Definition 1. (Distance)
Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

The *distance* of A and B is the number

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This definition is motivated by the Pythagorean Theorem.

Example 1. (Find the Distance Between Two Points)
Let $A = (1, 6)$ and $B = (-3, 2)$. Find the distance from A to B.

Solution. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$; thus $x_1 = 1$, $y_1 = 6$, $x_2 = -3$, $y_2 = 2$. Plug this into the formula to get

$$d = \sqrt{(-3-1)^2 + (2-6)^2} = \sqrt{(-4)^2 + (-4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}.$$

The distance is

$$\boxed{d = 4\sqrt{2}.}$$

□

Example 2. (Show that a Triangle is Isosceles)
Show that the triangle with vertices $(0, 0)$, $(7, -1)$, and $(4, 3)$ is isosceles.

Solution. The length of one side of the triangle is the distance from $(4, 3)$ to $(0, 0)$. The length of another side of the triangle is the distance from $(4, 3)$ to $(7, -1)$.

We use the distance formula to compute the distance from $(4, 3)$ to $(0, 0)$ and get

$$d_1 = \sqrt{(4-0)^2 + (3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5.$$

We use the distance formula to compute the distance from $(4, 3)$ to $(7, -1)$ and get

$$d_2 = \sqrt{(4-7)^2 + (3-(-1))^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

This triangle has two sides of the same length, so it is isosceles. □

Example 3. (Show that a Triangle is Right)
Show that the triangle with vertices $(-2, 0)$, $(4, 0)$ and $(3, \sqrt{5})$ is right.

Solution. We compute the lengths of the sides, and see if they satisfy the the Pythagorean Theorem:

$$a^2 + b^2 = c^2.$$

The distance from $(-2, 0)$ to $(4, 0)$ is 6.

The distance from $(-2, 0)$ to $(3, \sqrt{5})$ is $\sqrt{(3-(-2))^2 + (\sqrt{5}-0)^2} = \sqrt{25+5} = \sqrt{30}$.

The distance from $(4, 0)$ to $(3, \sqrt{5})$ is $\sqrt{(3-4)^2 + (\sqrt{5}-0)^2} = \sqrt{1+5} = \sqrt{6}$.

Since $\sqrt{30}^2 + \sqrt{6}^2 = 36 = 6^2$, the converse of the Pythagorean Theorem says that we have a right triangle. □

Example 4. (Distance from a Point to a Line)
Find the distance from the point $(2, 5)$ to the line $y = -2x + 3$.

Solution. By distance, we mean the shortest distance. This occurs along a line perpendicular to the given line.

Step 1: Find the equation of the line through the given point and perpendicular to the given line. The slope of the perpendicular line is the negative reciprocal, which is $m = \frac{1}{2}$. The given point is $(x_0, y_0) = (2, 5)$. Thus the given line is

$$y = m(x - x_0) + y_0 = \frac{1}{2}(x - 2) + 5 = \frac{1}{2}x - 1 + 5 = \frac{1}{2}x + 4.$$

Thus the perpendicular line is

$$y = \frac{1}{2}x + 4.$$

Step 2: Find the intersection of the two lines.
The lines are $y = -2x + 3$ and $\frac{1}{2}x + 4$. Set the right hand sides equal, to get $-2x + 3 = \frac{1}{2}x + 4$. Multiply by 2 to get $-4x + 3 = x + 8$. Solve for x and see that $5x = -5$, so $x = -1$; this is the x-coordinate of the point of intersection. Plug this into either line to get the y-coordinate: $y = -2(-1) + 3 = 0$. So, the intersection is $(-1, 0)$.

Step 3: Find the distance between the two points.
The points are $(2, 5)$ and $(-1, 0)$. The distance is $d = \sqrt{3^2 + 5^2} = \sqrt{34}$. Thus, the distance from $(2, 5)$ to $y = -2x + 3$ is

$$\boxed{d = \sqrt{34}.}$$

□

Definition 2. (Midpoint)
Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

The *midpoint* of A and B is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 5. (Find the Midpoint Between Two Points)
Let $A = (1, 6)$ and $B = (-3, 2)$. Find the midpoint between A and B.

Solution. We line up this information with the formula. Let $x_1 = 1$, $y_1 = 6$, $x_2 = -3$, $y_2 = 2$.

Then midpoint between A and B is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{1 - 3}{2}, \frac{6 + 2}{2}\right) = \left(\frac{-2}{2}, \frac{8}{2}\right) = (-1, 4).$$

The midpoint is

$$\boxed{M = (-1, 4).}$$

□

Example 6. (Find the Area of a Triangle) Find the area of a triangle with vertices $A = (2, 0)$, $B = (0, 2)$, and $C = (7, 7)$.

Solution. The area of a triangle is $A = \frac{1}{2}bh$. Let the base be $\overline{AB}$; we have $b = AB = \sqrt{2^2 + 2^2} = 4\sqrt{2}$.

This is clearly an isosceles triangle. A perpendicular from vertex C hits the base at its midpoint, which is $M = (1, 1)$. The height of the triangle is the distance from C to M, which is $h = CM = \sqrt{(7 - 1)^2 + (7 - 1)^2} = 7\sqrt{2}$.

Thus the area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(4\sqrt{2})(7\sqrt{2}) = 28$. □