Algebra II — Lesson 0820 - Completing the Square

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1. Square Binomials

Let r be a number. We have computed that

$$(x+r)^2 = x^2 + 2rx + r^2.$$

How do you recognize a square binomial? If we have $x^2 + bx + c$, it is a square if there is a number r such that $b = 2r$ and $c = r^2$. Thus, $x^2 + bx + c$ is a perfect square if $c = \left(\frac{b}{2}\right)$.

Is it a perfect square?

- $x^2 + 6x + 9$ — Yes, since $6/2 = 3$ and $3^2 = 9$
- $x^2 - 8x + 12$ — No, since $8/2 = 4$ but $4^2 = 16 \neq 12$
- $x^2 - 12x + 36$ — Yes, since $12/2 = 6$ and $6^2 = 36$

Find c so that $x^2 + bx + c$ is a perfect square.

- $x^2 + 8x + c$ — Let $c = (8/2)^2 = 4^2 = 16$
- $x^2 - 10x + c$ — Let $c = (10/2)^2 = 5^2 = 25$
- $x^2 - 7x + c$ — Let $c = (7/2)^2 = \frac{49}{4}$

2. Solving Quadratics by Completing the Square

We start with an example.

Example 1. Solve $6x + x^2 - 11 = 0$.

Solution. First, we ask ourselves if we can factor this; that is, can we find two integers whose product is 11 and whose difference is 4. We see that no, no such integers exist. So we can't (easily) factor this.

Next, we complete the square in steps. The "thing that complete the square" is $(6/2)^2 = 9$. We will see what Step 1 is later.

- $x^2 + 6x - 11 = 0$ — Step 0: put the equation in standard form.
- $x^2 + 6x = 11$ — Step 2: put the constant on the right hand side.
- $x^2 + 6x + 9 = 11 + 9$ — Step 3: add the thing that completes the square to both sides.
- $(x+3)^2 = 20$ — Step 4: factor the left and add the right.
- $x + 3 = \pm\sqrt{20}$ — Step 5: take the square root of both sides.
- $x = -3 \pm \sqrt{20}$ — Step 6: put the constant on the right hand side.

The solution set is $\{-3 - \sqrt{20}, -3 + \sqrt{20}\}$. □

If the coefficient of x (b) is odd, we have to deal with fractions.

Example 2. Solve $x^2 - 5x + 2 = 0$.

Solution. Complete the square in steps. The thing that completes the square is $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

- $x^2 - 5x + 2 = 0$ Step 0: put the equation in standard form (this one already was).
- $x^2 - 5x = -2$ Step 2: put the constant on the right hand side.
- $x^2 - 5x + \frac{25}{=} - 2 + \dfrac{25}{4}$ Step 3: add the thing that completes the square to both sides.
- $\left(x - \dfrac{5}{2}\right)^2 = \dfrac{17}{4}$ Step 4: factor the left and add the right.
- $x - \dfrac{5}{2} = \pm\dfrac{\sqrt{17}}{2}$ Step 5: take the square root of both sides.
- $x = \dfrac{5}{2} \pm \dfrac{\sqrt{17}}{2}$ Step 6: put the constant on the right hand side.

The solution set is $\left\{\dfrac{5 \pm \sqrt{17}}{2}\right\}$. □

If the leading coefficient (a) is not one, the easiest thing to do (or at least, the easiest thing to describe) is to divide by a at the beginning.

Example 3. Solve $3x^2 - 14x - 5 = 0$.

Solution. Complete the square in steps.

- $3x^2 - 14x - 4 = 0$ Step 0: put the equation in standard form.
- $x^2 - \dfrac{14}{3}x - \dfrac{4}{3} = 0$ Step 1: divide through by the leading coefficient.
- $x^2 - \dfrac{14}{3}x = \dfrac{4}{3}$ Step 2: put the constant on the right hand side.
- $x^2 - \dfrac{14}{3}x + \frac{49}{9} = \dfrac{4}{3} + \dfrac{49}{9}$ Step 3: add the thing that completes the square to both sides.
- $\left(x - \dfrac{7}{3}\right)^2 = \dfrac{61}{9}$ Step 4: factor the left and add the right.
- $x - \dfrac{7}{3} = \pm\dfrac{\sqrt{61}}{3}$ Step 5: take the square root of both sides.
- $x = \dfrac{7 \pm \sqrt{41}}{3}$ Step 6: put the constant on the right hand side.

The solution set is $\left\{\dfrac{7 \pm \sqrt{41}}{3}\right\}$. □