REAL ANALYSIS
TOPIC 33 - THE BAIRE CATEGORY THEOREM

PAUL L. BAILEY

Abstract. We have previously explored various aspects of the real number system which relate to their algebra and their topology. We now wish to move towards understanding their measure, and we begin by trying to see more deeply into their underlying structure by examining some unusual subsets.

1. Open and Closed Sets

1.1. **Review of Basic Definitions.** Let's start at the very beginning, a very good place to start ... recall the definitions.

If $A \subset \mathbb{R}$, the *complement* of A is denoted A^c; that is, $A^c = \mathbb{R} \smallsetminus A$.

We say that $G \subset \mathbb{R}$ is *open* if, for every $x \in G$, there exist $\delta > 0$ such that $(x-\delta, x+\delta) \subset G$.

We say that $F \subset \mathbb{R}$ is *closed* if F^c is open.

We have seen that a set is closed if and only it is equal to its closure, and that a set is open if and only if it is equal to its interior. We have also seen that the closure of a set is the intersection of all closed sets which contain it, and that the interior of a set is the union of all open sets it contains.

Recall the *extended real numbers* are the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$. Any subset of $\mathbb{R}$ has a supremum and an infimum in $\overline{\mathbb{R}}$.

An *interval* is a subset of $\overline{\mathbb{R}}$ of one of these forms, where $a, b \in \overline{\mathbb{R}}$:

- Open Interval: $(a, b) = \{x \in \overline{\mathbb{R}} \mid a < x < b\}$
- Closed Interval: $[a, b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$
- Left Semiopen: $(a, b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$
- Left Semiopen: $(a, b] = \{x \in \overline{\mathbb{R}} \mid a < x \leq b\}$
- Right Semiopen: $[a, b) = \{x \in \overline{\mathbb{R}} \mid a \leq x < b\}$

The order on $\overline{\mathbb{R}}$ gives it a topology, and in this sense, $\overline{\mathbb{R}}$ is homeomorphic to $[0, 1]$. A subset of $\mathbb{R}\mathbb{R}$ is an interval if and only if it is connected and contains more than one point.

Finally, we will use the fact that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Date: February 26, 2019.

1.2. **Structure of Open Sets.**

Definition 1. Let $G \subset \mathbb{R}$ be an open set. A *component* of G is an open interval (a,b), with $a, b \in \overline{\mathbb{R}}$, such that $(a,b) \subset G$, $a \notin G$, and $b \notin G$.

Proposition 1. *Distinct components of an open set are disjoint.*

Proof. Let G be open and let (a_1, b_1) and (a_2, b_2) be distinct components of G; without loss of generality, we may assume that $a_1 < a_2$. Suppose that x is a point in their intersection. Then

$$a_1 < a_2 < x < b_1.$$

This shows that $a_2 \in (a_1, b_1) \subset U$, contradicting that a_2 is the endpoint of a component. □

Proposition 2. *Every open subset of* $\mathbb{R}$ *is a union of countably many disjoint intervals.*

Proof. Let $G \subset \mathbb{R}$ be open.

Every point in G is in a component of G. To see this, let $x_0 \in G$.

$$a = \inf\{x \in \mathbb{R} \mid [x, x_0] \subset G\} \quad \text{and} \quad b = \sup\{x \in \mathbb{R} \mid [x_0, x] \subset G\}.$$

Claim: $(a,b) \subset G$

Claim: $a \neq G$ and $b \in G$

Claim: There are countably many such components. □

1.3. **Structure of Closed Sets.**

Definition 2. Let $A \subset \mathbb{R}$. The *smallest closed interval* of A, denoted $\mathrm{sci}(A)$, is the interval $[a,b] \subset [-\infty, \infty]$, where $a = \inf A$ and $b = \sup A$.

Proposition 3. *Let F be a bounded closed set. Let $a = \inf F$ and $b = \sup F$. Then* $\mathrm{sci}(F) = [a,b]$, *and F is of the form*

$$F = [a,b] \smallsetminus G,$$

where G is a the union of a countable collection of disjoint open intervals contained in (a,b).

1.4. **Finite Intersection Property.**

Definition 3. Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}$.

We say that $\mathcal{C}$ is *bounded* if there exists $M > 0$ such that $C \subset [-M, M]$ for every $C \in \mathcal{C}$.

We say that $\mathcal{C}$ has the *finite intersection property* if the intersection of every finite subcollection from $\mathcal{C}$ is nonempty.

Proposition 4. *Let $\mathcal{F}$ be a bounded collection of closed sets with the finite intersection property. Then $\cap\mathcal{F}$ is nonempty.*

Proof. Let X be a bounded closed interval which contains $\cup\mathcal{F}$. We suppose that $\cap\mathcal{F}$ is empty; then the collection of complements of sets in $\mathcal{F}$ covers X. Since X is compact, this collection has a finite subcover, say $F_1^c, \ldots, F_n^c$, where $F_i \in \mathcal{F}$ for $i = 1, \ldots, n$.

Now $X \subset \cup_{i=1}^n F_i^c = (\cap_{i=1}^n F_i)^c$, and since $\cup\mathcal{F} \subset X$, this states that $\cap_{i=1}^n F_i = \varnothing$. However, since $\mathcal{F}$ has the finite intersection property, $\cap_{i=1}^n F_i$ is nonempty, so there exists an $x \in \cap_{i=1}^n F_i \subset X$. This contradiction proves the result. $\square$

Proposition 5. *Let $\mathcal{F} = \{F_n \mid n \in \mathbb{N}\}$ be a collection of nonempty bounded closed sets such that $F_k \supset F_{k+1}$. Show that $\cap_{n=1}^\infty F_n \neq \varnothing$.*

Proof. Let $\mathcal{C}$ be a finite subcollection from $\mathcal{F}$, and let $k = \max\{n \in \mathbb{N} \mid F_n\}$. Then $\cap\mathcal{C} = F_k$, which is nonempty. Thus $\mathcal{F}$ has the finite intersection property, and is thus nonempty by Proposition 4. $\square$

1.5. **Perfect Sets.**

Definition 4. Let $A \subset \mathbb{R}$. The *derived set* of A, denoted A', is the set of all accumulation points of A:

$$A' = \{x \in \mathbb{R} \mid \text{ every neighborhood of } x \text{ intersects } A \ \}.$$

We see have seen that every point in A is either an accumulation point of A, or is an isolated point of A. We note that A' may be bigger than A; for example, $\mathbb{Q}' = \mathbb{R}$.

Definition 5. A nonempty set $A \subset \mathbb{R}$ is *dense in itself* if every point in A is an accumulation point of A.

That is, a nonempty set A is dense in itself if and only if $A \subset A'$. Thus, A is dense in itself if and only if A has no accumulation points.

Definition 6. A nonempty subset $F \subset \mathbb{R}$ is called *perfect* if it is closed and dense in itself.

That is, a nonempty set A is perfect if and only if it is closed and contains no isolated points. Since a closed set contains all of its accumulation points, we have $A = A'$.

The following proof was outlined at

`https://mathcs.org/analysis/reals/topo/proofs/pfctuncb.html`.

Proposition 6. *Let $F \subset \mathbb{R}$ be a perfect set. Then F is uncountable.*

Proof. Since F is perfect, it cannot contain any isolated point. If F were finite, every point would be isolated, so clearly F is infinite.

We assume by way of contradiction F is countable, and let (x_n) be a surjective sequence of points from F. We define a subsequence of (x_n), and a corresponding sequence of neighborhoods, as follows.

Let $k_1 = 1$. Let $U_1 = (x_{k_1} - 1, x_{k_1} + 1)$. Since x_{k_1} is an accumulation point of F, there exist infinitely many members of F inside U_1. Let $k_2 = \min\{n \in \mathbb{N} \mid n > k_1 \text{ and } x_n \in U_1\}$. Then there exists an open neighborhood U_2 of x_{k_2} whose closure is contained in $U_1 \smallsetminus \{x_{k_1}\}$. Again, U_2 contains infinitely many points from F, so let $k_3 = \min\{n \in \mathbb{N} \mid n > k_2 \text{ and } x_n \in U_2\}$. Then there exists an open neighborhood U_3 of x_{k_3} whose closure is contained in $U_2 \smallsetminus \{x_{k_2}\}$.

Continue in this manner, and obtain a sequence of open sets (U_i) such that $\overline{U_i} \subset U_{i-1}$, U_i is an open neighborhood of x_{k_i}, and $x_n \notin U_i$ for $n < i$.

Let $C_i = \overline{U_i} \cap F$. Then $C_1 \supset C_2 \supset \cdots$ is a decreasing sequence of nonempty bounded closed sets. Let $C = \cap_{i=1}^{\infty} C_i$. By Proposition 5, C is nonempty; let $x \in C$. Now if $n \in \mathbb{N}$, $x_n \notin U_{n+1}$, so $x \neq x_n$ for any $n \in \mathbb{N}$. Thus, F is not countable. $\square$

Can this result be generalized to any topological space?

1.6. **Meager Sets.**

Definition 7. Let $A \subset \mathbb{R}$. We say that A is *nowhere dense* if every point of closure of A is a boundary point of $\overline{A}$.

That is, A is nowhere dense if $\overline{A} \subset \partial\overline{A}$. Since $\overline{A} = A^\circ \cup \partial A$, this implies that $\overline{A}$ has no interior points, which implies that $\overline{A}$ does not contain an interval.

Proposition 7. *Let $A \in \mathbb{R}$. If A is nowhere dense, then A^c is dense in itself.*

Proof. Exercise. $\square$

The converse of the above proposition in not necessarily true. For example, let $A = \mathbb{Q}$. Then the set of irrational, $\mathbb{Q}^c$, is dense in itself, but $\mathbb{Q}$ is not nowhere dense, since $\overline{\mathbb{Q}} = \mathbb{R}$, but the boundary of $\mathbb{R}$ is empty.

Definition 8. A nonempty set $A \subset \mathbb{R}$ is *meager*, or *of the first category*, if A is a union of countably many countable sets.

The following proof combines ideas from Proposition 6 and the textbook *Introduction to Real Variable Theory*, by Saxena Shah.

Proposition 8 (Baire Category Theorem)**.** *The set $\mathbb{R}$ is not meager.*

Proof. Suppose that $\mathbb{R}$ is meager. Then there exist a countable collection $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$ of nowhere dense sets such that $\mathbb{R} = \cup\mathcal{A}$. But for each $A \in \mathcal{A}$, $A \subset \overline{A}$, so

$$\mathbb{R} = \cup_{n=1}^{\infty} A_n \subset \cup_{n=1}^{\infty} \overline{A_n} \subset \mathbb{R},$$

which shows that $\mathbb{R} = \cup_{n=1}^{\infty} \overline{A_n}$. We now use this characterization to find a point in $\mathbb{R}$ which is not in the union of the closures of the sets in $\mathcal{A}$.

Since A_1 is nowhere dense, $\overline{A_1}$ does not equal $\mathbb{R}$, so there exists a real number $x_1 \notin \overline{A_1}$. Moreover, x_1 is not a boundary point of the closed set $\overline{A_1}$, so there exists an open neighborhood U_1 of x_1 such that $\overline{U_1} \cap \overline{A_1} = \varnothing$.

Since A_2 is meager, it is impossible that $\overline{A_2}$ contains the open set U_1. So, there exists $x_2 \in U_1$ which is not in the closure of A_2. Thus, there exists an open neighborhood U_2 of x_2 with $U_2 \subset U_1$ and $\overline{U_2} \cap \overline{A_2} = \varnothing$.

Continuing in the fashion, we obtain an infinite sequence of points (x_n), with corresponding open neighborhoods (U_n), such that $\overline{U_n} \subset U_{n-1}$, and $\overline{A_m} \cap \overline{U_n} = \varnothing$ for $m \leq n$.

Let $C = \cap_{n=1}^{\infty} \overline{U_n}$; since each of the sets $\overline{U_n}$ is nonempty, so it the intersection, by Proposition 5; say $x \in C$. But C is disjoint from every $A \in \mathcal{A}$; so $x \notin \cup\mathcal{A}$. This shows that $\cup\mathcal{A} \neq \mathbb{R}$. Thus, $\mathbb{R}$ is not meager. $\square$

Definition 9. The *Cantor set* is defined inductively by repeatedly removing the "open middle third" from a set of intervals, starting with the closed unit interval $[0, 1]$. To be precise, set $F_0 = [0, 1]$, and for $k \geq 1$, let F_k be the set obtained from F_{k-1} by removing the open middle third of each of the maximal intervals in F_{k-1}.

For example:

- $F_0 = [0, 1]$
- $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
- $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
- and so forth

This creates a decreasing sequence of sets:

$$F_0 \supset F_1 \supset \cdots \supset F_{k-1} \supset F_k \supset F_{k+1} \supset \cdots .$$

The *Cantor set* is the intersection of this sequence:

$$C = \cap_{k=0}^{\infty} F_k.$$

Proposition 9. *The Cantor set C has these properties:*

(a) *C is closed and dense in itself, and is therefore perfect*
(b) *C is uncountable*
(c) *C is nowhere dense*
(d) *C is obtained from an interval of length one by removing intervals whose lengths add to one.*

It is this last property we now wish to focus on.

2. Problems

Problem 1. There are two ways to generalize the Cantor set:

- Remove the open middle $\dfrac{1}{n}^{\text{th}}$ from each component, at each stage;
- Remove subintervals of length $\dfrac{1}{n^k}$ at stage k.

Try this for $n = 4$ and see what you get.

Department of Mathematics and CSci, BASIS Scottsdale
Email address: paul.bailey@basised.com