Algebra II — Lesson 11 - Locus

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1. The Real Line

We have discussed how there exists a one-to-one correspondence between points on a line and decimal expansions. For this reason, we defined real numbers as decimal expansions.

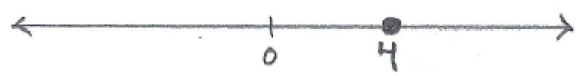
The set of all real numbers is denoted $\mathbb{R}$. Geometrically, we view $\mathbb{R}$ as a line.

2. Locus in One Variable

The *locus* of an equation or inequality in one variable (say x) is the set of all real numbers, which, when plugged into the equation, make it true. For most purposes, the word "locus" is synonymous with "solution set".

Example 1. Find the locus of the equation $3x - 8 = 4$. Sketch this set.

Solution. Add 8 to both sides to get $3x = 12$. Divide both sides by 3 to get $x = 4$. The locus is $\{4\}$.



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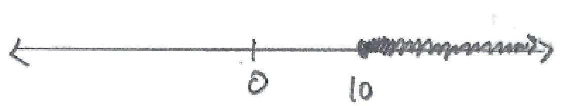
Example 2. Find the locus of the equation $5x - 8 \geq 3x + 12$. Sketch this set.

Solution. Add 8 to both sides to get $5x \geq 3x + 20$. Subtract $3x$ from both sides to get $2x \geq 20$. Divide both sides by 2 to get $x \geq 10$. The locus is $\{x \in \mathbb{R} \mid x \geq 10\}$.



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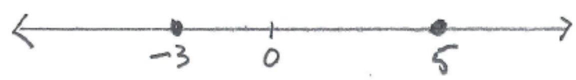
Example 3. Find the locus of the equation $x^2 - 2x - 15 = 0$. Sketch this set.

Solution. Factor to get $(x + 3)(x - 5) = 0$. Thus $x + 3 = 0$ or $x - 5 = 0$. Therefore $x = -3$ or $x = 5$. The locus is $\{-3, 5\}$.



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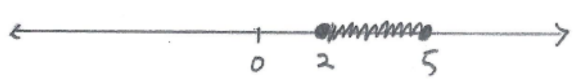
Example 4. Find the locus of the equation $x^2 - 7x + 10 \leq 0$. Sketch this set.

Solution. Factor to get $(x - 2)(x - 5) \leq 0$. The left hand side is the product of two expressions. If the expressions have different signs, then the result is negative; otherwise it is positive. There are three cases:

- If $x > 5$, then $x - 2$ and $x - 5$ are both positive, so the product is positive.
- If $x < 2$, then $x - 2$ is negative and $x - 5$ is negative.
- If x between 2 and 5, $x - 2$ is positive and $x - 5$ is negative.

Thus the locus is $\{x \in \mathbb{R} \mid 2 \leq x \leq 5\}$.



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3. Ordered Pairs

An *ordered pair* consists of two elements in a specific order. The ordered pair which has element a first and element b second is denoted (a, b).

Ordered pairs have the defining property

$$(a, b) = (c, d) \quad \text{if and only if} \quad a = b \text{ and } c = d.$$

4. The Cartesian Plane

The set of all ordered pairs of real numbers is denoted $\mathbb{R}^2$. That is,

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

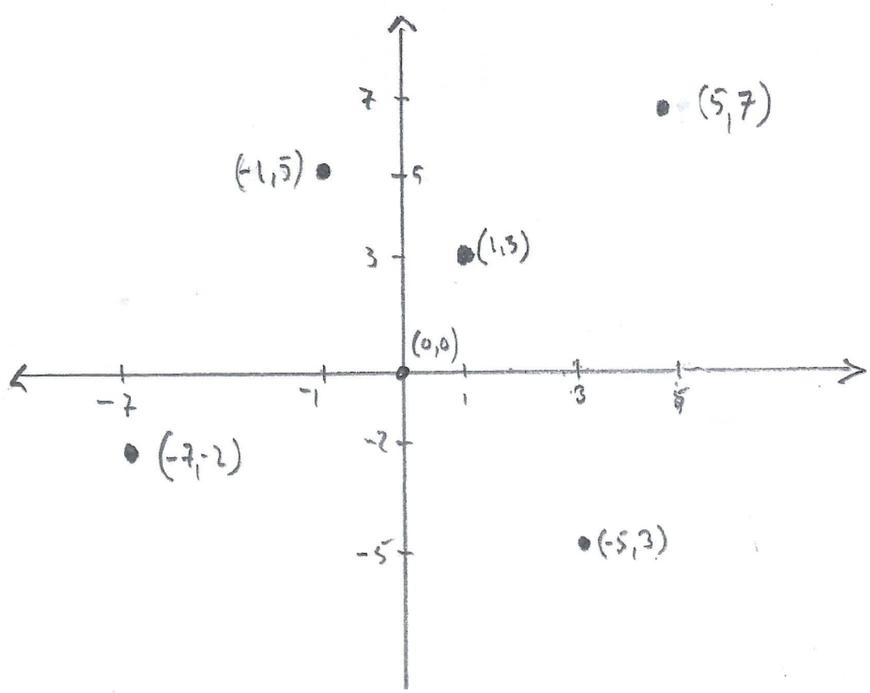
Geometrically, we view $\mathbb{R}^2$ as a plane.

As you have seen in previous courses, take two lines (copies of $\mathbb{R}$) and place them at right angles to obtain the *cartesian coordinate system*. This is a plane on which every point is labeled with an ordered pair of real numbers. The horizontal line is usually the x-axis, and the vertical line is usually the y-axis.

5. Plotting Points in $\mathbb{R}^2$

When you plot a few points, it is not necessary to be overly accurate; you just have to estimate the distances along the axes for the points you want to plot. For example, this is how one could draw the following points:

$$(0, 0), (1, 3), (5, 7), (-1, 5), (3, -5), (-7, -2)$$



6. Locus in Two Variables

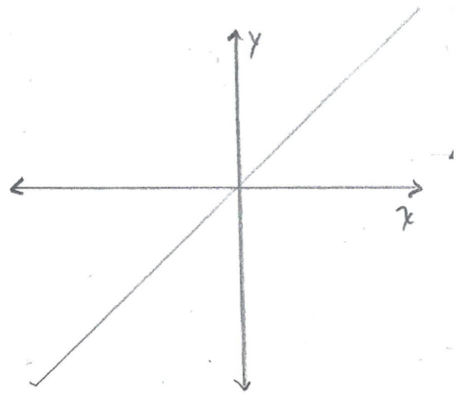
The *locus* of an equation or inequality in two variables (say x and y) is the set of all ordered pairs (x, y) of real numbers, which, when plugged into the equation, make it true.

Example 5. Find the locus of the equation $y = x$.

Solution. The locus is the set

$$\{(x, y) \in \mathbb{R}^2 \mid y = x\}.$$

The *graph* is a picture of this set in the cartesian plane; it is the diagonal line through the origin on which the x and y coordinates are equal.



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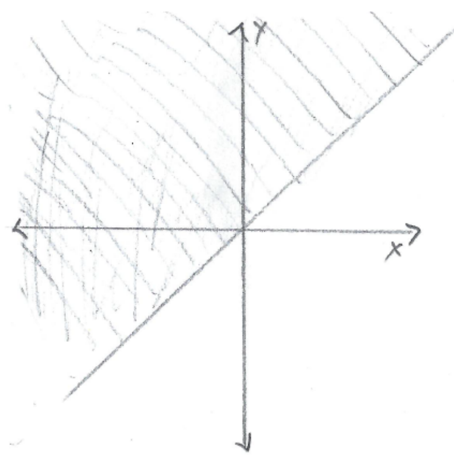
Example 6. Find the locus of the inequality $y \geq x$.

Solution. The locus is the set

$$\{(x, y) \in \mathbb{R}^2 \mid y \geq x\}.$$

The *graph* is a picture of this set in the cartesian plane; it is the shaded half plane above and including the diagonal line through the origin on which the x and y coordinates are equal.



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