PRINCIPLES OF ANALYSIS
SOLUTIONS TO PROBLEM SET D

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Problem 1. Let $\mathcal{C}_{[a,b]}$ denote the set of all continuous functions $f : [a,b] \to \mathbb{R}$. Let $X = \mathcal{C}_{[a,b]}$ and for $f, g \in X$ define

$$\rho(f,g) = \int_a^b |f(x) - g(x)| \, dx.$$

Show that (X, ρ) is a metric space (Hint: open a calculus book and look up properties of integration).

Solution. In order to prove this, we will need these properties of integration:

Lemma 1. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable, and let $c \in \mathbb{R}$. Then*

- **(a)** $\int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$;
- **(b)** $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$.

Lemma 2. *Let $f : [a,b] \to [,\infty)$ be a continuous function. If $\int_a^b f(x) \, dx = 0$, then $f(x) = 0$ for every $x \in [a,b]$.*

Lemma 3. *Let $f, g : [a,b] \to [0,\infty)$ be continuous functions. If $f(x) \le g(x)$ for every $x \in [a,b]$, then $\int_a^b f(x) \, dx \le \int_a^b g(x) \, dx$.*

We have $\rho(f,f) = \int_a^b |f - f| \, dx = \int_a^b 0 \, dx = 0$; moreover, Lemma 2 tells us that if the integral of a nonnegative continuous function is zero, then that function is the zero function; thus $\rho(f,g) = 0 \Rightarrow |f - g| = 0 \Rightarrow f = g$. Thus **(M1)** follows.

Since $|f - g| = |g - f|$, clearly $\rho(f,g) = \rho(g,f)$. Thus **(M2)** follows.

Let $f, g, h \in \mathcal{C}[a,b]$. Then $|f(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)|$ for every $x \in [a,b]$, by the triangle inequality for $\mathbb{R}$. Then by Lemmas 3 and 1,

$$\begin{aligned}
\rho(f,h) &= \int_a^b |f(x) - h(x)| \, dx \\
&\le \int_a^b \Big(|f(x) - g(x)| + |g(x) - h(x)| \Big) \, dx \\
&= \int_a^b |f(x) - g(x)| \, dx + \int_a^b |g(x) - h(x)| \, dx \\
&= \rho(f,g) + \rho(g,h).
\end{aligned}$$

□

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Problem 2. Let (X, ρ) be a metric space, and let $G = \operatorname{diam}(A)$ with respect to ρ. Define a function

$$\widehat{\rho} : X \times X \to \mathbb{R} \quad \text{by} \quad \widehat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

- **(a)** Show that $\widehat{\rho}$ is a metric on X. (Hint: let $a = \rho(x, y)$, $b = \rho(y, z)$ and $c = \rho(x, z)$. Write what you have to show, clear denominators, and work backwards.)

Let $H = \operatorname{diam}(X)$ with respect to $\widehat{\rho}$.

- **(b)** Show that $H \leq 1$.
- **(c)** Show that if $G = \infty$, then $H = 1$.
- **(d)** Show that if X is finite, then $H = \frac{G}{1+G}$.

Solution. Let $x, y, z \in X$; we wish to show that

$$\widehat{\rho}(x, z) \leq \widehat{\rho}(x, y) + \widehat{\rho}(y, z).$$

Let $a = \rho(x, y)$, $b = \rho(y, z)$ and $c = \rho(x, z)$. Then we wish to show that $a, b, c \geq 0$ and $c \leq a + b$ imply

$$\frac{c}{1 + c} \leq \frac{a}{1 + a} + \frac{b}{1 + b}.$$

Now

$$\begin{aligned}
c \leq a + b &\Rightarrow c \leq (a + b) + (2ab + abc) \qquad \text{since } a, b, c \geq 0 \\
&\Rightarrow c + ac + bc + abc \leq (a + b) + (2ab + abc) + (ac + bc + abc) \\
&\Rightarrow c(1 + a + b + ab) \leq a(1 + b + c + bc) + b(1 + a + c + ac) \\
&\Rightarrow c(1 + a)(1 + b) \leq a(1 + b)(1 + c) + b(1 + a)(1 + c) \\
&\Rightarrow \frac{c}{1 + c} \leq \frac{a}{1 + a} + \frac{b}{1 + b}.
\end{aligned}$$

Let $x, y \in X$. Then

$$\widehat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)} < \frac{\rho(x, y)}{\rho(x, y)} = 1;$$

thus $H = \sup\{\widehat{(\rho)}(x, y) \mid x, y \in X\} \leq 1$.

Suppose that $G = \infty$, and let $\epsilon > 0$. Then there exist $x, y \in X$ such that $\rho(x, y) > \frac{1}{\epsilon} - 1$, so that $1 + \rho(x, y) > \frac{1}{\epsilon}$. Now

$$1 + \rho(x, y) > \frac{1}{\epsilon} \Leftrightarrow \frac{1}{1 + \rho(x, y)} < \epsilon \Leftrightarrow 1 - \frac{\rho(x, y)}{1 + \rho(x, y)} < \epsilon \Leftrightarrow 1 - \widehat{\rho}(x, y) < \epsilon.$$

Since this is true for every epsilon, $H = \sup\{\widehat{(}\rho(x, y) \mid x, y \in X\} \geq 1$. Combined with part **(b)**, we have $H = 1$.

Suppose that X is finite. Then the set $\{\rho(x, y) \mid x, y \in X\}$ is also finite, and thus has a maximum, and this maximum is equal to G. Then there exist $a, b \in X$ such that $\rho(a, b) = G$. Since $f(x) = \frac{x}{1+x}$ is an increasing function, $\rho(a, b) \geq \rho(c, d)$ implies that $\widehat{\rho}(a, b) \geq \widehat{\rho}(c, d)$. Thus

$$\frac{G}{1 + G} = \widehat{\rho}(a, b) = \max\{\widehat{\rho}(x, y) \mid x, y \in X\} = H.$$

$\square$

Definition 1. Let (X, ρ) and (Y, χ) be metric spaces, and let $f : X \to Y$ be a function. We say that f is a *homometry* if $\chi(f(x), f(y)) = \rho(x, y)$. A bijective homometry is an *isometry*. An injective homometry is an *isometric embedding*.

Problem 3. Let X be a set containing 4 points together with the discrete metric $\rho : X \to \mathbb{R}$ given by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise.} \end{cases}$$

(a) Show that X can be isometrically embedded in $\mathbb{R}^3$.
(b) Show that X cannot be isometrically embedded in $\mathbb{R}^2$.

Solution.
(a) Let $\alpha = \sqrt{2}$, and map the four points of X onto the set

$$\{(\alpha, 0, 0), (0, \alpha, 0), (0, 0, \alpha), (\alpha, \alpha, \alpha)\}.$$

It is easy to verify that the distance between any two of these points is 1, so this must be an isometric embedding.
(b) It suffices to show that there is no collection of four points in $\mathbb{R}^2$ such that the distance between any two of them is equal to 1. To see this, assume that there is such a set. By translation and rotation of the plane, we may assume that three of the points are $(0, 0)$, $(0, 1)$, and $(1, 0)$. Let (x, y) be the fourth point. Then (x, y) is a solution to this system of equations:
(1) $x^2 + y^2 = 1$;
(2) $(x - 1)^2 + y^2 = 1$;
(3) $x^2 + (y - 1)^2 = 1$.

The intersection of circles (1) and (2) is $\{(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})\}$. The intersection of circles (1) and (3) is $\{(\pm\frac{\sqrt{3}}{2}, \frac{1}{2})\}$. Since there is no common point of intersection, there is no solution to the system of three equations, and no such (x, y) exists. □

Problem 4. Let X be a set and define a metric ρ on X by $\rho(x, y) = 1$ if $x \neq y$. Let (a_n) be a sequence in X.

(a) Show that $p \in X$ is a limit point of (a_n) if and only if

$$\exists N \in \mathbb{N} \mid n \geq N \Rightarrow a_n = p.$$

(b) Show that $q \in X$ is a cluster point of (a_n) if and only if

$$\forall N \in \mathbb{N} \exists n \geq N \mid a_n = q.$$

Solution. Note that for $x \in X$, we have $B(x, \frac{1}{2}) = \{x\}$.
(a) Suppose that p is a limit point of (a_n). Then there exists $N \in \mathbb{N}$ such that $\rho(a_n, p) < \frac{1}{2}$ for $n \geq N$. Then $a_n = p$ for $n \geq N$, which proves the forward direction of the implication. The other direction of the implication is obvious.
(b) Suppose that q is a cluster point of (a_n). Then for every $N \in \mathbb{N}$ there exists $n \geq N$ such that $\rho(a_n, q) < \frac{1}{2}$. Such an a_n must equal q, since q is the only point within $\frac{1}{2}$ of q. Thus for every $N \in \mathbb{N}$ there exists $n \geq N$ such that $a_n = q$. Again, the reverse direction of the implication is obvious. □

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