Calculus AB
Riemann Integrals
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Definition 1. Let $a, b \in \mathbb{R}$ with $a < b$.
A *partition* of the closed interval $[a, b]$ is a finite set

$$P = \{x_0, x_1, x_2, \dots, x_n\}$$

with the property that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$. We view P as indicating a way of breaking the interval $[a, b]$ into n subintervals. The width of the i^{th} subinterval is $\Delta x_i = x_i - x_{i-1}$, for $i = 1, \dots, n$.
The *norm* of the partition P is

$$||P|| = \max\{\Delta x_i \mid i = 1, \dots, n\}.$$

A *choice set* for P is a finite set

$$C = \{c_1, c_2, \dots, c_n\}$$

such that $c_i \in [x_{i-1}, x_i]$, for $i = 1, \dots, n$. Note that this implies

$$c_1 < c_2 < \cdots < c_n.$$

Let $f : [a, b] \to \mathbb{R}$. The *Riemann sum* associated to a partition P and a choice set C for P is

$$R(f, P, C) = \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

We say that f is *Riemann integrable with integral* I if there exists a real number $I \in \mathbb{R}$ such that, for every positive real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every partition P and choice set C of P,

$$\|P\| < \delta \quad \Rightarrow \quad |R(f, P, C) - I| < \epsilon.$$

If f is Riemann integrable with integral I, we write

$$\int_a^b f(x)\,dx.$$

This is read, "the integral from a to b of $f(x)\,dx$".

The Riemann integral represents the area between the graph of f and the x-axis. Note that this is *signed area*; that is, area below the x-axis is counted as negative.

Remark 1 (Properties of the Riemann Integral)**.** Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be integrable.
Let $c, d \in [a, b]$ with $a \le c \le d \le b$. Let $k \in \mathbb{R}$.

- **(a)** f is integrable on $[c, d]$
- **(b)** $\int_a^a f(x)\,dx = 0$
- **(c)** $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- **(d)** $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
- **(e)** $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
- **(f)** $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- **(g)** if $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b f(x)\,dx$

Theorem 1 (Fundamental Theorem of Calculus, Part I). **(FTC I)**
Let $f : [a, b] \to \mathbb{R}$ be integrable. Define a function

$$F : [a, b] \to \mathbb{R} \quad \text{by} \quad F(x) = \int_a^x f(t)\, dt.$$

Then F is differentiable at x for $x \in (a, b)$, and $F'(x) = f(x)$.

Reason. Consider

$$F(x + h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt = \int_x^{x+h} f(t)\, dt.$$

Now $\int_x^{x+h} f(t)\, dt$ is the area under the graph of f from x to $x + h$. Since f is continuous, it is clear that, for very small h, this area is approximately the area of the rectangle whose height is $f(x)$ and whose width is h; that is,

$$\int_x^{x+h} f(t)\, dt \approx f(x)h.$$

Thus, for very small h,

$$F'(x) \approx \frac{F(x + h) - F(x)}{h} = \frac{\int_x^{x+h} f(t)\, dt}{h} \approx \frac{f(x)h}{h} = f(x).$$

These approximations become precise as h approaches zero, so

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = f(x).$$

$\square$

Theorem 2 (Fundamental Theorem of Calculus, Part II). **(FTC II)**
Let $f : [a, b] \to \mathbb{R}$ and suppose that F is an antiderivative for f on (a, b). Then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Proof. Let $G(x) = \int_a^x f(t)\, dt$. Then by FTC I, G is differentiable on (a, b), and $G'(x) = F'(x) = f(x)$. Since F and G have the same derivative, they differ by a constant. Thus there exists a constant $C \in \mathbb{R}$ such that

$$G(x) = F(x) + C \qquad \text{for all } x \in [a, b].$$

Plugging in $x = a$, we have $G(a) = F(a) + C$. But $G(a) = \int_a^a f(x)\, dx = 0$, so $F(a) = -C$, so

$$G(x) = F(x) - F(a).$$

Finally, plug in $x = b$ to get $G(b) = F(b) - F(a)$, so

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

$\square$