PRINCIPLES OF ANALYSIS
SOLUTIONS TO MIDTERM

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1. Portion A - Take Home

Problem 1. (Computation) Let $p \in \mathbb{Z}$ be a positive prime integer and let

$$f(x) = x^3 - x^2 - 21x + p.$$

Suppose that $f(x)$ has a positive rational root. Find p, and then find all roots of $f(x)$.

Solution. By the rational roots theorem, the only possibly positive rational roots are 1 and p. Now $f(1) = 1 - 1 - 21 + p$, so if this equal zero, we get that $p = 21$, which is not prime, so this is not a possibility. Thus assume that $f(p) = 0$. This gives $p^3 - p^2 - 20p = 0$, so $p^2 - 1 - 20 = 0$, which factors as $(p-5)(p+4) = 0$. Thus $p = 5$.

Divide $f(x)$ by $(x-5)$ to find that $f(x) = (x-5)(x^2 + 4x - 1)$. Apply the quadratic formula to this quadratic to find that the entire solution set to the equation $f(x) = 0$ is

$$\{5, -2 + \sqrt{5}, -2 - \sqrt{5}\}.$$

□

Problem 2. (Computation)
Define a sequence in $\mathbb{R}$ by $a_1 = 1$ and

$$a_{n+1} = \sqrt{a_n + 1}.$$

(a) Show that $0 < a_n < a_{n+1} < 2$ for all $n \in \mathbb{N}$.
(b) Show that the sequence (a_n) converges.
(c) Find $\lim a_n$.

Solution.
(a) For $n = 1$, we have $0 < 1 < \sqrt{2} < 2$. Let $n > 1$ and assume that $0 < a_{n-1} < a_n < 2$. Adding 1 gives $0 < a_{n-1} + 1 < a_n + 1 < 3$. Taking the square root gives

$$0 < 1 < a_n = \sqrt{a_{n-1} + 1} < \sqrt{a_n + 1} = a_{n+1} < \sqrt{3} < 2.$$

(b) By part **(a)**, (a_n) is a bounded increasing sequence, and so it converges.
(c) Let $L = \lim a_n$. Then $L \geq 0$, and $L = \sqrt{L+1}$, so $L^2 - L - 1 = 0$. Then only nonnegative root to this quadratic is $L = \frac{1+\sqrt{5}}{2}$. □

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Problem 3. (Theory)
Let (X, ρ) be a metric space and let $D \subset X$. Recall that the *closure of* D, denoted by $\overline{D}$, is the intersection of all closed subsets of X which contain D. Set

$$A = \{x \in X \mid x \text{ is an accumulation point of } D\};$$
$$B = \{x \in X \mid x \text{ is an isolated point of } D\};$$

Show that

(a) A is closed;
(b) $A \cap B = \varnothing$;
(c) $A \cup B = \overline{D}$.

Lemma 1. *Let $U, F \subset X$, with U open and F closed. Then $F \smallsetminus U$ is closed.*

Proof of Lemma. Since U is open, $X \smallsetminus U$ is closed. Now $F \smallsetminus U = F \cap (X \smallsetminus U)$, which is the intersection of two closed sets, and is therefore closed. □

Solution.
(a) It suffices to show that $X \smallsetminus A$ is open. Let $x \in X \smallsetminus A$. Then x is not an accumulation point of D. Thus their exists a deleted open neighborhood U of x which does not intersect D. Suppose that $a \in U$ for some $a \in A$. Then U is an open neighborhood of a, and a is an accumulation point of D, so U intersects D, which it doesn't. Thus no such a exists, and U does not intersect A. This shows that every point in $X \smallsetminus A$ has an open neighborhood contained in $X \smallsetminus A$, so $X \smallsetminus A$ is open. Thus A is closed.
(b) Let $a \in A$. Then every deleted neighborhood of a intersects D. Thus a is not an isolated point of D. Therefore $A \cap B = \varnothing$.
(c) We show that $A \cup B \subset \overline{D}$, and that $\overline{D} \subset A \cup B$.
($\subset$) Let $c \in A \cup B$. Then $c \in A$ or $c \in B$. If $c \in B$, then $c \in D$; clearly $D \subset \overline{D}$, so $c \in \overline{D}$. Thus suppose that $c \in A$, so that c is an accumulation point of D. Let F be a closed set containing D. Suppose that $c \in X \smallsetminus F$. Now $X \smallsetminus F$ is open, so there exists an open neighborhood U of c contained in $X \smallsetminus F$. That is, $U \cap F = \varnothing$, and since $D \subset F$, we have $U \cap D = \varnothing$. This is impossible, since c is an accumulation point. Thus we must have $c \in F$. Since c is in every closed set that contains D, we have $c \in \overline{D}$.
($\supset$) Let $x \in X$ and suppose that $x \notin A \cup B$; we show that $x \notin \overline{D}$. Since $x \notin B$, we see that $x \notin D$. Moreover, since $x \notin A$, there exists an open neighborhood U of x such that $U \subset X \smallsetminus D$, that is, $U \cap D = \varnothing$. Now let F be any closed set which contains D. By the lemma, $F \smallsetminus U$ is a closed set which contains D, and $x \notin F$. Thus $x \notin \overline{D}$. □

Problem 4. (Example)
If $v_1 = (x_1, y_1, z_1) \in \mathbb{R}^3$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, the *vector sum* of v_1 and v_2 is

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

If $v = (x, y, z) \in \mathbb{R}^3$ and $a \in \mathbb{R}$. the *scalar product* of a and v is

$$av = (ax, ay, az),$$

The *standard basis* for $\mathbb{R}^3$ is

$$\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}.$$

If $v = (x, y, z) \in \mathbb{R}^3$, then

$$v = xe_1 + ye_2 + ze_3.$$

The *closed unit ball* in $\mathbb{R}^3$ is the metric subspace of $\mathbb{R}^3$ defined by

$$\mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

The *unit sphere* in $\mathbb{R}^3$ is the metric subspace of $\mathbb{R}^3$ defined by

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Find a subset $A \subset (\mathbb{R} \smallsetminus \mathbb{D}^3)$ satisfying:

(1) every point in A is isolated;
(2) the accumulation points of A form the vertices of a regular octahedron in $\mathbb{S}^2$.

Solution. Let

$$A = \Big\{ \pm \Big(1 - \frac{1}{n}\Big) e_i \mid n \in \mathbb{N} \text{ and } i = 1, 2, 3 \Big\}.$$

□

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