HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC III
THE GOLDEN RATIO

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Abstract. We discuss the relationships between the Golden Ratio, Constructibility, regular polygons, and regular solids.

1. The Golden Section

Let A and B be points in a plane. A *section* of $\overline{AB}$ is a point C in the interior of $\overline{AB}$. Consider the case where $|AC| \geq |CB|$; here are various ratios of the lengths of the segments that can be explored, for example $\frac{|AB|}{|AC|}$ and $\frac{|AC|}{|BC|}$.

A *golden section* of $\overline{AB}$ is section C of $\overline{AB}$ which satisfies

$$\frac{|AB|}{|AC|} = \frac{|AC|}{|BC|}.$$

In this case, the common value of these fractions is known as the *golden ratio*; this clearly does not depend on the length of $\overline{AB}$. Thus the golden ratio is a specific, well-defined number which we denote by the Greek letter φ.

Let $x = |AB|$, $y = |AC|$, and $z = |CB|$. In the case of a golden section, $\frac{x}{y} = \frac{y}{z}$, so that $xz = y^2$. Moreover, $x = y + z$, and substituting this into the previous equation and rearranging, we obtain

$$y^2 - zy - z^2 = 0.$$

Then the quadratic formula gives

$$y = \frac{z \pm \sqrt{z^2 + 4z^2}}{2} = z\frac{1 \pm \sqrt{5}}{2}.$$

Since $\sqrt{5} > 1$ and y cannot be negative, one of these solutions is spurious. In the ratio $\frac{y}{z}$, the z's cancel and we obtain

$$\boxed{\varphi = \frac{1 + \sqrt{5}}{2}.}$$

What percentage of a given line segment is a golden section?

$$\frac{y}{x} = \frac{1}{\phi} = \frac{2}{\sqrt{5} + 1} = \frac{2(\sqrt{5} - 1)}{5 - 1} = \frac{\sqrt{5} - 1}{2} \approx 0.62;$$

also,

$$\frac{z}{x} = \frac{x - y}{x} = 1 - \frac{y}{x} = 1 - \frac{\sqrt{5} - 1}{2} = \frac{3 - \sqrt{5}}{2} \approx 0.38.$$

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2. Recreational Appearances of the Golden Ratio

We see that the golden ratio is the positive solution to the polynomial equation $x^2 - x - 1$. In particular,

$$\varphi^2 = \varphi + 1.$$

Moreover, dividing this equation by φ and subtracting 1 from both sides yields

$$\frac{1}{\varphi} = \varphi - 1.$$

So here we have a number whose square is obtained by adding 1, and whose inverse is obtained by subtracting 1.

Consider the continued square root

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}.$$

Assuming that this pattern is meaningful and represents a number, let x be that number. Then clearly $x > 0$. Squaring $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$ yields

$$x^2 = 1 + \sqrt{1 + \sqrt{1 + \dots}} = 1 + x.$$

Thus x satisfies $x^2 - x - 1 = 0$, so $x = \varphi$.

Consider the continued fraction

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}.$$

Again assume that this pattern represents some number x; we see that

$$x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} = 1 + \frac{1}{x}.$$

Multiplying both sides by x gives $x^2 = x + 1$. Thus again we see that $x = \varphi$.

Let's attempt to make this example more precise by restating it using the language of sequences. We wish to construct a (hopefully convergent) sequence $(a_n)_{n \in \mathbb{N}}$ such that each a_n is a fraction representing an approximation of the above continued fraction, with increasing accuracy, so that the limit would be the inescapable meaning of the above continued fraction. Let's begin with $1 + \frac{1}{2}$, and at each stage replace 2 with $1 + \frac{1}{2}$. We obtain

$$\begin{aligned}
a_1 &= 1 + \frac{1}{2} &&= \frac{3}{2} \\
a_2 &= 1 + \frac{1}{1 + \frac{1}{2}} &&= \frac{5}{3} \\
a_3 &= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} &&= \frac{8}{5} \\
a_4 &= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}} &&= \frac{13}{8}
\end{aligned}$$

and so forth. Can you guess the value of a_5? Does this relate to anything else you have previously seen?

3. Construction of the Golden Section

We now describe how to construct a golden section of a given line segment. The idea is to construct a right triangle such that one leg is twice as long as the other, so that by the Pythagorean theorem, the hypotenuse will contain a square root of 5.

Proposition 1. *A golden section is constructible.*

Construction. We are given line segment $\overline{AB}$; we construct a point Z between A and B such that $\frac{|AB|}{|AZ|} = \frac{|AZ|}{|ZB|}$, or equivalently, such that $|AZ| = \frac{\sqrt{5}-1}{2}$.

- **(a)** Let D be the point of intersection of line AB and circle $A - B$ which is not B.
- **(b)** Let E be the midpoint of $\overline{DA}$.
- **(c)** Let F be the point of intersection of circle $A - B$ and the line through A perpendicular to A.
- **(d)** Let Z be the point of intersection of line AB and circle $E - F$ which lies on $\overline{AB}$.

To see that this succeeds, scale our situation so that $|AB| = 1$. Then $|DA| = 1$, so $|EA| = \frac{1}{2}$. Also, $|FA| = 1$, so by the Pythagorean Theorem, $|EF| = |EZ| = \sqrt{1^2 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$. Thus $|AZ| = |EZ| - |EA| = \frac{\sqrt{5}-1}{2}$. □

4. The Golden Rectangle

Consider a rectangle $ABDC$ such that sides $\overline{AB}$ and $\overline{CD}$ are the longer sides, with length x, and that sides $\overline{AC}$ and $\overline{BD}$ are shorter, with length y. Let E and F lie on $\overline{AB}$ and $\overline{CD}$, respectively, so that $AEFC$ is a square. We call rectangle $ABDC$ a *golden rectangle* if rectangle $ABDC$ is similar to rectangle $FEBD$.

Suppose that rectangle $ABCD$ is golden, and let $z = |EB|$; then $x = y + z$. By similarity, we have $\frac{x}{y} = \frac{y}{z}$, which leads to $y^2 - zy - z^2 = 0$. We see that E and F cut $\overline{AB}$ and $\overline{BD}$ in a golden section, and $\frac{x}{y} = \varphi$. Thus a golden rectangle is constructible as a rectangle build on a golden section.

Proposition 2. *A golden rectangle is constructible.*

Construction. We are given point A and B which form one side of the rectangle.

- **(a)** Let C be a golden section of $\overline{AB}$, with longer side $\overline{AC}$.
- **(b)** Let D be the point of intersection of circle $A - C$ and the line through A which is perpendicular to AB.
- **(c)** Let E be the point of intersection of the line through B which is perpendicular to B, and the line through D which is parallel to AB.

Then $ABED$ is a golden rectangle. □

5. The Golden Triangle

Consider an isosceles triangle $\triangle ABC$, where $\angle ABC = \angle ACB$. Let D be the point of intersection of line AC and a bisector of angle $\angle ABC$. We call $\triangle ABC$ a *golden triangle* if $\triangle ABC$ is similar to $\triangle BDC$.

Suppose that $\triangle ABC$ is golden, and let $x = |AB| = |AC|$ and $y = |BC|$. Then $\triangle BDC$ is isosceles, and $|BD| = |BC| = y$. Also $\angle BAC = \angle ABD$, so $\triangle DAB$ is also an isosceles triangle, and $|AD| = |BD| = y$. Let $z = |DC|$; then $x = y + z$. By similarity, we have $\frac{x}{y} = \frac{y}{z}$; therefore, as before, $\frac{x}{y} = \varphi$.

We may compute the angles of a golden triangle as follows. Let $\alpha = \angle BAC$ and $\beta = \angle ABC = \angle ACB$, so that $\beta = 2\alpha$. Then $5\alpha = 180°$, so $\alpha = 36°$ and $\beta = 72°$.

This allows us to compute $\cos 72°$; construct a right triangle $\triangle AEB$ by letting E be the midpoint of $\overline{BC}$. Then $|BE| = \frac{y}{2}$, so

$$\cos\beta = \frac{y}{2x} = \frac{1}{2\varphi}.$$

Since $\frac{1}{\varphi} = \varphi - 1$, conclude that

$$\boxed{\cos 72° = \frac{-1+\sqrt{5}}{4}.}$$

This fact will help us in the construction of a golden triangle.

Proposition 3. *A golden triangle is constructible.*

Construction. We are given points A and D; we construct points B and C so that $\triangle ABC$ is golden, with base $\overline{AB}$.

- **(a)** Let B be a golden section of $\overline{AD}$, we longer side $\overline{AB}$.
- **(b)** Let C be the point of intersection of the circle $A - D$ and the line through B which is perpendicular to AD.

□

6. Construction of a Regular Pentagon

We are given two points O and A, and we wish to construct a regular pentagon inscribed in the circle $O - A$ such that A is one of the vertices. If Z is a vertex adjacent to A, then $\angle AOZ = \frac{360^\circ}{5} = 72^\circ$. Thus if we can construct on $\overline{OA}$ a section Y such that $|OY| = \cos 72^\circ = \frac{-1+\sqrt{5}}{4}$, we will be well on our way to construction of the regular pentagon. We have seen that this is possible; we repeat the construction here.

Proposition 4. *A regular pentagon is constructible.*

Construction. We are given point O and A with $|OA| = r$. For simplicity and without loss of generality, assume that $r = 1$.

- **(a)** Let B be the point of intersection of line OA and circle $O - A$ which is not A.
- **(b)** Let C be the midpoint of $\overline{BO}$.
- **(c)** Let D be a point of intersection of the line through O which is perpendicular to OA, and the circle $O - A$.
- **(d)** Let E be the point of intersection of line OA and circle $C - D$.
- **(e)** Let F be the midpoint of $\overline{OE}$.
- **(f)** Let Z be the point of intersection of circle $O - A$ and the line through F which is perpendicular to OA.

Then $\angle AOZ = 72^\circ$, so that $\overline{AZ}$ is the side of a regular pentagon inscribed in circle $O - A$. The other sides are now attainable. $\square$

7. The Golden Pentagram

The *diagonals* of a regular pentagon are the line segments between non-adjacent edges. There are five such diagonals; there union is known as the *golden pentagram*. This star-shaped figure was used as the logo of the Pythagorean brotherhood.

Let A, B, C, D, and E be the vertices of a regular pentagon, labeled in counterclockwise order. Label the points of intersection of the diagonals as follows: $G \in \overline{AC} \cap \overline{BE}$, $H \in \overline{BD} \cap \overline{CA}$, $I \in \overline{CE} \cap \overline{DB}$, and $J \in \overline{DA} \cap \overline{EC}$.

We wish to show that $\triangle ACD$ is a golden triangle, and that polygon $FGHIJ$ is another regular pentagon.

Let $\alpha = \angle CAD$, $\beta = \angle ACD$, $\gamma = \angle BAC$, and $\delta = \angle BAE$.

By the formula for the angles of a regular polygon, we have

$$\delta = 180°(1 - \frac{2}{n}) = 108°.$$

Since pentagon $ABCDE$ is regular, the Side-Angle-Side Theorem implies that

$$\triangle ABC \cong \triangle BCD \cong \triangle CDE \cong \triangle DEA \cong \triangle EAB,$$

where the symbol $\cong$ means "is congruent to"; moreover, these are all isosceles triangles. Thus $\gamma = \angle ABE$. This shows that $\triangle FAB$ is similar to $\triangle ABE$, which is isosceles; thus $\angle AFB = \delta$, so $2\gamma + \delta = 180°$, which gives

$$\gamma = \frac{180° - \delta}{2} = 36°.$$

We also have $\gamma = \angle DAE$, so $\angle BAE = \delta = 2\gamma + \alpha$, so

$$\alpha = \delta - 2\gamma = \gamma = 36°.$$

Now $\overline{AC} = \overline{AD}$ because $\triangle BAC \cong \triangle EAD$, so $\triangle ACD$ is isosceles. Thus $\alpha + 2\beta = 180°$, so

$$\beta = \frac{180° - \alpha}{2} = 72°.$$

Thus $\triangle CAD$ is a golden triangle.

Similarly, one finds other golden triangles in this diagram; we see that

$$\triangle ACD \cong \triangle BDC \cong \triangle CEA \cong \triangle DAB \cong \triangle EBC.$$

We also see that

$$\triangle ABG \cong \triangle BCH \cong \triangle CDI \cong \triangle DEJ \cong \triangle EAF,$$

and

$$\triangle AFJ \cong \triangle BGF \cong \triangle CHG \cong \triangle DIH \cong \triangle EJI$$

are sets of congruent golden triangles. From this, polygon $FGHIJ$ is a regular pentagon.

8. Incommensurability

Let A, B, C, and D be points in a plane. We say that $\overline{AB}$ and $\overline{CD}$ are *commensurable* if there exists a line segment $\overline{EF}$ and positive integers m and n such that

$$|AB| = m\overline{EF} \quad \text{and} \quad |CD| = n\overline{EF};$$

thus $\frac{|AB|}{|CD|} = \frac{m}{n}$. The Pythagoreans assumed in their proofs that any two line segments as commensurable. Suppose that $\overline{CD} = 1$; then this assumption amounts to

$$|AB| = \frac{m}{n} \in \mathbb{Q},$$

that is, the length of any line segment is a rational number.

Thus for the Pythagoreans, it must have been quite a shock to realize that not all constructible numbers are rational. This may have been discovered during contemplation of the golden pentagram, and follows.

Continue notation from the previous section. Since $\triangle ACD$ is golden, we have $|AC|/|CD| = \varphi$. Now $|CD| = |CI|$ and $\triangle CDI$ is golden, so $|CD|/|DI| = \varphi$. But then $\triangle DIH$ is golden, so $|IH|/|DH| = \varphi$. At this point, we notice that $\overline{HI}$ is an edge of the regular pentagon $FGHIJ$, and the diagonals of this pentagon have length $|DH|$. If we inscribe another pentagram in this pentagon, we see that this chain of equalities will continue forever.

Now if all line segments are commensurable, there exists a line segment $\overline{MN}$ such that $|AC|$ and $|CD|$ are in integer multiples of $|MN|$. Now $|AI| = |CD|$, so $|AI|$ is also an integer multiple of $|MN|$. This shows that

$$|DI| = |AD| - |AI|$$

is also a multiple of $|mn|$. Repeating this argument shows that $|HI|$ is an integer multiple of $|MN|$, and this continues into the smaller pentagon.

We can continue this process, getting smaller and smaller pentagons with smaller and smaller edges, but each edge will be an integral multiple of some line segment $\overline{MN}$ of fixed length. Perhaps it was this contradiction which first demonstrated the existence of irrational numbers.

9. Regular Solids

Recall that a *polygon* is a plane figure bounded by line segments. A plane region is *convex* if, given any to points in the interior, the line segment between these points is contained in the interior.

Recall that a *polyhedron* is a space figure bounded by polygons. The bounding polygons are called *faces*, the bounding line segments of these polygons are called *edges*, and the endpoints of these line segments are called *vertices*. Again, a space region is *convex* if, given any to points in the interior, the line segment between these points is contained in the interior.

A polyhedron is *regular* if

- **(a)** it is convex;
- **(b)** its faces of congruent regular polygons;
- **(c)** its vertices have the same number of attached edges.

Regular polyhedra are also known as regular solids, or as Platonic solids. We wish to classify the regular solids.

First, we decide what the possibilities are, and then we describe the construction of each possibility.

The key to deciding the possibilities is to realize that if multiple faces come together are a vertex, there must be at least three faces, and the sum of the angles which come together must be less than $360°$.

The following chart indicates the possibilities. The first column represents the number of sides of the polygonal faces. By the formula angle $= 180°(1 - \frac{2}{n})$, we compute the internal angles of a regular n-gon. Then we see how many faces can come together are a vertex.

Sides	Angle/Vertex	Faces/Vertex	Total Angle	Possible?
3	$60°$	3	$180°$	Yes
3	$60°$	4	$240°$	Yes
3	$60°$	5	$300°$	Yes
3	$60°$	≥ 6	$\geq 360°$	No
4	$90°$	3	$270°$	Yes
4	$90°$	≥ 4	$\geq 360°$	No
5	$108°$	3	$324°$	Yes
5	$108°$	≥ 4	$\geq 432°$	No
≥ 6	$\geq 120°$	≥ 3	$\geq 360°$	No

So we have five possibilities; there are at most five regular solids (up to similarity). Next we demonstrate that each of the five possibilities exist.

10. Construction of the Regular Solids

We wish to construct each regular solid using Euclidean tools (even through we analyze the construction using analytic geometry). It suffices to construct the vertices in $\mathbb{R}^3$ from the set $Q = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3$.

First start with 3 squares coming together at a vertex. This will form a a solid with six sides which we may call a Hexahedron, but is usually known as a cube, The cube is easily constructed from the set Q; for example, $(1,1,0)$ is the intersection of a line on the xy-plane perpendicular to the x-axis through $(1,0,0)$, and a line on the xy-plane perpendicular to the y-axis through $(0,1,0)$. The complete vertex set is

$$\text{Cube} = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}.$$

Next construct a regular solid with 3 equilateral triangles coming together at each vertex; this solid will have 4 faces, and is thus known as a regular tetrahedron. We see tetrahedra embedded in the cube by drawing line segments diagonally across the faces; this will create two sets of vertices of regular tetrahedra. One of these sets is

$$\text{Tetrahedron} = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}.$$

This will produce regular faces if all of the edges have the same length. Computation shows that indeed, the edges have length $\sqrt{2}$.

Now we wish to produce a regular solid with 4 equilateral triangles coming together at each vertex; this solid will have 8 faces, and so it is known as a regular octahedron. To construct a regular octahedron, take its vertices to be the set of centers of the faces of the cube; this will give 6 vertices; take the cube to have sides of length two to simplify the situation, and find that

$$\text{Octahedron} = \{(1,1,0),(1,0,1),(0,1,1),(1,1,2),(1,2,1),(2,1,1)\}.$$

The lengths of the edges of this solid are also $\sqrt{2}$.

We note that if we take the centers of the faces of an octahedron as vertices for a solid, we obtain a cube; that is, the 8 vertices of the cube correspond to the 8 faces of the octahedron, just as the 6 vertices of the octahedron correspond to the 6 faces of the cube. We say that the cube and the octahedron are *dual* polyhedra. Note that the dual of the tetrahedron is another tetrahedron; it is *self-dual.*

Next we construct a solid with five triangular faces coming together at each vertex, which has twenty faces and as such is known as an icosahedron. To do this, embed three golden rectangles with sides of length 1 and φ in $\mathbb{R}^3$ on the coordinate planes so that the center of each rectangle is the origin.

Let $\alpha = \frac{1}{2}$ and let $\beta = \frac{1+\sqrt{5}}{4} = \frac{\varphi}{2}$. Set

$$\text{Icosahedron} = \{(0,\pm\alpha,\pm\beta),(\pm\alpha,\pm\beta,0),(\pm\beta,0,\pm\alpha)\},$$

this set contains 12 points, and produces a solid with 20 triangular faces. For example, one of the faces has vertices $A = (\beta,\alpha,0)$, $B = (\beta,-\alpha,0)$, and $C = (\alpha,0,\beta)\}$. That $|AB| = 1$ is clear, and that $|AC| = |BC|$ is also clear. To see that this is an equilateral triangle, we compute

$$
\begin{aligned}
|AC| &= \sqrt{(\alpha-\beta)^2 + (0-\alpha)^2 + (\beta-0)^2} \\
&= \sqrt{2\alpha^2 + 2\beta^2 - 2\alpha\beta} \\
&= \sqrt{\frac{1}{2} + \frac{\varphi^2}{2} - \frac{\varphi}{2}} \\
&= \sqrt{\frac{1 + (\varphi+1) - \varphi}{2}} \\
&= 1.
\end{aligned}
$$

Thus indeed, we have constructed a regular triangle, so we have a regular icosahedron.

Finally, we consider the case of three regular pentagons coming together at a vertex; this produces a polyhedron with twelve faces known as a dodecahedron. We can obtain this as the dual of the icosahedron; that is, let the vertex set be the set of centers of the faces of a regular icosahedron.

We investigate this vertex set. The center of an equilateral triangle in space is obtained by averaging the coordinates of the vertices; that is, the center of the equilateral triangle $\triangle A_1A_2A_3$, where $A_i = (x_i, y_i, z_i)$, is

$$\Big(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\Big).$$

In our case, we obtain two types of triangles: those who share a side with one of the golden rectangles, and those whose vertices come from three different golden rectangles. There are twelve of the former and eight of the latter.

The first twelve are easy to see: there are six sides of length 1 on the rectangles, and two triangles sharing each such side. The eight others are obtained by noticing that only certain combinations are possible. Here is a complete list, with all coordinates multiplied by three, $\gamma = \alpha + 2\beta$, and $s_1, s_2, s_3 \in \{\pm 1\}$.

$$
\begin{aligned}
\text{Dodecahedron} = \{&(\pm\gamma, 0, \pm\beta), (\pm\beta, \pm\gamma, 0), (0, \pm\beta, \pm\gamma), \\
&(s_1\beta, s_2\alpha, 0), (0, s_2\beta, s_3\alpha), (s_1\alpha, 0, s_3\beta)\}.
\end{aligned}
$$

11. Exercises

Exercise 1. Consider the sequence (a_n) of real numbers defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \sqrt{1 + a_n}.$$

Assuming that (a_n) converges, find $\lim_{n\to\infty} a_n$. To prove that (a_n) converges, show that (a_n) is bounded and increasing.

Exercise 2. Consider the sequence (a_n) of real numbers defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{1}{1 + a_n}.$$

Assuming that (a_n) converges, find $\lim_{n\to\infty} a_n$.

Exercise 3. Consider a pyramid with four triangular sides and a square base. Let h be the height of the pyramid. Let s be the height of a triangular side, let a be half the length of its base, so that the area of the triangular side is sa. Show that if $h^2 = sa$, then the slope of the pyramid, $\frac{s}{a}$, is equal to the Golden Ratio.

Exercise 4. Consider the regular solids inscribed in a unit sphere.

- **(a)** Find the lengths of the line segments for each solid.
- **(b)** Find the area of a face of each solid.
- **(c)** Find the angle between the faces of each solid.
- **(d)** Find the radius of the inscribed sphere of each solid.
- **(e)** Find the volume of each solid.

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