If G is a group, H is a subgroup of G, and K is a subgroup of H, then K is a subgroup of G.

If G is a group, and H and K are subgroups of G, then their intersection $H \cap K$ is a subgroup of G.

A permutation $\alpha \in S_n$ is called *even* if it can be written as a product of an even number of transpositions; otherwise it is called *odd*. Exactly half of the permutations in S_n are even.

Set
$$A_n = \{\alpha \in S_n \mid \alpha \text{ is even}\}.$$
Then A_n is a subgroup of S_n, called the *alternating subgroup*.

Let H be a subgroup of S_n. Then either H consists of even permutations or exactly half of the permutations in H are even. We prove this now.

Problem 1. Let $H \leq S_n$. Show that either $H \leq A_n$ or $|H| = 2|H \cap A_n|$.

Solution. If G is a group, $x \in G$, and $Y \subset G$, define
$$xY = \{xy \mid y \in Y\}.$$

Suppose that H is not contained in A_n, and let $\alpha \in H \setminus A_n$. Let K denote the set of even permutation in H, and let L denote the set of odd permutations in H. Then $K \cup L = H$, and $K \cap L = \varnothing$. Thus $|H| = |K| + |L|$.

Since H is closed under composition, $\alpha K \subset H$ and $\alpha L \subset H$. Since the product of an odd and and even permutation is odd, and the product of two odd permutations is even, we see that actually $\alpha K \subset L$ and $\alpha L \subset K$. Consider the map
$$f_1 : K \to L \text{ given by } f_1(\kappa) = \alpha\kappa,$$
and the map
$$f_2 : L \to K \text{ given by } f_2(\lambda) = \alpha\lambda.$$
Since left multiplication by α^{-1} produces an inverse for these maps, it is clear that both are injective. Therefore there exists a bijective functions $K \to L$, showing that these two sets have the same cardinality. This shows that $|H| = 2|K|$. □

Let $\rho, \tau \in S_n$ be given by
$$\rho = (1 \quad 2 \quad \dots \quad n) \quad \text{and} \quad \tau = \begin{cases} (2 \quad n)(3 \quad n-1)\dots(\frac{n+1}{2} \quad \frac{n+3}{2}) & \text{if } n \text{ is odd;} \\ (2 \quad n)(3 \quad n-1)\dots(\frac{n}{2} \quad \frac{n}{2}+2) & \text{if } n \text{ is even.} \end{cases}$$

Set
$$D_n = \{\epsilon, \rho, \rho^2, \dots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \dots, \tau\rho^{n-1}\} \subset S_n.$$
Then D_n is a subgroup of S_n, called the *dihedral subgroup*. The proof that this is a subgroup follows from the identity $\tau\rho = \rho^{n-1}\tau$. Clearly, $|D_n| = 2n$.

Set $K_n = D_n \cap A_n$. Then K_n is a subgroup of S_n, and either $K_n = D_n$ or K_n is exactly half of D_n. Thus $|K_n| = 2n$, or $|K_n| = n$.

Problem 2. Let $n = 4$.

(a) Compute ρ and τ in this case.

(b) Show that K_4 is a noncyclic abelian subgroup of S_4.

Solution. We have
$$\rho = \texttt{(1 2 3 4)} \quad \text{and} \quad \tau = \texttt{(2 4)}.$$
Also,
$$K_4 = \{\epsilon, \texttt{(1 2)(3 4)}, \texttt{(1 3)(2 4)}, \texttt{(1 4)(2 3)}\}.$$
Every element of K_4 has order two, so K_4 is not cyclic (of order four). Computation shows that the group is abelian. □

Problem 3. Let $n = 5$.

(a) Compute ρ and τ in this case.

(b) Show that $K_5 = D_5$.

Solution. We have

$$\rho = \texttt{(1 2 3 4 5)} \quad \text{and} \quad \tau = \texttt{(2 5)(3 4)}.$$

Since ρ is an even permutation, so are all of its powers. Moreover, all of the reflections fix exactly one point and transpose two pairs of points; thus they are even. Since each member of K_5 is even, we have $K_5 = D_5$. $\square$

Problem 4. Let $n = 7$.

(a) Compute ρ and τ in this case.

(b) Show that K_7 is a cyclic subgroup of S_7.

Solution. We have

$$\rho = \texttt{(1 2 3 4 5 6 7)} \quad \text{and} \quad \tau = \texttt{(2 7)(3 6)(4 5)}.$$

Since ρ is even, all of its powers are in K_7. Moreover, since n is odd, each of the reflections fixes exactly one point, and so is the product of three transpositions, and is therefore odd. So, $K_7 = \langle \rho \rangle$ is cyclic. $\square$

Problem 5. Try to generalize the previous problems: what can you say about K_n in the following cases?

(a) $n \equiv 0 \pmod 4$

(b) $n \equiv 1 \pmod 4$

(c) $n \equiv 2 \pmod 4$

(d) $n \equiv 3 \pmod 4$

Solution. We will discuss **(a)** and **(c)** together.

(a) and **(c)** Suppose n is even. Then ρ is an odd permutation, but ρ^2 is even, and exactly half of $\langle \rho \rangle$ is contained in A_n.

Also, τ fixes two points, so τ moves $(n/2) - 2$ pairs of points; and $\tau\rho$ does not fix any points (it is reflection through the midpoints of opposite edges), so $\tau\rho$ moves $n/2$ pairs of points.

If $n \equiv 0 \pmod 4$, then $n/2$, so $\tau\rho$ is even, and τ is odd, in which case $K_n = \langle \rho^2, \tau\rho \rangle$.

If $n \equiv 2 \pmod 4$, then $n/2 - 2$ is even, so τ is even, and $\tau\rho$ is odd, in which case $K_n = \langle \rho^2, \tau \rangle$.

In either case, exactly half of the reflections are in K_n.

It turns that in either case, K_n is isomorphic to $D_{(n/2)}$. We will be able to show this more accurately when we have more tools.

(b) Suppose that $n \equiv 1 \pmod 4$. Since n is odd, ρ is an even permutation. All reflections fix exactly one point, so they move $\dfrac{n-1}{2}$ pairs of points. Since $n \equiv 1 \pmod 4$, $\dfrac{n-1}{2}$ is even, so all of these reflections are in A_n. Thus, $D_n \subset A_n$, and $K_n = D_n$.

(c) Suppose that $n \equiv 3 \pmod 4$. Since n is odd, ρ is an even permutation. All reflections fix exactly one point, so they move $\dfrac{n-1}{2}$ pairs of points. Since $n \equiv 3 \pmod 4$, $\dfrac{n-1}{2}$ is odd, so none of these reflections are in A_n. Thus, $K_n = \langle \rho \rangle$ is cyclic. $\square$