PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §11

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Exercise 1 (11.7)**.** Let (r_n) be an enumeration of the set $\mathbb{Q}$. Show that there exists a subsequence (r_{n_k}) such that $\lim_{k\to\infty} r_{n_k} = +\infty$.

Proof. Set

$$E_m = \{n \in \mathbb{N} \mid n > m\};$$
$$A_k = \{n \in \mathbb{N} \mid r_n > k\}.$$

Suppose that $E_m \cap A_k = \varnothing$ for some $k, m \in \mathbb{N}$. Then $r_n > k \Rightarrow n \leq m$; but there are only finitely many such n, and there are infinitely many rationals greater than k; this contradicts that (r_n) is an enumeration of the rationals. Thus for every $k, m \in \mathbb{N}$, we have $E_m \cap A_k \neq \varnothing$.

The Well-Ordering Principle states that every nonempty set of natural numbers has a minimum. Set $n_1 = 0$ and let $n_{k+1} = \min E_{n_k} \cap A_k$. Then n_k is an increasing sequence of natural numbers, so (r_{n_k}) is a subsequence of (r_n). Also $r_{n_k} > k$ for every $k \in \mathbb{N}$, so $r_{n_k} \to \infty$ as $k \to \infty$. $\square$

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Exercise 2 (11.8.(a))**.** Let (s_n) be a sequence of real numbers. Show that $\liminf s_n = -\limsup(-s_n)$.

Proof. For each $m \in \mathbb{N}$, let $T_m = \{s_n \mid n \geq m\}$; this is the "$n^{\text{th}}$ tail". For any set $A \subset \mathbb{R}$, let $-A = \{-a \mid a \in A\}$. Clearly $A = -(-A)$. Then we have seen that $\inf A = -\sup(-A)$ in Exercise 4.9.

Note that $\limsup(-s_n) = \lim_{m\to\infty}(\sup(-T_m))$. Then

$$\begin{aligned}\liminf s_n &= \lim_{m\to\infty}(\inf T_m)\\ &= \lim_{m\to\infty}(\inf(-(-T_m)))\\ &= \lim_{m\to\infty}(-\sup(-T_m))\\ &= -\lim_{m\to\infty}(\sup(-T_m))\\ &= -\liminf -s_n.\end{aligned}$$

□

Exercise 3 (11.8.(b))**.** Let (t_k) be a monotonic subsequence of $(-s_n)$ converging to $\limsup(-s_n)$. Show that $(-t_k)$ is a monotonic subsequence of (s_n) converging to $\liminf s_n$.

Proof. Since t_k is a subsequence of $(-s_n)$, there exists an increasing sequence (n_k) in $\mathbb{N}$ such that $t_k = -s_{n_k}$ for all $k \in \mathbb{N}$. Then $-t_k = s_{n_k}$, so $(-t_k)$ is a subsequence of (s_n).

Since t_k is monotone, we have either $t_{k+1} \leq t_k$ for all $k \in \mathbb{N}$, or $t_{k+1} \geq t_k$ for all $k \in \mathbb{N}$. Thus either $-t_{k+1} \geq -t_k$ for all $k \in \mathbb{N}$, or $-t_{k+1} \leq -t_k$ for all $k \in \mathbb{N}$. Thus $(-t_k)$ is monotone.

By part (a), $\liminf(s_n) = -\limsup(-s_n)$. Thus $\lim(-t_k) = -\lim t_k = -\limsup(-s_n) = \liminf s_n$. □

Problem 1. Let (s_n) be a sequence of real numbers which converges to $s \in \mathbb{R}$. Let $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$. Show that (σ_n) converges to s.

Proof. Let $\tau_n = \sigma_n - s$. It suffices to show that (τ_n) converges to zero. Note that

$$\tau_n = \frac{1}{n}\sum_{i=1}^{n} s_i - \frac{ns}{n} = \frac{1}{n}\sum_{i=1}^{n}(s_i - s).$$

Let $N_0 \in \mathbb{N}$ be so large that $|s_n - s| < \frac{\epsilon}{2}$ for all $n > N_0$. Let $M = \sum_{i=1}^{N} |s_i - s|$. Then for $n > N_0$, we have

$$\begin{aligned}
|\tau_n| &\leq \frac{M}{n} + \frac{1}{n}\sum_{i=N_0+1}^{n} |s_n - s| && \text{by } \Delta\text{-inequality} \\
&< \frac{M}{n} + \frac{1}{n}(n - N_0)\frac{\epsilon}{2} && \text{summing } n - N_0 \text{ small numbers} \\
&< \frac{M}{n} + \frac{\epsilon}{2} && \text{since } \frac{n - N_0}{n} \leq 1.
\end{aligned}$$

Now select $N \in \mathbb{N}$ with $N > N_0$ which is so large that $\frac{M}{n} < \frac{\epsilon}{2}$. Then for $n > N$, we have $|\tau_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that $|\tau_n| \to 0$ as $n \to \infty$. Thus $\lim \tau_n = 0$. $\square$

Problem 2. Let (a_n) and (b_n) be a sequences of real numbers we converge to a and b respectively. Let

$$\mu_n = \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_{n-1} b_2 + a_n b_1}{n}.$$

Show that (μ_n) converges to ab.

Proof. Let $\nu_n = \mu_n - ab$. It suffices to show that (ν_n) converges to zero.

Since (a_i) is a convergent sequence, is bounded; select $M > 0$ such that $|a_i| \leq M$. Also note that for any sequence (s_i), we have $\sum_{i=1}^n s_{n-i+1} = \sum_{i=1}^n s_i$; this follows from inductive use of commutativity.

Now

$$\begin{aligned}
|\nu_n| &= \frac{1}{n} | \sum_{i=1}^n a_i b_{n-i+1} - \frac{nab}{n} | \\
&= \frac{1}{n} | \sum_{i=1}^n (a_i b_{n-i+1} - ab) | \\
&\leq \frac{1}{n} \sum_{i=1}^n |a_i b_{n-i+1} - ab| \\
&= \frac{1}{n} \sum_{i=1}^n |a_i b_{n-i+1} - a_i b + a_i b - ab| \\
&\leq \frac{\sum_{i=1}^n |a_i b_{n-i+1} - a_i b|}{n} + \frac{\sum_{i=1}^n |a_i b - ab|}{n} \\
&\leq M \frac{\sum_{i=1}^n |b_{n-i+1} - b|}{n} + b \frac{\sum_{i=1}^n |a_i - a|}{n} \\
&= M \frac{\sum_{i=1}^n |b_i - b|}{n} + b \frac{\sum_{i=1}^n |a_i - a|}{n}.
\end{aligned}$$

Let $\tau_n = M \frac{\sum_{i=1}^n |b_i - b|}{n} + b \frac{\sum_{i=1}^n |a_i - a|}{n}$. By the previous problem,

$$\begin{aligned}
\lim_{n \to \infty} \tau_n &= M \lim \frac{\sum_{i=1}^n |b_i - b|}{n} + b \lim \frac{\sum_{i=1}^n |a_i - a|}{n} \\
&= M \cdot 0 + b \cdot 0 \\
&= 0.
\end{aligned}$$

Since $0 \leq |\nu_n| \leq \tau_n$ and $\lim \tau_n = 0$, we have $|\nu_n| \to 0$ so $\lim \nu_n = 0$. $\square$

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