Complex Analysis | Homework 21 - Integration Solutions | Name:
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Problem 1. Let $a = 0$ and $b = 1+i$. Let γ be the path from a to b given by $\gamma(t) = a + t(b-a)$ for $t \in [0,1]$.

(a) Let $f(z) = 1$. Compute $\int_\gamma f(z)\,dz$.

(b) Let $f(z) = z$. Compute $\int_\gamma f(z)\,dz$.

(c) Let $f(z) = z^2$. Compute $\int_\gamma f(z)\,dz$.

Solution. With $b = 1+i$, we have $b^2 = 2i$ and $b^3 = 2i - 2$. With $a = 0$, the path from a to b is

$$\gamma(t) = bt \quad \text{and} \quad \gamma'(t) = b.$$

(a) With $f(z) = 1$,

$$\int_\gamma f(z)\,dz = \int_\gamma dz = \int_0^1 \gamma'(t)\,dt = \int_0^1 b\,dt = b\int_0^1 dt = b\Big[t\Big]_0^1 = b = 1+i.$$

(b) With $f(z) = z$,

$$\int_\gamma f(z)\,dz = \int_\gamma z\,dz = \int_0^1 \gamma(t)\gamma'(t)\,dt = \int_0^1 b^2 t\,dt = b^2\int_0^1 t\,dt = b^2\left[\frac{t^2}{2}\right]_0^1 = \frac{b^2}{2} = i$$

(c) With $f(z) = z^2$,

$$\int_\gamma f(z)\,dz = \int_\gamma z^2\,dz = \int_0^1 (\gamma(t))^2\gamma'(t)\,dt = \int_0^1 b^3t^2\,dt = b^3\int_0^1 t^2\,dt = b^3\left[\frac{t^3}{3}\right]_0^1 = \frac{b^3}{3} = -\frac{2}{3} + \frac{2}{3}i.$$

□

Problem 2. Let $a = 0$ and $b = 1+i$. Let γ be the path from a to b given by $\gamma(t) = t + it^2$ for $t \in [0,1]$.

(a) Let $f(z) = 1$. Compute $\int_\gamma f(z)\,dz$.

(b) Let $f(z) = z$. Compute $\int_\gamma f(z)\,dz$.

(c) Let $f(z) = z^2$. Compute $\int_\gamma f(z)\,dz$.

Solution. With $b = 1+i$, we have $b^2 = 2i$ and $b^3 = 2i - 2$. The path from a to b is

$$\gamma(t) = t + it^2 \quad \text{and} \quad \gamma'(t) = 1 + 2it.$$

(a) With $f(z) = 1$,

$$\int_\gamma f(z)\,dz = \int_\gamma dz = \int_0^1 \gamma'(t)\,dt = \int_0^1 (1+2it)\,dt = \int_0^1 dt + i\int_0^1 2t\,dt = \Big[t\Big]_0^1 + i\Big[t^2\Big]_0^1 = 1+i.$$

(b) With $f(z) = z$, we first note that

$$\gamma(t)\gamma'(t) = (t+it^2)(1+2it) = (t-2t^3) + i(2t^2+t^2) = (t-2t^3) + i(3t^3).$$

Then

$$\int_\gamma f(z)\,dz = \int_\gamma z\,dz = \int_0^1 \gamma(t)\gamma'(t)\,dt = \int_0^1 (t+it^2)(1+2it)\,dt =$$

$$\int_0^1 (t-2t^3)\,dt + i\int_0^1 3t^2\,dt = \left[\frac{t^2}{2} - \frac{t^3}{2}\right]_0^1 + i\Big[t^3\Big]_0^1 = 0 + i = i$$

(c) With $f(z) = z^2$, we first note that

$$(\gamma(t))\gamma'(t) = (t+it^2)^2(1+2it) = (t^2-t^4+2it^3)(1+2it) = (t^2-t^4-4t^4)+i(2t^3-2t^5+2t^3) = (t^2-5t^4)+i(4t^3-2t^5).$$

Then

$$\int_\gamma f(z)\,dz = \int_\gamma z^2\,dz = \int_0^1 (\gamma(t))^2\gamma'(t)\,dt = \int_0^1 (t^2 - t^4 + 2it^3)(1+2it)\,dt =$$

$$\int_0^1 (t^2 - 5t^4)\,dt + i\int_0^1 (4t^3 - 2t^5)\,dt = \Big[\frac{1}{3}t^3 - t^5\Big]_0^1 + i\Big[t^4 - \frac{1}{3}t^6\Big]_0^1 = -\frac{2}{3} + \frac{2}{3}i$$

□

Problem 3. Let $a \in \mathbb{C}$ and let γ parameterize the circle of radius 1 about 0 by $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

(a) Let $f(z) = 1$. Compute $\int_\gamma f(z)\,dz$.

(b) Let $f(z) = z$. Compute $\int_\gamma f(z)\,dz$.

(c) Let $f(z) = \dfrac{1}{z}$. Compute $\int_\gamma f(z)\,dz$.

Solution. Our contour is a circle given by $\gamma(t) = e^{it} = \cos t + i\sin t$. Thus

$$\gamma'(t) = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}.$$

Also note that if $a, b, c \in \mathbb{R}$,

$$\int_a^b e^{cit}\,dt = \int_a^b \cos ct + i\sin ct\,dt = \frac{1}{c}\Big[\sin ct - i\cos ct\Big]_a^b = \frac{1}{ci}e^{cit}\Big]_a^b.$$

(a) With $f(z) = 1$,

$$\int_\gamma f(z)\,dz = \int_\gamma dz = \int_0^{2\pi} \gamma'(t)\,dt = \int_0^{2\pi} ie^{it}\,dt = i\int_0^{2\pi} e^{it}\,dt = i\Big[\frac{1}{i}e^{it}\Big]_0^{2\pi} = e^{2\pi i} - e^0 = 0.$$

(b) With $f(z) = z$,

$$\int_\gamma f(z)\,dz = \int_\gamma z\,dz = \int_0^{2\pi} \gamma(t)\gamma'(t)\,dt = \int_0^{2\pi} ie^{2it}\,dt = i\int_0^{2\pi} e^{2it}\,dt = i\Big[\frac{1}{2i}e^{2it}\Big]_0^{2\pi} = \frac{1}{2}(e^{4\pi i} - e^0) = 0.$$

(c) With $f(z) = \dfrac{1}{z}$,

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)\,dt = \int_0^{2\pi} i\,dt = i\Big[t\Big]_0^{2\pi} = 2\pi i.$$

□