PRINCIPLES OF ANALYSIS
SOLUTIONS TO PROBLEM SET B

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Problem 1. Let $a, b \in \mathbb{R}$. Show that $a^2 \leq b^2 \Leftrightarrow |a| \leq |b|$.

Solution. We have

$$\begin{aligned} a^2 \leq b^2 &\Leftrightarrow b^2 - a^2 \geq 0 \\ &\Leftrightarrow |b|^2 - |a|^2 \geq 0 \\ &\Leftrightarrow (|b| - |a|)(|b| + |a|) \geq 0 \\ &\Leftrightarrow (|b| - |a|) \geq 0 \quad \text{(since } |b| + |a| \geq 0 \text{ for all } a, b) \\ &\Leftrightarrow |a| \leq |b|. \end{aligned}$$

□

Problem 2. Let $a, b \in \mathbb{R}$. Show that $||a| - |b|| \leq |a - b|$.

Solution. For every $x \in \mathbb{R}$, we have $x \leq |x|$. Thus, using the previous problem, we have

$$\begin{aligned} ab \leq |ab| &\Rightarrow -|ab| \leq -ab \\ &\Rightarrow |a|^2 - 2|ab| + |b|^2 \leq |a|^2 - 2ab + |b|^2 \\ &\Rightarrow |a|^2 - 2|ab| + |b|^2 \leq a^2 - 2ab + b^2 \\ &\Rightarrow (|a| - |b|)^2 \leq (a - b)^2 \\ &\Rightarrow ||a| - |b|| \leq |a - b|. \end{aligned}$$

□

Problem 3. Let S and T be sets of positive real numbers which are bounded above. Suppose that $S \cap T \neq \varnothing$. Show that $\inf S \leq \sup T$.

Solution. Since $S \cap T$ is nonempty, select $x \in S \cap T$. Then $x \in S$, so $\inf S \leq x$. Also, $x \in T$, so $x \leq \sup T$. By transitivity of order, $\inf S \leq \sup T$. □

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To complete Problem 4, we would like to be able to confidently take sqaure roots. We now use the completeness axiom to prove that for every nonnegative real number a there exists a unique nonnegative real number b such that $b^2 = a$, which we denote by $\sqrt{a}$.

The plan of the proof is as follows. We wish to find a set of rational numbers such that its supremum is the square root of a. The natural set to consider is

$$S = \{x \in \mathbb{Q} \mid x^2 < a\}.$$

We are using $\mathbb{Q}$ here primarily for aesthetic reasons: we wish to construct an irrational number from the rational ones using the Completeness Axiom.

Let $b = \sup S$. We wish to show that $b^2 = a$. Thus we try to show that $b^2 \leq a$ and $a \leq b^2$. Each of these inequalities presents its own difficulties.

To show that $b^2 \leq a$, we note that we can select an $s \in S$ as close to b as we like; thus their squares will be as close to b^2 as we like. If $a < b^2$, then one of these squares will be bigger than a, a contradiction.

To show that $a \leq b^2$, assume that $b^2 < a$ and find a rational whose square is between b^2 and a. To do this, we first show that the set of square integers between 0 and 1 is dense in $[0, 1]$ by breaking up the interval into pieces whose endpoints are square rationals with denominators n^2 for any $n \in \mathbb{N}$. If n is large enough, the distance between any two of these endpoints is less than $\beta - \alpha$.

The next two propositions help with the inequality $b^2 \leq a$.

Proposition 1. *Let $S \subset \mathbb{R}$ be a set of real numbers which is bounded above, and let $b = \sup S$. Then for every $n \in \mathbb{N}$ there exists $s \in S$ such that $b - s < \frac{1}{n}$.*

Proof. Otherwise, $b - \frac{1}{n}$ is an upper bound for S. □

Proposition 2. *Let $x, y \in \mathbb{R}$ such that $0 \leq x$. Suppose that for every $n \in \mathbb{N}$, we have $0 \leq x \leq \frac{y}{n}$. Then $x = 0$.*

Proof. We prove the contrapositive.

Suppose that $x > 0$. We wish to show that there exists $n \in \mathbb{N}$ such that $\frac{y}{n} < x$.

Now either $y \leq 0$ or $y > 0$.

If $y \leq 0$, then $\frac{y}{n} \leq 0 < x$ for any $n \in \mathbb{N}$.

If $y > 0$, then $0 < \frac{y}{x}$, so there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \frac{x}{y}$. Thus $\frac{y}{n} < x$. □

The next three propositions will give us the inequality $a \le b^2$.

Proposition 3. *Let $q \in \mathbb{Q}$ be a positive rational number. Then there exists $n \in \mathbb{N}$ such that $1 - (\frac{n-1}{n})^2 < q$.*

Proof. Since $q \in \mathbb{Q}$, there exist $l, m \in \mathbb{Z}$ such that $q = \frac{l}{m}$, and since $q > 0$, we may choose $l, m > 0$. Thus $\frac{1}{m} \le q$.

Let $n = 2m$. Then

$$\frac{2}{n} - \frac{1}{n^2} = \frac{1}{m} - \frac{1}{4m} < \frac{1}{m} \le q;$$

so $-q < -\frac{2}{n} + \frac{1}{n^2}$. Adding 1 to both sides gives

$$1 - q < 1 - \frac{2}{n} + \frac{1}{n^2} = \frac{n^2 - 2n + 1}{n^2} = (\frac{n-1}{n})^2.$$

Therefore $1 - (\frac{n-1}{n})^2 < q$. □

Proposition 4. *Let $n, i \in \mathbb{N}$ with $0 < i < n$. Then*

$$(\frac{i}{n})^2 - (\frac{i-1}{n})^2 < 1 - (\frac{n-1}{n})^2.$$

Proof. Since $i < n$, we have $2i - 1 < 2n - 1$. Then

$$i^2 - (i-1)^2 = 2i - 1 < 2n - 1 = n^2 - (n-1)^2.$$

The result follows upon dividing by n^2. □

Proposition 5. *Let $\alpha, \beta \in \mathbb{Q}$ with $0 < \alpha < \beta$. Then there exists $\gamma \in \mathbb{Q}$ such that $\alpha < \gamma^2 < \beta$.*

Proof. First assume that $0 < \alpha < \beta < 1$.

Let $q = \beta - \alpha$; note that $q > 0$. By Proposition 3, there exists $n \in \mathbb{N}$ such that $1 - (\frac{n-1}{n})^2 < q$. Let i be the smallest integer such that $\beta < (\frac{i}{n})^2$; since $\beta < 1$, such an integer exists, and and $i \le n$. Then $(\frac{i-1}{n})^2 < \beta$. Now by Proposition 4,

$$\beta - \alpha > 1 - (\frac{n-1}{n})^2 > (\frac{i}{n})^2 - (\frac{i-1}{n})^2 > \beta - (\frac{i-1}{n})^2;$$

subtracting β from both sides and multiplying by -1 gives

$$\alpha < (\frac{i-1}{n})^2.$$

Letting $\gamma = \frac{i-1}{n}$, we have $\alpha < \gamma^2 < \beta$.

Now drop the assumption that $\beta < 1$. Then there exists a natural number n such that $\beta < n^2$. Then $0 < \frac{\alpha}{n^2} < \frac{\beta}{n^2} < 1$, so there exists $\gamma \in \mathbb{Q}$ such that $\frac{\alpha}{n^2} < \gamma^2 < \frac{\beta}{n^2}$. Therefore $\alpha < n^2\gamma^2 < \beta$. □

Proposition 6. *Let $a \in \mathbb{R}$ with $a \geq 0$, and let*

$$S = \{x \in \mathbb{Q} \mid x^2 < a\}.$$

Then S is bounded above, and $(\sup S)^2 = a$.

Proof. Let $s \in S$. If $|s| \leq 1$, then $s < 1 + a$. If $|s| > 1$, then $s < s^2 < a < 1 + a$. In either case, $1 + a$ is greater than s, so $1 + a$ is an upper bound for the set S. Thus $\sup S$ exists; let $b = \sup S$. Since $0 \in S$, we know that $b \geq 0$. We show that $a \leq b^2$ and that $b^2 \leq a$.

Suppose that $b^2 < a$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $b^2 < q < a$. By the Proposition 5, there exists $s \in \mathbb{Q}$ such that $b^2 < s^2 < a$. By definition of S, $s \in S$. But by Problem 1, $b^2 < s^2 \Rightarrow b < s$, so b is not an upper bound for S. This contradiction shows that $a \leq b^2$.

Since $b = \sup S$, Proposition 1 tells us that for every $n \in \mathbb{N}$ there exists $s \in S$ such that $b - s < \frac{1}{n}$. Then $(b - s)(b + s) < \frac{b+s}{n}$. So

$$0 \leq b^2 - a < b^2 - s^2 < \frac{b + s}{n} < \frac{2b}{n}.$$

By Proposition 2, we have $b^2 - a = 0$, so $b^2 = a$. □

Proposition 7. *Let $a \in \mathbb{R}$ be a nonnegative real number. Then there exists a unique nonnegative real number $b \in \mathbb{R}$ such that $b^2 = a$.*

Proof. By Proposition 6, the polynomial equation $f(x) = x^2 - a$ has a root, say c. Then $-c$ is also a root, since $(-c)^2 = c^2 = a$. By a corollary to the division algorithm for polynomials, there are at most two roots. We see that if we let $b = |c|$, it is a unique positive root of $f(x)$. □

This justifies the notation $\sqrt{a}$. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, and define $f : \mathbb{R}_+ \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Then Proposition 7 shows that f is well-defined and injective; Problem 1 shows that f is increasing.

Problem 4. Let S be a bounded set of positive real numbers, and let

$$T = \{t \in \mathbb{R} \mid t = s^2 \text{ for some } s \in S\}.$$

Show that T is bounded above, and that $\sup T = (\sup S)^2$.

Solution. Since S is bounded above, $\sup S$ exists.

Let $t \in T$. Then $t = s^2$ for some $s \in S$. We have $0 \leq s \leq \sup S$; by Problem 1, $t = s^2 \leq (\sup S)^2$. Then T is bounded above, so $\sup T$ exists and $\sup T \leq (\sup S)^2$.

Let $s \in S$, and let $t = s^2$. Then $t \leq \sup T$, so $s = \sqrt{t} \leq \sqrt{\sup T}$. Thus $\sqrt{\sup T}$ is an upper bound for S, so $\sup S \leq \sqrt{\sup T}$. Thus $(\sup S) \leq \sup T$. $\square$

Problem 5. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers, and let $A = \{a_n \mid n \in \mathbb{Z}^+\}$. Show that $\lim_{n\to\infty} a_n \leq \sup A$.

Solution. Let $L = \lim_{n\to\infty} a_n$, and suppose, by way of contradiction, that $L > \sup A$. Let $\epsilon = \frac{L - \sup A}{2}$. By definition of limit, there exists $N \in \mathbb{Z}^+$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. In particular, $L - \epsilon < a_N$. But $L - \epsilon = \sup A + \epsilon$, so $\sup A < a_N$. Since $a_N \in A$, this is a contradiction. $\square$

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