Real Analysis — Homework 5 Solutions
Dr. Paul L. Bailey — Saturday, September 29, 2018

Definition 1. Let (a_n) be a sequence of real numbers. We say that $p \in \mathbb{R}$ is a *limit point* of (a_n) if

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \ni n \geq N \Rightarrow |a_n - p| < \epsilon.$$

Problem 1. Let (a_n) be a sequence of real numbers, and let $p, q \in \mathbb{R}$. Show that if p and q are limit points for (a_n), then $p = q$.

Solution. Suppose not. Then without loss of generality, $q > p$. Let $\epsilon = \dfrac{q-p}{2}$, which is positive, and let $N \in \mathbb{N}$ be so large that $n \geq N$ implies $|a_n - q| < \epsilon$. Then for $n \geq N$, $|a_n - p| > \epsilon$, so p is not a limit point. □

Definition 2. Let (a_n) be a sequence of real numbers. We say that $q \in \mathbb{R}$ is a *cluster point* of (a_n) if

$$\forall \epsilon > 0 \, \forall N \in \mathbb{N} \, \exists n \geq N \ni |a_n - q| \leq \epsilon.$$

Problem 2. Give an example of a sequence (a_n) which has exactly three cluster points.

Solution. Let $a_n = \sin \dfrac{2\pi n}{3}$; then $A = \{a_n \mid n \in N\} = \{0, \pm \dfrac{\sqrt{3}}{2}\}$ contains exactly three values, each of which occurs infinitely often. Thus, it has each of these as a cluster point, and no other cluster points. □

Problem 3. Let (a_n) be a sequence of real numbers. Show that if p is a limit point of (a_n), then p is a cluster point of (a_n).

Proof. Suppose that (a_n) converges to p. Let $\epsilon > 0$ and $N \in \mathbb{N}$; considering the definition of cluster point, we need to find $n \geq N$ such that $|a_n - p| < \epsilon$. Since (a_n) converges to p, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - p| < \epsilon$. Let $n = \max\{N, N_1\}$. Then $|a_n - p| < \epsilon$. □

Definition 3. Let (a_n) be a sequence of real numbers. A *subsequence* of (a_n) is a sequence of the form (a_{n_k}), where (n_k) is a strictly increasing sequence of natural numbers.

Definition 4. Let (a_n) be a sequence of real numbers. We say that $s \in \mathbb{R}$ is a *subsequential limit* of (a_n) if there exists a subsequence of (a_n) which converges to s.

Problem 4. Give an example of a sequence (a_n) and four subsequences which converge to different subsequential limits.

Proof. Let a_n denote the remainder when n is divided by 4. Then $(a_n) = 1, 2, 3, 0, 1, 2, 3, 0, \dots$. Then (a_{4n}), (a_{4n+1}), (a_{4n+2}), and (a_{4n+3}) are four distinct subsequences which converge to 0, 1, 2, and 3, respectively. □

Problem 5. Let (a_n) be a sequence of real numbers and let $q \in \mathbb{R}$. Show that q is a cluster point of (a_n) if and only if q is a subsequential limit of (a_n).

Solution. To prove an if and only if statement, we prove each direction of implication.

($\Rightarrow$) Suppose that q is a cluster point. We wish to construct a subsequence (a_{n_k}) of (a_n) whose limit is q. Set

$$E_m = \{n \in \mathbb{N} \mid n > m\}; \qquad A_k = \{n \in \mathbb{N} \mid a_n \in (q - \frac{1}{k}, q + \frac{1}{k})\};$$

where $m, k \in \mathbb{N}$. Since q is a cluster point, for every $m, k \in \mathbb{N}$, there exists $n > m$ such that $|a_n - q| < \frac{1}{k}$. Now $n > m$ means that $n \in E_m$, and $|a_n - q| < \frac{1}{k}$ means that $a_n \in (q - \frac{1}{k}, q + \frac{1}{k})$, so $a_n = A_k$. Thus $E_m \cap A_k \neq \varnothing$.

By the Well-Ordering Principle, every nonempty subset of $\mathbb{N}$ has a minimum. Let $n_1 = \min A_1$ and let $n_{k+1} = \min E_{n_k} \cap A_k$. Then n_k is an increasing sequence of natural numbers, because $n_{k+1} \in E_{n_k} \Rightarrow n_{k+1} > n_k$. Also $a_{n_k} \in (q - \frac{1}{k}, q + \frac{1}{k})$, so $\lim_{k \to \infty} a_{n_k} = q$. Therefore q is a subsequential limit.

($\Leftarrow$) Suppose that (a_{n_k}) is a subsequence which converges to q. Let $\epsilon > 0$ and let $N \in \mathbb{N}$. We wish to show that (a_n) clusters at q, so it suffices to find $n > N$ such that $|a_n - q| < \epsilon$.

Since (a_{n_k}) converges to q, there exists $K \in \mathbb{N}$ such that $|a_{n_k} - q| < \epsilon$ for all $k > K$. Let $n = \max\{N, n_K\} + 1$. Then $|a_n - q| < \epsilon$, so (a_n) clusters at q. □