COMPLEX ANALYSIS
TOPIC XVI: SEQUENCES

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ABSTRACT. We outline the development of sequences in $\mathbb{C}$, starting with open and closed sets, and ending with the Bolzano-Weierstrauss Theorem for complex numbers. Some propositions are formulated as problems.

1. Topology of $\mathbb{C}$

1.1. Open Sets.

Definition 1. Let $u \in \mathbb{C}$. The *open ball around u of radius r* is

$$B_r(u) = \{z \in \mathbb{C} \mid |z - u| < r\}.$$

Let $U \subset \mathbb{C}$. We say that U is *open* if for every $u \in U$ there exists $r > 0$ such that $B_r(u) \subset U$.

Problem 1. Let $U \subset \mathbb{C}$. Show that U is open if and only if U is the union of a collection of open balls.

Problem 2. Let $U, V \subset \mathbb{R}$ be open sets. Show that $U \cap V$ is an open set.

Definition 2. Let X be a set. A *collection of subsets* of X is a set $\mathcal{C}$ whose members are subsets of X. We will use the following notation.

- $\cup\mathcal{C} = \{x \in X \mid x \in C \text{ for some } C \in \mathcal{C}\}$
- $\cap\mathcal{C} = \{x \in X \mid x \in C \text{ for all } C \in \mathcal{C}\}$

Problem 3. Let $\mathcal{T}$ denote the collection of all open subsets of $\mathbb{C}$.

(a) Show that $\varnothing \in \mathcal{T}$ and $\mathbb{C} \in \mathcal{T}$.
(b) Show that if $\mathcal{C} \subset \mathcal{T}$, then $\cup\mathcal{C} \in \mathcal{T}$.
(c) Show that if $\mathcal{C} \subset \mathcal{T}$ and $\mathcal{C}$ is finite, then $\cap\mathcal{C} \in \mathcal{T}$.

The collection $\mathcal{T}$ is known as the *topology* of $\mathbb{C}$.

Definition 3. Let $z \in \mathbb{C}$. A *neighborhood* of z is a subset of $\mathbb{C}$ which contains an open set which contains z.

Problem 4. Let $z \in \mathbb{C}$ and let $A, B \subset \mathbb{C}$ be neighborhoods of z. Show that $A \cap B$ is a neighborhood of z.

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1.2. **Closed Sets.**

Definition 4. Let $F \subset \mathbb{C}$.
We say that F is *closed* if its complement $\mathbb{C} \smallsetminus F$ is open.

Problem 5. Let $\mathcal{F}$ denote the collection of all closed subsets of $\mathbb{C}$.

(a) Show that $\varnothing \in \mathcal{F}$ and $\mathbb{C} \in \mathcal{F}$.
(b) Show that if $\mathcal{C} \subset \mathcal{F}$, then $\cap\mathcal{C} \in \mathcal{F}$.
(c) Show that if $\mathcal{C} \subset \mathcal{F}$ and $\mathcal{C}$ is finite, then $\cup\mathcal{C} \in \mathcal{F}$.

Definition 5. Let $A \subset \mathbb{C}$.
We say that A is *bounded* if there exists $M \in \mathbb{R}$ such that $A \subset B_M(0)$.
We say that A is *compact* if it is closed and bounded.

The word compact has a more general definition in a more general settings, but in the larger sense, a subset of $\mathbb{C}$ is compact if and only if it is closed and bounded. Hence, for our purposes, we simply take it as a definition.

1.3. **Classification of Points.**

Definition 6. Let $A \subset \mathbb{C}$.
An *interior point* of A is a point $z \in A$ such that A contains a neighborhood of z. The *interior* of A is the set of interior points of A and is denoted A°.

Proposition 1. *Let $A \subset \mathbb{C}$. Then:*

(a) *A is open if and only if $A = A^\circ$;*
(b) *A is open if and only if every point in A is an interior point;*
(c) *The interior of A is the union of all open sets which are contained in A.*

Definition 7. Let $A \subset \mathbb{C}$. A *closure point* of A is a point $z \in \mathbb{C}$ such that every neighborhood of z intersects A. The *closure* of $A \subset X$ is the set of closure points of A and is denoted $\overline{A}$.

Proposition 2. *Let $A \subset \mathbb{C}$. Then:*

(a) *A is closed if and only if $A = \overline{A}$;*
(b) *A is closed if and only if every point in A is an closure point;*
(c) *Then $\overline{A}$ is the intersection of the closed subsets of $\mathbb{C}$ which contain A.*

Definition 8. Let $A \subset \mathbb{C}$. A *boundary point* of A is a point $z \in \mathbb{C}$ such that every neighborhood of z intersects A and A^c. The *boundary* of A is the set of boundary points of A and is denoted ∂A.

Proposition 3. *Let $A \subset \mathbb{C}$. Then*

(a) $\partial A = \overline{A} \smallsetminus A^\circ$*;*
(b) $\partial A = \overline{A} \cap \overline{A^c}$*;*
(c) $\partial A = \partial A^c$*;*
(d) $\overline{A} = A \cap \partial A$*;*
(e) $A^\circ = A \smallsetminus \partial A$*;*
(f) $\partial(\partial A) \subset \partial A$*;*
(g) $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$.

Definition 9. Let $A \subset \mathbb{C}$. An *accumulation point* of A is a point $z \in \mathbb{C}$ such that every deleted neighborhood of z intersects A. The *derived set* of A is the set of accumulation points of A and is denoted A'.

2. Sequences

Definition 10. Let X be a set. A *sequence* in X is a function $a : \mathbb{N} \to X$.

We may write a_n to mean $a(n)$. Next we specify notation to indicate the entire sequence as opposed to a specific member of the range.

We often think of a sequence as an infinitely long tuple of elements from X, so it looks like $(a_1, a_2, a_3, \dots)$. This is written more succinctly as $(a_n)_{n=1}^{\infty}$, or $(a_n)_{n\in\mathbb{N}}$, or simply (a_n).

In the case $X \subset \mathbb{C}$, we call this a sequence of complex number numbers. We are also interested in sequences of real numbers, which become an important special case.

Definition 11. Let (a_n) be a sequence in a set X.

We say that (a_n) is *injective* if $a_m = a_n \Rightarrow m = n$.

The *image* of (a_n) is

$$\{a_n\} = \{x \in X \mid x = a_n \text{ for some } n \in \mathbb{N}\}.$$

The N^{th} *tail* of (a_n) is

$$\{a_n : N\} = \{x \in X \mid x = a_n \text{ for some } n \geq N\}.$$

3. Limits of Sequences

Definition 12. Let (a_n) be a sequence of complex numbers and let $p \in \mathbb{C}$. We say that the sequence *converges* to p if

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \ni n \geq N \Rightarrow |a_n - p| < \epsilon.$$

If (a_n) converges to p, we call p the *limit* of the sequence, and write $\lim a_n = p$.

Problem 6. Let (a_n) be a sequence of complex numbers and let $p \in \mathbb{C}$.
Show that the following conditions are equivalent:
(L1) For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - p| < \epsilon$.
(L2) For every neighborhood U of p there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$.
(L3) Every neighborhood of p contains a tail of (a_n).
(L4) Every neighborhood of p contains a_n for all but finitely many $n \in \mathbb{N}$.

Solution.
(**L1** $\Rightarrow$ **L2**) Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - p| < \epsilon$. Let U be a neighborhood of p. Then there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset U$. Let N be so large that $|a_n - p| < \epsilon$ whenever $n \geq N$. Then for $n \geq N$, we have $a_n \in B_\epsilon(p) \subset U$.

(**L2** $\Rightarrow$ **L3**) Suppose that for every neighborhood U of p there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$. Let U be a neighborhood of p and let N be so large that $n \geq N \Rightarrow a_n \in U$. Then $\{a_n \mid n \geq N\} \subset U$, so U contains the N^{th} tail of (a_n).

(**L3** $\Rightarrow$ **L4**) Suppose that every neighborhood U of p contains a tail of (a_n). Let U be a neighborhood of p and let $N \in \mathbb{N}$ such that $\{a_n \mid n \geq N\} \subset U$. If $a_n \notin U$ for some $n \in \mathbb{N}$, then $a_n \notin \{a_n \mid n \geq N\}$, so $n < N$. There are only finitely many such n.

(**L4** $\Rightarrow$ **L1**) Suppose that every neighborhood of p contains a_n for all but finitely many n. Let $\epsilon > 0$. Then $B_\epsilon(p)$ is a neighborhood of p, so $a_n \in B_\epsilon(p)$ for all but finitely many $n \in \mathbb{N}$. The maximum of a finite set of natural numbers always exists. Let $N = 1 + \max\{n \in \mathbb{N} \mid a_n \notin B_\epsilon(p)\}$. Then for $n > N$, we have $|a_n - p| < \epsilon$. $\square$

Problem 7. Let (a_n) be an injective sequence and let $p \in \mathbb{C}$. Show that (a_n) converges to p if and only if every neighborhood of p contains a for all but finitely many $a \in \{a_n\}$.

Problem 8. Find an example of a noninjective sequence (a_n) of real numbers, and a real number p, such that every neighborhood of p contains all but finitely many points in $\{a_n\}$, but (a_n) does not converge to p.

4. Cluster Points of Sequences

Definition 13. Let (a_n) be a sequence of real numbers and let $q \in \mathbb{C}$.
We say that the sequence *clusters* at q if

$$\forall \epsilon > 0 \, \forall N \in \mathbb{N} \, \exists n \geq N \ni |a_n - q| < \epsilon.$$

If (a_n) clusters at q, we call q a *cluster point* of (a_n).

Problem 9. Let (a_n) be a sequence of real numbers and let $q \in \mathbb{C}$.
Show that the following conditions are equivalent:
(C1) For every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - q| < \epsilon$.
(C2) For every neighborhood U of q and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $a_n \in U$.
(C3) Every neighborhood of q intersects every tail of (a_n).
(C4) Every neighborhood of q contains a_n for infinitely many $n \in \mathbb{N}$.

Solution.
(**C1** $\Rightarrow$ **C2**) Suppose that for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - q| < \epsilon$. Let U be a neighborhood of q and let $N \in \mathbb{N}$. Then there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset U$; thus there exists $n \geq N$ such that $|a_n - q| < \epsilon$. But this says that $a_n \in B_\epsilon(q)$, so $a_n \in U$.

(**C2** $\Rightarrow$ **C3**) Suppose that for every neighborhood U of q and every $N \in \mathbb{N}$ there exists $n > N$ such that $a_n \in U$. Let U be a neighborhood of q and let $\{a_n \mid n \geq N\}$ be an arbitrary tail of (a_n). Then for some $n \geq N$, we have $a_n \in U$. But $a_n \in \{a_n \mid n \geq N\}$, so $a_n \in \{a_n \mid n \geq N\} \cap U$, and $\{a_n \mid n \geq N\}$ intersects U.

(**C3** $\Rightarrow$ **C4**) Suppose that every neighborhood of q intersects every tail of (a_n). Let U be a neighborhood of q. Suppose by way of contradiction that U contains a_n for only finitely many $n \in \mathbb{N}$. Let m be the largest natural number such that $a_m \in U$. Then $\{a_n : m+1\}$ is a tail of (a_n) which does not intersect U; this is a contradiction.

(**C4** $\Rightarrow$ **C1**) Suppose that every neighborhood of q contains a_n for infinitely many $n \in \mathbb{N}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$. Then $U = B_\epsilon(q)$ is a neighborhood of q, and U contains a_n for infinitely many $n \in \mathbb{N}$. One such n must be larger than N; if $n \in \mathbb{N}$ such that $a_n \in U$, then $|a_n - q| < \epsilon$. □

Problem 10. Let (a_n) be a sequence of complex numbers and let $p \in \mathbb{C}$. Show that if (a_n) converges to p, then (a_n) clusters at p, and p is the only cluster point.

Solution. Suppose that (a_n) converges to p. Then every neighborhood of p contains a_n for all but finitely many n. Thus there are infinitely many n such that a_n is in the neighborhood. By Problem 9 **(d)**, (a_n) clusters at p.

To see that p is the only cluster point, let $q \in X$, $q \neq p$; we show that (a_n) does not cluster at q. Let $\epsilon = \frac{|p-q|}{2}$ and let $U = B_\epsilon(p)$ and $V = B_\epsilon(q)$. Then U and V are disjoint neighborhoods of p and q respectively.

Let A be a tail of (a_n) such that $A \subset U$. Since $U \cap V = \varnothing$, we have $A \cap V = \varnothing$, so V is a neighborhood of q which does not intersect A. Thus (a_n) does not cluster at q, by 9 **(c)**. □

Problem 11. Find an example of a sequence (a_n) of real numbers and a real number $q \in \mathbb{C}$ such that (a_n) clusters at q but does not converge to q.

Problem 12. Let (a_n) be an injective sequence and let $q \in \mathbb{C}$. Show that (a_n) clusters at q if and only if every neighborhood of q contains a for infinitely many $a \in \{a_n\}$.

Problem 13. Find an example of a noninjective sequence (a_n) of real numbers, a real number q, and a neighborhood U of q, such that (a_n) clusters at q but U contains only finitely many points from $\{a_n\}$.

Problem 14. Let (a_n) be a bounded sequence of real numbers, and set

$$C = \{q \in \mathbb{C} \mid q \text{ is a cluster point of } (a_n)\}.$$

Show that C is closed and bounded.

5. Bounded Sequences

Definition 14. A sequence (a_n) of complex number is *bounded* if there exists a $R > 0$ such that $|a_n| < R$ for all $n \in \mathbb{N}$.

Problem 15. Let (a_n) be a convergent sequence of complex numbers. Then (a_n) is bounded.

6. Arithmetic of Sequences

Problem 16. Let (a_n) and (b_n) be sequences in $\mathbb{C}$. Suppose that $\lim a_n = L$ and $\lim b_n = M$ for some $L, M \in \mathbb{C}$. Apply the definition to show that the sequence $(a_n + b_n)$ converges to $L + M$.

Solution. Let $\epsilon > 0$.

Let N_1 be so large that $n \geq N_1$ implies $|a_n - L| < \epsilon/2$.

Let N_2 be so large that $n \geq N_2$ implies $|b_n - M| < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$.

Then, for $n \geq N$, we have

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Problem 17. Let (a_n) be a sequences in $\mathbb{C}$ and let $c \in \mathbb{C}$. Suppose that $\lim a_n = L$ for some $L \in \mathbb{C}$. Apply the definition to show that the sequence (ca_n) converges to cL.

Problem 18. Let (a_n) and (b_n) be sequences in $\mathbb{C}$. Suppose that $\lim a_n = L$ and $\lim b_n = M$ for some $L, M \in \mathbb{C}$. Apply the definition to show that the sequence $(a_n \cdot b_n)$ converges to LM.

Solution. Let $\epsilon > 0$.

Since (b_n) converges, it is bounded; let $B > 0$ be so large that $|b_n| < B$ for all $n \in \mathbb{N}$.

Let N_1 be so large that $n \geq N_1$ implies $|a_n - L| < \dfrac{\epsilon}{2B}$.

Let N_2 be so large that $n \geq N_2$ implies $|b_n - M| < \dfrac{\epsilon}{2|L|}$.

Let $N = \max\{N_1, N_2\}$.

Then, for $n \geq N$, we have

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - Lb_n + Lb_n - LM| \\ &\leq |a_n b_n - Lb_n| + |Lb_n - LM| \\ &= |a_n - L||b_n| + |L||b_n - M| \\ &< \frac{\epsilon}{2B} \cdot B + |L|\frac{\epsilon}{2|L|} \\ &= \epsilon. \end{aligned}$$

□

7. Cauchy sequences

Definition 15. Let (a_n) be a sequence of complex numbers. We say that (a_n) is a *Cauchy sequence* if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \ni m, n \geq N \Rightarrow |a_m - a_n| < \epsilon.$$

Proposition 4. *Let (a_n) be a sequence of complex numbers. If (a_n) converges, then (a_n) is a Cauchy sequence.*

Proof. Let $\lim a_n = L$.

Let $\epsilon > 0$, and let N be so large that $n \geq N$ implies that $|a_n - L| < \epsilon$.

Then, for $m, n \geq N$, we have

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Definition 16. A subset of $X \subset \mathbb{C}$ is called *complete* if every Cauchy sequence in X converges to a point in X.

Theorem 1. *The set $\mathbb{C}$ is complete.*

Proof. This follows from the fact that $\mathbb{R}$ is complete, which requires a formal definition of the real numbers. □

8. Subsequences

Definition 17. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence in a set X. A *subsequence* of (a_n) is a sequence $b : \mathbb{N} \to X$ which can be expressed as a composition $b = a \circ n$, where $n : \mathbb{N} \to \mathbb{N}$ is an increasing sequence of natural numbers. For each natural number k, we write n_k instead of $n(k)$; thus $b(k) = a(n(k)) = a(n_k) = a_{n_k}$. Thus we may write (a_{n_k}) is indicate a subsequence of (a_n).

Definition 18. Let (a_n) be a sequence of real numbers. A *subsequential limit* of (a_n) is a real number $q \in \mathbb{C}$ such that there exists a convergent subsequence (a_{n_k}) whose limit is q.

Problem 19. Let (a_n) be a sequence of real numbers and let $q \in \mathbb{C}$. Show that q is a cluster point of (a_n) if and only if q is a subsequential limit of (a_n).

Problem 20 (Bolzano-Weierstrauss Theorem)**.** Show that every bounded sequence of complex numbers has a convergent subsequence.

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