MATH 2753 LINEAR ALGEBRA — PROF. PAUL BAILEY

Project 1 Solutions
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Computing with Vectors

Problem 1. Find the distance from the point $(6, -2)$ to the line $4x + y = 12$ in $\mathbb{R}^2$.

Solution. The normal vector to the line is $\vec{n} = (4, 1)$. Let $x = 0$ to see that $P = (0, 12)$ is a point on the line. Let $Q = (6, -2)$. Form the vector $\vec{v} = Q - P = (6, -14)$. The distance from the point to the line is the projection of $\vec{v}$ onto $\vec{n}$:

$$d = \text{proj}_{\vec{n}}\vec{v} = \frac{\vec{v} \cdot \vec{n}}{|n|} = \frac{24 - 14}{\sqrt{16 + 1}} = \frac{10}{\sqrt{17}}.$$

□

Problem 2. Find the distance from the point $(4, 1, -3)$ to the plane $2x + 3y - z = 2$ in $\mathbb{R}^3$.

Solution. The normal vector to the plane is $\vec{n} = (2, 3, -1)$. Let y and z be zero to see that $P = (1, 0, 0)$ is a point on the plane. Let $Q = (4, 1, -3)$. Form the vector $\vec{v} = Q - P = (3, 1, -3)$. The distance from Q to the plane is

$$d = \text{proj}_{\vec{n}}\vec{v} = \frac{\vec{v} \cdot \vec{n}}{|n|} = \frac{6 + 3 + 3}{\sqrt{4 + 9 + 1}} = \frac{12}{\sqrt{14}}.$$

□

Problem 3. Find the distance from the point $(5, 2, 1)$ to the line $(4 + 2t, 1 - t, -3 + 3t)$ in $\mathbb{R}^3$.

We offer two solutions for this.

Solution 1. The direction vector for the line is $\vec{w} = (2, -1, 3)$, and $P = (4, 1, -3)$ is a point on the line. Let $Q = (5, 2, 1)$. Form the vector $\vec{v} = Q - P = (1, 1, 4)$.

The distance from the point to the line is the length of one leg of a right triangle whose hypotenuse is the vector $\vec{v}$ starting at point P. The other length of the other leg is the projection of $\vec{v}$ onto $\vec{w}$. Now

$$\text{proj}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{|w|} = \frac{2 - 1 + 12}{\sqrt{4 + 1 + 9}} = \frac{13}{\sqrt{14}}.$$

Thus the distance d is

$$d = \sqrt{|\vec{v}|^2 - (\text{proj}_{\vec{w}}\vec{v})^2} = \sqrt{(1 + 1 + 16) - \frac{169}{14}} = \sqrt{\frac{252}{14} - \frac{169}{14}} = \sqrt{\frac{83}{14}}.$$

□

Solution 2. The direction vector for the line is $\vec{w} = (2, -1, 3)$, and $P = (4, 1, -3)$ is a point on the line. Let $Q = (5, 2, 1)$. Form the vector $\vec{v} = Q - P = (1, 1, 4)$. If θ is the angle between $\vec{v}$ and $\vec{w}$, then $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$; thus the distance we seek is

$$d = |\vec{v}| \sin\theta = \frac{|\vec{v} \times \vec{w}|}{|\vec{w}|}.$$

Now

$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix} = (3 + 4, 8 - 3, -1 - 2) = (7, 5, -3).$$

Thus

$$d = \frac{|(7, 5, -3)|}{|(2, -1, 3)|} = \frac{\sqrt{49 + 25 + 9}}{\sqrt{4 + 1 + 9}} = \sqrt{\frac{83}{14}}.$$

□

Problem 4. Find the distance between the lines $(2+2t, 1-3t, -4+t)$ and $(-2+3t, 5-4t, 8+t)$ in $\mathbb{R}^3$.

Solution. Seeing that the lines are not parallel and assuming that they do not intersect, we conclude that there is a unique pair of parallel planes, each containing one of the lines. To find a normal vector for either of these planes, we cross the direction vectors of the lines.

Let $\vec{v} = (2, -3, 1)$ and $\vec{w} = (3, -4, 1)$. Set

$$\vec{n} = \vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 3 & -4 & 1 \end{bmatrix} = (-3+4, 3-2, -8+9) = (1, 1, 1).$$

The (shortest) distance between the lines is the distance between the parallel planes. To find the distance between parallel planes, take a point on each plane, subtract them to get a vector, then project this vector onto the common normal vector.

Let $P = (2, 1, -4)$ and $Q = (-2, 5, 8)$. Set $\vec{x} = Q - P = (-4, 4, 12)$. The distance we seek is

$$d = \operatorname{proj}_{\vec{n}} \vec{x} = \frac{\vec{x} \cdot \vec{n}}{|\vec{n}|} = \frac{-4+4+12}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

$\square$

Problem 5. Consider the points $A = (1, 2, 5)$, $B = (-1, 1, 4)$, $C = (3, 1, -4)$, and $D = (6, 2, 7)$ in $\mathbb{R}^3$. Let $\mathcal{P}$ be the plane passing through A, B, and C, and let $\mathcal{L}$ be the line perpendicular to $\mathcal{P}$ which passes through D. Let E be the point of intersection of $\mathcal{P}$ and $\mathcal{L}$. Find E.

Solution. Let $\vec{v} = B - A = (-2, -1, -1)$ and $\vec{w} = C - A = (2, -1, -9)$. A normal vector for the plane through A, B, and C is

$$\vec{x} = \vec{v} \times \vec{w} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -1 \\ 2 & -1 & -9 \end{bmatrix} = (9-1, -2-18, 2+2) = (8, -20, 4).$$

Actually, one quarter of this is also normal, so set $\vec{n} = \frac{1}{4}\vec{x} = (2, -5, 1)$. Thus, the equation of the plane through A, B, and C is

$$(P - A) \cdot \vec{n} = 0, \qquad \text{or} \qquad 2x - 5y + z = -3.$$

The line in the direction of $\vec{n}$ through D is

$$P = D + t\vec{n}, \qquad \text{or} \qquad (x, y, z) = (6+2t, 2-5t, 7+t).$$

Substitute this into the equation of the plane and solve for t:

$$2(6+2t) - 5(2-5t) + (7+t) = -3 \quad \Rightarrow \quad 12 + 4t - 10 + 25t + 7 + t = -3 \quad \Rightarrow \quad 30t = -12 \quad \Rightarrow \quad t = -\frac{2}{5}.$$

Thus

$$E = \left(6 + 2\left(\frac{-2}{5}\right), 2 - 5\left(\frac{-2}{5}\right), 6 + \left(\frac{-2}{5}\right)\right) = \left(\frac{26}{5}, 4, \frac{33}{5}\right).$$

$\square$

Problem 6. Consider the points $A = (1, 5, -1)$, $B = (2, 1, 4)$, and $C = (-1, 3, 2)$. There is a unique circle in $\mathbb{R}^3$ passing through these three points. Find its center.

We offer two solutions to this problem.

Solution 1. **(a)** Let PQ denote the distance between the points P to Q. An observation which is essential to this this solution is that, in this case, $AC = BC$. The approach is to form an isosceles triangle whose apex is the center and whose base vertices lie on the circle.

Let D be the center of the circle. Then $\triangle CDB$ is isosceles, and its side length is the radius of the circle, which is

$$CD = \frac{BC}{2\cos(\angle DCB)}.$$

Since $AC = BC$, we have $\angle DCB = \frac{1}{2}\angle ACB$, and this latter angle we can compute.

Let $\vec{v} = A - C = (2, 2, -3)$ and $\vec{w} = B - C = (3, -2, 2)$. If $\theta = \angle DCB$, then $2\theta = \angle ACB = \angle(\vec{v}, \vec{w})$; thus

$$\cos 2\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{6 - 4 - 6}{\sqrt{4 + 4 + 9}\sqrt{9 + 4 + 4}} = -\frac{4}{17}.$$

Therefore

$$\cos\theta = \sqrt{\frac{1 + \cos(2\theta)}{2}} = \sqrt{\frac{13}{34}}.$$

Thus, the radius r of the circle is

$$r = \frac{|\vec{w}|}{2\cos\theta} = \frac{\sqrt{17}\sqrt{34}}{2\sqrt{13}} = \frac{17\sqrt{2}}{2\sqrt{13}}.$$

Let $\vec{x} = \vec{v} + \vec{w} = (5, 0, -1)$; the line through C with direction vector $\vec{x}$ passes through the center of the circle. We unitize this vector and follow it for a distance r; thus the center is

$$D = C + r\frac{\vec{x}}{|\vec{x}|} = (-1, 3, 2) + \frac{17\sqrt{2}}{2\sqrt{13}\sqrt{26}}(5, 0, -1) = (-1, 3, 2) + \frac{17}{26}(5, 0, -1) = \left(\frac{59}{26}, 3, \frac{35}{26}\right).$$

□

Solution 2. The center of the circle lies on planes which are perpendicular to the midpoints of the line segments connecting the given points.

Let $P = (A + B)/2 = (3/2, 3, 3/2)$ and $Q = (B + C)/2 = (1/2, 2, 3)$; these are the midpoints we will use.

Let $\vec{v} = A - B = (1, -4, 5)$ and $\vec{w} = C - B = (-3, 2, -2)$.

Compute $\vec{x} = -\vec{v} \times \vec{w} = (2, 13, 10)$; this is a normal vector for the plane on which the circle lies.

The plane through P perpendicular to $\vec{v}$ is $\mathcal{P}_1 : x - 4y + 5z = -3$.

The plane through Q perpendicular to $\vec{w}$ is $\mathcal{P}_2 : 6x - 4y + 4z = 7$.

The plane through B perpendicular to $\vec{x}$ is $\mathcal{P}_3 : 2x + 13y + 10z = 57$.

The center of the circle is the unique point of intersection of these three planes.

Now $\mathcal{P}_3 - 2\mathcal{P}_1 : 21y = 63$, so $y = 3$. Substitute this into $\mathcal{P}_1$ and $\mathcal{P}_2$ to get $x + 5z = 9$ and $6x + 4z = 19$. Plug $x = 9 - 5z$ into the second equation to get $54 - 30z + 4z = 19$, so $-26z = -35$, so $z = 35/26$. Then $x = 9 - 5(35/26) = 59/26$. Thus the center is

$$D = \left(\frac{59}{26}, 3, \frac{35}{26}\right).$$

□