CATEGORY THEORY
CATEGORY V - TOPOLOGICAL SPACES

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Definition 1 (Objects)**.** Let X be a set. A *topology* on X is a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ such that

(T1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
(T2) $\mathcal{U} \subset \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$;
(T3) $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ finite $\Rightarrow \cap\mathcal{U} \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*.

A subset $A \subset X$ is called *open* if $A \in \mathcal{T}$, and is called *closed* if $X \smallsetminus A \in \mathcal{T}$.

Example 1. Let X be a set and let $\mathcal{T} = \{\varnothing, X\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *trivial* topology on X.

Example 2. Let X be a set and let $\mathcal{T} = \mathcal{P}(X)$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *discrete* topology on X.

Example 3. Let X be a set and let $\mathcal{T} = \{A \subset X \mid X \smallsetminus A \text{ is finite }\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *cofinite* topology on X.

Example 4. Let X be a set and let $\mathcal{T} = \{A \subset X \mid X \smallsetminus A \text{ is countable }\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *cocountable* topology on X.

Definition 2. Let X be a set. A *tower* of subsets of X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ which contains the empty set and the entire set and is totally ordered by inclusion.

Example 5. Let X be a set and $\mathcal{T}$ a tower of subsets of X. Then $\mathcal{T}$ is a topology on X, called a *tower topology*.

Example 6. Let X be a totally ordered set. For $a \in X$, set

$$L_a = \{x \in X \mid x < a\} \quad \text{and} \quad R_a = \{x \in X \mid x > a\}.$$

Set

$$\mathcal{L} = \{L_a \mid a \in X\} \cup \{\varnothing, X\} \quad \text{and} \quad \mathcal{R} = \{R_a \mid a \in X\} \cup \{\varnothing, X\}.$$

Then $\mathcal{L}$ is a topology on X, called the *left order topology*, and $\mathcal{R}$ is a topology on X, called the *right order topology*.

Example 7. Let (X, ρ) be a metric space. Let $U \subset X$ and say that U is *open* if for every $u \in U$ there exists $\epsilon > 0$ such that $x \in U$ whenever $\rho(x, u) < \epsilon$. Let $\mathcal{T}$ denote the collection of open sets. Then $(X, \mathcal{T})$ is a topological space.

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Definition 3 (Subobjects)**.** Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. The *relative topology* on Y with respect to $\mathcal{T}$ is

$$\mathcal{U} = \{U \subset X \mid U = O \cap Y \text{ for some } O \in \mathcal{T}\}.$$

Then $\mathcal{U}$ is a topology on Y, and the topological space $(Y, \mathcal{U})$ is called a *subspace* of the topological space $(X, \mathcal{T})$.

Definition 4 (Neighborhoods)**.** Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. An *open neighborhood* of x is an open set $U \in \mathcal{T}$ with $x \in U$. A *neighborhood* of x is a subset of X which contains an open neighborhood of X.

Definition 5 (Morphisms)**.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Let $f : X \to Y$ and let $x \in X$. We say that f is *continuous at x* if for every neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subset V$.

Let $A \subset X$. We say that f is *continuous on A* if f is continuous at x for every $x \in A$. We say that f is *continuous* if f is continuous on X.

We will show that the composition of continuous functions is continuous. Thus, topological spaces together with continuous functions form a category.

Proposition 1. *A function $f : X \to Y$ is continuous if and only if the preimage of every open set in Y is open in X.*

Proposition 2. *The composition of continuous functions is continuous.*

Definition 6. Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective function which is continuous with a continuous inverse. Two spaces are said to be *homeomorphic* if there exists a homeomorphism between them.

So, homeomorphism is isomorphism in the category of topological spaces.

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