Definition 1. Let $f : A \to B$.
We say that f is *injective* if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, for all $a_1, a_2 \in A$.
We say that f is *surjective* if $\forall b \in B \, \exists a \in A \ni f(a) = b$.
We say that f is *bijective* if f is injective and surjective.

Apply the following standard approaches for basic proofs about sets.

- To show that $f : A \to B$ is injective, select two arbitrary elements in A, and show that they are in B. Start with "Let $a_1, a_2 \in A$, and suppose that $f(a_1) = f(a_2)$." End with "Therefore, $a_1 = a_2$."
- To show that $f : A \to B$ is surjective, pick an arbitrary element of B, and find an element of A that is mapped to it. Start with "Let $b \in B$." End with "Therefore, $f(a) = b$."
- To show that $f : A \to B$ is bijective, show that it is injective, then show that it is surjective.

Problem 1. Let X be a set and let $A, B \subset X$.

(a) Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(b) Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.

(c) Find an example such that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

Solution. **(a)** We show this using an if and only if approach; that is, we show that $x \in \mathcal{P}(A \cap B)$ if and only if $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

$$\begin{aligned} C \in \mathcal{P}(A \cap B) &\Leftrightarrow C \subset A \cap B \\ &\Leftrightarrow \forall c \in C : (c \in A \text{ and } c \in B) \\ &\Leftrightarrow (\forall c \in C : c \in A) \text{ and } (\forall c \in C : c \in B) \\ &\Leftrightarrow C \subset A \text{ and } C \subset B \\ &\Leftrightarrow C \in \mathcal{P}(A) \text{ and } C \in \mathcal{P}(B) \\ &\Leftrightarrow C \in \mathcal{P}(A) \cup \mathcal{P}(B). \end{aligned}$$

(b) Here we adapt the above approach, but use implication instead of logical equivalence, where necessary.

$$\begin{aligned} C \in \mathcal{P}(A) \cup \mathcal{P}(B) &\Leftrightarrow C \in \mathcal{P}(A) \text{ or } C \in \mathcal{P}(B) \\ &\Leftrightarrow C \subset A \text{ or } C \subset B \\ &\Leftrightarrow (\forall c \in C : c \in A) \text{ or } (\forall c \in C : c \in B) \\ &\Rightarrow \forall c \in C : (c \in A \text{ or } c \in B) \\ &\Leftrightarrow C \subset A \cup B \\ &\Leftrightarrow C \in \mathcal{P}(A \cup B). \end{aligned}$$

(c) Let $A = \{1\}$ and $B = \{2\}$. Then $\{1, 2\} \in \mathcal{P}(A \cup B)$, but $\{1, 2\} \neq \mathcal{P}(A) \cup \mathcal{P}(B)$. □

Problem 2. Let X be a set. Define a function $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $A \mapsto X \smallsetminus A$. Show that ϕ is bijective.

Solution. First, we note that if $A \subset X$, then $X \smallsetminus (X \smallsetminus A) = A$. To see this, let $x \in X$. Then $x \in X \smallsetminus (X \smallsetminus A)$ if and only if $x \notin X \smallsetminus A$, so it is not that case that x is not in A, so that x must be in A.

Define $\phi(A) = X \smallsetminus A$. Note that $\phi(\phi(A)) = X \smallsetminus (X \smallsetminus A) = A$.

(Injective) Let $A_1, A_2 \in \mathcal{P}(X)$, and suppose that $\phi(A_1) = \phi(A_2)$. Since ϕ is a function, we may apply ϕ to both sides and get $\phi(\phi(A_1)) = \phi(\phi(A_2))$, i.e., $A_1 = A_2$.

(Surjective) Let $A \in \mathcal{P}(X)$. Let $B = \phi(A)$. Then $\phi(B) = \phi(\phi(A)) = A$. Thus ϕ is bijective. □

Problem 3. Let X be a set and let $T = \{0, 1\}$. Show that there is a correspondence between the sets $\mathcal{P}(X)$ and $\mathcal{F}(X, T)$, by finding a bijective function

$$\Phi : \mathcal{P}(X) \to \mathcal{F}(X, T).$$

Solution. For each $A \subset X$, define a function

$$\phi_A : X \to T \quad \text{by} \quad \phi_A(x) = \begin{cases} 0 & \text{if } x \notin A\ ; \\ 1 & \text{if } x \in A\ . \end{cases}$$

Define a function

$$\Phi : \mathcal{P}(X) \to \mathcal{F}(X, T) \quad \text{by} \quad \Phi(A) = \phi_A.$$

Then Φ is bijective.

(Injectivity) Let $A, B \in \mathcal{P}(X)$ such that $\Phi(A) = \Phi(B)$. Then $\phi_A = \phi_B$. Now

$$x \in A \Leftrightarrow \phi_A(x) = 1 \Leftrightarrow \phi_B(x) = 1 \Leftrightarrow x \in B,$$

so $A = B$, and Φ is injective.

(Surjectivity) Let $\phi \in \mathcal{F}(X, T)$. Then $\phi : X \to T$. Define $A \subset X$ by $A = \phi^{-1}(1)$. Then $\phi_A(x) = \phi$. $\square$

Problem 4. Let X be as set.

(a) Find an injective function $\phi : X \to \mathcal{P}(X)$.

(b) Show that there does not exist a surjective function $\phi : X \to \mathcal{P}(X)$.

Solution. **(a)** Define $\phi : X \to \mathcal{P}(X)$ by $\phi(x) = \{x\}$. This is clearly injective.

(b) Let $\phi : X \to \mathcal{P}(X)$. We show that ϕ is not surjective. Define a set $A \subset X$

$$A = \{x \in X \mid x \notin \phi(x)\}.$$

We now observe that A is not in the range of ϕ. To see this, suppose by way of contradiction that $x \in X$ such that $\phi(x) = A$. Now if $x \in A$, then $x \notin \phi(x) = A$, a contradiction. On the other hand, if $x \notin A$, then it is not the case that $x \in \phi(x) = A$, so x is not in A, again a contradiction. In either case, it is impossible that $\phi(x) = A$. Thus A is not in the range of ϕ, and ϕ is not surjective.

$\square$