

REAL ANALYSIS

TOPIC 34 - ALGEBRAS OF SETS

PAUL L. BAILEY

1. SEQUENCES OF SETS

Definition 1. Let X be a set. A *sequence of subsets of X* is a function $A : \mathbb{N} \rightarrow \mathcal{P}(X)$. We write A_n to mean $A(n)$, and we write (A_n) to indicate the entire sequence.

If $\mathcal{A} \subset \mathcal{P}(X)$, a *sequence in \mathcal{A}* is a sequence of subsets of X such that $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$.

Let (A_n) be a sequence of subsets of X . There is a corresponding collection of subsets of X , say $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$. The reader should note a couple of distinctions between these objects: the sets in (A_n) come in a specific order, whereas the sets in \mathcal{A} have no order. Also, the same set may appear multiple time in the sequence (A_n) , whereas there is no notion of the multiplicity of a member of \mathcal{A} . However, we should note that unions and intersections may be written in two ways:

$$\cup \mathcal{A} = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \cap \mathcal{A} = \bigcap_{n=1}^{\infty} A_n.$$

If $A \subset X$, we let $A^c = X \setminus A$. That is, the ambient set X is assumed to be understood in our notation.

The following properties are relatively easy to see.

Proposition 1 (Distributive Laws). *Let (A_n) be a sequence of subsets of a set X . Let $B \subset X$. Then*

- (a) $\left(\bigcup_{n=1}^{\infty} A_n \right) \cap B = \bigcup_{n=1}^{\infty} (A_n \cap B);$
- (b) $\left(\bigcap_{n=1}^{\infty} A_n \right) \cup B = \bigcap_{n=1}^{\infty} (A_n \cup B).$

Proposition 2 (DeMorgan's Laws). *Let (A_n) be a sequence of subsets of a set X . Then*

- (a) $\left(\bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c;$
- (b) $\left(\bigcap_{n=1}^{\infty} A_n \right)^c = \bigcup_{n=1}^{\infty} A_n^c.$

Proposition 3. Let (A_n) be a sequence of functions from a set X . Let $f : X \rightarrow Y$. Then

- (a) $f\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f(A_n);$
- (b) $f\left(\bigcap_{n=1}^{\infty} A_n\right) \subset \bigcap_{n=1}^{\infty} f(A_n).$

Proof.

(a) (\subset) Let $y \in f(\bigcup_{n=1}^{\infty} A_n)$. Then $y = f(x)$ for some $x \in \bigcup_{n=1}^{\infty} A_n$. There exists $n \in \mathbb{N}$ such that $x \in A_n$, so $y \in f(A_n)$. Thus $y \in \bigcup_{n=1}^{\infty} f(A_n)$.

(a) (\supset) Let $y \in \bigcup_{n=1}^{\infty} f(A_n)$. Then $y \in f(A_n)$ for some $n \in \mathbb{N}$, so $y = f(x)$ for some $x \in A_n$. Now $x \in \bigcup_{n=1}^{\infty} A_n$, so $f(x) \in f(\bigcup_{n=1}^{\infty} A_n)$.

(b) (\subset) Let $y \in f(\bigcap_{n=1}^{\infty} A_n)$. Then $y = f(x)$ for some $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x \in A_n$ for every $n \in \mathbb{N}$, so $y = f(x) \in f(A_n)$ for every $n \in \mathbb{N}$. Thus $y \in \bigcap_{n=1}^{\infty} f(A_n)$. \square

Can you find an example where the inverse inclusion of (b) above does not hold?

Proposition 4. Let (A_n) be a sequence of functions from a set X . Let $g : Y \rightarrow X$. Then

- (a) $g^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} g^{-1}(A_n);$
- (b) $g^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} g^{-1}(A_n).$

Proof.

(a) (\subset) Let $x \in g^{-1}(\bigcup_{n=1}^{\infty} A_n)$, and let $y = g(x)$. Then $y \in \bigcup_{n=1}^{\infty} A_n$, so $y \in A_n$ for some $n \in \mathbb{N}$. Thus $x \in g^{-1}(A_n)$, so $x \in \bigcup_{n=1}^{\infty} g^{-1}(A_n)$.

(a) (\supset) Let $x \in \bigcup_{n=1}^{\infty} g^{-1}(A_n)$, and let $y = g(x)$. Then $x \in g^{-1}(A_n)$ for some $n \in \mathbb{N}$, so $y \in A_n$. Then $y \in \bigcup_{n=1}^{\infty} A_n$, so $x \in g^{-1}(\bigcup_{n=1}^{\infty} A_n)$.

(b) (\subset) Let $x \in g^{-1}(\bigcap_{n=1}^{\infty} A_n)$, and let $y = g(x)$. Then $y \in \bigcap_{n=1}^{\infty} A_n$, so $y \in A_n$ for every $n \in \mathbb{N}$. Thus $x \in g^{-1}(A_n)$ for every $n \in \mathbb{N}$, so $x \in \bigcap_{n=1}^{\infty} g^{-1}(A_n)$.

(b) (\supset) Let $x \in \bigcap_{n=1}^{\infty} g^{-1}(A_n)$, and let $y = g(x)$. Then $x \in g^{-1}(A_n)$ for every $n \in \mathbb{N}$, so $y \in A_n$ for every $n \in \mathbb{N}$. Then $y \in \bigcap_{n=1}^{\infty} A_n$, so $x \in g^{-1}(\bigcap_{n=1}^{\infty} A_n)$. \square

2. MONOTONE SEQUENCES

Definition 2. Let (A_n) be a sequence of subsets of a set X .

We say that (A_n) is *increasing* (or *nondecreasing*, or *expanding*) if $A_k \subset A_{k+1}$, for all $k \in \mathbb{N}$.

We say that (A_n) is *decreasing* (or *nonincreasing*, or *contracting*) if $A_k \supset A_{k+1}$, for all $k \in \mathbb{N}$.

We say that (A_n) is *monotone* if it is either increasing or decreasing.

Problem 1. Let (A_n) be a sequence of subsets of a set X .

- (a) Show that if (A_n) is increasing, then $\bigcap_{n=k}^{\infty} A_n = A_k$.
- (b) Show that if (A_n) is decreasing, then $\bigcup_{n=k}^{\infty} A_n = A_k$.
- (c) Show that if (A_n) is decreasing if and only if (A_n^c) is increasing sequence.

Problem 2. Let (A_n) be a sequence of subsets of a set X . Define two new sequences of sets,

- $\underline{A}_n = \bigcap_{j=n}^{\infty} A_j$.
- $\overline{A}_n = \bigcup_{j=n}^{\infty} A_j$.
- (a) Show that (\underline{A}_n) is an increasing sequence of sets.
- (b) Show that (\overline{A}_n) is a decreasing sequence of sets.

Problem 3. Let (A_n) be a sequence of subsets of a set X .

- (a) Show that, for all $n \in \mathbb{N}$, we have

$$\underline{A}_n \subset A_n \subset \overline{A}_n.$$

- (b) Find a sequence of sets (A_n) such that
 - $A_i \neq A_j$ for $i \neq j$, and
 - $\underline{A}_i \subsetneq A_i \subsetneq \overline{A}_i$.

3. LIMITS OF SEQUENCES OF SETS

Definition 3. Let (A_n) be a sequence of subsets of a set X .

The *limit inferior* of (A_n) is

$$\liminf A_n = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j.$$

The *limit superior* of (A_n) is

$$\limsup A_n = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j.$$

An alternative notation is used by some books: let $\underline{\lim} A_n = \liminf A_n$ and $\overline{\lim} A_n = \limsup A_n$. We may call $\underline{\lim} A_n$ the *lower limit* and $\overline{\lim} A_n$ the *upper limit*.

Problem 4. Let (A_n) be a sequence of subsets of a set X .

- (a) Show that $\underline{\lim} A_n = \lim \underline{A}_n$.
- (b) Show that $\overline{\lim} A_n = \lim \overline{A}_n$.

Proposition 5. Let (A_n) be a sequence of subsets of a set X . Show that

- (a) $\liminf A_n = \{x \in X \mid x \in A_n \text{ for all but finitely many } n \in \mathbb{N}\};$
- (b) $\limsup A_n = \{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\};$
- (c) $\liminf A_n \subset \limsup A_n$.

Proof.

(a) (\subset) Suppose that $x \in A_n$ for all but finitely many n . Then, let $N \in \mathbb{N}$ be so large that $x \in A_n$ for $n \geq N$. Then $x \in \bigcap_{j=N}^{\infty} A_j = \underline{A}_N$, so $x \in \bigcup_{i=1}^{\infty} \underline{A}_i = \liminf A_n$.

(a) (\supset) Suppose that $x \in \liminf A_n$. Then $x \in \bigcup_{i=1}^{\infty} \underline{A}_i$, so $x \in \underline{A}_i$ for some $i \in \mathbb{N}$. But $\underline{A}_i = \bigcap_{j=i}^{\infty} A_j$, so $x \in A_j$ for all $j \geq i$. Thus $x \in A_n$ for all but finitely many n .

(b) (\subset) Suppose that $x \in A_n$ for infinitely many n . Then for every $i \in \mathbb{N}$, there exists $n \geq i$ such that $n \geq i$ implies $x \in A_n$. Thus for every $i \in \mathbb{N}$, $x \in \bigcup_{j=i}^{\infty} A_i = \overline{A}_i$. Thus $x \in \bigcap_{i=1}^{\infty} \overline{A}_i = \limsup A_n$.

(b) (\supset) Suppose that $x \in \limsup A_n$. Then $x \in \bigcap_{i=1}^{\infty} \overline{A}_i$, so $x \in \overline{A}_i = \bigcup_{j=i}^{\infty} A_j$ for all i . Thus, for every $i \in \mathbb{N}$, there exists $n \geq i$ such that $x \in A_n$, which implies that $x \in A_n$ for infinitely many $n \in \mathbb{N}$.

(c) Let $x \in \liminf A_n$. Then $x \in A_n$ for all but finitely many $n \in \mathbb{N}$; since \mathbb{N} is infinitely, this implies that $x \in A_n$ for infinitely many $n \in \mathbb{N}$, so $x \in \limsup A_n$. \square

Problem 5. Let $X = \mathbb{R}$. Define a sequence (A_n) of subsets X by

$$A_n = \begin{cases} \left[0, \frac{1}{n}\right] & \text{if } n \text{ is odd;} \\ [0, n] & \text{if } n \text{ is even.} \end{cases}$$

Find $\liminf A_n$ and $\limsup A_n$.

Problem 6. Let $X = [0, 1] \subset \mathbb{R}$. Define a sequence (A_n) of subsets X by

$$A_n = \left\{ \frac{m}{n} \mid m \in \mathbb{Z} \text{ and } 0 \leq m \leq n \right\}.$$

Find $\liminf A_n$ and $\limsup A_n$.

Problem 7. Let $X = \mathbb{R}$. Define a sequence (A_n) of subsets X by

$$a_n = 4 \sin^2 \frac{2\pi n}{3} \text{ and } A_n = [a_n - 1, a_n + 1].$$

Find $\liminf A_n$ and $\limsup A_n$.

Problem 8. Let (A_n) be a sequence of subsets of a set X . Show that

$$\liminf A_n = (\limsup A_n^c)^c.$$

Definition 4. Let (A_n) be a sequence of subsets of a set X .

We say that (A_n) *converges* if $\liminf A_n = \limsup A_n$. In this case, the *limit* of (A_n) is

$$\lim A_n = \liminf A_n = \limsup A_n.$$

If we claim that $\lim A_n = L$, we mean that (A_n) converges, and that the limit of (A_n) is L .

Problem 9. Let (A_n) be a sequence of subsets of a set X .

(a) Show that if (A_n) is decreasing, then $\lim A_n = \bigcap_{i=1}^{\infty} A_i$.

(b) Show that if (A_n) is increasing, then $\lim A_n = \bigcup_{i=1}^{\infty} A_i$.

Problem 10. Let (A_n) and (B_n) be sequences of subsets of a set X . Show that

$$(\liminf A_n \cup \liminf B_n) \subset \liminf (A_n \cup B_n) \subset (\liminf A_n \cup \overline{\liminf B_n}) \subset \overline{\liminf (A_n \cup B_n)} \subset (\overline{\liminf A_n} \cup \overline{\liminf B_n}).$$

4. SIGMA ALGEBRAS

Definition 5. Let X be a set and let $\mathcal{A} \subset \mathcal{P}(X)$. We say that \mathcal{A} is an *algebra* of subsets of X if

- (A0) $X \in \mathcal{A}$;
- (A1) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$;
- (A2) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$, where $A^c = X \setminus A$.

Proposition 6. Let \mathcal{A} be an algebra of subsets of X . Then

- (A3) $A, B \in \mathcal{A}$ implies $A \cap B \in \mathcal{A}$.

Proof. Let $A, B \in \mathcal{A}$. Then $A^c, B^c \in \mathcal{A}$ by (A2), and $A^c \cup B^c \in \mathcal{A}$ by (A1). The by DeMorgan's Law and (A2) again,

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{A}.$$

□

Proposition 7. Let \mathfrak{A} be a collection of algebras of subsets of a set X . Then $\cap \mathfrak{A}$ is an algebra of subsets of X .

Proof. Let $A, B \in \cap \mathfrak{A}$. Then $A, B \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{A}$. Since each \mathcal{A} in \mathfrak{A} is an algebra, $A \cup B$ and A^c are in \mathcal{A} , for every \mathcal{A} in \mathfrak{A} . So, $A \cup B$ and A^c are in $\cap \mathfrak{A}$. □

Definition 6. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. The *algebra generated by \mathcal{C}* is

$$\langle \mathcal{C} \rangle = \cap \{ \mathcal{A} \subset \mathcal{P}(X) \mid \mathcal{A} \text{ is an algebra which contains } \mathcal{C} \}.$$

One sees that the algebra generated by \mathcal{C} is the smallest algebra which contains all the sets in \mathcal{C} .

Proposition 8. Let \mathcal{A} be an algebra of subsets of X , and let (A_n) be a sequence of sets in \mathcal{A} . Then there exists a sequence (B_n) of sets in \mathcal{A} such that $B_j \cap B_k = \emptyset$ if $j \neq k$, and

$$\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i.$$

Proof. Define

$$B_n = A_n \setminus \left(\cup_{i=1}^{n-1} A_i \right).$$

Since $B_n \subset A_n$, it is clear that

$$\cup_{i=1}^{\infty} B_i \subset \cup_{i=1}^{\infty} A_i.$$

Let $x \in \cup_{i=1}^{\infty} A_i$. Then $x \in A_i$ for some i ; let n denote the smallest positive integer such that $x \in A_n$. Then $x \in A_n \setminus (\cup_{i=1}^{n-1} A_i)$, so $x \in B_n$. Thus $x \in \cup_{i=1}^{\infty} B_i$, so

$$\cup_{i=1}^{\infty} A_i \subset \cup_{i=1}^{\infty} B_i,$$

which implies that

$$\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i.$$

Suppose that $x \in B_j \cap B_k$ for some $j < k$; then $x \in B_j$, so $x \in A_j$. But then $x \in \cup_{i=1}^{j-1} A_i$, so $x \notin A_k \setminus (\cup_{i=1}^{k-1} A_i) = B_k$, a contradiction. Thus $B_j \cap B_k = \emptyset$. □

Definition 7. Let X be a set and let $\mathcal{A} \subset \mathcal{P}(X)$. We say that \mathcal{A} is a σ -algebra of subsets of X if

- (S1) if $\mathcal{C} \subset \mathcal{A}$ is countable, then $\cup \mathcal{C} \in \mathcal{A}$;
- (S2) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

That is, a σ -algebra is an algebra which is not only closed under finite unions, but is also closed under countable unions.

Proposition 9. Let \mathcal{A} be a σ -algebra of subsets of X . Then

- (A3) if $\mathcal{C} \subset \mathcal{A}$ is countable, then $\cap \mathcal{C} \in \mathcal{A}$.

Proof. DeMorgan's Law also applies to infinite collections; let $\mathcal{C} \subset \mathcal{A}$ be countable. Then

$$\cap \mathcal{C} = \cap_{A \in \mathcal{C}} A = (\cup_{A \in \mathcal{C}} A^c)^c.$$

Now if $A \in \mathcal{C}$, then $A \in \mathcal{A}$, so $A^c \in \mathcal{A}$. Thus $\cup_{A \in \mathcal{C}} A^c$ is a countable union of sets in \mathcal{A} , and so is in \mathcal{A} . Thus its complement $\cap \mathcal{C}$ is in \mathcal{A} . \square

Proposition 10. Let \mathfrak{A} be a collection of σ -algebras of subsets of a set X . Then $\cap \mathfrak{A}$ is a σ -algebra of subsets of X .

Proof. Exercise. \square

Definition 8. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. The σ -algebra generated by \mathcal{C} , denoted $\langle \mathcal{C} \rangle$, is the intersection of all σ -algebras which contain \mathcal{C} .

We see that $\langle \mathcal{C} \rangle$ is necessarily a σ -algebra, and is the smallest σ -algebra which contains all of the sets in the collection \mathcal{C} .

Proposition 11. Let \mathcal{A} be a σ -algebra of subsets of a set X . Let (A_n) be a sequence in \mathcal{A} . Then

- (a) $\underline{A}_n, \overline{A}_n \in \mathcal{A}$;
- (b) $\liminf A_n, \limsup A_n \in \mathcal{A}$.

Proof. Since $\underline{A}_n = \cup_{i=n}^{\infty} A_i$ is a union of countable collection from \mathcal{A} , we know that $\underline{A}_n \in \mathcal{A}$. Also, $\overline{A}_n = \cap_{i=n}^{\infty} A_i^c$ is the intersection of a countable collection, so $\overline{A}_n \in \mathcal{A}$.

Now $\liminf A_n = \cup_{i=1}^{\infty} \underline{A}_i$, so $\liminf A_n$ is a countable union of sets in \mathcal{A} , so $\liminf A_n \in \mathcal{A}$. Similarly, $\limsup A_n = \cap_{i=1}^{\infty} \overline{A}_i$, so $\limsup A_n$ is a countable intersection of sets in \mathcal{A} , and so is in \mathcal{A} . \square