Real Analysis | Homework 4 | Name:
Dr. Paul L. Bailey | Friday, September 14, 2018

Due Tuesday, September 18, 2018. Write solutions neatly on $8.5 \times 11$ printer paper. Stable this sheet to the front of your solutions.

**Problem 1. (Dominance Theorem)**
Let $(s_n)$ and $(t_n)$ be convergent sequences of real numbers such that $s_n \leq t_n$ for all $n \in \mathbb{N}$. Show that $\lim s_n \leq \lim t_n$.

*Proof.* Let $S = \lim s_n$ and $T = \lim t_n$, and suppose bwoc that $S > T$.

Let $N_1 \in \mathbb{N}$ be so large that $n \geq N$ implies $|s_n - S| < \dfrac{S-T}{2}$ for all $n \geq N_1$.

Let $N_2 \in \mathbb{N}$ be so large that $n \geq N$ implies $|t_n - T| < \dfrac{S-T}{2}$ for all $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$.

Note that $s_N > \dfrac{S+T}{2}$. To see this, write $|s_N - S| < \dfrac{S-T}{2}$. Then $-\dfrac{S-T}{2} < s_N - S < \dfrac{S-T}{2}$. Add $S$ to the first inequality to get $\dfrac{S+T}{2} = S - \dfrac{S-T}{2} < s_N$. Similarly, $t_N < \dfrac{S+T}{2}$.

Now we have $t_N < \dfrac{S+T}{2} < s_N$, contradicting the premise. □

**Problem 2. (Squeeze Theorem)**
Let $(a_n)$, $(b_n)$, and $(c_n)$ be sequences of real numbers such that $a_n \leq b_n \leq c_n$. Suppose that $(a_n)$ and $(c_n)$ both converge to $L \in \mathbb{R}$. Show that $(b_n)$ converges to $L$.

*Solution.* Let $\epsilon > 0$.
Let $N_1$ be so large that $n \geq N_1$ implies $|a_n - L| < \epsilon$.
Let $N_2$ be so large that $n \geq N_2$ implies $|c_n - L| < \epsilon$.
Let $N = \max\{N_1, N_2\}$.
Then, for $n \geq N$, we have $-\epsilon < a_n - L < \epsilon$, so $L - \epsilon < a_n < L + \epsilon$. Similarly, $L - \epsilon < c_n < L + \epsilon$. Thus

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon.$$

From this, $|b_n - L| < \epsilon$. □

**Problem 3** (Ross 9.5)**.** Consider the recursively defined sequence given by $t_1 = 1$ and $t_{n+1} = \dfrac{t_n^2 + 2}{2t_n}$. Assume that $(t_n)$ converges, and find the limit.

*Solution.* Let $L = \lim t_n$. It is clear that $\lim a_{n+1} = L$. Taking the limit of both sides of the recursive equation gives

$$L = \lim \frac{t_n^2 + 2}{2t_n} = \frac{(\lim t_n)^2 + 2}{2 \lim t_n};$$

the latter equal sign follows from the rules for the arithmetic of sequences which we have previously proven. This gives the equation $L = \dfrac{L^2+1}{L}$, which we solve to $L$ to obtain $L = \sqrt{2}$. □

**Problem 4** (Ross 9.15)**.** Let $a$ be a positive real number. Explain why $\displaystyle\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

*Proof.* Let $\epsilon > 0$. By the Archimedean Principle, there exists $M \in \mathbb{N}$ such that $M > a$. Let $B = \dfrac{a^M}{M!}$. Let $N \in \mathbb{N}$ be so large that $N > \dfrac{aB}{\epsilon}$. In this case, we have $\dfrac{a}{N} < \dfrac{\epsilon}{B}$. Then, for $n \geq N$, we have

$$\left|\frac{a^n}{n!}\right| = B\Big(\frac{a}{M+1} \cdots \frac{a}{n}\Big) \leq B\Big(\frac{a}{M+1} \cdots \frac{a}{N}\Big) < B\frac{a}{N} = B\frac{\epsilon}{B} < \epsilon.$$

□

**Problem 5** (Challenge)**.** Let $(s_n)$ be a sequence of real numbers which converges to $s \in \mathbb{R}$. Let

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}.$$

Show that $(\sigma_n)$ converges to $s$.

*Solution.* Let $\tau_n = \sigma_n - s$. It suffices to show that $(\tau_n)$ converges to zero. Note that

$$\tau_n = \frac{1}{n}\sum_{i=1}^{n} s_i - \frac{ns}{n} = \frac{1}{n}\sum_{i=1}^{n}(s_i - s).$$

Let $N_0 \in \mathbb{N}$ be so large that $|s_n - s| < \frac{\epsilon}{2}$ for all $n > N_0$. Let $M = \sum_{i=1}^{N} |s_i - s|$. Then for $n > N_0$, we have

$$\begin{aligned}
|\tau_n| &\leq \frac{M}{n} + \frac{1}{n}\sum_{i=N_0+1}^{n} |s_n - s| && \text{by } \Delta\text{-inequality} \\
&< \frac{M}{n} + \frac{1}{n}(n - N_0)\frac{\epsilon}{2} && \text{summing } n - N_0 \text{ small numbers} \\
&< \frac{M}{n} + \frac{\epsilon}{2} && \text{since } \frac{n - N_0}{n} \leq 1.
\end{aligned}$$

Now select $N \in \mathbb{N}$ with $N > N_0$ which is so large that $\frac{M}{n} < \frac{\epsilon}{2}$. Then for $n > N$, we have $|\tau_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that $|\tau_n| \to 0$ as $n \to \infty$. Thus $\lim \tau_n = 0$. $\square$