PRINCIPLES OF ANALYSIS
METRIC SPACES I - METRICS

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Abstract. The document introduces the initial definitions regarding metric spaces, subspaces, open and closed sets, and product spaces.

1. Metric Spaces

Definition 1. Let X be a set. A *metric* on X is a function

$$\rho : X \times X \to \mathbb{R}$$

satisfying

(M1) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$ (Positivity);
(M2) $\rho(x, y) = \rho(y, x)$ (Symmetry);
(M3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (Triangle Inequality).

The pair (X, ρ) is called a *metric space*.

Example 1. Let X be any set and define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise .} \end{cases}$$

Then ρ is a metric on X, called the *discrete metric*, and (X, ρ) is called a *discrete metric space*.

Proof. In order to prove that (X, ρ) is a metric space, we need to demonstrate **(M1)**, **(M2)**, and **(M3)**. We take the opportunity of this simple example to discuss some general assumptions we can make in the course of such a proof.

(M1) We have two things to verify; that the image of ρ consists only of nonnegative reals, and that $\rho(x, x) = 0$ for every $x \in X$. The first is immediate upon inspection of the definition, and in general won't need to be mentioned unless there is some doubt. The second is directly provided by the definition.

(M2) Let $x, y \in X$. If $x = y$, then $\rho(x, y) = 0 = \rho(y, x)$. Note that if **(M1)** if already verified, then this case need not be considered.

Thus suppose that x and y are distinct. Then $\rho(x, y) = 1 = \rho(y, x)$.

(M3) Let $x, y, z \in X$. If $x = z$, and **(M1)** is already verified, then this condition says that $0 \leq \rho(x, y) + \rho(y, z)$, which is true. If $x = y$ or $y = z$, this statement becomes an immediate equality. So again, we can assume that x, y, and z are distinct. Then

$$\rho(x, z) = 1 < 2 = \rho(x, y) + \rho(y, z).$$

□

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Example 2. Let $x = \mathbb{R}$ and define $\rho(x, y) = |x - y|$. Then (X, ρ) is a metric space.

Proof. We address **(M1)**, **(M2)**, and **(M3)**.

(M1) The absolute value is always nonnegative, and the absolute value of zero is zero, so $\rho(x, x) = |x - x| = |0| = 0$. On the other hand, if $x \neq y$, then $x - y \neq 0$, so $\rho(x, y) = |x - y| \neq 0$.

(M2) Let $x, y \in \mathbb{R}$; without loss of generality, assume that $x > y$. Then $\rho(x, y) = |x - y| = x - y = -(y - x) = |y - x| = \rho(y, x)$.

(M3) Let $x, y, z \in \mathbb{R}$; we have seen that $|a + c| \leq |a + b| + |b + c|$ for every $a, b, c \in \mathbb{R}$. Set $a = x - y$, $b = 0$, and $c = y - z$. Then $a + c = x - y$, $a + b = x - y$, and $b + c = y - z$. Thus

$$\rho(x, z) = |x - z| \leq |x - y| + |y - z| = \rho(x, y) + \rho(y, z).$$

□

Example 3. Let $X = \mathbb{R}^k$ and define

$$\rho(x, y) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2},$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$.

Remark. The positivity and symmetry of ρ are clear, but the proof of the triangle inequality is involved, and appears in the last section, where it is generalized to the product of a finite number of metric spaces. □

Example 4. Let $\mathbb{R}^\infty$ denote the set of all sequences of real numbers that are eventually zero, that is, sequences (x_n) such that $x_n = 0$ for all but finitely many n. Let $X = \mathbb{R}^\infty$ and for $x, y \in X$, define

$$\rho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

where $x = (x_n)$ and $y = (y_n)$. TThis make sense, since there are only finitely many nonzero summands. Then (X, ρ) is a metric space.

Example 5. Let ℓ^2 denote the set of all sequences of real numbers (x_n) that satisfy the converge criterion

$$\sum_{i=1}^{\infty} x_i^2 < 0.$$

Let $X = \ell^2$ and for $x, y \in X$, define

$$\rho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

where $x = (x_n)$ and $y = (y_n)$. That this series converges follows from the inequality

$$(a \pm b)^2 \leq 2(a^2 + b^2),$$

which the reader is welcome to verify. Then (X, ρ) is a metric space.

Example 6. Let $X = \mathbb{Q}$ and let p be a positive prime integer. For each $x \in \mathbb{Q}$, there exists unique $m, n, \alpha \in \mathbb{Z}$ such that $x = p^\alpha \frac{m}{n}$, where $\gcd(m, n) = 1$ and p does not divide m or n. The *p-adic norm* of x is $|x|_p = \frac{1}{p^\alpha}$. Set $\rho(x, y) = |x - y|_p$. Then (X, ρ) is a metric, known as the *p-adic metric* on $\mathbb{Q}$. Here one can show that not only does ρ satisfy the triangle inequality, but also the stronger inequality $|x - y|_p \leq \max\{|x|_p, |y|_p\}$.

Problem 1. Let $\mathcal{F}_{[a,b]}$ denote the set of all bounded functions $f : [a, b] \to \mathbb{R}$. Let $X = \mathcal{F}_{[a,b]}$ and for $f, g \in X$ define

$$\rho(f, g) = \max\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

Show that (X, ρ) is a metric space.

Problem 2. Let $\mathcal{C}_{[a,b]}$ denote the set of all continuous functions $f : [a, b] \to \mathbb{R}$. Let $X = \mathcal{C}_{[a,b]}$ and for $f, g \in X$ define

$$\rho(f, g) = \int_a^b |f - g| \, dx.$$

Show that (X, ρ) is a metric space.

Solution. In order to prove this, we will need these properties of integration:

Lemma 1. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable, and let $c \in \mathbb{R}$. Then*

(a) $\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$;
(b) $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$.

Lemma 2. *Let $f : [a, b] \to [, \infty)$ be a continuous function. If $\int_a^b f(x)\, dx = 0$, then $f(x) = 0$ for every $x \in [a, b]$.*

Lemma 3. *Let $f, g : [a, b] \to [0, \infty)$ be continuous functions. If $f(x) \leq g(x)$ for every $x \in [a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$.*

We have $\rho(f, f) = \int_a^b |f - f|\, dx = \int_a^b 0\, dx = 0$; moreover, Lemma 2 tells us that if the integral of a nonnegative continuous function is zero, then that function is the zero function; thus $\rho(f, g) = 0 \Rightarrow |f - g| = 0 \Rightarrow f = g$. Thus **(M1)** follows.

Since $|f - g| = |g - f|$, clearly $\rho(f, g) = \rho(g, f)$. Thus **(M2)** follows.

Let $f, g, h \in \mathcal{C}[a, b]$. Then $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$ for every $x \in [a, b]$, by the triangle inequality for $\mathbb{R}$. Then by Lemmas 3 and 1,

$$\begin{aligned}
\rho(f, h) &= \int_a^b |f(x) - h(x)|\, dx \\
&\leq \int_a^b \Big(|f(x) - g(x)| + |g(x) - h(x)|\Big)\, dx \\
&= \int_a^b |f(x) - g(x)|\, dx + \int_a^b |g(x) - h(x)|\, dx \\
&= \rho(f, g) + \rho(g, h).
\end{aligned}$$

□

2. Subspaces

Definition 2. Let $f : A \to B$ be a function, and let $C \subset A$. The *restriction* of f to C is a function

$$f \restriction_C : C \to B \quad \text{given by} \quad f \restriction_C (c) = f(c).$$

It is the same function with a restricted domain.

Definition 3. Let (X, ρ) be a metric space and let $A \subset X$. Define a function $\rho_A : A \times A \to \mathbb{R}$ by $\rho_A = \rho \restriction_{A \times A}$. Then ρ_A is a metric on A, and (A, ρ_A) is called a *subspace* of (X, ρ).

Example 7. Let $X = \mathbb{R}^2$, and define

$$\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \quad \text{by} \quad \rho(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where $p_i = (x_i, y_i)$. Then ρ is the standard metric on $\mathbb{R}^2$.

Define

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We call $\mathbb{S}^1$ the *unit circle*. It inherits the metric $\rho_{\mathbb{S}^1}$ from $(\mathbb{R}^2, \rho)$.

Define

$$\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

We call $\mathbb{D}^2$ the *(closed) unit disk*, and $(\mathbb{D}^2, \rho_{\mathbb{D}^2})$ is a metric space.

Example 8. Let $\mathbb{S}^1$ be the unit circle, and let ρ be as in Example 7. We may define a metric

$$\alpha : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R} \quad \text{by} \quad \alpha(p_1, p_2) = 2 \arcsin(\rho(p_1, p_2)).$$

where $p_1, p_2 \in \mathbb{S}^1$. Then $\alpha(p_1, p_2)$ is the angle, measured in radians, from p_1 to the origin and then to p_2; this is the arclength of the shortest path between these two points.

This produces a different metric on $\mathbb{S}^1$. In due course, we will investigate the relationship between these metrics and related consequences for the structure of the metric space.

Example 9. Let $X = \mathbb{R}^3$, and define

$$\rho : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \quad \text{by} \quad \rho(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where $p_i = (x_i, y_i, z_i)$. Then ρ is the standard metric on $\mathbb{R}^3$.

Define

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 + z^2 = 1\}.$$

We call $\mathbb{S}^2$ the *unit sphere*, and $(\mathbb{S}^2, \rho_{\mathbb{S}^2})$ is a metric space.

Define

$$\mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

We call $\mathbb{D}^2$ the *(closed) unit ball*, and $(\mathbb{D}^3, \rho_{\mathbb{D}^3})$ is a metric space.

3. Diameter

Definition 4. Let (X,ρ) be a metric space, and let $A \subset X$. Define the *diameter* of A with respect to ρ to be

$$\operatorname{diam}(A) = \sup\{\rho(a,b) \mid a, b \in A\};$$

by convention, the diameter of an empty set is zero. Note that $\operatorname{diam}(A)$ is an extended real number which can be ∞.

We say that A is *bounded* if $\operatorname{diam}(A) < \infty$.

Proposition 1. *Let (X,ρ) be a metric space, and let $A, B \subset X$. Then*

(a) $\operatorname{diam}(A) = 0 \Leftrightarrow |A| \leq 1$;
(b) $A \subset B \Rightarrow \operatorname{diam}(A) \leq \operatorname{diam}(B)$;
(c) $A \cap B \neq \varnothing \Rightarrow \operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B)$.

Proof. Recall that $|A|$ is the *cardinality* of A, and is defined to be the number of elements in A. If A contains at least two distinct elements, the distance between them is positive, so the diameter of A is greater than 0. On the other hand, if A contains exactly one element, say $A = \{a\}$, then $\{\rho(a,b) \mid a, b \in A\} = \{\rho(a,a)\} = \{0\}$, and the supremum of this set is zero.

Suppose $A \subset B \subset X$. Set $S_A = \{\rho(a_1,a_2) \mid a_1, a_2 \in A\}$, and $S_B = \{\rho(b_1,b_2) \mid b_1, b_2 \in B\}$. Clearly $S_A \subset S_B$, so $\operatorname{diam}(A) = \sup(S_A) \leq \sup(S_B) = \operatorname{diam}(B)$.

Finally, suppose that $A, B \subset X$ and that $A \cap B \neq \varnothing$. Suppose that $\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B)$, and let $\epsilon = \frac{1}{2}(\operatorname{diam}(A \cup B) - (\operatorname{diam}(A) + \operatorname{diam}(B)))$. Then, from the definition of diameter, there exist points $c_1, c_2 \in A \cup B$ such that $\operatorname{diam}(A \cup B) - \rho(c_1,c_2) > \epsilon$. □

Problem 3. Let (X,ρ) be a metric space, and let $G = \operatorname{diam}(A)$ with respect to ρ. Define a function

$$\widehat{\rho} : X \times X \to \mathbb{R} \quad \text{by} \quad \widehat{\rho}(x,y) = \frac{\rho(x,y)}{1 + \rho(x,y)}.$$

(a) Show that $\widehat{\rho}$ is a metric on X.

Let $H = \operatorname{diam}(X)$ with respect to $\widehat{\rho}$.

(b) Show that $H \leq 1$.
(c) Show that if $G = \infty$, then $H = 1$.
(d) Show that if X is finite, then $H = \frac{G}{1+G}$.

Solution. Let $x, y, z \in X$; we wish to show that

$$\widehat{\rho}(x,z) \leq \widehat{\rho}(x,y) + \widehat{\rho}(y,z).$$

Let $a = \rho(x,y)$, $b = \rho(y,z)$ and $c = \rho(x,z)$. Then we wish to show that $a, b, c \geq 0$ and $c \leq a + b$ imply

$$\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Now

$$\begin{aligned}
c \leq a + b &\Rightarrow c \leq (a+b) + (2ab + abc) \quad \text{ since } a, b, c \geq 0 \\
&\Rightarrow c + ac + bc + abc \leq (a+b) + (2ab+abc) + (ac+bc+abc) \\
&\Rightarrow c(1+a+b+ab) \leq a(1+b+c+bc) + b(1+a+c+ac) \\
&\Rightarrow c(1+a)(1+b) \leq a(1+b)(1+c) + b(1+a)(1+c) \\
&\Rightarrow \frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.
\end{aligned}$$

Let $x, y \in X$. Then

$$\widehat{\rho}(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} < \frac{\rho(x,y)}{\rho(x,y)} = 1;$$

thus $H = \sup\{\widehat{(\rho)}(x,y) \mid x, y \in X\} \leq 1$.

Suppose that $G = \infty$, and let $\epsilon > 0$. Then there exist $x, y \in X$ such that $\rho(x,y) > \frac{1}{\epsilon} - 1$. Now

$$\begin{aligned}
\rho(x,y) > \frac{1}{\epsilon} - 1 &\Leftrightarrow 1 + \rho(x,y) > \frac{1}{\epsilon} \\
&\Leftrightarrow \frac{1}{1+\rho(x,y)} < \epsilon \\
&\Leftrightarrow 1 - \frac{\rho(x,y)}{1+\rho(x,y)} < \epsilon \\
&\Leftrightarrow 1 - \widehat{\rho}(x,y) < \epsilon.
\end{aligned}$$

Since this is true for every epsilon, Thus $H = \sup\{\widehat{(}\rho(x,y) \mid x, y \in X\} \geq 1$. Combined with part **(b)**, we have $H = 1$.

Suppose that X is finite. Then the set $\{\rho(x,y) \mid x, y \in X\}$ is also finite, and thus has a maximum, and this maximum is equal to G. Then there exist $a, b \in X$ such that $\rho(a,b) = G$. Since $f(x) = \frac{x}{1+x}$ is an increasing function, $\rho(a,b) \geq \rho(c,d)$ implies that $\widehat{\rho}(a,b) \geq \widehat{\rho}(c,d)$. Thus

$$\frac{G}{1+G} = \widehat{\rho}(a,b) = \max\{\widehat{\rho}(x,y) \mid x, y \in X\} = H.$$

$\square$

4. Open Sets

Definition 5. Let (X, ρ) be a metric space. Let $x_0 \in X$ and let $\delta > 0$. Set

$$B(x_0, \delta) = \{x \in X \mid \rho(x, x_0) < \delta\};$$

this is known as an *open ball about x_0 of radius δ.*

Let $U \subset X$. We say that U is *open* if

$$\forall u \in U \exists \delta > 0 \mid B(x_0, \delta) \subset U.$$

Proposition 2. *Let (X, ρ) be a metric space, and let $A \subset X$. Then A is open if and only if A can be expressed as a union of open balls.*

Proof. Suppose that A is open; then for every $a \in A$ there exists $\delta_a > 0$ such that $B(a, \delta_a) \subset A$. Then

$$A = \cup_{a \in A} B(a, \delta_a),$$

so A is a union of open balls.

On the other hand, suppose that A is the union of open balls. Let $a \in A$. Then $a \in B(x, \delta)$ for some $x \in X$ and $\delta > 0$, where $B(x, \delta) \subset A$. Then $B(a, \delta - \rho(x, a)) \subset B(x, \delta)$. To see this, let $b \in B(a, \delta - \rho(x, a))$. Then the triangle inequality implies that

$$\rho(x, b) \leq \rho(x, a) + \rho(a, b) \leq \rho(x, a) + (\delta - \rho(x, a)) = \delta.$$

Thus $B(a, \delta) \subset B(x, \delta) \subset A$, and A satisfies the definition of an open set. $\square$

Proposition 3. *Let (X, ρ) be a metric space. Then*

- **(a)** *The sets $\varnothing$ and X are open.*
- **(b)** *The union of any collection of open subsets of X is open.*
- **(c)** *The intersection of any finite collection of open subsets of X is open.*

Proof. The empty set vacuously satisfies the condition for openness; every $x \in \varnothing$ has an open ball contained in $\varnothing$, because there is no $x \in \varnothing$. If $x \in X$, then $B(x, 1) \subset X$ by definition of $B(x, 1)$.

Suppose that $\{U_\alpha \mid \alpha \in I\}$ is a collection of open subsets of X indexed by the indexing set I. Let $U = \cup_{\alpha \in I} U_\alpha$. Let $x \in U$. Then $x \in U_\alpha$ for some $\alpha \in I$. Since U_α is open, $B(x, \delta) \subset U_\alpha$ for some $\delta > 0$. Then $B(x, \delta) \subset U$, since $U_\alpha \subset U$. Thus U is open.

Suppose that $\{U_1, \dots, U_n\}$ is a finite collection of open subsets of X. Let $U = \cap_{i=1}^n U_i$, and let $x \in U$. Then $x \in U_i$ for $i = 1, \dots, n$. Since each of these is open, there exist positive real number $\delta_1, \dots, \delta_n$ such that $x \in B(x, \delta_i)$ for $i = 1, \dots, n$.

Set $\delta = \min\{\delta_1, \dots, \delta_n\}$. Then $B(x, \delta) \subset U_i$ for $i = 1, \dots, n$. Thus $B(x, \delta) \subset \cap_{i=1}^n U_i = U$. In this way, we see that U is open. $\square$

Definition 6. Let X be a set. A *topology* on X is a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ satisfying

- **(T1)** $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
- **(T2)** if $\mathcal{U} \subset \mathcal{T}$, then $\cup \mathcal{U} \in \mathcal{T}$;
- **(T3)** if $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ is finite, then $\cap \mathcal{U} \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. The pair $(X, \mathcal{T})$ is called a *topological space*.

Observation 1. If (X, ρ) is a metric space, then the collection of open subsets of X is a topology on X.

5. Closed Sets

Definition 7. Let (X,ρ) be a metric space. Let $F \subset X$. We say that F is *closed* if $X \setminus F$ is open.

Warning 1. Just because a set is not open does not mean that it is closed. For example, $[1,2) \subset \mathbb{R}$ is neither.

Proposition 4. *Let (X,ρ) be a metric space. Then*

- **(a)** *The sets $\varnothing$ and X are closed.*
- **(b)** *The intersection of any collection of closed subsets of X is closed.*
- **(c)** *The union of any collection of closed subsets of X is closed.*

Proof. Recall *DeMorgan's Laws*, which state that the union of complements is the complement of the intersection, and the intersection of complements is the complement of the union. Then Proposition 4 follows from Proposition 3 and DeMorgan's Laws. $\square$

6. Interior, Closure, and Boundary

Definition 8. Let (X,ρ) be a metric space, and let $A \subset X$. The *interior* of A is the union of all open subsets of A:

$$A^\circ = \bigcap_{\substack{U \subset A \\ U \text{ is open}}} U.$$

Since the union of open sets is open, this is clearly the largest open subset of A.

Proposition 5. *Let (X,ρ) be a metric space and let $A \subset X$. Then A is open if and only if $A = A^\circ$.*

Definition 9. Let (X,ρ) be a metric space, and let $A \subset X$. The *closure* of A is the intersection of all closed subsets of X which contain A:

$$\overline{A} = \bigcap_{\substack{A \subset F \\ F \text{ is closed}}} F.$$

Since the intersection of closed sets is closed, this is clearly the smallest closed subset of X which contains A.

Proposition 6. *Let (X,ρ) be a metric space and let $A \subset X$. Then A is closed if and only if $A = \overline{A}$.*

Definition 10. Let (X,ρ) be a metric space and let $A \subset X$. The *boundary* of A is $\partial A = \overline{A} \setminus A^\circ$.

7. Neighborhoods

Definition 11. Let (X, ρ) be a metric space and let $x \in X$. A *basic open neighborhood* of x is an open ball of the form $B(x, \delta)$ for some $\delta > 0$. An *open neighborhood* of x is any open subset of X which contains x. A *neighborhood* of x is any subset of X which contains an open neighborhood of x.

Problem 4. Let (X, ρ) be a metric space. Let $x \in X$ and let $A, B \subset X$ be neighborhoods of x. Show that $A \cap B$ is a neighborhood of x.

Solution. Since A and B are neighborhoods of x, each contains an open set which contains x; say $x \in U \subset A$ and $x \in V \subset B$ with U and V open. Then $U \cap V$ is open, contains x, and is a subset of $A \cap B$. Thus $A \cap B$ is a neighborhood. □

Definition 12. If A and B are sets, we say that A *intersects* B if $A \cap B \neq \varnothing$.

Let (X, ρ) be a metric space. Let $A \subset X$ and $x \in X$.

We say that x is an *interior point* of A if there exists a neighborhood of x which is contained in A.

We say that x is a *closure point* of A if for every neighborhood of x intersects A.

We say that x is a *boundary point* of A if for every neighborhood of x intersects both A and $X \smallsetminus A$.

Proposition 7. *Let (X, ρ) be a metric space. Let $A \subset X$ and $x \in X$. Then*

- **(a)** *x is an interior point of A if and only if $x \in A^\circ$;*
- **(b)** *x is a closure point of A if and only if $x \in \overline{A}$;*
- **(c)** *x is a boundary point of A if and only if $x \in \partial A$.*

Definition 13. Let (X, ρ) be a metric space. Let $A \subset X$ and let $x \in X$.

A *deleted neighborhood* of x is a subset $V \subset X$ such that $V = U \smallsetminus \{x\}$ for some neighborhood U of x.

We say that x is an *isolated point* of A if every deleted neighborhood of x is contained in $X \smallsetminus A$.

We say that x is an *accumulation point* of A if every deleted neighborhood of x intersects A.

Proposition 8. *Let (X, ρ) be a metric space. Let $A \subset X$. Set*

$$B = \{x \in X \mid x \text{ is an isolated point of } A\};$$

$$C = \{x \in X \mid x \text{ is an accumulation point of } A\}.$$

Then $\overline{A} = B \cup C$.

8. Product Metric Spaces

The definition of distance in $\mathbb{R}^k$ has been computed using $k-1$ applications of the Pythagorean Theorem; it is clearly the definition that we want. However, in order to apply results to $\mathbb{R}^k$ that we have proven from the metric axioms, we need to first prove that $\mathbb{R}^k$ is indeed a metric space. This involves a demonstration of the triangle inequality; that is, given

$$x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k), z = (z_1, \ldots, z_k),$$

we need to show that

$$\sqrt{\sum_{j=1}^{k}(x_j - z_j)^2} \leq \sqrt{\sum_{j=1}^{k}(x_j - y_j)^2} + \sqrt{\sum_{j=1}^{k}(y_j - z_j)^2}.$$

Proving this directly would make use of the triangle inequality

$$|a - c| \leq |a - b| + |b - c|$$

in $\mathbb{R}$, and an application of the Cauchy-Schwartz Inequality (below). With approximately the same effort, we can generalize this result to construct the product of a finite number of arbitrary metric spaces. The definition of distance in the product space is motivated by our previous use of the Pythagorean Theorem.

Theorem 1. *Let $(X_1, \rho_1), \ldots, (X_n, \rho_n)$ be a finite collection of metric spaces. Let $X = \times_{k=1}^{n} X_k$, and define $\rho : X \times X \to \mathbb{R}$ by*

$$\rho(x, y) = \sqrt{\sum_{k=1}^{n} \rho_k(x_k, y_k)^2},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, and $x_k, y_k \in X_k$ for $k = 1, \ldots, n$. Then (X, ρ) is a metric space.

We call ρ the *product metric* on X. The difficulty of the proof of this proposition lies in the triangle inequality, a computation which we will break into several intermediate results.

Lemma 4. *Let $a_k, b_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Then*

$$\sum_i \sum_j (a_i b_j - a_j b_i)^2 = 2 \sum_{i \neq j} (a_i^2 b_j^2 - a_i a_j b_i b_j).$$

Proof. Note that

$$(a_i b_j - a_j b_i)^2 = a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i a_j b_i b_j.$$

Then

$$\begin{aligned}\sum_i \sum_j (a_i b_j - a_j b_i)^2 &= \sum_i \sum_j (a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i a_j b_i b_j) \\ &= 2 \sum_{i \neq j} (a_i^2 b_j^2 - a_i a_j b_i b_j).\end{aligned}$$

□

Lemma 5. *Let $a_k, b_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Then*

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^2 = \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2.$$

Proof. Compute that

$$\begin{aligned}
\sum_k a_k^2 \sum_k b_k^2 &= \sum_k a_k^2 b_k^2 + 2\sum_{i\neq j} a_i^2 b_j^2 \\
&= \Big(\sum_k a_k^2 b_k^2 + 2\sum_{i\neq j} a_i a_j b_i b_j\Big) - 2\sum_{i\neq j} a_i a_j b_i b_j + 2\sum_{i\neq j} a_i^2 b_j^2 \\
&= \Big(\sum_k a_k b_k\Big)^2 + 2\Big(\sum_{i\neq j} a_i^2 b_j^2 - \sum_{i\neq j} a_i a_j b_i b_j\Big).
\end{aligned}$$

Subtracting the equation of Lemma 4 to both sides implies the result. □

Lemma 6. Cauchy-Schwartz Inequality
Let $a_k, b_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Then

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2.$$

Proof. This follows from Lemma 5 by noting that $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2$ is always nonnegative. □

Lemma 7. *Let $a_k, b_k, c_k \in \mathbb{R}$ be positive for $k = 1, \ldots, n$. Then*

$$\sqrt{\sum_{k=1}^{n} a_k^2} \leq \sqrt{\sum_{k=1}^{n} b_k^2} + \sqrt{\sum_{k=1}^{n} c_k^2}.$$

Proof. For $k = 1, \ldots, n$, we have $a_k \leq b_k + c_k$, so $a_k^2 \leq b_k^2 + c_k^2 + 2b_k c_k$. Thus

$$(*) \qquad \sum_{k=1}^{n} a_k^2 \leq \sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n} c_k^2 + 2\sum_{k=1}^{m} b_k c_k.$$

Now by Lemma 6, we have

$$\Big(\sum_{k=1}^{n} b_k c_k\Big)^2 \leq \sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2.$$

Take the square root of both sides to obtain

$$\Big(\sum_{k=1}^{n} b_k c_k\Big) \leq \sqrt{\sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2}.$$

Combine this with inequality (*) to obtain

$$\sum_{k=1}^{n} a_k^2 \leq \sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n} c_k^2 + 2\sqrt{\sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2}$$

Taking the square root of both sides produces the result. □

Proof of Theorem 1. The positivity of ρ is clear from the use of positive square root in the definition, and the symmetry is given by the symmetry of the metric on the constituent spaces. Thus is suffices to demonstrate the triangle inequality. Let $a_k = \rho(x_k, z_k)$, $b_k = \rho(x_k, y_k)$, and $c_k = \rho(y_k, z_k)$. By the triangle inequality in the constituent spaces, we have $a_k \leq b_k + c_k$ for $i = 1, \ldots, n$. Apply Lemma 7 to obtain the result. $\square$

Example 10. Let (X, ρ) be a metric space, and let X^k denote the cartesian product of k copies of X, endowed with the product metric.

Example 11. Let $\mathbb{R}^k$ denote the cartesian product of k copies of the real line. The product metric on $\mathbb{R}^k$ as defined in Example 3 is the same as that defined in Theorem 1.

Example 12. Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2\}$ be the unit circle, considered as a metric subspace of $\mathbb{R}^2$. Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, endowed with the product metric. Then $\mathbb{T}$ is a torus.

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