is a quadratic expression; if $ax^2 + bx + c = x^2 + (r + s)x + rs$, then $a = 1$, $b = r + s$, and $c = rs$. The coefficient of x is the sum of the roots, and the constant term is the product of the roots. Repeat this with different signs to see the following.

Proposition 1. *Let r and s be numbers and let x be a variable. Then*

- $(x + r)(x + s) = x^2 + (r + s)x + rs$
- $(x - r)(x - s) = x^2 - (r + s)x + rs$
- $(x + r)(x - s) = x^2 + (r - s)x + rs$

Let us do this with numbers.

- $(x + 3)(x + 5) = x^2 + 8x + 15$ Observe that $8 = 3 + 5$ and $15 = 3 \times 5$.
- $(x - 3)(x - 5) = x^2 - 8x + 15$ Here, 8 has a minus sign since 3 and 5 have minus signs.
- $(x - 3)(x + 5) = x^2 + 2x - 15$ Now 15 has a minus sign since 3 and 5 have opposite signs.
- $(x + 3)(x - 5) = x^2 - 2x - 15$ In this case, 2 has a minus sign because $3 < 5$ so $3 - 5 < 0$.

3. Solving Quadratics by Factoring

The symbol $\Rightarrow$ means "implies". So, $p \Rightarrow q$ means "p implies q", which means "if p, then q".

If the product of two numbers is zero, then one of them must be zero. We write this as

$$ab = 0 \quad \Rightarrow \quad a = 0 \text{ or } b = 0.$$

This remains true with binomials:

$$(x - r)(x - s) = 0 \quad \Rightarrow \quad x - r = 0 \text{ or } x - s = 0 \quad \Rightarrow \quad x = r \text{ or } x = s.$$

So, to solve a quadratic equation, if we can factor it and find the roots; the roots are the solutions. This is easiest when $a = 1$.

Example 1. Solve $x^2 - 8x + 15 = 0$.

Solution. We ask, "can we find two numbers whose product is 15 and whose sum is 8?" Yes, they are 3 and 5. Thus

$$x^2 - 8x + 15 = (x - 3)(x - 5) = 0 \quad \Rightarrow \quad x = 3 \text{ or } x = 5.$$

Note that since $b = -8$ is negative, we have two negative signs in the factored form, and this produces positive solutions. The solution set is $\{3, 5\}$. □

Example 2. Solve $x^2 + 11x + 28 = 0$.

Solution. We ask, "are there two numbers whose product is 28 and whose sum is 11?" Yes, they are 4 and 7. Thus

$$x^2 + 11x + 15 = (x+4)(x+7) = 0 \quad \Rightarrow \quad x = -4 \text{ or } x = -7.$$

In this case, the positive 11 leads to plus signs inside the binomials, which in turn leads to negative solutions. □

Example 3. Solve $x^2 + 5x - 24 = 0$.

Solution. Since the constant term is negative, we look for a difference instead of a sum.

We ask, "are there two numbers whose product is 24 and whose difference is 5?" Yes, they are 3 and 8. Because the coefficient of x is positive, the larger number in the binomials is positive. Thus

$$x^2 + 5x - 24 = (x-3)(x+8) = 0 \quad \Rightarrow \quad x = 3 \text{ or } x = -8.$$

□

Example 4. Solve $x^2 - x - 72 = 0$.

Solution. We ask, "are there two numbers whose product is 72 and whose difference is 1?" Yes, they are 8 and 9. Because the coefficient of x is negative, the larger number in the binomials is negative. Thus

$$x^2 - x - 72 = (x-9)(x+8) = 0 \quad \Rightarrow \quad x = 9 \text{ or } x = -8.$$

□

If $a \neq 1$, it is more difficult to factor the quadratic expression, although sometimes we can do it.

Example 5. Solve $2x^2 + 9x - 35 = 0$.

Solution. Let's use the age-old go-to of "guess and check".

We look for something of the form $(2x \pm p)(x \pm q)$, where $pq = 35$. It seems improbable that p or q is 35, so lets try 5 and 7. Also, $2q - p = \pm 9$. If $q = 5$, we would get $10 - 7 = 3 \neq 9$; however, if we try $q = 7$, we get $14 - 5 = 9$ ... so good! We see that

$$2x^2 + 9x - 35 = (2x-5)(x+7) = 0 \quad \Rightarrow \quad 2x - 5 = 0 \text{ or } x + 7 = 0 \quad IMP \quad x = \frac{5}{2} \text{ or } x = -7.$$

The solution set is $\left\{\frac{5}{2}, -7\right\}$. □

We note that we can use factoring to solve equations that we previously solved by extraction of roots.

Example 6. Solve $x^2 - 81 = 0$.

Solution. We know that $a^2 - b^2 = (a+b)(a-b)$. Since $81 = 9^2$,

$$x^2 - 81 = (x+9)(x-9) = 0 \quad \Rightarrow \quad x = -9 \text{ or } x = 9.$$

The solution set is $\{\pm 9\}$. □

4. Exercises

Problem 1. Solve the following quadratic equations by either extracting roots or factoring.

(a) $x^2 - 10x + 25 = 0$
(b) $x^2 - 10x + 9 = 0$
(c) $x^2 - 10x - 24 = 0$
(d) $x^2 - 9x = 0$
(e) $2x^2 - 7 = 0$