AP Calculus AB — Responses 0406

Dr. Paul L. Bailey — Sunday, April 5, 2020

I will write up some solutions to the problems on the practice test that caused some trouble.

Problem 1 (# 7)**.** If $\int_4^{-10} g(x)\,dx = -3$ and $\int_4^6 g(x)\,dx = 5$, then $\int_{-10}^6 g(x)\,dx = ?$

Solution. We know:

- $\int_a^b g(x)\,dx = -\int_b^a g(x)\,dx$
- $\int_a^b g(x)\,dx = \int_a^c g(x)\,dx + \int_c^b g(x)\,dx$

If you reverse the order of integration, you negative the value of the integral. So, we know that $\int_{-10}^4 g(x)\,dx = 3$. Thus

$$\int_{-10}^6 g(x)\,dx = \int_{-10}^4 g(x)\,dx + \int_4^b g(x)\,dx = 3 + 5 = \boxed{8}.$$

□

Problem 2 (# 14)**.** The weight of a population of yeast is given by a differentiable function y, where $y(t)$ is measured in grams and t is measured in days. The weight of the yeast population increases according to the equation $\frac{dy}{dt} = ky$, where k is a constant. At time $t = 0$, the weight of the yeast population is 120 grams and is increasing at the rate of 24 grams per day. Find $y(t)$.

Solution. This is a separable differential equation we have seen before. Separate the variable to get

$$\frac{dy}{y} = k\,dt.$$

Slap integral signs in front to get

$$\int \frac{dy}{y} = \int k\,dt.$$

Take antiderivatives and arrive at

$$\ln y = kt + C.$$

Solve for y and you see that

$$y = e^{kt+C} = e^C e^{kt}.$$

If $t = 0$, we have $y = e^C$, so e^C is the initial amount, so

$$y = A_0 e^{kt},$$

where A_0 is the initial amount. In our case, $A_0 = 120$, so

$$y = 120e^{kt}.$$

To solve for k, we use the extra piece of information which is given; the initial rate of growth is 24. That is $y'(0) = 24$. By the chain rule,

$$y'(t) = 120ke^{kt}, \text{ so } y'(0) = 24 = 120ke^0, \text{ so } k = \frac{24}{120} = \frac{1}{5}.$$

Put it together to get

$$\boxed{y(t) = 120e^{t/5}}.$$

□

Problem 3 (# 16)**.** Let f be a function defined by $f(x) = -3+6x^2-2x^3$. What is the largest open interval on which the graph of f is both concave up and increasing?

Solution. We know that f is increasing when $f'(x) > 0$, and f is concave up when $f''(x) > 0$. So, we draw a sign chart for f' and f''. First, look over these guidelines from the College Board regarding sign charts on the AP examination.

`https://apcentral.collegeboard.org/courses/resources/sign-charts-ap-calculus-exams`

Okay, we have

$$f(x) = -2x^3 + 6x^2 - 3, \text{ so } f'(x) = -6x^2 + 12x, \text{ and } f''(x) = -12x + 12.$$

Now $f'(x) = -6x(x-2)$ and $f''(x) = -12(x-1)$. The sign charts are

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f'      - - - - - - - 0 + + + 2 - - - - - - -
f''     + + + + + + + + + + 1 - - - - - - - -
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(Sorry, I couldn't think of a better way to typeset this.)

We see that f' and f'' are both positive on the interval $(0,1)$ and nowhere else. Thus, the answer is

$$\boxed{(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}.}$$

□

Problem 4 (# 20)**.** let $f(x) = \dfrac{x-2}{2|x-2|}$. Which is true?

(A) $\lim_{x\to 2} f(x) = \frac{1}{2}$

(B) f has a removable discontinuity at $x = 2$

(C) f has a jump discontinuity at $x = 2$.

(D) f has a discontinuity due to a vertical asymptote at $x = 2$.

Solution. Let's rewrite f:

$$f(x) = \begin{cases} \dfrac{1}{2} & \text{if } x > 2 \\ -\dfrac{1}{2} & \text{if } x < 2 \end{cases}$$

So f jumps from $-\dfrac{1}{2}$ to $\dfrac{1}{2}$ at $x = 2$. That's a jump discontinuity.

A removable discontinuity is a hole in the graph. That's not what is happening here. □

Problem 5 (# 21)**.** Let $f(x) = \ln x$. Compute $\displaystyle\lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$.

Solution. We recognize that

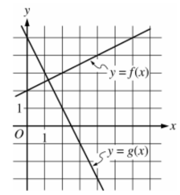
$$f'(3) = \frac{f(x)-f(3)}{x-3}.$$

For $f(x) = \ln x$, we have $f'(x) = \dfrac{1}{x}$, so

$$\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = f'(3) = \boxed{\frac{1}{3}}.$$

□

Problem 6 (# 25)**.** The figure below shows that graphs of the functions f and g.



Let $h(x) = f(x)g(x)$. Find $h'(2)$.

Solution. By examining the graphs, computing slopes, and observing y-intercepts, we find that

$$f(x) = \frac{1}{2}x + 2 \qquad \text{and} \qquad g(x) = -2x + 5.$$

We see that $f'(x) = \dfrac{1}{2}$ and $g'(x) = 2$.

There is no need to multiply the functions; let's just use the product rule and plug in. We have

$$h'(2) = f'(2)g(2) + f(2)g'(2) = \frac{1}{2}(1) + 3(-2) = \frac{1}{2} - 6 = \boxed{-\frac{11}{2}}.$$

□

Problem 7 (#26)**.** Compute $\lim_{x\to\infty} \dfrac{\ln(e^{3x} + x)}{x}$.

Solution. This is of the form $\dfrac{\infty}{\infty}$, so we use L'Hospital's rule. Now $\dfrac{d}{dx}\ln(e^{3x}+x) = \dfrac{1}{e^{3x}+x}(3e^{3x}+1)$, which for large x is approximately $\dfrac{3e^{3x}}{e^{3x}} = 3$.

Thus

$$\lim_{x\to\infty} \frac{\ln(e^{3x} + x)}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(e^{3x} + x)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{3e^{3x} + 1}{e^{3x} + x} = 3.$$

□

Problem 8 (# 28)**.** An isosceles right triangle with legs of length s has area $A = \dfrac{1}{2}s^2$. At the instant when $s = \sqrt{32}$ centimeters, the area of the triangle is increasing at a rate of 12 square centimeters per second. At what rate is the length of the hypotenuse of the triangle increasing, in centimeters per second, at that instant?

Solution. Let h be the length of the hypotenuse.

If s is the length of a side of an isosceles right triangle, then its hypotenuse is $h = \sqrt{2}s$. We seek $\dfrac{dh}{dt} = \sqrt{2}\dfrac{ds}{dt}$, when $s = \sqrt{32} = 4\sqrt{2}$.

If your a golf announcer, you whisper. That's what you do.

If you are a dumb victim in a B-grade horror movie, you make stupid decisions. That's what you do.

If you are a Calculus student doing a related rates problem, you find a good formula and differentiate it with respect to time. That's what you do.

In this case,

$$A = \frac{1}{2}s^2 \qquad \text{so} \qquad \frac{dA}{dt} = \frac{1}{2}(2s)\frac{ds}{dt}.$$

Since $\dfrac{dA}{dt} = 12$, $s = \sqrt{32}$, and $\dfrac{ds}{dt} = \dfrac{1}{\sqrt{2}}\dfrac{dh}{dt}$, we have

$$12 = (4\sqrt{2})\frac{1}{\sqrt{2}}\frac{dh}{dt}, \text{ so } \frac{dh}{dt} = \frac{12}{4} = \boxed{3}.$$

□

Problem 9 (# 39)**.** The number of bacteria in a container increases at the rate of $R(t)$ bacteria per hour. If there are 1000 bacteria at time $t = 0$, write an expression for the number of bacteria in the container at time $t = 3$ hours.

Solution. The integral of rate of change is total amount of change.

Here, the integral of rate of change from $t = 0$ to $t = 3$ is the integral of the rate from $t = 0$ to $t = 3$. So, the total amount at time $t = 3$ is the amount at time $t = 0$, plus the amount of change between $t = 0$ to $t = 3$. This is

$$\text{Amount at time } 3 = \text{Amount at time } 0 + \text{Amount changed from time } 0 \text{ to time } 3 = 1000 + \int_0^3 R(t)\,dt.$$

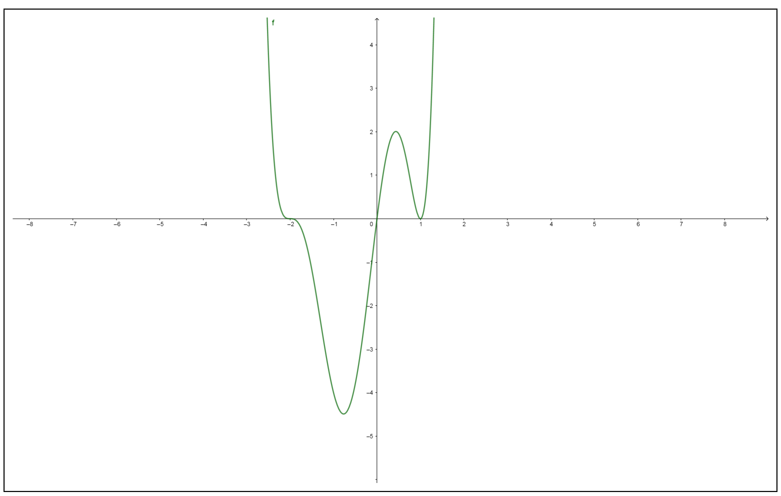
□

Problem 10 (# 41)**.** Consider the twice-differentiable functions f, g, and h, we second derivatives

- $f''(x) = x(x-1)^2(x+2)^3$
- $g''(x) = x(x-1)^2(x+2)^3 + 1$
- $h''(x) = x(x-1)^2(x+2)^3 - 1$

Which of the function f, g, and h have a graph with exactly two points of inflection?

Solution. The first step is to graph $f''(x)$. We know it crosses touches the x-axis at 1, 0, and -2; it crosses at 0 and -2, but is positive and just bounces off the x-axis at $x = 1$. This is the graph:



There is an inflection point if the concavity changes; that is, there is an inflection point if the second derivative changes sign. We see that this occurs exactly twice for f, but what about g and h?

If we shift f up by one to get g, there are exactly two x-intercepts, both are crossings. So, g has exactly two inflection points.

However, if we shift f down by 1 to get h, the "bouncing off" at $x = 1$ becomes two crossings, and h have 4 points of inflection.

So, the answer is f and g.

□