CATEGORY THEORY
TOPIC II: COLLECTIONS

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1. Collections of Sets

We do not disallow the possibility that a set may be an element of another set. In fact, this idea is very useful. For example, we may talk about the set of lines in a plane, even though each line is a set of points in the plane. The set of lines is a set of subsets of the points in the plane. It is common to call sets whose elements are subsets of a given set a *collection* of subsets.

Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. Then the *intersection* and *union* of the sets in the collection are defined by

- $\cap\mathcal{C} = \{x \in X \mid x \in C \text{ for all } C \in \mathcal{C}\}$;
- $\cup\mathcal{C} = \{x \in X \mid x \in C \text{ for some } C \in \mathcal{C}\}$.

Thus $\cap\mathcal{C}$ is the intersection of all the sets in $\mathcal{C}$ and $\cup\mathcal{C}$ is their union.

Example 1. Let $A = \{n \in \mathbb{N} \mid n < 25\}$, $O = \{n \in A \mid n \text{ is odd}\}$, $P = \{n \in A \mid n \text{ is prime}\}$, and $S = \{n \in A \mid n \text{ is a square}\}$. Let $\mathcal{C} = \{O, P, S\}$. Then

- $\cap\mathcal{C} = \varnothing$, because no square is a prime;
- $\cup\mathcal{C} = \{2, 3, 4, 5, 7, 9, 11, 13, 15, 16, 17, 19, 21, 23\}$.

□

Example 2. Let $A = \{n \in \mathbb{N} \mid n < 1000\}$. For each $d \in \mathbb{N}$, define

$$D_d = \{n \in A \mid n = dm \text{ for some } m \in \mathbb{N}\}.$$

Let $\mathcal{D} = \{D_p \mid p \text{ is prime and } p \leq 7\}$. Find $\cap\mathcal{D}$.

Solution. The set D_d is the set of positive multiples of d which are less then 1000. The set $\mathcal{D}$ is the collection of all D_p such that p is a prime which is less than 7. Thus $\mathcal{D} = \{D_2, D_3, D_5, D_7\}$. Then $\cap\mathcal{D}$, being the intersection of these sets, is the set of natural numbers less than 1000 which are multiples of 2, 3, 5, and 7. Such a number must be a multiple of 210. Also, any multiple of 210 which is less than 1000 is in all four sets. Thus $\cap\mathcal{D} = \{210, 420, 630, 840\}$. □

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2. Collections of Functions

We may also consider sets whose members are functions.

Example 3. Let X be a set and let $\mathrm{Sym}(X)$ be the set of all bijective functions on X. Then $\mathrm{Sym}(X)$ is a collection of functions. $\square$

If A and B are sets, we may speak of the set of all functions from A to B. We shall denote this set by $\mathcal{F}(A, B)$:

$$\mathcal{F}(A, B) = \{f : A \to B\}.$$

Example 4. Let $A = \{1, 2\}$ and $B = \{5, 6, 7\}$. Then $\mathcal{F}(A, B)$ contains the following functions:

- $1 \mapsto 5$ and $2 \mapsto 5$;
- $1 \mapsto 5$ and $2 \mapsto 6$;
- $1 \mapsto 5$ and $2 \mapsto 7$;
- $1 \mapsto 6$ and $2 \mapsto 5$;
- $1 \mapsto 6$ and $2 \mapsto 6$;
- $1 \mapsto 6$ and $2 \mapsto 7$;
- $1 \mapsto 7$ and $2 \mapsto 5$;
- $1 \mapsto 7$ and $2 \mapsto 6$;
- $1 \mapsto 7$ and $2 \mapsto 7$.

Also $\mathcal{F}(B, A)$ contains the following functions:

- $5 \mapsto 1$, $6 \mapsto 1$, $7 \mapsto 1$;
- $5 \mapsto 1$, $6 \mapsto 1$, $7 \mapsto 2$;
- $5 \mapsto 1$, $6 \mapsto 2$, $7 \mapsto 1$;
- $5 \mapsto 1$, $6 \mapsto 2$, $7 \mapsto 2$;
- $5 \mapsto 2$, $6 \mapsto 1$, $7 \mapsto 1$;
- $5 \mapsto 2$, $6 \mapsto 1$, $7 \mapsto 2$;
- $5 \mapsto 2$, $6 \mapsto 2$, $7 \mapsto 1$;
- $5 \mapsto 2$, $6 \mapsto 2$, $7 \mapsto 2$.

$\square$

Example 5. Let $\mathcal{F} = \mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the set of all real valued functions of a real variable:

$$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}.$$

Let $\mathcal{D}$ denote the set of all differentiable functions in $\mathcal{F}$:

$$\mathcal{D} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}.$$

Note that $\mathcal{D} \subset \mathcal{F}$.

The differentiation operator is a function

$$\frac{d}{dx} : \mathcal{D} \to \mathcal{F}.$$

Not every function is the derivative of a function, so $\frac{d}{dx}$ is not surjective. Since two functions which differ by a constant have the same derivative, $\frac{d}{dx}$ is not injective. $\square$

3. Power Sets

Let X be a set. The *power set* of X is denoted $\mathcal{P}(X)$ and is defined to be the set of all subsets of X:

$$\mathcal{P}(X) = \{A \mid A \subset X\}.$$

Here are a few examples:

- $X = \varnothing \Rightarrow \mathcal{P}(X) = \{\varnothing\}$;
- $X = \{0\} \Rightarrow \mathcal{P}(X) = \{\varnothing, \{0\}\}$;
- $X = \{0,1\} \Rightarrow \mathcal{P}(X) = \{\varnothing, \{0\}, \{1\}, X\}$;
- $X = \{0,1,2\} \Rightarrow \mathcal{P}(X) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, X\}$.

and so forth. Here are some properties:

- $Y \subset X \Rightarrow \mathcal{P}(Y) \subset \mathcal{P}(X)$;
- $\cap\mathcal{P}(X) = \varnothing$;
- $\cup\mathcal{P}(X) = X$.

Let X be any set and let $T = \{0,1\}$. A given function $f : X \to T$ may be viewed as a subset of X by thinking of f as saying, for a given element, whether or not it is in the subset. The element 1 is thought of as "ON" or "TRUE" and the element 0 is thought of as "OFF" or "FALSE". Specifically, given $f : X \to T$, define A to the preimage of 1:

$$A = \{a \in A \mid f(a) = 1\};$$

that is, $A = f^{-1}[\{1\}]$.

On the other hand, given a subset of X, we can construct a function

$$\chi_A : X \to T$$

by defining

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in a. \end{cases}$$

This is just the characteristic function of the subset A.

Thus the power set of X corresponds to the set of functions from X into T in a natural way. Another way of stating this is that there exists a bijective function between $\mathcal{P}(X)$ and $\mathcal{F}(X,T)$.

4. Partitions

Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. We say that $\mathcal{C}$ *covers* X if $\cup\mathcal{C} = X$. We say that the sets in $\mathcal{C}$ are *collectively disjoint* if $\cap\mathcal{C} = \varnothing$. If for every two distinct sets $C, D \in \mathcal{C}$, we have $C \cap D = \varnothing$, we say that the members of $\mathcal{C}$ are *mutually disjoint* (or *pairwise disjoint*). If the sets of a collection are mutually disjoint, then they are collectively disjoint, but the converse of this is not necessarily true.

Example 6. Let $X = \{1, 2, 3\}$ and let $\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then

$$\cup\mathcal{C} = (\{1, 2\} \cup \{2, 3\}) \cup \{2, 3\} = \{1, 2, 3\} \cup \{2, 3\} = \{1, 2, 3\} = X,$$

so the sets in $\mathcal{C}$ cover X. Also

$$\cap\mathcal{C} = (\{1, 2\} \cap \{1, 3\}) \cap \{2, 3\} = \{1\} \cap \{2, 3\} = \varnothing,$$

so the sets in $\mathcal{C}$ are collectively disjoint. They are not, however, mutually disjoint.

Let $\mathcal{D} = \{\{1, 2\}, \{3\}\}$. Then $\mathcal{D}$ covers X with mutually disjoint sets. $\square$

A *partition* of X is a collection of mutually disjoint nonempty subsets of X which covers X. The members of a partition are called *blocks*.

Let $\mathcal{C} \subset \mathcal{P}(X)$. Then $\mathcal{C}$ is a partition of X if

(P0) $C \in \mathcal{C} \Rightarrow C \neq \varnothing$

(P1) $C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cap C_2 = \varnothing \vee C_1 = C_2$

(P2) $\cup\mathcal{C} = X$

Suppose that $\mathcal{C}$ is a partition of X. If $x \in X$, then there is a unique $A \in \mathcal{C}$ such that $x \in A$; x is certainly in one of them, because X is covered by the members of $\mathcal{C}$; x is in no more than one, for otherwise the ones containing x would overlap and not be disjoint. Put another way, every $x \in X$ is in exactly one of the members of $\mathcal{C}$.

Example 7. Let x be a point in a space and let $S(x, r)$ be a sphere of radius r with center x. Then the collection

$$\mathcal{S} = \{S(x, r) \mid r \in \mathbb{R} \text{ and } r \geq 0\}$$

is a partition of space; the blocks of this partition are spheres centered at x. This is true since each point in space has a unique distance from the point x. $\square$

Example 8. Let C be the set of cards in a deck and let S be the set of suits. That is, C contains 52 elements and $S = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$. There is a natural function $f : C \to S$ which sends a given card to its suit. The preimage of a suit under f is the set of cards in that suit, for example:

$$f^{-1}[\spadesuit] = \{2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, \mathrm{J}\spadesuit, \mathrm{Q}\spadesuit, \mathrm{K}\spadesuit, \mathrm{A}\spadesuit\}.$$

Let $\mathcal{S} = \{f^{-1}[s] \mid s \in S\}$. Then $\mathcal{S}$ is a collection of subsets of C, each subset consisting of all the cards in a given suit. It is clear that $\mathcal{S}$ covers C and that the sets within $\mathcal{S}$ are mutually disjoint. Thus $\mathcal{S}$ is a partition of C. This is a general phenomenon: functions induce partitions on their domains. We will explore this in depth later.

One more thing to notice here. There are as many elements in $\mathcal{S}$ as there are in S. Indeed, in some philosophical way, $\mathcal{S}$ is *essentially the same* as the set S. $\square$

5. Partitions Given by Functions

Let X and Y be sets and let $f : X \to Y$ be surjective. Let $y \in Y$. The *fiber over* y is $f^{-1}(y)$, the preimage of y.

The function f induces a partition of the set X is a natural way, by taking the collection of fibers. Let

$$\overline{X} = \{A \subset X \mid A = f^{-1}(y) \text{ for some } y \in Y\}.$$

Then $\overline{X}$ is a partition of X.

For each $x \in X$, let $\overline{x} = f^{-1}(f(x))$. Clearly, $\overline{x}$ is the fiber over $f(x)$, and $x \in \overline{x}$. That is, $\overline{x}$ is the set of elements in X which are mapped to $f(x)$ by f. As x ranges over all of X, we see that the blocks of the partition $\overline{X}$ are the subsets of X of the form $\overline{x}$. That is,

$$\overline{X} = \{A \in \mathcal{P}(X) \mid A = \overline{x} \text{ for some } x \in X\}.$$

The *canonical function* induced by f is

$$\beta : X \to \overline{X} \quad \text{given by} \quad \beta(x) = \overline{x}.$$

We use the greek letter β to remind us that this is the BAR function.

We attempt to define a function $\overline{f} : \overline{X} \to Y$ by setting $\overline{f}(\overline{x}) = f(x)$. The problem with this is that the definition appears to depend on which element in $\overline{x}$ we pick. If x' is another element in $\overline{x}$, then x' is in the fiber over $f(x)$, so $f(x') = f(x)$. Thus our definition does not depend on the particular element in $\overline{x}$ we pick; we say that the function $\overline{f}$ is "well-defined".

Proposition 1. *Let X and Y be sets and let $f : X \to Y$ be surjective. Let*

$$\overline{X} = \{A \subset X \mid A = f^{-1}(y) \textit{ for some } y \in Y\}.$$

Let

$$\beta : X \to \overline{X} \quad \textit{given by} \quad \beta(x) = \overline{x}.$$

Set

$$\overline{f} : \overline{X} \to Y \quad \textit{given by} \quad \overline{f}(\overline{x}) = f(x).$$

Then $\overline{f}$ is well-defined, and

$$\overline{f} \circ \beta = f.$$

This result is the *Isomorphism Theorem in the Category of Sets*. When we see the analogous theorem in other categories, it will be quite useful. The situation is represented by the following "commutative diagram".

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\beta"'] & & Y \\ & \overline{X} \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Saying that the diagram commutes means that an element "flows" through the arrows from one set to another in a manner which does not depend on the route.

6. Exercises

Exercise 1. Design a collection $\mathcal{C}$ of subsets of $\mathbb{N}$ which has all of the following properties:

(1) $\mathcal{C}$ covers $\mathbb{N}$ ($\cup\mathcal{C} = \mathbb{N}$);
(2) distinct sets in $\mathcal{C}$ are disjoint ($C, D \in \mathcal{C}$ and $C \neq D \Rightarrow C \cap D = \varnothing$);
(3) each set $C \in \mathcal{C}$ contains infinitely many elements;
(4) $\mathcal{C}$ contains exactly 7 subsets of $\mathbb{N}$.

Recall that we have given the name "partition" to collections of sets satisfying the first two properties.

Exercise 2. Let $\mathbb{R}$ be the set of real numbers.
(a) Find a collection of subsets of $\mathbb{R}$ which covers $\mathbb{R}$ but whose members are not collectively disjoint.
(b) Find a collection of subsets of $\mathbb{R}$ which covers $\mathbb{R}$ and whose members are collectively disjoint but not mutually disjoint.
(c) Find three different partitions of $\mathbb{R}$, each containing a different number of blocks.

Exercise 3. Let $X = \{1, 2, 3, 4, 5\}$ and let $Y = \{1, 2, 3\}$. Find a five different partitions of the set $\mathcal{F}(X, Y)$, each of which contains three blocks.

Exercise 4. Let X be a set and let $A, B \subset X$.
(a) Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(b) Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.
(c) Find an example such that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

Exercise 5. Let X be a set. Find an injective function $\phi : X \to \mathcal{P}(X)$.

Exercise 6. Let X be as set. Show that there does not exist a surjective function $\phi : X \to \mathcal{P}(X)$.
(Hint: select an arbitrary function $\phi : X \to \mathcal{P}(X)$, and construct a set in $\mathcal{P}(X)$ which is not in the image of ϕ.)

Exercise 7. Let X be a set. Define a function $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $A \mapsto X \setminus A$. Show that ϕ is bijective.

Exercise 8. Let X be a set and let $T = \{0, 1\}$. Show that there is a correspondence between the sets $\mathcal{P}(X)$ and $\mathcal{F}(X, T)$.

Exercise 9. Let X be a set containing n elements. Count the size of the set $\mathcal{P}(X)$.

Exercise 10. Let A and B be sets containing m and n elements respectively. Count the size of the set $\mathcal{F}(A, B)$.

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