Algebra II — Lesson 0509
Dr. Paul L. Bailey — Monday, May 9, 2022

1. Review

Recall our definitions regarding probability. We perform some sort of "experiment", such as rolling a die. An *outcome* is result of the experiment; in this case, the outcome is an integer between one and six. The *sample space* is the set of all possible outcomes. An *event* is a subset of the sample space.

The *cardinality* of a set is the number of things in it. The cardinality of the set A is denoted $|A|$.

The *probability* of event E is defined to be

$$P(E) = \frac{|E|}{|S|}.$$

For a positive integer n, we define n factorial as the product of positive integers less than or equal to n:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

This is the number of ways of rearranging n things.

The number of *permutations* (ordered subsets) of n things taken k at a time is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

The number of *combinations* (unordered subsets) of n things taken k at a time is

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

The number of combinations is typically referred to as "n choose k", and is also written

$$\binom{n}{k} = C(n,k).$$

The *cartesian product* of the sets A and B is

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

More generally, the cartesian product of sets $A_1, \dots, A_n$ is the set of ordered n-tuples whose i^{th} entry is from the set A_i.

2. Counting

2.1. **Sampling.** Suppose we have a set of n objects, and we wish to select k objects from this set. Our choice of objects is called a sample, and the act of choosing them is called sampling. There are a few different methods of sampling of interest.

- *Ordered sampling*: We keep track of the order in which the objects are selected. Thus, if we select the same objects but in a different order, this is consider a different sample.
- *Unordered sampling*: The order in which objects are select does not matter. The same set of objects, selected in a different order, would be considered the same sample.
- *With replacement*: We could select an object, note which one was selected, and then return the object so that it could potentially be selected again. That is, we could allow repetitions.
- *Without replacement*: Once we select an object, we may not select it again.

This gives four different combinations of these two ideas. Each one produces a different number of distinct samples. We look at each.

We wish to count the number of ways to select k things from a set of n things. Note that when we are counting with replacement, it is possible that $n < k$, but if we are counting without replacement, it is necessary that $k \leq n$.

2.2. **Ordered Sampling with Replacement.** To count the number of ways to select k things from a set of n things with order and replacement, we repetitively apply the multiplication principle. We imagine filling k slots with objects from the set. There are n choices for slot 1, and n choices for slot 2, so there are n^2 choices for the first two. There are n choices for the third, and n choices for the forth, up to n choices for the k^{th}; we continue to multiply by n and obtain

$$\text{Ordered with Replacement Selection gives } n^k \text{ samples.}$$

Note that in this case, it is possible that $n < k$.

If A is the set of n things, this form of counting is modeled by

$$A^k = \{ \text{ ordered } k\text{-tuples from } A \ \} \quad \text{where} \quad |A^k| = n^k.$$

2.3. **Ordered Sampling without Replacement.** To count the number of ways to select k things from a set of n things with order but without replacement, we first select one for the first slot, and have n choices; now, however, we have one less choice for the second slot, so we multiply by $n-1$. There are $n-2$ choices for the third slot, and so forth. We do this k times, multiplying as we go, and see that

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

to obtain

$$\text{Ordered without Replacement Selection gives } \frac{n!}{(n-k)!} \text{ samples.}$$

An ordered list of distinct objects is called a *permutation*. The number of permutations of k objects from a set of n objects is denoted $P(n,k)$.

If A is a set of n things, this form of counting is modeled by

$$A_p^k = \{ \text{ ordered } k \text{ tuples from } A \text{ with distinct entries } \} \quad \text{where} \quad |A_p^k| = \frac{n!}{(n-k)!}.$$

2.4. **Unordered Sampling without Replacement.** Here, we use the last result. Consider an unordered sample of size k, chosen without duplicates from a set of n objects. There are $k!$ ways to arrangement this sample into an ordered list, which implies that in our above computation of order sample without replacement, each such set was counted $k!$ times. Thus, we divide the number of ordered samples without replacement by $k!$, to obtain

$$\text{Unordered without Replacement Selection gives } \frac{n!}{k!(n-k)!} \text{ samples.}$$

An unordered set of distinct objects is called a *combination*. The number of combinations of k objects from a set of n objects is denoted $C(n,k)$, or $\binom{n}{k}$. The latter notation is normally read n *choose* k. Note that

$$C(n,k) = \binom{n}{k} \quad \text{and} \quad P(n,k) = \binom{n}{k}k!.$$

If A is a set of n things, this form of counting is modeled by

$$\mathcal{P}(A)_k = \{ \text{ subsets of } A \text{ of cardinality } k \ \} \quad \text{where} \quad |\mathcal{P}(A)_k| = \frac{n!}{k!(n-k)!}.$$

2.5. **Unordered Sampling with Replacement.** This is the most difficult form of counting to understand, and we postpone discussing it for now. The formula for it is given in the chart below.

2.6. **Summary.** Our four counting techniques produce four formulae, which we summarize here. If we select k objects from a set of n objects, we recapitulate the number of ways to do this with each counting technique.

	Ordered	Unordered
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$