AP Calculus AB | Activity 0827 | Name:
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Problem 5. Let $g(x) = \sqrt{100 - x^2}$ and $h(x) = \dfrac{1}{x^2 - 25}$. Find the domain of the given function.

(a) $f(x) = g(x) + h(x)$

(b) $f(x) = \dfrac{g(x)}{h(x)}$

(c) $f(x) = g(h(x))$

(d) $f(x) = h(g(x))$

Solution. First we determine the domains of g and h.

The domain of g is the solution to $100 - x^2 \geq 0$, so $x^2 \leq 100$, so $|x| \leq 10$. Thus

$$\mathrm{dom}(g) = [-10, 10].$$

The domain of h is the solution to $x^2 - 25 \neq 0$, so $x^2 \neq 25$, so $x \neq \pm 5$. Thus

$$\mathrm{dom}(h) = \mathbb{R} \smallsetminus \{-5, 5\}.$$

(a) The domain of the sum is the intersection of the domains:

$$\mathrm{dom}(g + h) = \mathrm{dom}(g) \cap \mathrm{dom}(h).$$

Therefore,

$$\mathrm{dom}(g + h) = [-10, 10] \cap (\mathbb{R} \smallsetminus \{-5, 5\}) = [-10, 10] \smallsetminus \{-5, 5\} = [-10, -5) \cup (-5, 5) \cup (5, 10].$$

(b) The domain of a quotient is given by

$$\mathrm{dom}\left(\frac{g}{h}\right) = (\mathrm{dom}(g) \cap \mathrm{dom}(h)) \smallsetminus \{x \in \mathrm{dom}(h) \mid h(x) = 0\}.$$

Since $h(x) \neq 0$ for all $x \in \mathrm{dom}(h)$, we have

$$\mathrm{dom}\left(\frac{g}{h}\right) = [-10, -5) \cup (-5, 5) \cup (5, 10].$$

(d) The domain of a composition is given by

$$\mathrm{dom}(h \circ g) = \{x \in \mathrm{dom}(g) \mid g(x) \in \mathrm{dom}(h)\}.$$

That is, we need $x \in [-10, 10]$ and $\sqrt{100 - x^2} \neq 5$. Solving the inequality, we have

$$100 - x^2 \neq 25 \;\Rightarrow\; x^2 \neq 75 \;\Rightarrow\; x \neq \pm\sqrt{75}.$$

Thus,

$$\mathrm{dom}(h \circ g) = [-10, 10] \smallsetminus \{\pm\sqrt{75}\}.$$

(c) The domain of a composition is given by

$$\operatorname{dom}(g \circ h) = \{x \in \operatorname{dom}(h) \mid h(x) \in \operatorname{dom}(g)\}.$$

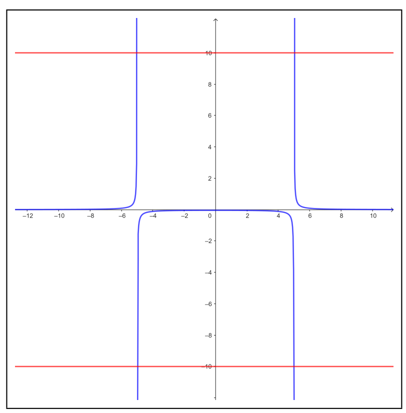
In this case, we need $x \neq \pm 5$ and $\left|\dfrac{1}{x^2-25}\right| \leq 10$. So we have to solve this inequality. This is significantly harder than part **(d)**, as we shall see.

Try

$$\begin{aligned}
\left|\frac{1}{x^2-25}\right| \leq 10 &\Leftrightarrow -10 \leq \frac{1}{x^2-25} \leq 10 \\
&\Leftrightarrow -\frac{1}{10} \leq x^2 - 25 \leq \frac{1}{10} \\
&\Leftrightarrow \frac{249}{10} \leq x^2 \leq \frac{251}{10} \\
&\Leftrightarrow \sqrt{24.9} \leq x \leq \sqrt{25.1} \\
&\Leftrightarrow x \in [\sqrt{24.9}, \sqrt{25.1}]
\end{aligned}$$

But this is wrong!!! Why? Because when you take the reciprocal of both sides of an inequality, you have to reverse the inequality. Moreover, you have to restrict yourself of both sides being positive or both sides being negative, or you will get all tangled up.

In order to see what is going on here, let us graph the situation.



We are looking to remove the x values corresponding to the parts of the graph outside of the two horizontal lines, at $y = \pm 10$. By symmetry, it suffices to find this for positive values of x. Now, assuming $x > 0$, we have

$$\begin{aligned}
\left|\frac{1}{x^2-25}\right| > 10 &\Leftrightarrow |x^2 - 25| < 0.1 \\
&\Leftrightarrow -0.1 < x^2 - 25 < 0.1 \\
&\Leftrightarrow 24.9 < x^2 \leq 25.1 \\
&\Leftrightarrow \sqrt{24.9} \leq x < \sqrt{25.1} \\
&\Leftrightarrow x \in (\sqrt{24.9}, \sqrt{25.1})
\end{aligned}$$

These points are *not* in the domain. We remove the mirror interval on the negative side to see that

$$\operatorname{dom}(g \circ h) = \mathbb{R} \setminus ((-\sqrt{25.1}, -\sqrt{24.9}) \cup (\sqrt{24.9}, \sqrt{25.1})).$$

$\square$