Problem 1. (Points in $\mathbb{R}^2$)
Consider the points $A(2,5)$, $B(6,1)$, and $C(-7,3)$.
Let $\vec{v}$ be the vector from A to B, and let $\vec{w}$ be the vector from A to C.

(a) Compute $\vec{v}$ and $\vec{w}$.

(b) Find the cosine of the angle $\angle BAC$.

(c) Find the area of the triangle $\triangle ABC$.

(d) Find the general equation of the line $\overleftrightarrow{AB}$.

(e) Find the distance from the point C to the line $\overleftrightarrow{AB}$.

Solution. Compute.

(a) $\vec{v} = B - A = \langle 4, -4\rangle$
$\vec{w} = C - A = \langle -9, -2\rangle$

(b) $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|} = \dfrac{-7}{\sqrt{170}}$

(c) The signed area of parallelogram is $ad - bc = -8 + 36 = -28$, so area of triangle is 14.

(d) A normal vector is $\vec{n} = \langle 1, 1\rangle$, so the equation is $x + y = 7$.

(e) The distance is $|\mathrm{proj}_{\vec{n}}\vec{w}| = \dfrac{11}{\sqrt{2}}$.

$\square$

Problem 2. (Lines, Planes, and Spheres in $\mathbb{R}^3$)
Let A be the locus of the equation

$$x^2 + y^2 + z^2 = 4x + 6y + 12z.$$

Let B be the locus of the equation

$$x - 4y + 8z = 11.$$

Let $C = A \cap B$.

(a) A is a sphere. Find its center and radius.

(b) B is a plane. Find a normal vector for B. Find the vector equation of a line through the center of A and perpendicular to B.

(c) C is a circle. Find its center.

Solution. Complete the square to rewrite the sphere's equation as $(x-2)^2 + (y-3)^2 + (z-6)^2 = 7^2$. So, the center is $(2,3,6)$ and the radius is 7.

Read off the normal vector as $\vec{n} = \langle 1, -4, 8\rangle$. This is also the direction vector for the line, so the line is $\vec{r}(t) = (2,3,6) + t\langle 1,-4,8\rangle = \langle 2+t, 3-4t, 6+8t\rangle$.

The center of the circle is clearly the intersection of the plane and the line. Plug the line into the plane to get $(2+t) - 4(3-4t) + 8(6+8t) = 11$, and solve for t to get $t = -\dfrac{1}{3}$. The point of intersection is $\vec{r}(-1/3) = \left\langle \dfrac{5}{3}, \dfrac{13}{3}, \dfrac{10}{3}\right\rangle$.

$\square$

Problem 3. (Lines and Planes)
Compute the indicated value(s).

(a) Find the parametric equations of the line passing through the points $P(2,5,1)$ and $Q(1,4,2)$.

(b) Find the standard equation of a plane which contains the line from part **(a)** and passes through the point $R(4,2,1)$.

(c) Find the distance from the point $S(-2,4,-5)$ to the plane from part **(b)**.

Solution. Compute.

(a) A direction vector for the line is $\vec{v} = Q - P = \langle -1,-1,1\rangle$, so the line is $\vec{r}(t) = \langle 2-t, 5-t, 1+t\rangle$.

(b) Let $\vec{w} = R - P = \langle 2,-3,0\rangle$. A normal vector for the plane is $\vec{n} = \vec{v}\times\vec{w} = \langle 3,2,5\rangle$. An equation for the plane is $3x + 2y + 5z = 21$.

(c) Let $\vec{x} = S - P = \langle -4,-1,-6\rangle$. The distance is $|\mathrm{proj}_{\vec{n}}\vec{x}| = \dfrac{44}{\sqrt{38}}$.

□

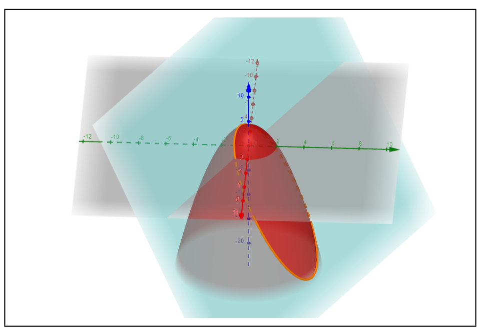
Problem 4. (Circles of Intersection) Let $A = \{(x,y,z)\in\mathbb{R}^3 \mid z = 4 - x^2 - y^2\}$ and $B = \{(x,y,z)\in\mathbb{R}^3 \mid 2x + 4y + z = 0\}$ Let $C = A\cap B$.

(a) Draw a (very rough) sketch of this situation which includes the set C. Describe A, B, and C.

(b) The projection of C onto the xy-plane is a circle. Find its center and radius.

(c) Find the intersection of C with the xz-plane.

Solution. The surface A is a circular paraboloid, B is a plane, and C is an ellipse.



Any point on the intersection satisfies the equations $z = 4 - x^2 - y^2$ and $2x + 4y + z = 0$. Plug $z = -2x - 4y$ into the first equation and rearrange to get $x^2 - 2x + y^2 - 4y = 4$. By eliminating z, we obtain the projection onto the xy-plane. Now complete the square to get $(x-1)^2 + (y-2)^2 = 9$. So, the projection is a circle with center $(1,2)$ and radius 3.

On the xz-plane, $y = 0$. The equations become $z = 4 - x^2$ and $2x + z = 0$. So, $z = -2x$, and $x^2 - 2x - 4 = 0$. By QF, $x = 1 \pm \sqrt{5}$, so we have to points of intersection, $(1+\sqrt{5}, 0, -2-2\sqrt{5})$ and $(1-\sqrt{5}, 0, -2+2\sqrt{5})$. □

Problem 5. (Distance from Point to Line in $\mathbb{R}^2$)
Consider the points $A(3, -2)$, $B(6, 2)$, and $C(7, 11)$.
Let $\vec{v}$ be the vector from A to B, $\vec{w}$ be the vector from A to C, and $\vec{x}$ a vector which is perpendicular to $\vec{v}$.

(a) Find expressions for $\vec{v}$, $\vec{w}$, and $\vec{x}$.

(b) Find the scalar projection of $\vec{w}$ onto $\vec{x}$.

(c) Find the distance from the point C to the line $\overleftrightarrow{AB}$. Justify your answer.

Solution. Compute.

(a) $\vec{v} = B - A = \langle 3, 4 \rangle$
$\vec{w} = C - A = \langle 4, 13 \rangle$
$\vec{x} = \langle 4, -3 \rangle$

(b) $\text{proj}_{\vec{x}}\vec{w} = -\dfrac{23}{5}$

(c) Clearly $\vec{v}$ is a direction vector for the line. Since $\vec{x} \perp \vec{v}$, we see that $\vec{x}$ is a normal vector for the line. Thus the distance is the absolute value of the projection above, that is, $d = \dfrac{23}{5}$.

□

Problem 6. (Spheres)
Find an equation of a sphere if one of its diameters has endpoints $A(2, 1, 4)$ and $B(4, 3, 10)$.

Solution. Let M be the midpoint of the diameter, which is the center of the center. Then $M = (3, 2, 7)$. The radius is the distance from M to A, so $r = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$. Thus the equation is

$$\boxed{(x-3)^2 + (y-2)^2 + (z-7)^2 = 11.}$$

□

Problem 7. (Parallelepipeds)
Find the volume of the parallelepiped determined by the vectors $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 2, 3, 1 \rangle$, and $\vec{c} = \langle -1, 0, z \rangle$. Find z such that these vectors are coplanar.

Solution. The volume is the triple scalar project:

$$V = \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & 0 & z \end{bmatrix} = -z + 7.$$

The vectors are coplanar when $V = 0$, that is, when $z = 7$. □

Problem 8. (Planes)
Find an equation for the plane consisting of all points that are equidistant from the two points $(1, 1, 0)$ and $(0, 1, 1)$.

Solution. The midpoint $M = \left(\dfrac{1}{2}, 1, \dfrac{1}{2}\right)$ is a point on the plane. A normal vector for the plane is the difference of the points, so let $\vec{n} = \langle 1, 0, -1 \rangle$. The plane is $x - z = 0$. □

Problem 9. (Quadric Surfaces)
The intersection of the quadric surface $x^2 + y^2 + z = 15$ and the plane $2x + 6y + z = 1$ is a curve whose projection onto the xy-plane is a circle. Find the center and radius of the circle.

Solution. From the plane, $z = 1 - 2x - 6y$. Plug this into the quadric surface to get $x^2 + y^2 + 1 - 2x - 6y = 15$. Complete the square to get $(x-1)^2 + (y-3)^2 = 24$. So, the center is $(1, 3, 0)$, and the radius in $\sqrt{24}$. □

Problem 10. (Intersection Of Planes)
Let A be the plane given by $x + 2y + 3z = 6$ and B be the plane given by $3x + 2y + z = 6$.
Let $L = A \cap B$ be the line of intersection of A and B. Let $P_0 = (1, 1, 1)$ and note that $P_0 \in L$.
Find the equation of the plane which is perpendicular to L and passes through the point P_0, expressed in the form $ax + by + cz = d$.

Solution. Another point on the line of intersection is $Q = (0, 2, 0)$, so the vector $\vec{v} = Q - P_0 = \langle 1, -1, 1\rangle$ is a direction vector for this line, and hence is a normal vector for the plane. Since $(1, 1, 1)$ is a point on the plane, its equation is

$$\boxed{x - y + z = 1.}$$

□

Problem 11. (Distance from Point to Plane in $\mathbb{R}^3$)
Let A be the plane in $\mathbb{R}^3$ with equation $x + 2y + 3z = 6$. Let Q be the point in $\mathbb{R}^3$ given by $Q = (7, -2, 4)$. Find the distance from the point Q to the plane A.

Solution. A normal vector for the plane is $\vec{n} = \langle 1, 2, 3\rangle$. A point on the plane is $P = (6, 0, 0)$. Let $\vec{w} = Q - P = \langle 1, -2, 4\rangle$. The distance is

$$d = \text{proj}_{\vec{n}}\vec{w} = \frac{9}{\sqrt{14}}.$$

□

Problem 12. (Distance from Point to Line in $\mathbb{R}^3$)
Let A be the plane in $\mathbb{R}^3$ with equation $x + 2y + 3z = 6$. Let Q be the point in $\mathbb{R}^3$ given by $Q = (7, -2, 4)$. Let B be the plane in $\mathbb{R}^3$ with equation $z = 0$. Find the distance from the point Q to the line $A \cap B$.

Solution. The locus of the equation $z = 0$ is the xy-plane. The intersection of the line and the xy-plane is the line $x + 2y = 6$ (with $z = 0$). A direction vector for this line is $\langle 2, -1, 0\rangle$. A point on this line is $P = (6, 0, 0)$. Let $\vec{w} = Q - P = \langle 1, -2, 4\rangle$. Let R denote the point on the line which is closest to Q. The distance from P to R is

$$c = \text{proj}_{\vec{v}}\vec{w} = \frac{4}{\sqrt{5}}.$$

By the Pythagorean Theorem, the distance from Q to R is

$$d = \sqrt{|\vec{w}|^2 - c^2} = \sqrt{\frac{89}{5}}.$$

The distance □

Problem 13. (Paths in $\mathbb{R}^3$)
Consider the path

$$\vec{r} : [0, 2\pi] \to \mathbb{R}^3 \quad \text{be given by} \quad \vec{r}(t) = \langle \cos t, \sin t, t^2 + \pi t\rangle.$$

(a) Find the time t when the curve intersects the plane $z = \dfrac{10\pi^2}{9}$.

(b) Find the point P where the curve intersects the plane $z = \dfrac{10\pi^2}{9}$.

(c) Find the velocity vector as a function of t.

(d) Find the acceleration vector as a function of t.

(e) Find the time t where the velocity vector is perpendicular to the acceleration vector.

Problem 14. A curve is perpendicular to a surface at a point of intersection if a tangent vector of the curve is parallel to the normal vector of the surface at that point. Find a level surface of $f(x, y, z) = z - (y+1)^2 + x^2$ which is perpendicular to the curve traced out by $\vec{r}(t) = \langle t, t, e^t\rangle$, and the point of intersection.
(Hint: use partials to find the normal to the level surface at a value k. Compare this normal to the velocity vector to find t. Now find k.)

Problem 15. Let $\vec{r}(t) = \langle t, t, t^2 + 3\rangle$ be the position of a particle at time t. Find the minimum distance between the particle and the sphere $(x-5)^2 + (y-5)^2 + z^2 = 9$, the time t when it occurs, and the closest point on the sphere at this time.
(Hint: this may be done in many ways.)

Problem 16. Let $\vec{r}(t) = \langle t, t, t^2 + 3\rangle$ be the position of a particle at time t. Find the minimum distance between the particle and the sphere $(x-5)^2 + (y-5)^2 + z^2 = 9$, the time t when it occurs.

Problem 17. Find the minimal distance between the lines $\alpha(s) = \langle 2+s, 3-2s, 5+4s\rangle$ and $\beta(t) = \langle t, 2t, 3t\rangle$.

Problem 18. The function $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\vec{r}(t) = \left\langle t^2 - t, \dfrac{4}{t} \right\rangle$ represents a curve in $\mathbb{R}^2$. The function $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\vec{f}(x, y) = \langle x, y, xy+1\rangle$ represents a surface in $\mathbb{R}^3$. The composite function $\vec{f} \circ \vec{r} : \mathbb{R} \to \mathbb{R}^3$ given by $\vec{f} \circ \vec{r}(t) = \vec{f}(\vec{r}(t))$ represents a curve in $\mathbb{R}^3$ which lies on the surface. Find the tangent vector to the curve represented by $\vec{f} \circ \vec{r}$ at the point $(2, 2, 5)$.