Vector Calculus
Examination 2 Preview - Solutions

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Problem 2. (Ellipses)
The locus of the equation

$$4x^2 + 24x + 9y^2 - 36y + 36 = 0.$$

is an ellipse. Find its center, vertices, and foci.

Solution. Complete the square to arrive at

$$\frac{(x+3)^2}{9} + \frac{(y-2)^2}{4} = 1.$$

Now read off $a = 3$ and $b = 2$, so $c = \sqrt{a^2 - b^2} = \sqrt{5}$.

The axis is horizontal.
The center is $(h, k) = (-3, 2)$.
The vertices are $(h \pm a, k) = (-3 \pm 3, 2)$.
The covertices are $(h, k \pm b) = (-3, 2 \pm 2)$.
The foci are $(h \pm c, k) = (-3 \pm \sqrt{5}, 2)$. □

Problem 3. (Surfaces) Find an equation in three variables x, y, and z, whose locus in $\mathbb{R}^3$ is the following.

(a) A point.

(b) A line.

(c) A plane.

(d) The union of two planes.

(e) A hyperbolic cylinder.

(f) An elliptic paraboloid.

(g) A hyperbolic paraboloid.

(h) A cone.

(i) A one-sheeted hyperboloid.

(j) A two-sheeted hyperboloid.

Solution. There are many possible answers, these are some.

(a) A point: $x^2 + y^2 + z^2 = 0$ is the origin

(b) A line: $x^2 + y^2 = 0$ is the z-axis

(c) A plane: $z = 0$ is the xy-plane

(d) The union of two planes: $xy = 0$ is the union of the yz-plane and the xz-plane

(e) A hyperbolic cylinder: $x^2 - y^2 = 0$ - z is free

(f) An elliptic paraboloid: $z = x^2 + y^2$

(g) A hyperbolic paraboloid: $z = x^2 - y^2$

(h) A cone: $z^2 = x^2 + y^2$

(i) A one-sheeted hyperboloid: $x^2 + y^2 - z^2 = 1$

(j) A two-sheeted hyperboloid: $z^2 - x^2 - y^2 = 1$

□

Problem 4. (Dot and Cross Product)
Let $A = (3, 8, -2)$, $B = (-7, 3, 9)$, and $C = (2, -2, 10)$. Let $\vec{v}$ be the vector from A to B, and let $\vec{w}$ be the vector from A to C.

(a) Compute $\vec{v}$ and $\vec{w}$.

(b) Compute the dot product $\vec{v} \cdot \vec{w}$.

(c) Compute the scalar projection $\text{proj}_{\vec{w}}\vec{v}$.

(d) Compute the cross product $\vec{v} \times \vec{w}$.

Answers. We compute:

(a) $\vec{v} = \langle -10, -5, 11 \rangle$, $\vec{w} = \langle -1, -10, 12 \rangle$

(b) $\vec{v} \cdot \vec{w} = 10 + 50 + 132 = 192$

(c) $\text{proj}_{\vec{w}}\vec{v} = \dfrac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \dfrac{192}{\sqrt{245}}$

(d) $\vec{v} \times \vec{w} = \langle 50, 109, 95 \rangle$

□

Problem 5. (Lines and Planes)
Compute the indicated value(s).

(a) Find the parametric equations of the line passing through the points $P(5, -2, 8)$ and $Q(2, 4, 5)$.

(b) Find the standard equation of a plane which contains the line from part **(a)** and passes through the point $R(7, -2, 1)$.

(c) Find the distance from the point $S(-3, 1, 5)$ to the plane from part **(b)**.

Solution. **(a)** The ingredients for a parametric line are a point on the line, and a direction vector for the line.

A point on the line is $P_0 = (5, -2, 8)$. A direction vector for the line is $Q - P = \langle -3, 6, -3 \rangle$. Any vector in this direction will work, so divide by -3 to get $\vec{v} = 1, -2, 1$. So the line is the image of

$$\boxed{\vec{r} : \mathbb{R} \to \mathbb{R}^3 \quad \text{given by} \quad \vec{r}(t) = P_0 + t\vec{v} = \langle 5 + t, -2 - 2t, 8 + t \rangle.}$$

(b) The ingredients for the general equation of a plane are a point on the plane, and a normal vector for the plane.

A point on the plane is $P_0 = (5, -2, 8)$. Another vector on the plane is $\vec{w} = R - P = \langle 2, 0, -7 \rangle$. The normal vector is perpendicular to $\vec{v}$ and $\vec{w}$, so we cross these:

$$\vec{n} = \vec{v} \times \vec{w} = \langle 14, 9, 4 \rangle.$$

So the equation of the plane is $\vec{n} \cdot (x, y, z) = \vec{n} \cdot P_0$, that is,

$$\boxed{14x + 9y + 4z = 84.}$$

(c) Let $\vec{x} = S - P = \langle -8, 3, -3 \rangle$. Project this onto the normal vector to get

$$\text{proj}_{\vec{n}}\vec{x} = \frac{\vec{n} \cdot \vec{x}}{|\vec{n}|} = \frac{-97}{\sqrt{293}};$$

the distance is the absolute value of this, so the distance is

$$\boxed{d = \frac{97}{\sqrt{293}}.}$$

□

Problem 6. (Intersecting Planes)
Let A be the plane given by $7x + 2y + z = 8$ and B be the plane given by $x + 2y + 7z = 8$.
Let $L = A \cap B$ be the line of intersection of A and B. Find the equation of the plane which is perpendicular to L and passes through the point P_0, expressed in the form $ax + by + cz = d$.

Solution. Find two points on the line of intersection: let $P_0 = (1, 0, 1)$ and $Q = (0, 4, 0)$.

Find a direction vector for the line: let $\vec{v} = P_0 - Q = \langle 1, -4, 1\rangle$.

A normal vector for the plane perpendicular to this line is the direction vector of the line; that is, the normal vector for the plane is $\vec{n} = \vec{v}$.

The equation of the plane is $\vec{n} \cdot (P - P_0) = 0$; since $\vec{n} \cdot P_0 = 1 - 0 + 1 = 2$, this simplifies to

$$\boxed{x - 4y + z = 2.}$$

□

Problem 7. (Paths in $\mathbb{R}^2$)
Let $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ be given by $\vec{r}(t) = \langle \sec t, \tan t\rangle$.

(a) Find the velocity vector for $\vec{r}(t)$.

(b) Find the speed at time t.

(c) Find the speed at the time $t = \dfrac{\pi}{3}$.

(d) The coordinate parametric equations for $\vec{r}$ are $x = \sec t$ and $y = \tan t$. Use this to show that the image of $\vec{r}$ lies on a hyperbola in $\mathbb{R}^2$, and sketch the image of $\vec{r}$.

Solution. The velocity vector is $\vec{v}(t) = \vec{r}\,'(t) = \langle \sec t \tan t, \sec^2 t\rangle$.

The speed is $\sqrt{\sec^2 t \tan^2 t + \sec^4 t} = \sec t\sqrt{2\sec^2 t - 1}$.

Since $\sec \dfrac{\pi}{3} = 2$, we see that the speed at time $t = \dfrac{\pi}{3}$ is $2\sqrt{2 \cdot 4 - 1} = 2\sqrt{7}$.

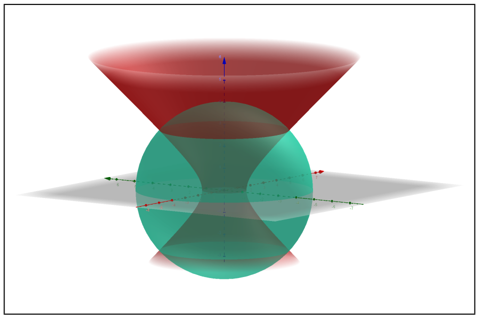
Note that $\sec^2 t = 1 + \tan^2 t$, so $\sec^2 t - \tan^2 t = 1$. For our path, $x = \sec t$ and $y = \tan t$, so $x^2 - y^2 = 1$. This is a hyperbola. □

Problem 9. (Intersecting Quadrics)
Let S be the sphere centered at the origin with radius 4. Let H be the hyperboloid with equation $x^2+y^2-z^2 = 1$. Let $C = S \cap H$; then C consists of two circles.

(a) Sketch the sphere, the hyperboloid, and their intersection in the same picture.

(b) Find the centers of the two circles.

Solution. The situation is shown below.



There centers of the circles are clearly on the z-axis, and so to find the centers, we need to find their z-coordinates. The equation of the sphere is $x^2 + y^2 + z^2 = 16$. Subtract the equation of the hyperboloid to get $2z^2 = 15$. So, $z = \pm\sqrt{\dfrac{15}{2}}$. Thus, the centers of the circles are

$$\boxed{(0, 0, \pm\sqrt{\frac{15}{2}}).}$$

$\square$

Problem 10. (Paths on Quadrics)
Consider a path given by

$$\vec{r} : \mathbb{R} \to \mathbb{R}^3 \quad \text{given by} \quad \vec{r} = \langle \sqrt{1+t^2}\cos t, \sqrt{1+t^2}\sin t, t \rangle.$$

(a) Show that $\dfrac{dz}{dt} = 1$.

(b) Show that the image of $\vec{r}$ is a subset of the one-sheeted hyperboloid with equation $x^2 + y^2 - z^2 = 1$.

(c) Sketch the image of $\vec{r}$.

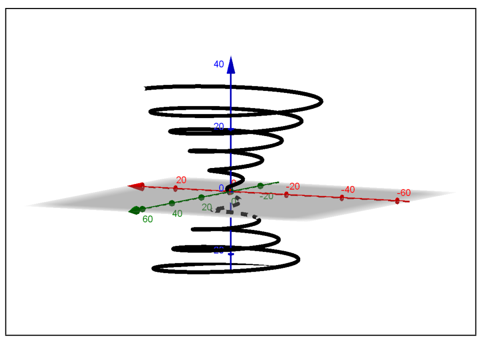
Solution. View the path as the trace of a particle in motion.

Since $z = t$, $\dfrac{dz}{dt} = 1$. This says that the particle rises at a constant rate.

Plug the coordinate functions of the path into the equation of the hyperboloid to see it they satisfy this equation at every time t. We get

$$(1+t^2)\cos^2 t + (1+t^2)\sin^2 t - t^2 = 1 = (1+t^2) - t^2 = 1.$$

To imagine the image of this path, realize that the radius is increasing with t.



$\square$

Problem 11. (Paths Intersect Quadrics)
Consider a path given by
$$\vec{s} : \mathbb{R} \to \mathbb{R}^3 \quad \text{given by} \quad \vec{s} = \langle 2t, 2t^2, t^3 \rangle,$$
and the one-sheeted hyperboloid with equation $x^2 - y^2 + z^2 = 1$. Find all times t when the path intersects the hyperboloid. Find a point where the path intersects the hyperboloid.

Solution. The coordinate parametric functions for $\vec{s}$ are $x = 2t$, $y = 2t^2$, and $z = t^3$. Plug these into the equation of the hyperboloid and solve for t. We get $(2t)^2 - (2t^2)^2 + (t^3)^2 = 1$, which may be rearranged to $t^6 - 4t^4 + 4t^2 - 1 = 0$. This is a cubic polynomial in t^2; that is, if we let $x = t^2$, our equation because
$$x^3 - 4x^2 + 4x - 1 = 0.$$
We see that $x = 1$ is a solution, so we use synthetic division to factor the polynomial. We find that $x^3 - 4x^2 + 4x - 1 = (x-1)(x^2 - 3x + 1)$. Use that quadratic formula to find that $x = 1$ or $x = \dfrac{3 \pm \sqrt{5}}{2}$. Since $x = t^2$, we have
$$\boxed{t = \pm 1 \quad \text{or} \quad t = \pm\sqrt{\frac{3 \pm \sqrt{5}}{2}}.}$$
An alternate solution method involves rewriting the equation thusly:
$$t^2(t^2-2)^2 - 1 = 0 \Rightarrow (t(t^2-2)-1)(t(t^2-2)+1) = 0 \Rightarrow (t+1)(t^2-t-1)(t-1)(t^2+t-1) = 0.$$
Here the solutions are
$$\boxed{t = \pm 1 \quad \text{or} \quad t = \frac{\pm 1 \pm \sqrt{5}}{2}.}$$
Are these the same? Or is there some error in our computation?

There are six times when the path intersects the hyperboloid. Plug in one of them, say $t = 1$, to find a point of intersection.
$$\boxed{\vec{s}(1) = \langle 2, 2, 1 \rangle.}$$
□

Problem 12. (Hyperboloids) [*Challenge*]
Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

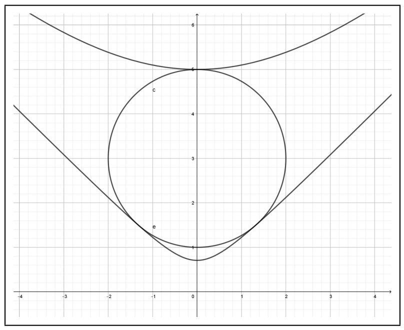
$$f(x, y, z) = x^2 + y^2 - z^2.$$

For $t \in \mathbb{R}$, the preimage of t is

$$f^{-1}(t) = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = t\}.$$

The preimage of t is a surface in $\mathbb{R}^3$. Find t such that $f^{-1}(t)$ is tangent to the sphere with equation $x^2 + y^2 + (z - 3)^2 = 4$.

Solution. If the sphere intersects the cone $x^2 + y^2 - z^2 = 0$, then the sphere will be tangent to a one-sheeted hyperboloid. However, one computes that the distance from the center of the sphere to the cone is $\frac{3}{2}\sqrt{2} > 2$, so the sphere is actually above the cone. Thus, it is tangent to two distinct two-sheeted hyperboloids. This is shown in yz-plane below.



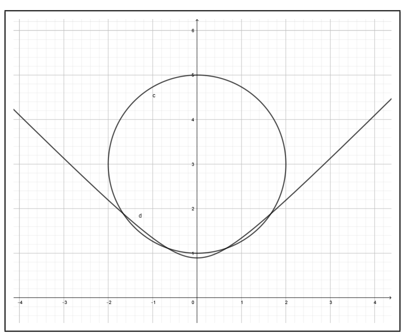
To find the value of t which produces these tangents, we subtract the equation of the hyperboloid from the equation of the sphere to arrive at

$$2z^2 - 6z + (5 + t).$$

Now it is clear the point of tangency of the higher hyperboloid occurs at $(0, 0, 5)$, so we plug $z = 5$ into the equation above and arrive at $50 - 30 + (5 + t)$, so

$$\boxed{t = -25.}$$

The second instance of tangency is more subtle to find. We view t as increasing from -25 towards 0. As this occurs, the hyperbola in the yz-plane will intersect the circle in two distinct z-values, as shown below.



We know that a quadratic equation has two solutions if the discriminant is positive, no solutions if the discriminant is negative, and a unique solution when the discriminant is zero. It is this unique solution we seek.

In our quadratic above, we have $a = 2$, $b = -6$, and $c = 5 + t$, so the discriminant in our case is $b^2 - 4ac = 36 - 8(5 + t)$. Set this to zero and solve to find that

$$\boxed{t = -\frac{1}{2}.}$$

$\square$