PRINCIPLES OF ANALYSIS
LECTURE 17 - CONNECTED AND COMPACT SETS

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1. Goal

We wish to prove the the continuous image of a connected set is connected, and that the continuous image of a compact set is compact.

Remark 1. Let X and Y be sets and let $f : X \to Y$ be a function. Let $A, B \subset X$ and $C, D \subset Y$. Then

- **(a)** $f^{-1}(f(A)) \supset A$ and equality holds if f is injective;
- **(b)** $f(f^{-1}(C)) \subset C$ and equality holds if f is surjective;
- **(c)** $f(A \cup B) = f(A) \cup f(B)$;
- **(d)** $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
- **(e)** $f(A \cap B) \subset f(A) \cap f(B)$ (give and example where equality fails);
- **(f)** $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

2. Connected Sets

A subset $A \subset \mathbb{R}$ is *disconnected* if there exist disjoint open sets $U_1, U_2 \subset \mathbb{R}$ with $A \cap U_1 \neq \varnothing$ and $A \cap U_2 \neq \varnothing$ such that $A \subset (U_1 \cup U_2)$. Otherwise, we say that A is *connected*.

Proposition 1. *Let $A \subset \mathbb{R}$. The following conditions on A are equivalent:*

- **(a)** *there exist $a_1, a_2 \in A$ and $c \notin A$ such that $a_1 < c < a_2$;*
- **(b)** $(\inf(A), \sup(A)) \subset A$*;*
- **(c)** *A is an interval;*
- **(d)** *A is connected.*

Proof. Exercise. $\square$

Proposition 2. *Let $f : E \to \mathbb{R}$ be a continuous function, and let $A \subset E$ be connected. Then $f(A)$ is connected.*

Proof. It suffices to show that if $f(A)$ is disconnected, then A is disconnected. Thus assume that $f(A)$ is disconnected, and let V_1 and V_2 be open subsets of $\mathbb{R}$ such that $f(A) \cap V_1 \neq \varnothing$, $f(A) \cap V_2 \neq \varnothing$, but $f(A) \subset (V_1 \cup V_2)$. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Then $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, but $A \subset (U_1 \cup U_2)$. Moreover, since f is continuous, U_1 and U_2 are open. Thus, A is disconnected. $\square$

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3. Compact Sets

Let $A \subset \mathbb{R}$. A *cover* of A is a collection $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ of subsets of $\mathbb{R}$ such that $A \subset \cup\mathcal{C}$.

Let $\mathcal{C}$ be a cover of $A \subset \mathbb{R}$. We say that $\mathcal{C}$ is an *open cover* if every member $U \in \mathcal{C}$ is an open subset of $\mathbb{R}$. We say that $\mathcal{C}$ is a *finite cover* if $\mathcal{C}$ is a finite set.

Note that the modifier *open* refers to the sets inside $\mathcal{C}$, whereas the modifier *finite* refers to the collection $\mathcal{C}$ itself.

A *subcover* of $\mathcal{C}$ is a subset $\mathcal{D} \subset \mathcal{C}$ such that $A \subset \cup\mathcal{D}$.

We say that A is *compact* if every open cover of A has a finite subcover.

Example 1. Let $A = \mathbb{Z}$. Let $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{Z}\}$. Then $\mathcal{C}$ is an open cover of $\mathbb{Z}$ with no finite subcover. Thus $\mathbb{Z}$ is not compact.

Example 2. Let $A = (0, 1)$. Let $I_n = (0, 1 - \frac{1}{n})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{Z}^+\}$. Then $\mathcal{C}$ is an open cover of $(0, 1)$ with no finite subcover. Thus $(0, 1)$ is not compact.

Proposition 3. *Let $A = \{a_1, \dots, a_n\}$ be a finite set. Then A is compact.*

Proof. Let $\mathcal{C}$ be an open cover of A. Then for each $a_i \in A$, there exists and open set $U_i \in \mathcal{C}$ such that $a_i \in U_i$. Then $A \subset \cup_{i=1}^n U_i$, and $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{C}$. Thus A is compact. $\square$

Proposition 4. *Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $[a, b]$.

Let $x \in [a, b]$ and let $U_x \in \mathcal{C}$ be an open set which contains x. Then there exists $\epsilon_x > 0$ such that $(x - \epsilon_x, x + \epsilon_x) \subset U_x$. Let

$$B = \{x \in [a, b] \mid [a, x] \text{ can be covered by a finite subcover of } \mathcal{C}\}.$$

Note that B is nonempty, since the closed interval $[a, a + \frac{\epsilon_a}{2}] \subset U_a$, and $\{U_a\}$ is a finite subcover of $\mathcal{C}$, so for example $a + \frac{\epsilon_a}{2} \in B$.

Let $z = \sup B$; clearly $a + \frac{\epsilon_a}{2} \leq z \leq b$. We claim that $z \in B$, and that $z = b$. To see this, let $\epsilon = \min\{\epsilon_z, z - a\}$. Then $z - \frac{\epsilon}{2} \in B$. Let $\mathcal{D}$ be a finite subcover of $\mathcal{C}$ which covers $[a, z - \frac{\epsilon}{2}]$. Then $\mathcal{D} \cup \{U_z\}$ covers $[a, z]$, so $z \in B$. Now suppose that $z < b$, and set $\delta = \min\{\epsilon, z - b\}$. Then $z < z + \frac{\delta}{2} < b$, and $\mathcal{D} \cup \{U_z\}$ covers $[a, z + \frac{\delta}{2}]$; since $z + \frac{\delta}{2} \in [a, b]$, this contradicts the definition of z. Thus $z = b$. This completes the proof. $\square$

Proposition 5. *Let $f : E \to \mathbb{R}$ be a continuous function, and let $A \subset E$ be a compact set. Then $f(A)$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $f(A)$. Define

$$\mathcal{B} = \{f^{-1}(V) \mid V \in \mathcal{C}\}.$$

Since A is compact, there exists a finite subset of $\mathcal{B}$, say $\mathcal{U} = \{U_1, \dots, U_n\}$, such that $A \subset \cup_{i=1}^n U_i$. Each U_i is the preimage of an open subset, say $U_i = f^{-1}(V_i)$. Then $f(U_i) \subset V_i$, and

$$f(A) \subset f(\cup_{i=1}^n U_i) = \cup_{i=1}^n f(U_i) \subset \cup_{i=1}^n V_i;$$

now $\mathcal{V} = \{V_1, \dots, V_n\}$ is a finite subcover of $\mathcal{C}$. This shows that $f(A)$ is compact. □

Proposition 6. *Let $A \subset \mathbb{R}$ be compact and let $F \subset A$ be closed. Then F is compact.*

Proof. Let $\mathcal{C}$ be an open cover of F. Let $U = \mathbb{R} \smallsetminus F$; since F is closed, U is open. Let $\mathcal{B} = \mathcal{C} \cup \{U\}$. Now $\mathcal{B}$ is an open cover of A. Since A is compact, let $\mathcal{U}$ be a finite subcover of A. Since $F \subset A$, then $\mathcal{U}$ is also a finite open cover of F. Let $\mathcal{V} = \mathcal{U} \smallsetminus \{U\}$; now $\mathcal{V}$ is still a finite open cover of F, and $\mathcal{V}$ is a subcover of $\mathcal{C}$. Thus F is compact. □

Theorem 1 (Heine-Borel Theorem)**.** *Let $A \subset \mathbb{R}$. Then A is compact if and only if A is closed and bounded.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that A is compact; we wish to show that A is closed and bounded.

Cover A with sets of the form $(-n, n)$, for $n \in \mathbb{Z}^+$. Since A is compact, there exists a finite subcover. This subcover contains an interval of maximum length, say $(-M, M)$, and clearly $A \subset (-M, M)$. Thus $A \subset [-M, M]$, and A is bounded.

To show that A is closed, we show that its complement is open. Let $B = \mathbb{R} \smallsetminus A$. Let $b \in B$. For each point $a \in A$, set $\epsilon_a = |b - a|/2$, $I_a = (a - \epsilon, a + \epsilon)$, and $J_a = (b - \epsilon, b + \epsilon)$. Let $\mathcal{I} = \{I_a \mid a \in A\}$. Then $\mathcal{I}$ is an open cover of A, and so it has a finite subcover $\{I_{a_1}, \dots, I_{a_n}\}$. The open set $\cup_{i=1}^n I_{a_i}$ contains A and is disjoint from the set $\cap_{i=1}^n J_{a_i}$, which is also open and contains b. Thus B is open.

($\Leftarrow$) Suppose that A is closed and bounded; we wish to show that A is compact. Since A is bounded, there exists $M > 0$ such that $A \subset [-M, M]$. The set $[-M, M]$ is a closed interval, and so it is compact by Proposition 4. Thus A is a closed subset of a compact set, and therefore is compact by Proposition 6. □

Proposition 7. *Let K be a compact set. Then* $\inf K \in K$ *and* $\sup K \in K$.

Proof. Since K is bounded, then $\sup K$ exists as a real number, say $b = \sup K$. Suppose $b \notin K$; then $\{(-\infty, b - \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$ is an open cover of K with no finite subcover, contradicting that K is compact. Thus $b \in K$. Similarly, $\inf K \in K$. □

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