PRINCIPLES OF ANALYSIS
CONTINUITY EXAMPLES

PAUL L. BAILEY

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $x_0 = 2$. Show that f is continuous at x_0.

Proof. Let $\epsilon > 0$; we may assume that $\epsilon < 4$. Let $\delta = \sqrt{x_0^2 + \epsilon} - x_0 = \sqrt{4 + \epsilon} - 2$. Thus $(\delta + 2)^2 = 4 + \epsilon$, so $\epsilon = \delta^2 + 4\delta$.

Suppose that $x \in (2 - \delta, 2 + \delta)$. Then $x + 2 < \delta + 4$, and

$$|f(x) - f(x_0)| = |x^2 - 4| = |x - 2|(x + 2) < \delta(4 + \delta) = \epsilon.$$

□

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Show that f is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. We wish to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

For simplicity, assume that $x_0 > 0$. Let $\delta = \sqrt[3]{x_0^3 + \epsilon} - x_0$. Solving for ϵ yields $\epsilon = (x_0 + \delta)^3 - x_0^3$.

Let $x \in (x_0 - \delta, x_0 + \delta)$. Then $x > 0$, and

$$\begin{aligned}
|f(x) - f(x_0)| &= |x^3 - x_0^3| \\
&= |x - x_0|(x^2 + x_0 x + x_0^2) \\
&< \delta((x_0 + \delta)^2 + x_0(x_0 + \delta) + x_0^2) \\
&= \delta(x_0^2 + 2x_0\delta + \delta^2 + x_0^2 + x_0\delta + x_0^2) \\
&= \delta(3x_0^2 + 3x_0\delta + \delta^2) \\
&= \epsilon.
\end{aligned}$$

□

Date: November 17, 2003.

Example 3. Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Show that f is continuous.

Motivation. Graph the curve $f(x) = \sqrt{x}$. Select arbitrary $x_0 \in \mathrm{dom}(f)$. Project up and to the right to find the point $\sqrt{x_0}$ on the y-axis. Draw an ϵ-band around this point. Project the intersection of this band with the graph of f onto the x-axis. Notice that the point on the left of this projection is closer to x_0 than is the point on the right. Let δ be one half of the distance between x_0 and the left endpoint of the inverse image of $[f(x_0) - \epsilon, f(x_0) + \epsilon]$. $\square$

Proof. Let $x_0 \in [0, \infty)$ and let $\epsilon > 0$; wlog assume that $\epsilon^2 \leq x_0$. If $x_0 = 0$, let $\delta = \epsilon^2$; clearly this will work. Otherwise set

$$\delta = \frac{1}{2}(x_0 - (\sqrt{x_0} - \epsilon)^2);$$

this is positive. Note that for $x \in \mathbb{R}$, $|x - x_0| = |\sqrt{x} - \sqrt{x_0}|(\sqrt{x} + \sqrt{x_0})$. Then if $|x - x_0| < \delta$, we have

$$\begin{aligned}
|\sqrt{x} - \sqrt{x_0}| &< \frac{\delta}{\sqrt{x} + \sqrt{x_0}} \\
&= \frac{x_0 - (x_0 - 2\sqrt{x_0}\epsilon + \epsilon^2)}{2(\sqrt{x} + \sqrt{x_0})} \\
&= \frac{\epsilon(2\sqrt{x_0} - \epsilon)}{2(\sqrt{x} + \sqrt{x_0})} \\
&< \epsilon\frac{(2\sqrt{x_0} - \epsilon)}{2\sqrt{x_0}} \\
&= \epsilon\Big(1 - \frac{\epsilon}{2\sqrt{x_0}}\Big) \\
&< \epsilon.
\end{aligned}$$

$\square$

Example 4. Show that every polynomial function is continuous.

Proof. This is tedious but obviously important. We build it gradually.

Claim 1: The constant function $f(x) = C$, where $C \in \mathbb{R}$, is continuous.
Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = 1$. Then if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = 0 < \epsilon$. Thus f is continuous in this case.

Claim 2: The identity function $f(x) = x$ is continuous.
Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = \epsilon$. Then if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = |x - x_0| < \delta = \epsilon$, so f is continuous in this case.

Claim 3: The function $f(x) = x^n$ is continuous.
By induction on n. For $n = 1$, the function $g(x) = x$ is the identity function, and so it is continuous. By induction, $h(x) = x^{n-1}$ is continuous. Then by the Continuous Arithmetic Proposition, $f = gh$ is continuous in this case.

Claim 4: The monomial function $f(x) = a_n x^n$ is continuous, where $a_n \in \mathbb{R}$ is constant.
By Claim 1, $g(x) = a_n$ is continuous, and by Claim 3, $h(x) = x^n$ is continuous, so there product $f = gh$ is continuous.

Claim 5: The polynomial function $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous.
By induction on n, the degree of the polynomial.

For $n = 0$, $f(x)$ is constant and therefore continuous.

Assume that $g(x) = a_0 + \cdots + a_{n-1} x^{n-1}$ is continuous. By Claim 4, $h(x) = a_n x^n$ is continuous. Then $f = g + h$ is continuous by the Continuous Arithmetic Proposition. □

Example 5. Show that every rational function is continuous.

Proof. Let f be a rational function. Then $f(x) = p(x)/q(x)$, where p and q are polynomial functions. Since p and q are continuous, then f is continuous on its domain by a Proposition from the arithmetic of continuous functions. □

Example 6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is discontinuous at every real number.

Proof. Let $x_0 \in \mathbb{R}$. To show that f is discontinuous at x_0, it suffices to find $\epsilon > 0$ such that for every $\delta > 0$, there exists $x \in (x_0 - \delta, x_0 + \delta)$ with $|f(x) - f(x_0)| \geq \epsilon$.

Let $\epsilon = \frac{1}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If x_0 is rational, let x_1 be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = 1 > \epsilon$; if x_0 is irrational, let x_2 be a rational in this interval, and we still have $|f(x_2) - f(x_0)| = 1 > \epsilon$. Thus f is not continuous at x_0. $\square$

Example 7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is continuous at $x = 0$ and discontinuous at all nonzero real numbers.

Proof. Let $x_0 \in \mathbb{R} \setminus \{0\}$; we show that f is discontinuous at x_0. Let $\epsilon = \frac{|x_0|}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If x_0 is rational, let x_1 be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = |x_0| > \epsilon$. If x_0 is irrational, let x_2 be a rational in this interval such that $|x_2| > |x_0|$ and we still have $|f(x_2) - f(x_0)| = |x_2| > |x_0| > \epsilon$. Thus f is not continuous at x_0.

Now we consider the behavior of f at zero. Let $\epsilon > 0$ and let $\delta = \epsilon$. Then if $|x - 0| < \delta$, we have $|f(x) - f(0)| = 0$ if x is irrational and $|f(x) - f(0)| = |x|$ if x is rational; in either case, $|f(x) - f(0)| \leq |x| < \delta = \epsilon$, so f is continuous at zero. $\square$

Example 8. If $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $r = \frac{p}{q}$. Define $q : \mathbb{Q} \to \mathbb{R}$ by

$$q(r) = \min\{q \in \mathbb{N} \mid r = \frac{p}{q} \text{ for some } p \in \mathbb{Z}\}.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q(x)} & \text{if } x \text{ is rational} \end{cases}$$

Show that f is discontinuous at every rational and continuous at every irrational.

Proof. Suppose that x_0 is rational. We wish to show that f is not continuous at x_0. It suffices to find $\epsilon > 0$ such that for every $\delta > 0$ there exists $x_1 \in (x_0 - \delta, x_0 + \delta)$ with $|x_0 - x_1| > \epsilon$.

Since x_0 is rational, we have $x_0 = \frac{p}{q(x_0)}$ for some $p \in \mathbb{Z}$. Let $\epsilon = \frac{1}{2q(x_0)}$ and let $\delta > 0$. Then $(x_0 - \delta, x_0 + \delta)$ contains an irrational number, say x_1; then $|x_0 - x_1| < \delta$ but $|f(x_0) - f(x_1)| = \frac{1}{q(r)} > \epsilon$. Thus f cannot be continuous at x_0.

Suppose that x_0 is irrational. Let $\epsilon > 0$. It suffices to find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$. Let a be the greatest integer which is less than x_0 and b be the least integer which is greater than x_0; then $b = a + 1$ and $x_0 \in [a, b]$.

For $q \in \mathbb{Q}$, there exist only finitely many points in the set $[a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}\}$ (in fact, this set contains no more than q points). Thus the set

$$D = [a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}, q \leq N\}$$

is finite (there are no more than $\frac{N(N+1)}{2}$ points in this set). Let

$$\delta = \min\{|x_0 - d| \mid d \in D\};$$

since this set is a finite set of positive real numbers, the minimum exists as a positive real number. Then $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Let $x \in (x_0 - \delta, x_0 + \delta)$. If x is irrational, we have $|f(x) - f(x_0)| = 0 < \epsilon$, and if x is rational, we have $|f(x) - f(x_0)| = \frac{1}{q(x)} < \frac{1}{N} < \epsilon$. Thus f is continuous at x_0. $\square$

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu