REAL ANALYSIS
SOLUTIONS TO ROSS §10

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Exercise 1 (10.6.(a))**.** Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < \frac{1}{2^n} \qquad \text{for all } n \in \mathbb{N}.$$

Show that (s_n) is a Cauchy sequence.

Lemma 1. *Let $m, n \in \mathbb{N}$ with $2 < m < n$. Then*

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{1}{2^m} < \frac{1}{m}.$$

Proof of Lemma. We prove the first inequality by induction on $k = n - m$. If $k = 1$, then our statement reads $\frac{1}{2^{m+1}} < \frac{1}{2^m}$, which is true.

Suppose that our proposition is true for differences of size $k - 1$. Then

$$\sum_{i=m+2}^{n} \frac{1}{2^i} < \frac{1}{2^{m+1}}.$$

Adding $\frac{1}{2^{m+1}}$ to both sides gives

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{2}{2^{m+1}} = \frac{1}{2^m}.$$

For the second inequality, it suffices to show that for $m > 2$ we have $m < 2^m$. For $m = 3$, we have $3 < 4$. By induction, $m - 1 < 2^{m-1}$. Then $m < 2^{m-1} + 1 < 2^{m-1} + 2^{m-1} = 2^m$. □

Proof of Exercise. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\frac{1}{\epsilon} < N$. Let $m, n > N$; assume that $n > m$. Then

$$\begin{aligned}
|s_n - s_m| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{m+1} - s_m| \\
&\leq |s_n - s_{n-1}| + \cdots + |s_{m+1} - s_m| \\
&< \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \\
&< \frac{1}{2^{m-1}} \\
&< \frac{1}{m-1} \leq \frac{1}{N} < \epsilon.
\end{aligned}$$

This shows that (s_n) is a Cauchy sequence. □

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Exercise 2 (10.6.(b))**.** Show that there exists a sequence (s_n) such that $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$, and yet (s_n) is not a Cauchy sequence.

Proof. Let $s_n = \sum_{i=1}^{n} \frac{1}{2n}$. Then for $n \in \mathbb{N}$ we have $|s_{n+1} - s_n| = \frac{1}{2n+2} < \frac{1}{n}$. However, this sequence is unbounded, so it cannot be a Cauchy sequence.

Let $M \in \mathbb{R}$, and let N be so large that $N > 2M$. We claim that $s_{2^N} > 1 + \frac{N}{2} > M$; this is because

$$\begin{aligned}
\sum_{i=1}^{2^N} \frac{1}{i} &= 1 + \sum_{j=1}^{N} \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} \\
&> 1 + \sum_{j=1}^{N} \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{2^j} \\
&= 1 + N(2^j - 2^{j-1})\frac{1}{2^j} \\
&= 1 + \frac{N}{2}.
\end{aligned}$$

□

Exercise 3 (10.7)**.** Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose that $\sup S \notin S$. Show that there is a nondecreasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Proof. Since $\sup S$ is the least upper bound for S, the set $(\sup S - \frac{1}{n}, \sup S) \cap S$ is nonempty, for otherwise $\sup S - \frac{1}{n}$ would be an upper bound for S. For every $n \in \mathbb{N}$, let $s_n \in (\sup S - \frac{1}{n}, \sup S) \cap S$; that is, $s_n \in S$ with $|s_n - \sup S| < \frac{1}{n}$. Then (s_n) is a sequence in S which clearly converges to $\sup S$. □

Exercise 4 (10.8)**.** Let (s_n) be a nondecreasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$. Show that (σ_n) is a nondecreasing sequence.

Remark. Note that if (σ_n) converges, we would be well-advised to view its limit as the average of the infinite number of numbers s_n. □

Proof. We show that $\sigma_n \leq \sigma_{n+1}$, i.e., $\sigma_{n+1} - \sigma_n \geq 0$. Since s_n is nondecreasing, an easy induction shows that

$$n s_{n+1} \geq \sum_{i=1}^{n} s_i.$$

Thus

$$\begin{aligned}
\sigma_{n+1} - \sigma_n &= \frac{\sum_{i=1}^{n+1} s_i}{n+1} - \frac{\sum_{i=1}^{n} s_i}{n} \\
&= \frac{n\sum_{i=1}^{n+1} s_i - (n+1)\sum_{i=1}^{n} s_i}{n(n+1)} \\
&= \frac{n s_{n+1} - \sum_{i=1}^{n} s_i}{n(n+1)} \\
&\geq 0.
\end{aligned}$$

□

Exercise 5 (10.10)**.** Let $s_1 = 1$ and $s_{n+1} = \frac{(s_n+1)}{3}$. Show that (s_n) converges and find the limit.

Proof. If s_n is positive, then so is s_{n+1}; since s_1 is positive, $s_n > 0$ for all $n \in \mathbb{N}$ by induction.

Next we show that $s_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Proceed by induction; $s_1 \geq \frac{1}{2}$, and assume that $s_n \geq \frac{1}{2}$. Then $s_{n+1} = \frac{s_n+1}{3} \geq \frac{3/2}{3} = \frac{1}{2}$.

Next we show that $s_{n+1} < s_n$. Since $s_n \geq \frac{1}{2}$, we have $1 \leq 2s_n$. Thus $\sigma_{n+1} = \frac{s_n+1}{3} \leq \frac{s_n+2s_n}{3} = s_n$.

Since (s_n) is nonincreasing, we see that $s_n \in [\frac{1}{2}, 1]$ for all $n \in \mathbb{N}$. So (s_n) is a bounded monotone sequence, so its limit exists. Let $s = \lim s_n$. We know that $\lim s_{n+1} = \lim s_n$, so $s = \lim s_{n+1} = \lim \frac{s_n+1}{3} = \frac{s+1}{2}$. Solving this for s gives $s = \frac{1}{2}$. □

Exercise 6 (10.12)**.** Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}]t_n$ for $n \geq 1$. Show that (t_n) converges, and find the limit.

Proof. Let $a_n = 1 - \frac{1}{(n+1)^2}$. Since $a_n > 0$ for all $n \in \mathbb{N}$ and $t_1 > 0$, we can see that $t_n > 0$ for all $n \in \mathbb{N}$. Since $a_n < 1$ for all $n \in \mathbb{N}$, we see that (t_n) is nonincreasing. Thus t_n is a bounded monotone sequence, so its limit exists; say $t = \lim t_n$. Then

$$t = \lim t_{n+1} = \lim a_n t_n = \lim a_n \lim t_n = t.$$

That was not helpful.

Claim: $t_n = \frac{n+1}{2n}$.

True for $n = 1$. By induction, $t_{n-1} = \frac{n}{2(n-1)}$. Then

$$\begin{aligned} t_n &= a_{n-1}t_{n-1} \\ &= (1 - \frac{1}{n^2})(\frac{n}{2(n-1)}) \\ &= \frac{n}{2(n-1)} - \frac{1}{2n(n-1)} \\ &= \frac{n^2-1}{2n(n-1)} \\ &= \frac{n+1}{2n}. \end{aligned}$$

Now we see that $t_n \to \frac{1}{2}$ as $n \to \infty$. □

Definition 1. Let $A \subset \mathbb{R}$ be an open interval. A function $f : A \to \mathbb{R}$ is called a *contraction* if there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| \leq M|a - b|$ for any $a, b \in U$.

Example 1. The following are contractions:

- $f(x) = mx + b$, where $m, b \in \mathbb{R}$; $U = \mathbb{R}$ and $M = |m|$;
- $f(x) = \sin(x)$ and $\cos(x)$; $U = \mathbb{R}$ and $M = 1$;
- $f(x) = \log(x)$; $U = (a, \infty)$ where $a > 0$ and $M = \frac{1}{a}$.
- $f(x) = \sqrt{x}$; $U = (a, \infty)$ where $a > 0$ and $M = \frac{1}{2\sqrt{a}}$;
- $f(x)$ is differentiable with bounded derivative on an open interval U; $M = \sup\{|f'(a)| : a \in U\}$.

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. Let (a_n) be a sequence of real numbers which converges to $L \in \mathbb{R}$. Show that $\lim f(a_n) = f(L)$.

Proof. Let $\epsilon > 0$. Since f is a contraction, there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| < M|a - b|$ for all $a, b \in \mathbb{R}$.

Since (a_n) converges to L, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{M}$ for all $n > N$. Since f is a contraction,

$$|f(a_n - f(L)| < M|a_n - L| < M\frac{\epsilon}{M} = \epsilon$$

for all $n > N$. Thus $f(a_n) \to f(L)$. □

Problem 2. For each $i \in \mathbb{N}$, let A_i be a set of real numbers which is bounded above by $M \in \mathbb{R}$. Let $B_n = \cup_{i=1}^{n} A_i$, and let $B_\infty = \cup_{i=1}^{\infty} A_i$. Let $b_n = \sup B_n$. Show that $\lim b_n = \sup B_\infty$.

Proof. For every $a \in B_\infty$, we have $a \leq M$; that is, B_∞ is bounded above and its supremum exists as a real number. Let $b = \sup B_\infty$.

Let $n \in \mathbb{N}$. Then $B_n \subset B_\infty$; by Exercise 4.7.(a), we have $b_n = \sup B_n \leq \sup B_\infty = b$. By Exercise 8.9.(b), $\lim b_n \leq b$.

Suppose that $\lim b_n < b$. Since $b = \sup B_\infty$, there exists $c \in B_\infty$ such that $\lim b_n < c < b$ (otherwise, $\frac{\lim b_n + b}{2}$ is a lower bound for B_∞, contradicting our definition of b). Then $c \in A_N$ for some $i \in \mathbb{N}$, so $c \in B_N$. Since $c \in B_N$, then $c \in B_n$ for all $n > N$. This says that $c \leq \sup B_n = b_n$ for all $n > N$. By Exercise 8.9.(a), we have $\lim b_n \geq c$, contradicting our choice of c. This shows that $\lim b_n \geq b$. □

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