COMPLEX ANALYSIS
TOPIC XVIII: COMPACT RIEMANN SURFACES
DRAFT (MORE TO COME!)

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Abstract. Our goal for the last month of the course is to introduce the concept of compact Riemann surfaces, and glimpse their role in the theory of meromorphic functions. This is an extremely advanced topic for high school students, so we focus on understanding the definitions. We begin by reviewing what we have already seen about topological spaces, then define connectedness and compactness sufficiently for our purposes. We then move on to locally Euclidean spaces, manifolds, Riemann surfaces, and genera. Finally, we discover the Riemann-Hurwitz formula for discovering the genus of a ramified cover.

1. Topology

Definition 1. A *topological space* is a set X together with a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ such that

(T1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
(T2) $\mathcal{U} \subset \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$;
(T3) $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ finite $\Rightarrow \cap\mathcal{U} \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.
A subset $A \subset X$ is called *open* if $A \in \mathcal{T}$, and is called *closed* if $X \smallsetminus A \in \mathcal{T}$.

The set of real numbers and the set of complex numbers are topological spaces, with the definitions of open sets we have already given. Also, $\mathbb{R}^n$ is a topological space, where the open sets are unions of open balls. We outline this next.

Definition 2. Let $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ be points in $\mathbb{R}^n$. The *distance* from p to q is

$$d(p, q) = \sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2}.$$

Let $r \in \mathbb{R}$, $r > 0$. The *ball of radius r about p* is

$$B_r(p) = \{q \in \mathbb{R}^n \mid |p - q| < r\}.$$

Let $U \subset \mathbb{R}^n$. We say that U is *open* if for every $u \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(u) \subset U$. The collection of open subsets of $\mathbb{R}^n$ is a topology on $\mathbb{R}^n$, making $\mathbb{R}^n$ a topological space.

For the purposes of topology, we view $\mathbb{C}$ as $\mathbb{R}^2$, with the extra structure of complex multiplication.

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2. Subspaces

Any subset of a topological space is naturally a topological space, with the subspace topology.

Definition 3. Let X be a topological space and let $A \subset X$. A subset $W \subset A$ is called *relatively open* if there exists a set $U \subset X$ which is open in X such that $W = A \cap U$. The set of relatively open subsets of A forms a topology on A, called the *subspace topology*.

Example 1. Let $I = [0, 1]$. This is a subspace of $\mathbb{R}$. Let $U = (-0.5, 0.5)$; this is an open set in $\mathbb{R}$. Thus the set $W = U \cap I = [0, 0.5)$ is relatively open in I. If we view I as a topological space, then W is an open set in I.

Next, we define some standard topological spaces; each is endowed with the subspace topology inherited from the appropriate version of $\mathbb{R}^n$. We may think of these as building blocks to form new topological spaces.

The *open n-ball* is

$$B^n = \{q \in \mathbb{R}^n \mid d(0, q) < 1\}.$$

The *closed n-ball* is

$$D^n = \{q \in \mathbb{R}^n \mid d(0, q) \leq 1\}.$$

The *n-sphere* is

$$S^n = \{q \in \mathbb{R}^{n+1} \mid d(0, q) = 1\}.$$

So, $B^1 = (-1, 1)$ is an open interval and $D^1 = [-1, 1]$ is a closed interval. Also, S^1 is a circle, but D^2 is a closed disk.

3. Classification of Points

The definitions of neighborhood, deleted neighborhood, closure point, interior point, boundary point, accumulation point, isolated point, all carry over from our previous discussions virtually unchanged into this more general context, as do the concepts of the closure, interior, and boundary of a set. We review this now.

Definition 4. Let X be a topological space, and let $p \in X$. A *neighborhood* of p is a set which contains an open set which contains p. A *deleted neighborhood* of p is a set of the form $N \smallsetminus \{p\}$, where N is a neighborhood of p.

Let $A \subset X$. We say that p is a *closure point* of A if every neighborhood of p intersects A. We say that p is a *interior point* of A if there exists a neighborhood of p which is contained in A. We say that p is a *boundary point* of A if every neighborhood of p intersects A and $X \smallsetminus A$. We say that p is an *accumulation point* of A if every deleted neighborhood of p intersects A. We say that p is an *isolated point* of A if there exists a neighborhood U of p such that $A \cap U = \{p\}$.

The *closure* of A, denoted $\overline{A}$, is the set of closure points of A. The *interior* of A, denoted A°, is the set of interior points of A. The *boundary* of A, denoted ∂A, is the set of boundary points of A.

We say that A is *discrete* if every point in A is isolated.

Let $B \subset A$. We say that B is *dense* in A if $\overline{B} = A$.

4. Continuity

Continuity and convergence may now be defined on any topological space.

Definition 5. Let X and Y be topological spaces, and let $f : X \to Y$. We say that f is *continuous at* $x \in X$ if, for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. We say that f is continuous if it is continuous at every point in the domain.

Proposition 1. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if the preimage of every open set in Y is open in X.*

Proof. We prove both directions of the implication.

($\Rightarrow$) Suppose that f is continuous at every point in X. Let $V \subset Y$ be open and let $U = f^{-1}(V)$; we wish to show that U is open in X.

For every $x \in U$, V is a neighborhood of $f(x)$, so there exists an open neighborhood U_x of x such that $f(U_x) \subset V$. But then $U_x \subset U$, and U is the union of such sets; thus U is open, and f is continuous. Suppose that f is continuous, and let $x_0 \in X$. Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V.

($\Leftarrow$) Conversely, suppose that the preimage of every open set in Y is open in X, and let $x_0 \in X$. We wish to show that f is continuous at x_0.

Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V. $\square$

Definition 6. Let X and Y be topological spaces and let $f : X \to Y$. We say that f is *open* if the image of every open set in X is open in Y. We say that f is *bicontinuous* if f is open and continuous.

Definition 7. Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective continuous function $f : X \to Y$ whose inverse is also continuous. We say that X and Y are *homeomorphic* if there exists a homeomorphism between them.

A homeomorphism between topological spaces preserves all of the features of the domain which can be described exclusively using open sets; we may call such features "topological". Because of this, we view two topological spaces as equivalent, or essentially the same, if they are homeomorphic. However, a space may have additional structure beyond its topology, which is not preserved by homeomorphism.

Definition 8. Let X be a topological space and let (x_n) be a sequence in X. We say that (x_n) *converges* to $L \in X$ if, for every neighborhood V of L there exists $N \in \mathbb{N}$ such that $x_n \in V$ whenever $n \geq N$.

Example 2. Let $I = (0, \infty)$. The function $\exp : \mathbb{R} \to I$ given by $\exp(x) = e^x$ is a homeomorphism, so it preserves all topological properties. For example, if x_0 is an boundary point of $A \subset \mathbb{R}$, then $f(x_0)$ is a boundary point of $f(A)$. If a sequence (x_n) converges to $L \in \mathbb{R}$, then the sequence $f(x_n)$ converges to $f(L) \in I$.

5. Connectedness and Compactness

The notions of connectedness and compactness are critical for topology, but the most general definitions can be a bit overwhelming. We give the formal definitions, and then give conditions which are a little more familiar and are equivalent in the cases we are interested in.

5.1. **Connectedness.** A space is connected if it has only one "piece". We state this formally as follows.

Definition 9. Let X be a nonempty topological space.

A *separation* of X is a pair of nonempty open sets $U, V \subset X$ such that $U \cap V = \varnothing$ and $U \cup V = X$.

We say that X is *connected* if there does not exist a separation of X.

A *component* of X is a maximal connected subset; that is, it is a connected subset which is not properly contained in a connected subset.

If we speak of a subset of X being connected, we mean that it is connected as topological space with the subspace topology. It is clear that a set is connected if and only if it has exactly one component.

We give some examples.

- A nonempty subset of $\mathbb{R}$ is connected if and only if it is a singleton or an interval.
- S^n is connected unless $n = 0$ (in which case S^0 is a set containing two points.
- No finite subset of $\mathbb{R}^n$ is connected.
- Consider the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. Then $f^{-1}(B_1(0))$ is connected, but $f^{-1}(B_1(2)$ has two components.

Proposition 2. *Let $f : X \to Y$ be continuous, and let $A \subset X$. If A is connected, then $f(A)$ is connected.*

Proof. We use a proof by contrapositive; assume that $f(A)$ is not connected. Then there exist disjoint open sets V_1 and V_2 with $f(A) \subset V_1 \cup V_2$, with $f(A) \cap V_1 \neq \varnothing$ and $f(a) \cap V_2 \neq \varnothing$. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$; since f is continuous, U_1 and U_2 are open. Since V_1 and V_2 are disjoint, so are U_1 and U_2. Moreover, $U_1 \cap A$ and $U_2 \cap A$ are nonempty. Finally, $A \subset U_1 \cap U_2$. □

Definition 10. Let X be a topological space. We say that X is *path-connected* if for every $x_1, x_2 \in X$, there exists a continuous function $\gamma : [1, 2] \to X$ such that $\gamma(1) = x_1$ and $\gamma(2) = x_2$.

Proposition 3. *If X is path-connected, then X is connected.*

5.2. **Compactness.** A space is compact if it is not *too* big, and if it doesn't have any "holes". This may be stated in multiple ways, which are equivalent for "well-behaved" spaces.

Definition 11. Let X be a topological space.

A *cover* of X is a collection of subsets of X whose union is X.

An *open cover* of X is a cover consisting of open sets.

A *finite cover* of X is a cover consisting of finitely many sets.

A *subcover* of a cover is a subset of the cover whose union is X.

We say that X is *compact* if every open cover has a finite subcover.

Note that in the phrase "every open cover has a finite subcover", the word finite is describing the collection which is the subcover, but the word open is describing the sets in the cover.

We give some examples.

- Open balls are not compact.
- Closed balls are compact.
- A punctured disk is not compact.
- The entire real line is not compact.

Proposition 4. *Let $f : X \to Y$ be continuous, and let $A \subset X$. If A is compact, then $f(A)$ is compact.*

Proof. Consider an open cover of $f(A)$. The collection of preimages of the sets in the cover form an open cover of A. Since A is compact, a finite subset of these cover A. The collection of images of these sets form a finite subcover of the original cover of $f(A)$. □

Theorem 1. (Heine-Borel Theorem) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

We have the following alternate variations of the definition of compactness, which are equivalent to the standard definition in most cases in which we are interested.

Definition 12. Let X be a topological space.

We say that X is *sequentially compact* if every sequence in X has a cluster point in X.

We say that X is *limit point compact* if every infinite subset of X has an accumulation point in X.

Theorem 2. (Bolzano-Weierstrauss Theorem) *A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.*

Corollary 1. *A subset of $\mathbb{R}^n$ is compact if and only if it is sequentially compact.*

6. Topological Manifolds

Definition 13. Let X be a topological space. We say that X is *locally Euclidean* if, for every point $x \in X$, there exists a neighborhood of x which is homeomorphic to B^n for some n.

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