PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §12

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Exercise 1 (12.2)**.** Let (s_n) be a sequence in $\mathbb{R}$.
Show that $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

Proof. Suppose that $\limsup |s_n| = 0$. Since $(|s_n|)$ is a sequence of nonnegative numbers, so is every subsequence. The limit of a convergent sequence of nonnegative numbers is nonnegative, and since $\liminf |s_n|$ is the limit of a subsequence, we have

$$0 \leq \liminf |s_n| \leq \limsup |s_n| = 0.$$

Thus $\liminf |s_n| = \limsup |s_n| = 0$, so $(|s_n|)$ converges to zero. Therefore (s_n) converges to zero.

Suppose that $\lim s_n = 0$. Then $\lim |s_n| = 0$; this tacitly implies that $(|s_n|)$ converges, so its limit superior is equal to its limit. That is, $\limsup |s_n| = 0$. □

Exercise 2 (12.4)**.** Let (s_n) and (t_n) be sequences in $\mathbb{R}$.
Show that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.

Proof. Recall that for any two sets $S, T \subset \mathbb{R}$, we define

$$S + T = \{s + t \mid s \in S, t \in T\}.$$

Let $S_m = \{s_n \mid n > m\}$, $T_m = \{t_n \mid n > m\}$, and $U_m = \{s_n + t_n \mid n > m\}$. We have $\sup(S_m + T_m) = \sup S_m + \sup T_m$ by Exercise 4.14. But $U_m \subset S_m + T_m$, so $\sup U_m \leq \sup S_m + \sup T_m$ by Exercise 4.7. Thus

$$\begin{aligned}
\limsup(s_n + t_n) &= \lim(\sup U_m) \\
&\leq \lim(\sup S_m + \sup T_m) \\
&= \lim(\sup S_m) + \lim(\sup T_m) \\
&= \limsup s_n + \limsup t_n.
\end{aligned}$$

□

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Exercise 3 (12.8)**.** Let (s_n) and (t_n) be bounded sequences over nonnegative real numbers.
Show that $\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n)$.

Excuse for Lemma. The proof of 12.4 can be used here almost without change, if we have the following fact. ☐

Lemma 1. *Let $S, T \subset \mathbb{R}$ be bounded sets of nonnegative real numbers. Define*

$$ST = \{st \mid s \in S, t \in T\}.$$

Then $\sup ST = (\sup S)(\sup T)$.

Proof of Lemma. Let $st \in ST$. Then $s \leq \sup S$ and $t \leq \sup T$. Since s and t are nonnegative, $st \leq (\sup S)(\sup T)$. Thus $\sup ST \leq (\sup S)(\sup T)$.

Suppose $\sup ST < (\sup S)(\sup T)$. Then $\sup ST / \sup T < \sup S$. Select $s \in S$ such that $\sup ST / \sup T < s < \sup S$. Then $\sup ST/s < \sup T$. Select $t \in T$ such that $\sup ST/s < t < \sup T$. Then $\sup ST < st$, a contradiction. ☐

Proof. Let $S_m = \{s_n \mid n > m\}$, $T_m = \{t_n \mid n > m\}$, and $U_m = \{s_n t_n \mid n > m\}$. We have $\sup(S_m T_m) = (\sup S_m)(\sup T_m)$ by our lemma. But $U_m \subset S_m T_m$, so $\sup U_m \leq \sup S_m \sup T_m$ by Exercise 4.7. Thus

$$\begin{aligned}
\limsup(s_n t_n) &= \lim(\sup U_m) \\
&\leq \lim(\sup S_m \sup T_m) \\
&= \lim(\sup S_m) \lim(\sup T_m) \\
&= \limsup(s_n) \limsup(t_n).
\end{aligned}$$

☐

Problem 1. Let $A \subset \mathbb{R}$ be a bounded nonempty set of positive real numbers. Let $B = \{\sqrt{a} \mid a \in A\}$.
Show that $\sqrt{\sup A} = \sup B$.

Proof. Since A is a bounded nonempty set of positive real numbers, the suprema involved exist as nonnegative real numbers. Thus we can take their square roots; we recall that for any nonnegative $x, y \in \mathbb{R}$, we have $x \leq y \Leftrightarrow \sqrt{x} \leq \sqrt{y}$.

Let $a \in A$. Then $\sqrt{a} \in B$, so $\sqrt{a} \leq \sup B$ Thus $a \leq (\sup B)^2$, so $(\sup B)^2$ is an upper bound for A. Since $\sup A$ is the least upper bound, we have $\sup A \leq (\sup B)^2$, so $\sqrt{\sup A} \leq \sup B$.

Let $b \in B$. Then $b = a^2$ for some $a \in A$. Thus $a \leq \sup A$, so $b = \sqrt{a} \leq \sqrt{\sup A}$. Thus $\sup B \leq \sqrt{\sup A}$.

Together, this gives that $\sqrt{\sup A} = \sup B$. ☐

Problem 2. Let (a_n) be a bounded sequence of positive real numbers.
Show that $\limsup \sqrt{a_n} = \sqrt{\limsup a_n}$.

Proof. Since (a_n) is bounded, so is $(\sqrt{a_n}$, and the limit superiors of these sequences exist as real numbers. For $m \in \mathbb{N}$, set

$$A_m = \{a_n \mid n \geq m\} \quad \text{and} \quad B_m = \{\sqrt{a_n} \mid n \geq m\}.$$

Note that $B_m = \{\sqrt{a_n} \mid a_n \in A_m\}$. Then

$$\begin{aligned}
\limsup \sqrt{a_n} &= \lim_{m\to\infty} [\sup B_m] && \text{by definition} \\
&= \lim_{m\to\infty} [\sqrt{\sup A_m}] && \text{by Problem 1} \\
&= \sqrt{\lim_{m\to\infty} [\sup A_m]} && \text{by Example 8.5} \\
&= \sqrt{\limsup a_n} && \text{by definition.}
\end{aligned}$$

□

Problem 3. Let $m, n \in \mathbb{N}$ such that $m < n$.
Show that $\sum_{i=m+1}^{n} \frac{1}{i!} < \frac{1}{m!}$.

Proof. Proceed by induction on $k = n - m$, which is the number of terms being added.

For $k = 1$, we have $\sum_{m+1}^{n} \frac{1}{i!} = \frac{1}{(m+1)}! < \frac{1}{m!}$.

By induction on k, we assume the result to be true if we are adding $k - 1 = n - (m + 2)$ terms; this gives us

$$\sum_{i=m+2}^{n} \frac{1}{i!} < \frac{1}{(m+1)!}.$$

Thus

$$\begin{aligned}
\sum_{i=m+1}^{n} \frac{1}{i!} &= \sum_{i=m+2}^{n} \frac{1}{i!} + \frac{1}{(m+1)!} \\
&< \frac{1}{(m+1)!} + \frac{1}{(m+1)!} \\
&= \frac{2}{(m+1)!} \\
&< \frac{m+1}{(m+1)!} \\
&= \frac{1}{m!}.
\end{aligned}$$

□

Problem 4. Let $a_n = \sum_{i=1}^{n} \frac{1}{i!}$.
Show that (a_n) is a Cauchy sequence.

Proof. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$. Let m and n be natural numbers greater than N, and assume wlog that $m < n$. Then

$$\begin{aligned}
|a_n - a_m| &= \sum_{i=1}^{n} \frac{1}{i!} - \sum_{i=1}^{m} \frac{1}{i!} && \text{because } a_n > a_m \\
&= \sum_{i=m+1}^{n} \frac{1}{i!} \\
&< \frac{1}{m} && \text{by Problem 3} \\
&< \frac{1}{N} \\
&< \epsilon.
\end{aligned}$$

□

Remark 1. Let (a_n) be a sequence of real numbers. If $m \in \mathbb{N}$, then (a_{mk}) is a subsequence of (a_n); that is, if $n_k = mk$, then

$$(a_{mk}) = (a_{n_k}) = (a_m, a_{2m}, a_{3m}, a_{4m}, \dots);$$

for example,

$$(a_{7k}) = (a_7, a_{14}, a_{21}, a_{28}, \dots).$$

Problem 5. Let (a_n) be a sequence of real numbers. Suppose that (a_{mk}) converges for every $m \in \mathbb{N}$, $m \geq 2$.
Show that $\lim a_{xk} = \lim a_{yk}$ for every $x, y \in \mathbb{N}$, $x, y \geq 2$.

Proof. Let $x, y \in \mathbb{N}$. Then (a_{xk}) converges to a real number, say L. Now (a_{xyk}) is a subsequence of (a_{xk}), so (a_{xyk}) also converges to L. But (a_{xyk}) is also a subsequence of (a_{yk}), which converges by hypothesis; every subsequence of (a_{yk}) converges to the same limit, so $\lim a_{yk} = \lim a_{xyk} = L$. □

Fact 1. There are infinitely many prime integers.

Problem 6. Construct a divergent sequence (a_n) of real numbers such that (a_{mk}) converges for every $m \in \mathbb{N}$, $m \geq 2$.

Proof. Define

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime ;} \\ 0 & \text{otherwise .} \end{cases}$$

Since there are infinitely many primes, $\limsup a_n = 1$. Since there are infinitely many nonprimes, $\liminf a_n = 0$. Thus (a_n) does not converge.

However, for any $m \in \mathbb{N}$ with $m \geq 2$, mk is not prime for $k \geq 2$, so $a_{mk} = 0$ for all $k \geq 2$. Thus $\lim_{k \to \infty} a_{mk} = 0$, and (a_{mk}) converges. □

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