COMPLEX ANALYSIS
TOPIC VIII: THE REAL EXPONENTIAL FUNCTION

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1. Exponents

Let a be a positive real number, and let x be a real number. We ask, what is the meaning of a^x?

1.1. **When x is a positive integer.** Let $n = x$, and assume that n is a positive integer. Then a^n is defined to mean the product of n numbers whose value is a:

$$a^n = \underbrace{a \times \cdots \times a}_{n \text{ times}}.$$

From this, we obtain two significant properties.

(E1) $a^{m+n} = a^m \cdot a^n$
(E2) $(a^m)^n = a^{mn}$

To see this, write

$$a^{m+n} = \underbrace{a \times \cdots \times a}_{m+n \text{ times}} = \underbrace{a \times \cdots \times a}_{m \text{ times}} \times \underbrace{a \times \cdots \times a}_{n \text{ times}} = a^m \times a^n.$$

and

$$(a^m)^n = (\underbrace{a \times \cdots \times a}_{m \text{ times}})^n = \underbrace{(\underbrace{a \times \cdots \times a}_{m \text{ times}}) \times \cdots \times (\underbrace{a \times \cdots \times a}_{m \text{ times}})}_{n \text{ times}} = \underbrace{a \times \cdots \times a}_{mn \text{ times}} = a^{mn}.$$

We wish to extend the meaning of a^x so that it is defined for any real number x, in such a way that the properties **(E1)** and **(E2)** remain true.

1.2. **When $x = 0$.** Consider the case when $x = 0$. We multiply a times a^0; whatever a^0 means, if property **(E1)** is to remain true, we have

$$aa^0 = a^1 a^0 = a^{1+0} = a^1 = a.$$

Dividing both sides by a gives

$$a^0 = 1.$$

1.3. **When x is a negative integer.** Consider the case when x is a negative integer, so that $x = -n$ for some positive integer n. For **(E1)** to remain true, we must have

$$a^n a^x = a^{n+x} = a^0 = 1.$$

In this case,

$$a^{-n} = \frac{1}{a^n}.$$

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1.4. **When x is rational.** Consider the case when $x = \frac{1}{n}$, where n is a positive integer. For **(E2)** to remain true, we must have

$$(a^{1/n})^n = a^{n/n} = a^1 = a.$$

Thus, $a^{1/n}$ is the unique number whose n^{th} power is a; that is,

$$a^{1/n} = \sqrt[n]{a}.$$

Consider the case when $x = \frac{m}{n}$, where m and n are positive integers. Then **(E2)** produces $a^{m/n} = (a^m)^{1/n}$, so

$$a^{m/n} = \sqrt[n]{a^m}.$$

1.5. **When x is irrational.** We now consider the case when x is irrational. This is the hardest step, and requires a limiting process of some sort.

Integers are obtained from natural numbers by algebraic considerations (defining subtraction), and rational numbers are obtained from integers by additional algebraic considerations (defining division); however, real numbers are obtained from rationals by geometric considerations (filling in gaps in the number line). So, defining a^x for irrational x may require order or distance.

There is an additional property of exponents which is important in this context:

(E3) if $1 < a$ and $r < s$, then $a^r < a^s$

This is true when x is any rational number, and we wish it to remain true for any real number.

We could attempt to define a^x as the limit of a sequence, as follows. We line up all of the rationals by the order relation $<$, and see that there are gaps in the line; so, too, we can line up all of the numbers of the form a^q where q is rational, and see that there are gaps in the line; we hope to fill these gaps by numbers of the form a^x, where x is irrational.

Let x be an irrational number, and define the sequence (x_n) by

$$x_n = \frac{\lfloor 10^{n-1} x \rfloor}{10^{n-1}},$$

so that (x_n) is a bounded increasing sequence of rational numbers, and

$$x = \lim_{n \to \infty} x_n.$$

This is a sequence of decimal estimates of x of increasing accuracy; it is an increasing sequence that converges to x.

Since x_n is rational, a^{x_n} is defined. Consider the sequence (a^{x_n}); by Property **(E3)**, this is an increasing sequence of real numbers which is bounded above by $a^{\lceil x \rceil}$. Thus, it converges. We define

$$a^x = \lim_{n \to \infty} a^{x_n}.$$

This definition extends the previous definitions in such a way as to preserve properties **(E1)**, **(E2)**, and **(E3)**. The problem with this approach is that it is difficult to prove these properties, as well as the other properties we would like. Life is easier if take advantage of integration, and start by defining log; we take this approach. Thus, assume that we have never seen a definition for e, e^x, $\log x$, or for a^x when x is irrational.

2. The (Real) Natural Logarithm

Definition 1. The (real) *natural logarithm* is the function

$$\log : (0,\infty) \to \mathbb{R} \quad \text{given by} \quad \log(x) = \int_1^x \frac{1}{t}\,dt.$$

Proposition 1. *The function* $\log$ *is differentiable on its domain, with*

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

Proof. This follows from the Fundamental Theorem of Calculus. $\square$

Proposition 2. *Let* $a, b \in \mathbb{R}$ *and* $r \in \mathbb{Q}$. *Then*

(a) $\log(1) = 0$;
(b) $\log(ab) = \log(a) + \log(b)$;
(c) $\log(a^r) = r\log(a)$.

Proof. We know that $\log(1) = \int_1^1 \frac{1}{t}\,dt = 0$.

Let x be a real variable. Then the chain rule gives

$$\frac{d}{dx}\log(ax) = \frac{1}{ax}\cdot a = \frac{a}{ax} = \frac{1}{x}.$$

Thus $\log(ax)$ and $\log(x)$ have the same derivative, and so they differ by a constant, say $\log(ax) = \log(x) + C$. If we set $x = 1$, we have $\log(a) = C$, whence $\log(ax) = \log(a) + \log(x)$. Now set $x = b$ to obtain $\log(ab) = \log(a) + \log(b)$.

For part **(b)**, we note that by the chain rule and the power rule (which has previously been shown for rational exponents), we have

$$\frac{d}{dx}\log(x^r) = \frac{1}{x^r}\cdot rx^{r-1} = \frac{rx^{r-1}}{x^r} = \frac{r}{x} = \frac{d}{dx}r\log(x).$$

So $\log(x^r)$ and $r\log(x)$ have the same derivative, and so they differ by a constant, say $\log(x^r) = r\log(x) + C$. If $x = 1$, we obtain $0 = 0 + C$, so $C = 0$. Setting $x = a$, we get $\log(a^r) = r\log(a)$. $\square$

Proposition 3. *The function* $\log : (0,\infty) \to \mathbb{R}$ *is bijective.*

Proof. One may use the divergence of the harmonic series $\sum \frac{1}{n}$ to show that log maps onto $\mathbb{R}$, and so is surjective.

Since $\frac{d}{dx}\log x = \frac{1}{x} > 0$ for $x \in \operatorname{dom}(\log) = (0,1)$, we see that log is increasing on its domain, and so is injective. Therefore, log is bijective. $\square$

3. The (Real) Exponential Function

Definition 2. The (real) *exponential function* is defined to be the function

$$\exp : \mathbb{R} \to (0, \infty)$$

which is the inverse of the natural logarithm; thus exp satisfies the defining property

$$\exp x = y \quad \Leftrightarrow \quad \log y = x.$$

Proposition 4. *Let* $a, b \in \mathbb{R}$ *and let* $r \in \mathbb{Q}$. *Then*

- **(a)** $\exp(0) = 1$*;*
- **(a)** $\exp(a + b) = \exp(a) \cdot \exp(b)$*;*
- **(b)** $\exp(a)^r = \exp(ar)$.

Proof. That $\exp(0) = 1$ is a restatement of the fact that $\log(1) = 0$.

We have

$$\log(\exp(a + b)) = a + b = \log(\exp(a)) + \log(\exp(b)) = \log(\exp(a) \cdot \exp(b)),$$

by properties of log. Since log is injective, this implies that $\exp(a + b) = \exp(a) \cdot \exp(b)$.

Similarly,

$$\log(\exp(a)^r) = r \log(\exp(a)) = ra = ar = \log(\exp(ar));$$

since log is injective, $\exp(a)^r = \exp(ar)$. □

Proposition 5. *The function* exp *is differentiable on its domain, with*

$$\frac{d}{dx} \exp x = \exp x.$$

Proof. We use implicit differentiation.

Let $y = \exp x$. Then $x = \log(y)$. Take the derivative of both sides to get

$$1 = \frac{d}{dx} x = \frac{d}{dx} \log y = \frac{1}{y} \frac{dy}{dx},$$

so $\dfrac{dy}{dx} = y$. That is, $\dfrac{d}{dx} \exp x = \exp x$. □

4. The Base a Exponential Function

Definition 3. Let $a \in \mathbb{R}$, $a > 0$, and $a \neq 1$.
The *base a exponential function* is

$$\exp_a : \mathbb{R} \to (0, \infty) \quad \text{given by} \quad \exp_a(x) = \exp(x \log a).$$

Proposition 6. *If $r \in \mathbb{Q}$, then $\exp_a(r) = a^r$.*

Proof. We have $\exp_a(r) = \exp(r \log a) = \exp(\log(a^r)) = a^r$. □

Definition 4. Define a^x, where a is a positive real number, by

$$a^x = \exp(x \log a).$$

Proposition 7. *The function $\exp_a$ is differentiable on its domain, and*

$$\frac{d}{dx} \exp_a(x) = \log(a) \exp_a(x).$$

Proof. Using the definition of $\exp_a$ and the chain rule, we have

$$\frac{d}{dx} \exp_a(x) = \frac{d}{dx} \exp(x \log a) = \exp(x \log a) \log a = \log(a) \exp_a(x).$$

□

Definition 5. The number $e \in \mathbb{R}$ is defined by

$$e = \exp(1).$$

Proposition 8. *The function* exp *satisfies*

$$\exp(x) = \exp_e(x) = e^x.$$

Proof. By definition, $e^x = \exp(x \log e) = \exp(x \cdot 1) = \exp(x)$. □

Proposition 9. *The function $\exp_a$ is bijective.*

Proof. If $a \in (0, 1)$, the $\log(a) < 0$, since $\log(1) = 0$ and log is increasing. In this case, $\exp_a$ is decreasing. Otherwise, $a > 1$, and $\log(a) > 0$, so $\exp_a$ is increasing. In either case, $\exp_a$ is injective. Is is clearly onto $(0, \infty)$, since exp is. So $\exp_a$ is bijective. □

5. The Base a Logarithm

Definition 6. Define the *base a logarithm* to be the function

$$\log_a : (0, \infty) \to \mathbb{R}$$

which is the inverse of $\exp_a$. Thus $\log_a$ has the defining property

$$\log_a(x) = y \quad \Leftrightarrow \quad a^y = x.$$

Proposition 10. *The function* $\log_a$ *is differentiable on its domain, with*

$$\frac{d}{dx} \log_a(x) = \frac{1}{\log(a)x}.$$

Proof. We use implicit differentiation. Let $y = \log_a(x)$, so that $x = \exp_a(y)$. Differentiating both sides of this equation gives

$$1 = \frac{d}{dx} x = \frac{d}{dx} \exp_a(y) = \log(a) \exp_a(y) \frac{dy}{dx} = \log(a) x \frac{dy}{dx}.$$

Thus $\dfrac{dy}{dx} = \dfrac{1}{\log(a)x}$. □

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