

# COMPLEX ANALYSIS

## TOPIC XVI: SEQUENCES

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ABSTRACT. We outline the development of sequences in  $\mathbb{C}$ , starting with open and closed sets, and ending with the Bolzano-Weierstrauss Theorem for complex numbers. Some propositions are formulated as problems.

### 1. TOPOLOGY OF $\mathbb{C}$

#### 1.1. Open Sets.

**Definition 1.** Let  $u \in \mathbb{C}$ . The *open ball* around  $u$  of radius  $r$  is

$$B_r(u) = \{z \in \mathbb{C} \mid |z - u| < r\}.$$

Let  $U \subset \mathbb{C}$ . We say that  $U$  is *open* if for every  $u \in U$  there exists  $r > 0$  such that  $B_r(u) \subset U$ .

**Problem 1.** Let  $U \subset \mathbb{C}$ . Show that  $U$  is open if and only if  $U$  is the union of a collection of open balls.

**Problem 2.** Let  $U, V \subset \mathbb{C}$  be open sets. Show that  $U \cap V$  is an open set.

**Definition 2.** Let  $X$  be a set. A *collection of subsets* of  $X$  is a set  $\mathcal{C}$  whose members are subsets of  $X$ . We will use the following notation.

- $\cup \mathcal{C} = \{x \in X \mid x \in C \text{ for some } C \in \mathcal{C}\}$
- $\cap \mathcal{C} = \{x \in X \mid x \in C \text{ for all } C \in \mathcal{C}\}$

**Problem 3.** Let  $\mathcal{T}$  denote the collection of all open subsets of  $\mathbb{C}$ .

- (a) Show that  $\emptyset \in \mathcal{T}$  and  $\mathbb{C} \in \mathcal{T}$ .
- (b) Show that if  $\mathcal{C} \subset \mathcal{T}$ , then  $\cup \mathcal{C} \in \mathcal{T}$ .
- (c) Show that if  $\mathcal{C} \subset \mathcal{T}$  and  $\mathcal{C}$  is finite, then  $\cap \mathcal{C} \in \mathcal{T}$ .

The collection  $\mathcal{T}$  is known as the *topology* of  $\mathbb{C}$ .

**Definition 3.** Let  $z \in \mathbb{C}$ . A *neighborhood* of  $z$  is a subset of  $\mathbb{C}$  which contains an open set which contains  $z$ .

**Problem 4.** Let  $z \in \mathbb{C}$  and let  $A, B \subset \mathbb{C}$  be neighborhoods of  $z$ . Show that  $A \cap B$  is a neighborhood of  $z$ .

### 1.2. Closed Sets.

**Definition 4.** Let  $F \subset \mathbb{C}$ .

We say that  $F$  is *closed* if its complement  $\mathbb{C} \setminus F$  is open.

**Problem 5.** Let  $\mathcal{F}$  denote the collection of all closed subsets of  $\mathbb{C}$ .

- (a) Show that  $\emptyset \in \mathcal{F}$  and  $\mathbb{C} \in \mathcal{F}$ .
- (b) Show that if  $\mathcal{C} \subset \mathcal{F}$ , then  $\cap \mathcal{C} \in \mathcal{F}$ .
- (c) Show that if  $\mathcal{C} \subset \mathcal{F}$  and  $\mathcal{C}$  is finite, then  $\cup \mathcal{C} \in \mathcal{F}$ .

**Definition 5.** Let  $A \subset \mathbb{C}$ .

We say that  $A$  is *bounded* if there exists  $M \in \mathbb{R}$  such that  $A \subset B_M(0)$ .

We say that  $A$  is *compact* if it is closed and bounded.

The word compact has a more general definition in a more general settings, but in the larger sense, a subset of  $\mathbb{C}$  is compact if and only if it is closed and bounded. Hence, for our purposes, we simply take it as a definition.

### 1.3. Classification of Points.

**Definition 6.** Let  $A \subset \mathbb{C}$ .

An *interior point* of  $A$  is a point  $z \in A$  such that  $A$  contains a neighborhood of  $z$ . The *interior* of  $A$  is the set of interior points of  $A$  and is denoted  $A^\circ$ .

**Proposition 1.** Let  $A \subset \mathbb{C}$ . Then:

- (a)  $A$  is open if and only if  $A = A^\circ$ ;
- (b)  $A$  is open if and only if every point in  $A$  is an interior point;
- (c) The interior of  $A$  is the union of all open sets which are contained in  $A$ .

**Definition 7.** Let  $A \subset \mathbb{C}$ . A *closure point* of  $A$  is a point  $z \in \mathbb{C}$  such that every neighborhood of  $z$  intersects  $A$ . The *closure* of  $A \subset X$  is the set of closure points of  $A$  and is denoted  $\overline{A}$ .

**Proposition 2.** Let  $A \subset \mathbb{C}$ . Then:

- (a)  $A$  is closed if and only if  $A = \overline{A}$ ;
- (b)  $A$  is closed if and only if every point in  $A$  is an closure point;
- (c) Then  $\overline{A}$  is the intersection of the closed subsets of  $\mathbb{C}$  which contain  $A$ .

**Definition 8.** Let  $A \subset \mathbb{C}$ . A *boundary point* of  $A$  is a point  $z \in \mathbb{C}$  such that every neighborhood of  $z$  intersects  $A$  and  $A^c$ . The *boundary* of  $A$  is the set of boundary points of  $A$  and is denoted  $\partial A$ .

**Proposition 3.** Let  $A \subset \mathbb{C}$ . Then

- (a)  $\partial A = \overline{A} \setminus A^\circ$ ;
- (b)  $\partial A = \overline{A} \cap \overline{A^c}$ ;
- (c)  $\partial A = \partial A^c$ ;
- (d)  $\overline{A} = A \cup \partial A$ ;
- (e)  $A^\circ = A \setminus \partial A$ ;
- (f)  $\partial(\partial A) \subset \partial A$ ;
- (g)  $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$ .

**Definition 9.** Let  $A \subset \mathbb{C}$ . An *accumulation point* of  $A$  is a point  $z \in \mathbb{C}$  such that every deleted neighborhood of  $z$  intersects  $A$ . The *derived set* of  $A$  is the set of accumulation points of  $A$  and is denoted  $A'$ .

## 2. SEQUENCES

**Definition 10.** Let  $X$  be a set. A *sequence* in  $X$  is a function  $a : \mathbb{N} \rightarrow X$ .

We may write  $a_n$  to mean  $a(n)$ . Next we specify notation to indicate the entire sequence as opposed to a specific member of the range.

We often think of a sequence as an infinitely long tuple of elements from  $X$ , so it looks like  $(a_1, a_2, a_3, \dots)$ . This is written more succinctly as  $(a_n)_{n=1}^{\infty}$ , or  $(a_n)_{n \in \mathbb{N}}$ , or simply  $(a_n)$ .

In the case  $X \subset \mathbb{C}$ , we call this a sequence of complex number numbers. We are also interested in sequences of real numbers, which become an important special case.

**Definition 11.** Let  $(a_n)$  be a sequence in a set  $X$ .

We say that  $(a_n)$  is *injective* if  $a_m = a_n \Rightarrow m = n$ .

The *image* of  $(a_n)$  is

$$\{a_n\} = \{x \in X \mid x = a_n \text{ for some } n \in \mathbb{N}\}.$$

The  $N^{\text{th}}$  *tail* of  $(a_n)$  is

$$\{a_n : N\} = \{x \in X \mid x = a_n \text{ for some } n \geq N\}.$$

## 3. LIMITS OF SEQUENCES

**Definition 12.** Let  $(a_n)$  be a sequence of complex numbers and let  $p \in \mathbb{C}$ . We say that the sequence *converges* to  $p$  if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \ni n \geq N \Rightarrow |a_n - p| < \epsilon.$$

If  $(a_n)$  converges to  $p$ , we call  $p$  the *limit* of the sequence, and write  $\lim a_n = p$ .

**Problem 6.** Let  $(a_n)$  be a sequence of complex numbers and let  $p \in \mathbb{C}$ .

Show that the following conditions are equivalent:

- (L1) For every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |a_n - p| < \epsilon$ .
- (L2) For every neighborhood  $U$  of  $p$  there exists  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow a_n \in U$ .
- (L3) Every neighborhood of  $p$  contains a tail of  $(a_n)$ .
- (L4) Every neighborhood of  $p$  contains  $a_n$  for all but finitely many  $n \in \mathbb{N}$ .

*Solution.*

(L1  $\Rightarrow$  L2) Suppose that for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |a_n - p| < \epsilon$ . Let  $U$  be a neighborhood of  $p$ . Then there exists  $\epsilon > 0$  such that  $B_\epsilon(p) \subset U$ . Let  $N$  be so large that  $|a_n - p| < \epsilon$  whenever  $n \geq N$ . Then for  $n \geq N$ , we have  $a_n \in B_\epsilon(p) \subset U$ .

(L2  $\Rightarrow$  L3) Suppose that for every neighborhood  $U$  of  $p$  there exists  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow a_n \in U$ . Let  $U$  be a neighborhood of  $p$  and let  $N$  be so large that  $n \geq N \Rightarrow a_n \in U$ . Then  $\{a_n \mid n \geq N\} \subset U$ , so  $U$  contains the  $N^{\text{th}}$  tail of  $(a_n)$ .

(L3  $\Rightarrow$  L4) Suppose that every neighborhood  $U$  of  $p$  contains a tail of  $(a_n)$ . Let  $U$  be a neighborhood of  $p$  and let  $N \in \mathbb{N}$  such that  $\{a_n \mid n \geq N\} \subset U$ . If  $a_n \notin U$  for some  $n \in \mathbb{N}$ , then  $a_n \notin \{a_n \mid n \geq N\}$ , so  $n < N$ . There are only finitely many such  $n$ .

(L4  $\Rightarrow$  L1) Suppose that every neighborhood of  $p$  contains  $a_n$  for all but finitely many  $n$ . Let  $\epsilon > 0$ . Then  $B_\epsilon(p)$  is a neighborhood of  $p$ , so  $a_n \in B_\epsilon(p)$  for all but finitely many  $n \in \mathbb{N}$ . The maximum of a finite set of natural numbers always exists. Let  $N = 1 + \max\{n \in \mathbb{N} \mid a_n \notin B_\epsilon(p)\}$ . Then for  $n > N$ , we have  $|a_n - p| < \epsilon$ .  $\square$

**Problem 7.** Let  $(a_n)$  be an injective sequence and let  $p \in \mathbb{C}$ . Show that  $(a_n)$  converges to  $p$  if and only if every neighborhood of  $p$  contains  $a$  for all but finitely many  $a \in \{a_n\}$ .

**Problem 8.** Find an example of a noninjective sequence  $(a_n)$  of real numbers, and a real number  $p$ , such that every neighborhood of  $p$  contains all but finitely many points in  $\{a_n\}$ , but  $(a_n)$  does not converge to  $p$ .

## 4. CLUSTER POINTS OF SEQUENCES

**Definition 13.** Let  $(a_n)$  be a sequence of real numbers and let  $q \in \mathbb{C}$ .

We say that the sequence *clusters* at  $q$  if

$$\forall \epsilon > 0 \forall N \in \mathbb{N} \exists n \geq N \ni |a_n - q| < \epsilon.$$

If  $(a_n)$  clusters at  $q$ , we call  $q$  a *cluster point* of  $(a_n)$ .

**Problem 9.** Let  $(a_n)$  be a sequence of real numbers and let  $q \in \mathbb{C}$ .

Show that the following conditions are equivalent:

- (C1) For every  $\epsilon > 0$  and every  $N \in \mathbb{N}$  there exists  $n \geq N$  such that  $|a_n - q| < \epsilon$ .
- (C2) For every neighborhood  $U$  of  $q$  and every  $N \in \mathbb{N}$  there exists  $n \geq N$  such that  $a_n \in U$ .
- (C3) Every neighborhood of  $q$  intersects every tail of  $(a_n)$ .
- (C4) Every neighborhood of  $q$  contains  $a_n$  for infinitely many  $n \in \mathbb{N}$ .

*Solution.*

(C1  $\Rightarrow$  C2) Suppose that for every  $\epsilon > 0$  and every  $N \in \mathbb{N}$  there exists  $n \geq N$  such that  $|a_n - q| < \epsilon$ . Let  $U$  be a neighborhood of  $q$  and let  $N \in \mathbb{N}$ . Then there exists  $\epsilon > 0$  such that  $B_\epsilon(q) \subset U$ ; thus there exists  $n \geq N$  such that  $|a_n - q| < \epsilon$ . But this says that  $a_n \in B_\epsilon(q)$ , so  $a_n \in U$ .

(C2  $\Rightarrow$  C3) Suppose that for every neighborhood  $U$  of  $q$  and every  $N \in \mathbb{N}$  there exists  $n > N$  such that  $a_n \in U$ . Let  $U$  be a neighborhood of  $q$  and let  $\{a_n \mid n \geq N\}$  be an arbitrary tail of  $(a_n)$ . Then for some  $n \geq N$ , we have  $a_n \in U$ . But  $a_n \in \{a_n \mid n \geq N\}$ , so  $a_n \in \{a_n \mid n \geq N\} \cap U$ , and  $\{a_n \mid n \geq N\}$  intersects  $U$ .

(C3  $\Rightarrow$  C4) Suppose that every neighborhood of  $q$  intersects every tail of  $(a_n)$ . Let  $U$  be a neighborhood of  $q$ . Suppose by way of contradiction that  $U$  contains  $a_n$  for only finitely many  $n \in \mathbb{N}$ . Let  $m$  be the largest natural number such that  $a_m \in U$ . Then  $\{a_n : m + 1\}$  is a tail of  $(a_n)$  which does not intersect  $U$ ; this is a contradiction.

(C4  $\Rightarrow$  C1) Suppose that every neighborhood of  $q$  contains  $a_n$  for infinitely many  $n \in \mathbb{N}$ . Let  $\epsilon > 0$  and  $N \in \mathbb{N}$ . Then  $U = B_\epsilon(q)$  is a neighborhood of  $q$ , and  $U$  contains  $a_n$  for infinitely many  $n \in \mathbb{N}$ . One such  $n$  must be larger than  $N$ ; if  $n \in \mathbb{N}$  such that  $a_n \in U$ , then  $|a_n - q| < \epsilon$ .  $\square$

**Problem 10.** Let  $(a_n)$  be a sequence of complex numbers and let  $p \in \mathbb{C}$ . Show that if  $(a_n)$  converges to  $p$ , then  $(a_n)$  clusters at  $p$ , and  $p$  is the only cluster point.

*Solution.* Suppose that  $(a_n)$  converges to  $p$ . Then every neighborhood of  $p$  contains  $a_n$  for all but finitely many  $n$ . Thus there are infinitely many  $n$  such that  $a_n$  is in the neighborhood. By Problem 9 (d),  $(a_n)$  clusters at  $p$ .

To see that  $p$  is the only cluster point, let  $q \in X$ ,  $q \neq p$ ; we show that  $(a_n)$  does not cluster at  $q$ . Let  $\epsilon = \frac{|p-q|}{2}$  and let  $U = B_\epsilon(p)$  and  $V = B_\epsilon(q)$ . Then  $U$  and  $V$  are disjoint neighborhoods of  $p$  and  $q$  respectively.

Let  $A$  be a tail of  $(a_n)$  such that  $A \subset U$ . Since  $U \cap V = \emptyset$ , we have  $A \cap V = \emptyset$ , so  $V$  is a neighborhood of  $q$  which does not intersect  $A$ . Thus  $(a_n)$  does not cluster at  $q$ , by 9 (c).  $\square$

**Problem 11.** Find an example of a sequence  $(a_n)$  of real numbers and a real number  $q \in \mathbb{C}$  such that  $(a_n)$  clusters at  $q$  but does not converge to  $q$ .

**Problem 12.** Let  $(a_n)$  be an injective sequence and let  $q \in \mathbb{C}$ . Show that  $(a_n)$  clusters at  $q$  if and only if every neighborhood of  $q$  contains  $a$  for infinitely many  $a \in \{a_n\}$ .

**Problem 13.** Find an example of a noninjective sequence  $(a_n)$  of real numbers, a real number  $q$ , and a neighborhood  $U$  of  $q$ , such that  $(a_n)$  clusters at  $q$  but  $U$  contains only finitely many points from  $\{a_n\}$ .

**Problem 14.** Let  $(a_n)$  be a bounded sequence of real numbers, and set

$$C = \{q \in \mathbb{C} \mid q \text{ is a cluster point of } (a_n)\}.$$

Show that  $C$  is closed and bounded.

## 5. BOUNDED SEQUENCES

**Definition 14.** A sequence  $(a_n)$  of complex number is *bounded* if there exists a  $R > 0$  such that  $|a_n| < R$  for all  $n \in \mathbb{N}$ .

**Problem 15.** Let  $(a_n)$  be a convergent sequence of complex numbers. Then  $(a_n)$  is bounded.

**Definition 15.** A sequence  $(a_n)$  of real number is *increasing* if  $a_{n+1} > a_n$  for all  $n \in \mathbb{N}$ , and it is *decreasing* if  $a_{n+1} < a_n$  for all  $n \in \mathbb{N}$ . It is *monotone* if it is either increasing or decreasing.

### Proposition 4. (Bounded Monotone Convergence Rule)

*A bounded monotone sequence of real numbers converges.*

*Reason.* This may be proven using the Dedekind completeness axiom of the real numbers, which says that every set of real numbers which is bounded above has a least upper bound.  $\square$

### Proposition 5. (Squeeze Law)

*If  $(a_n)$  and  $(b_n)$  both converge to  $L$ , and  $a_n \leq c_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $(c_n)$  converges to  $L$ .*

## 6. ARITHMETIC OF SEQUENCES

**Problem 16.** Let  $(a_n)$  and  $(b_n)$  be sequences in  $\mathbb{C}$ . Suppose that  $\lim a_n = L$  and  $\lim b_n = M$  for some  $L, M \in \mathbb{C}$ . Apply the definition to show that the sequence  $(a_n + b_n)$  converges to  $L + M$ .

*Solution.* Let  $\epsilon > 0$ .

Let  $N_1$  be so large that  $n \geq N_1$  implies  $|a_n - L| < \epsilon/2$ .

Let  $N_2$  be so large that  $n \geq N_2$  implies  $|b_n - M| < \epsilon/2$ .

Let  $N = \max\{N_1, N_2\}$ .

Then, for  $n \geq N$ , we have

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

**Problem 17.** Let  $(a_n)$  be a sequences in  $\mathbb{C}$  and let  $c \in \mathbb{C}$ . Suppose that  $\lim a_n = L$  for some  $L \in \mathbb{C}$ . Apply the definition to show that the sequence  $(ca_n)$  converges to  $cL$ .

**Problem 18.** Let  $(a_n)$  and  $(b_n)$  be sequences in  $\mathbb{C}$ . Suppose that  $\lim a_n = L$  and  $\lim b_n = M$  for some  $L, M \in \mathbb{C}$ . Apply the definition to show that the sequence  $(a_n \cdot b_n)$  converges to  $LM$ .

*Solution.* Let  $\epsilon > 0$ .

Since  $(b_n)$  converges, it is bounded; let  $B > 0$  be so large that  $|b_n| < B$  for all  $n \in \mathbb{N}$ .

Let  $N_1$  be so large that  $n \geq N_1$  implies  $|a_n - L| < \frac{\epsilon}{2B}$ .

Let  $N_2$  be so large that  $n \geq N_2$  implies  $|b_n - M| < \frac{\epsilon}{2|L|}$ .

Let  $N = \max\{N_1, N_2\}$ .

Then, for  $n \geq N$ , we have

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - Lb_n + Lb_n - LM| \\ &\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\ &= |a_n - L||b_n| + |L||b_n - M| \\ &< \frac{\epsilon}{2B} \cdot B + |L| \frac{\epsilon}{2|L|} \\ &= \epsilon. \end{aligned}$$

□

Let  $(a_n)$  be a sequence. We say  $(a_n)$  is *increasing* if  $a_m \leq a_n$  whenever  $m \leq n$ ; we say that  $(a_n)$  is *decreasing* if  $a_m \geq a_n$  whenever  $m \leq n$ ; we say that  $(a_n)$  is *monotone* if it is either increasing or decreasing. We say that  $(a_n)$  is *bounded* if there exists a positive real number  $B$  such that  $a_n \in [-B, B]$  for all  $n \in \mathbb{N}$ .

## 7. CAUCHY SEQUENCES

**Definition 16.** Let  $(a_n)$  be a sequence of complex numbers. We say that  $(a_n)$  is a *Cauchy sequence* if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \ni m, n \geq N \Rightarrow |a_m - a_n| < \epsilon.$$

**Proposition 6.** Let  $(a_n)$  be a sequence of complex numbers. If  $(a_n)$  converges, then  $(a_n)$  is a Cauchy sequence.

*Proof.* Let  $\lim a_n = L$ .

Let  $\epsilon > 0$ , and let  $N$  be so large that  $n \geq N$  implies that  $|a_n - L| < \epsilon$ .

Then, for  $m, n \geq N$ , we have

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

**Definition 17.** A subset of  $X \subset \mathbb{C}$  is called *complete* if every Cauchy sequence in  $X$  converges to a point in  $X$ .

**Theorem 1.** The set  $\mathbb{C}$  is complete.

*Proof.* This follows from the fact that  $\mathbb{R}$  is complete, which requires a formal definition of the real numbers. □

## 8. SUBSEQUENCES

**Definition 18.** Let  $a : \mathbb{N} \rightarrow \mathbb{R}$  be a sequence in a set  $X$ . A *subsequence* of  $(a_n)$  is a sequence  $b : \mathbb{N} \rightarrow X$  which can be expressed as a composition  $b = a \circ n$ , where  $n : \mathbb{N} \rightarrow \mathbb{N}$  is an increasing sequence of natural numbers. For each natural number  $k$ , we write  $n_k$  instead of  $n(k)$ ; thus  $b(k) = a(n(k)) = a(n_k) = a_{n_k}$ . Thus we may write  $(a_{n_k})$  to indicate a subsequence of  $(a_n)$ .

**Definition 19.** Let  $(a_n)$  be a sequence of real numbers. A *subsequential limit* of  $(a_n)$  is a real number  $q \in \mathbb{C}$  such that there exists a convergent subsequence  $(a_{n_k})$  whose limit is  $q$ .

**Problem 19.** Let  $(a_n)$  be a sequence of real numbers and let  $q \in \mathbb{C}$ . Show that  $q$  is a cluster point of  $(a_n)$  if and only if  $q$  is a subsequential limit of  $(a_n)$ .

**Problem 20** (Bolzano-Weierstrauss Theorem). Show that every bounded sequence of complex numbers has a convergent subsequence.