Real Analysis — Homework 3 Solutions

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Problem 1. Let F be an ordered field, $\alpha, \beta, \zeta \in F$, and $n \in \mathbb{N}$.
(a) Show that if $|\alpha| = |\beta|$, then $\beta = \pm\alpha$.
(b) Show that if $\alpha^n = 1$, then $\alpha = \pm 1$.
(c) Show that if $\alpha^n = \beta^n$, then $|\alpha| = |\beta|$.
(d) Show that the function $f : F \to F$ given by $f(x) = x^n - \zeta$ has at most two roots in F.

Lemma 1. *Let F be an ordered field, $a, b \in F$, and $n \in \mathbb{N}$. Then:*
(a) $(ab)^n = a^n b^n$;
(b) $|a|^n = |a^n|$;
(c) $|a| < 1 \Rightarrow |a|^n < 1$;
(d) $|a| > 1 \Rightarrow |a|^n > 1$.

Justification for Lemma.

(a) Apply field axiom **(M2)**, the commutative property of multiplication, and induction.

(b) Apply Ross Theorem 3.5 (ii), which states that $|ab| = |a| \cdot |b|$, and induction.

(c) This follows from ordered field axiom **(O5)** and induction: the base case is given, so we assume that $|a|^{n-1} < 1$. Since $0 < |a| < 1$, we multiply by $|a|$ to get $0 < |a|^n < |a|$; by order field axiom **(O3)**, transitivity of order, $|a|^n < 1$.

(d) This is similar to **(d)**. □

Solution to Problem 1.

(a) Consider all four cases of α positive or negative and β positive or negative, and the definition of $|\cdot|$.

(b) Either $|\alpha| < 1$, $|\alpha| > 1$, or $|\alpha| = 1$. If $|\alpha| < 1$, then $|\alpha|^n < 1$ by Fact (c); if $|\alpha| > 1$, then $|\alpha|^n > 1$ by Fact (d). Thus if $|\alpha|^n = 1$, we must have $|\alpha| = 1$. By part (a), we have $\alpha = \pm 1$.

(c) Suppose that $\alpha^n = \beta^n$. Then $\frac{\alpha^n}{\beta^n} = (\frac{\alpha}{\beta})^n = 1$ by Fact (a). So $\frac{\alpha}{\beta} = \pm 1$ by part (b). Thus $\alpha = \pm\beta$.

(d) Suppose that $\alpha \in F$ is a root of $f(x) = x^n - \zeta$. If β is another root, then $\alpha^n = \beta^n = \zeta$, so $\alpha = \pm\beta$. So the set of all possible roots is $\{\alpha, -\alpha\}$; that is, there are at most two roots. □

Problem 2. Let $A \subset \mathbb{R}$ be a bounded nonempty set of positive real numbers. Let $B = \{\sqrt{a} \mid a \in A\}$. Show that $\sqrt{\sup A} = \sup B$.

Solution. First we note that $1 \leq a < b \Leftrightarrow 1 \leq a^2 < b^2$; this follows from ordered field axiom **(O5)**. Next we assume that $\sup A \geq 1$; the case that $\sup A < 1$ is similar but requires the reversal of a few inequalities below.

Let $q = \sqrt{\sup A}$. Then $q^2 = \sup A$, so for every $a \in A$, we have $a \leq q^2$. Thus $\sqrt{a} \leq q$, so q is an upper bound for B.

Now suppose that u is another upper bound for B and let $a \in A$. Then $\sqrt{a} \in B$, so $\sqrt{a} \leq u$, so $a \leq u^2$. Thus u^2 is an upper bound for A, so $\sup A \leq u^2$. Thus $q = \sqrt{\sup A} \leq u$. This shows that q is the least upper bound for B; that is, $q = \sup B$. □

Problem 3. Let $s_n = \dfrac{4n^2+3}{3n^2+1}$. Find the smallest $N \in \mathbb{N}$ such that $\left|s_n - \dfrac{4}{3}\right| < \dfrac{1}{100}$ whenever $n \geq N$.

Solution. Add fractions to see that

$$\left|\frac{4n^2+3}{3n^2+1} - \frac{4}{3}\right| = \frac{5}{9n^2+3}.$$

This is less that $\dfrac{1}{100}$ if and only if $500 < 9n^2 + 3$, which is true if and only if $n > \sqrt{\dfrac{497}{9}}$. Thus set $N = \left\lceil \sqrt{\dfrac{497}{9}} \right\rceil$; since $\sqrt{49} < \sqrt{\dfrac{497}{9}} < \sqrt{64}$, we see that $N = 8$. $\square$

Problem 4. Let $a_n = \dfrac{n+1}{n}$, $b_n = \dfrac{2n+5}{n}$, and $c_n = a_n + b_n$.

(a) Find the smallest $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - 1| < \dfrac{1}{20}$.

(b) Find the smallest $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|b_n - 2| < \dfrac{1}{20}$.

(c) Find the smallest $N \in \mathbb{N}$ such that $n \geq N$ implies $|c_n - 3| < \dfrac{1}{10}$.

Solution. We see that

$$|a_n - 1| < \frac{1}{20} \Leftrightarrow \frac{1}{n} < \frac{1}{20} \Leftrightarrow 20 < n,$$

so let $N_1 = 21$. Also,

$$|b_n - 2| < \frac{1}{20} \Leftrightarrow \frac{5}{n} < \frac{1}{20} \Leftrightarrow 100 < n,$$

so let $N_2 = 101$. We know from the sum of sequences proof that $|c_n - 3| < \dfrac{1}{10}$ if $n \geq 101$, but we can do better. We see that

$$|c_n - 3| < \frac{1}{10} \Leftrightarrow \frac{6}{n} < \frac{1}{10} \Leftrightarrow 60 < n,$$

so let $N = 61$. $\square$

Problem 5. Let (a_n) be a convergent sequence of real numbers and let $A = \{a_n \mid n \in \mathbb{N}\}$.
Show that $\lim a_n \leq \sup A$.

Solution. First, we note that since (a_n) converges, it is bounded, so the supremum exists as a real number. Let $L = \lim a_n$ and $M = \sup A$, and suppose that $M < L$. Let $N \in \mathbb{N}$ be so large that $n \geq N$ implies that $|a_n - L| < L - M$. In particular, $L - a_N < L - M$, so $M < a_N$. This contradicts that M is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$. $\square$