Real Analysis
Examination 21 Practice Problems - Solutions

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Tuesday, November 20, 2018

Problem 1. Show that $I = [0,1]$ is a closed subset of $\mathbb{R}$.

Solution. For the sake of completeness, we discuss this in detail.

Unless stated otherwise, the assumed topology on $\mathbb{R}$ is the metric topology, given by the metric $d(x,y) = |x-y|$. In any topological space, a set is open if each point is interior; that is, a set U is open if for every $u \in U$ there exists $r > 0$ such that $B_r(u) \subset U$.

Open intervals in $\mathbb{R}$ are open; for example, let $U = (a,b)$ and let $u \in U$. Let $r = \min\{u-a, b-u\}$. Then $B_r(U) \subset U$. Similarly, $(-\infty, b)$ and (a, ∞) are open.

In any metric space, the union of open sets is open. To see this, let U and V be open sets in a metric space. Let $w \in U \cup V$. Then $w \in U$ or $w \in V$; without loss of generality, assume $w \in U$. Since U is open, there exists $r > 0$ such that $B_r(w) \subset U$, so $B_r(w) \subset U \cup V$.

To show that I is closed, we show that its complement is open. But $I^c = \mathbb{R} \smallsetminus I = (-\infty, 0) \cup (1, \infty)$ is the union of open intervals. Since open intervals are open, and the union of open sets is open, then I^c is open. □

Problem 2. Give an example of an infinite collection of open subsets of $\mathbb{R}$ whose intersection is not open.

Solution. Let $\mathcal{C} = \{(-\frac{1}{n}, \frac{1}{n}) \mid n \in \mathbb{N}\}$. Then $\mathcal{C}$ is a collection of open subsets of $\mathbb{R}$, but

$$\cap \mathcal{C} = \{0\},$$

which is not open. □

Problem 3. Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, 0)$. Show that f is an embedding.

Solution. An embedding is an injective, continuous, relatively open map.

Injective: Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. Then $(x_1, 0) = (x_2, 0)$. By the defining property of open pairs, $x_1 = x_2$. Thus f is injective.

Continuous: Let V be an open subset of $\mathbb{R}^2$, and let $U = f^{-1}(V)$; we wish to show that U is open in $\mathbb{R}$.

Let $u \in U$; it suffices to show that u is an interior point of U.

Let $v = f(u) = (u, 0) \in \mathbb{R}^2$. Now $v \in V$, and since V is open, there exists $r > 0$ such that $B_r(v) \subset V$. Let $W = \{(x, 0) \in \mathbb{R}^2 \mid v - r < x < v + r\} \subset V$, so $f^{-1}(W) = (u - r, u + r) \subset U$. Thus u is an interior point of U, and U is open.

Relatively Open: Let X denote the real axis in $\mathbb{R}^2$; clearly $X = f(\mathbb{R})$.

Let $U \subset \mathbb{R}$ be open in $\mathbb{R}$, and let $W = f(U)$; we wish to show that W is open relative to X.

Let $w \in W$; it suffices to find an open set $V \subset \mathbb{R}^2$ such that $V \cap X \subset W$.

Since $w \in f(U)$, there exists $u \in U$ such that $f(u) = w$. Since U is open, there exists $r > 0$ such that $(u - r, u + r) \subset U$. Then $f((u - r, u + r)) = (w - r, w + r) \subset W$. Let $V = B_r(w) \subset \mathbb{R}^2$; then $V \cap X = (w - r, w + r) \subset W$, so W is relatively open. □

Problem 4. Let X be a topological space and let $A \subset X$ be closed. Let $B \subset A$ be closed in the subspace topology on A. Show that B is closed in X.

Solution. Since B is relatively closed in A, $A \smallsetminus B$ is relatively open in A. So, there exists an open set $U \subset X$ such that $U \cap A = A \smallsetminus B$. Let $F = X \smallsetminus U$. Then F is closed in X, and $B = A \cap F$, which is the intersection of closed sets in X, and hence is itself closed in X. □

Problem 5. Let X be a topological space and let $A \subset X$. Show that $\overline{A}$ is the intersection of all closed sets in X which contain A.

Solution. Let $\mathcal{F}$ denote the set of all closed sets which contain A. We wish to show that $\overline{A} = \cap\mathcal{F}$. To show that two sets are equal, we show that each is contained in the other.

($\overline{A} \subset \cap\mathcal{F}$): It suffices to show that if $x \notin \cap\mathcal{F}$, then $x \notin \overline{A}$.

Suppose $x \notin \cap\mathcal{F}$. Then there exists a closed set F containing A such that $x \notin F$. Then $x \in F^c$, which is an open set disjoint from A. Then $x \notin \overline{A}$.

($\cap\mathcal{F} \subset \overline{A}$): It suffices to show that if $x \notin \overline{A}$, then $x \notin \cap\mathcal{F}$.

Suppose that $x \notin \overline{A}$. Then x is not a point of closure of A. Thus there exists an open neighborhood U of x such that $U \cap A = \varnothing$. Let $F = U^c$; then $x \notin F$, but F is a closed set containing A, so $F \in \mathcal{F}$. Thus $x \notin \cap\mathcal{F}$. □

Problem 6. Let X be a topological space and let $f : X \to \mathbb{R}$ be a continuous function. Show that the set

$$Z = \{x \in X \mid f(x) = 0\}$$

is closed in X.

Solution. One sees that $Z^c = \{x \in X \mid f(x) \neq 0\}$; that is, Z^c is the preimage of the set $\mathbb{R} \setminus \{0\}$. Since $\{0\}$ is closed in $\mathbb{R}$, $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$. Since f is continuous, $Z^c = f^{-1}(\mathbb{R} \setminus \{0\})$ is open in X. Thus Z is closed in X. □

Problem 7. Let X and Y be topological spaces, and let $f : X \to Y$ be continuous and injective. Show that if Y is Hausdorff, then so is X.

Proof. Let $x_1, x_2 \in X$ be distinct. We wish to find disjoint open neighborhoods of x_1 and x_2.

Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is injective, $y_1 \neq y_2$. Thus there exist disjoint open sets $V_1, V_2 \subset Y$ which are neighborhoods of y_1 and y_2, respectively. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$; then $x_1 \in U_1$ and $x_2 \in U_2$. Since f is injective, $U_1 \cap U_2 = \varnothing$. Moreover, since f is continuous, U_1 and U_2 are open; thus these sets are disjoint open neighborhoods of x_1 and x_2, respectively. □

Problem 8. Find a continuous increasing function $f : (0, \infty) \to (0, 1)$. Use f to show that $\mathbb{R}^n \cong B^n$.

Solution. Let $f(x) = \dfrac{x}{x+1} = 1 - \dfrac{1}{x}$. Clearly f has the stated properties.

Every point in $\mathbb{R}^n$ is of the form $r\vec{u}$, where $\vec{u}$ is a unit vector. Let $g : \mathbb{R}^n \to B^n$ be given by $r\vec{u} \mapsto f(r)\vec{u}$. Then g is a homeomorphism. □

Problem 9. Let X and Y be Frechet spaces. Show that $X \times Y$ is Frechet.

Solution. We know that the topology on $X \times Y$ is generated by sets of the form $U \times Y$ and $X \times V$, where U is open in X and V is open in Y. In particular, if $U \subset X$ and $V \subset Y$ are open, then $U \times V$ is open in $X \times Y$.

Let (x_1, y_1) and (x_2, y_2) be two points in $X \times Y$. Since X and Y are Frechet, there exist open sets $U_1 \subset X$ and $V_1 \subset Y$ such that $x_1 \in U_1$, $x_2 \notin U_1$, $y_1 \in V_1$, and $y_2 \notin V_1$. Then $(x_1, y_1) \in U_1 \times V_1$ and $(x_2, y_2) \notin U_1 \times V_1$. Similarly, we can find open sets $U_2 \subset X$ and $V_2 \subset Y$ such that $(x_2, y_2) \in U_2 \times V_2$ and $(x_1, y_1) \notin U_2 \times V_2$. Thus, $X \times Y$ is Frechet. □

Problem 10. Let X and Y be Hausdorff spaces. Show that $X \times Y$ is Hausdorff.

Solution. We know that the topology on $X \times Y$ is generated by sets of the form $U \times Y$ and $X \times V$, where U is open in X and V is open in Y. In particular, if $U \subset X$ and $V \subset Y$ are open, then $U \times V$ is open in $X \times Y$.

Let (x_1, y_1) and (x_2, y_2) be two points in $X \times Y$. Since X is Hausdorff, there exist disjoint open sets $U_1, U_2 \subset X$ such that $x_1 \in U_1$ and $x_2 \in U_2$. Since Y is Hausdorff, there exist disjoint open sets $V_1, V_2 \subset Y$ such that $y_1 \in V_1$ and $y_2 \in V_2$. Thus $U_1 \times V_1$ and $U_2 \times V_2$ are disjoint open sets in $X \times Y$ such that $(x_1, y_1) \in U_1 \times V_1$ and $(x_2, y_2) \in U_2 \times V_2$. Thus $X \times Y$ is Hausdorff. □

Problem 11. Let X be a topological space. Show that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \in X \times X\}$$

is closed in $X \times X$.

Solution. We show both directions of the implication.

($\Rightarrow$) Suppose that X is Hausdorff. To show that Δ is closed, we show that its complement is open. Let $(x_1, x_2) \in \Delta^c$. Then $x_1 \neq x_2$, so there exist disjoint open sets $U_1, U_2 \subset X$ such that $x_1 \in U_1$ and $x_2 \in U_2$. We claim that $(U_1 \times U_2) \cap \Delta = \varnothing$. To see this, suppose that $(x, y) \in (U_1 \times U_2) \cap \Delta$. By definition of diagonal, $y = x$, so $(x, x) \in (U_1 \times U_2) \cap \Delta$, which implies that $(x, x) \in U_1 \times U_2$. Then $x \in U_1$ and $x \in U_2$, which contradicts that these are disjoint sets. Thus we see that $U_1 \times U_2$ is an open neighborhood of (x_1, x_2) contained in Δ^c. This shows that every point in Δ^c is interior, so Δ^c is open, and Δ is closed.

($\Leftarrow$) Suppose that Δ is closed in $X \times X$. Let $x_1, x_2 \in X$ be distinct. Then $(x_1, x_2) \in \Delta^c$, which is open. Let U be an open set in Δ^c containing (x_1, x_2). Let U_1 be the projection of U onto the first copy of X in $X \times X$, and let U_2 be the projection onto the second copy. Since projection is an open map, U_1 and U_2 are open in X. Now $U_1 \cap U_2 = \varnothing$, since U avoids the diagonal. Also $x_1 \in U_1$ and $x_2 \in U_2$, so we have disjoint open neighborhoods of our two points, which shows that X is Hausdorff. $\square$