Problem 1. Let X be a Hausdorff space and let $K \subset X$ be compact. Let $y \in X \smallsetminus K$. Show that there exist disjoint open sets $U, V \subset X$ such that $y \in U$ and $K \subset V$.

Solution. Since X is Hausdorff, for every $k \in K$ there exist disjoint open sets U_k and V_k such that $y \in U_k$ and $k \in V_k$. Then $\{V_k \mid k \in K\}$ is an open cover of K, and since K is compact, there exist $k_1, \dots, k_r \in K$ such that $\{V_{k_i} \mid i = 1, \dots, r\}$ is a finite subcover. Let $U = \cap_{i=1}^r U_{k_i}$ and $V = \cup_{i=1}^r V_{k_i}$. Then $y \in U$ and $K \subset V$, and by DeMorgan's Laws, U and V are disjoint. Since V is the union of open sets, V is open, and since U is the intersection of finitely many open sets, U is also open. □

Problem 2. Let X be a Hausdorff space and let $K_1, K_2 \subset X$ be compact. Show that there exist disjoint open sets $U_1, U_2 \subset X$ such that $K_1 \subset U_1$ and $K_2 \subset U_2$.

Solution. By Problem 1, for every $y \in K_1$ there exist disjoint open sets U_y and V_y such that $y \in U_y$ and $K_2 \subset V_y$. Then $\{V_y \mid y \in K_1\}$ is an open cover of K_1, and since K_1 is compact, there exist $y_1, \dots, y_r \in K_1$ such that $\{V_{y_i} \mid i = 1, \dots, r\}$ is a finite subcover. Let $U_1 = \cup_{i=1}^r U_{y_i}$ and $U_2 = \cap_{i=1}^r V_{y_i}$. Then $K_1 \subset U_1$, $K_2 \subset U_2$, and by DeMorgan's Laws, U_1 and U_2 are disjoint. Since U_1 is the union of open sets, U_1 is open, and since U_2 is the intersection of finitely many open sets, U_2 is also open. □

Problem 3. Let $\mathbb{R}^\infty$ denote the set of all sequences of real numbers which are eventually zero, that is, sequences $\vec{x} = (x_n)$ such that $x_n = 0$ for all but finitely many n. Let $X = \mathbb{R}^\infty$ and for $\vec{x}, \vec{y} \in X$, define

$$d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

where $\vec{x} = (x_n)$ and $\vec{y} = (y_n)$. This make sense without considering convergence, since there are only finitely many nonzero summands. Then (X, d) is a metric space. Let $|\vec{x}| = d(\vec{x}, \vec{0})$. Show that

$$D = \{\vec{x} \in \mathbb{R}^\infty \mid |\vec{x}| \leq 1\}$$

is closed and bounded but not compact.

Solution. Clearly, D is bounded. If $\vec{x} \notin D$, then $r = |\vec{x}| > 1$. Let $s = r - 1$; then the triangle inequality shows that $B_s(\vec{x})$ is contained in the complement of D, which shows that D^c is open, so D is closed.

Let $\vec{e}_i$ denote the sequence which equals 1 in the i^{th} slot, and equals 0 elsewhere. Let $E = \{\vec{e}_i \mid i = 1, \dots, \infty\}$. Let $U = \mathbb{R}^\infty \smallsetminus E$. Then U is open, since every point in U is an interior point.

Let B_i denote the open ball of radius $1/2$ about $\vec{e}_i$. Let $\mathcal{C} = \{B_i \mid i = 1, \dots, \infty\} \cup \{U\}$. Then $\mathcal{C}$ is an open cover of D which has no finite subcover, which demonstrates that D is not compact. □

Problem 4. Use our results regarding compactness and continuity to prove the Intermediate Value Theorem:

Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) < y_0 < f(b)$. Then there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$.

Proof. Since $[a, b]$ is a compact and connected set, its image $f([a, b])$ is also compact and connected. A compact connected subset of $\mathbb{R}$ is a closed interval $[p, q]$, where by convention we allow $p = q$, so that $[p, p] = \{p\}$. Thus $p \leq f(a) < y_0 < f(b) \leq q$, so $y_0 \in [p, q] = f([a, b])$. Thus $y_0 = f(x_0)$ for some $x_0 \in [a, b]$. □

Problem 5. Use our results regarding compactness and continuity to prove the Extreme Value Theorem:

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there $c, d \in [a, b]$ such that f has a global minimum at c and f has a global maximum at d.

Solution. Since $[a, b]$ is a compact and connected set, its image $f([a, b])$ is also compact and connected. A compact connected subset of $\mathbb{R}$ is a closed interval $[p, q]$, where by convention we allow $p = q$, so that $[p, p] = \{p\}$. Thus $f(c) = p$ and $f(d) = q$, for some $c, d \in [a, b]$. Clearly, $p \leq f(x)$ and $q \leq f(x)$, for every $x \in [a, b]$, so we have a global minimum and maximum. □

Definition 1. Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be continuous.

We say that f is *continuous at* $x_0 \in X$ if

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x_2 \in X\ :\ |x_2 - x_0| < \delta\ \Rightarrow\ |f(x_2) - f(x_0)| < \epsilon.$$

We say that f is *continuous on* X if f is continuous at x_1 for every $x_1 \in X$.

We say that f is *uniformly continuous on* X if

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x_1, x_2 \in X\ :\ |x_2 - x_1| < \delta\ \Rightarrow\ |f(x_2) - f(x_1)| < \epsilon.$$

Thus, the difference between the concepts of general continuity versus uniform continuity is that for general continuity, the δ may depend on the point x_0, but for uniform continuity, the same δ works throughout X. Clearly, if f is uniformly continuous, then f is generally continuous.

Problem 6. Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be continuous. Show that if X is compact, then f is uniformly continuous.

Solution. Suppose that X is compact. Let $\epsilon > 0$. Since f is continuous, for every $a \in X$, there exists δ_a such that $|f(x) - f(a)| < \frac{\epsilon}{2}$ whenever $x \in X$ and $|x - a| < \delta_a$.

It is clear that the collection $\{B_{\delta_a/2}(a) \mid a \in X\}$ is an open cover of X, and since X is compact, there is a finite subset $A \subset X$ such that the collection $\mathcal{A} = \{B_{\delta_a/2}(a) \mid a \in A\}$ covers X. Let $\delta = \min\{\frac{\delta_a}{2} \mid a \in A\}$; since A is finite, we know that $\delta > 0$.

Let $x_1, x_2 \in X$ such that $|x_1 - x_2| < \delta$. Since $\mathcal{A}$ covers X, there exists $a \in A$ such that $x_1 \in B_{\delta_a/2}(a)$, so $|x_1 - a| < \delta_a/2$. By the triangle inequality, and since $\delta \leq \delta_a/2$, we have

$$|x_2 - a| \leq |x_1 - x_2| + |x_1 - a| < \delta + \delta_a/2 \leq \delta_a.$$

Thus $|x_1 - a| < \delta_a$, and $|x_2 - a| < \delta_a$, so

$$|f(x_1) - f(x_2)| \leq |f(x_1) - a| + |f(x_2) - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Problem 7. Let X and Y be sets, and let $f : X \to Y$. In each case, determine whether the definition of uniform continuity can be extended to the given circumstance.

(a) $X, Y \subset \mathbb{R}$

(b) $X, Y \subset \mathbb{R}^2$

(c) $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$

(d) X and Y are metric spaces

(e) X is a metric space and Y is a topological space

(f) X is a topological space and Y is a metric space

(g) X and Y are topological spaces