Vector Calculus
PRACTICE before Examination 2

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Problem 1. (Conics) In each case, give the definition of the word, and find an equation in two variables x and y whose locus is a conic of the given type.

(a) Circle

(b) Parabola.

(c) Ellipse.

(d) Hyperbola.

Problem 2. (Ellipses)
The locus of the equation

$$4x^2 + 24x + 9y^2 - 36y + 36 = 0.$$

is an ellipse. Find its center, vertices, and foci.

Problem 3. (Surfaces) Find an equation in three variables x, y, and z, whose locus in $\mathbb{R}^3$ is the following.

(a) A point.

(b) A line.

(c) A plane.

(d) The union of two planes.

(e) A hyperbolic cylinder.

(f) An elliptic paraboloid.

(g) A hyperbolic paraboloid.

(h) A cone.

(i) A one-sheeted hyperboloid.

(j) A two-sheeted hyperboloid.

Problem 4. (Dot and Cross Product)
Let $A = (3, 8, -2)$, $B = (-7, 3, 9)$, and $C = (2, -2, 10)$. Let $\vec{v}$ be the vector from A to B, and let $\vec{w}$ be the vector from A to C.

(a) Compute $\vec{v}$ and $\vec{w}$.

(b) Compute the dot product $\vec{v} \cdot \vec{w}$.

(c) Compute the scalar projection $\text{proj}_{\vec{w}}\vec{v}$.

(d) Compute the cross product $\vec{v} \times \vec{w}$.

Problem 5. (Lines and Planes)
Compute the indicated value(s).

(a) Find the parametric equations of the line passing through the points $P(5, -2, 8)$ and $Q(2, 4, 5)$.

(b) Find the standard equation of a plane which contains the line from part **(a)** and passes through the point $R(7, -2, 1)$.

(c) Find the distance from the point $S(-3, 1, 5)$ to the plane from part **(b)**.

Problem 6. (Intersecting Planes)
Let A be the plane given by $7x + 2y + z = 8$ and B be the plane given by $x + 2y + 7z = 8$.
Let $L = A \cap B$ be the line of intersection of A and B. Let $P_0 = (1, 1, 1)$ and note that $P_0 \in L$.
Find the equation of the plane which is perpendicular to L and passes through the point P_0, expressed in the form $ax + by + cz = d$.

Problem 7. (Paths in $\mathbb{R}^2$)
Let $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ be given by $\vec{r}(t) = \langle \sec t, \tan t \rangle$.

(a) Find the velocity vector for $\vec{r}(t)$.

(b) Find the speed at time t.

(c) Find the speed at the time $t = \dfrac{\pi}{3}$.

(d) The coordinate parametric equations for $\vec{r}$ are $x = \sec t$ and $y = \tan t$. Use this to show that the image of $\vec{r}$ lies on a hyperbola in $\mathbb{R}^2$, and sketch the image of $\vec{r}$.

Problem 8. (Curvature) Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b > 0$.

(a) Create a function $f : [-a, a] \to \mathbb{R}$ whose graph is the upper half of this ellipse.

(b) Use the formula

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

to find the curvature of the ellipse at the point $(0, b)$.

(c) Find the osculating circle to the ellipse at the point $(0, b)$.

Problem 9. (Intersecting Quadrics)
Let S be the sphere centered at the origin with radius 4. Let H be the hyperboloid with equation $x^2+y^2-z^2 = 1$. Let $C = S \cap H$; then C consists of two circles.

(a) Sketch the sphere, the hyperboloid, and their intersection in the same picture.

(b) Find the centers of the two circles.

Problem 10. (Paths on Quadrics)
Consider a path given by

$$\vec{r} : \mathbb{R} \to \mathbb{R}^3 \quad \text{given by} \quad \vec{r} = \langle \sqrt{1+t^2}\cos t, \sqrt{1+t^2}\sin t, t \rangle.$$

(a) Show that $\dfrac{dz}{dt} = 1$.

(b) Show that the image of $\vec{r}$ is a subset of the one-sheeted hyperboloid with equation $x^2 + y^2 - z^2 = 1$.

(c) Sketch the image of $\vec{r}$.

(d) Compute the position, velocity, speed, and unit tangent vector at time $t = 0$.

Problem 11. (Paths Intersect Quadrics)
Consider a path given by

$$\vec{s} : \mathbb{R} \to \mathbb{R}^3 \quad \text{given by} \quad \vec{s} = \langle 2t, 2t^2, t^3 \rangle,$$

and the one-sheeted hyperboloid with equation $x^2 - y^2 + z^2 = 1$. Find all times t when the path intersects the hyperboloid. Find a point where the path intersects the hyperboloid.

Problem 12. (Hyperboloids) [*Challenge*]
Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = x^2 + y^2 - z^2.$$

For $t \in \mathbb{R}$, the preimage of t is

$$f^{-1}(t) = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = t\}.$$

The preimage of t is a surface in $\mathbb{R}^3$. Find t such that $f^{-1}(t)$ is tangent to the sphere with equation $x^2 + y^2 + (z-3)^2 = 4$.