1. The Number e

1.1. **Periodic Compound Interest.** Suppose we invest 1000 dollars at an interest rate of 10 percent compounded annually. The amount we have invested remains the same until one year passes, at which point 10 percent of the amount is added to the total. If we let A_t denote the amount invested after t years, then

- $A_0 = 1000$
- $A_1 = 1000 + (0.1)1000 = 1100$
- $A_2 = 1100 + (0.1)1100 = 1210$
- $A_3 = 1210 + (0.1)1210 = 1331$

We see that the rate at which this grows increases year by year; but the pattern is obscure. It is actually easier to see the pattern if we think more generally.

Let r be the annual interest rate, A_0 the initial investment, and A_t the amount after t years. Then

- $A_1 = A_0 + rA_0 = A_0(1 + r)$
- $A_2 = A_1 + rA_1 = A_1(1 + r) = A_0(1 + r)^2$
- $A_3 = A_2 + rA_2 = A_2(1 + r) = A_0(1 + r)^3$
- $A_t = A_0(1 + r)^t$

Suppose that, instead of compounding annually, we compound quarterly; that is, every three months, or four times per year. Then, the periodic interest rate is the annual rate divided by four.

- $A_{1/4} = A_0 + (\frac{r}{4})A_0 = A_0(1 + \frac{r}{4})$
- $A_{1/2} = A_{1/4} + (\frac{r}{4})A_{1/4} = A_{1/4}(1 + \frac{r}{4}) = A_0(1 + \frac{r}{4})^2$
- $A_1 = A_0(1 + \frac{r}{4})^4$
- $A_t = A_0(1 + \frac{r}{4})^{4t}$

Generalize this further; let k denote the number of periods per year, so that we compound k times per year. Then, there are k times every year when we the amount in the account by $(1 + \frac{r}{k})$; these gives

$$A_t = A_0\left(1 + \frac{r}{k}\right)^{kt},$$

where r is the annual rate, k is the number of periods per year, and A_t is the amount after t years.

The more periods per year, the faster the amount grows, as this table demonstrates. We let the annual rate r be ten percent and the initial investment A_0 be one thousand. We compute the amount after five years for various values of k, to the nearest dollar:

k	A_0	A_1	A_2	A_3	A_4	A_5
1	1000	1100	1210	1331	1464	1611
2	1000	1103	1216	1340	1477	1629
4	1000	1104	1218	1345	1485	1639
12	1000	1105	1220	1348	1489	1645
365	1000	1105	1221	1350	1492	1649
8760	1000	1105	1221	1350	1492	1649

This table demonstrates two facts:

- as k increases, the investment grows faster;
- as k increases, the rate at which the investment grows faster slows down.

1.2. **Continuous Compound Interest.** We wish to define continuously compounded interest as the limit of periodically compounded interest as the k goes to infinity. Thus we fix A_0, r, and t, and attempt to understand the expression

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}.$$

To do this, we define a new variable n by $n = \frac{k}{r}$, so that $k = nr$ and $\frac{r}{k} = \frac{1}{n}$. Since r is fixed, n goes to infinity as k goes to infinity. We compute

$$\begin{aligned}\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} &= \lim_{n\to\infty} A_0\left(1+\frac{1}{n}\right)^{nrt}\\ &= \lim_{n\to\infty} A_0\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}\\ &= A_0\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}.\end{aligned}$$

This computation tells us that continuously compounded interest may be computed using an exponential function whose base is the limit of the sequence $(1+\frac{1}{n})^n$; it can be show that this is an increasing sequence which is bounded above by 3, so it converges. The number it converges to turns out to be so important in mathematics that we give it a special name.

Define

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Then, the equation which computes the amount A_t for continuously compounded interest is

$$A_t = A_0 e^{rt}.$$

We estimate e by computing a few values:

n	$(1+\frac{1}{n})^n$	estimate
1	$(2)^1$	2.000000
2	$(1.5)^2$	2.250000
4	$(1.25)^4$	2.441406
10	$(1.1)^{10}$	2.593742
100	$(1.01)^{100}$	2.704813
1000	$(1.001)^{1000}$	2.716923
10000	$(1.0001)^{10000}$	2.718145
100000	$(1.00001)^{100000}$	2.718268
∞	e	2.718281