PRINCIPLES OF ANALYSIS
SOLUTIONS TO PROBLEM SET C

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Problem 1. Show that $\{s_n\}_{n=1}^{\infty}$ converges, and find the limit.

(a) $s_n = \frac{2^n}{n!}$.

(b) $s_n = \sum_{i=1}^{n} \frac{1}{3^i}$.

Solutions.

(a) We show that the limit is 0. First note that for $n \geq 4$, $\frac{2^n}{n!} < 1$. This is easily seen by induction; for $n = 4$, $\frac{2^n}{n!} = \frac{16}{24} = \frac{2}{3}$. For $n > 4$, we have $\frac{2}{n} < 1$, so $\frac{2^n}{n!} = \frac{2^{n-1}}{(n-1)!}\frac{2}{n}$ is the product of two positive numbers which are less than 1. Now let $N \geq 5$ be so large that $\frac{1}{N} < \frac{\epsilon}{2}$. Then for $n \geq N$, $\frac{2^{n-1}}{(n-1)!} < 1$, and we have

$$|\frac{2^n}{n!}| < |2|n < \epsilon.$$

(b) It is easy to see that

$$\frac{1 - x^{n+1}}{1 - x} = \sum_{i=0}^{n} x^i.$$

Assume that $|x| < 1$ so that $\lim_{n\to\infty} x^n = 0$. Then

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=0}^{n} x^i &= \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} \\ &= \frac{1 - \lim_{n\to\infty} x^n}{1 - x} \\ &= \frac{1}{1 - x}.\end{aligned}$$

So in our case, $x = \frac{1}{3}$ and we start summing at $i = 1$ instead of $i = 0$.
Thus $\lim_{n\to\infty} s_n = \frac{1}{1-\frac{1}{3}} - 1 = \frac{1}{2}$. $\square$

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Problem 2. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence which converges to s. Set

$$t_n = \frac{\sum_{i=1}^{n} s_n}{n}.$$

Show that $\{t_n\}_{n=1}^{\infty}$ converges to s.

Solution. Let $u_n = t_n - s$. It suffices to show that $\{u_n\}_{n=1}^{\infty}$ converges to zero. Note that

$$u_n = \frac{1}{n}\sum_{i=1}^{n} s_i - \frac{ns}{n} = \frac{1}{n}\sum_{i=1}^{n}(s_i - s).$$

Let $N_0 \in \mathbb{N}$ be so large that $|s_n - s| < \frac{\epsilon}{2}$ for all $n > N_0$. Let $M = \sum_{i=1}^{N_0} |s_i - s|$. Then for $n > N_0$, we have

$$\begin{aligned} |u_n| &\leq \frac{M}{n} + \frac{1}{n}\sum_{i=N_0+1}^{n} |s_n - s| && \text{by } \Delta\text{-inequality} \\ &< \frac{M}{n} + \frac{1}{n}(n - N_0)\frac{\epsilon}{2} && \text{summing } n - N_0 \text{ small numbers} \\ &< \frac{M}{n} + \frac{\epsilon}{2} && \text{since } \frac{n - N_0}{n} \leq 1. \end{aligned}$$

Now select $N \in \mathbb{N}$ with $N > N_0$ which is so large that $\frac{M}{n} < \frac{\epsilon}{2}$. Then for $n > N$, we have $|u_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that $|u_n| \to 0$ as $n \to \infty$. Thus $\lim u_n = 0$. □

Problem 3. Let $s_1 = 1$ and set $s_{n+1} = \sqrt{2s_n}$.

(a) Show that $\{s_n\}_{n=1}^{\infty}$ is bounded.
(b) Show that $\{s_n\}_{n=1}^{\infty}$ is monotone.
(c) Find $\lim_{n\to\infty} s_n$.

Solution. Assuming that $\{s_n\}_{n=1}^{\infty}$ converges, set $s = \lim_{n\to\infty} s_n$; then $s = \sqrt{2s}$, so $s^2 - 2s = 0$, and s is either 0 or 2. We see that it isn't going to be 0, so we guess that it is 2, and that $\{s_n\}_{n=1}^{\infty}$ is monotone increasing. This motivates the solution.

(a) Note that $s_1 = 1 < 2$. By induction, $s_n < 2$. Thus $s_{n+1} = \sqrt{2s_n} < \sqrt{2 \cdot 2} = 2$. Then (s_n) is bounded above by 2. Since $s_n \geq 0$ for all n, $\{s_n\}_{n=1}^{\infty}$ is bounded.

(b) Note that $s_{n+1} = \sqrt{2s_n} > s_n \Leftrightarrow 2s_n > s_n^2 \Leftrightarrow 2 > s_n$; since the latter inequality has already been shown, we see that $\{s_n\}_{n=1}^{\infty}$ is monotone increasing. Therefore $\{s_n\}_{n=1}^{\infty}$ converges.

(c) Since $\{s_n\}_{n=1}^{\infty}$ converges, let $s = \lim_{n\to\infty} s_n$; then $\lim s_{n+1} = \lim s_n$, and taking the limit of both sides of the defining equation $s_{n+1} = \sqrt{2s_n}$, we get $s = \lim \sqrt{2s_n} = \sqrt{2 \lim s_n} = \sqrt{2s}$. Thus $s = 0$ or $s = 2$. But $s_n \geq 1$ for all $n \in \mathbb{N}$, so $\lim s_n \geq 1$. Thus $\lim s_n = 2$. □

Problem 4. Let $s_n = \sum_{i=1}^{n} \frac{1}{i!}$. Show that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Lemma 1. *Let $m, n \in \mathbb{N}$ such that $m < n$.*
Show that $\sum_{i=m+1}^{n} \frac{1}{i!} < \frac{1}{m!}$.

Proof of Lemma. Proceed by induction on $k = n - m$, which is the number of terms being added.

For $k = 1$, we have $\sum_{m+1}^{n} \frac{1}{i!} = \frac{1}{(m+1)}! < \frac{1}{m!}$.

By induction on k, we assume the result to be true if we are adding $k - 1 = n - (m + 2)$ terms; this gives us

$$\sum_{i=m+2}^{n} \frac{1}{i!} < \frac{1}{(m+1)!}.$$

Thus

$$\begin{aligned}
\sum_{i=m+1}^{n} \frac{1}{i!} &= \sum_{i=m+2}^{n} \frac{1}{i!} + \frac{1}{(m+1)!} \\
&< \frac{1}{(m+1)!} + \frac{1}{(m+1)!} \\
&= \frac{2}{(m+1)!} \\
&< \frac{m+1}{(m+1)!} \\
&= \frac{1}{m!}.
\end{aligned}$$

□

Solution to Problem. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$. Let m and n be natural numbers greater than N, and assume without loss of generality that $m < n$. Then

$$\begin{aligned}
|s_n - s_m| &= \sum_{i=1}^{n} \frac{1}{i!} - \sum_{i=1}^{m} \frac{1}{i!} && \text{because } s_n > s_m \\
&= \sum_{i=m+1}^{n} \frac{1}{i!} \\
&< \frac{1}{m!} && \text{by Lemma} \\
&< \frac{1}{N} \\
&< \epsilon.
\end{aligned}$$

□

Problem 5. Let $\{s_n\}_{n=1}^\infty$ be a bounded sequence of real numbers. Let $a = \liminf s_n$ and $b = \limsup s_n$. Show that for every $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $s_n \in (a - \epsilon, b + \epsilon)$.

Solution. Let $T_n = \{s_m \mid m \geq n\}$.

We have seen that $\{\sup T_n\}_{n=1}^\infty$ is a bounded decreasing sequence, so it converges; say $b = \lim_{n\to\infty}[\sup T_n] = \limsup s_n$. Then there exists $N_1 \in \mathbb{N}$ such that $|\sup T_n - b| < \epsilon$ for all $n > N_1$. Thus $\sup T_n < b + \epsilon$ for all $n > N_1$.

We have seen that $\{\inf T_n\}_{n=1}^\infty$ is a bounded increasing sequence, so it converges; say $a = \lim[\inf T_n] = \liminf s_n$. Then there exists $N_2 \in \mathbb{N}$ such that $|\inf T_n - a| < \epsilon$ for all $n > N_2$. Thus $\inf T_n > a - \epsilon$ for all $n > N_2$.

Let $N = \max\{N_1, N_2\}$. Then for all $n > N$,

$$a - \epsilon < \inf T_n \leq s_n \leq \sup T_n < b + \epsilon.$$

Thus $s_n \in (a - \epsilon, b + \epsilon)$ for all $n > N$. □

Problem 6. Define a sequence $\{u_n\}_{n=1}^\infty$ by $u_n = 2^k$ if $2^k \leq n < 2^{k+1}$. Define a sequence $\{t_n\}_{n=1}^\infty$ by $t_n = n - u_n$. Define a sequence $\{s_n\}_{n=1}^\infty$ by $s_n = \sin \frac{2\pi t_n}{u_n}$. Find the set of cluster points of $\{s_n\}_{n=1}^\infty$. Justify your answer.

Solution. Let $\epsilon > 0$. For every $x \in [0,1]$, there exists a positive integer n and a positive integer $k < 2^n$ such that $|x - \frac{k}{2^n}| < \epsilon$. Thus the set of cluster points of the sequence $\{\frac{t_n}{u_n}\}_{n=1}^\infty$ is $[0,1]$.

A cluster point is the same as a subsequential limit, and the limits of sequences are preserved by continuous functions. Thus the set of cluster points of $\{s_n\}_{n=1}^\infty$ is the image of $[0,1]$ under the function $x \mapsto \sin(2\pi x)$. This image is $[-1,1]$. □

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