Math 1545 Calculus II Project 1 Name:
Prof. Paul Bailey January 31, 2007

Complete these problems and turn in the solution by Thursday, February 8, 2007. Attach this page to the front of the solution. Solutions should be self explanatory and written in complete sentences.

You may work in teams of up to three people. A team should submit a single solution. Separate teams may not share solutions.

Cylindrical Bands versus Conical Bands

Let V denote volume, A area, and L length. The ideas $V = \int A$ works for volumes in space, and $A = \int L$ works for areas in a plane. However, $A = \int L$ fails to compute curved areas in space. We investigate why by considering the case of a cone.

We have seen that the surface area of a cone is

$$A = \pi r\sqrt{h^2 + r^2} = \pi r s,$$

where r is the base radius, h is the height, and s is the slant height. From this, we saw that

$$A = 2\pi \overline{r} s$$

is the area of a frustum of a cone, where r_2 is the larger radius, r_1 is the smaller radius, $\overline{r} = \frac{r_1+r_2}{2}$ is the average of these radii, and s is the slant height of the frustum. We may also call the frustum of a cone a "conical band".

We approximated the surface area of a surface of revolution as a some of the frustums of cones. Why didn't we simply use $A = \int L$? The root of this lies in the computation of the surface area of the cone itself, which we obtained not by integration, but by classical geometry. So, let's instead attempt to compute the surface area of the cone using integration with the idea $A = \int L$.

Consider a cone of height h and base radius r obtained by rotating the line segment $y = \frac{h}{r}x$, $x \in [0, r]$, around the y-axis. If we slice this with a plane parallel to the axis of rotation (a horizontal plane), we obtain a circle. The idea $A = \int L$ leads us to believe that if we integrate the circumferences of these circle (they are lengths), we should obtain the area.

By similar triangles, the circle at height y should have length $L(y) = 2\pi x = \frac{2\pi r}{h} y$. So, by this reasoning, the area of the cone would be

$$A = \int_0^h \frac{2\pi r}{h} y\,dy = \frac{2\pi r}{h}\left[\frac{y^2}{2}\right]_0^h = \pi r h;$$

however, this is not the area $\pi r s$ we obtained earlier; it is actually half of the area of the circular cylinder of radius r and height h. Why doesn't this work?

To explore further why we may think it should work, in the hope of discovering what goes wrong, we unravel why we may believe $A = \int L$ in the first place.

Argument 1. Consider a cone of height h and base radius r obtained by revolving the line segment $y = \frac{h}{r}x$, $x \in [0, r]$, around the y-axis. Let A denote the surface area of the cone.

Select a natural number n and let $\Delta y = \frac{h}{n}$, dividing the vertical line segment from the origin to $(0, h)$ into n segments of equal size. Let $y_i = i\Delta y$, for $i = 0, 1, \ldots, n$. The conical band between y_i and y_{i+1} has height Δy, larger radius $\frac{r}{h}y_{i+1}$, and smaller radius $\frac{r}{h}y_i$. The area of this frustum can be estimated by a circumscribing cylindrical band of the same height and larger radius; it can also be estimated by an inscribing cylindrical band of the same height but the smaller radius.

Estimate the surface area of the cone by circumscribing it with cylindrical bands of height Δy; these bands are outside the cone. Let O be the sum of the areas of these bands. Since the bands are outside the cone, $A < O$ (O depends on h, r, and n).

Estimate the surface area of the cone by inscribing it with cylindrical bands of height Δy; these bands are inside the cone. Let I be the sum of the areas of these bands. Since the bands are outside the cone, $I < A$ (I depends on h, r, and n).

If $\lim_{n\to\infty}(O - I) = 0$, then $\lim I = \lim O$, and since $I < A < O$, we must have, by the squeeze law (or "sandwich theorem"), $\lim O = A$.

Problem 1. Analyze the above argument are describe as thoroughly as you can its flaw, in these steps.

(a) Sketch a picture of the inscribing and circumscribing cylindrical bands.

(b) Write the Riemann sums O and I for an arbitrary n.

(c) Find $O - I$ for an arbitrary n, and show that $\lim_{n\to\infty}(O - I) = 0$.

(d) Write $\lim_{n\to\infty} O$ as an integral.

(e) Step **(f)** produces a contradiction, so some assumption in Argument 1 must be fallacious. Find it.

(f) Explore the conditions under which a cone has a smaller surface area than a cylinder of the same base radius and height.

(g) Explore the conditions under which a conical band has a smaller surface area than a cylindrical band with the same base radius and height.