REAL ANALYSIS
TOPIC 31 - CALCULUS REVIEW

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Abstract. We review the three main definitions from Calculus: limits, derivatives, and integrals, as presented in Thomas' Calculus. We will focus on real-valued functions of a real variable, but encourage the reader to consider potential generalizations to $\mathbb{R}^n$, metric spaces, or topological spaces.

1. Topology of $\mathbb{R}$

Let us restate the basic topological concepts we need for Calculus.

Let $U \subset \mathbb{R}$. We say that U is *open* if, for every $x \in U$, there exits $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$.

Let $F \subset \mathbb{R}$. We say that F is *closed* if its complement, $F^c = \mathbb{R} \smallsetminus F$, is open.

Let $p \in \mathbb{R}$.

A *neighborhood* of p is a subset of $\mathbb{R}$ which contains an open set which contains p. That is, a set $N \subset \mathbb{R}$ is a neighborhood of p if there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset N$.

A *deleted neighborhood* of p is a set of the form $N \smallsetminus \{p\}$, where N is a neighborhood of p.

Let $A \subset \mathbb{R}$.

We say that p is a *closure point* of A if every neighborhood of p intersects A.

We say that p is an *interior point* of A if there exists a neighborhood of p which is contained in A.

We say that p is a *boundary point* of A if every neighborhood of p intersects A and A^c.

We say that p is an *accumulation point* of A if every deleted neighborhood of p intersects A.

We say that p is an *isolated point* of A if $p \in A$ and there exists a deleted neighborhood of p which is disjoint from A. Thus if $a \in A$, then a is either an accumulation point of A, or a is an isolated point of A, but not both.

We will use the definitions and results regarding compactness and connectedness which we have previously explored, in the context of the real numbers. In particular, we will use the following results:

- **(a)** A compact subset of $\mathbb{R}$ is closed.
- **(b)** A connected subset of $\mathbb{R}$ is an interval.
- **(c)** A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.
- **(d)** A compact, connected subset of $\mathbb{R}$ is a bounded closed interval.

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2. Continuity

A *real-valued function of a real variable* is a function of the form $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$. We allow D to be any subset; it may have accumulation points which are not in D, and it may have isolated points. We will become particularly interested in situations where D is an unusual set. We review the basic facts of continuity we have already studied, in this context.

Definition 1. Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$. We say that f is *continuous* at $x_0 \in D$ if

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in D \, : \, |x - x_0| < \delta \, \Rightarrow \, |f(x) - f(x_0)| < \epsilon.$$

We say that f is *continuous* if f is continuous at every point in D.

Proposition 1. *Let $f : D \to \mathbb{R}$ be any function. If $x_0 \in D$ is an isolated point of D, then f is continuous at x_0.*

Proof. Exercise. □

We have seen the following consequences of the definition of continuity:

(a) The continuous image of a connected set is connected.
(b) The continuous image of a compact set is compact.

Theorem 1 (Intermediate Value Theorem (IVT)). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We have seen that the connected, compact subsets of $\mathbb{R}$ are closed intervals of finite length. We have also seen that the continuous image of a compact set is compact, and that the continuous image of a connected set is connected. Thus, since the domain $[a, b]$ is compact and connected, so is its image; thus, $f([a, b]) = [y, z]$ for some $y, z \in \mathbb{R}$. Since $yz < 0$, either $y < 0 < z$, so $0 \in [y, z]$. □

Theorem 2 (Extreme Value Theorem (EVT)). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f admits a minimum and maximum value on $[a, b]$; that is, there exist $c, d \in \mathbb{R}$ such that, for every $x \in [a, b]$,*

$$f(c) \leq f(x) \leq f(d).$$

Proof. The image of $[a, b]$ is a closed interval, so $f([a, b]) = [y, z]$ for some $y, z \in \mathbb{R}$. Thus y and z are in the image of f, so there exist $c, d \in [a, b]$ such that $f(c) = y$ and $f(d) = z$. The result follows. □

3. Limits

Definition 2. Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$. We say that the *limit* of f at x_0 is L if

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in D \, : \, 0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

We write $\lim_{x \to x_0} f(x) = L$ to mean that

- x_0 is an accumulation point of $\operatorname{dom}(f)$, and
- the limit of f at x_0 is L.

Observation 1. Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Then f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = L$.

Question 1. We have already discussed and kind of understand continuity; why isn't it enough? We do we need this new definition?

Next, we prove the standard arithmetic limit laws, summarized as follows:

- The limit of a constant is that constant.
- The limit of a sum is the sum of the limits.
- The limit of a product is the product of the limits.
- The limit of a quotient is the quotient of the limits.
- The limit operator commutes with continuous functions.

Proposition 2. *Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Let $C \in \mathbb{R}$ be constant. Define $f : D \to \mathbb{R}$ by $f(x) = C$ for all $x \in D$. Then*

$$\lim_{x \to x_0} f(x) = C.$$

Proof. Let $\epsilon > 0$ and let $\delta > 0$. Then if $x \in D$ and $0 < |x - x_0| < \delta$, we have $|f(x) - C| = |C - C| = 0 < \epsilon$. □

Proposition 3. *Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Suppose that $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$. Then*

$$\lim_{x \to x_0} (f + g)(x) = L + M.$$

Proof. Let $\epsilon > 0$.

Let $\delta_1 > 0$ be so small that $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \frac{\epsilon}{2}$.

Let $\delta_2 > 0$ be so small that $0 < |x - x_0| < \delta$ implies $|g(x) - M| < \frac{\epsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $x \in D$ with $0 < |x - x_0| < \delta$, we have

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Proposition 4. *Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Suppose that $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$. Then*

$$\lim_{x \to x_0} (fg)(x) = LM.$$

Proof. Let $\epsilon > 0$.

Let $\delta_0 > 0$ be so small that $0 < |x - x_0| < \delta_0$ implies $|g(x) - M| < 1$. Suppose that $0 < |x - x_0| < \delta_0$. Then $-1 + M < g(x) < 1 + M$, and also, $-1 - M < -g(x) < 1 - M$. Now let $B = \max\{|1 - M|, |1 + M|\}$; therefore $|f(x)| < B$.

Let $\delta_2 > 0$ be so small that $0 < |x - x_0| < \delta_2$ implies $|g(x) - M| < \frac{\epsilon}{2(|L| + 1)}$. We require the plus one in case $L = 0$.

Let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. Then, for $x \in D$ with $0 < |x - x_0| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x)g(x) - Lg(x)| + |Lg(x) - LM| \\ &\leq |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\epsilon}{2B} B + |L| \frac{\epsilon}{2(|L| + 1)} \\ &< \epsilon \end{aligned}$$

□

Proposition 5. *Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Suppose that $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, and that $M \neq 0$. Then*

$$\lim_{x \to x_0} \left(\frac{f}{g} \right)(x) = \frac{L}{M}.$$

Proof. It suffices to show that $\lim_{x \to x_0} \left(\frac{1}{g} \right)(x) = \frac{1}{M}$, as the stated result simply combines this with Proposition 4.

Let $\epsilon > 0$. Let $B = \frac{|M|}{2}$.

Let δ_0 be so small that $0 < |x - x_0| < \delta_0$ implies that $|g(x) - M| < \frac{|M|}{2}$. If $M > 0$, we have $\frac{M}{2} < g(x) - M$, so $0 < \frac{|M|}{2} = \frac{M}{2} < g(x) = |g(x)|$. If $M < 0$, we have $g(x) - M < -\frac{M}{2}$, so $g(x) < \frac{M}{2}$, so $|g(x)| = -g(x) > -\frac{M}{2} = \frac{|M|}{2}$. In either case, $|g(x)| > \frac{|M|}{2}$.

Let δ_1 be so small that $0 < |x - x_0| < \delta_1$ implies that $|g(x) - M| < \frac{\epsilon |M|^2}{2}$

Let $\delta = \min\{\delta_0, \delta_1\}$.

Then, for $x \in D$ with $0 < |x - x_0| < \delta$, we have

$$\begin{aligned} \left| \frac{1}{g(x)} - \frac{1}{M} \right| &= \left| \frac{M - g(x)}{g(x)M} \right| \\ &= |g(x) - M| \cdot \frac{1}{|g(x)||M|} \\ &< \frac{\epsilon |M|^2}{2} \cdot \frac{2}{|M|^2} \\ &= \epsilon. \end{aligned}$$

□

Corollary 1. *Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Let $c \in \mathbb{R}$. Suppose that $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$. Then*

(a) $\lim_{x \to x_0} (cf)(x) = cf$*;*
(b) $\lim_{x \to x_0} (-f)(x) = -f$*;*
(c) $\lim_{x \to x_0} (1/f)(x) = 1/L$*, when* $L \neq 0$*.*

Proof. We use the fact that the limit of a constant function is that constant.
Part (a) follows from Proposition 4 by changing f to c and g to f.
Part (b) follows from Part (a) by changing c to -1
Part (c) follows from Proposition **??** by changing f to 1 and g to f. □

Proposition 6. *Let $f : D \to \mathbb{R}$ and let $g : E \to \mathbb{R}$. Let x_0 be an accumulation point of D and let $\lim_{x \to x_0} f(x) = y_0$. Let y_0 be an accumulation point of E and let $\lim_{y \to y_0} g(y) = z_0$. Additionally, suppose that either*

(a) *$g(y_0) = z_0$, so that g is continuous at y_0, or*
(b) *there exists $\beta > 0$ such that $0 < |x - x_0| < \beta$ implies $f(x) \neq y_0$.*

Then $\lim_{x \to x_0} (g \circ f)(x) = z_0$*.*

Proof. Let $\epsilon > 0$.
Let $\delta > 0$ be so small that $0 < |y - y_0| < \delta$ implies $|g(y) - z_0| < \epsilon$.
Let $\gamma > 0$ be so small that $0 < |x - x_0| < \gamma$ implies $|f(x) - y_0| < \delta$.
Suppose **(a)** holds, and let $x \in D$ with $0 < |x - x_0| < \gamma$. Then $|f(x) - y_0| < \delta$. If $f(x) = y_0$, then $g(f(x)) = g(y_0) = z_0$, so $|g(f(x)) - z_0| = 0 < \epsilon$. If $f(x) \neq y_0$, then $0 < |f(x) - y_0| < \delta$, so $|g(f(x)) - z_0| < \epsilon$.
Otherwise, **(b)** holds. Let $\alpha = \min\{\beta, \gamma\}$. Then for $x \in D$, if $0 < |x - x_0| < \alpha$, then $f(x) - y_0 \neq 0$, so $0 < |f(x) - y_0| < \delta$, which implies that $|g(f(x)) - z_0| < \epsilon$. □

The next proposition relates our previous work with the definition currently under consideration. Is is in fact possible to prove some of the theorems regarding the arithmetic of series using the next definition, but we thought it informative to proceed as we did.

Proposition 7. *Let $D \subset \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$. Then $\lim_{x \to x_0} f(x) = L$ if and only if for every sequence (x_n) in D which converges to L, the sequence $(f(x_n))$ converges to $f(L)$.*

Proof. We prove both directions of the implication.
($\Rightarrow$) Suppose that $\lim_{x \to x_0} f(x) = L$, and let (x_n) be a sequence in D which converges to x_0. We wish to show that $(f(x_n))$ converges to L.
Let $\epsilon > 0$. Let $\delta > 0$ be so small that for $x \in D$ such that $0 < |x - x_0| < \delta$, we have $|f(x) - L| < \epsilon$. Let $N \in \mathbb{N}$ be so large that $n \geq N$ implies $|x_n - x_0| < \delta$. Then, for $n \geq N$, we have $0 < |x_n - x_0|$, so $|f(x_n) - L| < \epsilon$. Thus $(f(x_n))$ converges to L.
($\Leftarrow$) Here we prove the contrapositive. That is, assume that the limit of f at x_0 is not L, and construct a sequence (x_n) which converges to x_0 such that $(f(x_n))$ avoids L.
Since the limit of f at x_0 is not L, there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ we that $|(|f(x_n) - L) > \epsilon$. As is clear to all those who observe the same, the sequence (x_n) converges to x_0, but $(f(x_n))$ does not converge to L. □

4. Differentiability

Definition 3. Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$. Let $a \in D$ be an accumulation point of D. We say that f is *differentiable at a* if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, set

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a};$$

we call $f'(a)$ the *derivative* of f at a.

We wish to vary the point at which we are taking the derivative, and for notational clarity, we to call it x. We rephrase the definition by replacing $x - a$ with h, and then replacing x_0 with x. It is clear, then, that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Let $D' \subset D$ be the set of all points in D at which f is differentiable. We obtain a function

$$f : D' \to \mathbb{R} \quad \text{given by } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Proposition 8. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$. If f is differentiable at a, then f is continuous at a.*

Proof. We use the fact that f is continuous at a if and only if $\underset{x \to a}{f}(x) = f(a)$.

Suppose that f is differentiable at a. Note that

$$f(a+h) = f(a) + f(a+h) - f(a) = f(a) + \frac{f(a+h) - f(a)}{h} \cdot h.$$

Take the limit as $h \to 0$ of both sides; the limit laws we have previously derived show that

$$\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a) + \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h = f(a) + f'(a) \cdot 0 = f(a).$$

Thus, f is continuous at a. □

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