PRINCIPLES OF ANALYSIS
METRIC SPACES II - COMPLETENESS

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Abstract. This document discusses sequences, subsequences, bounded sequences, and Cauchy sequences. In the process, the Bolzano-Weierstrass property and the completeness property of metric spaces are discussed. We show that these properties of a metric space carry over to products.

1. Sequences

Definition 1. Let X be a set. A *sequence* in X is a function $a : \mathbb{N} \to X$. We write a_n instead of $a(n)$, and we write $(a_n)_{n\in\mathbb{N}}$ or simply (a_n) to denote the entire sequence.

One can think of a sequence as an ordered tuple with infinity many entries; hence the notation.

Definition 2. Let (X, ρ) be a metric space and let (a_n) be a sequence in X. Let $p \in X$. We say that (a_n) *converges to* p, and write $\lim_{n\to\infty} a_n = p$, if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \mid n \geq N \Rightarrow \rho(a_n, p) < \epsilon.$$

If (a_n) converges to p, we call p a *limit point* of (a_n).

Definition 3. Let (X, ρ) be a metric space and let (a_n) be a sequence in X. Let $q \in X$. We say that (a_n) *clusters at* q if

$$\forall \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N \mid \rho(a_n, q) < \epsilon.$$

If (a_n) clusters at q, we call q a *cluster point* of (a_n).

Example 1. Let $X = \mathbb{R}$ and $\rho(x, y) = |x - y|$. Then our new definitions for convergence and clustering become identical to our previous definitions for this particular case.

Problem 1. Let $\mathbb{S}^1$ be the unit circle together with the subspace metric inherited from $\mathbb{R}^2$. Let (a_n) be the sequence in $\mathbb{S}^1$ defined by

$$a_n = \Big(\cos\frac{2\pi n}{6}, \sin\frac{2\pi n}{6}\Big).$$

Find the cluster points of (a_n).

Solution. The sequence (a_n) takes exactly the six values

$$\{(\pm 1, 0), (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})\}.$$

Each of these values occurs infinitely often, so this is the set of cluster points. $\square$

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Problem 2. Let X be a set and define a metric ρ on X by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise .} \end{cases}$$

Let (a_n) be a sequence in X.

(a) Show that $p \in X$ is a limit point of (a_n) if and only if

$$\exists N \in \mathbb{N} \mid n \geq N \Rightarrow a_n = p.$$

(b) Show that $q \in X$ is a cluster point of (a_n) if and only if

$$\forall N \in \mathbb{N} \exists n \geq N \mid a_n = q.$$

Solution. In a discrete metric space, the singleton set $\{x\}$ is a neighborhood of x. Now p is a limit point if and only if $\exists N \in \mathbb{N} \mid n \geq N$ implies that a_n is in $\{p\}$; this happens exactly when $a_n = p$ for $n \geq N$. Thus **(a)**. Clearly **(b)** is similar. □

Definition 4. Let (X, ρ) be a metric space and let (a_n) be a sequence from X. For each $N \in \mathbb{N}$, the N^{th} *tail* of (a_n) is defined to be the set

$$\{a_n \mid n \geq N\} = \{x \in X \mid x = a_n \text{ for some } n \geq N\}.$$

Proposition 1. *Let (X, ρ) be a metric space, (a_n) a sequence from X, and $p \in X$. Then the following conditions are equivalent:*
(L1) *For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \rho(a_n, p) < \epsilon$.*
(L2) *For every neighborhood U of p there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$.*
(L3) *Every neighborhood of p contains a tail of (a_n).*
(L4) *Every neighborhood of p contains a_n for all but finitely many $n \in \mathbb{N}$.*

Proof.
(**L1** $\Rightarrow$ **L2**) Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \rho(a_n, p) < \epsilon$. Let U be a neighborhood of p. Then there exists $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. Let N be so large that $\rho(a_n, p) < \epsilon$ whenever $n \geq N$. Then for $n \geq N$, we have $a_n \in B(p, \epsilon) \subset U$.

(**L2** $\Rightarrow$ **L3**) Suppose that for every neighborhood U of p there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$. Let U be a neighborhood of p and let N be so large that $n \geq N \Rightarrow a_n \in U$. Then $\{a_n \mid n \geq N\} \subset U$, so U contains the N^{th} tail of (a_n).

(**L3** $\Rightarrow$ **L4**) Suppose that every neighborhood U of p contains a tail of (a_n). Let U be a neighborhood of p and let $N \in \mathbb{N}$ such that $\{a_n \mid n \geq N\} \subset U$. If $a_n \notin U$ for some $n \in \mathbb{N}$, then $a_n \notin \{a_n \mid n \geq N\}$, so $n < N$. There are only finitely many such n.

(**L4** $\Rightarrow$ **L1**) Suppose that every neighborhood of p contains a_n for all but finitely many n. Let $\epsilon > 0$. Then $B(p, \epsilon)$ is a neighborhood of p, so $a_n \in B(p, \epsilon)$ for all but finitely many $n \in \mathbb{N}$. Let $N = 1 + \max\{n \in \mathbb{N} \mid a_n \notin B(p, \epsilon)\}$. Then for $n > N$, we have $\rho(a_n, p) < \epsilon$. □

Proposition 2. *Let (X,ρ) be a metric space, (a_n) a sequence from X, and $q \in X$. Then the following conditions are equivalent:*
(C1) *For every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $\rho(a_n, q) < \epsilon$.*
(C2) *For every neighborhood U of q and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $a_n \in U$.*
(C3) *Every neighborhood of q intersects every tail of (a_n).*
(C4) *Every neighborhood of q contains a_n for infinitely many $n \in \mathbb{N}$.*

Proof.
(**C1** $\Rightarrow$ **C2**) Suppose that for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $\rho(a_n, q) < \epsilon$. Let U be a neighborhood of q and let $N \in \mathbb{N}$. Then there exists $\epsilon > 0$ such that $B(p, \epsilon) \subset U$; thus there exists $n \geq N$ such that $\rho(a_n, q) < \epsilon$. But this says that $a_n \in B(q, \epsilon)$, so $a_n \in U$.

(**C2** $\Rightarrow$ **C3**) Suppose that for every neighborhood U of q and every $N \in \mathbb{N}$ there exists $n > N$ such that $a_n \in U$. Let U be a neighborhood of q and let $\{a_n \mid n \geq N\}$ be an arbitrary tail of (a_n). Then for some $n \geq N$, we have $a_n \in U$. But $a_n \in \{a_n \mid n \geq N\}$, so $a_n \in \{a_n \mid n \geq N\} \cap U$, and $\{a_n \mid n \geq N\}$ intersects U.

(**C3** $\Rightarrow$ **C4**) Suppose that every neighborhood of q intersects every tail of (a_n). Let U be a neighborhood of q. Suppose bwoc that U contains a_n for only finitely many $n \in \mathbb{N}$. Let m be the largest natural number such that $a_m \in U$. Then $[a_n : m+1]$ is a tail of (a_n) which does not intersect U; this is a contradiction.

(**C4** $\Rightarrow$ **C1**) Suppose that every neighborhood of q contains a_n for infinitely many $n \in \mathbb{N}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$. Then $U = B(q, \epsilon)$ is a neighborhood of q, and U contains a_n for infinitely many $n \in \mathbb{N}$. One such n must be larger than N; if $n \in \mathbb{N}$ such that $a_n \in U$, then $\rho(a_n, q) < \epsilon$. $\square$

Proposition 3. *Let (X,ρ) be a metric space, (a_n) a sequence from X, and $p \in X$. If (a_n) converges to p, then (a_n) clusters at p, and p is the only cluster point.*

Proof. Suppose that (a_n) converges to p. Then every neighborhood of p contains a_n for all but finitely many n. Thus there are infinitely many n such that a_n is in the neighborhood. By Proposition 2 **(d)**, (a_n) clusters at p.

To see that p is the only cluster point, let $q \in X$, $q \neq p$; we show that (a_n) does not cluster at q. Let $\epsilon = \frac{\rho(p,q)}{2}$ and let $U = B(p, \epsilon)$ and $V = B(q, \epsilon)$. Then U and V are disjoint neighborhoods of p and q respectively.

Let A be a tail of (a_n) such that $A \subset U$. Since $U \cap V = \varnothing$, we have $A \cap V = \varnothing$, so V is a neighborhood of q which does not intersect A. Thus (a_n) does not cluster at q, by 2 **(c)**. $\square$

Problem 3. Find an example of a sequence (a_n) of real numbers and a real number $q \in \mathbb{R}$ such that (a_n) clusters at q but does not converge to q.

Solution. Let $a_n = (-1)^n$. Then (a_n) clusters at 1. to see this, let U be a neighborhood of 1, and note that for all even n, of which there are infinitely many, we have $a_n = 1 \in U$. By **C4**, (a_n) clusters at 1.

However, (a_n) does not converge to 1, because for all odd n, of which there are infinitely many, we have $a_n = -1 \notin U$. Since **L4** is not satisfied, (a_n) does not converge to 1. $\square$

2. Subsequences

Definition 5. Let (X, ρ) be a metric space and let (a_n) be a sequence in X, where $a : \mathbb{N} \to X$ is the function defining (a_n). A *subsequence* of (a_n) is the composition $a \circ n$ of a with a strictly increasing sequence $n : \mathbb{N} \to \mathbb{N}$ of positive integers. Let $n_k = n(k)$, and denote the subsequence by (a_{n_k}).

Proposition 4. *Let (X, ρ) be a metric space and let (a_n) be a sequence in X, Then $q \in X$ is a cluster point of (a_n) if and only if (a_n) has a subsequence which converges to q.*

3. Bounded Sequences

Definition 6. Let (X, ρ) be a metric space and let (a_n) be a sequence in X. We say that (a_n) is *bounded* if there exists a point $c \in X$ and a positive real number $R > 0$ such that $\rho(a_n, c) \leq R$ for all $n \in \mathbb{N}$.

Definition 7. Let (X, ρ) be a metric space. We say that X has the *Bolzano-Weierstrass property* if every bounded sequence in X has a convergent subsequence.

Example 2. We have already shown that $\mathbb{R}$ has the Bolzano-Weierstrass property.

Proposition 5. *Let (X, ρ) be a metric space. Then X has the Bolzano-Weierstrass property if and only if every sequence has a cluster point.*

Proof. This follows immediately from Proposition 4. $\square$

Proposition 6. *Let (X, ρ) be a metric space. Then X has the Bolzano-Weierstrass property if and only if every bounded infinite subset of X has an accumulation point.*

Proof. Suppose that X has the Bolzano-Weierstrass property. Then every bounded sequence in X has a cluster point. Let $A \subset X$ be a bounded infinite set. Since A is infinite, there exists an injective function $a : \mathbb{N} \to A$. This produces a sequence (a_n). This sequence is bounded, so it has a cluster point, say $q \in X$.

We claim that q is an accumulation point of A. To see this, let U be a neighborhood of q. Since q is a cluster point, U contains a_n for infinitely many n. Since a is injective, at $a_n = q$ for at most one n. Thus $U \smallsetminus \{q\}$ contains a_n for some n, and $a_n \in A$. Thus U intersects A, and q is a cluster point.

Suppose that every bounded infinite subset of X has an accumulation point. Let (a_n) be a sequence in A. Let $B = \{a_n \mid n \in \mathbb{N}\}$. If B is finite, then there exists $b \in B$ such that $b = a_n$ for infinitely many n. In this case, b is a cluster point of A. On the other hand, if B is infinite, it has an accumulation point, and this accumulation point will be a cluster point of (a_n). $\square$

4. Cauchy Sequences

Definition 8. Let (X, ρ) be a metric space and let (a_n) be a sequence in X. We say that (a_n) is a *Cauchy sequence* if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \mid m, n \geq N \Rightarrow \rho(a_m, a_n) < \epsilon.$$

Definition 9. Let (X, ρ) be a metric space. We say that X is *complete* if every Cauchy sequence in X converges.

This definition of completeness appears different than the completeness axiom which we use to obtain the reals from the rationals. We now relabel that definition.

Definition 10. Let S be an ordered set. We say that S has the *supremum property* if every subset of S which is bounded above has a least upper bound. We say that S has the *infimum property* if every subset of S which is bounded below has a greatest lower bound.

We have already shown that a sequence in $\mathbb{R}$ converges if and only if it is a Cauchy sequence. We now show that for subsets of $\mathbb{R}$, the supremum and infimum properties are equivalent to the new completeness property; in this way, the new definition is a generalization of the old one.

Proposition 7. *Let $A \subset \mathbb{R}$. Then A is a complete metric subspace of $\mathbb{R}$ if and only if A has the supremum and infimum properties.*

Proof. Suppose that A is a complete metric subspace of $\mathbb{R}$. Then every Cauchy sequence in A converges to a point in A. Let $B \subset A$ be bounded above; Then B has a supremum in the reals, say $x = \sup B$. Then for each $n \in \mathbb{N}$, there exists $b_n \in B$ such that $x - b_n < \frac{1}{2^n}$. Then for $m < n$, we have $|b_n - b_m| \frac{1}{2^n}$. Therefore (b_n) is a Cauchy sequence, which converges to a point in A. But clearly $\lim b_n = a$, so $\sup B = x \in A$. Similarly, B has the infimum property.

On the other hand, suppose that A has the supremum and infimum properties, and let (a_n) be a Cauchy sequence in A. Then (a_n) converges in $\mathbb{R}$, say to $x \in \mathbb{R}$. Let $u_n = \inf\{a_m \mid m \geq n\}$. Since A has the infimum property, $u_n \in A$ for every $n \in \mathbb{N}$. Also, (u_n) is an increasing sequence which converges to x, so $x = \sup\{u_n \mid n \in \mathbb{N}\}$. Since A has the supremum property, this is also in A. Thus every Cauchy sequence in A converges to a point in A. □

Proposition 8. *Let (X, ρ) be a metric space and let (a_n) be a Cauchy sequence in X. Then (a_n) is bounded.*

Proof. Since (a_n) is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\rho(a_m, a_n) < 1$. Let $M = \max\{\rho(a_i, a_N) \mid i < N\} \cup \{1\}$. Then $\rho(a_n, a_N) < M$ for every $n \in \mathbb{N}$. □

Proposition 9. *Let (X, ρ) be a metric space and let (a_n) be a Cauchy sequence in X. If (a_n) has a subsequence converging to $p \in X$, then (a_n) converges to p.*

Proof. Suppose that (a_{n_k}) is a subsequence of (a_n) which converges to $p \in X$. Let $\epsilon > 0$, and let K be so large that $k \geq K$ implies that $\rho(a_{n_k}, p) < \frac{\epsilon}{2}$. Let M be so large that $m, n \geq M$ implies $\rho(a_m, a_n) < \frac{\epsilon}{2}$. Let $N = \max\{K, M\}$. Then for $n \geq N$, we have

$$\rho(a_n, p) \leq \rho(a_n, a_N) + \rho(a_N, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore (a_n) converges to p. □

Proposition 10. *Let (X, ρ) be a metric space. If X has the Bolzano-Weierstrass property, then X is complete.*

Proof. Suppose that X has the Bolzano-Weierstrass property, and let (a_n) be a Cauchy sequence. By Proposition 8, (a_n) is bounded, and so has a convergent subsequence. By Proposition 9, (a_n) converges. Thus X is complete. □

We have seen that if a metric space has the Bolzano-Weierstrass property, then it is complete. One may conjecture that these properties are equivalent. The following counterexample shows this is not the case.

Example 3. Let X be any set any consider the discrete metric on X such that the distance between distinct points equals 1. In this space, Cauchy sequences are eventually constant, and so they converge. Thus X is complete. However, *every* sequence in X is bounded, so X has the Bolzano-Weierstrass property if and only if X is finite.

Next we would like to show the following propositions.

Proposition 11. *A sequence converges in $\mathbb{R}^k$ if and only if each of the coordinate sequences converges. A sequence is Cauchy in $\mathbb{R}^k$ if and only if each of the coordinate sequences is Cauchy. The metric space $\mathbb{R}^k$ is complete.*

Proposition 12. Bolzano-Weierstrass Theorem
Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Discussion. Proposition 11 is a lemma for Proposition 12, which is a generalization of the Bolzano-Weierstruass Theorem which we have already shown for $\mathbb{R}$ (the case $k = 1$). However, these propositions can be generalized even further, and we postpone the proofs for this more general context, which we take up next. □

5. Product Space Inheritance

Proposition 13. *Let $(X_1,\rho_1),\dots,(X_k,\rho_k)$ be a finite collection of metric spaces. Let $X = \times_{i=1}^k X_i$, and let $\rho : X\times X \to \mathbb{R}$ be the product metric on X. Then*

- **(a)** *A sequence is bounded in X if and only if each of the coordinate sequences is bounded.*
- **(b)** *A sequence converges in X if and only if each of the coordinate sequences converges.*
- **(c)** *A sequence is Cauchy in X if and only if each of the coordinate sequences is Cauchy.*
- **(d)** *The metric space X is complete if and only if each of the spaces X_i is complete.*
- **(e)** *The metric space X has the Bolzano-Weierstrass property if and only if each of the spaces X_i has the Bolzano-Weierstrass property.*

Preliminary Observation. Now suppose that $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$ are points in X, where $x_j, y_j \in X_i$. Observe that, since all metrics are positive, we have

$$\rho_j(x_j,y_j) \le \sqrt{\sum_{i=1}^k \rho(x_i,y_i)} = \rho(x,y) \le \sqrt{k}\max\{\rho(x_i,y_i) \mid i = 1,\dots,k\}.$$

□

Notation. A point in X is an k-tuple with entries for X_1 through X_k. If we denote these entries with subscripts, we must find another place to indicate the position of such an k-tuple in a sequence. Thus let $(x^{(n)})$ denote a sequence in X, where

$$x^{(n)} = (x_1^{(n)},\dots,x_k^{(n)}),$$

where $x_i^{(n)} \in X_i$. □

Proof of **(a)**. This follows from the observation. □

Proof of **(b)**. Suppose that $(x_i^{(n)})$ converges for $i = 1,\dots,k$, say to $L_i \in X_i$. Let $L = (L_1,\dots,L_k)$. Let $\epsilon > 0$. Let N be so large that $\rho_i(x_i^{(n)}, L_i^{(n)}) < \frac{1}{k}\epsilon^2$ for $n \ge N$. Then for $n \ge N$ we have

$$\rho(x_n, L) = \sqrt{\sum_{i=1}^k \rho(x_i^{(n)}, L_i)} < \sqrt{\sum_{i=1}^k \frac{1}{k}\epsilon^2} = \sqrt{k(\frac{1}{k}\epsilon^2)} = \epsilon.$$

Therefore $\lim x^{(n)} = L$, and in particular, $(x^{(n)})$ converges.

Suppose that $(x^{(n)})$ converges, say to $L = (L_1,\dots,L_k)$. Let $\epsilon > 0$ and let n be so large that $\rho(x^{(n)}, L) < \epsilon$ for $n \ge N$. Then for i between 1 and k we have

$$\rho_i(x_i^{(n)}, L_i) \le \rho(x^{(n)}, L) < \epsilon.$$

Thus $\lim_{n\to\infty} x_i^{(n)} = L_i$, and in particular, the sequence $(x_i^{(n)})$ converges. □

Proof of **(c)**. Suppose that $(x_i^{(n)})$ is a Cauchy sequence for $i = 1, \ldots, k$. Let $\epsilon > 0$ and let N be so large that $m, n \geq N$ implies

$$\rho_i(x_i^{(m)}, x_i^{(n)}) < \frac{\epsilon}{\sqrt{k}}$$

for all $i = 1, \ldots, k$. Then by the observation, we have

$$\rho(x^{(m)}, x^{(n)}) \leq \epsilon.$$

Suppose that $(x^{(n)})$ is a Cauchy sequence. Let $\epsilon > 0$. Let N be so large that $m, n \geq N$ implies $\rho(x^{(m)}, x^{(n)}) < \epsilon$. Then for $m, n \geq N$, we have

$$\rho_i(x_i^{(m)}, x_i^{(n)}) \leq \rho(x^{(m)}, x^{(n)}) < \epsilon,$$

we says that the coordinate sequence $(x_i^{(n)})$ is a Cauchy sequence. □

Proof of **(d)**. We know that a metric space is complete if and only if each of its Cauchy sequences converges.

Suppose each space (X_i, ρ_i) is complete, and consider a Cauchy sequence in X. Each of the coordinate sequences are Cauchy by part **(b)**, so each converges since X_i is complete. Then the original sequence converges by part **(a)**, so X is complete.

On the other hand, suppose that (X, ρ) is complete, and let $i \in \{1, \ldots, k\}$. Consider a Cauchy sequence in X_i. Construct a sequence in X by selecting a constant $a_i \in X_i$ in every coordinate other than the i^{th}. These are all Cauchy sequences in the coordinate spaces, so the construct sequence in X converges. Thus the original sequence in X_i converges, and X_i is complete. □

Proof of **(e)**. Suppose that X_i has the Bolzano-Weierstrass property for $i = 1, \ldots, k$. Then each bounded sequence in X_i has a convergent subsequence. Given a bounded sequence in X, each of the coordinate sequences is bounded, and has a convergent subsequence. Select a convergent subsequence X_1 for the first coordinate subsequence, and take the corresponding subsequence in X. Now select a convergent subsequence in X_2 for the second coordinate subsequence of the new sequence in X, and again take the corresponding subsequence in X. Continue this process k times, and arrive at a sequence in X such that every subsequence converges. This sequence is a subsequence of the original sequence in X, and it converges. Thus X has the Bolzano-Weierstrass property.

Suppose that X has the Bolzano-Weierstrass property. Let $i \in \{1, \ldots, k\}$ and let consider a bounded sequence in X_i. Construct a sequence in X by selecting a constant $a_i \in X_i$ in every coordinate other than the i^{th}. This is bounded in X, and so has a convergent subsequence. The i^{th} coordinate sequence of this subsequence converges in X_i, and is a subsequence of the original sequence in X_i. Thus X_i has the Bolzano-Weierstrass property. □

Corollary 1. *The space $\mathbb{R}^k$ is complete and has the Bolzano-Weierstrass property.*

Example 4. Consider $\mathbb{R}^\infty$, whose points are all infinite tuples of real numbers with all but finitely many entries equal to zero. Construct a sequence $(x^{(n)})$ in $\mathbb{R}^\infty$ by setting

$$(\dagger) \qquad x_i^{(n)} = \begin{cases} 1 & \text{if } i = n; \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x^{(n)})$ is bounded (it is completely contained inside the closed unit ball), yet has no convergent subsequence. Thus $\mathbb{R}^\infty$ does not have the Bolzano-Weierstrass property. Note that the sequence above is not a Cauchy sequence.

However, consider this example. Construct a sequence $(y^{(n)})$ in $\mathbb{R}^\infty$ by setting

$$y_i(n) = \begin{cases} \frac{1}{2^i} & \text{if } i \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

This is a Cauchy sequence in $\mathbb{R}^\infty$ which does not converge in $\mathbb{R}^\infty$. So this space is not complete.

Example 5. Let ℓ^2 be the space of sequences (x_n) in $\mathbb{R}$ with the convergence criterion $\sum_{i=1}^\infty x_i^2 < 0$. Then $\mathbb{R}^\infty$ is a subspace of ℓ^2, and the sequence ($\dagger$) from Example 4 does not have a convergent subsequence in ℓ^2.

However, ℓ^2 is complete. To show this, proceed as follows. Consider a Cauchy sequence $(x_i^{(n)})$ in ℓ^2. Show that the coordinate sequences are Cauchy, and so the converge in $\mathbb{R}$; say that $(x_i^{(n)})$ converges to x_i for each i. Next see that the sequence (x_i) is in ℓ^2.

Clearly there is some relationship between the Bolzano-Weierstrass property and completeness. We need the concept of *compactness* to illuminate this further.

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