Linear Algebra Exercises E Solutions
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Definition 1. Let V be a vector space and let $W_1, W_2 \leq V$. We say that V is a *direct sum* of W_1 and W_2, and write
$$V = W_1 \oplus W_2,$$
if

(D1) $V = W_1 + W_2$;

(D2) $W_1 \cap W_2 = \{0\}$.

Problem 1. Let V be a subspace of $\mathbb{R}^n$ and let $W \leq V$. The *perp space* of W is
$$\widehat{W} = \{v \in V \mid v \cdot w = 0 \text{ for all } w \in W\}.$$

(a) Show that $\widehat{W} \leq V$.

(b) Show that $V = W \oplus \widehat{W}$.

Solution. To show that $\widehat{W}$ is a subspace of V, we must verify properties **(S0)**, **(S1)**, and **(S2)**.

(S0) Since $0 \cdot w = 0$ for every $w \in W$, $0 \in \widehat{W}$.

(S1) Let $v_1, v_2 \in \widehat{W}$, and let $w \in W$. Then $v_1 \cdot w = 0$ and $v_2 \cdot w = 0$. Adding these equations gives $(v_1 \cdot w) + (v_2 \cdot w) = 0$. By linearity of dot product, it follows that $(v_1 + v_2) \cdot w = 0$. This is true for every $w \in W$; thus $v_1 + v_2 \in \widehat{W}$.

(S2) Let $v \in \widehat{W}$ and $a \in \mathbb{R}$. Select $w \in W$. Now $v \cdot w = 0$; scalar multiply both sides of this equation by a to get $a(v \cdot w) = 0$. By linearity of dot product, $(av) \cdot w = 0$. Thus $av \in \widehat{W}$.

Next, we show that $W \cap \widehat{W} = \{0\}$. Let $v \in W \cap \widehat{W}$; then $|v|^2 = v \cdot v = 0$. The only vector with modulus zero is the zero vector, so $v = 0$.

Finally, we need to show that $V = W + \widehat{W}$. To do this, we use the Gram-Schmidt theorem to declare the existence of an orthonormal basis $\{w_1, \dots, w_m\}$ for W. Let $v \in V$, $y = \sum_{i=1}^{m}(v \cdot w_i)w_i$, and $x = v - y$. Clearly $y \in W$ and $v = y + x$; we only need to show that $x \in \widehat{W}$. That is, for $w \in W$, we show that $x \cdot w = 0$.

Now w is a linear combination of the basis vectors $w_1, \dots, w_m$, so by linearity of dot product, it suffices to show that $x \cdot w_j = 0$ for $j = 1, \dots, m$. Thus let j be between 1 and m and compute:

$$\begin{aligned}
x \cdot w_j &= v \cdot w_j - \sum_{i=1}^{m}(v \cdot w_i)w_i \cdot w_j \\
&= v \cdot w_j - (v \cdot w_j)w_j \cdot w_j \quad \text{because } w_i \cdot w_j = 0 \text{ for } i \neq j \\
&= v \cdot w_j - (v \cdot w_j) \quad \text{because } |w_j| = 1 \\
&= 0
\end{aligned}$$

$\square$

Problem 2. Let $w_1, w_2 \in \mathbb{R}^3$ be given by $w_1 = (1, -2, -1)$ and $w_2 = (3, 1, 1)$. Let $W = \operatorname{span}\{w_1, w_2\}$.

(a) Show that $w_1 \perp w_2$.

(b) Find a vector w_3 such that $\widehat{W} = \operatorname{span}\{w_3\}$.

(c) Write $x = (6, 8, 15)$ as a linear combination of the vectors w_1, w_2, w_3.

Solution. To show that two vectors are perpendicular, use the dot product. We have

$$w_1 \cdot w_2 = 3 - 2 - 1 = 0 \quad \Rightarrow \quad w_1 \perp w_2.$$

To find a vector perpendicular to two vectors, use the cross product. We have

$$w_3 = w_1 \times w_2 = \det \begin{bmatrix} i & j & k \\ 1 & -2 & -1 \\ 3 & 1 & 1 \end{bmatrix} = (-2+1)i - (1+3)j + (1+6)k = (-1, -4, 7).$$

Now W is a plane in $\mathbb{R}^3$, so $\widehat{W}$ consists of a line, and is spanned by a single nonzero vector. Since w_3 is perpendicular to both w_1 and w_2, it must generate $\widehat{W}$. So $\{(-1, -4, 7)\}$ is a basis for $\widehat{W}$.

To write $b = (6, 8, 15)$ as a linear combination of the basis $\{w_1, w_2, w_3\}$, we form the matrix $A = [w_1|w_2|w_3]$ by placing the the basis vectors in columns, form the column vector $x = (a_1, a_2, a_3)$, and solve the matrix equation $Ax = b$ for x. This will produce $Ax = a_1w_1 + a_2w_2 + a_3w_3 = b$, so that b is written as a linear combination of the basis vectors.

We form the augmented matrix and find the standard row-echelon form via forward elimination:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ -2 & 1 & -4 & 8 \\ -1 & 1 & 7 & 15 \end{array}\right] \underset{\mathsf{R_3+R_1}}{\xrightarrow{\mathsf{R_2+2R_1}}} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 0 & 7 & -6 & 20 \\ 0 & 4 & 6 & 21 \end{array}\right] \xrightarrow{\mathsf{R_3-(4/7)R_2}} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 0 & 7 & -6 & 20 \\ 0 & 0 & \frac{66}{7} & \frac{67}{7} \end{array}\right]$$

$$\underset{\mathsf{(1/7)R_2}}{\xrightarrow{\mathsf{(7/66)R_3}}} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 0 & 1 & -\frac{6}{7} & \frac{20}{7} \\ 0 & 0 & 1 & \frac{67}{66} \end{array}\right] \underset{\mathsf{R_1+R_3}}{\xrightarrow{\mathsf{R_2+(6/7)R_3}}} \left[\begin{array}{ccc|c} 1 & 3 & 0 & \frac{463}{66} \\ 0 & 1 & 0 & \frac{41}{11} \\ 0 & 0 & 1 & \frac{67}{66} \end{array}\right]$$

$$\xrightarrow{\mathsf{R_1-3R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{25}{6} \\ 0 & 1 & 0 & \frac{41}{11} \\ 0 & 0 & 1 & \frac{67}{66} \end{array}\right]$$

Therefore,

$$(6, 8, 15) = -\frac{25}{6}w_1 + \frac{41}{11}w_2 + \frac{67}{66}w_3.$$

$\square$

Problem 3. Let V and W be the subspaces of $\mathbb{R}^5$ given by:

$$V = \text{span}\{(1,0,1,0,1),(0,1,0,1,1),(1,1,0,1,0),(0,1,1,1,0)\};$$
$$W = \text{span}\{(1,2,0,1,2),(0,1,2,0,1),(2,0,1,2,0),(0,1,2,1,0)\}.$$

(a) Find a basis of each of the following spaces: V, W, $V+W$, $V\cap W$.

(b) Find a basis for a subspace U of W such that $\mathbb{R}^5 = U \oplus V$.

Solution. Let $v_1 = (1,0,1,0,1)$, $v_2 = (0,1,0,1,1)$, $v_3 = (1,1,0,1,0)$, $v_4 = (0,1,1,1,0)$, $w_1 = (1,2,0,1,2)$, $w_2 = (0,1,2,0,1)$, $w_3 = (2,0,1,2,0)$, and $w_4 = (0,1,2,1,0)$. Row reducing the matrices $[v_1|v_2|v_3|v_4]$ and $[w_1|w_2|w_3|w_4]$ shows that neither of these matrices has a free column; thus the sets $\{v_1,v_2,v_3,v_4\}$ and $\{w_1,w_2,w_3,w_4\}$ are linearly independent, and form a basis for V and W, respectively.

Now form the matrix $A = [v_1|v_2|v_3|v_4|w_1|w_2|w_3|w_4]$ and row reduce. The resulting matrix in reduced row echelon form is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & -\frac{4}{3} & \frac{7}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & -\frac{4}{3} & \frac{7}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & \frac{5}{3} & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \end{bmatrix}.$$

This shows that the first five columns of A are linearly independent, and since $\dim(\mathbb{R}^5) = 5$, they must be a basis. Therefore $B = \{v_1,v_2,v_3,v_4,w_4\}$ is a basis for $V+W = \mathbb{R}^5$. Furthermore, it is clear that if $U = \text{span}\{w_1\}$, then $V+W = \mathbb{R}^5 = U \oplus V$.

The formula $\dim(V+W) = \dim(V) + \dim(W) - \dim(V\cap W)$ becomes $5 = 4+4-\dim(V\cap W)$, so $\dim(V\cap W) = 3$.

Now $R = UA$ for the invertible matrix U which is the product of the elementary invertible matrices corresponding to the row operations which produced R from A via Gaussian elimination. Matrix R shows how to express Uw_2, Uw_3, and Uw_4 as a linear combination of the standard basis; pulling this linear combination back through U^{-1} expresses w_2, w_3, and w_4 as linear combinations from the basis B. Specifically,

$$3w_2 = v_1 - 4v_2 - 4v_3 + 5v_4 + w_1;$$
$$3w_3 = 5v_1 + 7v_2 + 7v_3 - 2v_4 - 2w_1;$$
$$3w_4 = v_1 - v_2 - v_3 + 5v_4.$$

Therefore,

$$3w_2 - w_1 \in V\cap W;$$
$$3w_3 + 2w_1 \in V\cap W;$$
$$3w_4 \in V\cap W.$$

Set $x_1 = 3w_2 - w_1$, $x_2 = 3w_3 + 2w_1$, and $x_3 = w_4$. Suppose that

$$a_1x_1 + a_2x_2 + a_3x_3 = 0;$$

then

$$(2a_2 - a_1)w_1 + 3a_1w_2 + 3a_2w_3 + a_3w_4 = 0,$$

so $a_1 = a_2 = a_3 = 0$ by linear independence of the set $\{w_1,w_2,w_3,w_4\}$. Thus the set $\{x_1,x_2,x_3\}$ is also linearly independent, and so is a basis for $V\cap W$. $\square$

Problem 4. Let $R, S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations which have the following effects:

- R rotates by $60°$ around the x-axis;
- S projects onto the xy-plane;
- T reflects across the xz-plane.

Find a basis for the image and the kernel of $T \circ S \circ R : \mathbb{R}^3 \to \mathbb{R}^3$.

Solution. For convenience, let's call the matrices corresponding to R, S, and T by the same names. Then

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}; \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We compute that

$$T \circ S \circ R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

One quickly sees that columns one and two are basic and three is free; thus a basis for the image is the first two columns. Any scalar multiple of a basis vector can be substituted, so we have

$$\text{image basis} = \{(1, 0, 0), (0, 1, 0)\};$$

this is precisely what our geometric intuition tells us: the image is the xy-plane.

Now put the matrix in reduced row echelon form to obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution readoff produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus we have

$$\text{kernel basis} = \{(0, \sqrt{3}, 1)\}.$$

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