

# COMPLEX ANALYSIS

## TOPIC XII: CAUCHY'S INTEGRAL FORMULA

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### 1. REVIEW

**Definition 1.** Let  $D$  be an open subset of  $\mathbb{C}$  and let  $f : D \rightarrow \mathbb{C}$ . We say that  $f$  is *analytic* at  $z_0 \in D$  if  $f$  is differentiable at every point in a neighborhood of  $z_0$ . We say that  $f$  is analytic on  $D$  if  $f$  is differentiable at every point in  $D$ .

**Definition 2.** Let  $D$  be an open subset of  $\mathbb{C}$  and let  $f : D \rightarrow \mathbb{C}$  be analytic. Let  $\gamma : [a, b] \rightarrow D$  be a piecewise smooth path. The *path integral of  $f$  along  $\gamma$*  is

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt.$$

**Theorem 1.** Let  $D$  be an open connected set and let  $f : D \rightarrow \mathbb{C}$ . Suppose that  $f$  admits a primitive  $F$  in  $D$ . Let  $z_1, z_2 \in D$ . Then for every piecewise smooth path  $\gamma : [a, b] \rightarrow D$  with  $\gamma(a) = z_1$  and  $\gamma(b) = z_2$ ,

$$\int_{\gamma} f(z) dz = F(z_2) - F(z_1).$$

**Theorem 2.** Let  $D$  be an open subset of  $\mathbb{C}$ . Let  $f : D \rightarrow \mathbb{C}$  be analytic on  $D$ . Let  $\alpha$  and  $\beta$  be homotopic paths in  $D$ . Then

$$\int_{\alpha} f(z) dz = \int_{\beta} f(z) dz.$$

**Corollary 1.** Let  $D$  be an open, connected, simply connected subset of  $\mathbb{C}$ . Let  $f$  be analytic on  $D$ . Then for every simple closed curve  $C$  in  $D$ ,

$$\int_C f(z) dz = 0.$$

## 2. TECHNICAL LEMMAS

To continue our development, we need a bound on the size of an integral.

**Proposition 1.** *Let  $\alpha : [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth path in  $\mathbb{C}$ . The arclength of  $\alpha$  is*

$$L = \int_a^b |\alpha'(t)| dt.$$

*Proof.* We partition the domain of  $\alpha$  into  $n$  pieces of equal size, by setting

$$\Delta t = \frac{b-a}{n} \quad \text{and} \quad t_i = a + i\Delta t.$$

An estimate for the length of the arc is

$$L \approx \sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| = \sum_{i=1}^n \left| \frac{\alpha(t_i) - \alpha(t_{i-1})}{\Delta t} \right| \Delta t.$$

Taking the limit as  $n \rightarrow \infty$  we obtain

$$L = \lim_{\Delta t \rightarrow 0} \sum_{i=1}^n \left| \frac{\alpha(t_i) - \alpha(t_{i-1})}{\Delta t} \right| \Delta t = \int_a^b |\alpha'(t)| dt.$$

□

**Proposition 2.** Let  $\alpha : [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth path in  $\mathbb{C}$ . Then

$$\left| \int_a^b \alpha(t) dt \right| \leq \int_a^b |\alpha(t)| dt.$$

*Proof.* This is certainly true if the integral on the left hand side is zero, so let us assume that it is not; thus, set  $re^{i\theta} = \int_a^b \alpha(t) dt$ . Then  $|\int_a^b \alpha(t) dt| = r$ , and

$$\begin{aligned} r &= \int_a^b e^{-i\theta} \alpha(t) dt && \text{since } e^{i\theta} \text{ is constant} \\ &= \operatorname{Re} \left( \int_a^b e^{-i\theta} \alpha(t) dt \right) && \text{since the integral is real} \\ &= \int_a^b \operatorname{Re}(e^{-i\theta} \alpha(t)) dt && \text{by the definition of the integral} \\ &\leq \int_a^b |e^{-i\theta} \alpha(t)| dt && \text{since } \operatorname{Re}(z) \leq |z| \text{ for } z \in \mathbb{C} \\ &= \int_a^b |e^{-i\theta}| |\alpha(t)| dt && \text{by a property of modulus} \\ &= \int_a^b |\alpha(t)| dt && \text{since } |e^{-i\theta}| = 1 \end{aligned}$$

□

**Proposition 3.** Let  $D$  be an open subset of  $\mathbb{C}$  and let  $f : D \rightarrow \mathbb{C}$  be continuous. Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a piecewise smooth path in  $D$ . Suppose that  $L$  is the arclength of  $\gamma$ , and that suppose  $f(z) \leq M$  for all  $z \in D$ . Then

$$\left| \int_{\gamma} f(z) dz \right| \leq ML.$$

*Proof.* Behold:

$$\begin{aligned} \left| \int_{\gamma} f(z) dz \right| &\leq \int_{\gamma} |f(z)| dz && \text{by Proposition 2} \\ &= \int_{\gamma} |f(\gamma(t)) \gamma'(t)| dt && \text{by the definition of path integration} \\ &= \int_{\gamma} |f(\gamma(t))| |\gamma'(t)| dt && \text{by a property of modulus} \\ &\leq \int_{\gamma} M |\gamma'(t)| dt && \text{by a property of Riemann integration} \\ &= M \int_{\gamma} |\gamma'(t)| dt && \text{by a property of Riemann integration} \\ &= ML && \text{by Proposition 1} \end{aligned}$$

□

## 3. ANTIDERIVATIVES

**Theorem 3.** *Let  $D$  be an open connected subset of  $\mathbb{C}$ , and let  $f : D \rightarrow \mathbb{C}$  be continuous. The following are equivalent.*

- (a) *The function  $f$  admits a primitive in  $D$ .*
- (b) *The integral of  $f$  between two points in  $D$  is path independent.*
- (c) *The integral of  $f$  along a closed curve in  $D$  is zero.*

*Proof.* We have previously seen why (b) and (c) are equivalent, and that (a) implies (b). It remains to see why (b) implies (a).

Suppose that integration of  $f$  in  $D$  is path independent. That is, for any  $z_1, z_2 \in D$  and any paths  $\alpha$  and  $\beta$  from  $z_1$  to  $z_2$ , we have  $\int_{\alpha} f(z) dz = \int_{\beta} f(z) dz$ . Thus we write such an integral as  $\int_{z_1}^{z_2} f(z) dz$ .

Select  $z_0 \in D$ . Note that since  $D$  is connected, there is a path from  $z_0$  to every  $z \in D$ . Define a function

$$F : D \rightarrow \mathbb{C} \quad \text{by} \quad F(z) = \int_{z_0}^z f(s) ds;$$

the value of  $F$  is independent of the path we trace from  $z_0$  to  $z$ .

Let  $z \in D$ . Since  $D$  is open, there is an  $\delta > 0$  such that  $B_{\delta}(z) \subset D$ . Let  $h \in \mathbb{C}$  with  $|h| < \delta$ . Then the line segment from  $z$  to  $z + h$  is contained in  $D$ . Let  $\alpha : [0, 1] \rightarrow D$  given by  $\alpha(t) = z + ht$  parameterize this line segment. Then  $\alpha'(t) = h$ , so

$$\int_{\alpha} ds = \int_0^1 h dt = h.$$

Let  $\epsilon$  be any positive real number. Since  $f$  is continuous, we can select  $\delta$  above such that  $|s - z| < \delta$  implies  $|f(s) - f(z)| < \epsilon$ . Then, by Proposition 3,

Now, by path independence,

$$F(z + h) - F(z) = \int_{z_0}^{z+h} f(s) ds - \int_{z_0}^z f(s) ds = \int_z^{z+h} f(s) ds.$$

Since  $\int_z^{z+h} ds = h$  and  $f(z)$  is constant with respect to  $s$ , we have

$$f(z) = \frac{f(z)}{h} \int_z^{z+h} ds = \frac{1}{h} \int_z^{z+h} f(z) ds,$$

so that

$$\frac{F(z + h) - F(z)}{h} - f(z) = \frac{1}{h} \int_z^{z+h} (f(s) - f(z)) ds.$$

Let  $\epsilon$  be any positive real number. Since  $f$  is continuous, we can select  $\delta$  above such that  $|s - z| < \delta$  implies  $|f(s) - f(z)| < \epsilon$ . Then, by Proposition 3,

$$\left| \frac{F(z + h) - F(z)}{h} - f(z) \right| < \frac{1}{h}(|h|\epsilon),$$

so that

$$\left| \frac{F(z + h) - F(z)}{h} - f(z) \right| < \epsilon.$$

Since  $\epsilon$  is arbitrary,

$$\lim_{h \rightarrow 0} \left| \frac{F(z + h) - F(z)}{h} - f(z) \right| = 0.$$

We conclude that  $F'(z) = f(z)$ . □

## 4. WINDING NUMBERS

The Jordan Curve Theorem indicates that a simple closed curve in a plane partitions its complement into two components, an inside and an outside. Considering that the curve can be very complex, this turns out to be more difficult to prove than expected. The notions of the homotopy and winding numbers give alternate formulations of inside and outside.

**Proposition 4.** *Let  $z_0 \in \mathbb{C}$  and let  $C$  be a circle of radius 1 centered at  $z_0$ , with positive orientation. Then*

$$\int_C \frac{dz}{z - z_0} = 2\pi i.$$

*Proof.* Let  $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$  be given by  $\gamma(t) = z_0 + e^{it}$ . Then  $\gamma$  is a parameterization of  $C$ , and  $\gamma'(t) = ie^{it}$ , so

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{1}{\gamma(t) - z_0} \gamma'(t) dt = \int_0^{2\pi} \frac{ie^{it}}{(z_0 + e^{it}) - z_0} dt = i \int_0^{2\pi} dt = 2\pi i.$$

□

The function  $f(z) = \frac{1}{z - z_0}$  may be used to give information about curves. It is clear that if we go around the circle  $C$  centered at  $z_0$  twice, the integral  $\int_C f(z) dz$  will be  $4\pi i$ , and if we go around once in the opposite direction, the integral will be  $-2\pi i$ ; in general, if we go around the circle  $k$  times, the integral will be  $2k\pi i$ . Moreover, any loop  $\gamma$  which is homotopic to a circular loop which wraps  $k$  times around  $z_0$  will have  $\int_\gamma f(z) dz = 2k\pi i$ .

**Definition 3.** Let  $\gamma : I \rightarrow \mathbb{C}$  be a loop in  $\mathbb{C}$ , and let  $z_0 \in \mathbb{C}$ . The *winding number* (or *index*) of  $\gamma$  about  $z_0$  is

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} dz.$$

The domain of analyticity of  $f(z) = \frac{1}{z - z_0}$  is  $D = \mathbb{C} \setminus \{z_0\}$ . It is intuitively clear that any simple closed curve in  $\mathbb{C}$  which does not pass through  $z_0$  is homotopic to a circle in  $D$ . If  $z_0$  is inside such a circle, the curve is further homotopic to a circle of radius one centered at  $z_0$ , in which case the integral of  $f(z)$  around the circle is  $2\pi i$ . On the other hand, if  $z_0$  is not inside the circle, the curve is homotopic to a constant in  $D$ , so the integral is zero. Given a simple closed curve  $C \subset \mathbb{C}$  and a point  $z_0 \in \mathbb{C} \setminus C$ , we have these methods to detect whether the point is inside the curve.

- $z_0$  is an element of the bounded component of  $\mathbb{C} \setminus C$ ;
- a (nontangential) ray from  $z_0$  to  $\infty$  intersects the curve an odd number of times;
- $C$  is homotopic to a constant in  $\mathbb{C} \setminus \{z_0\}$ ;
- $n(\gamma, z_0) = \pm 1$ .

If  $C$  is a simple closed curve, then the winding number tells whether  $z_0$  is inside  $n(\gamma, z_0) = \pm 1$  or outside  $n(\gamma, z_0) = 0$  the curve. However, if  $C$  is the image of any loop,  $n(\gamma, z_0)$  tells how many times  $C$  wraps around  $z_0$ . The situation regarding “inside” or “outside” is less clear if the curve is not simple. Indeed, we could define  $z_0$  to be outside of a closed curve if the curve is homotopic to a constant in  $\mathbb{C} \setminus \{z_0\}$ . However, it turns out that there exist loops  $\gamma$  such that  $n(\gamma, z_0) = 0$ , and yet  $\gamma$  is not homotopic to a constant in  $\mathbb{C} \setminus \{z_0, z_1\}$ , where  $z_1$  is some other point.

## 5. RATIONAL FUNCTIONS

To review, we note that one can compute the next proposition directly from the definition.

**Result 1. (Logarithms)** Let  $C$  be a positively oriented circle centered at  $z_0 \in \mathbb{C}$ . Then

$$\int_C \frac{1}{z - z_0} dz = 2\pi i.$$

The following extension of this fact follows from the fact that  $f(z) = \frac{1}{(z - z_0)^n}$  has an antiderivative in its domain, if  $n > 1$ .

**Result 2. (Power functions)** Let  $C$  be a positively oriented circle centered at  $z_0 \in \mathbb{C}$ , and let  $k \in \mathbb{Z}$ . Then

$$\int_C (z - z_0)^k dz = \begin{cases} 2\pi i & \text{if } k = -1 ; \\ 0 & \text{if } k \neq -1 . \end{cases}$$

In particular,

**Result 3. (Polynomials)** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a polynomial, and let  $C$  be any closed curve in  $\mathbb{C}$ . Then

$$\int_C f(z) dz = 0.$$

Combine these with homotopy, winding numbers, and the principle of partial fraction decomposition to obtain the following.

**Result 4. (Rational Functions)** Let  $f(z)$  be a rational function. Let  $z_1, z_2, \dots, z_n$  denote the poles of  $f$  of multiplicity 1. Then

$$f(z) = g(z) + \sum_{j=1}^n \frac{A_j}{z - z_j},$$

where  $g(z)$  is either a polynomial or a rational function whose poles have multiplicity exceeding 1. Let  $C$  be an closed curve in  $\mathbb{C}$ . Then

$$\int_C f(z) dz = 2\pi i \sum_{j=1}^n A_j n(C, z_j),$$

where

$$n(C, z_j) = \frac{1}{2\pi i} \int_C \frac{1}{z - z_j} dz$$

is the winding number of  $C$  about  $z_j$ .

## 6. CAUCHY'S INTEGRAL FORMULA

The technique used to show that having an antiderivative is equivalent to path independence may be modified to provide the following striking integration formula. We begin with a lemma, which really contains the main argument.

**Theorem 4. (Cauchy's Integral Formula)** *Let  $f$  be analytic on and inside a simple closed curve  $C$  with positive orientation. Let  $z_0$  be inside of  $C$ . Then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

*Proof.* Because of path independence within  $C$ , we may assume that  $C$  is a circle of radius  $\rho$  centered at  $z_0$ . Thus the length of  $C$  is  $L = 2\pi\rho$ .

Let  $\epsilon$  be any positive real number and let  $M = \frac{\epsilon}{L}$ . Since  $f$  is differentiable at  $z_0$ , we can select  $\delta$  such that  $|z - z_0| < \delta$  implies  $\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < M$ . Then, by Proposition 3,

$$\left| \int_C \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) dz \right| \leq LM = \epsilon.$$

Since  $\epsilon$  is arbitrary, we see that

$$\int_C \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) dz = 0,$$

so

$$\int_C \frac{f(z) - f(z_0)}{z - z_0} dz = \int_C f'(z_0) dz = 0,$$

since  $f'$  has a primitive inside  $C$ . This shows that

$$\int_C \frac{f(z)}{z - z_0} dz = \int_C \frac{f(z_0)}{z - z_0} dz = f(z_0) \int_C \frac{dz}{z - z_0} = \frac{f(z_0)}{2\pi i}.$$

□

**Theorem 5. (Cauchy's Integral Extension)** *Let  $f$  be analytic on and inside a simple closed curve  $C$  with positive orientation. Let  $z_0$  be inside of  $C$ . Let  $n$  be a positive integer. Then the  $n^{\text{th}}$  derivative of  $f$  at  $z_0$  exists, and*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

*Reason.* Under the circumstances of this theorem, for reasons we do not investigate, it is possible to “differentiate under the integral sign”. That is, we apply the differentiation operator  $\frac{d}{dz}$  to each side of Cauchy's Integral Formula and obtain

$$\begin{aligned} f'(z_0) &= \frac{d}{dz} \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_C \frac{d}{dz} \left[ \frac{f(z)}{z - z_0} \right] dz \\ &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz \end{aligned}$$

Consider this the base case; the form for  $n > 1$  follows similarly using induction. □

## 7. MORERA'S THEOREM

Morera's Theorem can be considered to be a converse of the Cauchy-Goursat theorem.

**Theorem 6. (Morera's Theorem)** *Let  $D \subset \mathbb{C}$  be an open set and let  $f : D \rightarrow \mathbb{C}$  be continuous. Suppose that*

$$\int_C f(z) dz = 0$$

*for every simple closed contour  $C$  in  $D$ . Then  $f$  is analytic on  $D$ .*

*Proof.* This follows from Theorem 3 and Theorem 5 as follows.

Since the integral around every closed curve is zero, we know that integration of  $f$  in  $D$  is independent of path. Thus we can construct an antiderivative  $F$  for  $f$  in  $D$ . Now  $F$  is differentiable at every point in  $D$ , since  $F'(z) = f(z)$  for all  $z \in D$ . Thus  $F$  is analytic in  $D$ , and now Cauchy's Integral Extension implies that  $F' = f$  is also differentiable.  $\square$

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