PRINCIPLES OF ANALYSIS
SOLUTIONS TO PROBLEM SET A

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Abstract. This document contains solutions to Problem Set A. These are not the only correct solutions. In most cases, there is more than one sensible approach to a given proof.

Problem 1. Let A, B, and C sets. Show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Solution. To show that two sets are equal, we show that each is contained in the other. To show that a set is a subset of another set, pick an arbitrary element from the first set and show that it is in the second.

Let $x \in (A \cap B) \cup C$. Then $x \in (A \cap B)$ or $x \in C$. Suppose $x \in (A \cap B)$. Then $x \in A$ and $x \in B$. Then $x \in A \cup C$ and $x \in B \cup C$. Thus $x \in (A \cup C) \cap (B \cup C)$. On the other hand, if $x \in C$, then again $x \in A \cup C$ and $x \in B \cup C$, so $x \in (A \cup C) \cap (B \cup C)$. Thus $(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$.

Let $x \in (A \cup C) \cap (B \cup C)$. Then $x \in (A \cup C)$ and $x \in (B \cup C)$. Then $x \in A$ or $x \in C$, and simultaneously $x \in B$ or $x \in C$. If $x \in C$, then $x \in (A \cap B) \cup C$. On the other hand, if $x \notin C$, then $x \in A$ and $x \in B$, so $x \in A \cap B$, in which case we still have $x \in (A \cap B) \cup C$. Therefore $(A \cup C) \cap (B \cup C) \subset (A \cap B) \cup C$. $\square$

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Problem 2. Let $f : X \to Y$ be a function.
(a) Show that f is surjective if and only if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.
(b) Show that f is injective if and only if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

Solution.
(a) ($\Rightarrow$) Suppose that f is surjective. Then, for every $y \in Y$ there exists $x_y \in X$ such that $f(x_y) = y$. Define $g : Y \to X$ by $g(y) = x_y$. Then for every $y \in Y$ we have $f \circ g(y) = f(g(y)) = f(x_y) = y$, and $f \circ g = \mathrm{id}_Y$.
(a) ($\Leftarrow$) Suppose that there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. Let $y \in Y$; then $f(g(y)) = y$, that is, the image of $g(y)$ under f is y, and since y is arbitrary, so f is surjective.
(b) ($\Rightarrow$) Suppose that f is injective. For for every y in the image $f(X)$, there exists a unique x_y in X such that $f(x_y) = y$. Select $z \in X$. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x_y & \text{if } y \in f(X); \\ z & \text{otherwise.} \end{cases}$$

Then for every $x \in X$ we have $g \circ f(x) = g(f(x)) = x$, and $g \circ f = \mathrm{id}_X$.
(b) ($\Leftarrow$) Suppose that there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, so f is injective. □

Problem 3. Let X be a set and let $T = \{0, 1\}$. Show that there is a natural bijective correspondence between the sets $\mathcal{P}(X)$ and $\mathcal{F}(X, T)$.

Solution. For each $A \subset X$, define a function

$$\psi_A : X \to T \quad \text{by} \quad \psi_A(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in A. \end{cases}$$

Define a function

$$\Psi : \mathcal{P}(X) \to \mathcal{F}(X, T) \quad \text{by} \quad \Psi(A) = \psi_A.$$

Define a function

$$\Phi : \mathcal{F}(X, T) \to \mathcal{P}(X) \quad \text{by} \quad \Phi(f) = f^{-1}(1).$$

Now it is clear that $\Phi \circ \Psi = \mathrm{id}_{\mathcal{P}(X)}$ and $\Psi \circ \Phi = \mathrm{id}_{\mathcal{F}(X,T)}$; that is, $\Phi = \Psi^{-1}$, so Ψ is bijective. □

Let $\mathcal{C} \subset \mathcal{P}(X)$. Recall that $\mathcal{C}$ is a partition of X if

(P0) $\varnothing \notin \mathcal{C}$;
(P1) $\cup\mathcal{C} = X$; (Covering property)
(P2) if $C_1, C_2 \in \mathcal{C}$ and $C_1 \cap C_2 \neq \varnothing$, then $C_1 = C_2$. (Disjointness property)

Problem 4. Let X be a set and let $\mathcal{C} = \{C_1, \dots, C_m\}$ and $\mathcal{D} = \{D_1, \dots, D_n\}$ be partitions of X. Define

$$\mathcal{E} = \{C_i \cap D_j \mid C_i \in \mathcal{C}, D_j \in \mathcal{D}\} \smallsetminus \{\varnothing\}.$$

(a) Show that $\mathcal{E}$ is a partition of X.
(b) Describe the equivalence relation induced by $\mathcal{E}$ in terms of the equivalence relations induced by $\mathcal{C}$ and $\mathcal{D}$.

Solution. For $\mathcal{E}$ to be a partition, it needs to cover X by a collection of disjoint sets.

Covering: We wish to show that $\cup\mathcal{E} \subset X$. Let $x \in X$. Since $\mathcal{C}$ and $\mathcal{D}$ are partitions, each covers X, so there exists i, j such that $x \in C_i$ and $x \in D_j$. Then $x \in C_i \cap D_j$. Thus $x \in \cup\mathcal{E}$.

Disjoint: Let $E_1 = C_{i_1} \cap D_{j_1}$ and $E_2 = C_{i_2} \cap D_{j_2}$ be arbitrary members of $\mathcal{E}$. Suppose that $x \in E_1 \cap E_2$. Then

$$x \in (C_{i_1} \cap D_{j_1}) \cap (C_{i_2} \cap D_{j_2}) = (C_{i_1} \cap C_{i_2}) \cap (D_{j_1} \cap D_{j_2}).$$

Therefore $C_{i_1} = C_{i_2}$ and $D_{j_1} = D_{j_2}$, which implies that $E_1 = E_2$.

Let the equivalence relations given by $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be denoted by $\cong_{\mathcal{C}}$, $\cong_{\mathcal{D}}$, and $\cong_{\mathcal{E}}$, respectively. Then $a \cong_{\mathcal{E}} b$ if and only if $a \cong_{\mathcal{C}} b$ and $a \cong_{\mathcal{D}} b$. $\square$

Problem 5. Show that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \in \mathbb{N}$.

Solution. For $n = 1$, we have $2n - 1 = 1 = 1^2$, so the formula is true in this case. By induction, assume that $\sum_{i=1}^{n-1}(2i-1) = (n-1)^2$. Then

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) &= (n-1)^2 + 2n - 1 \\ &= n^2 - 2n + 1 + 2n - 1 \\ &= n^2.\end{aligned}$$

$\square$

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