PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §8

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Exercise 1 (8.1.(a))**.** Show that $\lim[(-1)^n/n] = 0$.

Proof. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then $\frac{1}{\epsilon} < N$. Then for $n > N$, we have $|\frac{(-1)^n}{n} - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$. □

Exercise 2 (8.1.(c))**.** Show that $\lim[(2n-1)/(3n+2)] = 2/3$.

Proof. Let $\epsilon > 0$. Now

$$\begin{aligned} |\frac{2n-1}{3n+2} - \frac{2}{3}| < \epsilon &\Leftrightarrow |\frac{-7}{6n+6}| < \epsilon \\ &\Leftrightarrow \frac{7}{\epsilon} < 6n+6 \\ &\Leftrightarrow \frac{7-6\epsilon}{6\epsilon} < n. \end{aligned}$$

Let $N \in \mathbb{N}$ such that $N \geq \frac{7-6\epsilon}{6\epsilon}$. Then for $n > N$, we have $|\frac{2n-1}{3n+3} - \frac{2}{3}| < \epsilon$. Therefore, $\lim[(2n-1)/(3n+2)] = 2/3$. □

Exercise 3 (8.2.(b))**.** Show that $\lim[(7n-19)/(3n+7)] = 7/3$.

Proof. Let $\epsilon > 0$. Now

$$\begin{aligned} |\frac{7n-19}{3n+7} - \frac{7}{3}| < \epsilon &\Leftrightarrow |\frac{-106}{6n+21}| < \epsilon \\ &\Leftrightarrow \frac{106}{\epsilon} < 6n+21 \\ &\Leftrightarrow \frac{7-21\epsilon}{6\epsilon} < n. \end{aligned}$$

Let $N \in \mathbb{N}$ such that $N \geq \frac{7-21\epsilon}{6\epsilon}$. Then for $n > N$, we have $|\frac{7n-19}{3n+7} - \frac{7}{3}| < \epsilon$. Therefore, $\lim[(7n-19)/(3n+7)] = 7/3$. □

Date: October 28, 2005.

Exercise 4 (8.1.(d))**.** Show that $\lim[(2n+4)/(5n+2)] = 2/5$.

By this point, we clearly see the generalization:

Lemma 1. *Let $a, b, c, d \in \mathbb{R}$ with $cd > 0$. Then* $\lim \frac{an+b}{cn+d} = \frac{a}{c}$.

Proof. Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $N \geq \frac{|bc-ad|-cd\epsilon}{c^2\epsilon}$. Then if $n > N$, we have

$$\begin{aligned}\frac{|bc-ad|-cd\epsilon}{c^2\epsilon} < n \Rightarrow \frac{|bc-ad|}{\epsilon} &< c^2n + cd \\ \Rightarrow \frac{|bc-ad|}{c^2n+cd} &< \epsilon \quad \text{because } c^2n + cd > 0 \\ \Rightarrow \left|\frac{acn+bc-acn-ad}{c^2n+cd}\right| &< \epsilon \\ \Rightarrow \left|\frac{an+b}{cn+d} - \frac{a}{c}\right| &< \epsilon.\end{aligned}$$

Thus $\lim \frac{an+n}{cn+d} = \frac{a}{c}$. □

Exercise 5 (8.3.(e))**.** Show that $\lim \frac{\sin n}{n} = 0$.

This gets tiresome pretty fast unless you invent some more general results. Lets try to prove the following lemma, which clearly applies to this case.

We say that a sequence (a_n) is *bounded* if there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Lemma 2. *Let (a_n) and (b_n) be sequences such that (a_n) is bounded and (b_n) converges to zero. Then* $\lim a_n b_n = 0$.

Proof of Lemma. Let $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Since $\lim b_n = 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|b_n - 0| < \frac{\epsilon}{M}$. Then for $n > N$, we have

$$|a_n b_n - 0| = |a_n||b_n| \leq M\frac{\epsilon}{M} = \epsilon.$$

Thus $\lim a_n b_n = 0$. □

Proof of Exercise. Let $a_n = \sin n$ and let $b_n = \frac{1}{n}$. Then $|a_n| \leq 1$ for all $n \in \mathbb{N}$, so (a_n) is bounded. Moreover, $\lim b_n = 0$. Then $\lim \frac{\sin n}{n} = \lim a_n b_n = 0$ by our lemma. □

Exercise 6 (8.3)**.** Let (a_n) be a sequence of nonnegative real numbers such that $\lim a_n = 0$. Show that $\lim \sqrt{a_n} = 0$.

Proof. Let $\epsilon > 0$. Since $\lim a_n = 0$ and all of the a_n are positive, there exists a natural number $N \in \mathbb{N}$ such that $a_n < \epsilon^2$ for all $n > N$. Then $\sqrt{a_n} < \epsilon$ for all $n > N$. Thus $\lim \sqrt{a_n} = 0$. □

Exercise 7 (8.4)**.** Oops, already unintentionally did this as a lemma. See the benefits of generalization?

Exercise 8 (8.5.(a))**. Squeeze Law**
Let (a_n), (b_n), and s_n be sequences of real numbers such that $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$. Suppose that $\lim a_n = \lim b_n = s$. Show that $\lim s_n = s$.

Proof. Let $\epsilon > 0$. Note that for any $n \in \mathbb{N}$, since $a_n < s_n < b_n$ we have

$$|s_n - a_n| = s_n - a_n \leq b_n - a_n = |b_n - a_n|.$$

Since $\lim a_n = s$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - s| < \frac{\epsilon}{3}$ for $n > N_1$.
Since $\lim b_n = s$, there exists $N_2 \in \mathbb{N}$ such that $|b_n - s| < \frac{\epsilon}{3}$ for $n > N_2$.
Let $N = \max\{N_1, N_2\}$. Now for $n > N$, we have

$$|b_n - a_n| = |b_n - s + s - a_n| \leq |b_n - s| + |a_n - s| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Then for $n > N$, we have

$$|s_n - s| = |s_n - a_n + a_n - s| \leq |s_n - a_n| + |a_n - s| \leq |b_n - a_n| + |a_n - s| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that $\lim s_n = s$. □

Exercise 9 (8.5.(b))**.** Suppose that (s_n) and (t_n) are sequences in $\mathbb{R}$ such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Show that $\lim s_n = 0$.

Proof. Since $|s_n| \leq t_n$, we have $-t_n \leq s_n \leq t_n$.
Let $\epsilon > 0$ and let N be so large that $|t_n - 0| < \epsilon$ for $n > N$. Since

$$|t_n - 0| = |t_n| = |-t_n| = |-t_n - 0|,$$

then $|-t_n - 0| < \epsilon$ for $n > N$. Thus $\lim -t_n = 0$.
The result follows by the Squeeze Law. □

Exercise 10 (8.7.(a))**.** Show that the sequence (a_n), where $a_n = \cos(\frac{n\pi}{3})$, does not converge.

Proof. If q is the statement, "$\forall\epsilon > 0 \exists N \in \mathbb{N} \ni n > N \Rightarrow |a_n - L| < \epsilon$", what is $\neg q$? It is the statement "$\exists\epsilon > 0 \ni \forall N \in \mathbb{N} \exists n > N \ni |a_n - L| \geq \epsilon$". To show this is to show that the limit is not L. To show that a sequence has no limit, select an arbitrary L and show $\neg q$.

Let $L \in \mathbb{R}$. We show that (a_n) does not converge to L. Let $\epsilon = 1$, and let $N \in \mathbb{N}$. It suffices to find $n > N$ such that $|a_n - L| \geq \epsilon$.

Note that $a_{6N} = 1$. If $|a_{6N} - L| \geq \epsilon$, we are done; so suppose that $|a_{6N} - L| < \epsilon$; that is, suppose that $|1 - L| < 1$. Thus $L \geq 0$. Then $a_{6N+3} = -1$, and

$$|a_{6N+3} - L| = |-1 - L| = |1 + L| \geq 1 = \epsilon.$$

□

Exercise 11 (8.8.(b))**.** Let $a_n = \sqrt{n^2+n} - n$. Show that $\lim a_n = \frac{1}{2}$.

Solution 1. Let $\epsilon > 0$. We wish to find $N \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n > N$, the distance between a_n and $\frac{1}{2}$ is less than ϵ.

First we manipulate the expression of a_n. The goal here is to isolate n; that is, to create an equal expression with only one n in it, so that we can later solve for it. We have:

$$\begin{aligned} a_n &= \sqrt{n^2+n} - n \\ &= \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} \\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\ &= \frac{n}{\sqrt{n^2+n}+n} \\ &= \frac{1}{\sqrt{1+\frac{1}{n}}+1}. \end{aligned}$$

Since $\sqrt{1+\frac{1}{n}} > 1$, we have $a_n < \frac{1}{2}$. Thus

$$\begin{aligned} |a_n - \frac{1}{2}| < \epsilon &\Leftrightarrow \frac{1}{2} - a_n < \epsilon \\ &\Leftrightarrow \frac{1-2\epsilon}{2} < a_n \\ &\Leftrightarrow \frac{2}{1-2\epsilon} > \sqrt{1+\frac{1}{n}} + 1 \\ &\Leftrightarrow (\frac{1+2\epsilon}{1-2\epsilon})^2 > 1 + \frac{1}{n} \\ &\Leftrightarrow \frac{1}{(\frac{1+2\epsilon}{1-2\epsilon})^2 - 1} < n. \end{aligned}$$

Let $N \in \mathbb{N}$ be a natural number such that

$$N \geq \frac{1}{(\frac{1+2\epsilon}{1-2\epsilon})^2 - 1}.$$

Then for $n > N$, we have $|a_n - \frac{1}{2}| < \epsilon$. $\square$

That was rather messy and complicated. Here is another way.

Exercise 12 (8.8.(b))**.** Let $a_n = \sqrt{n^2+n} - n$. Show that $\lim a_n = \frac{1}{2}$.

Solution 2. For any $n \in \mathbb{N}$, $a_n < 1$. To see this, note that

$$\sqrt{n^2+n} - n < 1 \Leftrightarrow n^2 + n < (n+1)^2 = n^2 + 2n + 1.$$

This latter inequality is always true.

Now

$$\begin{aligned}
|\sqrt{n^2+n} - n - \frac{1}{2}| &= |\frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2}| \\
&= \frac{\sqrt{n^2+n} - n}{2\sqrt{n^2+n} + 2n} \\
&\leq \frac{1}{2\sqrt{n^2+n} + 2n} \text{ because numerator was } \leq 1 \\
&\leq \frac{1}{4n} \text{ because denominator was } \geq 2\sqrt{n^2} + 2n.
\end{aligned}$$

Let $N \in \mathbb{N}$ be so large that $N > \frac{1}{4\epsilon}$. Then for $n > N$, we have

$$|a_n - \frac{1}{2}| \leq \frac{1}{4n} < \frac{1}{4N} < \epsilon.$$

□

Here are some review problems.

Problem 1. Show that $n + 8 < n^2$ for all natural numbers $n \geq 4$.

Proof. For $n = 4$, we have $n + 8 = 12 < 16 = n^2$.

Assume that $n > 4$ and that $(n-1) + 8 < (n-1)^2$. Then $n + 7 < n^2 - 2n + 1$, so

$$\begin{aligned} n + 8 &< n^2 - 2n + 2 \\ &< n^2 - 8 + 2 \text{ because } n > 4 \\ &= n^2 - 6 < n^2. \end{aligned}$$

□

Problem 2. Let S and T be two bounded sets in $\mathbb{R}$ such that $S \subset T$. Show that $\inf T \leq \inf S$.

Proof. Let $s \in S$. Then $s \in T$, so $\inf T \leq s$. This shows that $\inf T$ is a lower bound for S. Thus $\inf T \leq \inf S$. □

Problem 3. Let $a_n = \sqrt{n^2 + 1} - n$.
Show that $\lim a_n = 0$.

Proof. Note that

$$\begin{aligned} a_n &= \frac{n^2 + 1 - n^2}{\sqrt{n^2+1} + n} \\ &= \frac{1}{\sqrt{n^2+1} + n} \\ &< \frac{1}{\sqrt{n^2} + n} \\ &= \frac{1}{2n} \\ &< \frac{1}{n}. \end{aligned}$$

Let $\epsilon > 0$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Let $n > N$; since a_n is positive, we have

$$|a_n - 0| = a_n < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Thus $\lim a_n = 0$. □

Problem 4. Let (s_n) and (t_n) be sequences of real numbers which converge to s and t respectively. Show that if $s_n \leq t_n$ for all but finitely many $n \in \mathbb{N}$, then $s \leq t$.

Proof. We prove the contrapositive statement:
If $s < t$, then $s_n < t_n$ for infinitely many $n \in \mathbb{N}$.

Suppose that $s > t$, and let $\epsilon = \frac{s-t}{2}$. Let N_1 be so large that $|s_n - s| < \epsilon$ and let N_2 be so large that $|t_n - t| < \epsilon$. Let $N = \max\{N_1, N_2\}$.

Let $n > N$. Then

$$|t_n - t| < \epsilon \Rightarrow t_n - t < \epsilon \Rightarrow t_n < \epsilon + t,$$

and

$$|s_n - s| < \epsilon \Rightarrow s - s_n < \epsilon \Rightarrow s_n > \epsilon - s.$$

Then for all $n > N$,

$$t_n < \epsilon + t = \epsilon - s < s_n.$$

Therefore, $t_n < s_n$ for infinitely many $n \in \mathbb{N}$. □

Problem 5. Determine whether or not the sequence converges. If it converges, find the limit and show that the sequence converges to this limit. If it does not converge, show that it does not converge.

(a) $a_n = (-1)^n \frac{n}{n^2-1}$
This converges to zero.
Let $\epsilon > 0$ and let $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon} + 1$. Then for $n > N$,

$$\begin{aligned}
|a_n - 0| &= |(-1)^n \frac{n}{n^2-1}| \\
&= \frac{n}{n^2-1} \\
&< \frac{n+1}{n^2-1} \\
&< \frac{1}{n-1} \\
&< \frac{1}{N-1} \\
&< \epsilon.
\end{aligned}$$

(b) $b_n = \frac{n^2-4}{n+1}$
This does not converge. To show this, we show that it is unbounded by selecting arbitrary $M \in \mathbb{R}$ and finding $n \in \mathbb{N}$ such that $a_n > M$.

Let $M \in \mathbb{R}$. Then

$$b_n = \frac{n^2-4}{n+1} > \frac{n^2-4}{n+2} = \frac{n-2}{;}$$

if $n > M + 2$, then $b_n > M$.

Problem 6. Let S and T be bounded sets of positive real numbers. Suppose that $S \cap T \neq \varnothing$.
Show that $\inf S \leq \sup T$.

Proof. Let $x \in S \cap T$. Then $x \in S$, so $\inf S \leq x$. Also $x \in T$, so $x \leq \sup T$. Thus $\inf S \leq \sup T$. □

Problem 7. Let (a_n) be a convergent sequence of real numbers, and let $A = \{a_n \mid n \in \mathbb{N}\}$.
Show that $\lim a_n \leq \sup A$.

Proof. Let $L = \lim a_n$, and suppose that $\sup A < L$. Let $\epsilon = L - \sup A$, and let N be so large that $|a_n - L| < \epsilon$ for $n > N$. In particular, $|a_{N+1} - L| < \epsilon$, so $L - a_{N+1} < L - \sup A$; therefore $a_{N+1} > \sup A$. But since $a_{N+1} \in A$, this is a contradiction. □

Here are some extra problems to spark your interest. The first one has a short solution, but the next two are meant to be mind-benders.

Definition 1. Let $A \subset \mathbb{R}$ be an open interval. A function $f : A \to \mathbb{R}$ is called a *contraction* if there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| \leq M|a - b|$ for any $a, b \in U$.

Example 1. The following are contractions:

- $f(x) = mx + b$, where $m, b \in \mathbb{R}$; $U = \mathbb{R}$ and $M = |m|$;
- $f(x) = \sin(x)$ and $\cos(x)$; $U = \mathbb{R}$ and $M = 1$;
- $f(x) = \log(x)$; $U = (a, \infty)$ where $a > 0$ and $M = \frac{1}{a}$.
- $f(x) = \sqrt{x}$; $U = (a, \infty)$ where $a > 0$ and $M = \frac{1}{2\sqrt{a}}$;
- $f(x)$ is differentiable with bounded derivative on an open interval U; $M = \sup\{|f'(a)| : a \in U\}$.

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. Let (a_n) be a sequence of real numbers which converges to $L \in \mathbb{R}$. Show that $\lim f(a_n) = f(L)$.

Are the following problems true? See if you can decide.

Problem 9. For each $i \in \mathbb{N}$, let A_i be a set of real numbers which is bounded above by $M \in \mathbb{R}$. Let $B_n = \cup_{i=1}^{n} A_i$, and let $B_\infty = \cup_{i=1}^{\infty} A_i$. Let $b_n = \sup B_n$. Show that $\lim b_n = \sup B_\infty$.

Problem 10. For pair of natural numbers $i, j \in \mathbb{N}$, let $a_{i,j} \in \mathbb{R}$. Let n range over $\mathbb{N}$; for a fixed i, we obtain a sequence $(a_{i,n})$, and for a fixed j, we obtain a sequence $(a_{n,j})$. Suppose that all of these sequences converge, and let $b_i = \lim_{n\to\infty} a_{i,n}$ and $c_j = \lim_{n\to\infty} a_{n,j}$. Suppose that the sequences (b_i) and (c_j) converge, and let $\lim_{i\to\infty} b_i = b$ and $\lim_{j\to\infty} c_j = c$. Show that $b = c$.

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