AP Calculus AB — Dr. Paul L. Bailey

Responses 0407 — Monday, April 6, 2020

Problem 1. Let f be a function that is twice differentiable for all real numbers. The table below gives values of f for selected points in the closed interval $2 \leq x \leq 13$.

x	2	3	5	8	13
$f(x)$	1	4	-2	3	6

(A) Estimate $f'(4)$. Show the work that leads to your answer.

Solution. The derivative of f at $x = 4$ is approximately equal to the slope of a nearby secant line. The best we can do in this case is to compute the average rate of change between 3 and 5. Thus

$$f'(4) \approx \frac{f(5) - f(3)}{5 - 3} = \frac{-2 - 4}{2} = -3.$$

$\square$

(B) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \leq x \leq 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \leq 4$. Use the secant line for the graph of f on $5 \leq x \leq 8$ to show that $f(7) \geq \frac{4}{3}$.

Solution. Since $f(5) = -2$ and $f'(5) = 3$, the line tangent to the graph of f at $x = 5$ is

$$\ell_1(x) = 3(x - 5) - 2.$$

Since $f''(x) < 0$ for $x \in (5, 8)$, the graph of f is concave down on this interval, which implies that the tangent line ℓ_1 lies above the graph of f. Thus, $f(7) \leq \ell_1(7) = 3(7 - 5) - 2 = 4$.

The secant line from $x = 5$ to $x = 8$ has slope $\dfrac{f(8) - f(5)}{8 - 5} = \dfrac{3 - (-2)}{3} = \dfrac{5}{3}$, so the line is

$$\ell_2(x) = \frac{5}{3}(x - 5) - 2.$$

Since $f''(x) < 0$ for $x \in (5, 8)$, the graph of f is concave down on this interval, which implies that the secant line ℓ_2 lies below the graph of f. Thus, $f(7) \geq \ell_2(7) = \frac{5}{3}(7 - 5) - 2 = \frac{4}{3}$. $\square$

(C) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\displaystyle\int_2^{13} f(x)\, dx$.

Solution. The left Riemann sum is

$$f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + f(x_2)\Delta x_3 + f(x_3)\Delta x_4 = 1(3-2) + 4(5-3) - 2(8-5) + 3(13-8) = 1 + 8 - 6 + 15 = 18.$$

$\square$