AP Calculus AB | Quiz 0507 | Name:
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Problem 1. Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c) = -1$.

Solution. Since f is continuous on $[2, 5]$ and differentiable on $(2, 5)$, the Mean Value Theorem says that the exists $c \in (2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{2 - 5}{5 - 2} = -1.$$

$\square$

(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k) = 0$.

Solution. By the Chain Rule, $g'(x) = f'(f(x))f'(x)$. Now $g'(2) = f'(f(2))f'(2) = f'(5)f'(2)$, and $g'(5) = f'(f(5))f'(5) = f'(2)f'(5)$. By the Commutative Property of Multiplication, $g'(2) = g'(5)$. By Rolle's Theorem, there exists $c \in (2, 5)$ such that $g'(c) = 0$. $\square$

(c) Show that if $f''(x) = 0$ for all x, then the graph of g does not have a point of inflection.

Solution. Suppose that $f''(x) = 0$ for all x. Then, by the product rule,

$$g''(x) = f''(f(x))f'(x) + f'(f(x))f''(x) = 0 \cdot f'(x) + f'(f(x)) \cdot 0 = 0$$

for all x. Thus g is twice differentiable everywhere, but the sign of g'' never changes. Therefore g has no point of inflection. $\square$

(d) Let $h(x) = f(x) - x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r) = 0$.

Solution. The difference of continuous functions is continuous, so h is continuous. Note that $h(2) = 5 - 2 > 0$, and $h(5) = 2 - 5 < 0$. Since the sign of h changes between 2 and 5, the Intermediate Value Theorem states that there must be $r \in (2, 5)$ such that $h(r) = 0$. $\square$