PRINCIPLES OF ANALYSIS
SOLUTIONS TO PROBLEM SET C

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Problem 1. Let $f(x) = x^3 - 2x^2 - 22x + 35$.

(a) Use the Rational Roots Theorem to find one rational root a of $f(x)$.
(b) Divide $f(x)$ by $(x - a)$ to obtain a quadratic polynomial $g(x)$.
(c) Find all real roots of $f(x)$.

Solution. Based on the information supplied by the leading and constant coefficients, the set of possible rational roots is $\{\pm 1, \pm 5, \pm 7, \pm 35\}$. Trying these possibilities, we see that $f(5) = 0$. Polynomial division produces $f(x) = (x - 5)(x^2 + 3x - 7)$. Apply the quadratic formula to $x^2 + 3x - 7$ to find that the entire solution set to $f(x) = 0$ is $\{5, \frac{-3 \pm \sqrt{37}}{2}\}$. □

Problem 2. Let (a_n) and (b_n) be bounded sequences of nonnegative real numbers.
Define a sequence (c_n) by $c_n = a_n b_n$.

(a) Show that (c_n) is a bounded sequence of nonnegative real numbers.
(b) Show that $\sup\{c_m \mid m \geq n\} \leq \sup\{a_m \mid m \geq n\} \cdot \sup\{b_m \mid m \geq n\}$.
(c) Show that $\limsup c_n \leq \limsup(a_n) \cdot \limsup(b_n)$.
(d) Show that if (a_n) converges to L and $b_n \leq M$ for every $n \in \mathbb{N}$, then $\limsup c_n \leq LM$.

Solution.
(a) Since (a_n) and (b_n) are bounded sequences of nonnegative numbers, there exist $A, B \in \mathbb{R}$ such that $0 \leq a_n \leq A$ and $0 \leq b_n \leq B$. Thus $0 \leq a_n b_n = c_n \leq AB$, so (c_n) is a bounded sequence of nonnegative numbers.
(b) Let $n \in \mathbb{N}$; then for every $m \geq n$, we have $c_m = a_m b_m \leq \sup\{a_m \mid m \geq n\} \cdot \sup\{b_m \mid m \geq n\}$, so the latter product is an upper bound for the set $\{c_m \mid m \geq n\}$. Since the supremum is the least upper bound, we have

$$\sup\{c_m \mid m \geq n\} \leq \sup\{a_m \mid m \geq n\} \cdot \sup\{b_m \mid m \geq n\}.$$

(c) Since (a_n), (b_n), and (c_n) are bounded, the sequences expressed in the above inequality are also bounded and monotone, so they converge. Taking the limit of both sides gives

$$\begin{aligned}\limsup c_n &= \lim \sup\{c_m \mid m \geq n\} \\ &\leq \lim(\sup\{a_m \mid m \geq n\} \cdot \sup\{b_m \mid m \geq n\}) \\ &= \lim \sup\{a_m \mid m \geq n\} \cdot \lim \sup\{b_m \mid m \geq n\}) \\ &= \limsup(a_n) \cdot \limsup(b_n).\end{aligned}$$

(d) Suppose that (a_n) converges to L and that $b_n \leq M$ for every $n \in \mathbb{N}$. Then $\limsup(a_n) = L$, and $\limsup(b_n) \leq M$. Thus $\limsup(c_n) \leq LM$. □

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Problem 3. Let (a_n) and (b_n) be bounded sequences of real numbers. Define a sequence (c_n) by $c_n = a_n b_n$. Show that if $\limsup a_n$ and $\limsup b_n$ are negative,

$$\limsup c_n = \liminf(a_n) \cdot \liminf(b_n).$$

Solution. We know that for any nonempty set $A \subset \mathbb{R}$, we have

$$\sup\{-a \mid a \in A\} = -\inf A.$$

Suppose $\limsup a_n$ and $\limsup b_n$ are negative. Then there exists $N \in \mathbb{N}$ such that $a_n < 0$ and $b_n < 0$ for $n \geq N$. Then $\{-a_m \mid m \geq n\}$, $\{-b_m \mid m \geq n\}$, and $\{c_m \mid m \geq n\}$ are sets of nonnegative numbers, for $n \geq N$. As in the previous problem,

$$\begin{aligned}
\limsup c_n &= \lim(\sup\{-a_m \mid m \geq n\}) \lim(\sup\{-b_m \mid m \geq n\}) \\
&= \lim(-\inf\{a_m \mid m \geq n\}) \lim(-\inf\{b_m \mid m \geq n\}) \\
&= (-\liminf a_n)(-\liminf b_n) \\
&= \liminf a_n \cdot \liminf b_n.
\end{aligned}$$

□

Problem 4. Find statements similar to that of Problem 3 for the following cases:

(a) $\liminf a_n > 0$ and $\liminf b_n > 0$;
(b) $\liminf a_n > 0$ and $\limsup b_n < 0$;
(c) $\liminf a_n < 0$ and $\limsup a_n > 0$ but $\liminf b_n > 0$.

Solution. In each case, assume (a_n) and (b_n) are bounded, and that $c_n = a_n b_n$.
(a) If $\liminf a_n > 0$ and $\liminf b_n > 0$, then $\liminf c_n \geq \liminf a_n \cdot \liminf b_n$;
(b) If $\liminf a_n > 0$ and $\limsup b_n < 0$, then $\limsup c_n \leq \liminf a_n \cdot \liminf b_n$;
(c) If $\liminf a_n < 0$ and $\limsup a_n > 0$ but $\liminf b_n > 0$, then $\limsup c_n \leq \limsup a_n \cdot \limsup b_n$. □

Definition 1. A *linear fractional transformation* is a function of the form

$$f(x) = \frac{ax + b}{cx + d},$$

where $a, b, c, d \in \mathbb{R}$.

Problem 5. Let $a, b, c \in \mathbb{R}$ be positive numbers, and consider the linear fractional transformation

$$f(x) = \frac{ax + b}{cx + a}.$$

Select $x_1 \in \mathbb{R}$, and apply f repeatedly (infinitely many times) to x_1 to create a sequence defined by

$$x_{n+1} = f(x_n).$$

Whether or not the sequence (x_n) clearly depends on a, b, c and x_1.

Show that if (x_n) converges to a positive number, then

$$\lim_{n \to \infty} x_n = \sqrt{\frac{b}{c}}.$$

Proof. If (x_n) converges to L, then $\lim x_n = \lim x_{n+1} = L$. Taking the limit of both sides of $x_{n+1} = f(x_n)$, we get

$$L = \lim \left(\frac{ax + b}{cx + a} \right) = \frac{aL + b}{cL + a}.$$

Solving for L yields $L = \pm\sqrt{\frac{b}{c}}$. So if L is positive, it must be $\sqrt{\frac{b}{c}}$. □

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