# REAL ANALYSIS
# SOLUTIONS TO ROSS §3

PAUL L. BAILEY

A *field* is a set $F$ together with a pair of binary operations, normally called addition and multiplication, satisfying these nine axioms, for all $a, b, c \in F$:

| | | |
|---|---|---|
| **(A1)** | $a + (b + c) = (a + b) + c$ | (Associativity of Addition) |
| **(A2)** | $a + b = b + a$ | (Commutativity of Addition) |
| **(A3)** | $\exists 0 \in F \, \forall a \in F$ such that $a + 0 = a$ | (Additive Identity) |
| **(A4)** | $\forall a \in F \, \exists -a \in F$ such that $a + (-a) = 0$ | (Additive Inverses) |
| **(M1)** | $a(bc) = (ab)c$ | (Associativity of Multiplication) |
| **(M2)** | $ab = ba$ | (Commutativity of Multiplication) |
| **(M3)** | $\exists 1 \in F \, \forall a \in f$ such that $a \cdot 1 = a$ | (Multiplicative Identity) |
| **(M4)** | $\forall a \in F \, \exists a^{-1} \in F$ such that $aa^{-1} = 1$ | (Multiplicative Inverses) |
| **(DL)** | $a(b + c) = ab + ac$ for all $a, b, c \in F$ | (Distributivity) |

An *ordered field* is a field $F$ together with a relation $\leq$ satisfying these additional axioms, for all $a, b, c \in F$:

| | | |
|---|---|---|
| **(O1)** | $a \leq b$ or $b \leq a$ | (Definiteness) |
| **(O2)** | $a \leq b$ and $b \leq a$ implies $a = b$ | (Antisymmetry) |
| **(O3)** | $a \leq b$ and $b \leq c$ implies $a \leq c$ | (Transitivity) |
| **(O4)** | $a \leq b$ implies $a + c \leq b + c$ | |
| **(O5)** | $a \leq b$ and $0 \leq c$ implies $ac \leq bc$ | |

A *complete ordered field* is an ordered field $F$ which satisfies this additional axiom:

**(CA)** Every subset of $F$ which is bounded above has a supremum.

**Proposition 1.** *Let $F$ be a field. Suppose that $o_1$ and $o_2$ are additive identities. Then $o_1 = o_2$. Thus the element $0$ from Axiom* **(A3)** *is unique.*

*Proof.* Since $o_1$ and $o_2$ are additive identities, we have $x + o_1 = x$ and $x + o_2 = x$ for all $x \in F$.

$$\begin{aligned} o_1 &= o_1 + o_2 && \text{because } o_2 \text{ is an additive identity} \\ &= o_2 + o_1 && \text{by Axiom } \mathbf{(A2)} \\ &= o_2 && \text{because } o_1 \text{ is an additive identity.} \end{aligned}$$

□

**Proposition 2.** *Let $F$ be a field and let $a \in F$. Suppose that $i_1$ and $i_2$ are additive inverses for $a$. Then $i_1 = i_2$. Thus the element $-a$ from Axiom* **A4** *is unique.*

*Proof.* Since $i_1$ and $i_2$ are additive inverses of $a$, $a + i_1 = 0$ and $a + i_2 = 0$. Then $a + i_1 = a + i_1$. Adding $-a$ to both sides on the left gives $i_1 = i_2$. □

*Date*: January 23, 2019.

**Proposition 3.** *Let $F$ be a field. Suppose that $o_1$ and $o_2$ are multiplicative identities. Then $o_1 = o_2$. Thus the element $1$ from Axiom* **(M3)** *is unique.*

*Proof.* Analogous to the proof of Proposition 1. □

**Proposition 4.** *Let $F$ be a field and let $a \in F \setminus \{0\}$. Suppose that $i_1$ and $i_2$ are additive inverses for $a$. Then $i_1 = i_2$. Thus the element $a^{-1}$ from Axiom* **(M4)** *is unique.*

*Proof.* Analogous to the proof of Proposition 2. □

**Proposition 5.** *Let $F$ be a field and let $a \in F$. Then $-a = (-1)a$.*

*Proof.* We have

$$\begin{aligned} a + (-1)a &= a \cdot 1 + a(-1) && \text{by Axiom } \textbf{(M3)} \text{ and Axiom } \textbf{(M2)} \\ &= a(1 + (-1)) && \text{by Axiom } \textbf{(DL)} \\ &= a \cdot 0 && \text{by Axiom } \textbf{(A4)} \\ &= 0 && \text{by Theorem } \textbf{(ii)}. \end{aligned}$$

Thus $(-1)a$ is the unique additive inverse of $a$, namely $-a$. □

**Proposition 6.** *Let $F$ be a field and let $1 \in F$ be the multiplicative identity. Then $(-1)^2 = 1$.*

*Proof.* For any $a \in F$, if $-a$ is the additive inverse of $a$, then $a$ is the additive inverse of $-a$; thus $a = -(-a) = (-1)(-a) = (-1)(-1)a = (-1)^2 a$. In particular, if $a = 1$, we get $1 = (-1)^2$. □

**Exercise 1** (3.3.(a))**.** Let $F$ be a field and let $a, b \in F$.
Show that $(-a)(-b) = ab$.

*Proof.* We have

$$\begin{aligned} (-a)(-b) &= (-1)a(-1)b && \text{by Proposition 5} \\ &= (-1)^2 ab && \text{by Axiom } \textbf{(M2)} \\ &= ab && \text{by Proposition 6.} \end{aligned}$$

□

**Exercise 2** (3.3(b))**.** Let $F$ be a field and let $a, b, c \in F$. Show that $ac = bc$ and $c \neq 0$ imply $a = b$.

*Proof.* Since $c \neq 0$, it has a multiplicative inverse $c^{-1}$, which is also nonzero. Multiply both sides of $ac = bc$ on the right by $c^{-1}$ to obtain $a = b$. □

**Exercise 3** (3.4(a))**.** Let $F$ be an ordered field with $0, 1 \in F$. Show that $0 < 1$.

*Proof.* First suppose that $0 = 1$. By our definition of field, $F$ contains at least one additional element, say $a \in F \setminus \{0\}$. Then

$$\begin{aligned} 0 &= 0 \cdot a && \text{by Theorem 3.1(ii)} \\ &= 1 \cdot a && \text{since } 0 = 1 \\ &= a && \text{by Axioms } \textbf{(M2)} \text{ and } \textbf{(M3)}. \end{aligned}$$

This contradicts our assumption on $a$; therefore, $0 \neq 1$.

By Axiom (M3), $1^2 = 1$; thus by Theorem 3.2(iv), we have $0 \leq 1^2 = 1$. Since $0 \leq 1$ and $0 \neq 1$, we may write $0 < 1$. □

**Exercise 4** (3.4(b))**.** Let $F$ be an ordered field, and let $a, b \in F \setminus \{0\}$. Show that $0 < a < b$ implies $0 < b^{-1} < a^{-1}$.

*Proof.* Suppose $0 \leq a$. Assume $a^{-1} \leq 0$. Then by Theorem 3.2(ii), $a \cdot a^{-1} \leq 0 \cdot a^{-1}$, which says that $1 \leq 0$. This contradicts Theorem 3.2(v); thus $0 \leq a^{-1}$. Similarly, $0 \leq b^{-1}$ (this uses Axiom O3 ... how?). Now

$$\begin{aligned} a \leq b \Rightarrow ab^{-1} \leq bb^{-1} & \qquad \text{by Axiom } \textbf{(O4)} \\ \Rightarrow ab^{-1} \leq 1 & \qquad \text{by Axiom } \textbf{(M4)} \\ \Rightarrow b^{-1}a \leq 1 & \qquad \text{by Axiom } \textbf{(M2)} \\ \Rightarrow b^{-1}aa^{-1} \leq 1 \cdot a^{-1} & \qquad \text{by Axiom } \textbf{(O4)} \\ \Rightarrow b^{-1} \cdot 1 \leq a^{-1} \cdot 1 & \qquad \text{by Axiom } \textbf{(M4)} \text{ and } \textbf{(M2)} \\ \Rightarrow b^{-1} \leq a^{-1} & \qquad \text{by Axiom } \textbf{(M3)} \end{aligned}$$

□

**Exercise 5** (3.5.(a))**.** Show that $|b| \leq a$ if and only if $-a \leq b \leq a$.

*Proof.* Note that $0 \leq |b|$, so if $|b| \leq a$, then $a$ is nonnegative. Also, if $-a \leq b \leq a$, we have $-a \leq a$, so $0 \leq 2a$, and $0 \leq a$. Thus we may assume that $a$ is nonnegative.

($\Rightarrow$) Suppose that $|b| \leq a$. Since $|b| \geq 0$, we know that $a \geq 0$ by transitivity of order, which is Axiom O3.

*Case 1:* $b \geq 0$.
Then $|b| = b$, so $b \leq a$, and by 3.2.(i), $-a \leq -b$. But $-b \leq b$, so $-a \leq -b \leq b \leq a$.

*Case 2:* $b < 0$.
Then $|b| = -b$, so $-b \leq a$; thus $-a \leq b$, whence $-a \leq b \leq -b \leq a$.

($\Leftarrow$) Suppose that $-a \leq b \leq a$. Multiplying the left equality by $-1$ gives $-b \leq a$. Either $|b| = b$ or $|b| = -b$, and in either case, $|b| \leq a$. □

**Exercise 6** (3.5.(b))**.** Let $F$ be an ordered field and let $a, b \in F$. Show that $||a| - |b|| \leq |a - b|$.

First we note that while Theorem 3.5 is stated for the reals $\mathbb{R}$, it is true for any ordered field, since the proof uses only the axioms of an ordered field.

Although this exercise can be proved directly, instead we use a lemma. The lemma may be more important than the exercise, since it gives us a *technique* which allows us to solve other problems. The technique is demonstrated in the proof of the exercise.

**Lemma 1.** *Let $F$ be an ordered field and let $a, b, x \in F$. Then*
**(a)** $x^2 = |x|^2$.
**(b)** $|a| \leq |b| \Leftrightarrow a^2 \leq b^2$.

*Proof of Lemma.*
**(a)** By Thm 3.2.(iv), $0 \leq x^2$, so $x^2 = |x^2|$. By Thm 3.5.(ii), $|x^2| = |x|^2$.
**(b)** Note that $0 \leq |a| + |b|$. Using part **(a)**,

$$\begin{aligned} a^2 \leq b^2 &\Leftrightarrow |a|^2 \leq |b|^2 \\ &\Leftrightarrow 0 \leq |b|^2 - |a|^2 \\ &\Leftrightarrow 0 \leq (|b| - |a|)(|b| + |a|) \\ &\Leftrightarrow 0 \leq |b| - |a| \text{ by O5} \\ &\Leftrightarrow |a| \leq |b|. \end{aligned}$$

□

*Proof of Exercise.* By the Lemma part (b), it suffices to show that

$$(|a| - |b|)^2 \leq (a - b)^2.$$

Now $ab \leq |ab|$, and since $0 \leq 2$, we have

$$2ab \leq 2|ab| = 2|a||b|.$$

Then $-2|a||b| \leq -2ab$. Adding $a^2 + b^2$ to both sides gives

$$a^2 - 2|a||b| + b^2 \leq a^2 - 2ab + b^2.$$

Thus by the Lemma part (a), we have

$$|a|^2 - 2|a||b| + |b|^2 \leq a^2 - 2ab + b^2.$$

Factoring this gives

$$(|a| - |b|)^2 \leq (a - b)^2.$$

□

**Exercise 7** (3.6.(b))**.** Let $F$ be an ordered field, and let $a_1, \dots, a_n \in F$. Show that $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$.

*Proof.* Note that in any ordered field, we have the triangle inequality, since its proof used only the axioms.

Proceed by induction on $n$.

Suppose that $n = 1$. The fact that $|a_1| \leq |a_1|$ follows from order Axiom **(O1)**.

By induction, we have

$$|a_1 + \cdots + a_{n-1}| \leq |a_1| + \cdots + |a_{n-1}|.$$

Adding $|a_n|$ to both sides gives

$$|a_1 + \cdots + a_{n-1}| + |a_n| \leq |a_1| + \cdots + |a_n|.$$

By the standard triangle inequality,

$$|(a_1 + \cdots + a_{n-1}) + a_n| \leq |a_1 + \cdots + a_{n-1}| + |a_n|.$$

By associativity of addition and transitivity of order, the result follows. □

**Exercise 8** (3.8)**.** Let $a, b \in \mathbb{R}$. Show that if $a \leq c$ for every $c \in \mathbb{R}$ such that $c > b$, then $a \leq b$.

*Proof.* It suffices to prove the contrapositive. That is, if we have two propositions $p$ and $q$, we may formulate another proposition $p \Rightarrow q$. This latter proposition is true if and only if the proposition $\neg q \Rightarrow \neg p$ is true, where $\neg p$ means "not $p$".

Putting this in our current setting, let $p$ be the proposition "$a \leq c$ for every $c \in \mathbb{R}$ such that $c > b$", and let $q$ be the proposition "$a \leq b$". Then $\neg q \Rightarrow \neg p$ may be written: "If $a > b$, then $a > c$ for some $c \in \mathbb{R}$ such that $c > b$". We prove this.

Suppose that $a > b$. Then $\frac{a}{2} > \frac{b}{2}$ by O5.

Let $c = \frac{a+b}{2}$. Then $c \in \mathbb{R}$, and

$$a = \frac{a}{2} + \frac{a}{2} > \frac{a}{2} + \frac{b}{2} = c > \frac{b}{2} + \frac{b}{2} = b.$$

□

Note that the above proof does not rely on the Completeness Axiom of §4; thus it should be true in any ordered field. All we need to do is to show that $\frac{1}{2}$ exists in any ordered field.

Let $F$ be an ordered field. By Axiom M3, there exists an element $1 \in F$. By definition, we assume that $F$ has at least two elements. Under this condition, we can prove that $0 \neq 1$ in $F$; for suppose that $0 = 1$ in $F$ and $F$ has at least two distinct elements, say 0 and $a$. Then $a = a \cdot 1 = a \cdot 0 = 0$, a contradiction. So we always assume that a field has at least two elements.

Now define $2 = 1 + 1$ in $F$. Since $F$ is ordered, $0 < 1$ implies that $1 < 2$, so $0 < 2$, and 2 is not equal to zero. Thus $\frac{1}{2}$ exists in $F$.

We conclude that, in any ordered field, there exists an element strictly between any two distinct elements. In particular, an ordered field cannot be a finite field.

Every set admits an order relation satisfying axioms O1 through O3, but there are fields which do not admit an ordering which is compatible with their operations.

An example of this is the complex numbers. Let

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}.$$

This is indeed a field; where addition is given by $(a+ib)+(c+id) = (a+c)+i(b+d)$, and multiplication is given by $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$. The multiplicative inverse of $a + ib$ is $\frac{a-ib}{a^2+b^2}$.

It is a relatively easy algebraic fact than any polynomial of degree $n$ over a field $F$ has at most $n$ roots in $F$. It is a deep algebraic theorem that every polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$.

In particular, let's look at the polynomial $f(x) = x^3 - 1$. Let $\alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then $\alpha^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, and $\alpha^3 = 1$ (just multiply it out to check this). Thus $f$ has exactly three roots in $\mathbb{C}$. This cannot happen in an ordered field, as the next problem demonstrates.

**Fact 1.** Let $F$ be an ordered field, $a, b \in F$, and $n \in \mathbb{N}$. Then:
**(a)** $(ab)^n = a^n b^n$;
**(b)** $|a|^n = |a^n|$;
**(c)** $|a| < 1 \Rightarrow |a|^n < 1$;
**(d)** $|a| > 1 \Rightarrow |a|^n > 1$.

One may use induction to prove any of these facts.

**Problem 1.** Let $F$ be an ordered field, $\alpha, \beta, \zeta \in F$, and $n \in \mathbb{N}$.
**(a)** Show that if $|\alpha| = |\beta|$, then $\beta = \pm\alpha$.
**(b)** Show that if $\alpha^n = 1$, then $\alpha = \pm 1$.
**(c)** Show that if $\alpha^n = \beta^n$, then $|\alpha| = |\beta|$.
**(d)** Show that the polynomial $f(x) = x^n - \zeta$ has at most two roots in $F$.

*Proof.*

**(a)** Consider all four cases of $\alpha$ positive or negative and $\beta$ positive or negative, and the definition of $|\cdot|$.

**(b)** Either $|\alpha| < 1$, $|\alpha| > 1$, or $|\alpha| = 1$. If $|\alpha| < 1$, then $|\alpha|^n < 1$ by Fact (c); if $|\alpha| > 1$, then $|\alpha|^n > 1$ by Fact (d). Thus if $|\alpha|^n = 1$, we must have $|\alpha| = 1$. By part (a), we have $\alpha = \pm 1$.

**(c)** Suppose that $\alpha^n = \beta^n$. Then $\frac{\alpha^n}{\beta^n} = (\frac{\alpha}{\beta})^n = 1$ by Fact (a). So $\frac{\alpha}{\beta} = \pm 1$ by part (b). Thus $\alpha = \pm\beta$.

**(d)** Suppose that $\alpha \in F$ is a root of $f(x) = x^n - \zeta$. If $\beta$ is another root, then $\alpha^n = \beta^n = \zeta$, so $\alpha = \pm\beta$. So the set of all possible roots is $\{\alpha, -\alpha\}$; that is, there are at most two roots. $\square$

Now the polynomial $f(x) = x^3 - 1$ has three roots in $\mathbb{C}$; therefore, $\mathbb{C}$ is not an ordered field. In other words, there is no possible way to put an order on $\mathbb{C}$ which is compatible with the algebraic structure.

Department of Mathematics & CSci, BASIS Scottsdale
*Email address*: paul.bailey@basised.com