PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §5

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Let $+\infty$ and $-\infty$ be formal symbols; we may write ∞ instead of $+\infty$.

The *extended real numbers* is a set, denoted $\overline{\mathbb{R}}$, defined as the set of real numbers together with these symbols:

$$\overline{\mathbb{R}} = \mathbb{R} \cap \{+\infty, -\infty\}.$$

Every set $A \subset \mathbb{R}$ of real numbers, whether or not it is bounded, has a supremum and an infimum in $\overline{\mathbb{R}}$; if A is not bounded above, we define $\sup A = +\infty$, and if A is not bounded below, we define $\inf A = -\infty$. Indeed, we may think of $\overline{\mathbb{R}}$ as the set of all possible suprema and infima of subsets of the real numbers.

It is the algebra of $\overline{\mathbb{R}}$ which is interesting and useful. If $A, B \subset \mathbb{R}$, define

$$A + B = \{x \in \mathbb{R} \mid x = a + b \text{ for some } a \in A, b \in B\},$$

$$-A = \{x \in \mathbb{R} \mid x = -a \text{ for some } a \in A\},$$

and

$$AB = \{x \in \mathbb{R} \mid x = ab \text{ for some } a \in A, b \in B\}.$$

Suppose that A and B are bounded. Then

$$\sup A + \sup B = \sup(A + B)$$

and

$$\sup -A = -\inf A.$$

If $a \geq 0$ for every $a \in A$, we have

$$\sup A \sup B = \sup(AB).$$

The situation with infima is analogous.

We are motivated in defining the algebra of $\overline{\mathbb{R}}$ by the desire that these properties are preserved. Thus we define:

- $\infty + \infty = \infty$;
- $a\infty = \operatorname{sgn}(a)\infty$, where $a \in \mathbb{R} \setminus \{0\}$, and $\operatorname{sgn}(a)$ is ± 1.

Note that $\overline{\mathbb{R}}$ is not a field, because in a field, the cancellation law of addition holds.

Note that $\infty + (-\infty)$ and $\infty \cdot 0$ are undefined, so addition and multiplication are NOT binary operations on the set $\overline{\mathbb{R}}$.

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