Real Analysis — Dr. Paul L. Bailey

Homework 31 - Solutions — Tuesday, February 12, 2019

Name:

Problem 1. Let $f, g : D \to \mathbb{R}$ be continuous at an accumulation point $x_0 \in D$.

(a) Use limit laws to justify that $f + g$ and fg are continuous at x_0.

(b) Explain in detail how this leads to the conclusion that polynomials at continuous.

Solution. We have seen that f is continuous at x_0 if and only if $\lim x \to x_0 f(x) = f(x_0)$.

(a) Since f and g are continuous at x_0, $\lim_{x\to x_0} f(x) = f(x_0)$ and $\lim_{x\to x_0} g(x) = g(x_0)$. Since the limit of the sum is the sum of the limits when they exist, $\lim_{x\to x_0}(f + g)(x) = (f + g)(x_0)$ Thus $f + g$ is continuous at x_0.

The proof for product is completely analogous.

(b) We wish to show that every polynomial function is continuous.

This is tedious but obviously important. We build it gradually.

Claim 1: The constant function $f(x) = C$, where $C \in \mathbb{R}$, is continuous.

Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = 1$. Then if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = 0 < \epsilon$. Thus f is continuous in this case.

Claim 2: The identity function $f(x) = x$ is continuous.

Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = \epsilon$. Then if $|x-x_0| < \delta$, we have $|f(x)-f(x_0)| = |x-x_0| < \delta = \epsilon$, so f is continuous in this case.

Claim 3: The function $f(x) = x^n$ is continuous.

By induction on n. For $n = 1$, the function $g(x) = x$ is the identity function, and so it is continuous. By induction, $h(x) = x^{n-1}$ is continuous. Since the product of continuous functions is continuous, $f = gh$ is continuous in this case.

Claim 4: The monomial function $f(x) = a_n x^n$ is continuous, where $a_n \in \mathbb{R}$ is constant.

By Claim 1, $g(x) = a_n$ is continuous, and by Claim 3, $h(x) = x^n$ is continuous, so there product $f = gh$ is continuous.

Claim 5: The polynomial function $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous.

By induction on n, the degree of the polynomial.

For $n = 0$, $f(x)$ is constant and therefore continuous.

Assume that $g(x) = a_0 + \cdots + a_{n-1}x^{n-1}$ is continuous. By Claim 4, $h(x) = a_n x^n$ is continuous. Then $f = g + h$ is continuous, because the sum of continuous functions is continuous. □

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is discontinuous at every real number.

Solution. Let $x_0 \in \mathbb{R}$. Let $\epsilon = \dfrac{1}{2}$ and let $\delta > 0$.

Suppose x_0 is rational. Then $f(x_0) = 1$. But there exists $x \in (x_0 - \delta, x_0 + \delta)$ which is irrational, so $f(x) = 0$, and $|f(x) - f(x_0)| = |1 - 0| = 1 > \dfrac{1}{2} = \epsilon$. So f is not continuous at x_0.

Similarly, if x_0 is irrational, there exists $x \in (x_0 - \delta, x_0 + \delta)$ which is rational, and $|f(x) - f(x_0)| = |0 - 1| = 1 > \dfrac{1}{2} = \epsilon$. So f is not continuous at x_0. □

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is continuous at $x = 0$ and discontinuous at all nonzero real numbers.

Solution. Let $x_0 \in \mathbb{R} \smallsetminus \{0\}$; we show that f is discontinuous at x_0. Let $\epsilon = \frac{|x_0|}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If x_0 is rational, let x_1 be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = |x_0| > \epsilon$. If x_0 is irrational, let x_2 be a rational in this interval such that $|x_2| > |x_0|$ and we still have $|f(x_2) - f(x_0)| = |x_2| > |x_0| > \epsilon$. Thus f is not continuous at x_0.

Now we consider the behavior of f at zero. Let $\epsilon > 0$ and let $\delta = \epsilon$. Then if $|x-0| < \delta$, we have $|f(x) - f(0)| = 0$ if x is irrational and $|f(x) - f(0)| = |x|$ if x is rational; in either case, $|f(x) - f(0)| \le |x| < \delta = \epsilon$, so f is continuous at zero. □

Problem 4. If $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $r = \frac{p}{q}$. Define $q : \mathbb{Q} \to \mathbb{R}$ by

$$q(r) = \min\{q \in \mathbb{N} \mid r = \frac{p}{q} \text{ for some } p \in \mathbb{Z}\}.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q(x)} & \text{if } x \text{ is rational} \end{cases}$$

Show that f is discontinuous at every rational and continuous at every irrational.

Solution. Suppose that x_0 is rational. We wish to show that f is not continuous at x_0. It suffices to find $\epsilon > 0$ such that for every $\delta > 0$ there exists $x_1 \in (x_0 - \delta, x_0 + \delta)$ with $|x_0 - x_1| > \epsilon$.

Since x_0 is rational, we have $x_0 = \frac{p}{q(x_0)}$ for some $p \in \mathbb{Z}$. Let $\epsilon = \frac{1}{2q(x_0)}$ and let $\delta > 0$. Then $(x_0 - \delta, x_0 + \delta)$ contains an irrational number, say x_1; then $|x_0 - x_1| < \delta$ but $|f(x_0) - f(x_1)| = \frac{1}{q(r)} > \epsilon$. Thus f cannot be continuous at x_0.

Suppose that x_0 is irrational. Let $\epsilon > 0$. It suffices to find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$. Let a be the greatest integer which is less than x_0 and b be the least integer which is greater than x_0; then $b = a + 1$ and $x_0 \in [a, b]$.

For $q \in \mathbb{Q}$, there exist only finitely many points in the set $[a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}\}$ (in fact, this set contains no more than q points). Thus the set

$$D = [a, b] \cap \left\{\frac{k}{q} \mid k \in \mathbb{Z}, q \leq N\right\}$$

is finite (there are no more than $\frac{N(N+1)}{2}$ points in this set). Let

$$\delta = \min\{|x_0 - d| \mid d \in D\};$$

since this set is a finite set of positive real numbers, the minimum exists as a positive real number. Then $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Let $x \in (x_0 - \delta, x_0 + \delta)$. If x is irrational, we have $|f(x) - f(x_0)| = 0 < \epsilon$, and if x is rational, we have $|f(x) - f(x_0)| = \frac{1}{q(x)} < \frac{1}{N} < \epsilon$. Thus f is continuous at x_0. □

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

(1) $f(0) = 1$;

(2) f is differentiable at 0 and $f'(0) = 1$;

(3) $f(x + y) = f(x)f(y)$.

Show that f is differentiable on $\mathbb{R}$ and that $f'(x) = f(x)$ for every $x \in \mathbb{R}$.

Solution. As we apply the limit as h goes to zero, we see the $f(x)$ is a constant; thus

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{by definition of derivative at } x \\
&= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} && \text{by Property } \textbf{(3)} \\
&= f(x) \lim_{n\to\infty} \frac{f(h) - 1}{h} && \text{factoring out the constant } f(x) \\
&= f(x) \lim_{n\to\infty} \frac{f(h) - f(0)}{h} && \text{by Property } \textbf{(1)} \\
&= f(x)f'(0) && \text{by definition of derivative at 0} \\
&= f(x) && \text{by Property } \textbf{(2)}
\end{aligned}$$

□