Linear Algebra Exercises B
Solutions
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Exercise 1. Find the general equation of the line which is the set of all points in $\mathbb{R}^2$ equidistant between $(6, 9)$ and $(-4, 3)$.

Solution. The line in question is perpendicular to the line through the given points and passes through the midpoint between $P_1 = (6, 9)$ and $P_2 = (-4, 3)$. Let $\vec{n} = P_1 - P_2 = (10, 6)$; this is a direction vector for the line through the given points, and thus is normal to the line we seek. The midpoint is $P_0 = (1, 3)$. The normal equation of the new line is $(P - P_0) \cdot \vec{n} = 0$, or $P \cdot \vec{n} = P_0 \cdot \vec{n}$, which is $10x + 6y = 10 + 18 = 28$, which simplifies to

$$5x + 3y = 14.$$

□

Exercise 2. Find the general equation of the plane that is the set of all points in $\mathbb{R}^3$ equidistant to $(1, 3, 2)$ and $(5, -1, 4)$.

Solution. The plane in question is perpendicular to the line through the given points and passes through the midpoint between $P_1 = (1, 5, 2)$ and $P_2 = (5, -1, 4)$. A normal vector $\vec{n}$ for this plane is a direction vector for the line, so let $\vec{n} = P_2 - P_1 = (4, -6, 2)$. The midpoint is $P_0 = (3, 2, 3)$, and the equation of the plane is $(P - P_0) \cdot \vec{n} = 0$, which is equivalent to $4x - 6y + 2z = 12 - 12 + 6$, or

$$2x - 3y + z = 3.$$

□

Exercise 3. Find the parametric equations of the line in $\mathbb{R}^3$ which is the intersection of the planes given by $3x + 6y - 2z = 0$ and $x + 2y - z = -4$.

Solution. Multiply the second equation by -3 and add to find that for any point on the intersection of the two lines, $z = 12$. Plug this back into either of the original equations to see that for points of intersection of these planes, the x and y coordinates are related by the equation $x + 2y = 8$. If $x = 0$, then $y = 4$, and if $y = 0$, then $x = 8$.

Let $P_1 = (0, 4, 12)$ and $P_2 = (8, 0, 12)$; these are two points on the line. Set $\vec{v} = \frac{1}{4}(P_2 - P_1) = (2, -1, 0)$; this is a direction vector for the line. Thus the line is the parametric curve $P = P_1 + t\vec{v}$, or

$$x(t) = 2t, \quad y(t) = 4 - t, \quad z(t) = 12.$$

□

Exercise 4. Find the distance from the point $(6, 1)$ to the line $4x - 3y = 12$ in $\mathbb{R}^2$.

Solution. Let $P_1 = (6, 1)$ and $\vec{n} = (4, -3)$, and note that $\vec{n}$ is a normal vector for this line. To find a point on the line, set $y = 0$; then $4x = 12$, so $x = 3$, and the point $P_2 = (3, 0)$ is on the line. Now $\vec{v} = P_1 - P_2 = (3, 1)$ is the vector from P_2 to P_1; then the component of P_2 in the direction of $\vec{n}$ is

$$|\mathrm{proj}_{\vec{n}}\vec{v}| = \frac{\vec{v} \cdot \vec{n}}{|\vec{n}|} = \left| \frac{12 - 3}{\sqrt{16 + 9}} \right| = \frac{9}{5}.$$

□

Exercise 5. Find the distance from the point $(4, 1, -3)$ to the plane $6x - 2y + 3z = 6$ in $\mathbb{R}^3$.

Solution. Let $P_1 = (4, 1, -3)$ and $\vec{n} = (6, -2, 3)$ so that $\vec{n}$ is a normal vector for the plane. Again we find a point P_2 on the plane and project the vector $\vec{v}$ from P_2 to P_1 onto the normal vector $\vec{n}$.

We notice that $P_2 = (1, 0, 0)$ is on the plane, and is certainly a convenient point to work with. Let $\vec{v} = P_1 - P_2 = (3, 1, -3)$. The distance is

$$|\mathrm{proj}_{\vec{n}}\vec{v}| = \frac{\vec{n} \cdot \vec{v}}{|\vec{n}|} = \left|\frac{18 - 2 - 9}{\sqrt{36 + 4 + 9}}\right| = \frac{7}{\sqrt{49}} = 1.$$

□

Exercise 6. Find the distance from the point $(-7, 8, 3)$ to the line $(7 + t, 3 + 2t, 5 - 2t)$ in $\mathbb{R}^3$.

Solution. The direction vector of the given line is $\vec{v} = (1, 2, -2)$, and a point on the line is $P_2 = (7, 3, 5)$. Let $P_1 = (-7, 8, 3)$; the vector from P_2 to P_1 is $\vec{w} = P_1 - P_2 = (-14, 5, -2)$. If θ is the angle between the line from P_2 to P_1 and the given line, then the distance from P_1 to the line is $|\vec{w}||\sin\theta| = \frac{|\vec{v}\times\vec{w}|}{|\vec{v}|}$.

Now

$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ -14 & 5 & -2 \end{bmatrix} = (-4 + 10)\vec{i} - (-2 - 28)\vec{j} + (5 + 28)\vec{k} = (6, 30, 33) = 3(2, 10, 11).$$

Thus

$$\frac{|\vec{v} \times \vec{w}|}{|\vec{v}|} = \frac{3\sqrt{4 + 100 + 121}}{\sqrt{1 + 4 + 4}} = \frac{3\sqrt{225}}{\sqrt{9}} = 15.$$

□

Exercise 7. Find the distance between the lines $(2 + 2t, 1 - 3t, -4 + t)$ and $(-2 + 3t, 5 - 4t, 8 + t)$ in $\mathbb{R}^3$.

Solution. Let $P = (2, 1, -4)$ and $Q = (-2, 5, 8)$ so that P is on the first line and Q is on the second. The direction vectors of these lines are $\vec{v} = (2, -3, 1)$ and $\vec{w} = (3, -4, 1)$. These vectors are not parallel, so neither are the lines. Thus if the lines do not intersect, they do not lie the same plane. So, a plane which is parallel to both lines cannot contain both lines. It suffices to find a plane parallel to both lines and containing one of them, and to find the distance from the plane to the line, which equals the distance from the plane to any point on the line.

A plane parallel to both lines has a normal vector which is perpendicular to both lines. To find this normal vector, we cross the direction vectors:

$$\vec{n} = \vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 3 & -4 & 1 \end{bmatrix} = (-3 + 4)\vec{i} - (2 - 3)\vec{j} + (-8 + 9)\vec{k} = (1, 1, 1).$$

Let $\vec{x} = P - Q = (4, -4, -12)$. The distance between the lines is the absolute value of the scalar projection of $\vec{x}$ onto $\vec{n}$:

$$|\mathrm{proj}_{\vec{n}}\vec{x}| = \left|\frac{\vec{x} \cdot \vec{n}}{\vec{n}}\right| = \left|\frac{4 - 4 - 12}{\sqrt{3}}\right| = 4\sqrt{3}.$$

□

Exercise 8. Find the area of the triangle in $\mathbb{R}^2$ with vertices $(1, 2)$, $(5, -2)$, and $(-3, 5)$.

Solution. We have seen that the area of the parallelogram determined by the vectors (a, b) and (c, d) is $|ad - bc|$. Let $P = (1, 2)$, $Q = (5, -2)$, and $R = (-3, 5)$, and set $\vec{v} = Q - P = (4, -4)$ and $\vec{w} = R - P = (-4, 3)$. The area of the parallelogram is $|12 - 16| = 4$, and the area of the triangle is half of this, which is 2. □

Exercise 9. Find the area of the triangle in $\mathbb{R}^3$ with vertices $(1, 2, -1)$, $(4, 3, 2)$, and $(3, 4, 1)$.

Solution. We have seen that the area of the parallelogram determined by two vectors in $\mathbb{R}^3$ is the modulus of the cross product. Let $P = (1, 2, -1)$, $Q = (4, 3, 2)$, and $R = (3, 4, 1)$, and set $\vec{v} = Q - P = (3, 1, 3)$ and $\vec{w} = R - P = (2, 2, 2)$. The cross product is

$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix} = (2 - 6)\vec{i} - (6 - 6)\vec{j} + (6 - 2)\vec{k} = (-4, 0, 4).$$

The area of the triangle is

$$\frac{1}{2}|\vec{v} \times \vec{w}| = \frac{1}{2}\sqrt{16 + 16} = 2\sqrt{2}.$$

□

Exercise 10. Find the volume of the tetrahedron in $\mathbb{R}^3$ with vertices $(1, 0, 2)$, $(0, 4, 1)$, $(-2, 4, 0)$, and $(3, 3, 3)$.

Solution. The volume of the parallelopiped determined by three vectors in $\mathbb{R}^3$ is the scalar triple product, obtained by forming the matrix with the vectors in the rows and taking its determinant.

Let $P = (1, 0, 2)$, $Q = (0, 4, 1)$, $R = (-2, 4, 0)$, and $S = (3, 3, 3)$. Let $\vec{v} = Q - P = (-1, 4, -1)$, $\vec{w} = R - P = (-3, 4, -2)$, and $\vec{x} = S - P = (2, 3, 1)$. The volume is

$$\vec{v} \cdot (\vec{w} \times \vec{x}) = \det \begin{bmatrix} -1 & 4 & -1 \\ -3 & 4 & -2 \\ 2 & 3 & 1 \end{bmatrix} = (-1)(4 + 6) - 4(-3 + 4) + (-1)(-9 - 8) = -10 - 4 + 17 = 3.$$

□