COMPLEX ANALYSIS
TOPIC XVIII: COMPACT RIEMANN SURFACES
DRAFT (MORE TO COME!)

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Abstract. Our goal for the last month of the course is to introduce the concept of compact Riemann surfaces, and glimpse their role in the theory of meromorphic functions. This is an extremely advanced topic for high school students, so we focus on understanding the definitions. We begin by reviewing what we have already seen about topological spaces, then define connectedness and compactness sufficiently for our purposes. We then move on to locally Euclidean spaces, manifolds, Riemann surfaces, and genera. Finally, we discover the Riemann-Hurwitz formula for discovering the genus of a ramified cover.

1. Topology

Definition 1. A *topological space* is a set X together with a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ such that

(T1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
(T2) $\mathcal{U} \subset \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$;
(T3) $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ finite $\Rightarrow \cap\mathcal{U} \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.
A subset $A \subset X$ is called *open* if $A \in \mathcal{T}$, and is called *closed* if $X \smallsetminus A \in \mathcal{T}$.

The set of real numbers and the set of complex numbers are topological spaces, with the definitions of open sets we have already given. Also, $\mathbb{R}^n$ is a topological space, where the open sets are unions of open balls. We outline this next.

Definition 2. Let $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ be points in $\mathbb{R}^n$. The *distance* from p to q is

$$d(p,q) = \sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2}.$$

Let $r \in \mathbb{R}$, $r > 0$. The *ball of radius r about p* is

$$B_r(p) = \{q \in \mathbb{R}^n \mid |p - q| < r\}.$$

Let $U \subset \mathbb{R}^n$. We say that U is *open* if for every $u \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(u) \subset U$. The collection of open subsets of $\mathbb{R}^n$ is a topology on $\mathbb{R}^n$, making $\mathbb{R}^n$ a topological space.

For the purposes of topology, we view $\mathbb{C}$ as $\mathbb{R}^2$, with the extra structure of complex multiplication.

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2. Subspaces

Any subset of a topological space is naturally a topological space, with the subspace topology.

Definition 3. Let X be a topological space and let $A \subset X$. A subset $W \subset A$ is called *relatively open* if there exists a set $U \subset X$ which is open in X such that $W = A \cap U$. The set of relatively open subsets of A forms a topology on A, called the *subspace topology.*

Example 1. Let $I = [0, 1]$. This is a subspace of $\mathbb{R}$. Let $U = (-0.5, 0.5)$; this is an open set in $\mathbb{R}$. Thus the set $W = U \cap I = [0, 0.5)$ is relatively open in I. If we view I as a topological space, then W is an open set in I.

Next, we define some standard topological spaces; each is endowed with the subspace topology inherited from the appropriate version of $\mathbb{R}^n$. We may think of these as building blocks to form new topological spaces.

The *open n-ball* is
$$B^n = \{q \in \mathbb{R}^n \mid d(0, q) < 1\}.$$
The *closed n-ball* is
$$D^n = \{q \in \mathbb{R}^n \mid d(0, q) \leq 1\}.$$
The *n-sphere* is
$$S^n = \{q \in \mathbb{R}^{n+1} \mid d(0, q) = 1\}.$$
So, $B^1 = (-1, 1)$ is an open interval and $D^1 = [-1, 1]$ is a closed interval. Also, S^1 is a circle, but D^2 is a closed disk.

3. Classification of Points

The definitions of neighborhood, deleted neighborhood, closure point, interior point, boundary point, accumulation point, isolated point, all carry over from our previous discussions virtually unchanged into this more general context, as do the concepts of the closure, interior, and boundary of a set. We review this now.

Definition 4. Let X be a topological space, and let $p \in X$. A *neighborhood* of p is a set which contains an open set which contains p. A *deleted neighborhood* of p is a set of the form $N \setminus \{p\}$, where N is a neighborhood of p.

Let $A \subset X$. We say that p is a *closure point* of A if every neighborhood of p intersects A. We say that p is a *interior point* of A if there exists a neighborhood of p which is contained in A. We say that p is a *boundary point* of A if every neighborhood of p intersects A and $X \setminus A$. We say that p is an *accumulation point* of A if every deleted neighborhood of p intersects A. We say that p is an *isolated point* of A if there exists a neighborhood U of p such that $A \cap U = \{p\}$.

The *closure* of A, denoted $\overline{A}$, is the set of closure points of A. The *interior* of A, denoted A°, is the set of interior points of A. The *boundary* of A, denoted ∂A, is the set of boundary points of A.

We say that A is *discrete* if every point in A is isolated.

Let $B \subset A$. We say that B is *dense* in A if $\overline{B} = A$.

4. Bases and Subbases

Let X be any set; this set admits many different collections of subsets which satisfy the axioms of a topology If we have a collection of different topologies on X, we attempt to make a new topology on X by declaring that a given subset is open if and only if if is a member of each of the topologies in the collection. It is relatively easy to show that what we obtain in this manner is again a topology on X. We state this as a proposition.

Proposition 1. *Let X be a set. The intersection of topologies on X is a topology on X.*

We use this to produce the easiest definition of generated topologies; we wish to define the topology generated by a collection of subsets of X to be the coarsest (that is, with the fewest open sets) topology on X such that each of the sets in our collection is open.

Definition 5. Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. The *topology generated by* $\mathcal{C}$ is the intersection of all topologies on X which contain $\mathcal{C}$.

The topology generated by $\mathcal{C}$ may to constructed in stages as follows. First, take the collection of all finite intersections of all the sets in $\mathcal{C}$. Then, take the collection of all possible unions of sets obtained in above. There are definitions for these things.

Definition 6. Let X be a set and let $\mathcal{T}$ be a topology on X.

A *subbasis* for $\mathcal{T}$ is a collection of subsets of X which generated the topology $\mathcal{T}$.

A *basis* for $\mathcal{T}$ is a collection of subsets of X such that the collection of all possible unions of these subsets is $\mathcal{T}$.

For example, the collection of open interval of finite length is a basis for the standard topology on $\mathbb{R}$; also, the collection of all open disks in the complex plane is a basis for the standard topology on $\mathbb{C}$.

5. Discrete and Trivial Topologies

Definition 7. Let X be a set.

The *power set* of X, denoted $\mathcal{P}(X)$, is the collection of all subsets of X.

The *discrete topology* on X is the topology in which every subset of X is open; that is, $\mathcal{T} = \mathcal{P}(X)$. Thus the discrete topology is the topology generated by the collection of singleton sets.

The *trivial topology* on X is the topology in which the only open sets are the empty set and the whole space; that is, $\mathcal{T} = \{\varnothing, X\}$. Thus the trivial topology is the topology generated by the empty collection.

6. Continuity

Continuity and convergence may now be defined on any topological space.

Definition 8. Let X and Y be topological spaces, and let $f : X \to Y$. We say that f is *continuous at* $x \in X$ if, for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. We say that f is continuous if it is continuous at every point in the domain.

Proposition 2. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if the preimage of every open set in Y is open in X.*

Proof. We prove both directions of the implication.

($\Rightarrow$) Suppose that f is continuous at every point in X. Let $V \subset Y$ be open and let $U = f^{-1}(V)$; we wish to show that U is open in X.

For every $x \in U$, V is a neighborhood of $f(x)$, so there exists an open neighborhood U_x of x such that $f(U_x) \subset V$. But then $U_x \subset U$, and U is the union of such sets; thus U is open, and f is continuous. Suppose that f is continuous, and let $x_0 \in X$. Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V.

($\Leftarrow$) Conversely, suppose that the preimage of every open set in Y is open in X, and let $x_0 \in X$. We wish to show that f is continuous at x_0.

Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V. $\square$

Definition 9. Let X and Y be topological spaces and let $f : X \to Y$. We say that f is *open* if the image of every open set in X is open in Y. We say that f is *bicontinuous* if f is open and continuous.

Example 2. Continuous functions are characterized by having at least enough of open sets in the domain, and open maps are characterized by having a at least enough open sets in the range.

Let X be any set. Let $X_{\mathtt{T}}$ and $X_{\mathtt{D}}$ denote the topological space X together with the trivial or discrete topology, respectively. Let $f(x) = x$ be the identity map on X.

Then $f : X_{\mathtt{T}} \to X_{\mathtt{D}}$ is open but not continuous, because the range has more open sets than the domain. On the other hand, $f : X_{\mathtt{D}} \to X_{\mathtt{T}}$ is continuous, but not open, because the domain has more open sets than the range.

Definition 10. Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective continuous function $f : X \to Y$ whose inverse is also continuous. We say that X and Y are *homeomorphic* if there exists a homeomorphism between them.

A homeomorphism between topological spaces preserves all of the features of the domain which can be described exclusively using open sets; we may call such features "topological". Because of this, we view two topological spaces as equivalent, or essentially the same, if they are homeomorphic. However, a space may have additional structure beyond its topology, which is not preserved by homeomorphism.

7. Product Topology

Let X and Y be topological spaces; the set $X \times Y$ is the set of all ordered pairs of elements from X and Y:

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

We wish to put the "most natural" topology we can on $X \times Y$. Certainly, whatever we choose in this regard should conform with what we already experience with subsets of $\mathbb{R}^n$.

For example, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the standard cartesian plane. let $I = [0, 1] \subset \mathbb{R}$ be the closed unit interval. Then $I \times I$ is a square, and its topology should be that which it inherits as a subspace of $\mathbb{R}^2$. Let's list some more examples.

- $I \times I$ is a square;
- $I \times S^1$ is a cylinder;
- $S^1 \times S^1$ is a torus (the surface of a donut);
- $S^1 \times D^2$ is a solid torus (the entire donut).

Mathematicians think of these things in terms of mappings. The most primitive useful mappings on $X \times Y$ are the projections.

Let $p_X : X \times Y \to X$ be given by $(x, y) \mapsto x$ and $p_Y : X \times Y \to Y$ be given by $(x, y) \mapsto y$. These are called *projections* onto X and Y, respectively.

We wish to define the topology on $X \times Y$ to be the coarsest topology on $X \times Y$ such that the projection maps are continuous. What is required is that the preimages of open sets are open. So, for p_X to be open, we require that if U is open in X, the $p_X^{-1}(U) = U \times Y$ is open in $X \times Y$. A similar statement may be made regarding p_Y. Thus, a subbasis for the topology we seek is the collection of sets of the form $U \times Y$ and $X \times V$, and a basis for the topology is the collection of sets of the form $U \times V$, where U is open in X and V is open in Y.

Definition 11. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by sets of the form $U \times Y$ and $X \times V$, where U is open in X and V is open in Y.

8. Quotient Topology

Let X and Y be topological spaces. We know that a function $f : X \to Y$ is continuous if and only if the preimage of an open set in Y is open in X. So, in some sense, Y has at least as many open sets as it needs to the map to be continuous, and possibly more. On the other hand, we say the f is an open map if the image of an open set in X is open in Y. Here, we see that X has at least as many and potentially more open sets as it needs for f to be open. If f is bicontinuous, the number of open sets in X and Y is "just right" for f. Goldilocks would be proud.

Definition 12. Let X and Y be topological spaces, and let $f : X \to Y$ be a surjective function. We say that Y has the *quotient topology* with respect to f if

$$V \subset Y \text{ is open} \quad \Leftrightarrow \quad V = f(U) \text{ for some open } U \subset X.$$

the open sets is Y are exactly the images of the open sets in Y.

Now suppose that X is a topological space, Y is any set, and $f : X \to Y$ is a surjective function. We may *define* a topology on Y by declaring a subset of Y to be open if and only if its preimage in X is open.

One way the occurs is by creating an equivalence relation on X. That is, we partition X is disjoint subsets which cover X. Two elements from X are considered equivalent if the belong to the same set in the partition; these sets are called *equivalence classes*. Then, we view the set of equivalence classes as a set in its own right. These are the details.

Definition 13. Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. We say that $\mathcal{C}$ is a *partition* of X if

(P1) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$, then $C_1 \cap C_2 = \varnothing$;
(P2) $\cap \mathcal{C} = X$.

The members of $\mathcal{C}$ are called *blocks*.

Let $\mathcal{C}$ be a partition of X. Let $x_1, x_2 \in X$, we say that x_1 is *equivalent* to x_2, and write $x_1 \equiv x_2$, if $x_1, x_2 \in C$ for some $C \in \mathcal{C}$.

Let $a \in X$. The *equivalence class* of a is the set

$$\overline{a} = \{x \in X \mid a \equiv x\}.$$

The *quotient* of X by $\mathcal{C}$ is

$$\overline{X} = \{\overline{x} \mid x \in X\}.$$

Thus $\overline{X}$ is the set of equivalence class. There is a natural function from X to $\overline{X}$, given by sending a point x to the block that it is in:

$$\beta : X \to \overline{X} \quad \text{given by } \beta(x) = \overline{x}.$$

If X is a topological space, we then impose the quotient topology on $\overline{X}$.

9. Convergence

Definition 14. Let X be a topological space and let (x_n) be a sequence in X. We say that (x_n) *converges* to $L \in X$ if, for every neighborhood V of L there exists $N \in \mathbb{N}$ such that $x_n \in V$ whenever $n \geq N$.

Example 3. Let $I = (0, \infty)$. The function $\exp : \mathbb{R} \to I$ given by $\exp(x) = e^x$ is a homeomorphism, so it preserves all topological properties. For example, if x_0 is an boundary point of $A \subset \mathbb{R}$, then $f(x_0)$ is a boundary point of $f(A)$. If a sequence (x_n) converges to $L \in \mathbb{R}$, then the sequence $f(x_n)$ converges to $f(L) \in I$.

It may seem odd that the limit of a sequence is not necessary unique in every space. Next we give a condition that will ensure that the limit of a sequence is unique.

10. Hausdorff Spaces

It may seem odd that the limit of a sequence is not necessary unique in every space.

Example 4. Consider the "bug-eyed" line segment, constructed at follows. Let $X = (0, 1] \cup \{a, b\}$. Declare a subset U of X to be open if their exists an open set V in $\mathbb{R}$ such that

$$U = \begin{cases} X \cap (V \cup \{a, b\}) & \text{if } 0 \in V \ ; \\ X \cap V & \text{if } 0 \notin V \ . \end{cases}$$

Then the sequence (x_n), where $x_n = \dfrac{1}{n}$, converges to both a and b. The problem with this space is that the points a and b cannot be "separated".

We may construct this space using the quotient topology: Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1] \text{ and } y \in \{1, 2\}\}$. Define a relation on X by

$$(x_1, y_1) \equiv (x_2, y_2) \quad \Leftrightarrow \quad x_1 = x_2 \text{ and } x_1 \neq 0\}.$$

Then $\overline{X}$, with the quotient topology, it the "bug-eyed" line segment.

Definition 15. Let X be a topological space. We say that X is *Hausdorff* if for every distinct $x_1, x_2 \in X$ their exists neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \varnothing$.

Proposition 3. *Let X be a Hausdorff space and let (x_n) be a sequence in X which converges to L_1 and to L_2. Then $L_1 = L_2$.*

Proof. Suppose not. Since X is Hausdorff, there exists neighborhoods of U_1 of L_1 and U_2 of L_2 such that $U_1 \cap U_2 = \varnothing$. Since (x_n) converges to L_1, there exist $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U_1$. But then, for $n \geq N$, $x_n \notin U_2$. This contradicts that (x_n) converges to L_2. $\square$

11. Connectedness

A space is connected if it has only one "piece". We state this formally as follows.

Definition 16. Let X be a topological space.

A *separation* of X is a pair of nonempty open sets $U, V \subset X$ such that $U \cap V = \varnothing$ and $U \cup V = X$.

We say that X is *connected* if there does not exist a separation of X.

A *component* of X is a maximal connected subset; that is, it is a connected subset which is not properly contained in a connected subset.

If we speak of a subset of X being connected, we mean that it is connected as topological space with the subspace topology. It is clear that a set is connected if and only if it has exactly one component.

We give some examples.

- A nonempty subset of $\mathbb{R}$ is connected if and only if it is a singleton or an interval.
- S^n is connected unless $n = 0$ (in which case S^0 is a set containing two points.
- No finite subset of $\mathbb{R}^n$ is connected.
- Consider the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. Then $f^{-1}(B_1(0))$ is connected, but $f^{-1}(B_1(2)$ has two components.

Proposition 4. *Let $f : X \to Y$ be continuous, and let $A \subset X$. If A is connected, then $f(A)$ is connected.*

Proof. We use a proof by contrapositive; assume that $f(A)$ is not connected. Then there exist disjoint open sets V_1 and V_2 with $f(A) \subset V_1 \cup V_2$, with $f(A) \cap V_1 \neq \varnothing$ and $f(a) \cap V_2 \neq \varnothing$. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$; since f is continuous, U_1 and U_2 are open. Since V_1 and V_2 are disjoint, so are U_1 and U_2. Moreover, $U_1 \cap A$ and $U_2 \cap A$ are nonempty. Finally, $A \subset U_1 \cap U_2$. $\square$

Definition 17. Let X be a topological space. We say that X is *path-connected* if for every $x_1, x_2 \in X$, there exists a continuous function $\gamma : I \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

Proposition 5. *If X is path-connected, then X is connected.*

Proof. Suppose that X is not connected, and let $U_1, U_2 \subset X$ be disjoint nonempty open sets which cover X. Let $x_1 \in U_1$ and $x_2 \in U_2$. If X is path-connected, there exists a continuous function $\gamma : I \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. The image of γ is a subset of X which we denote by $\gamma(I)$. Since γ is continuous and I is connected, then $\gamma(I)$ is connected. But $\gamma I \cup U_1$ and $\gamma(I) \cup U_2$ is a separation of $\gamma(I)$, so $\gamma(I)$ cannot be connected; this contradiction implies that X is not path-connected. $\square$

12. Compactness

A space is compact if it is not *too* big, and if it doesn't have any "holes". This may be stated in multiple ways, which are equivalent for "well-behaved" spaces.

Definition 18. Let X be a topological space.
A *cover* of X is a collection of subsets of X whose union is X.
An *open cover* of X is a cover consisting of open sets.
A *finite cover* of X is a cover consisting of finitely many sets.
A *subcover* of a cover is a subset of the cover whose union is X.
We say that X is *compact* if every open cover has a finite subcover.

If we speak of a subset of X being compact, we mean that it is compact as topological space with the subspace topology.

Note that in the phrase "every open cover has a finite subcover", the word finite is describing the collection which is the subcover, but the word open is describing the sets in the cover.

We give some examples.

- Open balls are not compact.
- Closed balls are compact.
- A punctured disk is not compact.
- The entire real line is not compact.

Proposition 6. *A compact subset of a Hausdorff space is closed.*

Proposition 7. *Let $f : X \to Y$ be continuous, and let $A \subset X$. If A is compact, then $f(A)$ is compact.*

Proof. Consider an open cover of $f(A)$. The collection of preimages of the sets in the cover form an open cover of A. Since A is compact, a finite subset of these cover A. The collection of images of these sets form a finite subcover of the original cover of $f(A)$. $\square$

Theorem 1. (Heine-Borel Theorem) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

We have the following alternate variations of the definition of compactness, which are equivalent to the standard definition in most cases in which we are interested.

Definition 19. Let X be a topological space.
We say that X is *sequentially compact* if every sequence in X has a cluster point in X.
We say that X is *limit point compact* if every infinite subset of X has an accumulation point in X.

Theorem 2. (Bolzano-Weierstrauss Theorem) *A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.*

Corollary 1. *A subset of $\mathbb{R}^n$ is compact if and only if it is sequentially compact.*

13. Topological Manifolds

Definition 20. Let X be a topological space. We say that X is *locally Euclidean* if, for every point $x \in X$, there exists a neighborhood of x which is homeomorphic to B^n for some n.

A *topological manifold* is a locally Euclidean Hausdorff space.

If X is locally Euclidean and connected, then the dimension n is constant throughout the entire space, and is called the *dimension* of the manifold.

In order to compute with locally Euclidean spaces, we become more specific about these local homeomorphisms.

Definition 21. Let X be a topological space.

A *chart* on X is a bijective bicontinuous function $\psi : U \to B^n$, where $U \subset X$ is an open subset of X.

Let $\psi_1 : U_1 \to B^n$ and $\psi_2 : U_2 \to B^n$ be charts on X. The *transition function* given by these charts is

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2).$$

We say that ψ_1 and ψ_2 are *compatible* if the function

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$$

is a homeomorphism.

A collection of charts on X is said to *cover* X if every point on X is in the domain of one of the charts in the collection. An *atlas* on X is a collection of charts on X which cover X, such that every pair of charts in the collection are compatible.

We say that two atlases are *compatible* if their union is an atlas.

For a topological manifold, any two atlases are compatible. However, if we wish to put additional structure on our manifold, this may no longer be the case. For example, doing calculus on manifolds requires what is known as a differentiable manifold.

Definition 22. Let X be a topological manifold.

We say that two charts on X are *differentiably compatible* if the corresponding transition function is differentiable. A *differentiable atlas* is an atlas consisting of differentiably compatible charts.

The set of all differentiable atlases on X is partially ordered by inclusion. A *differentiable structure* on X is a maximal atlas with respect to this partial order.

A *differentiable manifold* is a topological space X together with a differentiable structure on X.

14. Riemann Surfaces

A Riemann surface is a topological 2-manifold, together with a complex structure. Put another way, a Riemann surface is a topology space such that each point has a neighborhood homoeomorphic to the open unit disk in the complex plane, such that the transition functions are analytic. We given the details now.

Definition 23. Let X be a topological space, and let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

A *complex chart* on X is a bijective bicontinuous function $\psi : U \to \Delta$, where $U \subset X$ is an open subset of X.

Let $\psi_1 : U_1 \to \Delta$ and $\psi_2 : U_2 \to \Delta$ be charts on X. The *transition function* given by these charts is

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2).$$

We say that ψ_1 and ψ_2 are *compatible* if the function

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$$

is analytic.

A collection of charts on X is said to *cover* X if every point on X is in the domain of one of the charts in the collection. An *atlas* on X is a collection of charts on X which cover X, such that every pair of charts in the collection are compatible.

We say that two atlases are *compatible* if their union is an atlas. Compatibility is an equivalence relation on the set of all atlases, and the union of compatible atlas is again an atlas. Thus the union of all atlases in an equivalence class is the maximal atlas in the class. A *complex structure* on X is a maximal atlas.

A *Riemann surface* is a topological space X together with a complex structure on X.

15. Compact Connected Orientable Surfaces

A compact orientable surface is a compact 2-manifold which can be embedded in $\mathbb{R}^3$. Every compact Riemann surface has the topology of a compact orientable surface. Up to homeomorphism, the topology of a compact connected orientable surface is completely characterized by a single nonnegative integer, called its genus. We describe this in more detail.

Let X be a compact connected orientable surface. The *genus* of X, denoted $g(X)$ or g_X, is (loosely speaking) the number of "holes" it has. This can be made more precise.

A *sphere* is S^2; this is a compact orientable surface of genus zero.

A *torus* is an orientable surface which may be created by taking a square, and identifying opposite sides using the quotient topology. This is a compact orientable surface of genus one.

Let X and Y be compact connected orientiable surfaces. Their *connected sum* of X and Y, denoted $X\#Y$, is the surface which results from removing an open disk from X and one from Y, and gluing together the two surfaces on the resulting circular boundaries using the quotient topology. The genus of the resulting space is given by

$$g(X\#Y) = g(X) + g(Y).$$

Every compact connected orientable surface may be constructed by using connected sums of spheres and tori. Thus, this inductively defines the concept of genus.

A *triangulation* of a compact surface is a net of triangles which cover the surface. Set

- $V =$ the number of vertices;
- $E =$ the number of edges;
- $F =$ the number of faces.

The *Euler characteristic* of the surface is

$$\chi = V - E + F.$$

We may relate this to genus as follows. Take two triangulated surfaces, X and Y, with Euler characteristics $\chi_X = V_X - E_X + F_X$ and $\chi_Y = V_Y - E_Y + F_Y$. Let $Z = X\#Y$. Note that Z by be formed by removing one face from each of the surfaces, and gluing together the exposed boundary triangles. This results in loosing two faces, and identifying two triangles into one. So, $V_Z = V_X + V_Y - 3$, $E_Z = E_X + E_Y - 3$, and $F_Z = F_X + F_Y - 2$. This results in

$$\chi_Z = \chi_X + \chi_Y - 2.$$

So, if the genus goes up by 1, the Euler characteristic goes down by 2. Thus, the genus g of the surface relates to the Euler characteristic via the formula

$$\chi = 2 - 2g.$$

16. Riemann-Hurwitz Formula

A *ramified cover* is a nonconstant analytic function $f : Y \to X$ between compact connected Riemann surfaces.

Let $f : Y \to X$ be a ramified cover. The *degree* of such a function is the maximum cardinality of a fiber over a point in x. Then f is n-to-1 over all but finitely many points in X. Also, f is injective in a neighborhood of p for all but finitely many $p \in Y$; otherwise, f is e-to-1 is a deleted neighborhood of p.

The *ramification index* of $p \in Y$, denoted $e(p)$ is an integer e such that f is e-to-1 in a neighborhood of p. If $e(p) > 1$, we say that f is *ramified* at p, and that p is a *ramification point* of f. A *branch point* of f is the image of a ramification point. The *total ramification* of the cover is

$$\mathrm{ram}(f) = \sum_{p \in Y} (e(p) - 1).$$

We wish to use triangulation to relate the genus of Y and X. Consider a triangulation of X which includes all of the branch points as vertices. The preimage of this triangulation is a triangulation of Y which includes all of the ramified points as vertices. The branch points of the cover are the points in X which are the images of the ramified points. The *total ramification* of the cover is

$$\mathrm{ram}(f) = \sum_{p \in Y} (e(p) - 1).$$

Proposition 8. (Riemann-Hurwitz Formula) *Let $f : Y \to X$ be a ramified cover of degree n. Let g_Y denote the genus of Y and g_X denote the genus of X. Then*

$$g_Y = 1 + n(g_X - 1) + \frac{1}{2}\,\mathrm{ram}(f).$$

Proof. Consider a triangulation of X which includes all of the branch points as vertices. The preimage of this triangulation is a triangulation of Y which includes all of the ramified points as vertices. Each face on X lifts to n faces on Y, and each edge on X lifts to n edges on Y. A vertex on X lifts to n vertices on Y, *unless the vertex is a branch point.* If the vertex x is a branch point, the number of points in the preimage is less by the amount of ramification over x; that is, if $F = f^{-1}(x)$ is the fiber over x, then $n = \sum_{y \in F} e(p)$, so $|F| = n - \sum_{y \in F}(e(p) - 1)$. Thus $F_Y = nF_X$, $E_Y = nE_X$, and $V_Y = nV_X - \sum_{y \in Y}(e(p) - 1)$, so

$$\chi_Y = n\chi_X - \mathrm{ram}(f) \quad \Rightarrow \quad 2 - 2g_Y = n(2 - 2g_X) - \mathrm{ram}(f).$$

Solving for g_Y gives the result. □

Corollary 2. *Let $f : Y \to \mathbb{C}_\infty$ be a ramified cover of degree n. Let g denote the genus of Y. Then*

$$g = 1 - n + \frac{1}{2}\,\mathrm{ram}(f).$$

Proof. Plug $g_X = 0$ into the Riemann-Hurwitz Formula. □

Example 5. Let $f : Y \to \mathbb{C}_\infty$ be a ramified cover of degree 2 with four branch points. Find g_Y.

Solution. We have $n = \deg(f) = 2$. Each of the ramification points p has $e(p) = 2$, and there are four of them. So,

$$g = 1 - n + \frac{1}{2}\operatorname{ram}(f) = 1 - 2 + \frac{1}{2}(4) = 1.$$

□

Example 6. Let $f : Y \to \mathbb{C}_\infty$ be a ramified cover of degree 5 with seven ramified points, each with ramification index 3. Find g_Y.

Solution. Here, $n = 5$ and $\operatorname{ram}(f) = 7(3 - 1) = 14$, so

$$g = 1 - 5 + \frac{1}{2}(14) = 3.$$

□

Example 7. Let $f : Y \to \mathbb{C}_\infty$ be a ramified cover of degree 5 with branch points x_1, x_2, and x_3. The points in the fiber over the branch points, and their ramification indices, are as follows:

- over x_1: $e(y_{1,1}) = 3$, $e(y_{1,2}) = 2$
- over x_2: $e(y_{2,1}) = 3$, $e(y_{2,2}) = 1$, $e(y_{2,3}) = 1$
- over x_3: $e(y_{3,1}) = 4$, $e(y_{3,2}) = 1$

Find g_Y.

Solution. The degree is $n = 5$ and the total ramification is

$$\operatorname{ram}(f) = ((3 - 1) + (2 - 1) + (3 - 1) + (4 - 1)) = 8.$$

Thus

$$g = 1 - n + \frac{1}{2}\operatorname{ram}(f) = 1 - 5 + \frac{1}{2}(8) = 0.$$

□

Example 8. Let $f : Y \to \mathbb{C}_\infty$ be a ramified cover of degree 8 with ten branch points. Eight of the branch points have fibers with one ramified point of index three. Two of the branch points have fibers with two ramified points, one of index three and the other of index four. Find g_Y.

Solution. The degree is $n = 8$ and the total ramification is

$$\operatorname{ram}(f) = 8(3 - 1) + 2(3 - 1) + 2(4 - 1) = 26.$$

So

$$g = 1 - n + \frac{1}{2}\operatorname{ram}(f) = 1 - 8 + 13 = 6.$$

□

17. Symmetry Groups

There is one last concept we wish to explore with respect to ramified covers: we address the issue or what covers of the Riemann sphere exist. We will explore enough definitions to state *Riemann's Existence Theorem*.

This approach begins with a ramified cover $f : Y \to X$, where $X = \mathbb{C}_\infty$. Remove all of the branch points from X, and their preimages from Y. Select a "base point" $x_0 \in X$, a create a loop starting at x_0 and going around exactly on of the branch points. Lift this path to a preimage; it will start at one point in the fiber over x_0, and will end in another. If the starting and ending points are different, then we have discovered a ramified point. To make this idea computational, we need a little bit of group theory.

Definition 24. Let X be a set. A *permutation* of X is a bijective function $\alpha : X \to X$.

The *Symmetry group* of X is the set of all permutations of X, and is denoted $\mathrm{Sym}(X)$:

$$\mathrm{Sym}(X) = \{\alpha : X \to X \mid \alpha \text{ is bijective.}$$

The set $\mathrm{Sym}(X)$ admits the binary operation of composition, which we view as a form of multiplication. Given $\alpha, \beta \in \mathrm{Sym}(X)$, we may denote the operation of composition by $\circ$, as in $\alpha \circ \beta$, or simply by juxtiposition, as in $\alpha\beta$. Also, α^n means α composed with itself n times. We define $\epsilon \in \mathrm{Sym}(X)$ by $\epsilon(x) = x$ for all $x \in X$.

Composition in $\mathrm{Sym}X$ satisfies these properties:

- **(G0)** The composition of bijective functions is bijective, so $\circ$ is closed on Sym.
- **(G1)** Composition is associative: $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$.
- **(G2)** Composition admits an identity: $\epsilon \circ \alpha = \alpha \circ \epsilon = \alpha$.
- **(G3)** Bijective functions have compositional inverses, given by $\alpha(x_1) = x_2 \Leftrightarrow \alpha^{-1}(x_2) = x_1$.

These properties indicate that $\mathrm{Sym}(X)$ is a mathematical object known as a *group*.

Note that if X is finite, the $\alpha : X \to X$ is injective if and only if it is surjective.

In the case that $X = \{1, 2, \ldots, n\}$, we use the notation

$$S_n = \mathrm{Sym}(X).$$

By enumerating the points in a finite set X, we can identify $\mathrm{Sym}(X)$ with S_n. So, studying S_n suffices for most purposes.

There are two standard ways to specify an arbitrary elements of S_n.

17.1. **Direct Notation.**

This first uses a $2 \times n$ matrix, with the domain points listed in the top row, and where they go listed in the second row. For example, set

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 3 & 7 & 6 & 4 \end{pmatrix}.$$

This indicates that $\alpha(1) = 2$, $\alpha(2) = 5$, ..., $\alpha(7) = 4$. We can plug in points on the right, such as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 3 & 7 & 6 & 4 \end{pmatrix}(4) = 3.$$

Note that in that the entries in the second row must be distinct, or else the function is not bijective.

We can multiply two such permutations by composition, which we indicate by juxtiposition. Note that multiplication is from left to right, as it is function composition. For example,

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 2 & 3 & 1 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 2 & 5 & 1 & 3 & 7 & 6 & 4\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 2 & 4 & 5 & 3 & 7 & 6 & 1\end{pmatrix}.$$

Notice that we automatically expanded the domain of the first entry, a priori in S_5, so as to view it in as a permutation S_7 which fixes 6 and 7.

The order in which we multiply permutation matters. That is, composition is *not* and associative operation. For example,

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}, \quad \text{but} \quad \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}.$$

Definition 25. Let $\alpha \in \mathrm{Sym}(X)$.

The *support* of α is

$$\mathrm{supp}(\alpha) = \{x \in X \mid \alpha(x) \neq x\}.$$

The *fixed set* of α is

$$\mathrm{fix}(\alpha) = \{x \in X \mid \alpha(x) = x\}.$$

The members of $\mathrm{fix}(\alpha)$ are called *fixed points* of α.

The *orbit* of $x_0 \in X$ under α is

$$\mathrm{orb}_\alpha(x_0) = \{x \in X \mid \alpha^n(x_0) = x \text{ for some } x \in \mathbb{N}\}.$$

The *length* of an orbit is its cardinality. An orbit is *trivial* if it has length one.

Thus we see that

$$x_0 \in \mathrm{fix}(\alpha) \Leftrightarrow x_0 \notin \mathrm{supp}(\alpha) \Leftrightarrow |\mathrm{orb}_\alpha(x)| > 1.$$

For example, let

$$\alpha = \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 2 & 5 & 7 & 4 & 1 & 6 & 3\end{pmatrix}.$$

Then $\mathrm{fix}(\alpha) = \{4, 6\}$, $\mathrm{supp}(\alpha) = \{1, 2, 3, 5, 7\}$, $\mathrm{orb}_\alpha(1) = \{1, 2, 5\}$, and $\mathrm{orb}_\alpha(3) = \{3, 7\}$. The trivial orbits are those containing 4 and 6.

The orbits of α form a partition of its domain. Seeing the orbits gives us a thorough idea of how the permutation behaves.

17.2. **Disjoint Cycle Notation.**

Definition 26. A *cycle* is a permutation whose support consists of one orbit. The *length* of the cycle is the length of this orbit.

We say that two cycles are *disjoint* if their supports have empty intersection.

Cycle notation writes as cycles as a finite ordered sequence, usually without commas, although in our definition we will use commas. Precisely, let

$$\alpha = (x_1, x_2, \dots, x_k).$$

Then

$$\alpha(y) = \begin{cases} x_{i+1} & \text{if } y = x_i \text{ and } i < k;\\ x_1 & \text{if } y = x_k;\\ y & \text{if } y \neq x_i \text{ for any } i.\end{cases}$$

So, if a number is not listed in the cycle, that number is fixed by the permutation.

Example 9. For example:

- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix} = \texttt{(1 2 5 3 4)}$
- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix} = \texttt{(1 2 3 4 5)}$

Note that `(1 3 5)` = `(3 5 1)` = `(5 1 3)`. It is standard to place the lowest number in a cycle in the first position.

Proposition 9. *Disjoint cycles commute. Every permutation can be written as a product of disjoint cycles.*

It is common to write permutations as a product (composition) of disjoint cycles.

Example 10. For example:

- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = \texttt{(1 2 3)(4 5)}$
- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 4 & 1 & 6 & 3 \end{pmatrix} = \texttt{(1 2 5)(3 7)}$

In the second case, since 4 and 6 are fixed, they are usually not written.

We can multiply cycles to put the result in disjoint cycle notation. Note that, since disjoint cycles commute, `(1 2 3)(4 5)` = `(4 5)(1 2 3)`. It is standard to place the disjoint cycles in increasing order of the lowest member of each cycle's support.

Example 11. Let $\alpha =$ `(1 2 5)(2 9 4)(1 3 6)`. Write α in disjoint cycle notation.

Solution. We start with 1, which we plug into the right side (remember, these cycles are functions). Proceed from right to left, asking in each case what the cycle does to the number.

(1) $1 \mapsto 3 \mapsto 3 \mapsto 3$
(3) $3 \mapsto 6 \mapsto 6 \mapsto 6$
(6) $6 \mapsto 1 \mapsto 1 \mapsto 2$
(2) $2 \mapsto 2 \mapsto 9 \mapsto 9$
(9) $9 \mapsto 9 \mapsto 4 \mapsto 4$
(4) $4 \mapsto 4 \mapsto 2 \mapsto 5$
(5) $5 \mapsto 5 \mapsto 5 \mapsto 1$

When we get back to where we started, we close off the resulting cycle. Here, we see that

$$\alpha = \texttt{(1 3 6 2 9 4 5)}.$$

□

18. Path Lifting

Let $I = [0,1]$ and let X be a topological space.

A *path* in X is a continuous function $\gamma : I \to X$. We call $\gamma(0)$ the *initial point*, and we call $\gamma(1)$ the *terminal point*, of the path.

A *loop* in X is a path $\gamma : I \to X$ such that $\gamma(0) = \gamma(1)$. In this case, we call $\gamma(0)$ the *basepoint* of the loop.

Let $f : Y \to X$ be a ramified cover, and let $\widetilde{\gamma} : I \to Y$ be a path in Y. Then $f \circ \widetilde{\gamma}$ is a path in X.

We wish to start with a path in the base space. Let $\gamma : I \to X$ be a path in X from x_0 to x. Let y_0 be any point in the preimage of x_0. Then there exists a unique path $\widetilde{\gamma} : I \to Y$ such that $\widetilde{\gamma}(0) = y_0$, and $f \circ \widetilde{\gamma} = \gamma$. We call $\widetilde{\gamma}$ a *lift* of γ, and this property is called *unique path lifting.*

Homotopies also lift in the following sense: if γ_1 is homotopic in X to γ_2, then $\widetilde{\gamma}_1$ is homotopic to $\widetilde{\gamma}_2$.

We focus on ramified covers whose base space is the Riemann sphere. We use path lifting to detect the index of ramification points in Y.

Let $f : Y \to X$, where $X = \mathbb{C}_\infty$, be a ramified cover. Let $B \subset X$ denote the set of branch points of f, and let $R = f^{-1}(B)$. Set $Y^\circ = Y \smallsetminus R$, $X^\circ = X \smallsetminus B$, and $f^\circ = f \restriction_{Y^\circ}$. Now Y° may be covered with open sets that map injectively and bicontinuously onto X; such a function is called a *topological cover.* The fiber over every point in X° is a consistent n, where $n = \deg(f)$.

Let $x_0 \in X^\circ$; we call x_0 a *basepoint.* Enumerate the fiber over x_0; that is, there are exactly n points over x_0, and we give each of them a number thusly: let $y_1, \ldots, y_n$ denote the points in the fiber over x_0. If λ is a loop based at x_0, and we lift λ to start at (for example) y_1, then the lift will terminate at another point in the fiber; it may terminate at y_1, but it may terminate at y_i for some $i \neq 1$. By lifting λ to different each point in the fiber, and noting the endpoint, we obtain a permutation of the fiber which may be written in cycle notation. We will denote this permutation by g. Then $g \in S_n$, where $n = \deg(f)$.

Suppose that λ is a loop based at x_0 which wraps once, clockwise, around a single branch point. The disjoint cycle decomposition of the corresponding permutation g indicates the ramification above the branch point. For example, if $g =$ `(1)(2 3)(4 5)(6 7 8)`, the fiber over the branch point has four points, one of which is unramified, two which are ramified to index 2, and one which is ramified to index 3.

Let $x_1, \ldots, x_r$ be the branch points of the cover. Let λ_j denote a loop on X° which starts at x_0 and goes around x_j clockwise exactly once, and does not go around any other branch point. We number these branch points so that it is possible to construct a single clockwise loop starting at x_0 and passing through the x_j's, in order. Path lifting of each of these loops gives a permutation of the fiber over x_0, which we denote by g_j. Let $\vec{g} = (g_1, g_2, \ldots, g_r)$ be the ordered r-tuple of permutations obtained in this way. We call $\vec{g}$ a *branch cycle description* of the cover.

If we concatenate consecutive paths and lift the concatenation, we will achieve a different permutation of the fiber over the basepoint. We may compute this permutation by multiplying the corresponding permutations of the paths we concatenated. Thus $\prod_{j=1}^{n} g_j$ is the product of the branch cycle description.

It is clear that if the path is contractible, then its corresponding permutation is the identity $\epsilon \in S_n$. However, a clever observer will notice that the concatenation of the paths λ_1 through λ_r is null-homotopic on the punctured Riemann sphere, since all of the branch points are inside this loop. The loop can be pulled backwards over the back side of the sphere to contract it to the basepoint. Thus,

$$\prod_{j=1}^{n} g_j = \epsilon.$$

We say that a subset of S_n acts *transitively* on $\{1, \dots, n\}$ if it is possible to get from i to j by some sequence of the permutations in the set, for any i and j. Any branch cycle description acts transitively if the cover is connected. This is because, if Y is connected, it is possible to draw a path between any two points in the fiber over x_0, whose image is a path in X°. and it can be shown that any loop in X° can be obtained by concatenation in λ_j's.

It turns out that every cover which can be described in this way exists. We finish by stating this.

Theorem 3. (Riemann's Existence Theorem)
Let n be a position integer. Let $g_1, \dots, g_r \in S_n$, which act transitively, such that $\prod_{j=1}^{n} g_j = \epsilon$. Let $x_1, \dots, x_r \in \mathbb{C}_\infty$ be distinct. Then there exists a cover $f : Y \to \mathbb{C}_\infty$ whose branch points are $x_1, \dots, x_r$, whose ramification is described by $\vec{g} = (g_1, \dots, g_r)$.

Example 12. Consider a cover $f : Y \to \mathbb{C}_\infty$ with branch cycle description

$$\vec{g} = (\texttt{(1 2 3)(4 5)}, \texttt{(2 4)}, \texttt{(2 5 1)}, \texttt{(1 3)(2 4)}, g_5).$$

(a) Find g_5.
(b) Find the genus of Y.

Solution. Since $\vec{g}$ is a branch cycle description, the ordered product of the permutation is ϵ, so g_5 is the inverse of the product of the rest of the terms. But

$$\prod_{j=1}^{4} g_j = \texttt{(1 2 3)(4 5)(2 4)(2 5 1)(1 3)(2 4)} = \texttt{(2 3 5)}.$$

So $g_5 = \texttt{(2 3 5)}^{-1} = \texttt{(2 5 3)}$.

We use the Riemann-Hurwitz formula to compute the genus. The sum of the ramification is $\mathrm{ram}(f) = 2 + 1 + 1 + 2 + 1 + 1 = 8$, so

$$g_Y = 1 - n + \frac{1}{2}\,\mathrm{ram}(f) = 1 - 5 + \frac{1}{2}(8) = 0.$$

□

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