GEOMETRY
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Summary 1 - Lines in the Coordinate Plane
Tuesday, November 17, 2020

Definition 1. (Real Line and the Cartesian Plane)
The set of all real numbers is denoted $\mathbb{R}$. Geometrically, the set of all real numbers may be viewed as a marked line; marked in the sense that we know where 0, 1, 2, and so forth, lie on the line. We sometimes call the set of all real numbers the *real line*.

The *cartesian plane* is the set of all ordered pairs of real numbers, viewed as the points on a plane. An ordered pair of real numbers may be graphed on a plane, as you have already learned. The set of all ordered pairs of real numbers has a standard name in mathematics: it is denoted $\mathbb{R}^2$.

Definition 2. (Slope)
Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. There is a unique line in $\mathbb{R}^2$ through A and B.

The *slope* of the line through A and B is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We can show, through the use of similar triangles, that the slope of a line does not depend on the two points on the line which are used to compute the slope.

Example 1. (Find the Slope of a Line between Two Points)
Let $A = (1, 6)$ and $B = (3, 2)$. Find the slope of the line through A and B.

Solution. Let $(x_1, y_1) = A$ and $(x_2, y_2) = B$; that is, $x_1 = 1$, $y_1 = 6$, $x_2 = 3$, and $y_2 = 2$. Plug this into the slope formula to find that the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 6}{3 - 1} = \frac{-4}{2} = -2.$$

This slope is

$$\boxed{m = -2.}$$

□

Definition 3. (Point-Slope Equation)
The *point-slope* form of the equation of a line with slope m through point (x_0, y_0) is

$$y = m(x - x_0) + y_0.$$

Here, the symbols m, x_0, and y_0 are constants. These are numeric values we plug into this formula to get the equation of a line. The symbols x and y are variables; they stay there; they do not change when you plug in m, x_0, and y_0.

Definition 4. (Slope-Intercept Equation)
The *slope-intercept* form of the equation of a line with slope m and y-intercept $(0, b)$ is

$$y = mx + b.$$

Example 2. (Find the Equation of a Line with a Given Slope through a Given Point)
Find the slope-intercept form of the equation of the line with slope $m = \frac{1}{2}$ through the point $(x_0, y_0) = (2, -4)$.

Solution. First, we plug m, x_0, and y_0 in to the point-slope equation $y = m(x - x_0) + y_0$ to get $y = \frac{1}{2}(x - 2) - 4$. Then we distribute and simply to arrive at

$$\boxed{y = \frac{1}{2}x - 5.}$$

□

Example 3. (Find the Equation of a Line through Two Points)
Let $A = (2, 3)$ and $B = (-1, 8)$. Find the slope-intercept form of the equation of $\overleftrightarrow{AB}$.

Solution. There are three steps.

Step 1: Find the slope m.
To use the formula, we identify $A = (x_1, y_1) = (2, 5))$ and $B = (x_2, y_2) = (-1, 8)$. Thus

$$m = \frac{y_2 - y_1}{x_2 - x_2} = \frac{8 - 3}{-1 - 2} = \frac{5}{-3} = -\frac{5}{3}.$$

If we had switched the roles of A and B, the formula would have given the same slope.

Step 2: Write the point-slope form of the equation of the line.
Identify one of the points to be (x_0, y_0). Either point will work. We let $(x_0, y_0) = (2, 3)$. The point-slope equation is $y = m(x - x_0) + y_0$, so we plug in $m = \frac{5}{3}$, $x_0 = 2$, and $y_0 = 3$, to get

$$y = \frac{5}{3}(x - 2) + 3.$$

Step 3: Write the slope-intercept form of the equation of the line.
To do this, use distribution to simplify the point-slope form: $y = \frac{5}{3}(x-2)+3 = \frac{5}{3}x - \frac{10}{3} + 3 = \frac{5}{3}x - \frac{1}{3}$; the slope-intercept form is

$$\boxed{y = \frac{5}{3}x - \frac{1}{3}.}$$

$\square$

Fact 1. (Parallel Line Slope)
The slope of a line parallel to a line with of the form $y = sx + c$ (with slope s) is

$$s = m.$$

Example 4. (Find the Equation of a Parallel Line through a Point)
Find the slope-intercept form of the equation of the line parallel to $y = 2x + 7$ through the point $(5, -2)$.

Solution. The slope of the original line is $s = 2$, so the slope of the parallel line is $m = s = 2$. The point is $(x_0, y_0) = (5, -2)$. Plug this into the point-slope equation to get $y = 2(x - 5) - 2$. Multiply this out to get $y = 2x - 10 - 2 = 2x - 12$. Thus the slope-intercept equation of the parallel line is

$$\boxed{y = 2x - 12.}$$

$\square$

Fact 2. (Perpendicular Line Slope)
The slope of a line perpendicular to a line of the form $y = sx + c$ (with slope s) is

$$m = -\frac{1}{s}.$$

You should ask, why?

Example 5. (Find the Equation of a Line Perpendicular Line through a Point)
Find the slope-intercept form of the equation of the line perpendicular to $y = 2x + 7$ through the point $(12, 4)$.

Solution. The slope of the original line is $s = 2$, so the slope of the perpendicular line is $m = -\frac{1}{s} = -\frac{1}{2}$. The point is $(x_0, y_0) = (12, 4)$. Plug this into the point-slope equation to get $y = -\frac{1}{2}(x - 12) + 4$. Multiply this out to get $y = -\frac{1}{2}x + 6 + 4 = -\frac{1}{2}x + 10$. Thus the slope-intercept equation of the perpendicular line is

$$\boxed{y = -\frac{1}{2}x + 10.}$$

$\square$