AP Calculus AB — Responses 03/20

Dr. Paul L. Bailey — Thursday, March 19, 2020

I am happy you all are engaged and asking thoughtful questions.

Problem 1. Compute $\int \frac{x}{x^2+1}\,dx$.

Solution. Let $u = x^2+1$ so that $du = 2x\,dx$. Then

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(x^2+1)+C.$$

□

Problem 2. Compute $\int \frac{x}{x^2}+1\,dx$.

Solution. We have

$$\int \frac{x}{x^2}+1\,dx = \frac{1}{2}\int \frac{1}{x}+1\,dx = \ln x + x + C.$$

□

Problem 3 (Thomas §8.1 # 48)**.** Compute $\displaystyle\int \frac{x^2}{x^2+1}\,dx$.

Solution. Add and subtract one from the top:

$$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx = x - \arctan x + C.$$

□

Question 1. On the top of page 482, where did the 5 and 1 for value of limits of integration come from?

Answer. The substitution theorem for definite integrals is

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

where $u = g(x)$.

It practice, this means the following. In §7.2 Example 2b, we wish to compute

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta.$$

The domain variable is θ, so the limits of integration are from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

We let $u = 3+2\sin\theta$ so that $du = 2\cos\theta\,d\theta$, and we may write

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta = \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{2}{u}\,du.$$

We now have a choice; we can take the antiderivative with respect to u, plug $u = 3+2\sin\theta$ back in, and plug in the limits:

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{2}{u}\,du = 2\ln u\Big|_{\theta=-\pi/2}^{\theta=\pi/2} = 2\ln(3+2\sin\theta)\Big|_{-\pi/2}^{\pi/2} = 2\ln 5 - 2\ln 1 = 2\ln 5.$$

Or, we can change the limits themselves by plugging the θ values into $u = 3+2\sin\theta$. If $\theta = -\pi/2$, then $u(\theta) = 1$, and if $\theta = \pi/2$, then $u(\theta) = 5$. So,

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{2}{u}\,du = \int_1^5 \frac{2}{u}\,du = 2\ln u\Big|_1^5 = 2\ln 5.$$

You may choose the approach you prefer in any particular problem. □

Question 2. I'm not too comfortable with reducing an improper fraction.

Proof. You wish to divide the denominator into the numerator. If we divide f into g, we get a quotient q and a remainder r such that $g = fq + r$. In this case, $\frac{g}{f} = q + \frac{r}{f}$, which may be easier to integrate. In some cases, this can be done with a simple trick, as in Problem 48 above.

Don't worry too much about this, you will cover it in detail in Calculus BC. □

Question 3. How do I do harder integration substitution problems because they tend to take me too long to do well on the multiple choice practice tests.

Answer. How does Drew Brees throw a 70 % completion rate every season? Practice, practice, practice. □

Question 4. What is the actual definition of exp?

Answer. Exp is a function $\exp : \mathbb{R} \to (0, \infty)$ given by $\exp(x) = e^x$.

That is, $\exp x$ and e^x mean the same thing.

Sometimes, it is notationally more convenient to write $\exp x$ instead of e^x.

For example, it is easier to write $\exp(x \log a)$ than it is to write $e^{x \ln a}$. It is easier to say "take exp of both sides of an equation" than to say "stick both sides of an equation into the exponent of e".

For the actual definition of either $\exp x$ or e^x, see the next answer. □

Question 5. Are there any other definitions that we can use for natural logarithm?

Proof. There are three basic approaches to defining e^x and/or the natural logarithm.

(a) First approach is to define $\log x = \int_1^x \frac{1}{t}\, dt$, and define $\exp x = e^x$ as the inverse function. This is what Thomas does in Chapter 7.

(b) The second approach is to define $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$, which is motivated by compounded interest. This is what we did in October.

(c) The third approach is to use an infinite series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots,$$

which has the advantage that it still works for complex numbers. This is how many theoretical mathematicians think about it. You will learn this in Calculus BC.

Henceforth, let us use Thomas' Chapter 7 definition, it is the cleanest for our purposes. □