COMPLEX ANALYSIS
TOPIC XV: MÖBIUS TRANSFORMATIONS

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1. Möbius Transformations

Which meromorphic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are invertible? Clearly they must be injective (one-to-one), so they have degree 1. That is, $f(z) = \dfrac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$. We would like to exclude from this the constant functions (which clearly are not injective).

To penetrate this question, let us use the fact that f is a constant function if and only if f' is identically zero. Compute

$$f'(z) = \frac{a(cz+d)-(az+b)c}{(cz+d)^2} = \frac{acz+d-acz-bc}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}.$$

Thus $f(z)$ is constant if and only if $ad - bc = 0$.

Definition 1. A *linear fractional transformation* is a function of the form

$$S : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad S(z) = \frac{az+b}{cz+d},$$

for some $a, b, c, d \in \mathbb{C}$. Such a function is called a *Möbius transformation* if $ad-bc \neq 0$.

Let $S(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation. We may compute the inverse of f in the standard way to be

$$f^{-1}(z) = -\frac{dz-b}{cz-a}.$$

In fact, a meromorphic function $\mathbb{C}_\infty \to \mathbb{C}_\infty$ is invertible if and only if it is a Möbius transformation. The reader who has been exposed to group theory will recognize that the set of all Möbius transformations form a group under the operation of function composition.

We note that the coefficients a, b, c, d are not unique; indeed,

$$\frac{az+b}{cz+d} = \frac{\lambda az+\lambda b}{\lambda cz+\lambda d},$$

for any $\lambda \in \mathbb{C}$. Actually, though, for any Möbius transformation S, there is a unique a, b, c, d such that $S(z) = \dfrac{az+b}{cz+d}$ and $ad - bc = 1$.

Möbius transformations are transformations of the Riemann sphere, and in this context, we note that

- $S(-\frac{d}{c}) = \infty$, with the caveat that if $c = 0$, then $S(\infty) = \infty$.
- $S(\infty) = \frac{a}{c}$

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2. Primitive Möbius transformations

A Möbius transformation $S(z) = \dfrac{az+b}{cz+d}$ is *primitive* if it matches one of the following four types.

- Translation $\quad S(z) = z + b$
- Dilation $\quad S(z) = az \quad$ where $a \in \mathbb{R}$ and $a > 0$
- Rotation $\quad S(z) = az \quad$ where $a = \operatorname{cis}\theta$ for some $\theta \in \mathbb{R}$
- Inversion $\quad S(z) = \frac{1}{z}$

A function of the form $S(z) = kz$, where k is an arbitrary complex number, may be viewed as a composition of a dilation and a rotation, since $k = r\operatorname{cis}\theta$ for $r = |k|$ and $\theta = \arg(k)$.

Proposition 1. *A Möbius transformation is a composition of translations, dilations, rotations, and inversions.*

Proof. Let $S(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation.

Suppose $c = 0$. Then $S(z) = \frac{a}{d}z + \frac{b}{d}$. Setting $S_1(z) = \frac{a}{d}z$ and $S_2(z) = z + \frac{b}{d}$, we see that $S = S_2 \circ S_1$.

On the other hand, if $c \neq 0$, we compute that that

$$S(z) = \frac{bc - ad}{c^2(z + \frac{d}{c})} + \frac{a}{c}.$$

Let $S_1(z) = z + \frac{d}{c}$, $S_2(z) = c^2 z$, $S_3(z) = \frac{1}{z}$, $S_4(z) = (bc - ad)z$, and $S_5(z) = z + \frac{a}{c}$. Then $S = S_5 \circ S_4 \circ S_3 \circ S_2 \circ S_1$. So S is a translation, following by a dilation/rotation, followed by inversion, followed by another dilation/rotation, followed by another translation. $\square$

Example 1. Find Möbius transformation $S(z) = \dfrac{az+b}{cz+d}$ which acts as the following sequence of transformations:

- Translate the plane so that 1 goes to 3
- Dilate the plane by a factor of 2
- Invert the sphere
- Rotate the sphere counterclockwise by 90°
- Translate the plane so that i goes to $2i$

Compute a, b, c, and d.

Solution. Let $S_1(z) = z + 2$, $S_2 = 2z$, $S_3(z) = \frac{1}{z}$, $S_4(z) = iz$, $S_5(z) = z + i$. Set $S = S_5 \circ S_4 \circ S_3 \circ S_2 \circ S_1$. Then

$$S(z) = \frac{i}{2z+4} + i = \frac{2iz + 5i}{2z+4}.$$

We have $a = 2i$, $b = 5i$, $c = 2$, and $d = 4$. $\square$

3. Circles

Recall the stereographic projection sends circles on the Riemann sphere to lines and circles on the complex plane.

It is also the case that Möbius transformations send circles to circles, with the understanding that a line in $\mathbb{C}$ can be considered to be a "circle through infinity". Thus, if $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ has the property that the image of any circle is a circle, then we will say that S "preserves circles". Now if S and T both perserve circles, it is clear that their composition $T \circ S$ also preserves circles.

Next, we use the fact that any Möbius transformation is a composition of primitive transformations of these types: translations, dilations, rotations, and the inversion. It is fairly obvious that the first three types preserve lines and circles in $\mathbb{C}$, so we focus on the inversion.

Let $f(z) = \dfrac{1}{z} = u + iv$. Direct computation shows that

$$\frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2},$$

so $u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$.

Consider the equation

$$\alpha(x^2+y^2) + \beta x + \gamma y = \delta,$$

where α, β, and γ are not all zero. The locus of any line or circle can be written in this form, any locus of this form is a line or a circle. Divide through by x^2+y^2 to get

$$\alpha + \beta\frac{x}{x^2+y^2} + \gamma\frac{y}{x^2+y^2} = \delta\frac{1}{x^2+y^2}.$$

We recognize u and v here; computation show that $u^2 + y^2 = \dfrac{1}{x^2+y^2}$. Thus this equation may be rearranged as

$$\alpha + \beta u - \gamma v = \delta(u^2+v^2),$$

which becomes

$$\delta(u^2+v^2) - \beta u + \gamma v = \alpha,$$

which is the equation of a circle. Thus inversion preserves circles, and therefore, all Möbius transformations preserve circles.

4. Fixed Points

Definition 2. Let $f : A \to A$. A *fixed point* of f is an element $a \in A$ such that $f(a) = a$.

We investigate the fixed points of a Möbius transformation. Suppose $S(z) = \dfrac{az+b}{cz+d}$. Then $S(w) = w$ means that $\dfrac{aw+b}{cw+d} = w$, so $aw + b = cw^2 + dw$, that is,

$$cw^2 + (d-a)w - b = 0.$$

Solving this equation leads to

$$w = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Clearly ∞ is a fixed point if and only if $c = 0$. In this case, S is linear, and there is a unique finite fixed point at $z = \dfrac{b}{d-a}$, unless $d = a$, in which case $b = 0$ and S is the identity given by $S(z) = z$.

Moreover, when S is not the identity, we see that S has at most two fixed points.

Now suppose that S and T are Möbius transformations which have the same values at three distinct points. Then $T^{-1} \circ S$ also will fix those three points, which implies that $T^{-1} \circ S$ is the identity, so $S = T$. This shows that a Möbius transformation is complete determined by its effect on any three points.

Example 2. Find the fixed points of $S(z) = \dfrac{z+2}{3z+5}$.

Solution. If $\dfrac{z+2}{3z+5} = z$, then

$$3z^2 + (5-1)z - 2 = 0,$$

so

$$z = \frac{-4 \pm \sqrt{16+24}}{6} = \frac{-2 \pm \sqrt{10}}{3}.$$

□

Example 3. Let $S(z) = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$ and $c = 1$, and the following properties:

- $S(0) = 0$;
- $S(1) = 1$;
- $S(\infty) = 2$.

Find a, b, c, and d.

Solution. Use each of these conditions:

- Since $S(0) = \dfrac{b}{d} = 0$, we see that $b = 0$, so $S(z) = \dfrac{az}{z+d}$.
- Since $S(1) = \dfrac{a}{1+d} = 1$, we see that $a = d + 1$.
- Since $S(\infty) = \dfrac{a}{1} = 2$, we see that $a = 2$, so $d = 1$.

Thus $a = 2$, $b = 0$, $c = 1$, $d = 1$; thus

$$S(z) = \frac{2z}{z+1}.$$

□

5. The Cross-Ratio

We know that a Möbius transformation is completely determined by its effect on any three distinct points in the Riemann sphere. In fact, there is exactly one Möbius transformation which sends any given ordered triple $(z_2, z_3, z_4) \in \mathbb{C}^3$ to another specified ordered triple $(w_2, w_3, w_4) \in \mathbb{C}^3$.

To see this, we first define a classical notation with historical roots, which we now describe. If A, B, C, D are points in an affine (think flat) plane, then their cross-ratio is the number

$$\mathfrak{R}(A, B, C, D) = \frac{AC}{AD} \Big/ \frac{BC}{BD},$$

where PQ represents the *signed* distance from P to Q. This number has important implication in projective geometry.

Definition 3. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$. The *cross ratio* of these points is

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}.$$

Beware, the (standard) notation here is ambiguous. The ordered tuple (z_1, z_2, z_3, z_4) is being a mapped to a value which is identified with the same notation. You have to discern which meaning is intended from the context.

Note that

- $(z_2, z_2, z_3, z_4) = 1$
- $(z_3, z_2, z_3, z_4) = 0$
- $(z_4, z_2, z_3, z_4) = \infty$

To understand the cross-ratio more fully, we select three points z_2, z_3, and z_4, which we wish to send to 1, 0, and ∞, respectively. The Möbius transformation

$$S(z) : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad S(z) = (z, z_2, z_3, z_4) = \frac{z - z_3}{z - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}$$

has this effect; that is,

- $S(z_2) = 1$
- $S(z_3) = 0$
- $S(z_4) = \infty$

Thus S is the *unique* bijective rational function $\mathbb{C}_\infty \to \mathbb{C}_\infty$ which sends the order triple (z_2, z_3, z_4) to the ordered triple $(1, 0, \infty)$.

Example 4. If $T(z) = \dfrac{az + b}{cz + d}$, find z_2, z_3, z_4, written in terms of a, b, c, d, such that $T(z) = (z, z_2, z_3, z_4)$.

Solution. Since $\dfrac{az_2 + b}{cz_2 + d} = 1$, we know that $az_2 + b = cz_2 + d$, so $(a - d)z_2 = d - b$, and $z_2 = \dfrac{d - b}{a - d}$.

Since $\dfrac{az_3 + b}{cz_3 + d} = 0$, we know that $az_3 + b = 0$, so $z_3 = -\dfrac{b}{a}$.

Since $\dfrac{az_4 + b}{cz_4 + d} = \infty$, we know that $cz_4 + d = 0$, so $z_4 = -\dfrac{d}{c}$. $\square$

6. Properties of Cross Ratio

It is convenient to suppress some of the traditional function notation when working with Möbius transformations. We view a Möbius transformation S as "acting on a point" $z \in \mathbb{C}$, and we write Sz to mean $S(z)$. Also, since the composition of Möbius transformations is a Möbius transformation, we write ST to mean $S \circ T$.

Proposition 2. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$. Then*

$$\overline{(z_1, z_2, z_3, z_4)} = (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}).$$

Proof. This follows from the definition of cross ratio, since conjugation splits on sums, products, and quotients. □

Proposition 3. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$, and let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Then*

$$(Sz_1, Sz_2, Sz_3, Sz_4) = (z_1, z_2, z_3, z_4).$$

Proof. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $T(z) = (z, z_2, z_3, z_4)$; then T is the unique Möbius transformation which sends the ordered triple (z_2, z_3, z_4) to the ordered triple $(1, 0, \infty)$.

Consider the Möbius transformation TS^{-1}; this sends the ordered triple (Sz_2, Sz_3, Sz_4) to $(1, 0, \infty)$, and so it is the unique Möbius transformation which does this. Hence $TS^{-1}(z) = (z, Sz_2, Sz_3, Sz_4)$ for all $z \in \mathbb{C}_\infty$, and in particular, $TS^{-1}(Sz_1) = (Sz_1, Sz_2, Sz_3, Sz_4)$. Thus

$$(Sz_1, Sz_2, Sz_3, Sz_4) = TS^{-1}Sz_1 = T(z_1) = (z_1, z_2, z_3, z_4).$$

□

Proposition 4. *Four points in $\mathbb{C}_\infty$ lie on the same circle if and only if their cross-ratio is real.*

Proof. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$. We wish to show that these points lie on the same circle if and only if

$$(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty.$$

Let C denote the circle which contains z_2, z_3, and z_4. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $T(z) = (z, z_2, z_3, z_4)$; then $T(C)$ is a circle which contains 1, 0, and ∞, so $T(C) = \mathbb{R}_\infty$, and $T^{-1}(\mathbb{R}_\infty) = C$.

Suppose that $z_1 \in C$. Then $T(z_1) \in \mathbb{R}_\infty$, that is, $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$.

On the other hand, suppose that $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty$. Then $T(z_1) \in \mathbb{R}_\infty$, so $z_1 = T^{-1}(T(z_1)) \in C$. □

7. Field of Definition

Definition 4. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. We say that T is *defined over* $\mathbb{R}$ if there exist $a, b, c, d \in \mathbb{R}$ such that $T(z) = \dfrac{az+b}{cz+d}$.

Note that it is possible that T is defined over $\mathbb{R}$, but that T is presented with nonreal coefficients. For example, $T(z) = \dfrac{2iz+3i}{5iz+7i}$ is defined over $\mathbb{R}$.

Proposition 5. *Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Then T is defined over $\mathbb{R}$ if and only if $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$.*

Proof. Suppose that T is defined over $\mathbb{R}$. Then $T(z) = \dfrac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{R}$. Let $x \in \mathbb{R}_\infty$. If $x = \infty$, then $T(x) = \dfrac{a}{c} \in \mathbb{R}_\infty$. If $T(x) = \infty$, then $T(x) \in \mathbb{R}_\infty$. Otherwise, $T(x) = \dfrac{ax+b}{cx+d}$ is a composition of the sums and products of real numbers, and hence is real.

Conversely, suppose that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Then $T^{-1}(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Let $z_2 = T^{-1}(1)$, $z_3 = T^{-1}(0)$, and $z_4 = T^{-1}(\infty)$, noting that $z_2, z_3, z_4 \in \mathbb{R}_\infty$. Then T is the unique Möbius transformation which maps the ordered triple (z_2, z_3, z_4) onto the ordered triple $(1, 0, \infty)$; this shows that

$$T(z) = \frac{z - z_3}{z - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}.$$

Since z_2, z_3, z_4 are real, we have written T using only real coefficients, and T is defined over $\mathbb{R}$. $\square$

Proposition 6. *Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Then T is defined over $\mathbb{R}$ if and only if*

$$T(\overline{z}) = \overline{T(z)}$$

for all $z \in \mathbb{C}$.

Proof. Suppose T is defined over $\mathbb{R}$. Write $T(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned} \overline{T(z)} &= \frac{\overline{a}\overline{z} + \overline{b}}{\overline{c}\overline{z} + \overline{d}} && \text{since conjugation splits on sums and products} \\ &= \frac{a\overline{z} + b}{c\overline{z} + d} && \text{since the coefficients are real} \\ &= T(\overline{z}). \end{aligned}$$

For the converse, we first note that $z \in \mathbb{R}$ if and only if $z = \overline{z}$. So, suppose that $T(\overline{z}) = \overline{T(z)}$ for all $z \in \mathbb{C}$. Let $x \in \mathbb{R}$. We have

$$T(x) = T(\overline{x}) = \overline{T(x)},$$

so $T(x)$ is real. Thus $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, so T is defined over $\mathbb{R}$. $\square$

8. Symmetry

Definition 5. Let C be a circle through $z_2, z_3, z_4 \in \mathbb{C}_\infty$. The points z and z^* are said to be *symmetric* with respect to C if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

Proposition 7. *The definition of symmetry is independent of the choice of z_2, z_3, z_4. That is, if the points z_2, z_3, z_4 lie on the same circle as w_2, w_3, w_4, then*

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} \quad \text{if and only if} \quad (z^*, w_2, w_3, w_4) = \overline{(z, w_2, w_3, w_4)}.$$

Proof. Let C be a circle in $\mathbb{C}_\infty$. Let z_2, z_3, z_4 be three distinct points on C, and let w_2, w_3, w_4 be another three distinct points on C. Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $T(z) = (z, z_2, z_3, z_4)$, and let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $S(z) = (z, w_2, w_3, w_4)$. We may rewrite our goal as

$$T(z^*) = \overline{T(z)} \Leftrightarrow S(z^*) = \overline{S(z)}.$$

Now T sends (z_2, z_3, z_4) to $(1, 0, \infty)$, and S sends (w_2, w_3, w_4) to $(1, 0, \infty)$, so $T(C) = \mathbb{R}_\infty$ and $S(C) = \mathbb{R}_\infty$. Thus $ST^{-1} : \mathbb{R}_\infty \to \mathbb{R}_\infty$, so ST^{-1} is defined over $\mathbb{R}$.

Let us assume that $T(z^*) = \overline{T(z)}$; we wish to show that $S(z^*) = \overline{S(z)}$. Now

$$S(z^*) = ST^{-1}(T(z^*)) = ST^{-1}(\overline{T(z)}) = \overline{ST^{-1}T(z)} = \overline{S(z)},$$

the pivotal third equal sign is attained by the fact that ST^{-1} is defined over $\mathbb{R}$. □

Proposition 8. *Let S be a Möbius transformation. Let $z \in \mathbb{C}_\infty$ and let z and z^* be symmetric with respect to a circle C. Then $S(z)$ and $S(z^*)$ are symmetric with respect to the circle $S(C)$.*

Proof. Let $z_2, z_3, z_4 \in \mathbb{C}$ be distinct points on the circle C. Let $T(z) = (z, z_2, z_3, z_4)$. Then $S(z_2)$, $S(z_3)$, and $S(z_4)$ are distinct point on the circle $S(C)$. Thus, by Proposition 3,

$$\begin{aligned}(Sz^*, Sz_2, Sz_3, Sz_4) &= (z^*, z_2, z_3, z_4)\\ &= \overline{(z, z_2, z_3, z_4)}\\ &= \overline{(Sz, Sz_2, Sz_3, Sz_4)}.\end{aligned}$$

This shows that Sz and Sz^* are symmetric with respect to C. □

Let C be a circle with center $a \in \mathbb{C}$ and radius $R \in \mathbb{R}$. Let w be a point outside the circle. Our next goal is to write w^* as a function of a, R, and w, and to find a geometric interpretation for this formula (so, we could find the symmetric point visually).

Proposition 9. *Let C be a circle in $\mathbb{C}$ with center a and radius R. Let $z \in \mathbb{C}$ and let z and z^* be symmetric with respect to a circle C. Let $z_2, z_3, z_4 \in C$. Then*

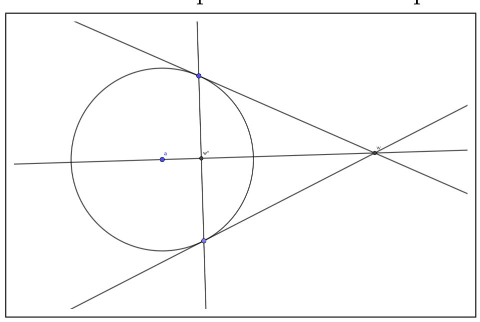
$$z^* = \frac{R^2}{\overline{z} - \overline{a}} + a.$$

Proof. We make repeated use of Propositions 2 and 3:

$$\begin{aligned}
\overline{(z, z_2, z_3, z_4)} &= \overline{(z - a, z_2 - a, z_3 - a, z_4 - a)} && \text{applying Prop 3 with } S(w) = w - a \\
&= (\overline{z - a}, \overline{z_2 - a}, \overline{z_3 - a}, \overline{z_4 - a}) && \text{applying Prop 2} \\
&= \left(\overline{z} - \overline{a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \frac{R^2}{z_4 - a}\right) && \text{using the fact that } \overline{w} = \frac{|w|^2}{w} \\
&= \left(\frac{R^2}{\overline{z} - \overline{a}}, z_2 - a, z_3 - a, z_4 - a\right) && \text{applying Prop 3 with } S(w) = \frac{R^2}{w} \\
&= \left(\frac{R^2}{\overline{z} - \overline{a}} + a, z_2, z_3, z_4\right) && \text{applying Prop 3 with } S(w) = w + a \\
&= (z^*, z_2, z_3, z_4) && \text{by definition of } z^*
\end{aligned}$$

Thus $z^* = \dfrac{R^2}{\overline{z} - \overline{a}} + a$. $\square$

There is a geometric interpretation for symmetry through a circle with center a and radius R. Let w be a point outside of the circle and draw the two lines through w and tangent to the circle. The midpoint between the points of tangency is w^*.



To see this, let b be a point of tangency. Then $\triangle abw^*$ is similar to $\triangle awb$. Thus $\dfrac{|w^* - a|}{|b - a|} = \dfrac{|b - a|}{|w - a|}$. Since the radius is R, this gives $|w - a||w^* - a| = R^2$, so $|w^* - a| = \dfrac{R^2}{|w - a|}$. Now w^* is on the line from a to w, and its distance from a is given above; we multiply both sides by the unit vector from a to w and add a; we find that

$$w^* = a + \frac{w - a}{|w - a|} \frac{R^2}{|w - a|} = a + \frac{R^2}{\overline{w} - \overline{a}}.$$

So, this is the correct interpretation for w^*.

9. Orientation

Let C be a circle in $\mathbb{C}$ with center a and radius R. Then C is parameterized by a path $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = a + Re^{\pi it}$.

Pick three distinct points on the circle, say z_i for $i = 2, 3, 4$. For each of these, there exists a unique $t \in [0, 2\pi]$ such that $z_i = \gamma(t_i)$. There are six possibilities, broken into two classes.

- Counterclockwise: $t_2 < t_3 < t_4$ $\quad t_3 < t_4 < t_2$ $\quad t_4 < t_2 < t_3$
- Clockwise: $t_2 < t_4 < t_3$ $\quad t_3 < t_2 < t_4$ $\quad t_4 < t_3 < t_2$

We say that the ordered triple (z_2, z_3, z_4) is either *clockwise* or *counterclockwise* depending on which of these classes it falls into.

Definition 6. An *orientation* of a circle is an equivalence class of ordered triples of points on the circle. Denote the equivalence class of (z_2, z_3, z_4) by $[z_2, z_3, z_4]$.

An *oriented circle* is a circle together with an orientation. If the orientation is counterclockwise, we say the circle is *positively oriented*, and that the orientation is a positive orientation; otherwise, the circle is *negatively oriented*, and the orientation is a negative orientation.

So, $[z_2, z_3, z_4]$ may be referred to as either "counterclockwise" (or positively oriented) or "clockwise" (or negatively oriented).

Definition 7. Let C be a circle in $\mathbb{C}$ and let $z_2, z_3, z_4 \in C$ be distinct points on the circle. Let $z \in \mathbb{C}$, not on the circle. We say that z is *to the right* of C with respect to the orientation $[z_2, z_3, z_4]$ if

$$\operatorname{Im}(z, z_2, z_3, z_4) > 0.$$

Otherwise it is *to the left* of C.

Example 5. Let C be an oriented circle, with orientation $[z_2, z_3, z_4]$, where $z_2 = 1$, $z_3 = i$, and $z_4 = -1$. According to our previous discussion, C is positively oriented. Let $z = 2$; we find whether z is to the right or to the left of z.

$$(z, z_2, z_3, z_4) = \frac{z - z_3}{z - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4} = \frac{2 - i}{2 + 1} \Big/ \frac{1 - i}{1 + 1} = \frac{2}{3} \cdot \frac{2 - i}{1 - i} = \frac{3 + i}{3},$$

so $\operatorname{Im}(z, z_2, z_3, z_4) = \dfrac{1}{3} > 0$, and z is on the right of C.

This indicates that a point is on the right of a positively oriented circle if and only if it is outside the circle; being outside a circle means being on the same side of the circle as ∞. We may verify this.

Example 6. Let C be an arbitrary circle with center $a \in \mathbb{C}$ and radius $R \in \mathbb{R}$. Let $z_2 = a + Ri$, $z_3 = a - R$, and $z_4 = a + R$. Then $[z_2, z_3, z_4]$ is a positive orientation for the circle. Let $z = \infty$. Compute

$$(z, z_2, z_3, z_4) = \frac{\infty - (a - R)}{\infty - (a + R)} \Big/ \frac{(a + iR) - (a - R)}{(a + iR) - (a + R)} = \frac{iR - R}{iR + R} = \frac{i - 1}{i + 1} = \frac{(i - 1)^2}{-2} = i.$$

Thus $\operatorname{Im}(z, z_2, z_3, z_4) = 1 > 0$, so ∞ is on the right of this positively oriented circle.

A Möbius transformation T sends the oriented circle C onto a circle oriented by Tz_2, Tz_3, and Tz_4. From the invariance of the cross-ratio, it follows that the left and right of C will correspond to the left and right of the image circle.

We know expand our viewpoint to include lines as circles through infinity.

Suppose that C is the real axis, so that $z_2, z_3, z_4 \in \mathbb{R}$. In this case, there exist $a, b, c, d \in \mathbb{R}$ such that

$$(z, z_2, z_3, z_4) = \frac{az+b}{cz+d}.$$

Compute that

$$\operatorname{Im}(z, z_2, z_3, z_4) = \frac{ad-bc}{|cz+d|^2} \cdot \operatorname{Im} z.$$

So, being to the right or left of C is equivalent to being in the upper or lower half plane if $\mathbb{R}_\infty$, depending on the orientation.

Example 7. Let T be the Möbius transformation which maps $(1, i, -1)$ to $(1, 0, \infty)$. Then $T(z) = (z, 1, i, -1)$, and $T(z)$ is to the right of $\mathbb{R}_\infty$, oriented by $[1, 0, \infty]$, if and only if z is to the right of the positively oriented unit circle. We have already computed that $T(2) = 1 + i\frac{1}{3}$; so $\operatorname{Im}(1 + i\frac{1}{3}, 1, 0, \infty) = \operatorname{Im}(2, 1, i, -1) = \frac{1}{3}$. Thus the upper half plane is to the right of the real line, when it is oriented from 1 to 0.

What can we say about orientation when the "circle" is a line? From this point of view, the line is a circle which passes through ∞, so we take $z_4 = \infty$. Now the orientation of the line is determined by an ordered pair of points.

Suppose L is a straight line. An orientation for L is $[z_2, z_3, \infty]$ where $z_2, z_3 \in L$. A point z is to the right if

$$\operatorname{Im} \frac{z - z_3}{z_2 - z_3} > 0.$$

These points lie in the right half plane determined by the oriented line $z = z_2 + (z_3 - z_2)t$.

Should we invent a meaning for a positively oriented line?

10. Angles

If we view two complex numbers $v, w \in \mathbb{C}$ as vectors, it is clear that the angle between these vectors is $\arg w - \arg v = \arg \frac{w}{v}$.

Let L_1 be the line through z_1 and z_3, and let L_2 be the line through z_2 and z_3. The angle between these lines is $\frac{z_1 - z_3}{z_2 - z_3}$.

Now if we have two oriented circles which intersect, they have well-defined tangent vectors at a point of intersection, and we define the angle between the circles to be the angle between these tangent vectors.

If the circles are tangent, the angle between them is zero. Consider two circles, or a line and a circle, which intersect in exactly one point, and let T be a Möbius transformation which sends that point to ∞. The images of the circles, or the line and the circle, now intersect only at ∞; that is, they are parallel lines.

Proposition 10. *Let C_1 and C_2 be circles in $\mathbb{C}_\infty$ which intersection at $z_3, z_4 \in \mathbb{C}_\infty$. Let $z_1 \in C_1$ and $z_2 \in C_2$ be other points. Let $[z_1, z_3, z_4]$ and $[z_2, z_3, z_4]$ be orientations for C_1 and C_2, respectively. Then the angle at z_3 between C_1 and C_2 is* $\arg(z_1, z_2, z_3, z_4)$.

Proof. First suppose that $z_4 = \infty$, so that C_1 and C_2 are lines. In this case,

$$\arg(z_1, z_2, z_3, z_4) = \arg \frac{z_1 - z_3}{z_2 - z_3},$$

which is the angle between the direction vector $z_1 - z_3$ and the direction vector $z_2 - z_3$, as desired.

Note that if we apply a Möbius transformation T to $\mathbb{C}_\infty$, we have

$$\angle C_1, C_2 = \arg(z_1, z_2, z_3, z_4) = \arg(Tz_1, Tz_2, Tz_3, Tz_4) = \angle T(C_1), T(C_2).$$

In general, let L_1 and L_2 be the oriented tangent lines to the circles C_1 and C_2, respectively, at the point z_3. The angle between L_1 and L_2 is the angle between C_1 and C_2. If we apply a Möbius transformation T which sends z_4 to ∞, the images of C_1 and L_1 are parallel lines, as are the images of C_2 and L_2. Thus the angle between $T(C_1)$ and $T(C_2)$ equals the angle between $T(L_1)$ and $T(L_2)$, so

$$\angle C_1, C_2 = \angle L_1, L_2 = \angle T(L_1), T(L_2) = \angle T(C_1), T(C_1) = \arg(Tz_1, Tz_2, Tz_3, Tz_4) = \arg(z_1, z_2, z_3, z_4),$$

the second to last equality is given by the fact that $T(C_1)$ and $T(C_2)$ are lines. $\square$

Proposition 11. *A Möbius transformation preserves angles between oriented circles.*

Proof. Since the angle is the argument of a cross ratio, and this cross ratio is preserved by a Möbius transformation, the angle is also preserved. $\square$

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