REAL ANALYSIS
TOPIC VIII - TOPOLOGICAL SPACES (DRAFT)

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Abstract. We define *topological spaces*, which are sets together with the additional structure provided by a collection of open sets. We give examples and develop their basic properties.

Contents

Date: February 14, 2015.

1. Topological Spaces

1.1. **Topological Spaces.**

1.1.1. *Topologies.*

Definition 1. A *topological space* is a set X together with a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ such that

- **(T1)** $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
- **(T2)** $\mathcal{U} \subset \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$;
- **(T3)** $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ finite $\Rightarrow \cap\mathcal{U} \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X. A subset $A \subset X$ is called *open* if $A \in \mathcal{T}$, and is called *closed* if $X \smallsetminus A \in \mathcal{T}$.

Example 1. Let X be a set and let $\mathcal{T} = \{\varnothing, X\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *trivial* topology on X.

Example 2. Let X be a set and let $\mathcal{T} = \mathcal{P}(X)$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *discrete* topology on X.

Example 3. Let X be a set and let $\mathcal{T} = \{A \subset X \mid X \smallsetminus A \text{ is finite }\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *cofinite* topology on X.

Example 4. Let X be a set and let $\mathcal{T} = \{A \subset X \mid X \smallsetminus A \text{ is countable }\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *cocountable* topology on X.

Definition 2. Let X be a set. A *tower* of subsets of X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ which contains the empty set and the entire set and is totally ordered by inclusion.

Example 5. Let X be a set and $\mathcal{T}$ a tower of subsets of X. Then $\mathcal{T}$ is a topology on X, called a *tower topology.*

Example 6. Let X be a totally ordered set. For $a \in X$, set

$$L_a = \{x \in X \mid x < a\} \quad \text{and} \quad R_a = \{x \in X \mid x > a\}.$$

Set

$$\mathcal{L} = \{L_a \mid a \in X\} \cup \{\varnothing, X\} \quad \text{and} \quad \mathcal{R} = \{R_a \mid a \in X\} \cup \{\varnothing, X\}.$$

Then $\mathcal{L}$ is a topology on X, called the *left order topology*, and $\mathcal{R}$ is a topology on X, called the *right order topology.*

1.1.2. *Cotopologies.*

Definition 3. Let X be a set and let $\mathcal{A} \subset \mathcal{P}(X)$.

The *cocollection* of $\mathcal{A}$ is defined to be the collection of complements of sets in $\mathcal{A}$.

Definition 4. Let X be a set and let $\mathcal{F}$ be a collection of subsets of X such that

- **(C1)** $\varnothing \in \mathcal{C}$ and $X \in \mathcal{C}$;
- **(C2)** $\mathcal{C} \subset \mathcal{F} \Rightarrow \cap\mathcal{C} \in \mathcal{F}$;
- **(C3)** $\mathcal{C} \subset \mathcal{F}$ and $\mathcal{C}$ finite $\Rightarrow \cup\mathcal{C} \in \mathcal{F}$.

Then $\mathcal{C}$ is called a *cotopology* on X.

Proposition 1. *Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$. Let $\mathcal{F}$ be the cocollection of $\mathcal{T}$. Then $\mathcal{T}$ is a topology on X if and only if $\mathcal{F}$ is a cotopology on X.*

Proof. The empty set is in $\mathcal{T}$ if and only if the entire set is in $\mathcal{F}$, and the entire set is in $\mathcal{T}$ if and only if the empty set is in $\mathcal{F}$.

By DeMorgan's laws, a union of sets in $\mathcal{T}$ is in $\mathcal{T}$ if and only if the intersection of their complements is in $\mathcal{F}$, and a finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$ if and only if the union of their complements is in $\mathcal{F}$. □

1.1.3. *Kuratowski Operators.*

Definition 5. Let X be a set. A *Kuratowski operator* on X is a function

$$K : \mathcal{P}(X) \to \mathcal{P}(X)$$

such that

- **(K1)** $K(\varnothing) = \varnothing$;
- **(K2)** $A \subset K(A)$;
- **(K3)** $K(K(A)) = K(A)$;
- **(K4)** $K(A \cup B) = K(A) \cup K(B)$.

Proposition 2. *Let X be a set and let K be a Kuratowski operator on X. If $A \subset B \subset X$, then $K(A) \subset K(B)$.*

Proof. Note that $B = A \cup (B \smallsetminus A)$, and by **(K4)**, we have $K(B) = K(A) \cup K(B \smallsetminus A)$. Thus $K(A) \subset K(B)$. □

Proposition 3. *Let X be a set and let K be a Kuratowski operator on X. Let $\mathcal{F} = K(\mathcal{P}(X))$ and let $\mathcal{T}$ be the cocollection of $\mathcal{F}$. Then $(X, \mathcal{T})$ is a topological space.*

Proof. It suffices to show that $\mathcal{F}$ is a cotopology.

By property **(K1)**, $K(\varnothing) = \varnothing$, so $\varnothing \in \mathcal{F}$.

Since $K(X) \in \mathcal{P}(X)$, $K(X) \subset X$. By property **(K2)**, $X \subset K(X) \subset X$. Thus $K(X) = X$ so $X \in \mathcal{F}$.

Let $\mathcal{C} \subset \mathcal{F}$. By property **(K2)**, we have $\cap\mathcal{C} \subset K(\cap\mathcal{C})$. By property **(K3)**, $K(C) = C$ for every $C \in \mathcal{C}$; also it is clear that $\cap\mathcal{C} \subset C$ for every $C \in \mathcal{C}$. By the previous proposition, we have $K(\cap\mathcal{C}) \subset K(C) = C$. Since C is arbitrary, $K(\cap\mathcal{C}) \subset \cap\mathcal{C}$. Thus $\cap\mathcal{C} = K(\cap\mathcal{C})$, so $\cap\mathcal{C} \in \mathcal{F}$.

Let $\mathcal{C}$ be a finite subset of $\mathcal{F}$. Then $\mathcal{C} = \{C_1, \dots, C_n\}$ for some $n \in \mathbb{N}$. By **(K4)** and induction, $\cup_{i=1}^n C_i = \cup_{i=1}^n K(C_i) = K(\cup_{i=1}^n C_i)$. Thus $\cup\mathcal{C} = K(\cup\mathcal{C})$, so $\cup\mathcal{C} \in \mathcal{F}$. □

1.2. **Neighborhoods.**

Definition 6. Let X be a topological space and let $x \in X$. A *neighborhood* of x is a subset $N \subset X$ such that there exists an open set $U \subset N$ with $x \in U$.

Remark 1. Let X be a topological space and let $x \in X$. If U is an open set containing x, then U is itself a neighborhood of x, and is referred to as an open neighborhood. Thus there exists at least one neighborhood of x; indeed, X is open and contains x.

We are interested in sets A whose intersection with neighborhoods of x are nonempty, in which case we say that A intersects the neighborhood. If $x \in A$, then every neighborhood of x intersects A; this is the less interesting case for us.

Clearly A intersects every neighborhood of x if and only if A intersects every open neighborhood of x; the forward direction is immediate and the reverse direction is given by considering a neighborhood which does not intersect A, which must contain an open neighborhood which does not intersect A.

Definition 7. A *deleted neighborhood* of x is a set of the form $N \setminus \{x\}$, where N is a neighborhood of x.

Remark 2. Let X be a topological space and let $x \in X$. If $N \setminus \{x\}$ is a deleted neighborhood of x which does not intersect A, then either N does not intersect A or there is an open set $U \subset N$ such that x is the only element of A in that open set.

1.2.1. *Closure Points.*

Definition 8. Let X be a topological space and let $A \subset X$.

A *closure point* of A is a point $x \in X$ such that every neighborhood of x intersects A.

The *closure* of $A \subset X$ is the set of closure points of A and is denoted $\overline{A}$.

Proposition 4. *Let X be a topological space and $A \subset X$. Then $\overline{A}$ is the intersection of the closed sets of X which contain A.*

Proposition 5. *Let X be a topological space and $A \subset X$. Then A is closed if and only if $A = \overline{A}$.*

Proposition 6. *Let X be a topological space. Then closure is a Kuratowski operator, that is, for $A, B \subset X$,*

- **(K1)** $\overline{\varnothing} = \varnothing$;
- **(K2)** $A \subset \overline{A}$;
- **(K3)** $\overline{\overline{A}} = \overline{A}$;
- **(K4)** $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

Proof. The first two are immediate from the definition.

From **(K2)** we have $\overline{A} \subset \overline{\overline{A}}$. Suppose that $x \in \overline{\overline{A}}$. Then every open neighborhood of x intersects $\overline{A}$. For any open neighborhood U of x, let $y \in U \cap \overline{A}$. Then every open neighborhood of y intersects A. Since U is an open neighborhood of y, U intersects A. Thus $x \in \overline{A}$.

Suppose that $x \notin \overline{A} \cup \overline{B}$. Then there exists a neighborhoods U, V of x such that $U \cap A = \varnothing$ and $V \cap B = \varnothing$. Then $U \cap V$ is a neighborhood of x such that $(U \cap V) \cap (A \cup B) = \varnothing$. So $x \notin \overline{(A \cup B)}$. Therefore $\overline{(A \cup B)} \subset \overline{A} \cup \overline{B}$.

Suppose that $x \in \overline{A} \cup \overline{B}$. Then every open neighborhood of x intersects A or B, so it intersects $A \cup B$. Thus $x \in \overline{A} \cup \overline{B}$, so $\overline{A} \cup \overline{B} \subset \overline{(A \cup B)}$. ☐

Proposition 7. *Let X be a topological space. If $A \subset B \subset X$, then $\overline{A} \subset \overline{B}$.*

Proof. Let $y \in \overline{A}$. Then every neighborhood of y intersects A. Since $A \subset B$, every neighborhood of y intersects B. Thus $y \in \overline{B}$. ☐

1.2.2. *Interior Points.*

Definition 9. Let X be a topological space and let $A \subset X$.

An *interior point* of A is a point $x \in X$ such that A contains a neighborhood of x.

The *interior* of A is the set of interior points of A and is denoted A°.

Proposition 8. *Let X be a topological space and let $A \subset X$. Then A° is the union of the open sets contained in A.*

Proposition 9. *Let X be a topological space and let $A \subset X$. Then A is open if and only if $A = A^\circ$.*

Proposition 10. *Let X be a topological space and let $A \subset X$. Then*

(a) $A^\circ = X \setminus \overline{(X \setminus A)}$;
(b) $\overline{A} = X \setminus (X \setminus A)^\circ$;
(c) $A \subset B \Rightarrow A^\circ \subset B^\circ$;
(d) $(A^\circ)^\circ = A^\circ$;
(e) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

1.2.3. *Boundary Points.*

Definition 10. Let X be a topological space and let $A \subset X$.

A *boundary point* of A is a point $x \in X$ such that every neighborhood of x intersects A and $X \setminus A$.

The *boundary* of A is the set of boundary points of A and is denoted ∂A.

Proposition 11. *Let X be a topological space and let $A \subset X$. Then*

(a) $\partial A = \overline{A} \setminus A^\circ$;
(b) $\partial A = \overline{A} \cap \overline{(X \setminus A)}$;
(c) $\partial A = \partial(X \setminus A)$;
(d) $\overline{A} = A \cap \partial A$;
(e) $A^\circ = A \setminus \partial A$;
(f) $\partial(\partial A) \subset \partial A$;
(g) $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$.

Proposition 12. *Let X be a topological space and let $A \subset X$. Then $\partial A = \varnothing$ if and only if A is both open and closed.*

Proof.

($\Rightarrow$) Suppose that $\partial A = \varnothing$. Then $\overline{A} \subset A^\circ$. But $A^\circ \subset A \subset \overline{A}$, so $A^\circ = A = \overline{A}$. Thus A is both open and closed.

($\Leftarrow$) Suppose that A is both open and closed. Then $A^\circ = A = \overline{A}$, so $\partial A = \overline{A} \setminus A^\circ = \varnothing$. ☐

1.2.4. *Limit Points.*

Definition 11. Let X be a topological space and let $A \subset X$.

A *limit point* of A is a point $x \in X$ such that every deleted neighborhood of x intersects A.

The *limit* of A is the set of limit points of A and is denoted A'.

Proposition 13. *Let X be a topological space and $A, B \subset X$.*

(a) $A \subset B \Rightarrow A' \subset B'$;
(b) $(A \cup B)' = A' \cup B'$;
(c) $\overline{A} = A \cup A'$.

Corollary 1. *A subset of a topological space is closed if and only if it contains all of its limit points.*

1.2.5. *Isolated Points.*

Definition 12. Let X be a topological space and let $A \subset X$.

An *isolated point* of A is a point $x \in A$ such that some deleted neighborhood of x is contained in $X \smallsetminus A$.

The *isolation* of A is the set of isolated points of A and is denoted $A^{\odot}$.

Proposition 14. *Let X be a topological space and $A \subset X$.*

- **(a)** $A^{\odot} \subset A$;
- **(b)** $A^{\odot} \subset \partial A$;
- **(c)** $\overline{A} = A' \sqcup A^{\odot}$.

1.3. **Refinements.**

1.3.1. *Refinements.*

Definition 13. Let X be a set and let $\mathcal{S}$ and $\mathcal{T}$ be topologies on X.

If $\mathcal{S} \subset \mathcal{T}$, we say that $\mathcal{S}$ is a *courser* topology than $\mathcal{T}$ and that $\mathcal{T}$ is a *finer* topology than $\mathcal{S}$.

Remark 3. The coarsest topology on a set is the trivial topology and the finest topology on a set is the discrete topology.

Proposition 15. *Let X be a set and let $\{\mathcal{T}_\alpha \mid \alpha \in A\}$ be a collection of topologies on X. Then $\mathcal{I} = \cap_{\alpha \in A} \mathcal{T}_\alpha$ is a topology on X.*

Proof. Since the empty set and the entire set are in every topology, they are in the intersection.

If $\mathcal{U} \subset \mathcal{I}$, then $\mathcal{U} \subset \mathcal{T}_\alpha$ for every α. Thus $\cup\mathcal{U} \in \mathcal{T}_\alpha$ for every α, so $\cup\mathcal{U} \in \mathcal{I}$.

If $\mathcal{U} \subset \mathcal{I}$, then $\mathcal{U} \subset \mathcal{T}_\alpha$ for every α. If $\mathcal{U}$ is a finite collection, $\cap\mathcal{U} \in \mathcal{T}_\alpha$ for every α, so $\cap\mathcal{U} \in \mathcal{I}$. ☐

1.3.2. *Generated Topologies.*

Definition 14. Let X be a set and let $\mathcal{A} \in \mathcal{P}(X)$.

The *topology generated by* $\mathcal{A}$ is the intersection of all the topologies on X which contain $\mathcal{A}$, and is denoted $\langle \mathcal{A} \rangle$.

Remark 4. The topology generated by a collection $\mathcal{A} \subset \mathcal{P}(X)$ is the coarsest topology on X in which all of the sets in $\mathcal{A}$ are open.

1.3.3. *Bases.*

Definition 15. Let X be a set.

A *basis* for a topology on X is a collection of subsets $\mathcal{B} \subset \mathcal{P}(X)$ such that

- **(B1)** $\cup\mathcal{B} = X$;
- **(B2)** if $B_1, B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ with $p \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Proposition 16. *Let X be a set and let $\mathcal{B}$ be a basis for a topology on X. Let $\mathcal{T}$ be the collection of unions of sets in $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X, and $\mathcal{T} = \langle \mathcal{B} \rangle$.*

Proof. We consider the empty set to be the empty union, so $\varnothing \in \mathcal{T}$. By basis property **(B1)**, the entire set is in $\mathcal{T}$. By definition of $\mathcal{T}$, any union of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

Let $U_1, U_2 \in \mathcal{T}$. Since U_1 and U_2 are the unions of basis sets, for every $p \in U_1 \cap U_2$ there exist basis sets $B_1 \subset U_1$ and $B_2 \subset U_2$ such that $p \in B_1 \cap B_2$. By basis property **(2)**, there exists a basis set $B_p \subset B_1 \cap B_2 \subset U_1 \cap U_2$ with $p \in B_p$. Then

$$U_1 \cap U_2 = \bigcap_{p \in U_1 \cap U_2} \{p\} \subset \bigcap_{p \in U_1 \cap U_2} B_p \subset U_1 \cap U_2,$$

so $U_1 \cap U_2 = \bigcap\limits_{p \in U_1 \cap U_2} B_p$ is open. □

Remark 5. Every topology on a set X has itself as a basis. Bases are not necessarily unique.

1.3.4. *Subbases.*

Definition 16. Let X be a set.

A *subbasis* for a topology on X is a collection of subsets $\mathcal{S} \subset \mathcal{P}(X)$ such that the collection of all finite intersections of sets in $\mathcal{S}$ form a basis for a topology on X.

Remark 6. Every basis is a subbasis.

Proposition 17. *Let X be a set and let $\mathcal{S} \subset \mathcal{P}(X)$. If $\cup \mathcal{S} = X$, then $\mathcal{S}$ is a subbasis for a topology on X. Let $\mathcal{B}$ be a basis which is the collection of all finite intersections of sets in $\mathcal{S}$. Then $\langle \mathcal{S} \rangle = \langle \mathcal{B} \rangle$.*

Proof. Basis property **(B1)** is given and basis property **(B2)** is obvious. The second claim follows immediately from the first. □

Remark 7. If $\mathcal{T}$ is a topology on X, then $\mathcal{S} \subset \mathcal{P}(X)$ is a subbasis for $\mathcal{T}$ if and only if $\langle \mathcal{S} \rangle = \mathcal{T}$.

1.3.5. *Equivalent Bases.*

Definition 17. Let X be a set.

Two subbases $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{P}(X)$ are *equivalent* if they generate the same topology on X.

Proposition 18. *Let X be a set with bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent if and only if for $p_1 \in B_1 \in \mathcal{B}_1$ there exists $B_2 \in \mathcal{B}_2$ such that $p_1 \in B_2 \subset B_1$ and for $p_2 \in B_2 \in \mathcal{B}_2$ there exists $B_1 \in \mathcal{B}_1$ such that $p_2 \in B_1 \subset B_2$.*

Proof. Let $\mathcal{T}_1 = \langle \mathcal{B}_1 \rangle$ and $\mathcal{T}_2 = \langle \mathcal{B}_2 \rangle$. Let $U \in \mathcal{T}_1$. Then $U = \cup_{\alpha \in A} B_\alpha$ for some $\{B_{1,\alpha} \mid \alpha \in A\} \subset \mathcal{B}_1$. For each $p \in U$ there exists $B_{1,\alpha}$ such that $p \in B_{1,\alpha}$. By hypothesis there exists $B_{2,\alpha} \in \mathcal{B}_2$ such that $p \in B_{2,\alpha} \subset B_{1,\alpha}$. Thus U is the union of such $B_{2,\alpha}$ and in $\mathcal{T}_2$. Therefore $\mathcal{T}_1 \subset \mathcal{T}_2$. Similarly, $\mathcal{T}_2 \subset \mathcal{T}_1$. □

Proposition 19. *Let X be a totally ordered set. For $a \in X$, let $L_a = \{x \in X \mid x < a\}$. and let $R_a = \{x \in X \mid x > a\}$. Then $\mathcal{S} = \{L_a \mid a \in X\} \cup \{R_a \mid a \in X\}$ is a subbasis for a topology on X.*

For $a, b \in X$, let $I_{a,b} = \{x \in X \mid a < x < b\}$. Let $\mathcal{B} = \{I_{a,b} \mid a, b \in X\}$. Then $\mathcal{B}$ forms the basis for a topology on X which is generated by the subbasis $\mathcal{S}$. The topology generated by $\mathcal{B}$ is called the total order topology *on X.*

Remark 8. The standard topology on the real numbers is a total order topology.

2. Continuous Functions

2.1. Continuous Functions.

Definition 18. Let X and Y be spaces and $f : X \to Y$.

We say that f is *continuous* if for every open set $V \subset Y$, $f^{-1}(V) \subset X$ is open.

Definition 19. Let X and Y be spaces and $f : X \to Y$ and let $x_0 \in X$.

We say that f is *continuous at* x_0 if for every neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that $f(U) \subset V$.

Proposition 20. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if f is continuous at every point in X.*

Proof. Suppose that f is continuous, and let $x_0 \in X$. Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V.

Conversely, suppose that f is continuous at every point in X. Let $V \subset Y$ be open and let $U = f^{-1}(V)$. For every $x \in U$, V is a neighborhood of $f(x)$, so there exists an open neighborhood U_x of x such that $f(U_x) \subset V$. But then $U_x \subset U$, and U is the union of such sets; thus U is open, and f is continuous. □

Proposition 21. *Let X and Y be spaces and $f : X \to Y$. If X has the discrete topology or Y has the trivial topology then f is continuous.*

Proposition 22. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if for every closed set $F \subset Y$, $f^{-1}(F) \subset X$ is closed.*

Proof.

($\Rightarrow$) Suppose that f is continuous. Let $F \subset Y$ be closed. Let $U = Y \smallsetminus F$; then U is open, so $f^{-1}(U)$ is open, so $X \smallsetminus f^{-1}(U)$ is closed. But $X \smallsetminus f^{-1}(U) = f^{-1}(Y \smallsetminus U) = f^{-1}(F)$.

($\Leftarrow$) Suppose that for every closed set $F \subset Y$, $f^{-1}(F)$ is closed in X. Let $U \subset Y$ be open; then $Y \smallsetminus U$ is closed in Y, so $f^{-1}(Y \smallsetminus U)$ is closed in X. Thus $f^{-1}(U) = X \smallsetminus f^{-1}(Y \smallsetminus U)$ is open in X. Therefore f is continuous. □

Proposition 23. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if for every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*

Proof.

($\Rightarrow$) Suppose that f is continuous. Let $A \subset X$ and let $y \in f(\overline{A})$. Then $y = f(x)$ for some point $x \in \bar{(}A)$. Let V be an open neighborhood of y. Then $f^{-1}(V)$ is open in X and contains x. Thus there exists $a \in A \cap f^{-1}(V)$, and $f(a) \in V$; that is, V intersects $f(A)$. Therefore $y \in \overline{f(A)}$, and $f(\overline{A}) \subset \overline{f(A)}$.

($\Leftarrow$) Suppose that for every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

Let $F \subset Y$ be closed and let $A = f^{-1}(F)$. Then $f(A) = F$, and since F is closed, $\overline{f(A)} = F$. Thus $F = f(A) \subset f(\overline{A}) \subset \overline{f(A)} = F$. This shows that $f(\overline{A}) = F$, so $\overline{A} \subset f^{-1}(F) = A$; since $A \subset \overline{A}$, we see that $A = \overline{A}$, so A is closed. Therefore f is continuous. □

Proposition 24. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Let $A \subset X$. Then*

(a) $f(A)^\circ \subset f(A^\circ)$*;*

(b) $f(A)^\odot \subset f(A^\odot)$*.*

Proof. Let $y \in f(A)^\circ$. Then $y \in f(A)$, so $y = f(x)$ for some $x \in X$. Also there exists an open set V in Y such that $y \in V \subset f(A)$. Since f is continuous, $f^{-1}(V) \subset A$ is an open neighborhood of x contained in A, so $x \in A^\circ$, and $y \in f(A^\circ)$. Therefore $f(A)^\circ \subset f(A^\circ)$; this proves **(a)**.

Let $y \in f(A)^\odot$). Then $y \in f(A)$, and there exists an open neighborhood V of y in Y such that $V \cap (f(A) \smallsetminus \{y\}) = \varnothing$. Then $f^{-1}(V) \cap A = \{y\}$, so $y \in f(A)^\odot$. Therefore $f(A)^\odot \subset f(A^\odot)$; this proves **(b)**. $\square$

2.2. **Open and Closed Maps.**

Definition 20. Let X and Y be spaces and let $f : X \to Y$.

We say that f is *open* if for every open set $U \subset X$, $f(U) \subset Y$ is open.

We say that f is *closed* if for every closed set $F \subset X$, $f(F) \subset Y$ is closed.

Proposition 25. *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Let $\mathcal{B}$ be a basis for X. If $f(B)$ is open in Y for every $B \in \mathcal{B}$, then f is an open map.*

Proof. Let $U \subset X$ be open. Then $U = \cup_{\alpha \in I} B_\alpha$ for some collection $\{B_\alpha \in \mathcal{B} \mid \alpha \in I\}$ of basis sets. Thus $f(U) = f(\cup_{\alpha \in I} B_\alpha) = \cup_{\alpha \in I} f(B_\alpha)$; since $f(B_\alpha)$ is open for every $\alpha \in I$, then so is $f(U)$. $\square$

Example 7. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$ and let $Y = \mathbb{R}$. Let $f : X \to Y$ by $f(x, y) = x$. Then f is a surjective continuous closed map which is not open. It is not open because, for example, the set $\{(0, y) \mid y \in (1, 2)\}$ is open in X but projects onto a point in $\mathbb{R}$.

Example 8. Let $X = \mathbb{R}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Define $f : X \to Y$ by $f(x) = (\cos x, \sin x)$. Then f is a surjective continuous open map which is not closed. It is open because it is open on a basis for the topology of $\mathbb{R}$ consisting of open intervals whose width is less than 2π. It is not closed because, for example, the set $\{x \in \mathbb{R} \mid x = 2\pi n + \frac{\pi}{2n}\}$ is closed in X but its image in Y has a limit point $(1, 0)$ which is not in the image.

Definition 21. Let X and Y be spaces and let $f : X \to Y$. We say that f is *bicontinuous* if it is both open and continuous.

Proposition 26. *Let $f : X \to Y$ be a bijective function between spaces. Then f is open if and only if f^{-1} is continuous.*

2.3. **Homeomorphisms.**

Definition 22. Let X and Y be spaces. A *homeomorphism* between X and Y is a function $f : X \to Y$ which is bijective, open, and continuous. If there exists a homeomorphism between X and Y, we say that X and Y are *homeomorphic.*

Proposition 27. *Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be topological spaces. Then X and Y are homeomorphic if and only if there exists an inclusion preserving bijection between $\mathcal{S}$ and $\mathcal{T}$.*

Proposition 28. *The empty space is unique. Up to homeomorphism, a one point space is unique.*

Proposition 29. *Let $S = \{0, 1\}$ and $\mathcal{T} = \{\varnothing, \{0\}, S\}$. Then $\mathcal{T}$ is a topology on S and $(S, \mathcal{T})$ is called the* Sierpinski space*.*

Proposition 30. *Up to homeomorphism, there are exactly three spaces with two elements.*

Proof. Let $X = \{a, b\}$. The trivial space and the discrete space on X are clearly distinct.

The only other possibilities for a topology on X are $\mathcal{T}_a = \{\varnothing, \{a\}, X\}$ are $\mathcal{T}_b = \{\varnothing, \{b\}, X\}$. Permuting a and b is a homeomorphism between $(X, \mathcal{T}_a)$ and $(X, \mathcal{T}_b)$. $\square$

Remark 9. In the above notation, sending a to 0 and b to 1 is a homeomorphism between $(X, \mathcal{T}_a)$ and the Sierpinski space. Homeomorphisms preserve all properties relevant to topology. For this reason, $(X, \mathcal{T}_a)$ and $(X, \mathcal{T}_b)$ may also be called the Sierpinski space.

3. Subspaces, Products and Quotients

3.1. **Subspaces.**

Definition 23. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$.

The *subspace topology* relative to Y is collection of subsets of Y

$$\mathcal{T}(Y) = \{U \cap Y \mid U \in \mathcal{T}\}.$$

Proposition 31. *Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. Then $(Y, \mathcal{T}(Y))$ is a topological space, called a* subspace *of $(X, \mathcal{T})$. The open sets of $(Y, \mathcal{T}(Y))$ are called* relatively open *in X and the closed sets are called* relatively closed *in X.*

Remark 10. Every subspace of a trivial, discrete, cofinite, or cocountable space is respectively trivial, discrete, cofinite, or cocountable.

Proposition 32. *Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$. Then $\mathcal{T}(Y) \subset \mathcal{T}$ if and only if $Y \in \mathcal{T}$.*

Definition 24. Let X be a space and let Y be a subspace of X. Then the map $i : Y \to X$ defined by $y \mapsto y$ is called the *inclusion map.*

Proposition 33. *Let X be a space and let Y be a subspace of X. Then the inclusion map $i : Y \to X$ is continuous.*

Definition 25. Let X and Y be spaces and let $f : X \to Y$.

We say that f is *relatively open* if for every open $U \subset X$, $f(U)$ is relatively open in $f(X)$.

We say that f is an *embedding* if f is injective, continuous, and relatively open.

Proposition 34. *Let $f : X \to Y$ be an embedding. Then X is homeomorphic to the subspace $f(X) \subset Y$.*

3.2. **Products.**

Definition 26. Let $\mathcal{X}$ be a nonempty family of topological spaces and let $\times\mathcal{X}$ be their Cartesian product. For $X \in \mathcal{X}$, let π_X be the projection from $\times\mathcal{X}$ onto X.

Let $\mathcal{S} = \{\pi_X^{-1}(U) \mid X \in \mathcal{X} \text{ and } U \text{ open in } X\}$. Then $\langle\mathcal{S}\rangle$ is called the *product topology* on $\times\mathcal{X}$.

A cartesian product of a family of spaces endowed with the product topology is called the *product space* of the family.

Proposition 35. *The product of a family of trivial spaces is trivial. The product of a family of discrete spaces is discrete if and only if the family is finite.*

Proposition 36. *Let $\mathcal{X} = \{X_\alpha \mid \alpha \in A\}$ be a nonempty family of spaces and let $\mathcal{B}$ be the set of cartesian products of one open set from each space such that all but finitely many of these open sets are the entire space. Then $\mathcal{B}$ is a basis for the product topology on $\times\mathcal{X}$.*

Proposition 37. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be endowed with the product topology. Then for every $X \in \mathcal{X}$, the projection function $\pi_X : \times\mathcal{X} \to X$ is continuous.*

Proposition 38. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be the cartesian product endowed with some topology. Suppose that for every $X \in \mathcal{X}$, the projection function $\pi_X : \times\mathcal{X} \to X$ is continuous. Then the topology on $\times\mathcal{X}$ is at least as fine as the product topology.*

3.3. **Quotients.**

Definition 27. Let $(X, \mathcal{T})$ be a topological space, Y a set, and $q : X \to Y$ a surjective function. Let $\mathcal{V} = \{V \subset Y \mid f^{-1}(V) \in \mathcal{T}\}$. Then $\mathcal{V}$ is a topology on Y, called the *quotient topology* on Y induced by q. The function q is called the or the *quotient projection* of X onto Y.

Proposition 39. *A quotient projection in continuous.*

Proposition 40. *The quotient topology is the finest topology such that the quotient projection is continuous.*

Proposition 41. *Let $\mathcal{X}$ be a family of topological spaces and let $\times\mathcal{X}$ be the product space. For $X \in \mathcal{X}$, let $\pi_X : \times\mathcal{X} \to X$ be the cartesian projection. Then X is endowed with the quotient topology induced by π_X.*

4. Connectedness

4.1. **Clopen Sets.**

Definition 28. Let X be a topological space and let $A \subset X$.

We say that A is *clopen* if A is both open and closed.

Proposition 42. *Let X be a topological space and let $A \subset X$. Then A is clopen if and only if for every $c \in \overline{A}$ there exists a neighborhood U of c such that $U \subset A$.*

4.2. **Separation.**

Definition 29. Let X be a space and let $A, B \subset X$.

We say that A and B are *separated* if there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$. This relationship is denoted $A|B$. The pair (U, V) is called a *separation* of (A, B).

Proposition 43. *In a trivial space two sets are separated if and only if one of them is empty. In a discrete space two sets are separated if and only if they are disjoint. In a cofinite space two nonempty sets are separated if and only if they are disjoint and finite.*

Proposition 44. *Let X be a space and let $A, B \subset X$. Then $A|B$ if and only if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.*

Proof.

($\Rightarrow$) Let (U, V) be a separation of (A, B). Then for any $b \in B$, V is a neighborhood of b which is disjoint from A. Thus b in not in $\overline{A}$. Similarly, A does not intersect $\overline{B}$.

($\Leftarrow$) Suppose $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. Let $U = X \setminus \overline{B}$ and $V = X \setminus \overline{A}$. Then U and V are open and form a separation of A and B. $\square$

Proposition 45. *Let X be a space with subsets A, B, and C. Then*

(a) $\varnothing|A$;
(b) $A|B \Rightarrow B|A$;
(c) $A|B$ *and* $C \subset A \Rightarrow C|B$;
(d) $A|B$ *and* $A|C \Rightarrow A|(B \cup C)$;

Proposition 46. *Let X be a space and let $A, B \subset X$. The following conditions are equivalent:*

i. $A|B$;
ii. *A and B are disjoint relatively closed subsets of $A \cup B$;*
iii. *A and B are disjoint relatively open subsets of $A \cup B$.*

Proof.

(i)$\Rightarrow$(ii) Suppose $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. Since $A \subset \overline{A}$, A and B are disjoint. Since no point in B is in the closure of A, every point in B has a neighborhood which is disjoint from A. Let V be the union of these neighborhoods. Then V is open, $B \subset V$, and $V \cap A = \varnothing$. Similarly there exists and open set U such that $A \subset U$ and $U \cap B = \varnothing$. Then $U \cap A = A$ is relatively open in $A \cup B$ and so is $V \cap B = B$.

(ii)$\Rightarrow$(iii) Suppose that A and B are disjoint relatively closed sets in $A \cup B$. Then $A \cup B \setminus B = A$ and $A \cup B \setminus A = B$ are relatively open.

(iii)$\Rightarrow$(i) Suppose that A and B are disjoint relatively open sets in $A \cup B$. Then there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, and $A \cap V = B \cap U = \varnothing$. Then no point of A is in the closure of B and vice versa. Thus $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. $\square$

4.3. **Connectedness.**

Definition 30. Let X be a space and let $A \subset X$.

We say that A is *separated* if it is the union of two nonempty separated sets, and A is *connected* if it is not separated.

Proposition 47. *Every subset of a trivial space is connected. A subset of a discrete space is connected if and only if it contains at most one element.*

Proposition 48. *Let X be a space. The following conditions are equivalent.*

i. *X is connected;*
ii. *X is not the disjoint union of two nonempty open sets;*
iii. *X is not the disjoint union of two nonempty closed sets;*
iv. *if $A \subset X$ is nonempty and clopen, then $A = X$.*

Proof. (i)$\Rightarrow$(ii) Suppose that $X = U \cup V$ with U, V open and disjoint. Then U and V are neighborhoods of the points they contain, and form a separation.

(ii)$\Rightarrow$(iii) Suppose that $X = C \cup F$ with C, F closed and disjoint. Let $U = X \setminus F$ and $V = X \setminus C$. Then U and V are open and disjoint and their union is X.

(iii)$\Rightarrow$(iv) Let C be a proper nonempty subset of X which is clopen. Then $F = X \setminus C$ is closed.

(iv)$\Rightarrow$(i) Suppose X is not connected. Then X is the union of nonempty separated sets A and B. Let (U, V) be a separation of (A, B). Then $A \subset U$, $B \cap U = \varnothing$, and $A \cup B = X$ together imply that $X = U \cup B$. Thus B is closed. Similarly, A is closed so B is also open. $\square$

Proposition 49. *Let X be a space and let $A, B, C \subset X$. Suppose that $A|B$, C is connected, and $C \subset A \cup B$. Then $C \subset A$ or $C \subset B$.*

Proposition 50. *Let $\mathcal{Q}$ be a collection of connected subsets of a space X and C a connected subset of X which is not separated from any member of $\mathcal{Q}$. Then $C \cup (\cup\mathcal{Q})$ is connected.*

Proof. Let $Y = C \cup (\cup\mathcal{A})$ and suppose that $Y = A \cup B$ for some separated sets A and B. Thus $C \subset A$ or $C \subset B$. Suppose, without loss of generality, that $C \subset A$. Then for $Q \in \mathcal{Q}$, since Q is connected, $Q \subset A$ or $Q \subset B$. If $Q \subset B$, then $C|Q$, contrary to our hypothesis. Thus $Q \subset A$. Thus $A = Y$ and $B = \varnothing$. Thus Y is connected. $\square$

Corollary 2. *Let $\mathcal{Q}$ be a collection of connected subsets of a space X. If $\cap\mathcal{Q}$ is nonempty, then $\cup\mathcal{Q}$ is connected.*

Proposition 51. *Let C be a connected subset of a space X and suppose that $C \subset A \subset \overline{C}$. Then A is connected. In particular, $\overline{C}$ is connected.*

Proof. Let U and V be disjoint open sets such that $A \subset U \cup V$. Then $C \subset U$ or $C \subset V$. Suppose, without loss of generality, that $C \subset U$. If $a \in A \cap V$, then V is a neighborhood of a which is disjoint from C and $a \notin \overline{C}$, contradicting our hypothesis. Thus $A \subset U$, and A cannot be separated. $\square$

Proposition 52. *The continuous image of a connected set is connected.*

Proof. Let $f : X \to Y$ be a continuous function Let $C \subset X$ and let $D = f(C)$. Suppose that $D \subset f(X) \subset Y$ is not connected. Then there exist disjoint open sets $V_1, V_2 \subset Y$ such that $V_1 \cap D$ and $V_2 \cap D$ are nonempty. Then $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ are disjoint open sets in X and $U_1 \cap C$ and $U_2 \cap C$ are nonempty. This implies that C is not connected. $\square$

Proposition 53. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be the Cartesian product endowed with the product topology. If every $X \in \mathcal{X}$ is connected, then $\times\mathcal{X}$ is connected.*

4.4. **Components.**

Proposition 54. *The connected subsets of a space X are partially ordered by inclusion.*

Definition 31. A *component* of a topological space is a maximal connected subset.

Proposition 55. *The only component of a connected space is the space itself. The components of a discrete space are the singleton sets.*

Proposition 56. *The components of a space partition the space.*

Proof. Let $x \in X$ and let C be the union of all connected subsets of X which contain x. Then C is connected and is clearly maximal. Thus X is the union of its components.

Let D be another component of X. If $C \cap D$ is nonempty, then $C \cup D$ is connected, and by the maximality of C, $D = C$. □

Proposition 57. *If C is a component of a space X, then C is closed.*

Proof. Since C is connected, so is $\overline{C}$. Since $C \subset \overline{C}$ and C is a maximal connected set, $C = \overline{C}$. □

Proposition 58. *Two distinct components of a space are separated.*

Proof. Let C and D be components of a space X and suppose that they are not separated. Then $\overline{C} \cap D$ or $C \cap \overline{D}$ is nonempty. Suppose, without loss of generality, that $\overline{C} \cap D$ is nonempty and let $x \in \overline{C} \cap D$. Since $C = \overline{C}$, $x \in C$. Thus $x \in C \cap D$ so $C = D$. □

Proposition 59. *If C is a component of a space X and $C \subset Y \subset X$, then C is a component of the subspace Y.*

4.5. **Dedekind Property.**

Definition 32. An ordered set X has the *Dedekind* property provided that for each decomposition $X = A \cup B$, where A and B are nonempty and $a < b$ whenever $a \in A$ and $b \in B$, either A contains a maximal element or B contains a minimal element, but not both.

Proposition 60. *A nonempty connected subset of an ordered space is infinite.*

Proposition 61. *An ordered space is connected if and only if it has the Dedekind property.*

Corollary 3. *Intervals of real numbers are connected.*

4.6. **Path Connectedness.**

Definition 33. Let X be a space. Let $I = [0, 1]$ be the unit interval in $\mathbb{R}$. A *path* in X is a continuous function $\gamma : I \to X$. The points $a = \gamma(0)$ and $b = \gamma(1)$ are called the *endpoints* of the path, and γ is referred to as a path between a and b.

Definition 34. A space X is *path connected* if for every two points in X there is a path between them.

Proposition 62. *If X is path connected, then X is connected.*

Proof. Suppose X is not connected. Then there exist disjoint nonempty open sets $U, V \subset X$ such that $U \cup V = X$. Let $a \in U$ and $b \in V$. Suppose γ is a path from a to b Then the image of γ is connected because I is. But $U \cap \gamma(I)$ and $V \cap \gamma(I)$ are disjoint and relatively open in $\gamma(I)$, so $\gamma(I)$ is separated, producing a contradiction. $\square$

Example 9. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1] \cap \mathbb{Q} \text{ and } y \in [0, 1]\}$. Then X is connected but not path connected.

5. Compactness

5.1. **Covers.**

Definition 35. Let X be a space and $A \subset X$. A *cover* of A is a collection of sets $\mathcal{E} \subset \mathcal{P}(X)$ such that $A \subset \cup\mathcal{E}$. If each set $E \in \mathcal{E}$ is open, then $\mathcal{E}$ is called and *open cover.* If only finitely many sets are in $\mathcal{E}$, then $\mathcal{E}$ is called a *finite cover.* Notice that the word finite applies to $\mathcal{E}$ whereas the word open applies to the sets in $\mathcal{E}$.

If $\mathcal{E}$ is a cover of A, a *subcover* of $\mathcal{E}$ is a subset of $\mathcal{E}$ which is itself a cover.

The set A is *compact* if every open cover of A contains a finite subcover.

Proposition 63. *The compact subsets of a discrete space are the finite subsets. In a trivial space or a cofinite space, every subset is compact.*

Proposition 64. *Finite subsets of any space are compact.*

Definition 36. Let X be a space and let $A \subset X$. A *basis cover* of a A is a cover by members of a fixed basis for the topology on X.

Proposition 65. *Let X be a space and $A \subset X$. Then A is compact if and only if every basis cover of A contains a finite subcover.*

Proof. The forward direction is immediate, because a basis cover is an open cover.

For the other direction, let $\mathcal{B}$ be a basis and suppose that every cover of A by members of $\mathcal{B}$ has a finite subcover. Let $\mathcal{U}$ be an open cover of A. Then each member of $U \in \mathcal{U}$ is the union of members of $\mathcal{B}$. Let $\mathcal{E}_U$ be a collection of basis sets such that $\cup\mathcal{E}_U = U$. Let $\mathcal{E}$ the be union of these collections. Then $\mathcal{E}$ is a basis cover and thus has a finite subcover $\mathcal{D}$. Each set in $\mathcal{D}$ is contained in one of the original open sets in $\mathcal{U}$; the collection of these sets is now a finite subcollection of $\mathcal{U}$ which covers A. $\square$

Definition 37. Let X be a space, $\mathcal{E} \subset \mathcal{P}(X)$, and $Y \subset X$. The *relative collection* of $\mathcal{E}$ on Y is
$$\mathcal{E} \cap Y = \{E \cap Y \mid E \in \mathcal{E}\}.$$

Proposition 66. *If X is a space and $A \subset Y \subset X$, then A is compact in X if and only if A is compact with respect to the subspace topology on Y.*

Proof. Let $\mathcal{U} \subset \mathcal{P}(X)$ be an open cover of A. Since $A \subset Y$, $\mathcal{U} \cap Y$ is an open cover of A in Y, and all such covers are of this form. Let $\mathcal{V} \subset \mathcal{U}$ so that $(\mathcal{V} \cap Y) \subset (\mathcal{U} \cap Y)$. Then $\mathcal{V}$ covers A if and only if $\mathcal{V} \cap Y$ covers A. $\square$

Proposition 67. *A closed subset of a compact set is compact.*

Proof. Let X be a space and let K be a compact subset of X. Let $F \subset K$ be closed. Let $\mathcal{U}$ be an open cover of F. Then $\mathcal{U} \cup \{X \setminus F\}$ is an open cover for K and thus has a finite subcover. If $X \setminus F$ is in this subcover of K, remove it; now we have a finite subcover of F. □

Proposition 68. *The continuous image of a compact set is compact.*

Proof. Let $f : X \to Y$ be a continuous function and let $K \subset X$ be compact. Let $\mathcal{V}$ be an open cover of $f(K)$. Let $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$. Then $\mathcal{U}$ is an open cover of K, and has a finite subcover $\mathcal{M}$. Suppose $|\mathcal{U}| = n$, and enumerate $\mathcal{U}$ so that $\mathcal{U} = \{U_i \mid i = 1, \dots, n\}$. For each i, select $V_i \in \mathcal{V}$ such that $U_i = f^{-1}(V_i)$. Let $\mathcal{W} = \{V_i \mid i = 1, \dots, n\}$. Then $\mathcal{W} \subset \mathcal{V}$ is a finite subcover of $f(K)$. □

Proposition 69. *Let $\mathcal{X}$ be a family of spaces and let $\times\mathcal{X}$ be the Cartesian product endowed with the product topology. If every $X \in \mathcal{X}$ is compact, then $\times\mathcal{X}$ is compact.*

5.2. **Finite Intersection Property.**

Definition 38. Let $\mathcal{C}$ be a collection of sets. We say that $\mathcal{C}$ has the *finite intersection property* if for every $\mathcal{D} \subset \mathcal{C}$,

$$|\mathcal{D}| < \infty \Rightarrow \cap\mathcal{D} \neq \varnothing.$$

Proposition 70. *A space X is compact if and only if every collection of closed subsets of X with the finite intersection property has nonempty intersection.*

Proof.

($\Rightarrow$) Let $\mathcal{F}$ be a collection of closed sets in X with the finite intersection property but empty intersection. Then $\mathcal{U} = \{X \setminus F \mid F \in \mathcal{F}\}$ is an open cover for X. Let $\mathcal{V} \subset \mathcal{U}$ be a finite subcollection. Then $\mathcal{C} = \{X \setminus V \mid V \in \mathcal{V}\}$ is a finite subcollection of $\mathcal{F}$ and so has nonempty intersection. Since $\cup\mathcal{V} = X \setminus \cap\mathcal{C}$, $\mathcal{V}$ is not a cover for X so X is not compact.

($\Leftarrow$) Suppose that X is not compact and let $\mathcal{U}$ be an open cover with no finite subcover. Let $\mathcal{F} = \{X \setminus U \mid U \in \mathcal{U}\}$. Let $\mathcal{C} \subset \mathcal{F}$ be a finite subcollection and let $\mathcal{V} = \{X \setminus C \mid C \in \mathcal{C}\}$. Then $\cap\mathcal{C} = X \setminus \cup\mathcal{V}$ and since $\mathcal{V}$ does not cover X, $\cap\mathcal{C}$ is nonempty. Thus $\mathcal{F}$ has the finite intersection property, but since $\mathcal{U}$ covers X, $\cap\mathcal{F} = \varnothing$. □

5.3. **Sequential Compactness.**

Definition 39. A space is called *sequentially compact* if every sequence of points in X has a cluster point.

Proposition 71. *A space is sequentially compact if and only if every sequence of distinct points has a cluster point.*

Proposition 72. *A space is sequentially compact if and only if every infinite subset has a limit point.*

Remark 11. There exist compact spaces which are not sequentially compact, and there exist sequentially compact spaces which are not compact. However, in a metric space, the notions are equivalent.

6. Separation

6.1. **Separation Axioms.**

Definition 40. The following conditions on a topological space X are known as *separation axioms*:
T_0: Given two points in X, at least one of them lies in an open set not containing the other.
T_1: Given two points in X, each of them lies in an open set not containing the other.
T_2: Given two points in X, there exist disjoint open sets, each containing exactly one of the points.
T_3: Given a point and a closed set in X not containing the point, there exist disjoint open sets, one containing the point and one containing the closed set.
T_4: Given two disjoint closed sets in X, there exist disjoint open sets, each containing exactly one of the closed sets.
T_5: Given two subsets $A, B \subset X$ with $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \varnothing$, there exist disjoint open sets, each containing exactly one of the subsets.

We say that X is a T_n space if X satisfies the T_n axiom for $n = 0, 1, 2, 3, 4, 5$.

Proposition 73. *The separation axioms are hierarchical in the following sense:*

$$\mathrm{T}_5 + \mathrm{T}_1 \Rightarrow \mathrm{T}_4 + \mathrm{T}_1 \Rightarrow \mathrm{T}_3 + \mathrm{T}_1 \Rightarrow \mathrm{T}_2 \Rightarrow \mathrm{T}_1 \Rightarrow \mathrm{T}_0.$$

Proposition 74. *The product of a family of* T_n *spaces is a* T_n *space for* $n = 0, 1, 2, 3$.

Proposition 75. *A nonempty trivial space is not a* T_0 *space but is* T_3 *and* T_4.

Proposition 76. *The natural numbers with the right order topology are a* T_0 *space but not a* T_1 *space.*

Proposition 77. *The Sierpinski space is a* T_0 *space which is not a* T_1 *space.*

Proposition 78. *An infinite cofinite space is a* T_1 *space but not a* T_2, T_3, *nor a* T_4 *space.*

Proposition 79. *Discrete spaces are* T_5 *spaces.*

6.2. **Closed Point Spaces.**

Definition 41. A T_1 space is called a *closed point space.*

Proposition 80. *A space X is a* T_0 *space if and only if* $\overline{\{x\}} = \overline{\{y\}} \Rightarrow x = y$ *for all* $x, y \in X$.

Proposition 81. *A space X is a* T_1 *space if and only if all of the singleton subsets of X are closed.*

Proposition 82. *A space X is a* T_1 *space if and only if all of the finite subsets of X are closed.*

Proposition 83. *A space X is a* T_1 *space if and only if singleton sets are equal to the intersection of all neighborhoods containing them.*

6.3. **Hausdorff Spaces.**

Definition 42. A T_2 space is called a *Hausdorff space.*

Proposition 84. *A space X is a Hausdorff space if and only if singleton sets are equal to the intersection of all closed neighborhoods containing them.*

Proposition 85. *A space X is a Hausdorff space if and only if for every pair of disjoint compact sets K_1, K_2 there exist disjoint open sets U_1, U_2 such that $K_1 \subset U_1$ and $K_2 \subset U_2$.*

Proposition 86. *A compact subset of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and K a compact subset of X. Let $y \in X \smallsetminus K$ and for each $x \in K$, let U_x and V_x be disjoint open sets such that $x \in U_x$ and $y \in V_x$. Then sets $\{U_x \mid x \in K\}$ cover K and thus have a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Then $V_y = \cap_{i=1}^n U_{x_i}$ is an open neighborhood of y which is disjoint from K. The complement of K in X is the union of all such sets, and is therefore open. Thus K is closed. $\square$

Proposition 87. *If X is a Hausdorff space then every sequence in X which has a limit point converges.*

Proof. Let X be a Hausdorff and let $x : \mathbb{N} \to X$ be a sequence in X which has a limit point p. Let $q \in X$. If p and q are distinct, there exist disjoint neighborhoods of p and q, and x is eventually in the neighborhood of p, and so is not eventually in the neighborhood of q. Therefore q is not a limit point. $\square$

Proposition 88. *Let X be a compact space and Y a Hausdorff space. Let $f : X \to Y$ be continuous and bijective. Then f is a homeomorphism.*

Proof. It suffices to show that f is an closed map. Since X is compact, every closed subset of X is compact. Its image in Y is compact because f is continuous. Since Y is Hausdorff, this image is closed. $\square$

Proposition 89. *If X is a space, Y is a Hausdorff space, and $f : X \to Y$ is continuous, then the set $D = \{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$.*

Proof. Let $Z = X \times Y$ and let $(x, y) \in Z \smallsetminus D$. Since Y is Hausdorff, there exist disjoint open neighborhoods U of $f(x)$ and V of y. Since $f^{-1}(U)$ contains x and is disjoint from $f^{-1}(V)$, x is not a limit point of $f^{-1}(V)$ and so $x \notin \overline{f^{-1}(V)}$. Let $O = f^{-1}(U) \smallsetminus \overline{f^{-1}(V)}$. Then O is an open set containing x so $(x, y) \in O \times V$. Also $O \times V$ is disjoint from D.

This shows that very point is the complement of D is contained in an open set disjoint from D. Therefore D is closed. $\square$

Corollary 4. *A space X is Hausdorff if and only if the diagonal of $X \times X$ is closed.*

Proof. Suppose X is Hausdorff. Then the identity map on X is continuous, so $D = \{(x, \mathrm{id}(x)) \mid x \in X\} = \{(x, x) \mid x \in X\}$ is closed.

Let X be a space and suppose that $D = \{(x, x) \mid x \in X\}$ is closed. Then for $y \neq x$ there exists of basis neighborhood $U \times V$ of (x, y) such that $U \times V \cap D = \varnothing$, where U and V are open subsets of X with $x \in U$ and $y \in V$.

Since $(x, x) \notin U \times V$, $x \notin V$. Since $(y, y) \notin U \times V$, $y \notin U$. Thus X is Hausdorff. $\square$

6.3.1. *The Tube Lemma.* We prove the following:

Proposition 90. *Let X and Y be topological spaces, with Y compact and Hausdorff. Let $f : X \to Y$. Then f is continuous if and only if the graph of f is closed.*

Definition 43. Let X and Y be topological spaces and let $x_0 \in X$. The *fiber* over x in $X \times Y$ is the set $\{x\} \times Y$. A *tube* over x in $X \times Y$ is a set of the form $A \times Y$, where $x \in A \subset X$.

Lemma 1 (Tube Lemma)**.** *Let X and Y be topological spaces, with Y compact. Let $x \in X$ and let G be an open subset which contains the fiber over x. Then G contains an open tube over x.*

Proof. Sets of the form $U \times V$, where U is open in X and V is open in Y, form a basis for the topology of $X \times Y$. Thus G is the union of such sets.

For each $y \in Y$, select a basis neighborhood $U_y \times V_y \subset G$ of (x, y) in the fiber over x. The V_y form a cover of Y, and thus have a finite subcover $V_1, \ldots, V_n$; let $U_1, \ldots, U_n$ be the corresponding subsets of X so that $U_i \times V_i$ cover the fiber over x. Let $U = \cap_{i=1}^n U_i$; then U is open in X and contains x. Now $\cup_{i=1}^n U \times V_i = U \times Y \subset G$ is a tube over x which is contained in G. □

Lemma 2. *Let X and Y be topological spaces, with Y compact. Then the projection $\pi_Y : X \times Y \to Y$ given by $(x, y) \mapsto y$ is a closed map.*

Proof. Let $F \subset X \times Y$ be a closed set in $X \times Y$, and let $x \in X \setminus f(F)$. Since the complement of F in $X \times Y$ is an open set which contains the fiber over x, it contains a tube over x. The projection of this tube onto X is an open set containing x which is disjoint from $f(F)$. Thus the complement of $f(F)$ in X is open. □

Proposition 91. *Let X and Y be topological spaces, with Y compact. Let $f : X \to Y$ be a function whose graph is closed. Then f is continuous.*

Proof. It suffices to show that the inverse image of a closed set is closed. Let C be a closed set in Y. Since projection is continuous, $\pi_Y^{-1}(C)$ is closed in $X \times Y$. Let F be the intersection of this inverse image with the graph of f. Then F is closed in $X \times Y$. Since Y is compact, the projection π_X is a closed map, so $\pi_X(F)$ is closed in X. But $\pi_X(F) = f^{-1}(C)$. □

Lemma 3. *Let X and Y be topological spaces. Let $f : X \to Y$ be a continuous function. Let $g : X \to X \times Y$ be the function given by $x \mapsto (x, f(x))$. Then g is continuous.*

Proof. Let G be an open set in $X \times Y$. Intersection G with the graph of f if $g^{-1}(G)$ is empty, then it is open, so assume that it is nonempty and let $x \in g^{-1}(G)$. Let $U \times V$ be a basis neighborhood of $(x, f(x))$ which is contained in G. Then $f^{-1}(V) \cap \pi_X(G)$ is open in X, $x \in f^{-1}(V) \cap \pi_X(G)$, and $f^{-1}(V) \cap \pi_X(G)$ is contained in $g^{-1}(G)$. □

Proposition 92. *Let X and Y be topological spaces, with Y Hausdorff. Let $f : X \to Y$ be a continuous function. Then the graph of f is closed.*

Proof. Let F be the graph of f. Let $(x, y) \in X \times Y \setminus F$. Then $y \neq f(x)$; let V_1 and V_2 be open neighborhoods in Y of y and $f(x)$ respectively which are disjoint. Let $g : X \to X \times Y$ be the map given by $x \mapsto (x, f(x))$. Then $U = g^{-1}(X \times V_2)$ is open in X, and $U \times V_1$ is an open neighborhood of (x, y) which does not intersect F. Thus the complement of F is open. □

6.4. **Regular Spaces.**

Definition 44. A T_3 space which is also a T_1 space is called *regular*.

Proposition 93. *A space X is regular if and only if every neighborhood of a point in X contains a closed neighborhood.*

Proposition 94. *A space X is a regular space if and only if for every pair of disjoint sets K, F such that K is compact and F is closed there exist disjoint open sets U, V such that $K \subset U$ and $F \subset V$.*

Proposition 95. *A compact Hausdorff space is regular.*

6.5. **Normal Spaces.**

Definition 45. A T_4 space which is also a T_1 space is called *normal*.

Proposition 96. *A space X is normal if and only if for every closed set F in X and every open set U in X containing F there exists an open set V such that $F \subset V \subset \overline{V} \subset U$.*

Proposition 97. *A compact regular space is normal.*

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