Vector Calculus — Responses 0407

Dr. Paul L. Bailey — Monday, April 6, 2020

Problem 1 (Thomas §16.4 # 10)**.** Let $\vec{F} = \langle \arctan(y/x), \ln(x^2+y^2) \rangle$. Let R be the region defined by the polar inequalities $1 \le r \le 2$ and $0 \le \theta \le \pi$. Let C be the boundary of R. Find the counterclockwise circulation and outward flux of $\vec{F}$ and over C.

Proof. We use the books notation: let $M = \arctan(y/x)$ and $N = \ln(x^2+y^2)$. Then

$$\frac{\partial M}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial M}{\partial y} = \frac{x}{x^2+y^2}, \quad \frac{\partial N}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial N}{\partial y} = \frac{2y}{x^2+y^2}.$$

Let $\vec{T}$ be the unit tangent vector for C, and let $\vec{n}$ be the outward normal.

To find the flux and flow, we use the two versions of Green's Theorem.

$$\begin{aligned}
\text{Flux} &= \int_C \vec{F} \cdot \vec{n}\, ds \\
&= \iint_R \operatorname{div} \vec{F}\, dA \\
&= \iint_R \Big(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \Big)\, dx\, dy \\
&= \iint_R \Big(-\frac{y}{x^2+y^2} + \frac{2y}{x^2+y^2} \Big)\, dx\, dy \\
&= \iint_R \frac{y}{x^2+y^2}\, dx\, dy \\
&= \int_0^\pi \int_1^2 \frac{r \sin\theta}{r^2} r\, dr\, d\theta \\
&= \int_0^\pi \int_1^2 \sin\theta\, dr\, d\theta \\
&= \Big[-\cos\theta \Big]_0^\pi \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\text{Flow} &= \int_C \vec{F} \cdot \vec{T}\, ds \\
&= \iint_R \operatorname{curl} \vec{F}\, dA \\
&= \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \Big)\, dx\, dy \\
&= \iint_R \Big(\frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2} \Big)\, dx\, dy \\
&= \iint_R \frac{x}{x^2+y^2}\, dx\, dy \\
&= \int_0^\pi \int_1^2 \frac{r \cos\theta}{r^2} r\, dr\, d\theta \\
&= \int_0^\pi \int_1^2 \cos\theta\, dr\, d\theta \\
&= \Big[\sin\theta \Big]_0^\pi \\
&= 0
\end{aligned}$$

$\square$

Question 1. How are the two forms of Green's Theorem equivalent conceptually?

Answer. Symbolically, they are equivalent. If $\vec{F} = \langle M, N \rangle$, then

Theorem 3 says

$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy.$$

Theorem 4 says

$$\int_C \vec{F} \cdot \vec{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

If we plug $\vec{G}_1 = \rangle -N, M\rangle$ in for $\vec{F}$ is Theorem 3, we get Theorem 4.

If we plug $\vec{G}_2 = \rangle M, -N\rangle$ in for $\vec{F}$ in Theorem 4, we get Theorem 3.

Conceptually, they aren't exactly "equivalent", although they are different manifestations of the same phenomenon, that being, that what happens on the boundary is determined by what happens in the interior. We will come to understand this better if we can absorb the proof. □