COMPLEX ANALYSIS
TOPIC XIV: THE RIEMANN SPHERE

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1. Topology

Just as algebra is the generalization of arithmetic, topology is the generalization of geometry. Topology uses the notion of open set to define neighborhoods, where a neighborhood of a point consists of other points which are somehow "close to" the given point. The concepts of limits and continuity can then be defined in terms of neighborhoods. Thus, these ideas may be generalized using the definition of topological space. We briefly discuss this.

Definition 1. A *topology* on a set X is a collection of subsets of X, whose members are called *open sets*, satisfying the following three properties:

(T1) The empty set $\varnothing$ and the entire set X are open;
(T2) The union of any number of open sets is open;
(T3) The intersection of finitely many open sets is open.

A *topological space* is a sets X together with a topology on X.

The set of real numbers and the set of complex numbers are topological spaces, with the definitions of open sets we have already given. Also, $\mathbb{R}^n$ is a topological space, where the open sets are unions of open balls. Any subset of a topological space is naturally a topological space, with the subspace topology.

Definition 2. Let X be a topological space and let $A \subset X$. A subset $W \subset A$ is called *relatively open* if there exists a set $U \subset X$ which is open in X such that $W = A \cap U$. The set of relatively open subsets of A forms a topology on A, called the *subspace topology.*

Continuity and convergence may now be defined on any topological space.

Definition 3. Let X and Y be topological spaces, and let $f : X \to Y$. We say that f is *continuous at* $x \in X$ if, for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. We say that f is continuous if it is continuous at every point in the domain.

Definition 4. Let X be a topological space and let (x_n) be a sequence in X. We say that (x_n) *converges* to $L \in X$ if, for every neighborhood V of L there exists $N \in \mathbb{N}$ such that $x_n \in V$ whenever $n \geq N$.

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2. The Riemann Sphere

Definition 5. The *Riemann sphere* is the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, where ∞ is a single point which we append to the complex plane.

A *neighborhood of* $a \in \mathbb{C}$ is a set $U \subset \mathbb{C}_\infty$ such that $a \in U$ and $\{z \in \mathbb{C} \mid |z| < \epsilon\} \subset U$, for some $\epsilon > 0$.

A *neighborhood of infinity* is a set $U \subset \mathbb{C}_\infty$ such that $\infty \in U$ and $\{z \in \mathbb{C} \mid |z| > M\} \subset U$ for some $M > 0$.

A subset $U \subset \mathbb{C}_\infty$ is *open* if every point in U admits a neighborhood which is completely contained in U.

This definition reveals the Riemann sphere to be a topological space. Thus continuous functions and convergent sequences exists for the Riemann sphere.

Definition 6. Let $D \subset \mathbb{C}_\infty$ be open. A function $f : D \to \mathbb{C}_\infty$ is *continuous* at $a \in \mathbb{C}_\infty$ if, for every neighborhood $V \subset \mathbb{C}_\infty$ of $f(a)$, there exists a neighborhood $U \subset D$ of a, such that $f(U) \subset V$.

Definition 7. Let (x_n) be a sequence in $\mathbb{C}_\infty$. We say that (x_n) *converges* to $L \in \mathbb{C}_\infty$ if, for every neighborhood V of L, there exists $N \in \mathbb{N}$ such $x_n \in V$ whenever $n \geq N$.

Some of our arithmetic carries over from $\mathbb{C}$ to $\mathbb{C}_\infty$. Let $*$ be a binary operation defined on $\mathbb{C}$ (suppose that $*$ is addition, multiplication, subtraction, or division). We wish to define $a * b$, where $a, b \in \mathbb{C}_\infty$. These operations are continuous on $\mathbb{C}$, and we wish to maintain that property. So, we define the operation using limits, and consider the operation on $\mathbb{C}_\infty$ to be well defined if it is independent of how the limit is constructed.

Definition 8. Let $a, b, c \in \mathbb{C}_\infty$. We say that $a * b = c$ if, for any $z_0 \in \mathbb{C}_\infty$ and any functions f and g,

$$\lim_{z \to z_0} f(z) = a \text{ and } \lim_{z \to z_0} g(z) = b \quad \Rightarrow \quad \lim_{z \to z_0} f(z) * g(z) = c,$$

where the limits approach z_0 through $z \neq \infty$.

Let a be a nonzero complex number. It is relatively easy to see that the following computation follow from the definition above.

- $-\infty = \infty$
- $\dfrac{a}{\infty} = 0$ and $\dfrac{a}{0} = \infty$
- $a + \infty = \infty$
- $a \cdot \infty = \infty$
- $\infty - \infty$, $\dfrac{\infty}{\infty}$, $0 \cdot \infty$, and $\dfrac{0}{0}$ are undefined

3. Stereographic Projection

In order to better visualize the Riemann sphere, we use the technique of *stereographic projection*, which maps the unit sphere in $\mathbb{R}^3$, minus the north pole, which is the point $(0, 0, 1)$, bijectively onto the complex plane. Then we use this to identify the unit sphere with the Riemann sphere, where the north pole becomes infinity.

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. Identify $\mathbb{C} = \{u + iv \mid u, v \in \mathbb{R}\}$ with the xy-plane in $\mathbb{R}^3$ via $u = x$ and $v = y$. Let $P = (0, 0, 1) \in \mathbb{R}^3$. We develop the mapping from $S^2 \setminus \{P\} \to \mathbb{C}$ via the process known as *stereographic projection.* This mapping identifies a point $Q \in \mathbb{C}$ with a point $R \in S^2$ by setting R to be the unique point in S^2 which is the intersection of the line through P and Q with S^2.

Given a point $Q = (u, v, 0)$, the line from P to Q can be parameterized by

$$\vec{r}(t) = P + (Q - P)t = \langle ut, vt, 1 - t\rangle.$$

The intersection of this line with S^2 is obtained by plugging $\vec{r}$ into the defining equation for the sphere:

$$(ut)^2 + (vt)^2 + (1 - t)^2 = 1 \Rightarrow u^2t^2 + v^2t^2 + t^2 = 2t.$$

Now $t = 0$ corresponds to P, since $\vec{r}(0) = P$; let R denote the other point of intersection of the line and the sphere. Then $R = \vec{r}(t)$ where

$$u^2t + v^2t + t = 2 \Rightarrow t = \frac{2}{u^2 + v^2 + 1}.$$

Plugging this t into $\vec{r}$ gives

$$R = \vec{r}(t) = \Big\langle \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \Big\rangle.$$

So, we may map a point $u + iv \in \mathbb{C}$ to the sphere S^2 via

- $x = \dfrac{2u}{u^2 + v^2 + 1}$
- $y = \dfrac{2v}{u^2 + v^2 + 1}$
- $z = \dfrac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$

Next we wish to find the inverse of this mapping; given a point $R = (x, y, z)$ on the Riemann sphere, we map it to a point $u + iv$ on the complex plane. A parameterization of the line from P to R is

$$\vec{r}(t) = P + (R - P)t = \langle xt, yt, (z - 1)t + 1\rangle.$$

This intersects the xy-plane if $(z - 1)t + 1 = 0$, in which case $t = \dfrac{1}{1 - z}$. Plugging this in to $\vec{r}$, and identifying (x, y) with (u, v), gives

$$\vec{r}\Big(\frac{1}{1 - z}\Big) = \Big\langle \frac{x}{1 - z}, \frac{y}{1 - z}, 0 \Big\rangle.$$

Thus

- $u = \dfrac{x}{1 - z}$
- $v = \dfrac{y}{1 - z}$

4. Circles

The next attribute of stereographic projection which we wish to explore is the fact that circles map to circles. This is under the understanding that a line is considered to be a "circle through infinity".

Consider an arbitrary circle on S^2. There is a unique plane $ax + by + cz = d$ such that the circle is the intersection of the S^2 and the plane. The images of a point in this intersection satisfy the equation obtained by plugging the stereographic projection formulas into the plane, thusly:

$$\frac{2au}{u^2 + v^2 + 1} + \frac{2bv}{u^2 + v^2 + 1} + \frac{c(u^2 + v^2 - 1)}{u^2 - v^2 + 1} = d.$$

Algebraic manipulation of this equation yields

$$(c - d)(u^2 + v^2) + 2au + 2bv - (c + d) = 0.$$

This is the equation of a line if $c = d$, otherwise it is the equation of a circle. Note that $c = d$ if and only if the point $(0, 0, 1)$ is on the plane; that is, if the circle passes through ∞ on the Riemann sphere.

5. Identification

We now identify the abstract Riemann sphere, with the topology previous stated, with the unit sphere in $\mathbb{R}^3$, using stereographic projection, where the north pole represents ∞. Under this identification, the following holds:

- Infinity is the north pole, and zero is the south pole.
- The open unit disk is the southern hemisphere.
- The unit circle in $\mathbb{C}$ is the equator of the unit sphere. These are the fixed points under stereographic projection, when $\mathbb{C}$ is identified with the xy-plane.
- The real line is the intersection of the xz-plane with the unit sphere.
- The imaginary axis is the intersection of the yz-plane with the unit sphere.

6. Meromorphic Functions

We wish to study continuous functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$. However, we also wish for such functions to be differentiable. This also requires definition: the standard definition already admits infinite derivatives; we extend the definition to say that $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is differentiable at ∞ if $f(1/z)$ is differentiable at 0, in a manner made precise using limits. Differentiable functions are continuous.

Definition 9. Let $D \subset \mathbb{C}_\infty$. A *meromorphic function* on D is a function $f : D \to \mathbb{C}_\infty$ which is analytic, that is, is differentiable at every point in D. The set of all meromorphic functions on D form a field, called the *function field* of D.

It can be shown that the only meromorphic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are rational functions; that is, the function field of $\mathbb{C}_\infty$ is the set of all rational functions. It is this topic we wish to explore further.

7. Polynomial Functions

A *polynomial function* is a function of the form

$$f(z) = \sum_{i=0}^{n} a_i x^i,$$

where n is a nonnegative integer, and $a_i \in \mathbb{C}$. We call n the *degree* of f, and write $\deg(f) = n$. The a_i's are called *coefficients.* We call a_0 the *constant coefficient* and a_n the *leading coefficient.*

A *zero* (or *root*) of f is a number r such that $f(r) = 0$. The Factor Theorem says that $f(r) = 0$ if and only if $(z - r)$ is a factor of $f(z)$. The *multiplicity* of r as a zero of f is the largest integer m such that $(z - r)^m$ is a factor of $f(z)$.

By the Fundamental Theorem of Algebra, f has exactly n zeros, counted with multiplicity. Indeed, there exist (not necessarily distinct) $r_1, r_2, \ldots, r_n \in \mathbb{C}$ such that

$$f(z) = a_n \prod_{i=0}^{n} (z - r_i).$$

We may view polynomial function as mapping from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. In this case, $f(\infty) = \infty$. Indeed, ∞ is may be viewed as a "pole of multiplicity n", since ∞ is the only point which maps to ∞, but there is a deleted neighborhood of infinity which is mapped n-to-1 onto a neighborhood of infinity.

The set of all polynomials with complex coefficients is denoted $\mathbb{C}[x]$.

8. Rational functions

A *rational function* is a function of the form $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are polynomials with complex coefficients, and h is not constantly zero. If $\deg(f) = m$ and $\deg(g) = n$, then the Fundamental Theorem of Algebra says that

$$f(z) = c \frac{\prod_{i=0}^{m} (z - r_i)}{\prod_{j=0}^{n} (z - s_j)},$$

for some complex numbers $c, r_i, s_j \in \mathbb{C}$. The r_i's are the *zeros* of f, and the s_j's are the *poles* of f, where the set of zeros is disjoint from the set of poles. The constant c is the ratio of the leading coefficients of g and h.

We view the rational function f as $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by extension using limits

- $f(z) = \infty$ if and only if z is a pole of f;
- $f(\infty) = \lim_{z \to \infty} f(z)$.

Thus

$$f(\infty) = \begin{cases} 0 & \text{if } \deg(g) < \deg(h); \\ c & \text{if } \deg(g) = \deg(h); \\ \infty & \text{if } \deg(g) > \deg(h), \end{cases}$$

where c is the ratio of the leading coefficients of g and h.

The set of all rational functions over $\mathbb{C}$ is denoted $\mathbb{C}(x)$.

9. Fibers

If $f : A \to B$ and $b \in B$, the *fiber of f over b* is the preimage of b:

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}.$$

We would like to investigate the fiber of a rational function.

Let $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are polynomials. Let $b \in \mathbb{C}$. The fiber over b is the set of solutions to the equation $f(z) = b$, that is, $\dfrac{g(z)}{h(z)} = b$. Equivalently, this equation becomes $g(z) = bh(z)$, or, $g(z) - bh(z) = 0$. Assuming that the leading coefficient of $g(z)$ does not equal the leading coefficient of $bh(z)$, the function $g(z) - bh(z)$ is a polynomial of degree $n = \max\{\deg(g(z)), \deg(h(z))\}$, and therefore has exactly n zeros, counted with multiplicity. Thus $g(z) - bh(z) = 0$ has at most n solutions, and has exactly n solutions except in the rare cases that it has multiple roots. So, the function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is n-to-1 almost everywhere.

The *degree* of $f(z) = \dfrac{g(z)}{h(z)}$ is defined to be the maximum of the degrees of g and h. A rational function of degree n is n-to-1 over almost every point in the range.

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