Problem 1. Consider the points $A = (1, 2, 5)$, $B = (-1, 1, 4)$, $C = (3, 1, -4)$, and $D = (6, 2, 7)$ in $\mathbb{R}^3$. Let $\mathcal{P}$ be the plane passing through A, B, and C, and let $\mathcal{L}$ be the line perpendicular to $\mathcal{P}$ which passes through D. Let E be the point of intersection of $\mathcal{P}$ and $\mathcal{L}$. Find E.

Solution. Let $\vec{v} = A - B = (2, 1, 1)$ and $\vec{w} = \frac{1}{4}(C - B) = (1, 0, -2)$. A vector normal to $\mathcal{P}$ is

$$\vec{n} = \vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} = ((-2 - 0)\vec{i} - (-4 - 1)\vec{j} + (0 - 1)\vec{k}) = (-2, 5, -1).$$

The equation of $\mathcal{P}$ is therefore $\vec{n} \cdot (x, y, z) = \vec{n} \cdot B = -2(-1) + 5(1) - 1(4) = 3$

$$-2x + 5y - z = 3.$$

To check this, we see if points A and C satisfy this equation:

$$-2(1) + 5(2) - 1(5) = -2 + 10 - 5 = 3 \quad \text{and} \quad -2(3) + 5(1) - 1(4) = -6 + 5 + 4 = 3;$$

they check out.

Since $\mathcal{L}$ is perpendicular to the plane, it is parallel to $\vec{n}$, and we may use $\vec{n}$ as a direction vector for $\mathcal{L}$. Since D is on $\mathcal{L}$, we see that the line is given parametrically as

$$\mathcal{L} : D + t\vec{n} = (6 - 2t, 2 + 5t, 7 - t).$$

To find where this line intersects the plane, we plug the ordinates of $\mathcal{L}$ into the equation of $\mathcal{P}$ and solve for t:

$$-2(6 - 2t) + 5(2 + 5t) - (7 - t) = 3 \Rightarrow 4t + 25t + t = 3 + 12 - 10 + 7 \Rightarrow 30t = 12 \Rightarrow t = \frac{2}{5}.$$

Thus at $t = \frac{2}{5}$, the line is on the plane at the point

$$\Big(6 - 2(\frac{2}{5}), 2 + 5(\frac{2}{5}), 7 - \frac{2}{5}\Big) = \Big(\frac{26}{5}, \frac{20}{5}, \frac{33}{5}\Big).$$

$\square$

Problem 2. Consider the points $A = (1, 2, 3)$, $B = (6, 9, -4)$, and $C = (2, 7, 4)$ in $\mathbb{R}^3$. Let $\mathcal{L}$ denote the set of all points in $\mathbb{R}^3$ which are equally distant to A, B, and C. Then $\mathcal{L}$ is a line. Find the parametric equations of $\mathcal{L}$.

Solution. The line we seek passes through the centroid of $\triangle ABC$ and is perpendicular to the plane on which this triangle lies.

We have seen that the centroid of a triangle is $\frac{2}{3}$ of the way from a given vertex to the midpoint of the opposite side. Selecting A as the vertex, we compute the midpoint of the opposite side to be $M = (4, 8, 0)$. The centroid is $D = A + \frac{2}{3}(M - A) = (1, 2, 3) + \frac{2}{3}(3, 6, -3) = (1, 2, 3) + (2, 4, -2) = (3, 6, 1)$.

To find the normal vector $\vec{n}$ of the plane, which is the direction vector of $\mathcal{L}$, let $\vec{v} = B - A = (5, 7, -7)$ and $\vec{w} = (C - A) = (1, 5, 1)$ so that

$$\vec{n} = \frac{1}{6}\vec{v} \times \vec{w} = \frac{1}{6}\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 7 & -7 \\ 1 & 5 & 1 \end{bmatrix} = \frac{1}{6}[(7 + 35)\vec{i} - (5 + 7)\vec{j} + (25 - 7)\vec{k}] = (7, -2, 3).$$

Then

$$\mathcal{L} : D + t\vec{n} = (3 + 7t, 6 - 2t, 1 + 3t).$$

□

Problem 3. The spheres

$$x^2 + y^2 + z^2 = 144 \quad \text{and} \quad (x - 3)^2 + (y - 4)^2 + (z - 12)^2 = 25$$

intersect in a circle. Find the center of the circle.

Solution. The first sphere is centered at $O = (0, 0, 0)$ and has radius $r = 12$; the second sphere is centered at $P = (3, 4, 12)$ and has radius $s = 5$. Let $\vec{w} = P - O = (3, 4, 12)$; the point we seek on on the line through O in the direction of $\vec{v}$.

Let $\mathcal{C}$ be the intersection of the spheres. If Q is a point on $\mathcal{C}$, let $\vec{v} = Q - O$, so that the point we seek is the tip of the vector projection of $\vec{v}$ onto $\vec{w}$.

By inspection, we see $Q = (0, 0, 12)$ is a point on $\mathcal{C}$. Thus let $\vec{v} = (0, 0, 12)$. The vector projection of $\vec{v}$ onto $\vec{w}$ is

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\vec{w} = \frac{144}{169}\vec{w} = \left(\frac{432}{169}, \frac{576}{169}, \frac{1728}{169}\right).$$

□

Problem 4. Find the volume of the parallelepiped determined by the vectors $\vec{x} = (1, 2, 3)$, $\vec{y} = (2, 3, 1)$, and $\vec{z} = (-1, 0, t)$, and find t such that these vectors are coplanar.

Solution. The volume is given by the triple scalar product:

$$V = \vec{x} \cdot (\vec{y} \times \vec{z}) = \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & 0 & t \end{bmatrix} = 1(3t - 0) - 2(2t + 1) + 3(0 + 3) = 3t - 4t - 2 + 9 = -t + 7.$$

The vectors are coplanar when this volume is zero, which happens when $t = 7$. □