Algebra II
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Lesson 0809 - Repeaters
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1. Characterization of Rational Numbers

We have defined the rational numbers as fractions, where the numerator and denominator are integers. We have defined real numbers as decimal expansions, which correspond to points on a line. Now we ask whether are not it makes sense to view rational numbers as real numbers; do they have a position on the number line?

The answer is yes; using decimal long division, we can find the decimal expansion of any rational number. Indeed, we have seen that rational numbers have decimal expansions that either terminate, or are repeating, with repeating part of length less than the denominator. This of course is under the assumption that the fraction is reduced; that is, in "lowest form", so that any common prime factors in the numerator and denominator have been cancelled.

If a decimal expansion terminates, we may view it as having a repeating zero at the end; for example, $\frac{1}{8} = 0.125 = 0.125\overline{0}$. With this understood, it is true and convenient to say that *all* rational numbers have a repeating decimal expansion. For brevity, let us call any number with a repeating decimal expansion a *repeater*.

The next question is whether or not this is a *complete characterization* of the rational numbers. Is every real number with a repeating decimal expansion a rational number?

The answer again is yes; we will be convince of this if we know how to convert a repeating decimal expansion into a fraction. There is a trick for doing this.

Example 1. Write $0.\overline{3}$ as a fraction.

Solution. Let $x = 0.\overline{3}$. We may write x as $x = 0.3\overline{3}$. Then $10x = 3.\overline{3}$. If we subtract x from $10x$, the repeating parts cancel:

$$\begin{aligned} 10x &= 3.\overline{3} \\ -\ x &= 0.\overline{3} \\ \hline 9x &= 3 \end{aligned}$$

Divide both sides by 9 to see that $x = \frac{3}{9} = \frac{1}{3}$. □

Example 2. Write $1.23\overline{4}$ as a fraction.

Solution. Let $x = 1.23\overline{4}$. Then $10x = 12.34\overline{4}$. Subtract these equations, and the repeating parts cancel:

$$\begin{aligned} 10x &= 12.34\overline{4} \\ -\ x &= 1.23\overline{4} \\ \hline 9x &= 11.11 \end{aligned}$$

Divide both sides by 9 to see that $x = \frac{11.11}{9} = \frac{1111}{900}$. □

More examples on the way ...