Category Theory	Homework 2 - Solutions	Name:
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Definition 1. A *partitioned set* is a pair $(X, \mathcal{C})$, where X is a set, and $\mathcal{C}$ is a partition of X.

Problem 1. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be partitioned sets, and let $f : X \to Y$.
Consider the following conditions on f.

(1) for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $f(C) \subset D$.

(2) for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $f(C) = D$.

(3) for every $D \in \mathcal{D}$ there exists $C \in \mathcal{C}$ such that $f(C) \subset D$.

(4) for every $D \in \mathcal{D}$ there exists $C \in \mathcal{C}$ such that $f(C) = D$.

(5) for every $D \in \mathcal{D}$ there exists a unique $C \in \mathcal{C}$ such that $f(C) \subset D$.

(6) for every $D \in \mathcal{D}$ there exists a unique $C \in \mathcal{C}$ such that $f(C) = D$.

Carefully answer the following questions.

(a) Do any of these imply that f is injective? surjective? bijective? Are any of these situations impossible?

(b) Do any of these conditions imply any of the other conditions? Which of the conditions is the weakest in this sense? Which is strongest?

Solution. **(a)** None of these imply injectivity. For example, let $X = \{1, 2\}$ and $Y = \{3\}$. Let $\mathcal{C} = \{X\}$ and $\mathcal{D} = \{Y\}$. Define $f : X \to Y$ by $f(x) = 3$ for $x = 1, 2$. Then f satisfies all six conditions, and is not injective. This also shows that each of the six situations is possible.

Types **(4)** and **(6)** imply surjectivity, since every D is covered by f, and $\mathcal{D}$ is a partition. The other types do not imply surjectivity.

(b) If $C = D$, then $C \subset D$; thus, it is clear that **(1)** $\Rightarrow$ **(2)**, **(3)** $\Rightarrow$ **(4)**, and **(5)** $\Rightarrow$ **(6)**. Moreover, if something exists uniquely, then it exists; thus **(3)** $\Rightarrow$ **(5)** and **(4)** $\Rightarrow$ **(6)**. The first two are independent of the last four, so there is no strongest or weakest condition.

□

Definition 2. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be partitioned sets, and let $f : X \to Y$. We will say that f is *partition preserving* if for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $f(C) \subset D$.

Problem 2. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $\mathcal{B} = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$ be a partition of X. How many partition preserving bijective functions $X \to X$ exist? Justify your answer.

Solution. Label the blocks A through D, in the order given. A bijective function will map blocks to blocks of the same size. Thus $\{1\}$ is fixed. There are $3! = 6$ permutations of D. There are $2! = 2$ permutations of each block B and C, and it is also possible to swap them. That makes $6 \cdot 2 \cdot 2 \cdot 2 = 48$ partition preserving permutations of X. □

Problem 3. Let $a, b, c \in \mathbb{Z}$ be positive integers. Show that

(a) $a \mid a$;

(b) $a \mid b$ and $b \mid a$ implies $a = b$;

(c) $a \mid b$ and $b \mid c$ implies $a \mid c$.

Solution. Recall that $x \mid y$ means $y = kx$ for some $k \in \mathbb{Z}$.

(a) Since $a = 1 \cdot a$, $a \mid a$.

(b) Suppose that $a \mid b$, and $b \mid a$. Then $b = ia$ and $a = jb$ for some $i, j \in \mathbb{Z}$. Thus $b = ijb$, so $ij = 1$, and since i and j are integers, we have $i, j = \pm 1$. But a and b are positive, so $i, j = 1$. Thus $a = b$.

(c) Suppose that $a \mid b$ and $b \mid c$. Then $b = ia$ and $c = jb$ for some $i, j \in \mathbb{Z}$. Thus $c = (ji)a$, and $ji \in \mathbb{Z}$, so $a \mid c$.

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