Real Analysis — Homework 21 - Solutions

Dr. Paul L. Bailey — Thursday, November 8, 2018

Problem 1. Show that if U is open in X and A is closed in X, then $U \smallsetminus A$ is open in X, and $A \smallsetminus U$ is closed in X.

Problem 2. Let X be any set, and let $Z \subset X$.

(a) Let $\mathcal{T}_1 = \{A \in \mathcal{P}(X) \mid Z \subset A\} \cup \{\varnothing\}$. Is $\mathcal{T}_1$ a topology on X?

(b) Let $\mathcal{T}_2 = \{A \in \mathcal{P}(X) \mid Z \subset A^c\} \cup \{X\}$. Is $\mathcal{T}_2$ a topology on X?

(c) Let $\mathcal{T}_3 = \{A \in \mathcal{P}(X) \mid Z \not\subset A\} \cup \{\varnothing, X\}$. Is $\mathcal{T}_3$ a topology on X?

Solution. **(a)** Yes. Clearly $Z \subset X$, so $X \in \mathcal{T}_1$. Also, if Z is a subset of each set in a collection, it is a subset of its union or its intersection.

(b) Yes. Clearly $Z \subset X = \varnothing^c$, so $\varnothing \in \mathcal{T}_2$. Also, if Z is a subset of the complement of each set in a collection, it is a subset of the complement of its union or its intersection, by DeMorgan's Law.

(c) No. For example, let $X = \{1, 2, 3, 4\}$ and $Z = \{2, 3\}$. Then $A = \{1, 2\} \in \mathcal{T}_3$ and $B = \{1, 3\} \in \mathcal{T}_3$, but $A \cup B = \{1, 2, 3\} \notin \mathcal{T}_3$.

$\square$

Definition 1. Topological spaces may be classified using the following separation axioms:

(T0) For every two points, there exists a neighborhood of one of the points that does not contain the other point.

(T1) For every two points, there exists a neighborhood of each point which does not contain the other point.

(T2) For every two points, there exist disjoint neighborhoods of each point.

Spaces which satisfy the T_0 axiom are called *Kolmogoroff spaces.*
Spaces which satisfy the T_1 axiom are called *Frechet spaces.*
Spaces which satisfy the T_2 axiom are called *Hausdorff spaces.*

Problem 3. Consider the collection of all topologies that exist on the set $X = \{a, b, c\}$.

(a) Find all topologies on X which produce Kolmogoroff spaces. Group them by equivalence.

(b) Find all topologies on X which produce Frechet spaces. Group them by equivalence.

(c) Find all topologies on X which produce Hausdorff spaces. Group them by equivalence.

Solution. We use the previously established notion. All topologies have $\varnothing$ and X, so don't write those. Also, write a to mean $\{a\}$ and ab to mean $\{a, b\}$. For example, write $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ as a, b, ab.

(a) There are nineteen, grouped into five equivalence classes, as follows:

- a, ab; a, ac; b, ab; b, bc; c, ac; c, bc
- a, b, ab; a, c, ac; b, c, bc
- a, b, ab, bc; a, b, ab, ac; a, c, ac, ab; a, c, ac, bc; b, c, bc, ab; b, c, bc, ac
- a, ab, ac; b, ab, bc; c, ac, bc
- a, b, c, ab, bc, ac (the discrete topology)

(b) Singleton sets are closed in a Frechet space. Since the union of closed sets is closed, in a finite Frechet space, every subset is the union of singleton sets, and so its closed. Thus in a finite Frechet space, the complement of every set is closed, so every set is open. Thus, a finite Frechet space has the discrete topology.

(c) Every Hausdorff space is Frechet, so a finite Hausdorff space has the discrete topology.

□

Problem 4. Let X be a set. In each case, justify your answer.

(a) If X is finite, is the trivial topology on X Kolmogoroff? Frechet? Hausdorff?

(b) If X is finite, is the cofinite topology on X Kolmogoroff? Frechet? Hausdorff?

(c) If X is finite, is the discrete topology on X Kolmogoroff? Frechet? Hausdorff?

(d) If X is infinite, is the trivial topology on X Kolmogoroff? Frechet? Hausdorff?

(e) If X is infinite, is the cofinite topology on X Kolmogoroff? Frechet? Hausdorff?

(f) If X is infinite, is the discrete topology on X Kolmogoroff? Frechet? Hausdorff?

Solution. Note that a Hausdorff space is Frechet, and a Frechet space is Kolmogoroff. So, we need only mention the strongest condition. We assume $|X| > 1$.

(a) Finite trivial: None.

(b) Finite cofinite: Hausdorff, since cofinite spaces are Frechet, and finite Frechet spaces are discrete, and thus Hausdorff.

(c) Finite discrete: Hausdorff.

(d) Infinite trivial: None.

(e) Infinite cofinite: Frechet, not Hausdorff.

(f) Infiinte discrete: Hausdorff.

□

Definition 2. Let X be a topological space. Let (x_n) be a sequence in X. Let $p \in X$. We say that (x_n) *converges* to p if

$$\text{for every neighborhood } U \text{ of } p \text{ there exists } N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow x_n \in U.$$

Problem 5. Let X be a topological space and let (x_n) be a sequence in X.

(a) Assume X is Hausdorff. Suppose that (x_n) converges to p and to q. Show that $p = q$.

(b) Find an example of a non-Hausdorff space X and a sequence (x_n) in X which converges to more than one point.

Solution. **(a)** Suppose $p \neq q$. Since X is Hausdorff, there exist disjoint open sets U_1 and U_2 such that $p \in U_1$ and $q \in U_2$.

Since (x_n) converges to p, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $x_n \in U_1$.

Since (x_n) converges to q, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $x_n \in U_2$.

Let $N = \max\{N_1, N_2\}$. Then $x_N \in U_1 \cap U_2$, contradicting that these sets are disjoint. This contradiction proves the claim.

(b) Let $X = \{a, b\}$, endowed with the trivial topology. Then every sequence in X converges to both a and to b.

□