Linear Algebra Exercises A
Solutions
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Exercise 5. Express i and j in terms of v and w.

(b) $v = 2i + 3j$, $w = i - j$

Solution. We wish to show that $\vec{i}$ and $\vec{j}$ are in the span of $\vec{v} = (2, 3)$ and $\vec{w} = (1, -1)$. To do this, we need to find $a, b, c, d \in \mathbb{R}$ such that

$$\vec{i} = a\vec{v} + b\vec{w} \quad \text{and} \quad \vec{j} = c\vec{v} + d\vec{w}.$$

The first equation amounts to solving the system of equations

$$\begin{aligned} 2a + b &= 1; \\ 3a - b &= 0. \end{aligned}$$

Adding these gives $5a = 1$, so $a = \frac{1}{5}$. Then the second equation produces $b = \frac{3}{5}$. Thus

$$\vec{i} = \frac{1}{5}\vec{v} + \frac{3}{5}\vec{w}.$$

Similarly,

$$\begin{aligned} 2c + d &= 0; \\ 3c - d &= 1. \end{aligned}$$

So again adding the equation gives $5c = 1$, which implies that $c = \frac{1}{5}$. The second equation then says $\frac{3}{5} - d = 1$, so $d = -\frac{2}{5}$. Therefore

$$\vec{j} = \frac{1}{5}\vec{v} - \frac{2}{5}\vec{w}.$$

$\square$

Exercise 8. Find the values for $x \in \mathbb{R}$ such that $v \perp w$.

(b) $v = (x, 12)$, $w = (x^2, -18)$

Solution. We know that $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$. Now $\vec{v} \cdot \vec{w} = x^3 - 216$, so we need to solve $x^3 - 216 = 0$. The unique real solution is $x = \sqrt[3]{216} = 6$. $\square$

Exercise 9. Find the values for $x \in \mathbb{R}$ such that the angle between $v = (1,1)$ and $w = (x,1)$ is $60°$.

Solution. We know that if θ is the angle between $\vec{v}$ and $\vec{w}$, then

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}.$$

Since $\cos 60° = \frac{1}{2}$, we need to solve the equation

$$\frac{x+1}{\sqrt{2}\sqrt{x^2+1}} = \frac{1}{2}.$$

Squaring both sides and multiplying by 2 gives

$$\frac{x^2+2x+1}{x^2+1} = \frac{1}{2}.$$

Cross multiplying and simplifying produces

$$x^2+4x+1 = 0.$$

The quadratic formula now gives

$$x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}.$$

Testing these in our original equation shows that squaring both sides introduced an extraneous solution; the single correct solution is

$$x = -2+\sqrt{3}.$$

□

Alternate Solution. The angle that $\vec{v}$ makes with the origin is $45°$, so the angle that $\vec{w}$ makes with the origin is $105°$. Now

$$\begin{aligned}
\cos 105° &= \cos 60° \cos 45° - \sin 60° \sin 45° &&= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4};\\
\sin 105° &= \cos 60° \sin 45° + \sin 60° \cos 45° &&= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}.
\end{aligned}$$

If $r = |w|$, the tip of $\vec{w}$ is at the point $(x,1) = (r\cos 105°, r\sin 105°)$. Thus $r\sin 105° = 1$, so $r = \frac{4}{\sqrt{2}+\sqrt{6}}$. Therefore

$$x = r\cos 105° = \frac{4}{\sqrt{2}+\sqrt{6}}\frac{\sqrt{2}-\sqrt{6}}{4} = \frac{(\sqrt{2}-\sqrt{6})^2}{2-6} = \frac{2+6-2\sqrt{12}}{-4} = -2+\sqrt{3}.$$

□

Exercise 10. Find the angle between the diagonal of a cube and one of its edges.

Solution. Select the cube determined by the vectors $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, and $\vec{k} = (0,0,1)$ in $\mathbb{R}^3$. The diagonal vector is $\vec{v} = \vec{i}+\vec{j}+\vec{k} = (1,1,1)$. If θ is the angle between $\vec{v}$ and $\vec{i}$, then

$$\cos\theta = \frac{\vec{v}\cdot\vec{i}}{|\vec{v}||\vec{i}|} = \frac{1}{\sqrt{3}}.$$

Thus $\theta = \arccos\frac{1}{\sqrt{3}} \approx 54.7°$. □