PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §1

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Remark 1. When using induction in a proof, it is perfectly okay to check the $n = 1$ case and then prove that $p_{n-1} \Rightarrow p_n$ for $n \geq 2$. I prefer this approach, because the proof seems to end with more of a "bang": the final conclusion is a statement of the proposition p_n as it was originally stated, instead of concluding with p_{n+1} and then noting that you are done.

Exercise 1 (1.1)**.** Show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Note that $n(n+1)(2n+1) = 2n^3 + 3n^2 + n$.

For $n = 1$, we have $1 = \frac{1}{6}(2+3+1) = \frac{1}{6}(2n^3 + 3n^2 + n)$.

Now assume that $n \geq 2$ and that the proposition is true for $n - 1$, and use this to prove that it is true for n. That is, assume that

$$\sum_{i=1}^{n-1} i^2 = \frac{(n-1)(n)(2n-1)}{6}.$$

Add n^2 to both sides and simplify the right hand side to get

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \frac{(n-1)(n)(2n-1)}{6} + n^2 \\
&= \frac{2n^3 - 4n^2 + n}{6} + \frac{6n^2}{6} \\
&= \frac{2n^3 + 2n^2 + n}{6} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

□

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Exercise 2 (1.2)**.** Show that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in \mathbb{N}$.

Proof. If $n = 1$, the left hand side is 3 and the right hand side is $4 - 1 = 3$.

By induction, we have

$$\sum_{i=1}^{n-1}(8i - 5) = 4(n-1)^2 - (n-1).$$

The left hand side simplifies to $4n^2 - 9n + 5$. Now add $8n - 5$ to both sides and get

$$\sum_{i=1}^{n}(8i - 5) = 4n^2 - n.$$

□

Exercise 3 (1.3)**.** Show that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$.

Proof. Recall that for any n, we have $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

By induction, $\sum_{i=1}^{n-1} i^3 = (\sum_{i=1}^{n-1} i)^2$, which is equal to $\frac{(n-1)^2 n^2}{4}$. Thus

$$\sum_{i=1}^{n-1} i^3 = \frac{n^4 - 2n^3 + n^2}{4}.$$

Adding n^3 to both sides yields

$$\begin{aligned}\sum_{i=1}^{n} i^3 &= \frac{n^4 - 2n^3 + n^2}{4} + n^3 \\ &= \frac{n^4 + 2n^3 + n^2}{4} \\ &= \frac{n^2(n+1)^2}{2^2} \\ &= \Big(\sum_{i=1}^{n} i\Big)^2.\end{aligned}$$

□

Exercise 4 (1.4(a))**.** Guess a formula for $1 + 3 + \cdots + (2n - 1)$.

Guess. Let $a_n = 1 + 3 + \cdots + (2n - 1)$. Then $a_1 = 1$, $a_2 = 4$, and $a_3 = 9$. These are the first three perfect squares. So our guess is that $a_n = n^2$. Testing this guess on $n = 4$, we see that $a_4 = 1 + 3 + 5 + 7 = 16$. Seems good. □

Exercise 5 (1.1.4(b))**.** Show that $1 + 3 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$.

Proof. This is immediate for $n = 1$.

By induction, we have $1 + 3 + \cdots + (2(n-1) - 1) = (n-1)^2$. That is, $1 + 3 + \cdots + (2n - 3) = n^2 - 2n + 1$. Adding $2n - 1$ to both sides gives us

$$1 + 3 + \cdots + (2n - 1) = n^2 - 2n + 1 + (2n - 1) = n^2.$$

□

Exercise 6 (1.1.6)**.** Show that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$.

Discussion. Comparing this to an example 2 in the book, we guess a generalization: that $x^n - y^n$ is divisible by $(x-y)$, where x and y are integers. This causes us to recall a familiar formula $\frac{1-x^n}{1-x} = \sum_{i=1}^{n} x^{i-1}$. If you recall, this is the basis for the geometric series: if $|x| < 1$ and we let n get large, since $\lim_{n\to\infty} x^n = 0$ we have

$$\frac{1}{1-x} = \sum_{i=1}^{\infty} x^{i-1}.$$

Now we try to guess the value of $\frac{x^n-y^n}{x-y}$. Using long division, we see that the first few terms are x^{n-1}, $x^{n-2}y$, $x^{n-3}y^2$, and so forth. Thus we guess that

$$\frac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}.$$

Convert this into summation notation and conjecture that

$$\frac{x^n - y^n}{x-y} = \sum_{i=1}^{n} x^{i-1}y^{n-i}.$$

This formula hasn't been proven. In fact, whenever one is reading mathematics and sees "dot dot dot", one is alerted to the fact that some sort of induction is being performed. Now we prove the following generalization, from which the exercise follows immediately. □

Proposition 1. *Let $x, y \in \mathbb{Z}$. Then*

$$x^n - y^n = (x-y)\sum_{i=1}^{n} x^{i-1}y^{n-i}.$$

Proof. For $n = 1$, this is obvious.

By induction, we have

$$x^{n-1} - y^{n-1} = (x-y)\sum_{i=1}^{n-1} x^{i-1}y^{n-1-i}.$$

Multiplying both sides by y, we get

$$x^{n-1}y - y^n = (x-y)\sum_{i=1}^{n-1} x^{i-1}y^{n-i}.$$

Now add $(x-y)x^{n-1}$ to both sides and arrive at

$$x^{n-1}y - y^n + x^n - yx^{n-1} = (x-y)\sum_{i=1}^{n-1} x^{i-1}y^{n-i} + (x-y)x^{n-1}.$$

This simplifies to our conclusion

$$x^n - y^n = (x-y)\sum_{i=1}^{n} x^{i-1}y^{n-i}.$$

□

Exercise 7 (1.7)**.** Show that $7^n - 6n - 1$ is divisible by 36 for all $n \in \mathbb{N}$.

Proof. Let $a_n = 7^n - 6n - 1$. Then $a_1 = 0$, which is divisible by 36.

By induction, a_{n-1} is divisible by 36; that is, there exists $k \in \mathbb{Z}$ such that $a_{n-1} = 36k$. Now $a_{n-1} = 7^{n-1} - 6(n-1) - 1 = 7^{n-1} - 6n + 5$. Thus $7^n - 42n + 35 = 7 \cdot 36k$; adding $36n + 36$ to both sides shows that $a_n = 7^n - 6n - 1 = 7 \cdot 36k + 36n + 36 = 36(7k + n + 1)$. Thus a_n is divisible by 36. □

Exercise 8 (1.9(a))**.** Decide for which integers the inequality $2^n > n^2$ is true.

Guess. Let $a_n = 2^n - n^2$. Then $a_0 = 1$, $a_1 = 1$, $a_2 = 0$, $a_3 = -1$, $a_4 = 0$, and $a_5 = 7$. Also $a_{-1} = -\frac{1}{2}$. So we conjecture that $a_n > 0$ for $n = 0, 1, 5, 6, 7, \dots$. □

Exercise 9 (1.9(b))**.** Show that $2^n > n^2$ for all $n \in \mathbb{N}$, $n \geq 5$.

Lemma 1. *Let $n \in \mathbb{N}$. If $n \geq 4$, then $n^2 > 4n - 2$.*

Proof of Lemma. For $n = 4$, we have $16 > 14$.

Let $n > 4$. By induction, $(n-1)^2 > 4(n-1) - 2$. That is, $n^2 - 2n + 1 > 4n - 6$. Adding $2n - 1$ to both sides gives

$$\begin{aligned} n^2 &> 6n - 7 \\ &= 4n + 2n - 7 \\ &> 4n + 8 - 7 \text{ because } n > 2 \\ &> 4n - 2. \end{aligned}$$

□

Proof of Exercise. This is true for $n = 5$.

Suppose that $n > 5$. By induction, $2^{n-1} > (n-1)^2$. Multiplying both sides by 2 and apply the lemma to get

$$\begin{aligned} 2^n &> 2(n-1)^2 \\ &= n^2 + n^2 - 4n + 2 \\ &> n^2 + (4n - 2) - 4n + 2 \text{ by the lemma} \\ &= n^2. \end{aligned}$$

□

Exercise 10 (1.10)**.** Show that $(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1) = 3n^2$ for all $n \in \mathbb{N}$.

Proof. The left hand side may be expressed as $\sum_{i=1}^n (2n + 2i - 1)$. Then by associativity and distributivity, and using the fact that $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$, we have

$$\begin{aligned} \sum_{i=1}^n (2n + 2i - 1) &= \sum_{i=1}^n (2n - 1) + 2\sum_{i=1}^n i \\ &= n(2n - 1) + n(n+1) \\ &= 3n^2. \end{aligned}$$

□

Exercise 11 (1.129(b))**.** Let $n, k \in \mathbb{N}$. Show that if $k \leq n$, then

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Proof. We have

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n!(n-k+1) + n!k}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

□

Exercise 12 (1.12(c))**. Binomial Theorem**
Show that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Proof. For $n = 0$, this is immediate.

By induction, we have

$$(a+b)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-i-1} b^i.$$

Multiplying both sides by $a + b$ yields

$$\begin{aligned}
(a+b)^n &= \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-i} b^i + \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-i-1} b^{i+1} \\
&= a^n + \sum_{i=1}^{n-1} \binom{n-1}{i} a^{n-i} b^i + \sum_{j=1}^{n-1} \binom{n-1}{j-1} a^{n-j} b^j + b^n \text{ (letting } j = i+1) \\
&= a^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] a^{n-k} b^k + b^n \text{ (letting } k = i = j) \\
&= a^n + \sum_{k=1}^{n-1} \binom{n}{k} a^{n-k} b^k + b^n \text{ (by part (b))} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.
\end{aligned}$$

□

Problem 1. The following statements are true for all $n \in \mathbb{N}$, as can be proved by induction:

- $\sum_{i=1}^{n}(2i-1) = n^2$;
- $\sum_{i=1}^{n}(4i-3) = 2n^2 - n$;
- $\sum_{i=1}^{n}(6i-5) = 3n^2 - 2n$.

(a) State a conjectured generalization of this pattern.
(b) Prove your conjecture.

Proposition 2. *Let* $m, n \in \mathbb{N}$. *Then* $\sum_{i=1}^{n}(2mi - 2m + 1) = mn^2 - mn + n$.

Proof. For $n = 1$, we have $6 - 5 = 1 = 3 - 2$.

By induction, we have $\sum_{i=1}^{n-1}(2mi - 2m + 1) = m(n-1)^2 - m(n-1) + (n-1)$. Simplifying yields

$$\sum_{i=1}^{n-1}(2mi - 2m + 1) = mn^2 - 3mn + 2m + n - 1.$$

Adding $2mn - 2m + 1$ to both sides gives

$$\sum_{i=1}^{n}(2mi - 2m + 1) = mn^2 - mn + n.$$

□

Proposition 3. *Let* $a, b \in \mathbb{Z}$ *and* $n \in \mathbb{N}$. *Then* $\sum_{i=1}^{n}(2ai - b) = an^2 - (b-a)n$.

Proof. Behold:

$$\begin{aligned}\sum_{i=1}^{n}(2ai - b) &= 2a\sum_{i=1}^{n} i - \sum_{i=1}^{n} b \\ &= 2a(\frac{n(n+1)}{2}) - bn \\ &= an^2 + an - bn \\ &= an^2 - (b-a)n.\end{aligned}$$

□

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