Definition 1. (Locus of an Equation)
The *cartesian plane* is the set of all ordered pairs of real numbers. These ordered pairs are the *points* in the plane.

Consider an equation in two variables, x and y. The *locus* of the equation is the set of points (x, y) in the cartesian plane which, when plugged into the equation, make it true.

The *equation of a line* is an equation in x and y such that a point is on the line if and only if it satisfies the equation.

Example 1. (Line Equation as a Test)
Consider the line $y = 2x - 5$. Determine if either of the points $(1, 7)$ or $(4, 3)$ are on the line.

Solution. View the equation as a test, which checks to see if a point is on the line. We plug the points into the equation to see if they make it true.

(a) $(1, 7)$ Since $7 = 2(1) - 5$ is false, the point IS NOT ON the line.
(b) $(4, 3)$ Since $4 = 2(3) - 5$ is true, the point IS ON the line.

□

The *equation of a circle* is an equation in x and y such that a point is on the circle if and only if it satisfies the equation. Consider a circle whose center is (h, k) and whose radius is r. If (x, y) is an arbitrary point on the circle, then the distance from (x, y) to (h, k) is r. The distance formula, which comes from the Pythagorean Theorem, states this as

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

Squaring both sides leads to the standard form of the equation of a circle.

Definition 2. (Equation of a Circle) The equation of a circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2.$$

Example 2. (Circle Equation as a Test) Consider the equation $(x-1)^2 + (y+2)^2 = 25$. Find its center and radius. Is either of the points $(3, 4)$ or $(5, 1)$ on the line?

Solution. Match the given equation to $(x-h)^2 + (y-k)^2 = r^2$. We see that $h = 1$, $k = -2$, and $r^2 = 25$ or $r = 5$ (the radius must be positive). Thus, the center is $(1, -2)$ and the radius is r.

Now use the equation as a test to check if a given point is on the circle. Plug the points into the equation to see if they make it true.

(a) $(3, 4)$ Since $(3-1)^2 + (4+2)^2 = 2^2 + 6^2 = 4 + 36 = 40 \neq 25$, the point IS NOT ON the circle.
(b) $(5, 1)$ Since $(5-1)^2 + (1+2)^2 = 4^2 + 3^2 = 16 + 9 = 25$, the point IS ON the circle.

□

Example 3. (Find the Center and Radius)
The locus of the equation $x^2 - 8x + y^2 + 14y = 12$ is a circle. Find its center and radius.

Solution. We complete the square. Add 16 and 49 to both sides, thusly:

$$x^2 - 8x + 16 + y^2 + 14y + 49 = 12 + 16 + 49.$$

Associate the appropriate terms, and add the right hand side:

$$(x^2 - 8x + 16) + (y^2 + 14y + 49) = 77.$$

Factor to get:

$$(x-4)^2 + (y+7)^2 = 77.$$

The center is $(4, -7)$ and the radius is $\sqrt{77}$. □

Example 4. (Find the Intersection of a Line and a Circle)
Find the points of intersection of the circle $(x-3)^2+(y-4)^2=72$ and the line $y=x+1$.

Solution. Substitute the equation of the line into the equation of the circle, and solve to x. We get:

$$\begin{aligned}(x-3)^2+((x+1)-4)^2=72 &\Rightarrow (x-3)^2+(x-3)^2=36\\ &\Rightarrow (x-3)^2=36\\ &\Rightarrow x-3=\pm 6\\ &\Rightarrow x=9 \text{ or } x=-3.\end{aligned}$$

Plug this into the line to get the corresponding y: if $x=9$, $y=x+1=10$, and if $x=-3$, $y=x+1=-2$. So, the two points of intersection are

$$\boxed{(9,10) \text{ and } (-3,-2).}$$

□