PRINCIPLES OF ANALYSIS
LECTURE 20 - DIFFERENTIATION

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Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in D$ be an accumulation point of D. Define a function

$$\widehat{f}_a : D \smallsetminus \{a\} \to \mathbb{R} \quad \text{by} \quad \widehat{f}_a(x) = \frac{f(x) - f(a)}{x - a}.$$

We say that f is *differentiable* at a if $\widehat{f}_a$ has a limit at a. In this case, we write

$$f'(a) = \lim_{x \to a} \widehat{f}_a(x),$$

and we call $f'(a)$ the *derivative* of f at a.

Proposition 1. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$. If f is differentiable at a, then f is continuous at a.*

Proof. Suppose that f is differentiable at a. Then $\widehat{f}_a$ has a limit at a. Now for $x \in D \smallsetminus \{a\}$, we have

$$f(x) = \widehat{f}_a(x)(x - a) + f(a).$$

The constituent functions of the right hand side have limits; thus the left hand side has a limit, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} \widehat{f}_a(x) \lim_{x \to a}(x - a) + \lim_{x \to a} f(a) = f'(a) \cdot 0 + f(a) = f(a).$$

Thus f is continuous at a. □

Proposition 2. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be differentiable at a. Then $f + g$ is differentiable at a, and $(f + g)'(a) = f'(a) + g'(a)$.*

Proof. Notice that

$$\begin{aligned}
(\widehat{f + g})_a(x) &= \frac{(f + g)(x) - (f + g)(a)}{x - a} \\
&= \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a} \\
&= \widehat{f}_a(x) + \widehat{g}_a(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{x \to a} (\widehat{f + g})_a(x) &= \lim_{x \to a} \widehat{f}_a(x) + \lim_{x \to a} \widehat{g}_a(x) \\
&= f'(a) + g'(a).
\end{aligned}$$

□

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Proposition 3. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ be differentiable at a and let $c \in \mathbb{R}$. Then cf is differentiable at a, and $(cf)'(a) = cf'(a)$.*

Proposition 4. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be differentiable at a. Then fg is differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.*

Proposition 5. *Let $D \subset \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be differentiable at a, with $g(a) \neq 0$. Then $\frac{f}{g}$ is differentiable at a, and $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$.*

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