HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC IX
CUBIC EQUATIONS AND QUARTIC EQUATIONS

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1. The Story

Various solutions for solving quadratic equations $ax^2 + bx + c = 0$ have been around since the time of the Babylonians. A few methods for attacking special forms of the cubic equation $ax^3+bx^2+cx+d = 0$ had been investigated prior to the discovery and development of a general solution to such equations, beginning in the fifteenth century and continuing into the sixteenth century, A.D. This story is filled with bizarre characters and plots twists, which is now outlined before describing the method of solution.

The biographical material here was lifted wholesale from the MacTutor History of Mathematics website, and then editted. Other material has been derived from Dunham's *Journey through Genius*.

1.1. **Types of Cubics.**

Zero and negative numbers were not used in fifteenth century Europe. Thus, cubic equations were viewed to be in different types, depending on the degrees of the terms and there placement with respect to the equal sign.

- $x^3 + mx = n$ "cube plus cosa equals number"
- $x^3 + mx^2 = n$ "cube plus squares equals number"
- $x^3 = mx + n$ "cube equals cosa plus number"
- $x^3 = mx^2 + n$ "cube equals squares plus number"

Mathematical discoveries at this time were kept secret, to be used in public "debates" and "contests". For example, the method of depressing a cubic (eliminating the square term by a linear change of variable) was discovered independently by several people. The more difficult problem of solving the depressed cubic remained elusive.

1.2. **Luca Pacioli.**

In 1494, Luca Pacioli published *Summa de arithmetica, geometria, proportioni et proportionalita*. The work gives a summary of the mathematics known at that time although it shows little in the way of original ideas. The work studies arithmetic, algebra, geometry and trigonometry and, despite the lack of originality, was to provide a basis for the major progress in mathematics which took place in Europe shortly after this time. The book admittedly borrows freely from Euclid, Boethius, Sacrobosco, Fibonacci, et cetera.

In this book, Pacioli states that the solution of the cubic is impossible.

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1.3. **Scipione del Ferro.** The first known mathematician to produce a general solution to a cubic equation is Sipione del Ferro. He knew that the problem of solving the general cubic could be reduced to solving the two cases $x^3+mx = n$ and $x^3 = mx + n$, where m and n are positive numbers, and del Ferro may have solved both cases; we do not know for certain, because his results were never published.

We know that del Ferro was appointed as a lecturer in arithmetic and geometry at the University of Bologna in 1496 and that he retained this post for the rest of his life. No writings of del Ferro have survived. We do know however that he kept a notebook in which he recorded his most important discoveries. This notebook passed to del Ferro's son-in-law Hannibal Nave when del Ferro died in 1526.

On his deathbed, del Ferro revealed at least part of his secret, the solution to the "cube plus cosa equals number" problem, to his student, Fior.

1.4. **Niccolo Tartaglia.** Niccolo Fontana, known as Tartaglia, was born in Brescia in 1499 or 1500. His father was murdered when he was six, and plunged the family into total poverty.

Niccolo was nearly killed as a teenager when, in 1512, the French captured his home town and put it to the sword. The twelve year old Niccolo was dealt horrific facial sabre wounds by a French soldier that cut his jaw and palate. He was left for dead and even when his mother discovered that he was still alive she could not afford to pay for any medical help. However, his mother's tender care ensured that the youngster did survive, but in later life Niccolo always wore a beard to camouflage his disfiguring scars and he could only speak with difficulty, hence his nickname Tartaglia, or stammerer.

He moved to Venice in 1534. As a lowly mathematics teacher in Venice, Tartaglia gradually acquired a reputation as a promising mathematician by participating successfully in a large number of debates.

Fior began to boast that he was able to solve cubics and a challenge between him and Tartaglia was arranged in 1535. In fact Tartaglia had previously discovered how to solve one type of cubic equation, the "cube + squares equals number" type. For the contest between Tartaglia and Fior, each man was to submit thirty questions for the other to solve. Fior was supremely confident that his ability to solve cubics would be enough to defeat Tartaglia but Tartaglia submitted a variety of different questions, exposing Fior as an, at best, mediocre mathematician. Fior, on the other hand, offered Tartaglia thirty opportunities to solve the "cube plus cosa" problem, since he believed that he would be unable to solve this type, as in fact had been the case when the contest was set up. However, in the early hours of February 13, 1535, inspiration came to Tartaglia and he discovered the method to solve 'cube equal to numbers'. Tartaglia was then able to solve all thirty of Fior's problems in less than two hours. As Fior had made little headway with Tartaglia's questions, it was obvious to all who was the winner. Tartaglia didn't take his prize for winning from Fior, however, the honor of winning was enough.

1.5. **Girolamo Cardano.** Girolamo or Hieronimo Cardano's name was Hieronymus Cardanus in Latin and he is sometimes known by the English version of his name Jerome Cardan. He was the illegitimate child of a lawyer/mathematician, Fazio Cardano. He was a brilliant physician and mathematician who loved to gamble, and generally had a fascinating, though often tragic, life. Among other travails, he was kept out of the College of Physicians because of his illegitimate birth; his wife died young; his favorite son was executed for the murder of his wife; his other son stole large sums of money from him; and he was jailed by the Inquisition.

In 1539,Cardan was a public lecturer of mathematics at the Piatti Foundation in Milan, and was aware of the problem of solving cubic equations; he had taken Pacioli at his word and assumed that, as Pacioli stated in the Suma published in 1494, solutions were impossible.

Cardan was greatly intrigued when he learned of the contest between Fior and Tartaglia, and he immediately set to work trying to discover Tartaglia's method for himself, but was unsuccessful. A few years later, in 1539, he contacted Tartaglia, through an intermediary, requesting that the method could be included in a book he was publishing that year. Tartaglia declined this opportunity, stating his intention to publish his formula in a book of his own that he was going to write at a later date. Cardan, accepting this, then asked to be shown the method, promising to keep it secret. Tartaglia, however, refused.

An incensed Cardan now wrote to Tartaglia directly, expressing his bitterness, challenging him to a debate but, at the same time, hinting that he had been discussing Tartaglia's brilliance with the governor of Milan, Alfonso d'Avalos, the Marchese del Vasto, who was one of Cardan's powerful patrons. On receipt of this letter, Tartaglia radically revised his attitude, realizing that acquaintance with the influential Milanese governor could be very rewarding and could provide a way out of the modest teacher's job he then held, and into a lucrative job at the Milanese court. He wrote back to Cardan in friendly terms, angling for an introduction to the Signor Marchese. Cardan was delighted at Tartaglia's new approach, and, inviting him to his house, assured Tartaglia that he would arrange a meeting with d'Avalos.

So, in March 1539, Tartaglia left Venice and travelled to Milan. To Tartaglia's dismay, the governor was temporarily absent from Milan but Cardan attended to his guest's every need and soon the conversation turned to the problem of cubic equations. Tartaglia, after much persuasion, agreed to tell Cardan his method, if Cardan would swear never to reveal it and furthermore, to only ever write it down in code so that on his death, nobody would discover the secret from his papers. The oath which Cardano swore is reportedly:

I swear to you, by God's holy Gospels, and as a true man of honor, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them.

Tartaglia divulged his formula in the form of a poem, to help protect the secret, should the paper fall into the wrong hands.

By the time he had reached Venice, Tartaglia was sure he had made a mistake in trusting Cardan and began to feel very angry that he had been induced to reveal his secret formula. When Cardan wrote to him in a friendly manner Tartaglia rebuffed his offer of continued friendship and mercilessly ridiculed his books on the merest trivialities.

1.6. Lodovico Ferrari. Lodovico Ferrari was sent, as a teenager, to be the servant of Cardano. However, when Cardano discovered that the boy could read and write, he made him his assistant, and quickly learned that Ferrari was quite talented. Ferrari became Cardano's mathematical apprentice.

Based on Tartaglia's formula, Cardan and Ferrari made remarkable progress finding proofs of all cases of the cubic and, even more impressively, solving the quartic equation. Tartaglia made no move to publish his formula despite the fact that, by now, it had become well known that such a method existed.

One of the first problems that Cardan hit was that the formula sometimes involved square roots of negative numbers even though the answer was a 'proper' number. In August 1539 Cardan wrote to Tartaglia:

I have sent to enquire after the solution to various problems for which you have given me no answer, one of which concerns the cube equal to an unknown plus a number. I have certainly grasped this rule, but when the cube of one-third of the coefficient of the unknown is greater in value than the square of one-half of the number, then, it appears, I cannot make it fit into the equation.

Indeed Cardan gives precisely the conditions here for the formula to involve square roots of negative numbers. Tartaglia by this time greatly regretted telling Cardan the method and tried to confuse him with his reply (although in fact Tartaglia, like Cardan, would not have understood the complex numbers now entering into mathematics):

... and thus I say in reply that you have not mastered the true way of solving problems of this kind, and indeed I would say that your methods are totally false.

Cardan and Ferrari travelled to Bologna in 1543 and learnt from Hannibal Nave that it had been del Ferro, not Tartaglia, who had been the first to solve the cubic equation. Cardan felt that although he had sworn not to reveal Tartaglia's method surely nothing prevented him from publishing del Ferro's formula. In 1545 Cardan published *Artis magnae sive de regulis algebraicis liber unus*, or *Ars magna*, as it is more commonly known, which contained solutions to both the cubic and quartic equations and all of the additional work he had completed on Tartaglia's formula. Del Ferro and Tartaglia are credited with their discoveries, as is Ferrari, and the story written down in the text.

It is to Cardan's credit that, although one could not expect him to understand complex numbers, he does present the first calculation with complex numbers in Ars Magna. Solving a particular cubic equation, he writes

Dismissing mental tortures, and multiplying $5+\sqrt{-15}$ *by* $5-\sqrt{-15}$*, we obtain* $25-(-15)$*. Therefore the product is* 40 *... and thus far does arithmetical subtlety go, of which this, the extreme, is, as I have said, as subtle as it is useless.*

1.7. Rapheal Bombelli. In 1572, Rapheal Bombelli wrote his book *Algebra*, in which explicitly uses negative numbers and zero. Moreover, he shows how manipulating complex numbers can help arrive at real solutions to cubic equations, thus demonstrating that, although they may be subtle, they are far from useless.

2. Solution of Quadratic Equations

Some version of the quadratic formula has been available to most advanced cultures of the last three thousand years. Let us review its derivation.

A polynomial is *monic* if the leading coefficient is 1. Two equations are *equivalent* if they have the same solution set.

Let $ax^2 + bx + c = 0$ be a general quadratic equation; Our method of solution is known as completing the square. First we produce an equivalent monic equation, and then we introduce a new term to create the square of a linear polynomial:

$$\begin{aligned} ax^2 + bx + c = 0 &\Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a} \\ &\Leftrightarrow x^2 + \frac{b}{a}x + \Big(\frac{b}{2a})^2 = (\frac{b}{2a}\Big)^2 - \frac{c}{a} \\ &\Leftrightarrow \Big(x + \frac{b}{2a}\Big)^2 = \frac{b^2 - 4ac}{4a^2} \\ &\Leftrightarrow x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\ &\Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

Notice that the fourth equation can be rewritten as

$$\Big(x + \frac{b}{2a}\Big)^2 + \frac{4ac - b^2}{4a^2} = 0.$$

Setting $y = x + \frac{b}{2a}$ and $n = \frac{4ac-b^2}{4a^2}$, rewrite this as

$$y^2 + n = 0.$$

This is *depressed quadratic equation*; the degree one term has be eliminated by the substitution $x \rightarrow y - \frac{b}{2a}$, which is known as a *linear change of variables.*

The *discriminant* of the quadratic equation is

$$\Delta = b^2 - 4ac;$$

this determines the number of real roots. There are three cases:

(a) if $b^2 - 4ac > 0$, there are two real roots;
(b) if $b^2 - 4ac = 0$, there is one real root;
(c) if $b^2 - 4ac < 0$, there are no real roots.

We point out here that negative numbers and there square roots were not accepted as actual solutions in antiquity. To some extent, this is justified, since the search for real solutions to such an equation was not significantly compromised by excluding these numbers. This situation changes upon consideration of cubic equations.

3. Depressing Cubic Equations

Let $f(x) = ax^3 + bx^2 + cx + d$ be a general cubic polynomial; we wish to solve $f(x) = 0$. If we can find one zero r of the polynomial $f(x)$, we can divide $(x - r)$ into $f(x)$ to obtain a quadratic polynomial, whose zeros can be found using the quadratic formula.

A *depressed cubic equation* of the type "cube plus cosa equals number" is an equation of the form

$$x^3 + mx = n.$$

We wish to take the general cubic equation $f(x) = 0$ and find a different depressed cubic equation whose solution will give us a solution to $f(x) = 0$.

Clearly we can divide by a to obtain a monic equation $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$; in this way, we can assume that $a = 1$. We wish to find a linear change of variable that will produce an equation which lacks the quadratic term.

To discover how to do this, substitute $y - h$ for x and multiply:

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(y-h)^3 + b(y-h)^2 + c(y-h) + d \\ &= a(y^3 - hy^2 + h^2y - h^3) + b(y^2 - 2hy + h^2) + c(y-h) + d \\ &= ay^3 - 3ahy^2 + 3ah^2y - h^3 + by^2 - 2hy + h^2 + cy - ch + d \\ &= ay^3 + (b - 3ah)y^2 + (3ah^2 - 2h)y + (d - ch). \end{aligned}$$

Now we see that if we set $h = \frac{b}{3a}$, we obtain the desired result of eliminating the quadratic term.

Thus, to solve cubic equations, we may assume that the equation is given in the form $x^3 + mx = n$.

4. Solving the Depressed Cubic

Consider the depressed cubic equation

$$x^3 + mx = n.$$

The key idea of the solution method is to write x as a difference of two quantities, $x = t - u$. We wish to find appropriate quantities $t, u \in \mathbb{R}$ such that $r = t - u$, where r is a solution to this equation.

By the binomial theorem, we have

$$(t-u)^3 = t^3 - 3t^2u + 3tu^2 - u^3.$$

Thus

$$\begin{aligned} t^3 - u^3 &= (t-u)^3 + 3t^2u - 3tu^2 \\ &= (t-u)^3 + 3tu(t-u). \end{aligned}$$

Note that if $x = t - u$, $m = 3tu$, and $n = t^3 - u^3$, this equation becomes $x^3 + mx = n$. This reduces the problem to solving the system of equations

$$m = 3tu; \tag{1}$$

$$n = t^3 - u^3. \tag{2}$$

Equation (1) gives

$$u = \frac{m}{3t}.$$

Substitute this into equation (2) to obtain

$$t^3 - \frac{m^3}{27t^3} = n.$$

Subtract n from both sides and multiply through by t^3 to get

$$t^6 - nt^3 - \frac{m^3}{27} = 0.$$

Now the quadratic formula gives

$$\begin{aligned} t^3 &= \frac{n \pm \sqrt{n^2 + \frac{4m^3}{27}}}{2} \\ &= \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}. \end{aligned}$$

Therefore,

$$t = \sqrt[3]{\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}.$$

Now $u^3 = t^3 - n$, so

$$u = \sqrt[3]{-\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}.$$

Finally, $x = t - u$, so

$$\boxed{x = \sqrt[3]{\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}}.$$

Upon examining the above equations, we see two $\pm$ signs in the expression for x, which may lead one to believe we have found four solutions to a cubic polynomial. However, to of them are the same.

Consider the solution to be in the form $x = \sqrt[3]{a \pm b} - \sqrt[3]{-a \pm b}$. The solution with two minus signs equals the solution with two negative signs, because factor out the negative sign gives

$$\sqrt[3]{a-b} - \sqrt[3]{-a-b} = (-1)\sqrt[3]{-a+b} - (-1)\sqrt[3]{a+b} = \sqrt[3]{a+b} - \sqrt[3]{-a+b}.$$

Memorizing this formula is unnecessary if we remember the technique. We let $x = t - u$, so that $x^3 + mx = n$ becomes $(t-u)^3 + 3tu(t-u) = t^3 - u^3$.

(1) Set $3tu = m$ and $t^3 - u^3 = n$.
(2) Solve the first equation for u to get $u = \frac{m}{3t}$.
(3) Plug this into the second equation to get $t^3 = n + (\frac{m}{3t})^3$.
(4) Multiply by t^3 to get $t^6 - nt^3 - (\frac{m}{3})^3 =$.
(5) Complete the square to get $t^3 = \frac{n}{2} \pm \Delta$.
(6) Use $u^3 = t^3 - n$ to get $u^3 = -\frac{n}{2} \pm \Delta$;
(7) Take cube root and set $x = t - u$.

Example 1. (A Typical Example)
Solve $x^3 + 15x = 22$.

Solution. Let $3tu = 15$ and $t^3 - u^3 = 22$. Then $u = \frac{5}{t}$, so $t^3 - \frac{125}{t^3} = 22$. Therefore $t^6 - 22t^3 - 125 = 0$, so by the quadratic formula,

$$t^3 = \frac{22 \pm \sqrt{484 + 500}}{2} = 11 \pm \sqrt{246}.$$

Then $u^3 = -11 \pm \sqrt{246}$, so

$$x = \sqrt[3]{11 \pm \sqrt{246}} - \sqrt[3]{-11 \pm \sqrt{246}}.$$

$\square$

Example 2. (Cardano's Example)
Solve $x^3 + 6x = 20$.

Solution. Let $3tu = 6$ and $t^3 - u^3 = 20$. Then $u = \frac{2}{t}$, so $t^3 - \frac{8}{t^3} = 20$, so $t^6 - 20t^3 - 8 = 0$. By the quadratic formula,

$$t^3 = \frac{20 \pm \sqrt{400 + 32}}{2} = 10 \pm \sqrt{108}.$$

Taking the positive value for t, applying $u^3 = t^3 - 20$, and taking the appropriate cube roots, we have

$$x = t - u = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}.$$

Note that this is a real number.

A more modern solution starts by setting $f(x) = x^3 + 6x - 20$ and seeking solutions to $f(x) = 0$. We note that $f(2) = 2^3 + 6(2) - 20 = 0$, so $x = 2$ is a solution. By the Factor Theorem, $(x-2)$ divides $f(x)$, and dividing we find that

$$f(x) = (x-2)(x^2 + 2x + 10) = (x-2)(x - (-1+3i))(x - (-1-3i)).$$

Note that the only real zero of f is 2; thus, we have no choice but to conclude that

$$\sqrt[3]{10+\sqrt{108}} - \sqrt[3]{-10+\sqrt{108}} = 2.$$

□

Example 3. (Bombelli's Example)
Solve $x^3 - 15x = 4$.

Solution. Let $3tu = -15$ and $t^3 - u^3 = 4$. Thus $u = -\frac{5}{t}$, so $t^3 + \frac{125}{t^3} = 4$. From this, $t^6 - 4t^3 + 125 = 0$, so, taking the positive square root, we have

$$t^3 = \frac{4+\sqrt{16-500}}{2} = 2 + \sqrt{4-125} = 2 + \sqrt{-121}.$$

At this point, instead of asserting the irrelevance of the problem, Bombelli continues as if $\sqrt{-1}$ is a perfectly acceptable quantity. He notes that

$$(2+\sqrt{-1})^3 = 8 + 12\sqrt{-1} - 6 - \sqrt{-1} = 2 + 11\sqrt{-1} = 2 + \sqrt{-121}.$$

Thus if $t = 2+\sqrt{-1}$, then $t^3 = 2+\sqrt{-121}$. Continuing, we have $u = -2+\sqrt{-11}$, so

$$x = t - u = (2+\sqrt{-121}) - (-2+\sqrt{-121}) = 4.$$

One verifies that 4 is indeed a solution to the original cubic equation; thus a real solution is attained by traversing through the realm of complex numbers. □

5. Depressing a Quartic Equation

The general quartic equation can be depressed in the same manner as the cubic. Consider

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$

We again want a linear change of variables that will eliminate the cubic term. Here, the substitution $x \mapsto (x - \frac{b}{4a}$ works.

Problem 1. Let

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Show that the substitution $x \mapsto (x - \frac{a_{n-1}}{na_n})$ eliminates the term of degree $(n-1)$.

6. Solving the Depressed Quartic

Consider the polynomial equation

$$x^4 + px^2 + qx + r = 0.$$

Complete the square to obtain

$$(x^2+p)^2 = px^2 - qx - r + p^2.$$

Let $y \in \mathbb{R}$ and add $2y(x^2+p) + y^2$ to both sides to get

$$(*) \qquad (x^2+p+y)^2 = (p+2y)x^2 + (p^2 - r + 2py + y^2).$$

The right hand side becomes a quadratic in x we can choose y so that it is a perfect square; this is done by making the discriminant equal to zero:

$$q^2 - 4(p - r + 2py + y^2) = 0.$$

Rewrite this as

$$(q^2 - 4p^3 + 4pr) + (8r - 16p^2)y - 20py^2 - 8y^3 = 0.$$

This is a cubic in y, and can be solved. With this value for y, take the square root of both sides of (*) to obtain a quadratic in x, which can also be solved.

7. Graphs of Cubics

7.1. **Backdrop.** Cardano had several cases for solving cubics, placing the monomials on the appropriate side of the equation to create positive coefficients. As it turns out, the sign of the constant term plays no role in the computation, so the main cases were:

(a) $x^3 = mx + n$ (cube equals cosa plus number)
(b) $x^3 + mx = n$ (cube plus cosa equals number)

Analytic geometry on in cartesian coordinates had not been invented at the time, so Cardano had no idea that these two cases correspond to distinctly different geometric interpretations for the graph of the cubic. We investigate this in modern notation using Calculus.

7.2. **The Leading Coefficient.** Consider the generic cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d.$$

If $a > 0$, then $\lim_{x\to-\infty} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) = \infty$; if $a < 0$, then $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to\infty} f(x) = -\infty$ Thus by the Intermediate Value Theorem, f has at least one real zero. By the Factor Theorem, if $f(r) = 0$, then $f(x) = (x - r)q(x)$, where q is a quadratic polynomial, which we can find by polynomial division, and thus find the other two zeros (which may be real or complex) using the quadratic formula.

We wish to solve $f(x) = 0$, and we realize that if we divide through by a, the zeros of the resulting polynomials are the same as the original. Thus, without loss of generality, we assume $a = 1$.

7.3. **The Square Coefficient.** Consider the polynomial

$$f(x) = x^3 + bx^2 + cx + d.$$

Note that $\lim_{x\to-\infty} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) = \infty$. Next we wish to discover the role of b in the graph of f.

By differentiation,

$$f'(x) = 3x^2 + 2bx + cand \quad f''(x) = 6x + 2b.$$

If $f''(x) = 0$, then $6x + 2b = 0$, so $x = -\frac{b}{3}$. Thus f has an inflection point at $(-\frac{b}{3}, f(-\frac{b}{3}))$. Also, $b = 0$ if and only if the inflection point of f lies on the y-axis. So, to eliminate the inflection point of f, we shift the graph of f right by $\frac{b}{3}$. The graph of $f(x - \frac{b}{3})$ is the graph of f shifted so that the inflection point is on the y-axis. If we find the zeros of $f(x - \frac{b}{3})$, we obtain the zeros of f subtracting $\frac{b}{3}$.

It turns out that

$$f(x - \frac{b}{3}) = x^3 - \frac{1}{3}(b^2 - 3c)x - \frac{1}{27}(b^3 - 3b^2 + 9bc - 27d).$$

Without loss of generality, we now assume that $b = 0$.

7.4. **The Cosa Coefficient.** Consider the polynomial

$$f(x) = x^3 + cx + d.$$

The graph of f has an inflection point on the y-axis. We wish to identify the role that c plays in the graph of f.

By differentiation,

$$f'(x) = 3x^2 + c.$$

Thus $f'(0)$ is the slope of the line tangent to the graph of f at its inflection point.

If $c = 0$, then f has a horizontal tangent at its inflection point.

If $c \geq 0$, then f is increasing and has no local minimum or maximum; in this case, f has exactly one x-intercept, and so, exactly one real zero and two complex zeros. This is the "cube + cosa = number" case.

If $c < 0$, we set $f'(x) = 0$ and obtain $x = \pm\sqrt{-\frac{c}{3}}$; thus f has a local maximum at $-\sqrt{-\frac{c}{3}}$ and a local minimum at $\sqrt{-\frac{c}{3}}$. This is the "cube = cosa + number" case. In this case, we sometimes have three distinct real zeros, and sometimes do not. We wish to discover a condition determining the number of real zeros.

7.5. **The Constant Coefficient.** Consider the polynomial

$$g(x) = x^3 + cx.$$

We have an excellent idea of its graph. It is an odd function, and thus has symmetry about the origin, and it has an inflection point at the origin. If $c \geq 0$, it is increasing, and if $c < 0$, it has zeros at $x = \pm\sqrt{c}$ with local extrema at $x = \pm\sqrt{-\frac{c}{3}}$.

If we shift this graph up by d (down if $d < 0$), the corresponding function is $f(x) = g(x) + d$.

Assume that $c < 0$. Let

$$h = g\Big(\sqrt{-\frac{c}{3}}\Big) = \frac{2c}{3}\sqrt{-\frac{c}{3}}.$$

This is the height (depth) of the local maximum (minimum). If we vertically shift the graph of g by less than h, we have three zeros; if we shift by h, we have one single zero and a double zero; if we shift by $d > h$, we have a unique zero. That is:

(a) $|d| < |h| \Rightarrow$ three distinct real zeros
(b) $|d| = |h| \Rightarrow$ one single real zero and one double real zero
(c) $|d| > |h| \Rightarrow$ a unique real zero and two complex zeros

Now

$$|d| < |h| \Leftrightarrow d^2 < h^2 - \frac{4c^2}{9}(-\frac{c}{3}) = -\frac{4c^3}{27} \qquad \Leftrightarrow \frac{d^2}{4} + \frac{c^3}{27}.$$

The *discriminant* of f is

$$D = \frac{d^2}{4} + \frac{c^3}{27}.$$

Note that if $c \geq 0$, then $D \geq 0$. Thus, for any c, we have

(a) $D < 0 \Rightarrow$ three distinct real zeros
(b) $D = 0 \Rightarrow$ one single real zero and one double real zero, or a triple zero
(c) $D > 0 \Rightarrow$ a unique real zero and two complex zeros

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