Vector Calculus — Responses 0330
Dr. Paul L. Bailey — Monday, March 30, 2020

I think this is all correct, please let me know if you find an error.

Problem 1. When evaluating a path integral, you are always asked to make sure the parametrization for the path is smooth. What does that mean and why is that a necessary condition before integrating?

Answer. According to Thomas, a path is *smooth* if it is differentiable, with continuous and nonvanishing derivatives. If this is the case, the path has a well-defined nonzero tangent vector at every point.

We will typically be using the formula

$$\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))|\vec{v}(t)|\,dt,$$

where $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$. So, $\vec{v}(t)$ must be at least piecewise differentiable to evaluate this integral. Moreover, work and flow integral require a vector field to be dotted with the velocity vector, so that velocity vector better exist. □

Problem 2. Consider the vector fields in $\mathbb{R}^2$ given by

$$\vec{F}(x,y) = \langle x, y\rangle \quad \text{and} \quad \vec{G}(x,y) = \langle -y, x\rangle.$$

Let C denote the unit circle, oriented counterclockwise.

(a) Draw a small sketch of each of these vector fields.

(b) Compare the flow of $\vec{F}$ versus $\vec{G}$ along C.

(c) Compare the flux of $\vec{F}$ versus $\vec{G}$ along C.

Write clearly, in complete sentences. Explain your reasoning.

Solution. The vector field $\vec{F}$ has vector that are point away from the origin, which get longer the farther away from the origin it gets. The vector field $\vec{G}$ has vectors that are perpendicular to the position vector at a point, so they look like they are going around the origin, and get longer the farther away from he origin they get.

The flow of $\vec{F}$ along the unit circle is zero, since the field vectors are perpendicular to the tangent vectors, so the dot products are all zero. The flow of $\vec{G}$ is maximal, since the field vectors are parallel to the tangent vectors. To compute this, parameterize the curve:

$$\vec{r} : [0, 2\pi] \to \mathbb{R}^2 \quad \text{given by } \vec{r}(t) = \langle \cos t, \sin t\rangle.$$

Then the velocity is

$$\vec{v}(t) = \langle -\sin t, \cos t\rangle.$$

Now

$$\vec{G}(\vec{r}(t)) = \langle -\sin t, \cos t\rangle,$$

so

$$\text{Flow}(G) = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\,dt = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi} \sin^2 t + \cos^2 t = 2\pi.$$

The flux of $\vec{G}$ along the unit circle is zero, again because the vectors are perpendicular, but the flux is maximal, since the vectors are parallel, and the flux is computed to be $\vec{F}$ is 2π. □

Problem 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous real valued function defined on $\mathbb{R}^2$.
Let C be a smooth curve in $\mathbb{R}^2$ parameterized by $\vec{r} : [0, 1] \to \mathbb{R}^2$ with parameter t. Compare and contrast the integrals below numerically (what "answer" do you get?), geometrically, and in words.

$$\int_C f(x, y)\, ds \qquad \text{versus} \qquad \int_0^1 f(\vec{r}(t))\, dt$$

Solution. These are different, unless the curve is parameterized by arclength, so the speed is one. This is because

$$\int_C f(x, y)\, ds = \int_0^1 f(\vec{r}(t))|\vec{v}(t)|\, dt \neq \int_0^1 f(\vec{r}(t))\, dt.$$

□

Problem 4. Let $\vec{F} = \langle -y, x\rangle$. Let C_1 denote the semicircle of radius 1 in the upper half plane from $(1, 0)$ to $(-1, 0)$ Let C_2 denote the line segment along the x-axis from $(1, 0)$ to $(-1, 0)$.

(a) Compute the flow of $\vec{F}$ along C_1.

(b) Compute the flow of $\vec{F}$ along C_2.

(c) Compare these values. Are they the same? Is $\vec{F}$ conservative?

Solution. The curve C_1 is parameterized by

$$\vec{r}_1(t) : [0, \pi] \to \mathbb{R}^2 \qquad \text{by} \qquad \vec{r}(t) = \langle \cos t, \sin t\rangle,$$

so

$$\vec{v}_1(t) = \langle -\sin t, \cos t\rangle.$$

We have

$$\mathrm{Flow}_{C_1} = \int_0^\pi \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^\pi dt = \pi.$$

The curve C_2 is parameterized by

$$\vec{r}_2 : [0, 1] \to \mathbb{R}^2 \qquad \text{by} \qquad \vec{r}(t) = (1, 0) + t\langle -2, 0\rangle,$$

so

$$\vec{v}_2(t) = \langle -2, 0\rangle.$$

Thus

$$\mathrm{Flow}_{C_2} = \int_0^1 \langle 0, 1 - 2t\rangle \cdot \langle -2, 0\rangle\, dt = 0.$$

The field is not conservative, because it does not admit path independence. □