AP Calculus AB — Responses 03/17

Dr. Paul L. Bailey — Tuesday, March 17, 2020

Hello all! In this document, I will go over some of the homework problems, and try to respond to some of your questions. That doesn't mean you should be completely enlightened after you read this, but hopefully we will inch a little closer. If I don't get to your question today, I will in the next installment!

Many of your problems refered to inverse trigonometric formulas were learned recently.

- $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$ $\qquad$ for $x \in [-1,1]$
- $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$ $\qquad \displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$ $\qquad$ for $x \in \mathbb{R}$
- $\dfrac{d}{dx}\operatorname{arcsec} x = \dfrac{1}{x\sqrt{x^2-x}}$ $\qquad \displaystyle\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec} x + C$ $\qquad$ for $x \in [1,\infty)$

Let's put these into words, describing what to look for inside the integral:

- $\arcsin x$: its a fraction with a constant top and whose bottom is the square root of a constant minus a square
- $\arctan x$: its a fraction with a constant top and whose bottom is a constant plus a square
- $\operatorname{arcsec} x$: its a fraction with a constant top and whose bottom is a variable times the square root of a square minus a constant

**Problem 1** (Thomas §8.1 # 1)**.** Compute $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$.

*Solution.* First we note that the top is not a constant; thus this is not an inverse trig function. Instead it is a relatively straightforward substitution.

We "look around" the integrand for one thing to be the approximate derivative of another thing. We see that $16x$ is the derivative of $8x^2$, so we could make $u = 8x^2$. But then we will have an annoying $+1$ hanging around, and we can get rid of it at the same time. So we proceed thusly:

Let $u = 8x^2 + 1$ so that $du = 16x$. Now the top is exactly the $du$, and we plug in:

$$\int \frac{16x\,dx}{\sqrt{8x^2+1}} = \int \frac{du}{\sqrt{u}} = \int u^{1/2}du = \frac{u^{3/2}}{(3/2)} + C = \frac{2}{3}(8x^2+1)^{3/2} + C.$$

□

**Problem 2** (Thomas §8.1 # 25)**.** Compute $\displaystyle\int \frac{9\,du}{1+9u^2}$.

*Solution.* The integrand is a fraction, where the top is constant and the bottom looks like a constant plus a square. So, I think it will be some sort of arctangent.

In this particular problem, I want to use a substitution, but the variable $u$ is already used. So, I will use $w$ instead.

I want the bottom to look like $1 + w^2$ and the top to look like 1. The nine that is already in the top will slide in and out of the integral:

$$\int \frac{9\,du}{1+9u^2} = 9\int \frac{du}{1+9u^2},$$

so I don't need to think to much about the 9, I just need to keep track of it.

I will *force* the bottom to be what I want by stating

Let $w^2 = 9u^2$. Then $w = 3u$, so $dw = 3du$.

Thus I need my top to be $3\,du$, not $9\,du$. I will pull out a 3, and plug in the $w^2$ and the $dw$, to write

$$\int \frac{9\,du}{1+9u^2} = 3\int \frac{3\,du}{1+9u^2} = 3\int \frac{dw}{1+w^2} = 3\arctan w + C = 3\arctan(x/3) + C.$$

□

**Problem 3** (Thomas §8.1 # 31)**.** Compute $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$.

*Solution.* I want $25x^2$ to be a variable squared.

Let $u^2 = 25x^2$. Then $u = 5x$, so $du = 5dx$. Pull out the 6, and put 5 next to the $dx$. When we put a 5 next to the $dx$, we have to divide by 5 on the outside.

That is,

$$\int \frac{6\,dx}{x\sqrt{25x^2-1}} = \frac{6}{5}\int \frac{5\,dx}{x\sqrt{25x^2-1}} = \frac{6}{5}\int \frac{du}{x\sqrt{u^2-1}}.$$

Now we want the $x$ on the bottom to be a $u$. Well, $u = 5x$, so if we pull the five back in, we get

$$\frac{6}{5}\int \frac{du}{x\sqrt{u^2-1}} = 6\int \frac{du}{5x\sqrt{u^2-1}} = 6\int \frac{du}{u\sqrt{u^2-1}} = 6\operatorname{arcsec} u + C = 6\operatorname{arcsec}(5x) + C.$$

□

Let's do one more. This one requires us to complete the square.

**Problem 4** (Thomas §8.1 # 40)**.** Compute $\displaystyle\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$.

*Solution.* Okay so this one looks a little odd. Since we are in section of problems entitled "Complete the square", I guess that is what I will try to do. Later, in the future, we will recognize to do this only because we have had lots of practice.

The bottom of the integrand is a radical containing a quadratic polynomial. So, if I successfully complete the square, I hope to get the denominator to be a square root of a constant minus a square. Then, I will achieve some sort of arcsine.

We have $2\theta - \theta^2$. We complete the square by adding and subtracting 1. Thus

$$\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-1+2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}}.$$

Now we solve the later integral with a substitution. I want $\sqrt{1-u^2}$ on the bottom, so

Let $u = \theta - 1$, so $du = d\theta$, so

$$\int \frac{d\theta}{\sqrt{1-(\theta-1)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin(\theta-1) + C.$$

□

Now let me address a few of your questions.

**Question 1.** What is the "fundamental period" of a function?

*Answer.* Let $f : \mathbb{R} \to \mathbb{R}$ be nonconstant. We say that $f$ is *periodic* if there exists a real number $T \in \mathbb{R}$ such that

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

The number $T$ is called a *period* of $f$. The *fundamental period* of $f$ is the smallest period of $f$.

Trigonometric functions are periodic. The fundamental period of sine and cosine is $2\pi$. The fundamental period of tangent is $\pi$.

We know that if we multiply $x$ by a positive constant, it compresses or stretches the graph of $f$ horizontally. The graph of $f(3x)$ is the graph of $f$, compressed horizontally by a factor of 3. The graph of $f(x/5)$ is the graph of $f$, stretched horizontally by a factor of 5.

So, the fundamental period is compressed or stretched similarly. The fundamental period of $\sin(5x)$ is $\dfrac{2\pi}{5}$; note that $5x = 2\pi$ when $x = \dfrac{2}{5}\pi$. The fundamental period of $\tan(x/3)$ is $3\pi$. □

**Question 2.** Can we review derivatives of inverse trig functions? How to use them?

*Answer.* Let's discuss inverse sine. We will just survey the main points. You may want to try to write something similar for arctan and arcsec.

**(a)** First, we define the arcsin function.

We know we have to pick a domain on which sine is injective. So, restrict sine's domain (and codomain), and call the inverse arcsin.

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1] \text{ so that } \arcsin : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \text{ where } \arcsin y = \theta \Leftrightarrow \sin\theta = y.$$

By the way, it is called "arcsin(y)" because it returns the actual arclength of the arc on the unit circle described by an angle $\theta$ whose sine is $y$.

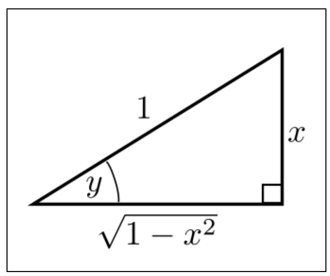
**(b)** Next, we compute the derivative of the arcsin function.

We compute the derivative of arcsine (and, in general, of an inverse function) using implicit differentiation.

Let $\arcsin x = y$. We want $\dfrac{dy}{dx}$.

Then $\sin y = x$, so $\dfrac{d}{dx}\sin y = \dfrac{d}{dx}x$, which is $\cos y \cdot \dfrac{dy}{dx} = 1$.

To compute $\cos y$, draw a "representative triangle":




We see from the diagram that the $\cos y = \sqrt{1-x^2}$.

Solve for $\dfrac{dy}{dx}$ to get

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

**(c)** An example of differentiation involving arcsin.

$$\frac{d}{dx}\arcsin(x^2+4) = \frac{1}{\sqrt{1-(x^2+1)^2}} \cdot (2x) = \frac{2x}{\sqrt{2x^2-x^4}}.$$

**(d)** An example of integration involving arcsin

To compute $\int \dfrac{2x}{\sqrt{1-x^4}}\,dx$, let $u = x^2$, so that $du = 2x\,dx$. Then

$$\int \frac{2x\,dx}{\sqrt{1-x^4}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin(x^2) + C.$$

$\square$

**Question 3.** What is L'Hospital's Rule.

*Answer.* We haven't covered that yet, and we will in the future, but *very* briefly: If you are taking the limit of a fraction but the top and the bottom both go to zero, then under good circumstances, we have

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

For example, if $f(x) = \sin x$ and $g(x) = x$, then

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

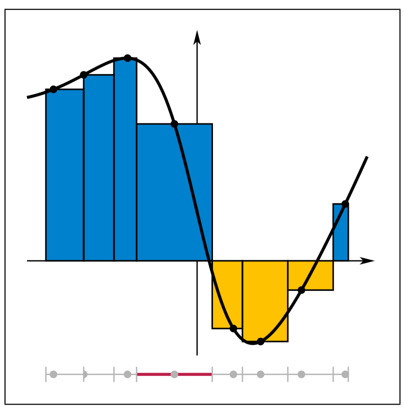
□

**Question 4.** What was that about the "norm" of a partition?

*Answer.* The norm of a partition is the length of the longest subinterval in the partition.

We estimate the area under the graph of a function by partitioning the domain into smaller pieces. We get a bunch of rectangles, whose width of the base is length of the piece, and whose height is gotten by picking a point in each piece and evaluating the function there. Now the sum of the areas of these rectangles will approximate the area under the curve. Moreover, the estimate will improve if we break the domain down into more pieces. However, if we leave one of the pieces to be relatively large, the corresponding rectangle will never get closer to the area under the curve over that base.

So, when we add more pieces to the partition, we have to make *all* of the pieces smaller. Since the norm of the partition is just the length of the longest subinterval in the partition, unless this norm gets constantly smaller as we refine the partition, the approximation will not sufficiently approach the actual area under the curve.



For example, in the picture above, notice how much the large blue rectangle misses the actual graph. If all the other rectangle are split into smaller pieces, but this large blue one just stays there, the norm of the partition will not decrease, and the error of the estimate can never go to zero.

This is why we define the area under the curve as the limit as the norm goes to zero: for $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta x_k = x_k - x_{k-1}$, and $c_k \in [x_{k-1}, x_k]$, the norm is

$$\|P\| = \max\{\Delta x_k \mid k = 1, \ldots, n\},$$

and we have

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

□

**Question 5.** Related rates ... aargh!

*Answer.* Ok well let's just keep doing these until it becomes more natural. And I even wrote up the solution to one, but this document has run out of room! So we'll just have to include it tomorrow. □