REAL ANALYSIS
TOPIC 34 - ALGEBRAS OF SETS

PAUL L. BAILEY

1. Sequences of Sets

Definition 1. Let X be a set. A *sequence of subsets of* X is a function $A : \mathbb{N} \to \mathcal{P}(X)$. We write A_n to mean $A(n)$, and we write (A_n) to indicate the entire sequence.

If $\mathcal{A} \subset \mathcal{P}(X)$, a *sequence in* $\mathcal{A}$ is a sequence of subsets of X such that $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$.

Let (A_n) be a sequence of subsets of X. There is a corresponding collection of subsets of X, say $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$. The reader should note a couple of distinctions between these objects: the sets in (A_n) come in a specific order, whereas the sets in $\mathcal{A}$ have no order. Also, the same set may appear multiple time in the sequence (A_n), whereas there is no notion of the multiplicity of a member of $\mathcal{A}$. However, we should note that unions and intersections may be written in two ways:

$$\cup \mathcal{A} = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \cap \mathcal{A} = \bigcap_{n=1}^{\infty} A_n.$$

If $A \subset X$, we let $A^c = X \smallsetminus A$. That is, the ambient set X is assumed to be understood in our notation.

The following properties are relatively easy to see.

Proposition 1 (Distributive Laws)**.** *Let (A_n) be a sequence of subsets of a set X. Let $B \subset X$. Then*

(a) $\left(\bigcup_{n=1}^{\infty} A_n \right) \cap B = \bigcup_{n=1}^{\infty} (A_n \cap B)$;

(b) $\left(\bigcap_{n=1}^{\infty} A_n \right) \cup B = \bigcap_{n=1}^{\infty} (A_n \cup B)$.

Proposition 2 (DeMorgan's Laws)**.** *Let (A_n) be a sequence of subsets of a set X. Then*

(a) $\left(\bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c$;

(b) $\left(\bigcap_{n=1}^{\infty} A_n \right)^c = \bigcup_{n=1}^{\infty} A_n^c$.

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Proposition 3. *Let (A_n) be a sequence of functions from a set X. Let $f : X \to Y$. Then*

(a) $f\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f(A_n)$;

(b) $f\left(\bigcap_{n=1}^{\infty} A_n\right) \subset \bigcap_{n=1}^{\infty} f(A_n)$.

Proof.

(a) ($\subset$) Let $y \in f(\cup_{n=1}^{\infty} A_n)$. Then $y = f(x)$ for some $x \in \cup_{n=1}^{\infty} A_n$. There exists $n \in \mathbb{N}$ such that $x \in A_n$, so $y \in f(A_n)$. Thus $y \in \cup_{n=1}^{\infty} f(A_n)$.

(a) ($\supset$) Let $y \in \cup_{n=1}^{\infty} f(A_n)$. Then $y \in f(A_n)$ for some $n \in \mathbb{N}$, so $y = f(x)$ for some $x \in A_n$. Now $x \in \cup_{n=1}^{\infty} A_n$, so $f(x) \in f(\cup_{n=1}^{\infty} A_n)$.

(b) ($\subset$) Let $y \in f(\cup_{n=1}^{\infty} A_n)$. Then $y = f(x)$ for some $x \in \cap_{n=1}^{\infty} A_n$. Then $x \in A_n$ for every $n \in bN$, so $y = f(x) \in f(A_n)$ for every $n \in \mathbb{N}$. Thus $y \in \cap_{n=1}^{\infty} f(A_n)$. □

Can you find an example where the inverse inclusion of **(b)** above does not hold?

Proposition 4. *Let (A_n) be a sequence of functions from a set X. Let $g : Y \to X$. Then*

(a) $g^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} g^{-1}(A_n)$;

(b) $g^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} g^{-1}(A_n)$.

Proof.

(a) ($\subset$) Let $x \in g^{-1}(\cup_{n=1}^{\infty} A_n)$, and let $y = g(x)$. Then $y \in \cup_{n=1}^{\infty} A_n$, so $y \in A_n$ for some $n \in \mathbb{N}$. Thus $x \in g^{-1}(A_n)$, so $x \in \cup_{n=1}^{\infty} g^{-1}(A_n)$.

(a) ($\supset$) Let $x \in \cup_{n=1}^{\infty} g^{-1}(A_n)$, and let $y = g(x)$. Then $x \in g^{-1}(A_n)$ for some $n \in \mathbb{N}$, so $y \in A_n$. Then $y \in \cup_{n=1}^{\infty} A_n$, so $x \in g^{-1}(\cup_{n=1}^{\infty} A_n)$.

(b) ($\subset$) Let $x \in g^{-1}(\cap_{n=1}^{\infty} A_n)$, and let $y = g(x)$. Then $y \in \cap_{n=1}^{\infty} A_n$, so $y \in A_n$ for every $n \in \mathbb{N}$. Thus $x \in g^{-1}(A_n)$ for every $n \in \mathbb{N}$, so $x \in \cap_{n=1}^{\infty} g^{-1}(A_n)$.

(b) ($\supset$) Let $x \in \cap_{n=1}^{\infty} g^{-1}(A_n)$, and let $y = g(x)$. Then $x \in g^{-1}(A_n)$ for every $n \in \mathbb{N}$, so $y \in A_n$ for every $n \in \mathbb{N}$. Then $y \in \cap_{n=1}^{\infty} A_n$, so $x \in g^{-1}(\cap_{n=1}^{\infty} A_n)$. □

2. Monotone Sequences

Definition 2. Let (A_n) be a sequence of subsets of a set X.

We say that (A_n) is *increasing* (or *nondecreasing*, or *expanding*) if $A_k \subset A_{k+1}$, for all $k \in \mathbb{N}$.

We say that (A_n) is *decreasing* (or *nonincreasing*, or *contracting*) if $A_k \supset A_{k+1}$, for all $k \in \mathbb{N}$.

We say that (A_n) is *monotone* if it is either increasing or decreasing.

Problem 1. Let (A_n) be a sequence of subsets of a set X.

(a) Show that if (A_n) is increasing, then $\cap_{n=k}^{\infty} A_n = A_k$.
(b) Show that if (A_n) is decreasing, then $\cup_{n=k}^{\infty} A_n = A_k$.
(c) Show that if (A_n) is decreasing if and only if (A_n^c) is increasing sequence.

Problem 2. Let (A_n) be a sequence of subsets of a set X. Define two new sequences of sets,

- $\underline{A}_n = \cap_{j=n}^{\infty} A_j$.
- $\overline{A}_n = \cup_{j=n}^{\infty} A_j$.

(a) Show that $(\underline{A}_n)$ is a increasing sequence of sets.
(b) Show that $(\overline{A}_n)$ is an decreasing sequence of sets.

Problem 3. Let (A_n) be a sequence of subsets of a set X.

(a) Show that, for all $n \in \mathbb{N}$, we have

$$\underline{A}_n \subset A_n \subset \overline{A}_n.$$

(b) Find a sequence of sets (A_n) such that
- $A_i \neq A_j$ for $i \neq j$, and
- $\underline{A}_i \not\subseteq A_i \not\subseteq \overline{A}_i$.

3. Limits of Sequences of Sets

Definition 3. Let (A_n) be a sequence of subsets of a set X.

The *limit inferior* of (A_n) is

$$\liminf A_n = \cup_{i=1}^{\infty} \cap_{j=i}^{\infty} A_j.$$

The *limit superior* of (A_n) is

$$\limsup A_n = \cap_{i=1}^{\infty} \cup_{j=i}^{\infty} A_j.$$

An alternative notation is used by some books: let $\underline{\lim}\, A_n = \liminf A_n$ and $\overline{\lim}\, A_n = \limsup A_n$. We may call $\underline{\lim}\, A_n$ the *lower limit* and $\overline{\lim}\, A_n$ the *upper limit.*

Problem 4. Let (A_n) be a sequence of subsets of a set X.

(a) Show that $\underline{\lim}\, A_n = \lim \underline{A}_n$.
(b) Show that $\overline{\lim}\, A_n = \lim \overline{A}_n$.

Proposition 5. *Let (A_n) be a sequence of subsets of a set X. Show that*

(a) $\liminf A_n = \{x \in X \mid x \in A_n$ *for all but finitely many* $n \in \mathbb{N}\}$*;*
(b) $\limsup A_n = \{x \in X \mid x \in A_n$ *for infinitely many* $n \in \mathbb{N}\}$*;*
(c) $\liminf A_n \subset \limsup A_n$.

Proof.

(a) ($\subset$) Suppose that $x \in A_n$ for all but finitely many n. Then, let $N \in \mathbb{N}$ be so large that $x \in A_n$ for $n \geq N$. Then $x \in \cap_{j=N}^{\infty} A_j = \underline{A}_N$, so $x \in \cup_{i=1}^{\infty} \underline{A}_i = \liminf A_n$.

(a) ($\supset$) Suppose that $x \in \liminf A_n$. Then $x \in \cup_{i=1}^{\infty} \underline{A}_i$, so $x \in \underline{A}_i$ for some $i \in \mathbb{N}$. But $\underline{A}_i = \cap_{j=i}^{\infty} A_j$, so $x \in A_j$ for all $j \geq i$. Thus $x \in A_n$ for all but finitely many n.

(b) ($\subset$) Suppose that $x \in A_n$ for infinitely many n. Then for every $i \in \mathbb{N}$, there exists $n \geq N$ such that $n \geq i$ implies $x \in A_n$. Thus for every $i \in \mathbb{N}$, $x \in \cup_{j=i}^{\infty} A_i = \overline{A}_i$. Thus $x \in \cap_{i=1}^{\infty} \overline{A}_i = \limsup A_n$.

(b) ($\supset$) Suppose that $x \in \limsup A_n$. Then $x \in \cap_{i=1}^{\infty} \overline{A}_i$, so $x \in \overline{A}_i = \cup_{j=i}^{\infty}$ for all i. Thus, for every $i \in \mathbb{N}$, there exists $n \geq i$ such that $x \in A_n$, which implies that $x \in A_n$ for infinitely many $n \in \mathbb{N}$.

(c) Let $x \in \liminf A_n$. Then $x \in A_n$ for all but finitely many $n \in \mathbb{N}$; since $\mathbb{N}$ is infinitely, this implies that $x \in A_n$ for infinitely many $n \in \mathbb{N}$, so $x \in \limsup A_n$. $\square$

Problem 5. Let $X = \mathbb{R}$. Define a sequence (A_n) of subsets X by

$$A_n = \begin{cases} \left[0, \dfrac{1}{n}\right] & \text{if } n \text{ is odd ;} \\ [0, n] & \text{if } n \text{ is even .} \end{cases}$$

Find $\liminf A_n$ and $\limsup A_n$.

Problem 6. Let $X = [0, 1] \subset \mathbb{R}$. Define a sequence (A_n) of subsets X by

$$A_n = \left\{ \frac{m}{n} \mid m \in \mathbb{Z} \text{ and } 0 \leq m \leq n \right\}.$$

Find $\liminf A_n$ and $\limsup A_n$.

Problem 7. Let $X = \mathbb{R}$. Define a sequence (A_n) of subsets X by

$$a_n = 4 \sin^2 \frac{2\pi n}{3} \text{ and } A_n = [a_n - 1, a_n + 1].$$

Find $\liminf A_n$ and $\limsup A_n$.

Problem 8. Let (A_n) be a sequence of subsets of a set X. Show that

$$\liminf A_n = (\limsup A_n^c)^c.$$

Definition 4. Let (A_n) be a sequence of subsets of a set X.

We say that (A_n) *converges* if $\liminf A_n = \limsup A_n$. In this case, the *limit* of (A_n) is

$$\lim A_n = \liminf A_n = \limsup A_n.$$

If we claim that $\lim A_n = L$, we mean that (A_n) converges, and that the limit of (A_n) is L.

Problem 9. Let (A_n) be a sequence of subsets of a set X.

(a) Show that if (A_n) is decreasing, then $\lim A_n = \cap_{i=1}^{\infty} A_i$.

(b) Show that if (A_n) is increasing, then $\lim A_n = \cup_{i=1}^{\infty} A_i$.

Problem 10. Let (A_n) and (B_n) be sequences of subsets of a set X. Show that

$$(\underline{\lim}\, A_n \cup \underline{\lim}\, B_n) \subset \underline{\lim}(A_n \cup B_n) \subset (\underline{\lim}\, A_n \cup \overline{\lim}\, B_n) \subset \overline{\lim}(A_n \cup B_n) \subset (\overline{\lim}\, A_n \cup \overline{\lim}\, B_n).$$

4. Sigma Algebras

Definition 5. Let X be a set and let $\mathcal{A} \subset \mathcal{P}(X)$. We say that $\mathcal{A}$ is an *algebra* of subsets of X if

(A0) $X \in \mathcal{A}$;
(A1) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$;
(A2) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$, where $A^c = X \setminus A$.

Proposition 6. *Let $\mathcal{A}$ be an algebra of subsets of X. Then*

(A3) $A, B \in \mathcal{A}$ *implies* $A \cap B \in \mathcal{A}$.

Proof. Let $A, B \in \mathcal{A}$. Then $A^c, B^c \in \mathcal{A}$ by **(A2)**, and $A^c \cup B^c \in \mathcal{A}$ by **(A1)**. The by DeMorgan's Law and **(A2)** again,

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{A}.$$

□

Proposition 7. *Let $\mathfrak{A}$ be a collection of algebras of subsets of a set X. Then $\cap\mathfrak{A}$ is an algebra of subsets of X.*

Proof. Let $A, B \in \cap\mathfrak{A}$. Then $A, B \in \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{A}$. Since each $\mathcal{A}$ in $\mathfrak{A}$ is an algebra, $A \cup B$ and A^c are int $\mathcal{A}$, for every $\mathcal{A}$ in $\mathfrak{A}$. So, $A \cup B$ and A^c are in $\cap\mathfrak{A}$. □

Definition 6. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. The *algebra generated by* $\mathcal{C}$ is

$$\langle \mathcal{C} \rangle = \cap\{\mathcal{A} \subset \mathcal{P}(X) \mid \mathcal{A} \text{ is an algebra which contains } \mathcal{C}\}.$$

One sees that the algebra generated by $\mathcal{C}$ is the smallest algebra which contains all the sets in $\mathcal{C}$.

Proposition 8. *Let $\mathcal{A}$ be an algebra of subsets of X, and let (A_n) be a sequence of sets in $\mathcal{A}$. Then there exists a sequence (B_n) of sets in $\mathcal{A}$ such that $B_j \cap B_k = \varnothing$ if $j \neq k$, and*

$$\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i.$$

Proof. Define

$$B_n = A_n \setminus \left(\cup_{i=1}^{n-1} A_n \right).$$

Since $B_n \subset A_n$, it is clear that

$$\cup_{i=1}^{\infty} B_i \subset \cup_{i=1}^{\infty} A_i.$$

Let $x \in \cup_{i=1}^{\infty} A_i$. Then $x \in A_i$ for some i; let n denote the smallest positive integer such that $x \in A_n$. Then $x \in A_n \setminus (\cup_{i=1}^{n-1} A_i)$, so $x \in B_n$. Thus $x \in \cup_{i=1}^{\infty} B_i$, so

$$\cup_{i=1}^{\infty} A_i \subset \cup_{i=1}^{\infty} B_i,$$

which implies that

$$\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i.$$

Suppose that $x \in B_j \cap B_k$ for some $j < k$; then $x \in B_j$, so $x \in A_j$. But then $x \in \cup_{i=1}^{k-1} A_i$, so $x \notin A_k \setminus (\cup_{i=1}^{k-1} A_n) = B_k$, a contradiction. Thus $B_j \cap B_k = \varnothing$. □

Definition 7. Let X be a set and let $\mathcal{A} \subset \mathcal{P}(X)$. We say that $\mathcal{A}$ is a *σ-algebra* of subsets of X if

- **(S0)** $X \in \mathcal{A}$;
- **(S1)** if $\mathcal{C} \subset \mathcal{A}$ is countable, then $\cup\mathcal{C} \in \mathcal{A}$;
- **(S2)** if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

That is, a σ-algebra is an algebra which is not only closed under finite unions, but is also closed under countable unions.

Proposition 9. *Let $\mathcal{A}$ be an σ-algebra of subsets of X. Then*

- **(S3)** *if $\mathcal{C} \subset \mathcal{A}$ is countable, then $\cap\mathcal{C} \in \mathcal{A}$.*

Proof. DeMorgan's Law also applies to infinite collections; let $\mathcal{C} \subset \mathcal{A}$ be countable. Then

$$\cap\mathcal{C} = \cap_{A\in\mathcal{C}} A = (\cup_{A\in\mathcal{C}} A^c)^c.$$

Now if $A \in \mathcal{C}$, then $A \in \mathcal{A}$, so $A^c \in \mathcal{A}$. Thus $\cup_{A\in\mathcal{C}} A^c$ is a countable union of sets in $\mathcal{A}$, and so is in $\mathcal{A}$. Thus its complement $\cap\mathcal{C}$ is in $\mathcal{A}$. $\square$

Proposition 10. *Let $\mathfrak{A}$ be a collection of σ-algebras of subsets of a set X. Then $\cap\mathfrak{A}$ is an σ-algebra of subsets of X.*

Proof. Exercise. $\square$

Definition 8. Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$. The *$\sigma$-algebra generated by* $\mathcal{C}$, denoted $\langle\mathcal{C}\rangle$, is the intersection of all σ-algebras which contain $\mathcal{C}$.

We see that $\langle\mathcal{C}\rangle$ is necessarily a σ-algebra, and is the smallest σ-algebra which contains all of the sets in the collection $\mathcal{C}$.

Proposition 11. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. Let (A_n) be a sequence in $\mathcal{A}$. Then*

- **(a)** $\underline{A}_n, \overline{A}_n \in \mathcal{A}$*;*
- **(b)** $\liminf A_n, \limsup A_n \in \mathcal{A}$.

Proof. Since $\underline{A}_n = \cup_{i=n}^{\infty}$ is a union of countable collection from $\mathcal{A}$, we know that $\underline{A}_n \in \mathcal{A}$. Also, $\overline{A}_n = \cup_{i=n}^{\infty} A_i$ is the union of a countable collection, so $\overline{A}_n \in \mathcal{A}$.

Now $\liminf A_n = \cup_{i=1}^{\infty} \underline{A}_n$, so $\liminf A_n$ is a countable union of sets in $\mathcal{A}$, so $\liminf A_n \in \mathcal{A}$. Similarly, $\limsup A_n = \cap_{i=1}^{\infty} \overline{A}_n$, so $\limsup A_n$ is a countable intersection of sets in $\mathcal{A}$, and so is in $\mathcal{A}$. $\square$

Department of Mathematics and CSci, BASIS Scottsdale
Email address: `paul.bailey@basised.com`