COMPLEX ANALYSIS
TOPIC XIV: RATIONAL FUNCTIONS

PAUL L. BAILEY

1. Topology

Just as algebra is the generalization of arithmetic, topology is the generalization of geometry. Topology uses the notion of open set to define neighborhoods, where a neighborhood of a point consists of other points which are somehow "close to" the given point. The concepts of limits and continuity can then be defined in terms of neighborhoods. Thus, these ideas may be generalized using the definition of topological space. We briefly discuss this.

Definition 1. A *topology* on a set X is a collection of subsets of X, whose members are called *open sets*, satisfying the following three properties:

- **(T1)** The empty set $\varnothing$ and the entire set X are open;
- **(T2)** The union of any number of open sets is open;
- **(T3)** The intersection of finitely many open sets is open.

A *topological space* is a sets X together with a topology on X.

The set of real numbers and the set of complex numbers are topological spaces, with the definitions of open sets we have already given. Also, $\mathbb{R}^n$ is a topological space, where the open sets are unions of open balls. Any subset of a topological space is naturally a topological space, with the subspace topology.

Definition 2. Let X be a topological space and let $A \subset X$. A subset $W \subset A$ is called *relatively open* if there exists a set $U \subset X$ which is open in X such that $W = A \cap U$. The set of relatively open subsets of A forms a topology on A, called the *subspace topology.*

Continuity and convergence may now be defined on any topological space.

Definition 3. Let X and Y be topological spaces, and let $f : X \to Y$. We say that f is *continuous at* $x \in X$ if, for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. We say that f is continuous if it is continuous at every point in the domain.

Definition 4. Let X be a topological space and let (x_n) be a sequence in X. We say that (x_n) *converges* to $L \in X$ if, for every neighborhood V of L there exists $N \in \mathbb{N}$ such that $x_n \in V$ whenever $n \geq N$.

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2. The Riemann Sphere

Definition 5. The *Riemann sphere* is the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, where ∞ is a single point which we append to the complex plane.

If $a \in \mathbb{C}$, a neighborhood of a in $\mathbb{C}_\infty$ is a set $U \subset \mathbb{C}_\infty$ such that $a \in U$ and $\{z \in \mathbb{C} \mid |z| < \epsilon\} \subset U$, for some $\epsilon > 0$.

A *neighborhood of infinity* is a set $U \subset \mathbb{C}_\infty$ such that $\infty \in U$ and $\{z \in \mathbb{C} \mid |z| > M\} \subset U$ for some $M > 0$.

A subset $U \subset \mathbb{C}_\infty$ is open if every point in U admits a neighborhood which is completely contained in U. This makes $\mathbb{C}_\infty$ into a topological space. Thus the notions of convergent sequences in $\mathbb{C}_\infty$, and continuous functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$, are defined.

Some of our arithmetic carries over from $\mathbb{C}$ to $\mathbb{C}_\infty$. Let $*$ be a binary operation defined on $\mathbb{C}$ (suppose that $*$ is addition, multiplication, subtraction, or division). We wish to define $a * b$, where $a, b \in \mathbb{C}_\infty$. These operations are continuous on $\mathbb{C}$, and we wish to maintain that property. So, we define the operation using limits, and consider the operation on $\mathbb{C}_\infty$ to be well defined if it is independent of how the limit is constructed.

Definition 6. Let $a, b, c \in \mathbb{C}_\infty$. We say that $a * b = c$ if, for any $z_0 \in \mathbb{C}_\infty$ and any functions f and g,

$$\lim_{z \to z_0} f(z) = a \text{ and } \lim_{z \to z_0} g(z) = b \quad \Rightarrow \quad \lim_{z \to z_0} f(z) * g(z) = c,$$

where the limits approach z_0 through $z \neq \infty$.

Let a be a nonzero complex number. It is relatively easy to see that the following computation follow from the definition above.

- $-\infty = \infty$
- $\frac{a}{\infty} = 0$ and $\frac{a}{0} = \infty$
- $a + \infty = \infty$
- $a \cdot \infty = \infty$
- $\infty - \infty$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, and $\frac{0}{0}$ are undefined

3. Meromorphic Functions

We wish to study continuous functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$. However, we also wish for such functions to be differentiable. This also requires definition: the standard definition already admits infinite derivatives; we extend the definition to say that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is differentiable at ∞ if $f(1/z)$ is differentiable at 0, in a manner made precise using limits. Differentiable functions are continuous.

Definition 7. Let $D \subset \mathbb{C}_\infty$. A *meromorphic function* on D is a function $f : D \to \mathbb{C}_\infty$ which is analytic, that is, is differentiable at every point in D. The set of all meromorphic functions on D form a field, called the *function field* of D.

It can be shown that the only meromorphic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are rational functions; that is, the function field of $\mathbb{C}_\infty$ is the set of all rational functions. It is this topic we wish to explore further.

4. Polynomial Functions

A *polynomial function* is a function of the form

$$f(z) = \sum_{i=0}^{n} a_i x^i,$$

where n is a nonnegative integer, and $a_i \in \mathbb{C}$. We call n the *degree* of f, and write $\deg(f) = n$. The a_i's are called *coefficients*. We call a_0 the *constant coefficient* and a_n the *leading coefficient*.

A *zero* (or *root*) of f is a number r such that $f(r) = 0$. The Factor Theorem says that $f(r) = 0$ if and only if $(z - r)$ is a factor of $f(z)$. The *multiplicity* of r as a zero of f is the largest integer m such that $(z - r)^m$ is a factor of $f(z)$.

By the Fundamental Theorem of Algebra, f has exactly n zeros, counted with multiplicity. Indeed, there exist (not necessarily distinct) $r_1, r_2, \ldots, r_n \in \mathbb{C}$ such that

$$f(z) = a_n \prod_{i=0}^{n} (z - r_i).$$

We may view polynomial function as mapping from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. In this case, $f(\infty) = \infty$. Indeed, ∞ is may be viewed as a "pole of multiplicity n", since ∞ is the only point which maps to ∞, but there is a deleted neighborhood of infinity which is mapped n-to-1 onto a neighborhood of infinity.

The set of all polynomials with complex coefficients is denoted $\mathbb{C}[x]$.

5. Rational functions

A *rational function* is a function of the form $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are polynomials with complex coefficients, and h is not constantly zero. If $\deg(f) = m$ and $\deg(g) = n$, then the Fundamental Theorem of Algebra says that

$$f(z) = c \frac{\prod_{i=0}^{m} (z - r_i)}{\prod_{j=0}^{n} (z - s_j)},$$

for some complex numbers $c, r_i, s_j \in \mathbb{C}$. The r_i's are the *zeros* of f, and the s_j's are the *poles* of f, where the set of zeros is disjoint from the set of poles. The constant c is the ratio of the leading coefficients of g and h.

We view the rational function f as $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by extension using limits

- $f(z) = \infty$ if and only if z is a pole of f;
- $f(\infty) = \lim_{z \to \infty} f(z)$.

Thus

$$f(\infty) = \begin{cases} 0 & \text{if } \deg(g) < \deg(h); \\ c & \text{if } \deg(g) = \deg(h); \\ \infty & \text{if } \deg(g) > \deg(h), \end{cases}$$

where c is the ratio of the leading coefficients of g and h.

The set of all rational functions over $\mathbb{C}$ is denoted $\mathbb{C}(x)$.

6. Fibers

If $f : A \to B$ and $b \in B$, the *fiber of f over b* is the preimage of b:

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}.$$

We would like to investigate the fiber of a rational function.

Let $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are polynomials. Let $b \in \mathbb{C}$. The fiber over b is the set of solutions to the equation $f(z) = b$, that is, $\dfrac{g(z)}{h(z)} = b$. Equivalently, this equation becomes $g(z) = bh(z)$, or, $g(z) - bh(z) = 0$. Assuming that the leading coefficient of $g(z)$ does not equal the leading coefficient of $bh(z)$, the function $g(z) - bh(z)$ is a polynomial of degree $n = \max\{\deg(g(z)), \deg(h(z))\}$, and therefore has exactly n zeros, counted with multiplicity. Thus $g(z) - bh(z) = 0$ has at most n solutions, and has exactly n solutions except in the rare cases that it has multiple roots. So, the function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is n-to-1 almost everywhere.

The *degree* of $f(z) = \dfrac{g(z)}{h(z)}$ is defined to be the maximum of the degrees of g and h. A rational function of degree n is n-to-1 over almost every point in the range.

7. Möbius Transformations

We next ask ourselves, which meromorphic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are invertible? Clearly they must be injective (one-to-one), so they have degree 1. That is, $f(z) = \dfrac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$. We would like to exclude from this the constant functions (which clearly are not injective).

To penetrate this question, let us use the fact that f is a constant function if and only if f' is identically zero. Compute

$$f'(z) = \frac{a(cz+d) - (az+b)c}{(cz+d)^2} = \frac{acz + d - acz - bc}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2}.$$

Thus $f(z)$ is constant if and only if $ad - bc = 0$.

Definition 8. A *linear fractional transformation* is a function of the form

$$S : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad S(z) = \frac{az+b}{cz+d},$$

for some $a, b, c, d \in \mathbb{C}$. Such a function is called a *Möbius transformation* if $ad - bc \neq 0$.

Let $S(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation. We may compute the inverse of f in the standard way to be

$$f^{-1}(z) = -\frac{dz - b}{cz - a}.$$

In fact, a meromorphic function $\mathbb{C}_\infty \to \mathbb{C}_\infty$ is invertible if and only if it is a Möbius transformation. The reader who has been exposed to group theory will recognize that the set of all Möbius transformations form a group under the operation of function composition.

We note that the coefficients a, b, c, d are not unique; indeed,

$$\frac{az+b}{cz+d} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d},$$

for any $\lambda \in \mathbb{C}$. Actually, though, for any Möbius transformation S, there is a unique a, b, c, d such that $S(z) = \dfrac{az+b}{cz+d}$ and $ad - bc = 1$.

Möbius transformations are transformations of the Riemann sphere, and in this context, we note that

- $S(-\frac{d}{c}) = \infty$, with the caveat that if $c = 0$, then $S(\infty) = \infty$.
- $S(\infty) = \frac{a}{c}$

8. Primitive Möbius transformations

A Möbius transformation $S(z) = \dfrac{az+b}{cz+d}$ is *primitive* if it matches one of the following four types.

- Translation $\quad S(z) = z + b$
- Dilation $\quad S(z) = az \quad$ where $a \in \mathbb{R}$ and $a > 0$
- Rotation $\quad S(z) = az \quad$ where $a = \operatorname{cis}\theta$ for some $\theta \in \mathbb{R}$
- Inversion $\quad S(z) = \frac{1}{z}$

A function of the form $S(z) = kz$, where k is an arbitrary complex number, may be viewed as a composition of a dilation and a rotation, since $k = r\operatorname{cis}\theta$ for $r = |k|$ and $\theta = \arg(k)$.

Proposition 1. *A Möbius transformation is a composition of translations, dilations, rotations, and inversions.*

Proof. Let $S(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation.

Suppose $c = 0$. Then $S(z) = \frac{a}{d}z + \frac{b}{d}$. Setting $S_1(z) = \frac{a}{d}z$ and $S_2(z) = z + \frac{b}{d}$, we see that $S = S_2 \circ S_1$.

On the other hand, if $c \neq 0$, we compute that that

$$S(z) = \frac{bc - ad}{c^2(z + \frac{d}{c})} + \frac{a}{c}.$$

Let $S_1(z) = z + \frac{d}{c}$, $S_2(z) = c^2 z$, $S_3(z) = \frac{1}{z}$, $S_4(z) = (bc - ad)z$, and $S_5(z) = z + \frac{a}{c}$. Then $S = S_5 \circ S_4 \circ S_3 \circ S_2 \circ S_1$. So S is a translation, following by a dilation/rotation, followed by inversion, followed by another dilation/rotation, followed by another translation. $\square$

Example 1. Find Möbius transformation $S(z) = \dfrac{az+b}{cz+d}$ which acts as the following sequence of transformations:

- Translate the plane so that 1 goes to 3
- Dilate the plane by a factor of 2
- Invert the sphere
- Rotate the sphere counterclockwise by 90°
- Translate the plane so that i goes to $2i$

Compute a, b, c, and d.

Solution. Let $S_1(z) = z + 2$, $S_2 = 2z$, $S_3(z) = \frac{1}{z}$, $S_4(z) = iz$, $S_5(z) = z + i$. Set $S = S_5 \circ S_4 \circ S_3 \circ S_2 \circ S_1$. Then

$$S(z) = \frac{i}{2z+4} + i = \frac{2iz + 5i}{2z+4}.$$

We have $a = 2i$, $b = 5i$, $c = 2$, and $d = 4$. $\square$

9. Stereographic Projection

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. Identify $\mathbb{C} = \{u + iv \mid u, v \in \mathbb{R}\}$ with the xy-plane in $\mathbb{R}^3$ via $u = x$ and $v = y$. Let $P = (0, 0, 1) \in \mathbb{R}^3$. We develop the mapping from $S^2 \setminus \{P\} \to \mathbb{C}$ via the process known as *stereographic projection*. This mapping identifies a point $Q \in \mathbb{C}$ with a point $R \in S^2$ by setting R to be the unique point in S^2 which is the intersection of the line through P and Q with S^2.

Given a point $Q = (u, v, 0)$, the line from P to Q can be parameterized by

$$\vec{r}(t) = P + (Q - P)t = \langle ut, vt, 1 - t\rangle.$$

The intersection of this line with S^2 is obtained by plugging $\vec{r}$ into the defining equation for the sphere:

$$(ut)^2 + (vt)^2 + (1 - t)^2 = 1 \Rightarrow u^2t^2 + v^2t^2 + t^2 = 2t.$$

Now $t = 0$ corresponds to P, since $\vec{r}(0) = P$; let R denote the other point of intersection of the line and the sphere. Then $R = \vec{r}(t)$ where

$$u^2t + v^2t + t = 2 \Rightarrow t = \frac{2}{u^2 + v^2 + 1}.$$

Plugging this t into $\vec{r}$ gives

$$R = \vec{r}(t) = \Big\langle \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \Big\rangle.$$

So, we may map a point $u + iv \in \mathbb{C}$ to the sphere S^2 via

- $x = \dfrac{2u}{u^2 + v^2 + 1}$
- $y = \dfrac{2v}{u^2 + v^2 + 1}$
- $z = \dfrac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$

Next we wish to find the inverse of this mapping; given a point $R = (x, y, z)$ on the Riemann sphere, we map it to a point $u + iv$ on the complex plane. A parameterization of the line from P to R is

$$\vec{r}(t) = P + (R - P)t = \langle xt, yt, (z - 1)t + 1\rangle.$$

This intersects the xy-plane if $(z - 1)t + 1 = 0$, in which case $t = \dfrac{1}{1 - z}$. Plugging this in to $\vec{r}$, and identifying (x, y) with (u, v), gives

$$\vec{r}\Big(\frac{1}{1 - z}\Big) = \Big\langle \frac{x}{1 - z}, \frac{y}{1 - z}, 0 \Big\rangle.$$

Thus

- $u = \dfrac{x}{1 - z}$
- $v = \dfrac{y}{1 - z}$

10. Circles

The next attribute of stereographic projection which we wish to explore is the fact that circles map to circles. This is under the understanding that a line is considered to be a "circle through infinity".

Consider an arbitrary circle on S^2. There is a unique plane $ax + by + cz = d$ such that the circle is the intersection of the S^2 and the plane. The images of a point in this intersection satisfy the equation obtained by plugging the stereographic projection formulas into the plane, thusly:

$$\frac{2au}{u^2+v^2+1} + \frac{2bv}{u^2+v^2+1} + \frac{c(u^2_+ v^2-1)}{u^2-v^2+1} = d.$$

Algebraic manipulation of this equation yields

$$(c-d)(u^2+v^2) + 2au + 2bv - (c+d) = 0.$$

This is the equation of a line if $c = d$, otherwise it is the equation of a circle. Note that $c = d$ if and only if the point $(0, 0, 1)$ is on the plane; that is, if the circle passes through ∞ on the Riemann sphere.

It is also the case that Möbius transformations send circles to circles, with the understanding that a line in $\mathbb{C}$ can be considered to be a "circle through infinity". Thus, if $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ has the property that the image of any circle is a circle, then we will say that S "preserves circles". Now if S and T both perserve circles, it is clear that their composition $T \circ S$ also preserves circles.

Next, we use the fact that any Möbius transformation is a composition of primitive transformations of these types: translations, dilations, rotations, and the inversion. It is fairly obvious that the first three types preserve lines and circles in $\mathbb{C}$, so we focus on the inversion.

Let $f(z) = \dfrac{1}{z} = u + iv$. Direct computation shows that

$$\frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2},$$

so $u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$.

Consider the equation

$$\alpha(x^2+y^2) + \beta x + \gamma y = \delta,$$

where α, β, and γ are not all zero. The locus of any line or circle can be written in this form, any locus of this form is a line or a circle. Divide through by $x^2 + y^2$ to get

$$\alpha + \beta\frac{x}{x^2+y^2} + \gamma\frac{y}{x^2+y^2} = \delta\frac{1}{x^2+y^2}.$$

We recognize u and v here; computation show that $u^2 + y^2 = \dfrac{1}{x^2+y^2}$. Thus this equation may be rearranged as

$$\alpha + \beta u - \gamma v = \delta(u^2+v^2),$$

which becomes

$$\delta(u^2+v^2) - \beta u + \gamma v = \alpha,$$

which is the equation of a circle. Thus inversion preserves circles, and therefore, all Möbius transformations preserve circles.

11. Fixed Points

Definition 9. Let $f : A \to A$. A *fixed point* of f is an element $a \in A$ such that $f(a) = a$.

We investigate the fixed points of a Möbius transformation. Suppose $S(z) = \dfrac{az+b}{cz+d}$. Then $S(w) = w$ means that $\dfrac{aw+b}{cw+d} = w$, so $aw + b = cw^2 + dw$, that is,

$$cw^2 + (d-a)w - b = 0.$$

Solving this equation leads to

$$w = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Clearly ∞ is a fixed point if and only if $c = 0$. In this case, S is linear, and there is a unique finite fixed point at $z = \dfrac{b}{d-a}$, unless $d = a$, in which case $b = 0$ and S is the identity given by $S(z) = z$.

Moreover, when S is not the identity, we see that S has at most two fixed points.

Now suppose that S and T are Möbius transformations which have the same values at three distinct points. Then $T^{-1} \circ S$ also will fix those three points, which implies that $T^{-1} \circ S$ is the identity, so $S = T$. This shows that a Möbius transformation is complete determined by its effect on any three points.

Example 2. Find the fixed points of $S(z) = \dfrac{z+2}{3z+5}$.

Solution. If $\dfrac{z+2}{3z+5} = z$, then

$$3z^2 + (5-1)z - 2 = 0,$$

so

$$z = \frac{-4 \pm \sqrt{16+24}}{6} = \frac{-2 \pm \sqrt{10}}{3}.$$

□

Example 3. Let $S(z) = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$ and $c = 1$, and the following properties:

- $S(0) = 0$;
- $S(1) = 1$;
- $S(\infty) = 2$.

Find a, b, c, and d.

Solution. Use each of these conditions:

- Since $S(0) = \dfrac{b}{d} = 0$, we see that $b = 0$, so $S(z) = \dfrac{az}{z+d}$.
- Since $S(1) = \dfrac{a}{1+d} = 1$, we see that $a = d + 1$.
- Since $S(\infty) = \dfrac{a}{1} = 2$, we see that $a = 2$, so $d = 1$.

Thus $a = 2$, $b = 0$, $c = 1$, $d = 1$; thus

$$S(z) = \frac{2z}{z+1}.$$

□

12. The Cross-Ratio

We know that a Möbius transformation is completely determined by its effect on any three distinct points in the Riemann sphere. In fact, there is exactly one Möbius transformation which sends any given ordered triple $(z_2, z_3, z_4) \in \mathbb{C}^3$ to another specified ordered triple $(w_2, w_3, w_4) \in \mathbb{C}^3$.

To see this, we first define a classical notation with historical roots, which we now describe. If A, B, C, D are points in an affine (think flat) plane, then their cross-ratio is the number

$$\mathfrak{R}(A, B, C, D) = \frac{AC}{BC} \Big/ \frac{AD}{BD},$$

where PQ represents the *signed* distance from P to Q. This number has important implication in projective geometry.

Definition 10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$. The *cross ratio* of these points is

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}.$$

Beware, the (standard) notation here is ambiguous. The ordered tuple (z_1, z_2, z_3, z_4) is being a mapped to a value which is identified with the same notation. You have to discern which meaning is intended from the context.

Note that

- $(z_2, z_2, z_3, z_4) = 1$
- $(z_3, z_2, z_3, z_4) = 0$
- $(z_4, z_2, z_3, z_4) = \infty$

To understand the cross-ratio more fully, we select three points z_2, z_3, and z_4, which we wish to send to 1, 0, and ∞, respectively. The Möbius transformation

$$S(z) : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad S(z) = (z, z_2, z_3, z_4) = \frac{z - z_3}{z - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}$$

has this effect; that is,

- $S(z_2) = 1$
- $S(z_3) = 0$
- $S(z_4) = \infty$

Thus S is the *unique* bijective rational function $\mathbb{C}_\infty \to \mathbb{C}_\infty$ which sends the order triple (z_2, z_3, z_4) to the ordered triple $(1, 0, \infty)$.

Example 4. If $T(z) = \dfrac{az + b}{cz + d}$, find z_2, z_3, z_4, written in terms of a, b, c, d, such that $T(z) = (z, z_2, z_3, z_4)$.

Solution. Since $\dfrac{az_2 + b}{cz_2 + d} = 1$, we know that $az_2 + b = cz_2 + d$, so $(a - d)z_2 = d - b$, and $z_2 = \dfrac{d - b}{a - d}$.

Since $\dfrac{az_3 + b}{cz_3 + d} = 0$, we know that $az_3 + b = 0$, so $z_3 = -\dfrac{b}{a}$.

Since $\dfrac{az_4 + b}{cz_4 + d} = \infty$, we know that $cz_4 + d = 0$, so $z_4 = -\dfrac{d}{c}$. $\square$

13. Properties of Cross Ratio

It is convenient to suppress some of the traditional function notation when working with Möbius transformations. We view a Möbius transformation S as "acting on a point" $z \in \mathbb{C}$, and we write Sz to mean $S(z)$. Also, since the composition of Möbius transformations is a Möbius transformation, we write ST to mean $S \circ T$.

Proposition 2. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$, and let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Then*

$$(Sz_1, Sz_2, Sz_3, Sz_4) = (z_1, z_2, z_3, z_4).$$

Proof. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $T(z) = (z, z_2, z_3, z_4)$; then T is the unique Möbius transformation which sends the ordered triple (z_2, z_3, z_4) to the ordered triple $(1, 0, \infty)$.

Consider the Möbius transformation TS^{-1}; this sends the ordered triple (Sz_2, Sz_3, Sz_4) to $(1, 0, \infty)$, and so it is the unique Möbius transformation which does this. Hence $TS^{-1}(z) = (z, Sz_2, Sz_3, Sz_4)$ for all $z \in \mathbb{C}_\infty$, and in particular, $TS^{-1}(Sz_1) = (Sz_1, Sz_2, Sz_3, Sz_4)$. Thus

$$(Sz_1, Sz_2, Sz_3, Sz_4) = TS^{-1}Sz_1 = T(z_1) = (z_1, z_2, z_3, z_4).$$

□

Proposition 3. *Four points in $\mathbb{C}_\infty$ lie on the same circle if and only if their cross-ratio is real.*

Proof. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$. We wish to show that these points lie on the same circle if and only if

$$(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty.$$

Let C denote the circle which contains z_2, z_3, and z_4. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $T(z) = (z, z_2, z_3, z_4)$; then $T(C)$ is a circle which contains 1, 0, and ∞, so $T(C) = \mathbb{R}_\infty$, and $T^{-1}(\mathbb{R}_\infty) = C$.

Suppose that $z_1 \in C$. Then $T(z_1) \in \mathbb{R}_\infty$, that is, $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$.

On the other hand, suppose that $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$. Then $T(z_1) \in \mathbb{R}_\infty$, so $z_1 = T^{-1}(T(z_1)) \in C$. □

14. Field of Definition

Definition 11. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. We say that T is *defined over* $\mathbb{R}$ if there exist $a, b, c, d \in \mathbb{R}$ such that $T(z) = \dfrac{az + b}{cz + d}$.

Note that it is possible that T is defined over $\mathbb{R}$, but that T is presented with nonreal coefficients. For example, $T(z) = \dfrac{2iz + 3i}{5iz + 7i}$ is defined over $\mathbb{R}$.

Proposition 4. *Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Then T is defined over $\mathbb{R}$ if and only if $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$.*

Proof. Suppose that T is defined over $\mathbb{R}$. Then $T(z) = \dfrac{az + b}{cz + d}$ for some $a, b, c, d \in \mathbb{R}$. Let $x \in \mathbb{R}_\infty$. If $x = \infty$, then $T(x) = \dfrac{a}{c} \in \mathbb{R}_\infty$. If $T(x) = \infty$, then $T(x) \in \mathbb{R}_\infty$. Otherwise, $T(x) = \dfrac{ax + b}{cx + d}$ is a composition of the sums and products of real numbers, and hence is real.

Conversely, suppose that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Then $T^{-1}(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Let $z_2 = T^{-1}(1)$, $z_3 = T^{-1}(0)$, and $z_4 = T^{-1}(\infty)$, noting that $z_2, z_3, z_4 \in \mathbb{R}_\infty$. Then T is the unique Möbius transformation which maps the ordered triple (z_2, z_3, z_4) onto the ordered triple $(1, 0, \infty)$; this shows that

$$T(z) = \frac{z - z_3}{z - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}.$$

Since z_2, z_3, z_4 are real, we have written T using only real coefficients, and T is defined over $\mathbb{R}$. $\square$

Proposition 5. *Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Then T is defined over $\mathbb{R}$ if and only if*

$$T(\overline{z}) = \overline{T(z)}$$

for all $z \in \mathbb{C}$.

Proof. Suppose T is defined over $\mathbb{R}$. Write $T(z) = \dfrac{az + b}{cz + d}$, where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned}
\overline{T(z)} &= \frac{\overline{a}\overline{z} + \overline{b}}{\overline{c}\overline{z} + \overline{d}} && \text{since conjugation splits on sums and products} \\
&= \frac{a\overline{z} + b}{c\overline{z} + d} && \text{since the coefficients are real} \\
&= T(\overline{z}).
\end{aligned}$$

For the converse, we first note that $z \in \mathbb{R}$ if and only if $z = \overline{z}$. So, suppose that $T(\overline{z}) = \overline{T(z)}$ for all $z \in \mathbb{C}$. Let $x \in \mathbb{R}$. We have

$$T(x) = T(\overline{x}) = \overline{T(x)},$$

so $T(x)$ is real. Thus $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, so T is defined over $\mathbb{R}$. $\square$

15. Symmetry

Definition 12. Let C be a circle through $z_2, z_3, z_4 \in \mathbb{C}_\infty$. The points z and z^* are said to be *symmetric* with respect to C if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

Proposition 6. *The definition of symmetry is independent of the choice of z_2, z_3, z_4. That is, if the points z_2, z_3, z_4 lie on the same circle as w_2, w_3, w_4, then*

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} \quad \textit{if and only if} \quad (z^*, w_2, w_3, w_4) = \overline{(z, w_2, w_3, w_4)}.$$

Proof. Let C be a circle in $\mathbb{C}_\infty$. Let z_2, z_3, z_4 be three distinct points on C, and let w_2, w_3, w_4 be another three distinct points on C. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $T(z) = (z, z_2, z_3, z_4)$, and let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $S(z) = (z, w_2, w_3, w_4)$. We may rewrite our goal as

$$T(z^*) = \overline{T(z)} \Leftrightarrow S(z^*) = \overline{S(z)}.$$

Now T sends (z_2, z_3, z_4) to $(1, 0, \infty)$, and S sends (w_2, w_3, w_4) to $(1, 0, \infty)$, so $T(C) = \mathbb{R}_\infty$ and $S(C) = \mathbb{R}_\infty$. Thus $ST^{-1} : \mathbb{R}_\infty \to \mathbb{R}_\infty$, so ST^{-1} is defined over $\mathbb{R}$.

Let us assume that $T(z^*) = \overline{T(z)}$; we wish to show that $S(z^*) = \overline{S(z)}$. Now

$$S(z^*) = ST^{-1}(T(z^*)) = ST^{-1}(\overline{T(z)}) = \overline{ST^{-1}T(z)} = \overline{S(z)},$$

the pivotal third equal sign is attained by the fact that ST^{-1} is defined over $\mathbb{R}$. □

Proposition 7. *Let S be a Möbius transformation, and let z and z^* be symmetric with respect to a circle C. Then $S(z)$ and $S(z^*)$ are symmetric with respect to the circle $S(C)$.*

Proof. Let $z_2, z_3, z_4 \in \mathbb{C}$ be distinct points on the circle C. Let $T(z) = (z, z_2, z_3, z_4)$. Then $S(z_2)$, $S(z_3)$, and $S(z_4)$ are distinct point on the circle $S(C)$. Thus, by Proposition 2,

$$\begin{aligned}(Sz^*, Sz_2, Sz_3, Sz_4) &= (z^*, z_2, z_3, z_4)\\ &= \overline{(z, z_2, z_3, z_4)}\\ &= \overline{(Sz, Sz_2, Sz_3, Sz_4)}.\end{aligned}$$

This shows that Sz and Sz^* are symmetric with respect to C. □

Department of Mathematics, BASIS Scottsdale
E-mail address: paul.bailey@basised.com