CATEGORY THEORY
TOPIC IX - GROUP ACTIONS

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1. Group Actions

Definition 1. A *group action* (G, X) is a group G together with a set X and a function $G \times X \to X$, $(g, x) \mapsto gx$, such that

(A1) $1x = x$ for all $x \in X$;

(A2) $(hg)x = h(gx)$ for all $g, h \in G$ and all $x \in X$.

If (G, X) is a group action, we say that G acts on X.

Proposition 1. *Let (G, X) be a group action. Define $\phi_g : X \to X$ by $\phi_g(x) = gx$ for each $g \in G$. Then ϕ_g is bijective, and*

$$\phi : G \to \mathrm{Sym}(X) \qquad \text{given by } g \mapsto \phi_g$$

is a group homomorphism.

Proof. We see that ϕ_g is injective by the associativity of the group action, viz,

$$gx = gy \Rightarrow x = 1x = (g^{-1}g)x = g^{-1}(gx) = g^{-1}(gy) = (g^{-1}g)y = 1y = y.$$

Also ϕ_g is surjective because for $x \in X$, $g(g^{-1}x) = x$. Thus ϕ_g is bijective.

To see that ϕ is a homomorphism, let $g, h \in G$; then

$$\phi_{gh}(x) = (gh)x = g(hx) = g(\phi_h(x)) = \phi_g \circ \phi_h(x).$$

□

Proposition 2. *Let $\phi : G \to \mathrm{Sym}(X)$ be a group homomorphism. Let $\phi_g : X \to X$ be the image of g in $\mathrm{Sym}(X)$. Define a function $G \times X \to X$ by $(g, x) \mapsto gx = \phi_g(x)$. Then (G, X) is a group action.*

Proof. Since ϕ is a homomorphism, $\phi(1) = \mathrm{id}_X$, so $1x = x$. Also

$$(gh)x = \phi_{gh}(x) = (\phi_g \circ \phi_h)(x) = \phi_g(\phi_h(x)) = g(hx).$$

□

Corollary 1. *If G acts on X, then every subgroup H acts on X by restriction of the associated homomorphism.*

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2. Examples

Example 1. Let X be a set. Then $\mathrm{Sym}(X)$ acts on X, and every subgroup of $\mathrm{Sym}(X)$ acts on X, in the obvious way.

Let G be a group. Then $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ act on G.

If $\phi : G \to \mathrm{Aut}(H)$ is a group homomorphism for some group H, then we say that G acts on H by automorphism.

Example 2. Let G be a group and let X be a set. Let $\phi : G \to \mathrm{Sym}(X)$ be a group action of G on X. Let $\mathcal{P}(X)$ be the set of all subsets of X, called the *power set* of X. There is an induced group action of G on the power set of X

$$\Phi : G \to \mathrm{Sym}(\mathcal{P}(X)) \text{ given by } \Phi_g : A \mapsto \phi_g(A).$$

Example 3. Let G and H be groups. Let $\phi : G \to \mathrm{Aut}(G)$ be a group action of G on H by automorphism. Let $\mathcal{S}(H)$ be the set of all subgroups of H. There is an induced group action of G on $\mathcal{S}(H)$

$$\Phi : G \to \mathrm{Sym}(\mathcal{S}(H)) \text{ given by } \Phi_g : U \mapsto \phi_g(U).$$

Example 4. Let G be a group. Then G acts on itself by conjugation. This action is induced by the homomorphism $\mathrm{inn} : G \to \mathrm{Inn}(G) \leq \mathrm{Sym}(G)$.

Let $\mathcal{S}(G)$ be the set of subgroups of G. Then G acts on $\mathcal{S}(G)$ by conjugation.

Example 5. Let G be a group. Then G acts on itself by left multiplication.

Example 6. The set of nonzero reals $\mathbb{R}^*$ is a group under multiplication which acts on the vector space $\mathbb{R}^n$ by scalar multiplication.

Let $\mathbf{GL}_n(\mathbb{R})$ be the set of invertible $n \times n$ matrices with real entries. Then $\mathbf{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication. The following subgroups of $\mathbf{GL}_n(\mathbb{R})$ also act on $\mathbb{R}^n$, each in its own geometric fashion:

- $\mathbf{SL}_n(\mathbb{R}) = \{A \in \mathbf{GL}_n(\mathbb{R}) \mid \det(A) = 1\}$;
- $\mathbf{AL}_n(\mathbb{R}) = \{A \in \mathbf{GL}_n(\mathbb{R}) \mid \det(A) > 0\}$;
- $\mathbf{DL}_n(\mathbb{R}) = \{A \in \mathbf{GL}_n(\mathbb{R}) \mid \det(A) = \pm 1\}$;
- $\mathbf{ZL}_n(\mathbb{R}) = \{A \in \mathbf{GL}_n(\mathbb{R}) \mid A = \lambda I \text{ for some } \lambda \in \mathbb{R}^*\}$;
- $\mathbf{GO}_n(\mathbb{R}) = \{A \in \mathbf{GL}_n(\mathbb{R}) \mid AA^t = I\}$;
- $\mathbf{SO}_n(\mathbb{R}) = \{A \in \mathbf{SL}n(\mathbb{R}) \mid AA^t = I\}$.

Example 7. Let V be a vector space over a field K and let $\mathrm{Aut}_K(V)$ be the group of linear transformations from V onto itself. Then $\mathrm{Aut}_K(V)$ acts on V.

If $\phi : G \to \mathrm{Aut}_K(V)$ is a group homomorphism, we say that G acts on V by linear transformation.

Example 8. Let X be a topological space and let $\mathrm{Homeo}(X) \leq \mathrm{Sym}(X)$ be the group of homeomorphisms from X onto itself. Then $\mathrm{Homeo}(X)$ acts on X.

If $\phi : G \to \mathrm{Homeo}(X)$ is a group homomorphism, we say that G acts on X by homeomorphism, or *continuously*.

Example 9. Let X be a smooth manifold and let $\mathrm{Diffeo}(X) \leq \mathrm{Sym}(X)$ be the group of diffeomorphisms from X onto itself. Then $\mathrm{Diffeo}(X)$ acts on X.

If $\phi : G \to \mathrm{Diffeo}(X)$ is a group homomorphism, we say that G acts on X by diffeomorphism, or *smoothly*.

3. Faithfulness and Simpleness

Definition 2. Let (G, X) be a group action.

The *kernel* of the action (G, X) is denoted $\ker(G, X)$ and is defined by

$$\ker(G, X) = \{g \in G \mid gx = x \text{ for all } x \in X\}.$$

Proposition 3. *The kernel of a group action (G, X) corresponds to the kernel of the induced homomorphism $\phi : G \to \mathrm{Sym}(X)$. Therefore, the kernel is a normal subgroup of G.*

Definition 3. Let (G, X) be a group action.

We say that the action is *faithful* if $\ker(G, X) = \{1\}$.

We say that the action is *trivial* if $\ker(G, X) = G$.

Remark 1. Some authors use the word *effective* instead of "faithful".

Proposition 4. *Let (G, X) be a group action with kernel K. Then (G, X) induces a group action $(G/K, X)$ which is faithful.*

Proof. If $\phi : G \to \mathrm{Sym}(X)$ is the homomorphism induced by the action, then ϕ factors through $\overline{\phi} : G/K \to \mathrm{Sym}(X)$ by the first isomorphism theorem, and G/K is isomorphic to its image in $\mathrm{Sym}(X)$. Thus the kernel of $\overline{\phi}$ is trivial, so the induced action $(G/H, X)$ is faithful. □

Definition 4. Let (G, X) be a group action and let $S \subset G$.

The *fixed set* of S is denoted $\mathrm{fix}(S)$ and is defined as

$$\mathrm{fix}(S) = \{x \in X \mid sx = x \text{ for all } s \in S\}.$$

If $S = \{s\}$ is a singleton, we may write $\mathrm{fix}(s)$ instead of $\mathrm{fix}(\{s\})$. The elements of $\mathrm{fix}(G)$ are called the *fixed points* of the action.

Proposition 5. *If (G, X) is a trivial group action, then $\mathrm{fix}(G) = X$.*

Definition 5. Let (G, X) be a group action.

We say that the action is *simple* if $\mathrm{fix}(G) = \varnothing$.

Proposition 6. *Let (G, X) be a group action and let $Y = X \smallsetminus \mathrm{fix}(G)$. Then G acts simply on Y.*

Example 10. When a group G acts on itself by conjugation, the kernel of the action is $Z(G)$. If G is abelian, this action is trivial. The fixed set of the action is also $Z(G)$.

When G acts on its set of subgroups by conjugation, the fixed points of the action are exactly the normal subgroups.

Example 11. When $\mathbf{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication, the only fixed point is the origin.

Warning 1. Let (G, X) be a simple group action and let $H \leq G$. We have a restricted action (H, X) which is not necessarily simple.

4. Orbits

Definition 6. Let (G, X) be a group action and let $x \in X$. The *orbit* of x under the action of G is the set

$$\mathrm{orb}(x) = \{y \in X \mid y = gx \text{ for some } g \in G\}.$$

More generally, if $A \subset X$ and $H \subset G$, then

$$\mathrm{orb}_H(A) = \{y \in X \mid y = ha \text{ for some } a \in A \text{ and some } h \in H\}.$$

In many cases it is natural to write Gx for $\mathrm{orb}(x)$, hA for $\mathrm{orb}_h(A)$, HA for $\mathrm{orb}_H(A)$, and so forth.

Proposition 7. *The orbits of an action partition the set X; that is, every element of X is in some orbit and if $x, y \in X$, then $\mathrm{orb}(x)$ and $\mathrm{orb}(y)$ are either identical or disjoint.*

Proof. First note that $x \in \mathrm{orb}(x)$ since $1x = x$. Thus the orbits cover X. Let $x, y \in X$ and suppose $z \in \mathrm{orb}(x) \cap \mathrm{orb}(y)$. Then there exist $g, h \in G$ such that $z = gx$ and $z = hy$. Thus $gx = hy$ so that $x = g^{-1}hy$. Thus $\mathrm{orb}(x) = \mathrm{orb}(g^{-1}hy) = \mathrm{orb}(y)$. $\square$

Example 12. Let G be a group and let $H \leq G$. Then H acts on G by left multiplication. The orbits of this action are the right cosets of H in G.

Example 13. When G acts on itself by conjugation, the orbits are called *conjugacy classes.*

When G acts on its set of subgroups by conjugation, members of the same orbit are called *conjugate subgroups.*

Example 14. The matrix group $\mathbf{SO}_n(\mathbb{R})$ consists of the linear transformations which preserve distance and orientation. When $\mathbf{SO}_n(\mathbb{R})$ acts on $\mathbb{R}^n$, the orbits are concentric spheres around the origin. The matrix group $\mathbf{ZL}_n(\mathbb{R})$ consists of the linear transformations which are dilations. When $\mathbf{ZL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$, the orbits are lines through the origin.

Remark 2. Let (G, X) be a group action, and suppose that X is a finite set. The cardinality of X is the sum of the cardinalities of the distinct orbits in X.

The fixed points of the action sit alone in their orbits. We may select one element from each orbit which is not the orbit of a fixed point and create a set R of representatives of the nonfixed orbits. This gives us the formula

$$|X| = |\mathrm{fix}(G)| + \sum_{x \in R} |\mathrm{orb}(x)|.$$

This seemingly benign observation will become very useful, for example in the proof of Cauchy's Theorem and the Sylow's Theorem.

5. Stabilizers

Definition 7. Let (G, X) be a group action and $x \in X$. The *stabilizer* of x is the subset of G defined by

$$\mathrm{stb}(x) = \{g \in G \mid gx = x\}.$$

The *pointwise stabilizer* of $A \subset X$ is defined by

$$\mathrm{stb}(A) = \{g \in G \mid ga = a \text{ for every } a \in A\}.$$

The *setwise stabilizer* of $A \subset X$ is defined by

$$\mathrm{stb}[A] = \{g \in G \mid ga \in A \text{ for every } a \in A\}.$$

Remark 3. It is clear that $g \in \mathrm{stb}(x) \Leftrightarrow x \in \mathrm{fix}(g)$.

Proposition 8. *Stabilizers are subgroups.*

Proof. Note that $\mathrm{stb}(a) = \mathrm{stb}(\{a\}) = \mathrm{stb}[\{a\}]$.

Let $g, h \in \mathrm{stb}[A]$ and let $a \in A$. Then $ha \in A$ so $gha \in A$ and $gh \in \mathrm{stb}[A]$. Also let $b = g^{-1}(a)$ so that $gb = a$. Since $gA = A$ and g permutes X, $b \in A$ so $g^{-1} \in \mathrm{stb}[A]$. Thus $\mathrm{stb}[A]$ is a subgroup.

Also, $\mathrm{stb}(A) = \cap_{a \in A} \mathrm{stb}(a)$, and the intersection of subgroups is a subgroup. □

Proposition 9. *Let (G, X) be a group action. Let $g \in G$ and $x, y \in X$. Then $\mathrm{stb}(gx) = g\mathrm{stb}(x)g^{-1}$.*

Proof. Since $x, y \in \mathrm{orb}(x)$, there exists $g \in G$ such that $gx = y$. But

$$h \in \mathrm{stb}(gx) \Leftrightarrow hgx = gx \Leftrightarrow g^{-1}hgx = x \Leftrightarrow h^g \in \mathrm{stb}(x) \Leftrightarrow h \in \mathrm{stb}(x)^{g^{-1}}.$$

□

Example 15. When G acts on itself by conjugation, the stabilizer of $g \in G$ is $C_G(g)$. The pointwise stabilizer of a subgroup $H \leq G$ is $C_G(H)$, and the setwise stabilizer is $N_G(H)$. When G acts on its set of subgroups by conjugation, the stabilizer of $H \leq G$ is $N_G(H)$.

Example 16. When $\mathbf{SO}3(\mathbb{R})$ acts on $\mathbb{R}^3$, the stabilizer of a vector $v \neq 0$ is the set of rotations around the axis which is the line through that vector. The setwise stabilizer of a plane through the origin is the one point stabilizer of a vector normal to that plane, which is also the pointwise stabilizer of the entire line in the normal direction.

Example 17. Let V be a vector space over a field K and let W be a subspace. When $\mathrm{Aut}_K(V)$ acts on V, the setwise stabilizer $\mathrm{stb}[W]$ is a collection of linear transformations which send W onto itself. By restriction these transformations to W, we obtain a map $\mathrm{stb}[W] \to \mathrm{Aut}_K(W)$ which is an epimorphism. The kernel of this epimorphism is the pointwise stabilizer $\mathrm{stb}(W)$.

Thus $\mathrm{stb}(W) \triangleleft \mathrm{stb}[W]$ and $\mathrm{Aut}_K(W) \cong \mathrm{stb}[W]/\mathrm{stb}(W)$.

Proposition 10. *Let (G, X) be a group action. Let $x \in X$ and $H = \mathrm{stb}(x)$. Let G/H be the left coset space of H in G. Let $\phi : G/H \to \mathrm{orb}(x)$ be given by $gH \mapsto gx$. Then ϕ is a bijection.*

Proof. The function ϕ is well defined and injective:

$$\begin{aligned} g_1 H = g_2 H &\Leftrightarrow g_1^{-1} g_2 \in H \\ &\Leftrightarrow g_1^{-1} g_2 x = x \\ &\Leftrightarrow g_1 x = g_2 x \\ &\Leftrightarrow \phi(g_1 H) = \phi(g_2 H) \end{aligned}$$

for any $g_1, g_2 \in G$.

The function ϕ is surjective:

$$\begin{aligned} y \in \mathrm{orb}(x) &\Rightarrow \exists g \in G \ni gy = x \\ &\Rightarrow \phi(g^{-1} H) = y. \end{aligned}$$

□

Corollary 2. *Let (G, X) be a group action. Let $x \in X$ and $H = \mathrm{stb}(x)$. Then $|G/H| = |\mathrm{orb}(x)|$.*

Proposition 11. *Let G be a group and let $S \subset G$. Then*

$$\{S^g \mid g \in G\} \leftrightarrow G/N_G(S)$$

is a one to one correspondence given by $gN_G(S) \mapsto S^g$.

Proof. When G acts on its power set by conjugation, $N_G(S)$ is the stabilizer of the set S and $X = \{S^g \mid g \in G\}$ is its orbit. The result follows from Proposition 10. □

Corollary 3. *Let G be a group and $g \in G$. Then*

$$g^G \leftrightarrow G/C_G(g)$$

is a one to one correspondence.

Proof. If $S = \{g\}$ is a singleton set, we have $N_G(g) = C_G(g)$. □

Remark 4. Recall that if X is finite and G acts on X, we have

$$|X| = |\mathrm{fix}(G)| + \sum_{x \in R} |\mathrm{orb}(x)|.$$

Let G be a group acting on itself by conjugation. Let R be a collection of representatives from the orbits of nonfixed points. In this case $\mathrm{fix}(G) = Z(G)$ and $|g^G| = |G/C_G(g)|$. By Lagrange's Theorem, $|G/C_G(g)| = [G : C_G(g)]$. This gives us the *class equation*:

$$|G| = |Z(G)| + \sum_{g \in R} [G : C_G(g)].$$

6. Transitivity

Definition 8. Let (G, X) be a group action.

We say that the action is *transitive* if for every $x, y \in X$ there exists $g \in G$ such that $gx = y$.

Proposition 12. *Let (G, X) be a group action. The following conditions are equivalent:*

- **i.** *for every $x, y \in X$ there exists $g \in G$ such that $gx = y$;*
- **ii.** *for every $x \in X$ the function $G \to X$ given by $g \mapsto gx$ is surjective;*
- **iii.** *for every $x \in X$ we have $\operatorname{orb}(x) = X$;*
- **iv.** *there exists $x \in X$ such that $\operatorname{orb}(x) = X$.*

Corollary 4. *Let (G, X) be a transitive group action. Then $|G| \geq |X|$.*

Proposition 13. *A group action (G, X) induces a transitive group action on each orbit in X.*

Proposition 14. *Let (G, X) be a transitive group action. Let $x \in X$ and let $H = \operatorname{stb}(x)$. Then $|G/H| = |X|$.*

Proof. By Corollary 2, $|G/H| = |\operatorname{orb}(x)|$. But since G is transitive, $\operatorname{orb}(x) = X$. □

Example 18. The action of a group on itself by automorphism is never transitive, since the identity is a fixed element.

The action of a group on itself by left translation is always transitive, because the equation $y = g(g^{-1}y)$.

Example 19. The action of $\mathbf{GL}_n(\mathbb{R})$ on the nonzero vectors in $\mathbb{R}^n$ is transitive, whereas the action of $\mathbf{SO}n(\mathbb{R})$ is not.

Example 20. Let X be a topological space. The action of $\operatorname{Homeo}(X)$ on X may or may not be transitive. For example, if X is a connected manifold, this action is transitive, but if X has a singularity, it is not.

7. Freeness

Definition 9. Let (G, X) be a group action.

We say that the action is *free* if $gx = hx \Rightarrow g = h$ for all $g, h \in G$ and any $x \in X$.

Proposition 15. *Let (G, X) be a group action. The following conditions are equivalent:*

- **i.** $gx = hx \Rightarrow g = h$ *for all* $g, h \in G$ *and any* $x \in X$*;*
- **ii.** *for every* $x \in X$ *the map* $G \to X$ *given by* $g \mapsto gx$ *is injective;*
- **iii.** $\operatorname{fix}(H) = \varnothing$ *for all nontrivial subgroups* $H \leq G$*;*
- **iv.** $\operatorname{fix}(g) = \varnothing$ *for all nontrivial elements* $g \in G$*;*
- **v.** $\operatorname{stb}(x) = \{1\}$ *for all points* $x \in X$*;*

Proof.

(i) $\Leftrightarrow$ (ii) This is immediate.

(i) $\Rightarrow$ (iii) Suppose that $gx = hx \Rightarrow g = h$ for all $g, h \in G$ and any $x \in X$.

Let A be a subgroup of G with a fixed point x. Let $a \in A$. Then $ax = x = 1x$, so $a = 1$; thus $A = \{1\}$.

(iii) $\Rightarrow$ (iv) Suppose that $\operatorname{fix}(A) = \varnothing$ for all nontrivial subgroups $H \leq G$.

Let $g \in G$, $g \neq 1$. Then $\langle g \rangle$ has no fixed points. But if g fixes an element, then so does every power of g, and so does $\langle g \rangle$. Thus g does not fix an element, so $\operatorname{fix}(g) = \varnothing$.

(iv) $\Rightarrow$ (v) Suppose that $\operatorname{fix}(g) = \varnothing$ for all nontrivial elements $g \in G$.

Let $x \in X$. We know that $1x = x$. But no other element of G fixes x, so $\operatorname{stb}(X) = \{1\}$.

(v) $\Rightarrow$ (i) Suppose that $\operatorname{stb}(x) = \{1\}$ for all points $x \in X$.

Let $x \in X$, so that $\operatorname{stb}(x) = \{1\}$, and let $g, h \in G$. Suppose that $gx = hx$. Then $h^{-1}gx = x$, so $h^{-1}g$ stabilizes x. Thus $h^{-1}g = 1$, so $g = h$. □

Corollary 5. *Let (G, X) be a free group action. Then $|G| \leq |X|$.*

Remark 5. Some authors use the word *semiregular* instead of "free".

Example 21. The action of a group on itself by left multiplication is free.

Example 22. When $\mathbf{SO}2(\mathbb{R})$ acts on $\mathbb{R}^2 \smallsetminus \{(0, 0)\}$, this action is free, since the only rotation which fixes a point is the identity rotation. However, the action of $\mathbf{SO}3(\mathbb{R})$ on $\mathbb{R}^3 \smallsetminus \{(0, 0, 0)\}$ is not free, since every rotation fixes the axis of rotation.

8. Regularity

Definition 10. Let (G, X) be a group action.

We say that the action is *regular* if for each $x, y \in X$ there exists a *unique* $g \in G$ such that $gx = y$.

Proposition 16. *Let (G, X) be a group action. The following conditions are equivalent:*

- **i.** *for each $x, y \in X$ there exists a* unique *$g \in G$ such that $gx = y$;*
- **ii.** *for every $x \in X$ the function $G \to X$ given by $g \mapsto gx$ is bijective;*
- **iii.** *the action is transitive and free.*

Corollary 6. *Let (G, X) be a regular group action. Then $|G| = |X|$.*

Proposition 17. *Let (G, X) be a transitive group action. Let $x \in X$ and let $H = \mathrm{stb}(x)$. If $H \triangleleft G$, then $H = \ker(G, X)$.*

Proof. Since G is transitive, all of the stabilizers are conjugate. But since $H \triangleleft G$, each stabilizer is exactly H. So H fixes every point in X. □

Proposition 18. *If G is abelian, transitive, and faithful, then G is regular.*

Proof. A one point stabilizer is normal since G is abelian. Thus it is the kernel of the action. But since G is faithful, this kernel is trivial. Therefore all of the stabilizers are trivial. □

Example 23. Let $X = \{1, 2, \ldots, n\}$. Let $\phi : G \to \mathrm{Sym}(X)$ be a transitive group action. Suppose that for some $y \in X$, we have $\ker(\phi) = \mathrm{stb}(y)$. Let $\sigma \in \mathrm{img}(\phi)$. What can be said about the cycle decomposition of σ?

Answer. Note $\mathrm{Sym}(X) = S_n$. Every permutation in S_n is the product of disjoint cycles.

Let $K = \ker(\phi)$. Since the stabilizers are conjugate, for any $x \in X$, we have $\ker(\phi) = \mathrm{stb}(x)$. Thus G/K acts regularly on X. Let $A = \phi(G)$ so that $G/K \cong A$. Then A acts regularly on X.

Let $\sigma \in A \smallsetminus \{\mathrm{id}\}$. We claim that the cycle decomposition of σ consists of cycles of identical length involving every element of X. Clearly σ has no fixed points, since A acts regularly on X; if σ has a fixed point, then it agrees with the identity at that point and thus is equal to the identity since A acts freely.

Let m be the length of the shortest cycle in the cycle decomposition of σ. Then σ^m fixes at least m points, and so must be the identity. Thus all of the cycles of σ have length m. □

9. Equivalence

Definition 11. Two group actions (G, X) and (H, Y) and *equivalent* if there exists a group isomorphism $\phi : G \to H$ and a bijection $f : X \to Y$ such that $f(gx) = \phi(g)f(x)$ for all $g \in G$ and $x \in X$.

$$X@ > g >> X$$

$$@VfVV@VVfV$$

$$Y@ >> \phi(g) > Y$$

Proposition 19. *Let G be a group with a subgroup H. The action of G on the left coset space of H by left multiplication is a group action.*

Proposition 20. *Let (G, X) be a transitive group action. Let $x \in X$ and let $H = \mathrm{stb}(x)$. Then (G, X) is equivalent to the action of G on the left coset space G/H by left multiplication.*

Proof. Let $Y = G/H$ be the left coset space. Let $\phi : Y \to X$ be given by $gH \mapsto gx$. By Proposition 10, ϕ is a bijection. Also for $g_1, g_2 \in G$,

$$\phi(g_1\overline{g_2}) = \phi(\overline{g_1g_2}) = g_1g_2x = g_1\phi(\overline{g_2}).$$

□

Proposition 21. *If (G, X) is a regular group action, then it is equivalent to the action of G on itself by left multiplication.*

Proof. By Proposition 20, since (G, X) is transitive the action of G is equivalent to the action of G on the left coset space of a single point stabilizer. But since (G, X) is also free, this stabilizer is trivial. □

10. Normalizer/Centralizer Connection

Proposition 22. *Let (G, X) be a transitive group action. Let $x \in X$ and let $H = \mathrm{stb}(x)$. Define*

$$\phi : N_G(H) \to \mathrm{Sym}(X) \quad \textit{by} \quad \phi_g(y) = zg^{-1}x,$$

where $zx = y$. Set $S = \mathrm{Sym}(X)$. Then

- **(a)** *ϕ is a well-defined group homomorphism;*
- **(b)** $\ker(\phi) = H$*;*
- **(c)** $\mathrm{img}(\phi) = C_S(G)$*;*
- **(d)** $N_G(H)/H \cong C_S(G)$*.*

Proof. Note that since G acts transitively, for every $y \in X$ there exists $z \in G$ such that $zx = y$. Suppose $z_1x = z_2x$. Then $z_2^{-1}z_1 \in H$. Since g normalizes H and H stabilizes x, we have $gz_2^{-1}z_1g^{-1}x = x$; thus $z_1g^{-1}x = z_2g^{-1}x$. Thus ϕ is well-defined.

Let $g_1, g_2 \in N_G(H)$. Then

$$\begin{aligned}\phi_{g_1g_2}(zx) &= zg_2^{-1}g_1^{-1}x \\ &= \phi_{g_1}(zg_2^{-1}x) \\ &= \phi_{g_1}(\phi_{g_2}(zx)).\end{aligned}$$

Thus ϕ is a homomorphism.

Let $g \in \ker(\phi)$. Then $g^{-1}x = x$, so $gx = x$ and $g \in H$. If $g \in H$, then $zg^{-1}x = zx$, and $g \in \ker(\phi)$. Thus $\ker(\phi) = H$.

Let $\sigma \in \mathrm{img}(\phi)$ and select $k \in G$ with $\phi(k) = \sigma$. Then $\sigma zx = zk^{-1}x$ and $\sigma^{-1}zx = z\sigma x$ for every $zx \in X$. Let $g \in G$. Then

$$\begin{aligned}g^{-1}\sigma^{-1}g\sigma zx &= g^{-1}\sigma^{-1}gzk^{-1}x \\ &= g^{-1}gzk^{-1}kx \\ &= x;\end{aligned}$$

thus $g^{-1}\sigma^{-1}g\sigma = \mathrm{id}_X$, and $\mathrm{img}(\phi) \subset C_S(G)$.

Let $\sigma \in C_S(G)$ and let $k = \sigma^{-1}$. Then for every $z \in G$, $\sigma zx = z\sigma x = zk^{-1}x$. Thus $\sigma = \phi(k)$, and $C_S(G) \subset \mathrm{img}(\phi)$. The last statement follows from the first isomorphism theorem. □

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