PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §10

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Exercise 1 (10.6.(a))**.** Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < \frac{1}{2^n} \qquad \text{for all } n \in \mathbb{N}.$$

Show that (s_n) is a Cauchy sequence.

Lemma 1. *Let $m, n \in \mathbb{N}$ with $2 < m < n$. Then*

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{1}{2^m} < \frac{1}{m}.$$

Proof of Lemma. We prove the first inequality by induction on $k = n - m$. If $k = 1$, then our statement reads $\frac{1}{2^{m+1}} < \frac{1}{2^m}$, which is true.

Suppose that our proposition is true for differences of size $k - 1$. Then

$$\sum_{i=m+2}^{n} \frac{1}{2^i} < \frac{1}{2^{m+1}}.$$

Adding $\frac{1}{2^{m+1}}$ to both sides gives

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{2}{2^{m+1}} = \frac{1}{2^m}.$$

For the second inequality, it suffices to show that for $m > 2$ we have $m < 2^m$. For $m = 3$, we have $3 < 4$. By induction, $m - 1 < 2^{m-1}$. Then $m < 2^{m-1} + 1 < 2^{m-1} + 2^{m-1} = 2^m$. □

Solution to Exercise. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\frac{1}{\epsilon} < N$. Let $m, n > N$; assume that $n > m$. Then

$$\begin{aligned}
|s_n - s_m| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{m+1} - s_m| \\
&\leq |s_n - s_{n-1}| + \cdots + |s_{m+1} - s_m| \\
&< \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \\
&< \frac{1}{2^{m-1}} \\
&< \frac{1}{m-1} \leq \frac{1}{N} < \epsilon.
\end{aligned}$$

This shows that (s_n) is a Cauchy sequence. □

Exercise 2 (10.6.(b))**.** Show that there exists a sequence (s_n) such that $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$, and yet (s_n) is not a Cauchy sequence.

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Solution. Let $s_n = \sum_{i=1}^{n} \frac{1}{2n}$. Then for $n \in \mathbb{N}$ we have $|s_{n+1} - s_n| = \frac{1}{2n+2} < \frac{1}{n}$. However, this sequence is unbounded, so it cannot be a Cauchy sequence.

Let $M \in \mathbb{R}$, and let N be so large that $N > 2M$. We claim that $s_{2^N} > 1 + \frac{N}{2} > M$; this is because

$$\begin{aligned}
\sum_{i=1}^{2^N} \frac{1}{i} &= 1 + \sum_{j=1}^{N} \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} \\
&> 1 + \sum_{j=1}^{N} \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{2^j} \\
&= 1 + N(2^j - 2^{j-1})\frac{1}{2^j} \\
&= 1 + \frac{N}{2}.
\end{aligned}$$

□

Exercise 3 (10.7)**.** Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose that $\sup S \notin S$. Show that there is a nondecreasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Solution. Since $\sup S$ is the least upper bound for S, the set $(\sup S - \frac{1}{n}, \sup S) \cap S$ is nonempty, for otherwise $\sup S - \frac{1}{n}$ would be an upper bound for S. For every $n \in \mathbb{N}$, let $s_n \in (\sup S - \frac{1}{n}, \sup S) \cap S$; that is, $s_n \in S$ with $|s_n - \sup S| < \frac{1}{n}$. Then (s_n) is a sequence in S which clearly converges to $\sup S$. □

Remark 1. Infinite processes are tricky, and one should be careful. The question is, how do we know that we can make infinitely many such choices simultaneously to invent to sequence (s_n)? There is a theorem called the Recursion Theorem, which can be proved by induction, which asserts that if we have an algorithm which *uniquely* identifies a_{n+1} given $a_1, \dots, a_n$, then there is a unique sequence (a_n). However, in the case above, there is no way to uniquely specify s_n; we merely know that one exists with the required property that it is in $\{s \in S \mid |s - \sup S| < \frac{1}{n}\}$.

The only proof that I have found which demonstrates the existence of such a sequence requires the (in)famous Axiom of Choice. The Axiom of Choice states that the cartesian product of an infinite number of nonempty sets is itself nonempty. Perhaps surprisingly, this cannot be shown using the other axioms of set theory.

Perhaps the Axiom of Choice is not required for this problem.

Some mathematicians do not except this axiom of set theory, because the work they are doing comes out cleaner without it. Other mathematicians lean heavily on the axiom. There is no need to worry, however, for we are among them.

Exercise 4 (10.8)**.** Let (s_n) be a nondecreasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$. Show that (σ_n) is a nondecreasing sequence.

Remark. Note that if (σ_n) converges, we would be well-advised to view its limit as the average of the infinite number of numbers s_n. □

Solution. We show that $\sigma_n \leq \sigma_{n+1}$, i.e., $\sigma_{n+1} - \sigma_n \geq 0$. Since s_n is nondecreasing, an easy induction shows that

$$ns_{n+1} \geq \sum_{i=1}^{n} s_i.$$

Thus

$$\begin{aligned}\sigma_{n+1} - \sigma_n &= \frac{\sum_{i=1}^{n+1} s_i}{n+1} - \frac{\sum_{i=1}^{n} s_i}{n} \\ &= \frac{n\sum_{i=1}^{n+1} s_i - (n+1)\sum_{i=1}^{n} s_i}{n(n+1)} \\ &= \frac{ns_{n+1} - \sum_{i=1}^{n} s_i}{n(n+1)} \\ &\geq 0.\end{aligned}$$

□

Exercise 5 (10.10)**.** Let $s_1 = 1$ and $s_{n+1} = \frac{(s_n+1)}{3}$. Show that (s_n) converges and find the limit.

Solution. If s_n is positive, then so is s_{n+1}; since s_1 is positive, $s_n > 0$ for all $n \in \mathbb{N}$ by induction.

Next we show that $s_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Proceed by induction; $s_1 \geq \frac{1}{2}$, and assume that $s_n \geq \frac{1}{2}$. Then $s_{n+1} = \frac{s_n+1}{3} \geq \frac{3/2}{3} = \frac{1}{2}$.

Next we show that $s_{n+1} < s_n$. Since $s_n \geq \frac{1}{2}$, we have $1 \leq 2s_n$. Thus $\sigma_{n+1} = \frac{s_n+1}{3} \leq \frac{s_n+2s_n}{3} = s_n$.

Since (s_n) is nonincreasing, we see that $s_n \in [\frac{1}{2}, 1]$ for all $n \in \mathbb{N}$. So (s_n) is a bounded monotone sequence, so its limit exists. Let $s = \lim s_n$. We know that $\lim s_{n+1} = \lim s_n$, so $s = \lim s_{n+1} = \lim \frac{s_n+1}{3} = \frac{s+1}{2}$. Solving this for s gives $s = \frac{1}{2}$. □

Exercise 6 (10.12)**.** Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}]t_n$ for $n \geq 1$. Show that (t_n) converges, and find the limit.

Proof. Let $a_n = 1 - \frac{1}{(n+1)^2}$. Since $a_n > 0$ for all $n \in \mathbb{N}$ and $t_1 > 0$, we can see that $t_n > 0$ for all $n \in \mathbb{N}$. Since $a_n < 1$ for all $n \in \mathbb{N}$, we see that (t_n) is nonincreasing. Thus t_n is a bounded monotone sequence, so its limit exists; say $t = \lim t_n$. Then

$$t = \lim t_{n+1} = \lim a_n t_n = \lim a_n \lim t_n = t.$$

That was not helpful.

Claim: $t_n = \frac{n+1}{2n}$.

True for $n = 1$. By induction, $t_{n-1} = \frac{n}{2(n-1)}$. Then

$$\begin{aligned} t_n &= a_{n-1} t_{n-1} \\ &= (1 - \frac{1}{n^2})(\frac{n}{2(n-1)}) \\ &= \frac{n}{2(n-1)} - \frac{1}{2n(n-1)} \\ &= \frac{n^2 - 1}{2n(n-1)} \\ &= \frac{n+1}{2n}. \end{aligned}$$

Now we see that $t_n \to \frac{1}{2}$ as $n \to \infty$. ∎

Definition 1. Let $A \subset \mathbb{R}$ be an open interval. A function $f : A \to \mathbb{R}$ is called a *contraction* if there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| \leq M|a - b|$ for any $a, b \in U$.

Example 1. The following are contractions:

- $f(x) = mx + b$, where $m, b \in \mathbb{R}$; $U = \mathbb{R}$ and $M = |m|$;
- $f(x) = \sin(x)$ and $\cos(x)$; $U = \mathbb{R}$ and $M = 1$;
- $f(x) = \log(x)$; $U = (a, \infty)$ where $a > 0$ and $M = \frac{1}{a}$.
- $f(x) = \sqrt{x}$; $U = (a, \infty)$ where $a > 0$ and $M = \frac{1}{2\sqrt{a}}$;
- $f(x)$ is differentiable with bounded derivative on an open interval U; $M = \sup\{|f'(a)| : a \in U\}$.

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. Let (a_n) be a sequence of real numbers which converges to $L \in \mathbb{R}$. Show that $\lim f(a_n) = f(L)$.

Proof. Let $\epsilon > 0$. Since f is a contraction, there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| < M|a - b|$ for all $a, b \in \mathbb{R}$.

Since (a_n) converges to L, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{M}$ for all $n > N$. Since f is a contraction,

$$|f(a_n - f(L)| < M|a_n - L| < M\frac{\epsilon}{M} = \epsilon$$

for all $n > N$. Thus $f(a_n) \to f(L)$. □

Problem 2. For each $i \in \mathbb{N}$, let A_i be a set of real numbers which is bounded above by $M \in \mathbb{R}$. Let $B_n = \cup_{i=1}^{n} A_i$, and let $B_\infty = \cup_{i=1}^{\infty} A_i$. Let $b_n = \sup B_n$. Show that $\lim b_n = \sup B_\infty$.

Proof. For every $a \in B_\infty$, we have $a \leq M$; that is, B_∞ is bounded above and its supremum exists as a real number. Let $b = \sup B_\infty$.

Let $n \in \mathbb{N}$. Then $B_n \subset B_\infty$; by Exercise 4.7.(a), we have $b_n = \sup B_n \leq \sup B_\infty = b$. By Exercise 8.9.(b), $\lim b_n \leq b$.

Suppose that $\lim b_n < b$. Since $b = \sup B_\infty$, there exists $c \in B_\infty$ such that $\lim b_n < c < b$ (otherwise, $\frac{\lim b_n + b}{2}$ is a lower bound for B_∞, contradicting our definition of b). Then $c \in A_N$ for some $i \in \mathbb{N}$, so $c \in B_N$. Since $c \in B_N$, then $c \in B_n$ for all $n > N$. This says that $c \leq \sup B_n = b_n$ for all $n > N$. By Exercise 8.9.(a), we have $\lim b_n \geq c$, contradicting our choice of c. This shows that $\lim b_n \geq b$. □

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