PRINCIPLES OF ANALYSIS
SOLUTIONS TO PROBLEM SET E

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Problem 1 (Exercise 4.25)**.** Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Suppose that there exists $M > 0$ such that for every $x \in (a, b)$, we have $|f'(x)| \leq M$.

(a) Show that if $x, y \in (a, b)$, then $|\frac{f(x)-f(y)}{x-y}| \leq M$.
(b) Show that f is uniformly continuous on (a, b).

Solution. Let $x, y \in (a, b)$, with $x > x$. Then by the Mean Value Theorem, there exists $c \in [y, x]$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. Therefore

$$\Big|\frac{f(x) - f(y)}{x - y}\Big| = |f'(c)| \leq M.$$

This proves part **(a)**.

Now let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{M}$. Let $x, y \in (a, b)$ such that $|x - y| < \delta$. Then

$$|f(x) - f(y)| \leq (x - y)M \leq \delta M = \epsilon.$$

□

Problem 2 (Exercise 4.35)**.** Let $f(x) = x^3 + 2x^2 - x + 1$. Find an equation for the line tangent to the graph of f^{-1} at the point $(3, 1)$.

Solution. We know that $f'(x) = 3x^2 + 4x - 1$. The roots of this quadratic function are $\frac{-2\pm\sqrt{7}}{3}$. Since $\sqrt{7} < 3$, the larger root is less that $\frac{1}{3}$. Therefore $f'(x)$ is nonzero on $(\frac{1}{3}, \infty)$, and f is invertible on this interval.

Let f^{-1} denote the inverse of f on $(\frac{1}{3}, \infty)$. Now $f(1) = 3$, so $(3, 1)$ is a point on the graph of f^{-1}. By the inverse function theorem,

$$(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{6},$$

so this is the slope of the tangent line. Thus the tangent line is of the form $y = \frac{1}{6}x + b$. Since $(3, 1)$ is on the line, we have $1 = \frac{1}{6} \cdot 3 + b$, so $b = \frac{1}{2}$, and the tangent line is

$$y = \frac{1}{6}x + \frac{1}{2}.$$

□

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Observation 1 (Alternate Definition)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. Define

$$Q : \mathbb{R} \smallsetminus \{0\} \to \mathbb{R} \quad \text{by} \quad Q(h) = \frac{f(x_0 + h) - f(x_0)}{h}.$$

Then f is differentiable at x_0 if and only if $\lim_{h \to 0} Q(h)$ exists, in which case $f'(x_0) = \lim_{h \to 0} Q(h)$.

Problem 3 (Exercise 4.39)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

(1) $f(0) = 1$;
(2) f is differentiable at 0 and $f'(0) = 1$;
(3) $f(x + y) = f(x)f(y)$.

Show that f is differentiable on $\mathbb{R}$ and that $f'(x) = f(x)$ for every $x \in \mathbb{R}$.

Solution. Let $x \in \mathbb{R}$. Then

$$\frac{f(x + h) - f(x)}{h} = \frac{f(x)f(h) - f(x)}{h} = f(x)\Big(\frac{f(h) - 1}{h}\Big).$$

Taking the limit as h goes to 0, and noting that $f(x)$ is a constant with respect to h, yields

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}.$$

Since f is differentiable at zero, this limit exists, and

$$f'(x) = f(x)f'(0) = f(x).$$

□

Definition 1. A function $f : [-b, b] \to \mathbb{R}$ is called *odd* if $f(x) = -f(-x)$ for every $x \in [-b, b]$.

Problem 4 (Exercise 5.14)**.** Let $f : [-b, b] \to \mathbb{R}$ be an odd function which is integrable on $[-b, b]$. Show that $\int_{-b}^{b} f\,dx = 0$.

Proof. Say that a partition P of $[-b, b]$ is *symmetric* if $0 \in P$ and $x \in P \Rightarrow -x \in P$. Suppose that P is symmetric; the number of points in P is odd, so enumerate them $-b = x_0 < x_1 < \cdots < x_m = 0 < x_{m+1} < \cdots < x_n = x_{2m} = b$. Under this enumeration, $-x_i = x_{2m-i}$.

Let x_{i-1} and x_i be adjacent points in $[-b, b] \cap P$. Let $c_i \in [x_{i-1}, x_i]$ such that $f(c_i) = M_f(P, i)$. Then $-f(-c_i) = m_f(P, 2m - i)$, and

$$\begin{aligned} U_f(P) &= \sum_{i=0}^{n} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=0}^{n} -m_f(P, 2m - i)(x_i - x_{i-1}) \\ &= \sum_{i=0}^{n} -m_f(P, 2m - i)(x_{2m-i} - x_{2m}) \\ &= -L_f(P). \end{aligned}$$

Let $\epsilon > 0$, and let P_0 be a partition of $[-b, b]$ such that

$$U_f(P_0) - L_f(P_0) < \epsilon.$$

Let $P_1 = \{-x \mid x \in P\}$, and let $P = P_0 \cup P_1$. Then P is a partition of $[-b, b]$, and P is a refinement of P_0, so $U_f(P) - L_f(P) < \epsilon$. Combine this with the fact that $U_f(P) = -L_f(P)$ to get $U_f(P) < \frac{\epsilon}{2}$. Now

$$-\frac{\epsilon}{2} < -U_f(P) = L_f(P) \leq \int_{-b}^{b} f\,dx \leq U_f(P) \leq \frac{\epsilon}{2}.$$

That is, for every $\epsilon > 0$,

$$\Big| \int_{-b}^{b} f\,dx \Big| < \epsilon.$$

Thus $\int_{-b}^{b} f\,dx = 0$. $\square$

Problem 5 (Exercise 5.27)**.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Define $h : [a, b] \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Show that h is integrable on $[a, b]$.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be integrable. Define a function $f^+ : D \to \mathbb{R}$ by*

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then $f^+ : [a, b] \to \mathbb{R}$ is also integrable.

Proof of Lemma. Let $\epsilon > 0$ and let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$ such that $U_f(P) - L_f(P) < \epsilon$. Then for every i we have $M_f(P, i) \geq M_{f^+}(P, i)$, and $m_f(P, i) \leq m_{f^+}(P, i)$; this implies that $M_{f^+}(P, i) - m_{f^+}(P, i) \leq M_f(P, i) - m_f(P, i)$. Thus

$$\begin{aligned} U_{f^+}(P) - L_{f^+}(P) &= \sum_{i=1}^{n} (M_{f^+}(P, i) - m_{f^+}(P, i))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_f(P, i) - m_f(P, i))(x_i - x_{i-1}) \\ &= U_f(P) - L_f(P) \\ &< \epsilon. \end{aligned}$$

This shows that f^+ is integrable. $\square$

Solution to Problem. Note that $h = (f - g)^+ + g$. Since f and g are integrable, so is $f - g$. Thus $(f - g)^+$ is integrable, and so is $h = (f - g)^+ + g$. $\square$

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