PRINCIPLES OF ANALYSIS
SOLUTIONS TO PROBLEM SET B

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Problem 1. Define a sequence (a_n) by setting $a_1 = 1$ and

$$a_{n+1} = \frac{a_n + 1}{3}$$

for $n \geq 1$.

(a) Write out the first few terms of the sequence, guess a formula for a_n as a function of n, and prove your assertion using induction.

(b) Find $\lim_{n\to\infty} a_n$ and prove your assertion using the limit theorems of §9.

Solution. Compute that

$$a_1 = 1, a_2 = \frac{2}{3}, a_3 = \frac{5}{9}, a_4 = \frac{14}{27}, a_5 = \frac{41}{81}.$$

We see a pattern: it appears that the denominator of a_n is 3^{n-1}, and the numerator is obtained from the denominator by adding 1 and dividing by 2. Thus we conjecture:

$$a_n = \frac{3^{n-1}+1}{2\cdot 3^{n-1}}.$$

To prove this by induction, we begin by establishing the base case: for $n = 1$, we know that $a_1 = 1$, whereas

$$\frac{3^{n-1}+1}{2\cdot 3^{n-1}} \,|_{n=1} = \frac{3^0+1}{2\cdot 3^0} = \frac{2}{2} = 1.$$

Thus our conjecture is true for $n = 1$.

Now assume that

$$a_{n-1} = \frac{3^{n-2}+1}{2\cdot 3^{n-2}}.$$

Then

$$\begin{aligned} a_n &= \frac{a_{n-1}+1}{3} \\ &= \frac{(3^{n-2}+1)/(2\cdot 3^{n-2})+1}{3} \\ &= \frac{3^{n-2}+1+2\cdot 3^{n-2}}{2\cdot 3^{n-1}} \\ &= \frac{3^{n-1}+1}{2\cdot 3^{n-1}}. \end{aligned}$$

We may rewrite this as

$$a_n = \frac{1}{2} + \frac{1}{2\cdot 3^{n-1}}.$$

Date: November 8, 2004.

Therefore

$$\begin{aligned}
\lim a_n &= \lim\left(\frac{1}{2}\right) + \lim\left(\frac{1}{2\cdot 3^{n-1}}\right) && \text{by Theorem 9.3}\\
&= \frac{1}{2} + \frac{3}{2}\lim\left(\frac{1}{3}\right)^n && \text{by Theorem 9.2}\\
&= \frac{1}{2} + \frac{3}{2}\cdot 0 && \text{by Example 9.7(b)}\\
&= \frac{1}{2}.
\end{aligned}$$

□

Problem 2. Define a sequence (b_n) by setting $b_1 = 1$ and

$$b_{n+1} = \frac{2b_n + 3}{b_n}$$

for $n \geq 1$. Show that (b_n) converges and find the limit.

Solution. We claim that $\lim b_n = 3$. To see this, it suffices to show that the sequence (c_n) given by $c_n = b_n - 3$ converges to zero. From the definition of (b_n), we have

$$c_{n+1} + 3 = \frac{2(c_n + 3) + 3}{(c_n + 3)},$$

so

$$c_{n+1} = \frac{2c_n + 6 + 3}{c_n + 3} - 3 = \frac{2c_n + 9 - 3c_n + 9}{c_n + 3} = \frac{-c_n}{c_n + 3}.$$

Now (c_n) converges to zero if and only if $(|c_n|)$ converges to zero. To show that $(|c_n|)$ converges to zero, it suffices to show that, for $n \geq 3$,

$$|c_n| < \frac{1}{2^{n-2}}.$$

We show this by induction. We have

$$c_1 = -2, c_2 = 2, c_3 = -\frac{2}{5},$$

and $|c_3| = \frac{2}{5} < \frac{1}{2}$. Thus the base case holds.

Note that if $|c_n| < 1$, then $c_n + 3 > 2$. Therefore $|c_{n+1}| = |\frac{-c_n}{c_n+3}| < \frac{|c_n|}{2}$. By induction, assume that

$$|c_{n-1}| \leq \frac{1}{2^{n-3}}.$$

Then

$$|c_n| < \frac{|c_{n-1}|}{2} < \frac{1}{2^{n-3}}\cdot\frac{1}{2} = \frac{1}{2^{n-2}}.$$

Thus $|c_n|$ converges to zero. □

Problem 3. Let (s_n) be a sequence satisfying

$$|s_{n+1} - s_n| < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Show that (s_n) is a Cauchy sequence, and is therefore convergent.

Lemma 1. *Let $m, n \in \mathbb{N}$ with $2 < m < n$. Then*

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{1}{2^m} < \frac{1}{m}.$$

Proof of Lemma. We prove the first inequality by induction on $k = n - m$. If $k = 1$, then our statement reads $\frac{1}{2^{m+1}} < \frac{1}{2^m}$, which is true.

Suppose that our proposition is true for differences of size $k - 1$. Then

$$\sum_{i=m+2}^{n} \frac{1}{2^i} < \frac{1}{2^{m+1}}.$$

Adding $\frac{1}{2^{m+1}}$ to both sides gives

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{2}{2^{m+1}} = \frac{1}{2^m}.$$

For the second inequality, it suffices to show that for $m > 2$ we have $m < 2^m$. For $m = 3$, we have $3 < 4$. By induction, $m - 1 < 2^{m-1}$. Then $m < 2^{m-1} + 1 < 2^{m-1} + 2^{m-1} = 2^m$. □

Proof of Problem. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\frac{1}{\epsilon} < N$. Let $m, n > N$; assume that $n > m$. Then

$$\begin{aligned}
|s_n - s_m| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{m+1} - s_m| \\
&\leq |s_n - s_{n-1}| + \cdots + |s_{m+1} - s_m| \\
&< \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \\
&< \frac{1}{2^{m-1}} \\
&< \frac{1}{m-1} \leq \frac{1}{N} < \epsilon.
\end{aligned}$$

This shows that (s_n) is a Cauchy sequence. □

Observation 1. Let (s_n) be a convergent sequence and let m be a (fixed) positive integer.
Then (s_{n+m}) is a convergent sequence, and

$$\lim_{n\to\infty} s_{n+m} = \lim_{n\to\infty} s_n.$$

Definition 1. Define a sequence (F_n) by setting $F_1 = 1$, $F_2 = 1$, and

$$F_{n+2} = F_n + F_{n+1}.$$

Then (F_n) is known as the *Fibonacci sequence*, after the 12^{th} century mathematician Fibonacci, who discovered the sequence while investigating the breeding of rabbits. We wish to prove that the golden ratio is the limit of the ratios of consecutive Fibonacci numbers.

Problem 4. Define a sequence (c_n) by setting $c_1 = 1$ and

$$c_{n+1} = 1 + \frac{1}{c_n}$$

for $n \geq 1$.

(a) Use induction to show that

$$c_n = \frac{F_{n+1}}{F_n}$$

for all $n \in \mathbb{N}$.

(b) Show that (c_n) is convergent by applying Problem 3 and induction.

(c) Find $\lim_{n\to\infty} c_n$, and prove your assertion using the limit theorems of §9.

Solution. For $n = 1$, we have $c_1 = 1 = \frac{1}{1} = \frac{F_2}{F_1}$. This establishes a base case for induction. Assume that

$$c_{n-1} = \frac{F_n}{F_{n-1}}.$$

Then

$$c_n = 1 + \frac{1}{(F_n/F_{n-1})} = 1 + \frac{F_{n-1}}{F_n} = \frac{F_{n-1} + F_n}{F_n} = \frac{F_{n+1}}{F_n}.$$

This proves part **(a)**.

To show that (c_n) converges, it suffices to show that

$$|c_{n+1} - c_n| < \frac{1}{2^{n-2}}.$$

To do this, we first show that $F_n F_{n+1} > 2^{n-1}$ for $n \geq 3$. For $n = 3$, we have $F_3 F_4 = 2 \cdot 3 > 4$. By induction, assume that $F_{n-1} F_n > 2^{n-2}$. Clearly (F_n) is a nondecreasing sequence, so

$$F_n F_{n+1} = F_n^2 + F_n F_{n-1} \geq 2 F_n F_{n-1} < 2^{n-1}.$$

Next we show that $|F_n F_{n+2} - F_{n+1}^2| = 1$ for $n \geq 1$. For $n = 1$, we have $|F_1 F_3 - F_1^2| = 2 - 1 = 1$. By induction, assume that $|F_n F_{n-1} - F_n^2| = 1$. Then

$$\begin{aligned}
|F_n F_{n+2} - F_{n+1}^2| &= |F_n(F_n + F_{n+1}) - F_{n+1}^2| \\
&= |F_n^2 + F_n F_{n+1} - F_{n+1}^2| \\
&= |F_n^2 - F_{n+1}(F_n + 1 - F_n)| \\
&= |F_n^2 - F_{n+1} F_{n-1}| \\
&= 1.
\end{aligned}$$

Now

$$\begin{aligned}
|c_{n+1} - c_n| &= |\frac{F_{n+2}}{F_{n+1}} - \frac{F_{n+1}}{F_n}| \\
&= |\frac{F_{n+2} F_n - F_{n+1}^2}{F_n F_{n+1}}| \\
&= |\frac{1}{F_n F_{n+1}}| \\
&< \frac{1}{2^{n-1}}.
\end{aligned}$$

Thus (c_n) converges by Problem 3; this completes **(b)**.

Since this converges, we let $L = \lim(c_n)$. Taking the limit of both sides of the definition of c_{n+1}, we obtain

$$L = 1 + \frac{1}{L}.$$

Thus

$$L^2 - L - 1 = 0.$$

The solutions to this quadratic equation are

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

□

Problem 5. Show that

$$F_n = \frac{1}{\sqrt{5}}\Big(\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big).$$

Solution. Recall that *golden ratio* is the positive solution to the equation

$$x^2 - x - 1 = 0;$$

the quadratic formula gives the roots as $\frac{1\pm\sqrt{5}}{2}$. Set

$$\Phi = \frac{1+\sqrt{5}}{2};$$
$$\Psi = \frac{1-\sqrt{5}}{2}.$$

then Φ and Ψ satisfy the above equation, which produces these identities:

- $\Phi + 1 = \Phi^2$;
- $\Phi - 1 = \frac{1}{\Phi}$;
- $\Psi + 1 = \Phi^2$;
- $\Psi - 1 = \frac{1}{\Psi}$;
- $\Psi = -\frac{1}{\Phi} = 1 - \Phi$;
- $\Phi - \Psi = \sqrt{5}$.

In light of this, what we wish to show can be rewritten as

$$F_n = \frac{1}{\sqrt{5}}\Big(\Phi^n - \Psi^n\Big).$$

We have $F_1 = 1$ and plugging 1 into the above expression produces

$$\frac{1}{\sqrt{5}}\Big(\Phi - \Psi\Big) = \frac{\sqrt{5}}{\sqrt{5}} = 1;$$

therefore the formula is true for $n = 1$.

By strong induction, assume that for $n \geq 3$ we have

$$F_{n-2} = \frac{1}{\sqrt{5}}\Big(\Phi^{n-2} - \Psi^{n-2}\Big);$$
$$F_{n-1} = \frac{1}{\sqrt{5}}\Big(\Phi^{n-1} - \Psi^{n-1}\Big),$$

Then

$$
\begin{aligned}
F_n &= F_{n-2} + F_{n-1} \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^{n-2} - \Psi^{n-2}\Big) + \frac{1}{\sqrt{5}}\Big(\Phi^{n-1} - \Psi^{n-1}\Big) \\
&= \frac{1}{\sqrt{5}}\Big((\Phi^{n-2} + \Phi^{n-1}) - (\Psi^{n-2} + \Psi^{n-1})\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^{n-2}(1 + \Phi) - \Psi^{n-2}(1 + \Psi)\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^{n-2}(\Phi^2) - \Psi^{n-2}(\Psi^2)\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^n - \Psi^n\Big).
\end{aligned}
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This completes the proof. $\square$

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