REAL ANALYSIS
SOLUTIONS TO ROSS §4

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Lemma 1. *Let F be a complete ordered field. Let $A \subset F$ and $x \in F$.*

(a) *If A is bounded above and $x < \sup A$, then $x < a$ for some $a \in A$.*
(b) *If A is bounded below and $x > \inf A$, then $x > a$ for some $a \in A$.*

Proof. We prove **(a)**; the proof of **(b)** is analogous.

Suppose that $x \geq \sup A$. Then x is an upper bound for A. By definition of supremum, $\sup A \leq x$. This is the contrapositive of what we wished to prove. □

Exercise 1 (4.5)**.** Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Show that if $\sup S$ belongs to S, then $\sup S = \max S$.

Proof. Let $u = \sup S$. Then $u \geq s$ for all $s \in S$. Since $u \in S$, we have $u = \max S$. □

Exercise 2 (4.6)**.** Let S be a nonempty bounded subsets of $\mathbb{R}$. Show that $\inf S \leq \sup S$. What can be said if $\inf S = \sup S$?

Proof. Since S is nonempty, there exists $s \in S$. Then $\inf S \leq s$ and $s \leq \sup S$. By transitivity of order, $\inf S \leq \sup S$.

If $\inf S = \sup S$, then S contains only one element. □

Exercise 3 (4.7.a)**.** Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show if $S \subset T$, the $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Proof. Let $s \in S$. Then $s \in T$, so $\inf T \leq s$. Thus $\inf T$ is a lower bound for S, so $\inf T \leq \inf S$. Similarly, $\sup S \leq \sup T$. That $\inf S \leq \sup S$ is true is exercise 6. □

Exercise 4 (4.7.b)**.** Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Proof. Either $\max\{\sup S, \sup T\} = \sup S$ or $\max\{\sup S, \sup T\} = \sup T$.

Suppose that $\max\{\sup S, \sup T\} = \sup S$; in this case, $\sup T \leq \sup S$. Since $S \subset S \cup T$, we have $\sup S \leq \sup(S \cup T)$ by part (a).

Now let $x \in S \cup T$. Then x is either in S or T. If $x \in S$, then $x \leq \sup S$. If $x \in T$, then $x \leq \sup T \leq \sup S$. Thus $\sup S$ is an upper bound for $S \cup T$. Therefore $\sup(S \cup T) \leq \sup S$.

Since $\sup S \leq \sup(S \cup T)$ and $\sup(S \cup T) \leq \sup S$, it follows that $\sup S = \sup(S \cup T)$.

Finally, if $\max\{\sup S, \sup T\} = \sup T$, the above proof is valid, with the roles of S and T reversed. □

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Exercise 5 (4.8(b))**.** Let S and T be nonempty subsets of $\mathbb{R}$ such that $s \leq t$ for every $s \in S$ and $t \in T$. Show that $\sup S \leq \inf T$.

Proof. Note that since S and T are nonempty, S is bounded above by an existing element of T and T is bounded below by an existing element of S. Thus $\sup S$ and $\inf T$ exist.

Suppose the conclusion is false; then $\inf T < \sup S$. By Lemma 1a, there exists $s \in S$ such that $\inf T < s$, By Lemma 1b, there exists $t \in T$ such that $t < s$. This is contrary to the assumption on S and T. □

Exercise 6 (4.10)**.** Show that if $a > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Proof. Let $b = \max\{a, \frac{1}{a}\}$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > b$. Since $a \leq b$, we have $a < n$. Also since $\frac{1}{a} \leq b$, we have $\frac{1}{a} < n$. Thus by Theorem 3.2.(vii), we have $\frac{1}{n} < a$. □

Exercise 7 (4.11)**.** Let $a, b \in \mathbb{R}$ such that $a < b$. Show that there exist infinitely many rational numbers between a and b.

Proof. Suppose not. The the set $S = (a, b) \cap \mathbb{Q}$ is finite, so it has a minimum, say $c = \min S$. But then Theorem 4.7 tells us that there exists $d \in \mathbb{Q}$ such that $a < d < c$. But then $d < b$, so $d \in S$. This contradicts that $c = \min S$. □

Define $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. For the purposes of the next exercise, assume that $\mathbb{I}$ is nonempty. We will show this in the appendix.

Exercise 8 (4.12)**.** Let $a, b \in \mathbb{R}$. Show that if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$.

Proof. Since $\mathbb{I}$ is nonempty, let $\alpha \in \mathbb{I}$.

Let $q \in \mathbb{Q}$. Then $q \in \mathbb{R}$, and since $\alpha \in \mathbb{R}$ and $\mathbb{R}$ is a field, $q + \alpha \in \mathbb{R}$. Suppose that $q + \alpha \in \mathbb{Q}$; say $q + \alpha = p \in \mathbb{Q}$. Then $\alpha = p - q$, and since p and q are both in $\mathbb{Q}$, then so is $p - q$, because $\mathbb{Q}$ is a field. This contradicts the assumption on α. Thus $q + \alpha$ is irrational.

By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $q \in \mathbb{Q}$ such that $a - \alpha < q < b - \alpha$. Then $a < q + \alpha < b$, and $q + \alpha$ is irrational. Therefore, there exists an irrational number between any two real numbers. □

Exercise 9 (4.14)**.** Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let

$$S = \{x \in \mathbb{R} \mid x = a + b \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup S = \sup A + \sup B$.
(b) Show that $\inf S = \inf A + \inf B$.

Proof. We prove **(a)**; the proof for **(b)** is symmetric. It suffices to show that $\sup S \leq \sup A + \sup B$ and that $\sup A + \sup B \leq \sup S$.

Let $s \in S$. Then $s = a + b$ for some $a \in A$ and $b \in B$. Then $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$. Thus $\sup A + \sup B$ is an upper bound for S, so $\sup S \leq \sup A + \sup B$.

Suppose that $\sup S < \sup A + \sup B$. Then $\sup S - \sup B < \sup A$, so there exists $a \in A$ such that $\sup S - \sup B < a$. From this, $\sup S - a < \sup B$, so there exists $b \in B$ such that $\sup S - a < b$. Let $s = a + b \in S$. We have $\sup S < s$, a contradiction. Therefore $\sup A + \sup B \leq \sup S$. □

Exercise 10 (4.15)**.** Let $a, b \in \mathbb{R}$. Show that if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

Proof. We prove the contrapositive.

Suppose that $a > b$. By exercise 4.10, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a - b$. Thus $b + \frac{1}{n} < a$. $\square$

Exercise 11 (4.16)**.** Show that $\sup\{r \in \mathbb{Q} \mid r < a\} = a$ for each $a \in \mathbb{R}$.

Proof. Let $a \in \mathbb{R}$, $A = \{r \in \mathbb{Q} \mid r < a\}$, and $s = \sup A$. We wish to show that $a = s$.

Suppose that $a < s$. Then there exists $r \in A$ such that $a < r < s$. This contradicts the definition of A.

On the other hand, suppose that $s < a$. By the density of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $s < r < a$. Then $r \in A$. This contradicts the definition of s.

The only remaining possibility is that $a = s$. $\square$

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