AP CALCULUS AB | Activity 0827 | Name:
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Problem 5. Let $g(x) = \sqrt{100 - x^2}$ and $h(x) = \dfrac{1}{x^2 - 25}$. Find the domain of the given function.

(a) $f(x) = g(x) + h(x)$

(b) $f(x) = \dfrac{g(x)}{h(x)}$

(c) $f(x) = g(h(x))$

(d) $f(x) = h(g(x))$

Solution. First we determine the domains of g and h.

The domain of g is the solution to $100 - x^2 \geq 0$, so $x^2 \leq 100$, so $|x| \leq 10$. Thus

$$\mathrm{dom}(g) = [-10, 10].$$

The domain of h is the solution to $x^2 - 25 \neq 0$, so $x^2 \neq 25$, so $x \neq \pm 5$. Thus

$$\mathrm{dom}(h) = \mathbb{R} \smallsetminus \{-5, 5\}.$$

(a) The domain of the sum is the intersection of the domains:

$$\mathrm{dom}(g + h) = \mathrm{dom}(g) \cap \mathrm{dom}(h).$$

Therefore,

$$\mathrm{dom}(g + h) = [-10, 10] \cap (\mathbb{R} \smallsetminus \{-5, 5\}) = [-10, 10] \smallsetminus \{-5, 5\} = [-10, -5) \cup (-5, 5) \cup (5, 10].$$

(b) The domain of a quotient is given by

$$\mathrm{dom}\left(\frac{g}{h}\right) = (\mathrm{dom}(g) \cap \mathrm{dom}(h)) \smallsetminus \{x \in \mathrm{dom}(h) \mid h(x) = 0\}.$$

Since $h(x) \neq 0$ for all $x \in \mathrm{dom}(h)$, we have

$$\mathrm{dom}\left(\frac{g}{h}\right) = [-10, -5) \cup (-5, 5) \cup (5, 10].$$

(c) The domain of a composition is given by

$$\mathrm{dom}(g \circ h) = \{x \in \mathrm{dom}(h) \mid h(x) \in \mathrm{dom}(g)\}.$$

In this case, we need $x \neq \pm 5$ and $\left|\dfrac{1}{x^2 - 25}\right| \leq 10$. Now

$$\begin{aligned}
\left|\frac{1}{x^2 - 25}\right| \leq 10 &\Leftrightarrow -10 \leq \frac{1}{x^2 - 25} \leq 10 \\
&\Leftrightarrow -\frac{1}{10} \leq x^2 - 25 \leq \frac{1}{10} \\
&\Leftrightarrow \frac{249}{10} \leq x^2 \leq \frac{251}{10} \\
&\Leftrightarrow \sqrt{24.9} \leq x \leq \sqrt{25.1} \\
&\Leftrightarrow x \in [\sqrt{24.9}, \sqrt{25.1}]
\end{aligned}$$

So,

$$\mathrm{dom}(g \circ h) = [\sqrt{24.9}, 5) \cup (5, \sqrt{25.1}].$$

(d) In this case, we require that $x \in \text{dom}(g)$ and $g(x) \in \text{dom}(h)$. That is, we need $x \in [-10, 10]$ and $\sqrt{100 - x^2} \neq 5$. Solving the inequality, we have

$$100 - x^2 \neq 25 \;\Rightarrow\; x^2 \neq 75 \;\Rightarrow\; x \neq \pm\sqrt{75}.$$

Thus,

$$\text{dom}(h \circ g) = [-10, 10] \smallsetminus \{\pm\sqrt{75}\}.$$

$\square$