# ARCHIMEDES AND THE CANNED SPHERE

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## 1. Introduction

1.0.1. *Historical Background.* Mathematics always precedes physics. The mathematics may be developed centuries before it is used, as is the case with Apollonius (ca. 240 B.C.) and his abstract treatise *Conic Sections*, which preceded the usage by Copernicus and Kepler of ellipses to explain the orbits of planets. Or the mathematics may be created by the physicist in order to solve his physical problems, as was the case with Newton. Either way, the physics cannot go forward until the mathematical model comes into existence.

Physics precedes engineering. What is physically possible is understood before what is physically practical can be implemented.

Thus one wonders what the world would be like if we kicked forward the pace of mathematical research during its reinvigoration in the fifteenth and sixteenth century, say by 200 years. Would the technology necessary to prevent famine have kept ahead of the growth of population? Would the great world wars have been prevented?

Between the ancient Greeks and the early Renaissance, European civilization was dominated first by the Roman Empire and then by the Holy Roman Church. Science as we understand it dissolved during this period, and much that was known to the Greeks was lost. After this long drought, during the 1400's, many brilliant people were being allowed to think and communicate their thoughts again, and a period of reconstruction began.

The details of calculus, necessary for the advancement of physics, were worked out during the late 1600's; thus it took nearly 200 years to advance to this point.

1.0.2. *Discovery of the Palimpsest.* In the first decade of the twentieth century, a Danish philologist Johan Ludvig Heiberg (1854-1928) discovered an ancient prayer book in a library in Constantinople. Barely visible behind the Latin text were Greek symbols. The book was an ancient parchment palimpsest.

Parchment is a material for the pages of a book, made from fine calf skin, sheep skin or goat skin. A palimpsest is a manuscript page, scroll, or book that has been written on, scraped off, and used again.

Heiberg came to realize that the Greek writing was a previously unknown work of the ancient Greek genius Archimedes, who lived in the third century BC, was written in the 10th century. In the 12th century it was imperfectly erased in order that a liturgical text could be written on the parchment, and Archimedes' work is still legible today. It was a book of nearly 90 pages before being made a palimpsest of 177 pages; the older leaves were folded so that each became two leaves of the liturgical book.

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Heiberg was not allowed to take the book from the library, so he had every page photographed. Using only a magnifying glass, he attempted to read the Greek text, and published what he could. Shortly thereafter it was translated into English by Thomas Heath.

1.0.3. *Contents of the Palimpsest.* This book is known as *The Method*, or *Method Concerning Mechanical Theorems*, and it describes the process by which Archimedes discovered many of his results.

Although the only mathematical tools at its author's disposal were what we might now consider secondary-school geometry, he used those methods with rare brilliance, explicitly using infinitesimals to solve problems that would now be treated by integral calculus. Among those problems were that of the center of gravity of a solid hemisphere, that of the center of gravity of a frustum of a circular paraboloid, and that of the area of a region bounded by a parabola and one of its secant lines. Contrary to historically ignorant statements found in some 20th-century calculus textbooks, he did not use anything like Riemann sums, either in the work embodied in this palimpsest or in any of his other works. For explicit details of the method used, see how Archimedes used infinitesimals.

Historian Reviel Netz of Stanford University, with technical assistance from several persons at the Rochester Institute of Technology, has been trying to fill in gaps in Heiberg's account. In Heiberg's time, much attention was paid to Archimedes' brilliant use of infinitesimals to solve problems about areas, volumes, and centers of gravity. Less attention was given to the Stomachion, a problem treated in the Palimpsest that appears to deal with a children's puzzle. Netz has shown that Archimedes found that the number of ways to solve the puzzle is 17,152. This is perhaps the most sophisticated work in the field of combinatorics in classical antiquity.

## 2. Archimedes Biography

**287 B.C.** Archimedes was born in Sicily around 287 B.C. His father was an astronomer and mathematician named Phidias. Unfortunately, little else is known about Archimedes' early life. It is believed, however, that Archimedes' family was a rich and noble one, perhaps related in some way to Hiero, King of Syracuse.

**269 B.C.** Archimedes travelled to Egypt to study at Alexandria. This city had been founded by Alexander the Great in 331 B.C, and by 300 BC was home to 500,000 people. Alexandria was also the home of Euclid, who lived from about 330 to 275 B.C. Euclid was a renowned mathematician and may best be remembered for his book, "The Elements" which was the most important geometry book in the world for over 2000 years. Archimedes undoubtedly studied this book along with others in the great library of Alexandria, which contained more than a million books in the form of scrolls of papyrus.

**263 B.C.** Archimedes returned to Syracuse after his studies in Alexandria and settled down to a life of study and research. He would typically sit for hours pondering geometry diagrams drawn in the sand floor of his home or on papyrus scrolls. His experimentations soon made him indispensable to King Hiero, and ultimately, to the rest of the world.

Archimedes' abilities were put to good use by King Hiero. In one case, the hold of a huge boat made for the King had become full of water after a heavy rain. Not sure how to remove the water from the ship, King Hiero asked Archimedes for assistance. Archimedes created what is now know as the "Archimedes Screw". It is a machine consisting of a hollow tube containing a spiral that could be turned by a handle at one end. When the lower end of the tube was put into the hold and the handle turned, water was carried up the tube and over the side of the ship. The "Archimedes Screw" soon became popular in Egypt as a device for irrigating fields and in other forms, is still in use today.

King Hiero had commissioned a new royal crown for which he provided solid gold to the goldsmith. But when the crown arrived, King Hiero was suspicious that the goldsmith only used some of the gold, kept the rest for himself and added silver to make the crown the correct weight. Archimedes was asked to determine whether or not the crown was pure gold without harming it in the process. Archimedes was perplexed but found inspiration while taking a bath. While noticing that the water overflowed from the tub when he lowered himself into it, he realized that he could measure the crown's density if he could determine the amount of water it displaced, or its "volume". Legend has it that Archimedes was so exuberant about his discovery that he ran down the streets of Syracuse naked shouting, "Eureka!" which meant "I've found it!" in Greek.

Archimedes found that the crown was indeed a fake proving that the goldsmith had cheated.

King Hiero relied on Archimedes' inventions for use in the military during a time when there was great competition for power in the Mediterranean region between Syracuse, Carthage and Rome. Putting his theories of levers and pulleys to work, Archimedes built other machines designed to defend Syracuse.

**216 B.C.** King Hiero died in the year 216 B.C. and was succeeded by his 15-year-old grandson Hieronymos. The new King formed an alliance with Hannibal, the ruler of Carthage, which alarmed the pro-Roman faction within Syracuse.

**215 B.C.** Hieronymos was assassinated in the Greek city of Leontini, ending his 13-month reign. After the assassination of Hieronymos, civil war erupted in Syracuse between the pro-Carthaginian and pro-Roman factions, during which most of Hiero's family was killed. The pro-Carthaginian faction was eventually victorious and two brothers of mixed Carthaginian-Syracusan descent, Hippokrates and Epikydes, took control of the city.

**214 B.C.** Marcellus led the Roman army in an invasion of Syracuse but they were thwarted by the ingenuity of Archimedes. Among his many inventions were the huge curved mirrors placed on top of the city walls. When the Roman fleet was in sight the mirrors were turned to reflect the Sun's rays onto the ships. The heat was so great that many ships burst into flames. Other ships were destroyed by huge boulders thrown by the catapults designed by Archimedes.

With the help of Archimedes' incredible machines, Syracuse was protected from the Roman army. One of these machines operated with great iron claws that could seize boats by the prow, draw them up into the air, and plunge them into the depths of the sea. Another projected huge wooden beams from the island's ramparts to gouge the hulls of enemy ships.

Unable to penetrate the devices which Archimedes had placed around the borders of Syracuse, Marcellus ultimately surrounded the city and prevented supplies from entering or leaving. The siege lasted over two years. Eventually, in 212 B.C., the Romans took advantage of an unguarded section of the city walls and invaded the city.

**212 B.C.** During the siege of Syracuse, a Roman soldier burst through Archimedes' door and demanded that the great military genius accompany him to the quarters of General Marcellus. Not realizing that the city had been invaded, Archimedes refused, claiming he had yet to finish a mathematical problem that presently occupied his attention. The soldier, in anger, struck the 75-year-old Archimedes dead.

Marcellus was distressed upon hearing the news of the death, and ordered that Archimedes be buried with honor. His tombstone was, as he wished, engraved with the geometrical diagram showing a sphere inside a cylinder, to remind the world of his great discoveries.

## 3. The Method's Journey

**4th century A.D.** During his life, Archimedes wrote out his theories on papyrus scrolls. Succeeding generations preserved his works by copying and recopying them onto other scrolls. Somewhere, in the fourth century A.D., scribes began to copy onto parchment, then bind them between wooden boards. This was the earliest version of what's known today as the "book".

**10th century A.D.** The Archimedes manuscript was copied onto parchment sheets and bound between wooden boards. Although manufactured more than a thousand years after the great mathematician's death, this book, which is now in the care of The Walters Art Gallery, is the earliest copy of Archimedes' treatises to survive.

**12th century A.D.** Parchment was scarce and it was common practice to reuse old manuscripts for newer writings. Apparently, the Archimedes text was taken apart, most likely in Constantinople, for this purpose. A scribe disassembled the manuscript and scraped off as much of the Archimedes text as he could. He cut the leaves in half along the inner fold and turned the page leaves 90 degrees before folding them in half. This scribe ruled fresh lines and copied new religious text onto the parchment, creating what's known as a "Palimpsest", or a text on parchment which has been overwritten with other text.

**12th - 19th century A.D.** Once the manuscript had become a religious text, it was considered a sacred document and cared for in the Holy Land, between Jerusalem and the Dead Sea. One of its homes was the monastery of Mar Saba, historically an intellectual and spiritual center for the Greek Church. The book was most likely used as religious text by the monastery's inhabitants for at least 400 years.

**Early 1800's** The palimpsest was moved from the monastery to the library of the Greek Patriarch in the Christian quarter of old Jerusalem. The book did not remain there long, however, as it continued to travel in the highest of religious circles. It is believed that the book travelled to the Church of the Holy Sepulchre, because it ultimately ended up in the Church's daughter house, the Metochion in Constantinople the city where the manuscript had first been created.

**1846 A.D,** Biblical scholar Constantine Tischendorf visited the Metochion Of The Holy Sepulchre to study the library's substantial collection of manuscripts. At the time, he claimed to find nothing of particular interest, except for a palimpsest dealing with mathematics. Though he didn't quite understand the importance of his discovery, he must have sensed the book's value, because he acquired one of its leaves now owned by the Cambridge University Library in England.

**1907 A.D.** Danish philologist Johan Heiburg meticulously transcribed the manuscript using nothing but a magnifying glass. It's not known whether Heiberg suspected the palimpsest's true origins at first, but he ultimately realized that this ancient manuscript was indeed a previously unknown treatise by Archimedes, the great mathematician. His great achievement and extraordinary find made headlines in the New York Times on July 16, 1907.

**1998 A.D.** The ownership of the palimpsest was disputed in federal court in New York in the case of the Greek Orthodox Patriarchate of Jerusalem versus Christie's, Inc. The plaintiff contended that the palimpsest had been stolen from one of its monasteries in the 1920s. Judge Kimba Wood decided in favor of Christie's Auction House on laches grounds.

**October 29, 1998** Christie's of New York held a much-publicized auction. The Archimedes Palimpsest was sold for two million dollars to an anonymous collector.

## 4. Archimedes Manuscripts

- *Sand Reckoner*
  Attempts to remedy for the inadequacies of the Greek numerical notation system by showing how to express the number of grains of sand required to fill the universe in a positional (base 100,000,000) numeral system.
- *Equilibrium of Planes* (two volumes)
  Find the centers of gravity of various plane figures and conics, and establishes the "law of the level".
- *Quadrature of the Parabola*
  Finds the area of any segment of a parabola.
- *Measurement of a Circle*
  Showed that the area constant was one quarter of the circumference constant $\pi$, and bound this constant between $3\frac{10}{71}$ and $3\frac{10}{70}$.
- *On the Sphere and the Cylinder* (in two volumes)
  Shows that the surface area of any sphere is $A = 4\pi r^2$, and that the volume of a sphere is $V = \frac{4}{3}\pi r^3$.
- *On Spirals*
  Develops the properties of the tangents to the "spiral of Archimedes", given in polar coordinates as $r = a\theta$.
- *On Conoids and Spheroids*
  Finds the volumes of solids of revolution.
- *On Floating Bodies* (two volumes)
  Find the positions that various solids will assume when floating in a fluid, and establishes "Archimedes' principle" (that the buoyant force on a submerged object is equal to the weight of the displaced fluid).
- *The Method*
  Describes the process of discovery in mathematics.

## 5. Precursors of Archimedes

5.1. **Pythagorean Irrational Numbers.** The Pythagoreans proved the existence of irrational numbers in the form of "incommensurable quantities". This tore at the fabric of their world view, based on the supremacy of whole numbers, and it is legend that the demonstrator of irrational numbers was thrown overboard at sea.

5.2. **Zeno's Paradoxes.** Zeno (ca. 450 B.C.) developed his famous "paradoxes of motion".

5.2.1. *The Dichotomy.* The first asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the half-way stage before it arrives at the goal. (Aristotle Physics, 239b11).

5.2.2. *Achilles and the Tortoise.* The [second] argument was called "Achilles," accordingly, from the fact that Achilles was taken [as a character] in it, and the argument says that it is impossible for him to overtake the tortoise when pursuing it. For in fact it is necessary that what is to overtake [something], before overtaking [it], first reach the limit from which what is fleeing set forth. In [the time in] which what is pursuing arrives at this, what is fleeing will advance a certain interval, even if it is less than that which what is pursuing advanced. And in the time again in which what is pursuing will traverse this [interval] which what is fleeing advanced, in this time again what is fleeing will traverse some amount. And thus in every time in which what is pursuing will traverse the [interval] which what is fleeing, being slower, has already advanced, what is fleeing will also advance some amount. (Simplicius(b) On Aristotle's Physics, 1014.10)

5.2.3. *The Arrow.* The third is that the flying arrow is at rest, which result follows from the assumption that time is composed of moments. He says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always in a now, the flying arrow is therefore motionless. (Aristotle Physics, 239b.30) Zeno abolishes motion, saying "What is in motion moves neither in the place it is nor in one in which it is not". (Diogenes Laertius Lives of Famous Philosophers, ix.72)

5.2.4. *The Stadium.* The fourth argument is that concerning equal bodies [AA] which move alongside equal bodies in the stadium from opposite directions – the ones from the end of the stadium [CC], the others from the middle [BB] – at equal speeds, in which he thinks it follows that half the time is equal to its double. And it follows that the C has passed all the As and the B half; so that the time is half. And at the same time it follows that the first B has passed all the Cs. (Aristotle Physics, 239b33)

5.3. **Eudoxus Method of Exhaustion.** Eudoxus (ca. 370 B.C.) is remembered for two major mathematical contributions: the *Theory of Proportion*, which filled the gaps in the Pythagoren theories created by the existence of incommensurable quantities, and the *Method of Exhaustion*, which dealt with Zeno's Paradoxes. This method is based on the proposition: *If from any magnitude there be subtracted a part not less than its half, from the remainder another part not less than its half, and so on, there will at length remain a magnitude less than any preassigned magnitude of the same kind.*

Archimedes credits Eudoxus with applying this method to find that the volume of "any cone is on third part of the cylinder which has the same base with the cone and equal height."

5.4. **Euclid's Elements.** Euclid of Alexandria (ca. 300 B.C.) wrote *The Elements*, which may be the second most published book in history (after the Bible). The work consists of thirteen books, summarizing much of the basic mathematics of the time, spanning plane and solid geometry, number theory, and irrational numbers.

Among the results in Euclid, we find

**Result 1.** *The circumferences of two circles are to each other as their diameters.*

Using modern notation, this says that if we are given two circles with diameters $D_1$ and $D_2$, and circumferences $C_1$ and $C_2$, then

$$\frac{C_1}{C_2} = \frac{D_1}{D_2}.$$

We can rearrange this to say

$$\frac{C_1}{D_1} = \frac{C_2}{D_2}.$$

From this, one may conclude that for any given circle, the ratio between the circumference and the diameter is a constant:

$$\frac{C}{D} = p, \qquad \text{so} \qquad C = pD.$$

We shall call $p$ the *circumference constant.*

Euclid later shows

**Result 2.** *The areas of two circles are to each other as the squares of their diameters.*

That is, if $A_1$ and $A_2$ represent the area of the circles, then

$$\frac{A_1}{D_1^2} = \frac{A_2}{D_2^2},$$

which says that there is an *area constant* for any circle:

$$\frac{A}{D^2} = k, \qquad \text{so} \qquad A = kD^2.$$

However, Euclid doesn't mention, and possibly doesn't realize, that $p$ and $k$ are related.

Later still, Euclid shows

**Result 3.** *The volumes of two spheres are to each other as the cubes of their diameters.*

Thus if $V_1$ and $V_2$ are the volumes of spheres of diameter $D_1$ and $D_2$, then

$$\frac{V_1}{D_1^3} = \frac{V_2}{D_2^3};$$

again, one sees that, for again given sphere, there is a *volume constant* $m$ such that

$$\frac{V}{D^3} = m, \qquad \text{so} \qquad V = mD^3.$$

## 6. Measurement of a Circle

**Proposition 1.** *The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*

Let be $C$ be the circumference, $r$ the radius, and $A$ the area of the circle. Let $T$ be the area of a right triangle with legs of length $r$ and $C$. Then $T = \frac{1}{2}rC$. Archimedes claims that $A = T$, so $A = \frac{1}{2}rC$.

**Lemma 1.** *Let $h$ be the apothem and let $Q$ be the perimeter of a regular polygon. Then the area of the polygon is*

$$P = \frac{1}{2}hQ.$$

*Proof.* Suppose the polygon has $n$ sides, each of length $b$. Clearly $Q = nb$. Then the area is subdivided into $n$ triangles of base $b$ and height $h$, so

$$P = n(\frac{1}{2}hb) = \frac{1}{2}hQ.$$

□

**Lemma 2.** *Consider a circle of area $A$ let $\epsilon > 0$. Then there exists an inscribed polygon with area $P_1$ and a circumscribed polygon with area $P_2$ such that*

$$A - \epsilon < P_1 < A < P_2 < A + \epsilon.$$

*Proof.* Archimedes simply says: "Inscribe a square, then bisect the arcs, then bisect (if necessary) the halvesand so on, until the sides of the inscribed polygonwhose angular points are the points o the division subtend segments whose sum is less than the excess of the area of the circle over the triangle." □

does not explicitly prove this

*Proof of Proposition.* By double reductio ad absurdum.

Suppose that $A > T$. Then $A - T > 0$, so there exists an inscribed regular polygon with area $P$ such that $A - P < A - T$. Thus $P > T$. If $Q$ is the perimeter and $h$ the apothem of the polygon, we have

$$P = \frac{1}{2}hQ < \frac{1}{2}rC = T,$$

a contradiction.

On the other hand, suppose that $A < T$. Then $T - A > 0$, so there exists a circumscribed polygon with area $P$ such that $P - A < T - A$. Thus $P < T$. However, if $Q$ is the perimeter and $h$ the apothem of the polygon, we have

$$P = \frac{1}{2}hQ > \frac{1}{2}rC = T,$$

a contradiction.

Therefore, as Archimedes writes, "since then the area of the circle is neither greater nor less than [the area of the triangle], it is equal to it." □

**Proposition 2.** *The ratio of the circumference of any circle to its diameter is less the* $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.

*Proof.* Inscribe a hexagon. Compute the area:

$$\pi = \frac{C}{D} > \frac{Q}{D} = \frac{6r}{2r} = 3.$$

Archimedes next doubles the number of vertices to obtain a regular dodecagon. The computation of its area requires accurate extraction of $\sqrt{3}$, which Archimedes estimates as

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780},$$

which is impressively close. The Archimedes continues with 24, 48, and finally 96 sides, at each stage extracting more sophisticated square roots.

Next circumscribe a hexagon and continue to 96 sides. □

## 7. On the Sphere and the Cylinder

The two volume work entitled *On the Sphere and the Cylinder* is Archimedes undisputed masterpiece, probably regarded by Archimedes himself as the apex of his career. These two volumes are constructed in a manner similar to Euclid's Elements, in that it proceeds from basic definitions and assumptions, through simpler known results, onto the new discoveries of Archimedes.

Among the results in this work are the following.

**Proposition 3.** *The surface of any sphere is equal to four times the greatest circle in it.*

*Technique of Proof.* Double reductio ad absurdum: assumption that the area is more leads to a contradiction, as does assumption that the area is less. □

Let us translate this into modern notation. Let $r$ be the radius of the sphere and let $S$ be its surface area. Then the radius of the greatest circle in it is $\pi r^2$. Thus Archimedes shows that

$$S = 4\pi r^2.$$

**Proposition 4.** *Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.*

Note that again, Archimedes has expressed the volume of the sphere in terms of the volume of a known solid; this is because the Greeks did not have modern algebraic notation. Using modern notation, we let $r$ be the radius of the $V$ be the volume of the sphere. The volume of the cone of radius $r$ and height $r$, as determined by Eudoxus, is $\frac{1}{3}\pi r^3$. Thus

$$V = \frac{4}{3}\pi r^3.$$

In this way, Archimedes found the relationship between the circumference constant $p$, the area constant $k$ (in *Measurement of a Circle*), and the volume constant $m$: We have

$$C = pD, \quad A = kD^2, \quad , \text{ and } V = mD^3,$$

and Archimedes has shown (in modern notation) that

$$\begin{aligned} &C = \pi D && \text{(that is, } p = \pi) \\ &A = \pi r^2 = \pi\Big(\frac{D}{2}\Big)^2 = \frac{\pi}{4}D && \text{(so } k = \frac{\pi}{4}) \\ &V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\Big(\frac{D}{2}\Big)^3 = \frac{\pi}{6}\pi D^3 && \text{(so } m = \frac{\pi}{6}) \end{aligned}$$

From here, Archimedes now describes an astounding discovery.

Suppose we have a sphere of radius $r$, surface area $S$, and volume $V$. Inscribe this sphere in a right circular cylinder, whose radius would also be $r$ and whose height would be $2r$. Then the surface area $A_{\text{cyl}}$ of the cylinder is simply the areas of the base and top circle, plus the area of the rectangle which forms the tube of the cylinder:

$$A_{\text{cyl}} = 2(\pi r^2) + (2\pi r)(2r) = 6\pi r^2.$$

Thus

$$A_{\text{cyl}} : A_{\text{sph}} = (6\pi r^2) : (4\pi r^2) = 3 : 2.$$

Moreover, the volume of the cylinder is the area of the circular base times the height:

$$V_{\text{cyl}} = (\pi r^2)(2r) = 2\pi r^3.$$

Again, we have

$$V_{\text{cyl}} : V_{\text{sph}} = (2\pi r^3) : (\frac{4}{3}\pi r^3) = 3 : 2.$$

This so impressed Archimedes that he requested that his tombstone be engraved with a sphere inscribed in a cylinder, together with the ration 3 : 2. Apparently, Marcellus, the conqueror of Syracuse, was so impressed with Archimedes, that he granted this wish.

## 8. Equilibrium of Planes

In the treatise *Equilibrium of Planes*, Archimedes establishes the *law of the lever.*

**Result 4.** *Let $W_1$ and $W_2$ be the respective weights of two objects placed on a lever on opposite sides of a fulcrum with respective distances $d_1$ and $d_2$. Then the objects balance if and only if*

$$d_1W_1 = d_2W_2.$$

This is proven by Archimedes, following three assumptions:

**(a)** Equal weights at equal distances from the fulcrum balance. Equal weights at unequal distances from the fulcrum do no balance, but the weight at the greater distance will tilt its end of the lever down.

**(b)** If, when two weights balance, we add something to one of the weights, they no longer balance. The side holding the weight we increased goes down.

**(c)** If, when two weights balance, we take something away from one, they no longer balance. The side holding the weight we did not change goes down.

## 9. The Method

An *infintesimal* is a number greater in absolute value than zero, yet smaller than any positive real number. A number $x \neq 0$ is an infintesimal if and only if every sum $|x| + \cdots + |x|$of finitely many terms is less than 1, no matter how large the finite number of terms. In that case, $\frac{1}{x}$ is larger than any positive real number.

Using modern techniques, it is possible to construct a field which contains the real numbers as a subfield, and which contains infinitesimals. However, it is not, and cannot be, complete. This last fact was known to Archimedes, and in fact is called the *Archimedean property* of the real numbers:

**Proposition 5** (Archimedean Property)**.** *Let $\mathbb{F}$ be a complete ordered field (for example, $\mathbb{F} = \mathbb{R}$). Let $a, b \in \mathbb{F}$. Then there exists a natural number $n \in \mathbb{N}$ such that $na > b$.*

*Proof.* Suppose not, then the set $X = \{na \mid n \in \mathbb{N}\}$ is bounded, and by completeness, it has a least upper bound, say $c$.

ETC. □

Thus, Archimedes worked under the premise that infinitesimals were appropriate for intuitive thought and discovery, but not for proof.

For example, in *Quadrature of a Parabola*, Archimedes uses the method of exhaustion to show that the area of a segment of a parabola is

$$\text{area segment} = \frac{4}{3}\text{area inscribed triangle.}$$

But the method of exhaustion is not how he discovered this formula. In fact, in *The Method*, he writes: *"certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge."*

As an example of this, consider the first proposition from the palimpsest.

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