AP Calculus AB — Responses 03/19

Dr. Paul L. Bailey — Wednesday, March 19, 2020

I will try to get to the rest of your questions tomorrow.

Question 1. Why would we use two variables for substitution? (Example 2 used "a" as well as another substituted variable)

Question 2. I'm worried about the use of "a" in this unit. How similar is it's function to "u" going into the next sections and under what circumstance is it used?

Answer. Usually (but not absolutely always), a is an unknown constant, and u is a variable.

In example 2, they are using a to match what you see to the formula in the box on the previous page. This formula is

$$\boxed{\int \frac{du}{a^2 - u^2} = \arctan u + C}$$

where a is constant and u is the variable of integration. We can simply set $u = x$ to write the exact same formula as

$$\boxed{\int \frac{dx}{a^2 - x^2} = \arctan x + C}$$

I will show you where that comes from.

We know that $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$, so $\int \frac{dx}{\sqrt{1-x^2}} = \arctan x + C$.

Now we wish to compute $\int \frac{dx}{\sqrt{a^2 - x^2}}$, where a is any fixed nonzero real number. Let $au^2 = x^2$, so that $u = \frac{x}{a}$ and $dx = a\,du$. Then

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\,du}{\sqrt{a^2 - a^2u^2}} = \int \frac{a\,du}{a\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arctan u + C.$$

For example,

$$\int \frac{dx}{\sqrt{9 - x^2}} = \arctan(x/3) + C.$$

$\square$

Question 3. What was the process for taking the derivative of a power with two variables?

Answer. I assume you mean these two formulas, where x is a variable and a is a constant:

- $\frac{d}{dx} x^a = ax^{a-1}$
- $\frac{d}{dx} a^x = \ln a \cdot a^x$

The first is just the power rule. The second comes from writing

$$a^x = \exp(x \log(a)),$$

where $\exp x = e^x$ and $\log x = \ln x$. We know this is true, since $a^x = \exp(\log(a^x)) = \exp(x \log(a))$; because exp and log are inverse functions.

Now

$$\frac{d}{dx} a^x = \frac{d}{dx} \exp(x \log(a)) = \exp(x \log(a)) \cdot \log(a),$$

the latter $\log(a)$ comes out by the chain rule. $\square$

Question 4. What does the +C after the antiderivative represent?

Answer. Since the derivative of a constant is zero, we see that (for example) $\dfrac{d}{dx}\sin x + 5 = \cos x$.

Recall Corollary 2 to the Mean Value Theorem. It says that if two function have the same derivative, then they differ by a constant. So, writing (for example) $\int \cos x\, dx = \sin x + C$ means that *any* function whose derivative is $\cos x$ is of the form $\sin x + C$, for some fixed real number C. □

Question 5. Related rates ... aargh!

Answer. Ok well let's just keep doing these until it becomes more natural. Here's one. □

Problem 1 (Thomas §3.7 # 17)**.** Sand falls from a conveyor belt at the rate of $10\text{m}^3/\text{min}$ onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast is the height changing when the pile is 4 m high?

Solution. When you are a student doing a related rates problem, you write down a formula, and implicitly differentiate with respect to time. Its what you do.

First, draw a picture.

Next, write down any pertinent equations, and identify the cheese.

The volume of a cone is $V = \dfrac{\pi}{3}r^2h$. From the problem, $\dfrac{dV}{dt} = 10$, and $h = \dfrac{3}{8}D = \dfrac{3}{4}r$. The cheese is $\dfrac{dh}{dt}$ when $h = 4$.

If we eliminate one of the variables from the volume equation, life will be easier. So, plug in $r = \dfrac{4}{3}h$ to get

$$V = \frac{\pi}{3}\left(\frac{4}{3}h\right)^2 h = \frac{16\pi}{27}h^3.$$

Now pull a GEICO: take $\dfrac{d}{dt}$ of both sides to get

$$\frac{dV}{dt} = \frac{16\pi}{27}(3h^2)\frac{dh}{dt} = \frac{16\pi}{9}h^2\frac{dh}{dt}.$$

This last $\dfrac{dh}{dt}$ comes from the chain rule, and is critical (it is the cheese!) so don't forget it.

Plug in $\dfrac{dV}{dt} = 10$ and $h = 4$ to get

$$10 = \frac{16\pi}{9}4^2\frac{dh}{dt}, \qquad \text{so } \frac{dh}{dt} = \frac{45}{128\pi}.$$

□

Question 6. What are some uses of the slope field diagrams we will discuss later?

Answer. They can help you visualize a solution to a differential equation. Imagine the slope field and a river flowing; if you drop a leaf at a given point, follow the slopes to imagine it moving. □

Question 7. How many weeks of preparation will we have before the AP?

Answer. If we can move through the material at a meaningful rate, we should have all of April to review. However, at this point we can't count on the AP exam being given on the scheduled date. □

Question 8. According to the prepbook, it says that we have learned all that we need to learn for AP Calc AB. Are we learning Calc BC stuff or is this just an extension of Calc AB?

Answer. We are not completely through the material. Please see the outline I sent on 03/16. □

Question 9. Which rules regarding integration and trigonometric functions will we have to be comfortable with for the AP?

Answer. We have just completed the approximate level of integration difficulty you will find on the AP. The AP assumes that you are fluent in trigonometry. □