AP Calculus AB — Responses 0403

Dr. Paul L. Bailey — Thursday, April 2, 2020

Problem 1. Compute $\lim_{x\to\infty} x\tan(1/x)$.

Solution. We have

$$\begin{aligned}
\lim_{x\to\infty} x\tan(1/x) &= \lim_{x\to\infty} \frac{\tan(1/x)}{(1/x)} \\
&= \lim_{u\to 0} \frac{\tan u}{u} \qquad \text{where } u = 1/x \\
&= \lim_{u\to 0} \Big(\frac{\sin u}{u}\cdot\frac{1}{\cos u}\Big) \\
&= \Big(\lim_{u\to 0} \frac{\sin u}{u}\Big)\cdot\Big(\lim_{u\to 0}\frac{1}{\cos u}\Big) \\
&= 1\cdot 1 = 1.
\end{aligned}$$

□

Question 1. When is the answer no solution for problems using MVT and Rolle's?

Answer. There would be "no solution" only if the hypothesis of the theorem does not hold.

For example, suppose the problem is:

Let $f(x) = |x|$, $a = -1$, $b = 2$. Find $c \in [a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Here, we wish to find $c \in [-1, 2]$ such that $f'(c) = \dfrac{f(2) - f(1)}{2 - (-1)} = \dfrac{1}{3}$.

The answer is, there is no solution. The reason is that part of the hypothesis for the Mean Value Theorem is that f be differentiable on $(a, b) = (-1, 2)$. Since f is not differentiable at $0 \in (-1, 2)$, we cannot expect the Mean Value Theorem's conclusion to hold on $[-1, 2]$. □

Question 2. Is there ever a L'Hôpital's Rule problem where the answer is DNE?

Answer. Yes. It is possible you start with an indeterminate form, but upon taking derivatives, it is no longer indeterminate.

For example, try to compute $\lim_{x\to 0} \dfrac{\sin x}{x^2}$. This is of the form $\dfrac{0}{0}$, but when you take derivatives, you get $\lim_{x\to 0} \dfrac{\cos x}{2x}$, which is not in the form $\dfrac{1}{0}$. The limit is $\pm\infty$. □

Question 3. is it better to use washers or shells?

Answer. It depends. For example, let R be the region in the first quadrant above $y = 0$ and below $y = 2x - x^2$.

If you revolve this region about the x-axis, it is easier to use disks, because the integral is just

$$V = \int_0^2 \pi(2x - x^2)^2\,dx.$$

However, if you revolve this region about the y-axis, you would have solve $y = 2x - x^2$ for y to get the radii of the washers. But if you use shells, the integral is

$$V = \int_0^2 2\pi x(2x - x^2)\,dx.$$

Basically, use the one for which you don't have to invert your equation. Typically, if you are revolving $y = f(x)$ around a horizontal line, washers are easier, but if you are revolving it around a vertical line, shells are easier.

College Board claims that shells are not on the AP examination, but I have found problems on the test that are easier with shells. □

Question 4. Why do we learn related rates before optimization problems?

Answer. This is an insightful question.

Thomas does have a few applications involving maximization by finding where the derivative is zero before Chapter 4, but only a few.

Section 3.6 covers implicit differentiation, which is required to do related rates problems. Even though related rates problems may seem hard, this is all the is required to approach them.

Section 4.1 through 4.4 cover the Extreme Value Theorem, Mean Value Theorem, and their corollaries. The first and second derivatives tests for classification of a critical point are in sections 4.3 and 4.4. You want these tools in order to approach the optimization problems, because you want to know for certain whether or not global extrema exist, and to classify them as minima, maxima, or neither. □

Question 5. Why does the inverse trigonometric function for arcsec have an absolute value?

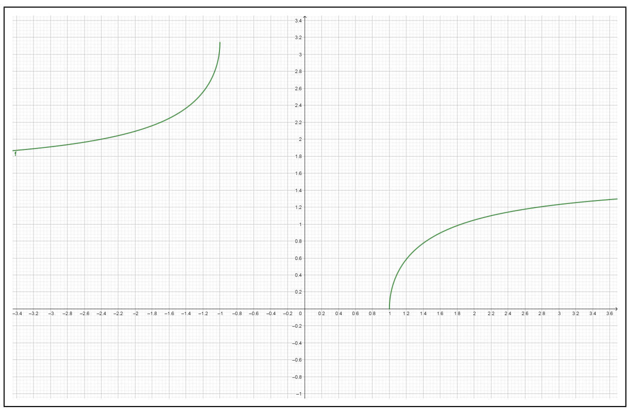
Answer. Because it can. We derived the formula

$$\int \operatorname{arcsec} x \, dx = \frac{1}{|x|\sqrt{x^2-1}},$$

but we need to understand arcsecant a little better to understand it. We chose a branch of inverse for secant; this means, we restricted secant to a domain on which it is injective. We obtained

$$\operatorname{arcsec} : (-\infty, -1] \cup [1, \infty) \quad \text{given by } \operatorname{arcsec}(x) = y \Leftrightarrow x = \sec(y).$$

The domain of arcsec is in two pieces, $(-\infty, 1] \cup [1, \infty)$. The graph looks like this:



Notice that the slope is positive on both sides of the graph.

You can't integrate arcsecant on any domain that contains part of $(-1, 1)$. So basically, the antiderivative on the piece from 1 to ∞ is $\dfrac{1}{x\sqrt{x^2-1}}$, and the antiderivative on the piece from $-\infty$ to -1 is $\dfrac{-1}{x\sqrt{x^2-1}}$ □

Question 6. What is the derivative of ln(X) and what does the graph look like?

Answer. The derivative of $\ln x$ is $\dfrac{1}{x}$, because $\ln x = \int_1^x \frac{1}{t}\, dt$.

To solve the integral $\int \ln x \, dx$ requires a technique called "integration by parts". It is Example 3 on page 563:

$$\int \ln x \, dx = x \ln x - x + C.$$

Of course, you can verify this formula by computing $\dfrac{d}{dx} x \ln x - x + C$. □

Question 7. On 4.6 # 21, how would the result be different if n was not a positive integer?

Answer. At this point in the book, we don't know how to compute $\dfrac{d}{dx}x^p$ if p is irrational. However we do know it if p is rational, so I suppose he could have written that instead. As you suggest, the result would be the same. □

Question 8. Which theory (s) would you use to solve 28?

Answer. Since square root is continuous, it commutes with the limit operator: $\lim \sqrt{f(x)} = \sqrt{\lim f(x)}$, assuming both limits exist. Similarly, $\lim \dfrac{1}{f(x)} = \dfrac{1}{\lim f(x)}$.

So,

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \lim_{x\to 0^+} \sqrt{\frac{x}{\sin x}} = \sqrt{\lim_{x\to 0^+} \frac{x}{\sin x}} = \sqrt{\frac{1}{\lim\limits_{x\to 0^+} \dfrac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1.$$

By the way, we don't need L'Hospital's Rule to compute $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$; we already proved that geometrically, and we needed it to compute that $\dfrac{d}{dx} \sin x = \cos x$. We wouldn't be able to use L'Hospital's Rule anyway on $\sin x$ without already knowing that. □

Question 9. Can you please summarize the methods of solving integrals please?

Answer. We really haven't learned very many techniques of integration, that is more of a BC topic. Basically, you have antiderivatives of basic functions, substitution, and tricks. A list of basic antiderivatives appears on page 554. We know all of these (and why they are true) except for the hyperbolic functions ($\sinh x$, $\cosh x$, etc.). Please review section 8.1 for more practice, and if you get stuck, please let me know (you can use email or Microsoft Teams). □