Problem 1 (Thomas §16.2 # 16)**.** Find the work done by $\vec{F} = \langle 6z, y^2, 12x \rangle$ along the path $\vec{r}(t) = \langle \sin t, \cos t, t/6 \rangle$.

Solution. The vectors along the path are given by plugging $\vec{r}$ into $\vec{F}$, thusly:

$$\vec{F}(\vec{r}(t)) = \langle t, \cos^2 t, 12 \sin t \rangle.$$

The derivative of $\vec{r}$ is

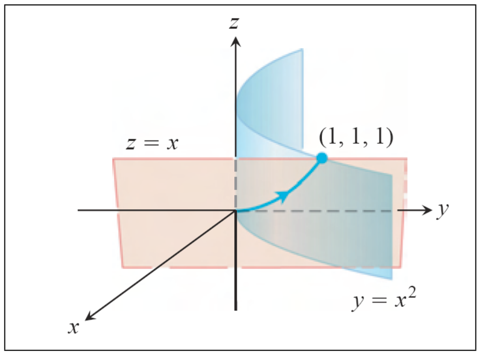
$$\vec{v}(t) = \langle \cos t, -\sin t, \frac{1}{6} \rangle.$$

So

$$\begin{aligned}
\text{Work} &= \int_0^{2\pi} \vec{F} \cdot \vec{v}\, dt \\
&= \int_0^{2\pi} \langle t, \cos^2 t, 12 \sin t \rangle \cdot \langle \cos t, -\sin t, \frac{1}{6} \rangle\, dt \\
&= \int_0^{2\pi} t \cos t - \cos^2 t \sin t + 2 \sin t\, dt \\
&= (t \sin t + \cos t) + (\frac{1}{3} \cos^3 t) - (2 \cos t) \Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

□

Problem 2 (Thomas §16.2 # 43)**.** The field $\vec{F}(x, y, z) = \langle xy, y, -yz \rangle$ is the velocity field of a flow in space. Find the flow from $(0, 0, 0,)$ to $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$.



Solution. Along the curve, we have $y = x^2$ and $z = x$, so the curve is parameterized by

$$\vec{r}(t) = \langle t, t^2, t \rangle,$$

whose derivative is

$$\vec{v})(t) = \langle 1, 2t, 1 \rangle.$$

Along this curve, the flow is

$$\vec{F}(\vec{r}(t)) = \langle t^3, t^2, -t^3 \rangle.$$

Now

$$\begin{aligned}
\text{Flow} &= \int_0^1 \vec{F} \cdot \vec{v}\, dt \\
&= \int_0^1 \langle t^3, t^2, -t^3 \rangle \cdot \langle 1, 2t, 1 \rangle\, dt \\
&= \int_0^1 t^3 + 2t^3 - t^3\, dt \\
&= \int_0^1 2t^3\, dt \\
&= \frac{t^4}{2}\bigg|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

□

Problem 3 (Thomas §16.2 # 44)**.** Find the flow of the field $\vec{F} = \nabla(xy^2z^3)$ along these paths.

(a) Once around the curve C, which is the ellipse which is the intersection of the plane $2x + 3y - z = 0$ and the cylinder $x^2 + y^2 = 12$, clockwise as viewed from above.

(b) Along the line segment from $(1, 1, 1)$ to $(2, 1, -1)$.

Attempted Solutions. I started to do these problems based on what was clear just from Section 16.2, and I will show you how far I got before I looked for an easier way. This will give an example of why theorems are useful.

(a) Let $\alpha = \sqrt{12}$.

Along C, we have $z = 2x + 3y$, so the curve is the image of the path

$$\vec{r}(t) = \langle \alpha \cos t, \alpha \sin t, 2\alpha \cos t + 3\alpha \sin t\rangle.$$

Its derivative is

$$\vec{v}(t) = \langle -\alpha \sin t, \alpha \cos t, -2\alpha \sin t + 3\alpha \cos t\rangle.$$

The gradient is

$$\vec{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle.$$

When we attempt to plug this into Along the path, this is

$$\vec{F}(\vec{r}(t)) = ...$$

I didn't really want to plug this in.

(b) The line segment from point A to point B is parameterized as $A + t(B - A)$ for $t \in [0, 1]$.

In our case,

$$\vec{r}(t) = (1, 1, 1) + t\langle 1, 0, -2\rangle = \langle 1 + t, 1, 1 - 2t\rangle,$$

whose derivative is

$$\vec{v}(t) = \langle 1, 0, -2\rangle.$$

The gradient is

$$\vec{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle.$$

Along the path, this is

$$\vec{F}(\vec{r}(t)) = \langle (1 - 2t)^3, 2(1 + t)(1 - 2t)^3, 3(1 + t)(1 - 2t)^3\rangle.$$

So

$$\begin{aligned}
\text{Flow} &= \int_0^1 \vec{F} \cdot \vec{v}\, dt \\
&= \int_0^1 \langle (1 - 2t)^3, 2(1 + t)(1 - 2t)^3, 3(1 + t)(1 - 2t)^3\rangle \cdot \langle 1, 0, -2\rangle\, dt \\
&= \int_0^1 (1 - 2t)^3 - 6(1 + t)(1 - 2t)^3\, dt \\
&= ...
\end{aligned}$$

At this point, I didn't what to multiply out this quartic polynomial. I could have, but ...

$\square$

Lemma 1. *Let $D \subset \mathbb{R}^n$ and let $I \subset \mathbb{R}$ be an interval. Let $f : D \to \mathbb{R}$ and let $\vec{F} : D \to \mathbb{R}^n$ be given by $\vec{F} = \nabla f$. Let $\vec{r} : I \to D$ be a path in D. Then*

$$\nabla f \cdot \frac{d\vec{r}}{dt} = \frac{d}{dt} f(\vec{r}(t)).$$

Proof. This is the chain rule:

$$\frac{d}{dt} f(\vec{r}(t)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt}.$$

□

Lemma 2. *Let $D \subset \mathbb{R}^n$ and let $a, b \in \mathbb{R}$ with $a < b$. Let $f : D \to \mathbb{R}$ and let $\vec{F} : D \to \mathbb{R}^n$ be given by $\vec{F} = \nabla f$. Let $\vec{r} : [a, b] \to D$ be a path in D. Then the flow of $\vec{F}$ along $\vec{r}$ is given b*

$$\int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Proof. From the previous lemma and the Fundamental Theorem of Calculus,

$$\text{Flow} = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} = \int_a^b \frac{d}{dt} f(\vec{r}(t)) = f(\vec{r}(b)) - f(\vec{r}(a)).$$

□

Problem 3 (Thomas §16.2 # 43 - Second Attempt)**.** The field $\vec{F}(x, y, z) = \langle xy, y, -yz \rangle$ is the velocity field of a flow in space. Find the flow from $(0, 0, 0,)$ to $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$.

Solution Thomas §16.2 # 44. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = xy^2 z^3$, so that $\vec{F} = \nabla f$. If we use the lemma, we don't actually need to compute the gradient, just the endpoints of the domain of the curve.

(a) Along C, we have $z = 2x + 3y$, so the curve is the image of the path

$$\vec{r} : [0, 2\pi] \to \mathbb{R}^3 \quad \text{given by} \quad \vec{r}(t) = \langle \alpha \cos t, \alpha \sin t, 2\alpha \cos t + 3\alpha \sin t \rangle.$$

So in this case, $a = 0$ and $b = 2\pi$.

Note that the ellipse is a closed loop, so $\vec{r}(0) = \vec{r}(2\pi) = (\alpha, 0, 2\alpha)$; that is, $\vec{r}(b) = \vec{r}(a)$, so

$$\boxed{\text{Flow} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0.}$$

(b) In this case,

$$\vec{r} : [0, 1] \to \mathbb{R}^3 \quad \text{is given by} \quad \vec{r}(t) = (1, 1, 1) + t\langle 1, 0, -2 \rangle = \langle 1 + t, 1, 1 - 2t \rangle,$$

so $a = 0$, $b = 1$, $\vec{r}(a) = (1, 1, 1)$, and $\vec{r}(b) = (2, 1, -1)$, so

$$\text{Flow} = f(\vec{r}(b)) - f(\vec{r}(a)) = -2 - 1 = -3.$$

□