REAL ANALYSIS
TOPIC 35 - MEASURE (PRELIMINARY))

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1. Extended Real Numbers

Recall that the *extended real numbers* consist of the real numbers together with two symbols for plus and minus infinity:

$$\overline{R} = \mathbb{R} \cup \{\pm\infty\}.$$

We write ∞ for $+\infty$. We order the set $\overline{\mathbb{R}}$ by defining $-\infty < x < \infty$ for all $x \in \mathbb{R}$. It then makes sense to write $\overline{\mathbb{R}} = [-\infty, \infty]$. When this set is endowed with the order topology, it is homeomorphic to $[0, 1]$.

We will use the following facts regarding extended real numbers without further comment.

- Every nonempty subset of $\overline{\mathbb{R}}$ has a supremum and an infimum in $\overline{\mathbb{R}}$.
- Every monotone sequence in $\overline{\mathbb{R}}$ has a limit in $\overline{\mathbb{R}}$.
- Every series of nonnegative real terms converges in $\overline{\mathbb{R}}$.

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2. Set Functions

We are interested in functions which associate an extended real number to each set in a collection of sets. This will allow us to generalize the notion of the length of an interval.

Let X be a set and let $\mathcal{C} \subset X$. A *set function* on $\mathcal{C}$ is a function

$$\gamma : \mathcal{C} \to \overline{\mathbb{R}}.$$

A set function may have one or more of the following properties, assuming that $\mathcal{C}$ is closed under the appropriate unions (finite or countable unions).

- *Monotone*: for $C_1, C_2 \in \mathcal{C}$ with $C_1 \subset C_2$,
$$\gamma(C_1) \leq \gamma(C_2).$$
- *Additive*: for $C_1, C_2 \in \mathcal{C}$ with $C_1 \cap C_2 = \varnothing$,
$$\gamma(C_1 \cup C_2) = \gamma(C_1) + \gamma(C_2).$$
- *Finitely additive*: for $C_1, \ldots, C_n \in \mathcal{C}$ with $C_i \cap C_j = \varnothing$ for $i \neq j$,
$$\gamma(\cup_{i=1}^{n} C_i) = \sum_{i=1}^{n} \gamma(C_i).$$
- *Countably additive*: for each sequence (C_n) in $\mathcal{C}$ with $C_i \cap C_j = \varnothing$ for $i \neq j$,
$$\gamma(\cup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} \gamma(C_i).$$
- *Subadditive*: for $C_1, C_2 \in \mathcal{C}$,
$$\gamma(C_1 \cup C_2) \leq \gamma(C_1) + \gamma(C_2).$$
- *Finitely subadditive*: for $C_1, dots, C_n \in \mathcal{C}$,
$$\gamma(\cup_{i=1}^{n} C_i) \leq \sum_{i=1}^{n} \gamma(C_i).$$
- *Countably additive*: for each sequence (C_n) in $\mathcal{C}$,
$$\gamma(\cup_{i=1}^{\infty} C_i) \leq \sum_{i=1}^{\infty} \gamma(C_i).$$

It is clear that additive implies finitely additive, by induction. Also, subadditive implies finitely subadditive. Also, countably additive implies additive, and countably subadditive implies subadditive.

Proposition 1. *Let $\mathcal{A}$ be an algebra of subsets of a set X, and let $\gamma : \mathcal{A} \to [0, \infty]$. If γ is additive, then γ is monotone and subadditive.*

Proof. Suppose that γ is additive.

First we show that γ is monotone. Let $A_1, A_2 \in \mathcal{A}$ with $A_1 \subset A_2$. Then $B = A_2 \smallsetminus A_1 = A_2 \cap A_1^c \in \mathcal{A}$. By additivity, $\gamma(A_2) = \gamma(A_1 \cup B) = \gamma(A_1) + \gamma(B)$, and since $\gamma(B)$ is nonnegative, $\gamma(A_2) \geq \gamma(A_1)$. Thus γ is subadditive.

Next we show that γ is subadditive. Let $A_1, A_2 \in \mathcal{A}$. Let $B = A_1 \cap A_2$. Now $A_1 \cup A_2 = (A_1 \smallsetminus B) \cup B \cup (A_2 \smallsetminus B) = (A_1 \cap B^c) \cup B \cup (A_2 \cap B^C) \in \mathcal{A}$. These sets are disjoint, so $\gamma(A_1 \cup A_2) = \gamma(A_1 \smallsetminus B) + \gamma(B) + \gamma(A_2 \smallsetminus B)$. $\square$

Proposition 2. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $\gamma : \mathcal{A} \to [0, \infty]$. If γ is countably additive, then γ is countably subadditive.*

Proof. Exercise. $\square$

Proposition 3. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $\gamma : \mathcal{A} \to [0, \infty]$. If γ is additive and countably subadditive, then γ is countably additive.*

Proof. Exercise. $\square$

3. Measures

Definition 1. Let X be a set and let $\mathcal{E}$ be a σ-algebra of subsets of X. A *measure* on $\mathcal{E}$ is a function $\mu : \mathcal{E} \to \overline{\mathbb{R}}$ such that

- **(M1)** $\mu(E) \geq 0$ for all $E \in \mathcal{E}$;
- **(M2)** $\mu(\varnothing) = 0$;
- **(M3)** (E_n) disjoint sequence in $\mathcal{E}$ implies $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

4. Length of Sets

It is plain that $\operatorname{sci}(A)$ is the intersection of all closed intervals which contain A.

Definition 2. Let $A \subset \overline{\mathbb{R}}$ be an interval. The *length* of A is

$$\ell(A) = b - a \qquad \text{where } a = \inf A \text{ and } b = \sup A.$$

Proposition 4. *Let $\mathcal{I}$ be a collection of pairwise disjoint subintervals of an interval J. Then the sum of the lengths of the intervals in $\mathcal{I}$ is bounded above by the length of J:*

$$\sum_{I \in \mathcal{I}} \ell(I) \leq \ell(J).$$

Proof. First assume that $\mathcal{I}$ is finite, say $\mathcal{I} = \{I_1, \dots, I_n\}$. Let $a_k = \inf I_k$, and $b_k = \sup I_k$. Let $a = \inf J$ and $b = \sup J$. Then

$$a \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n \leq b.$$

Thus

$$(b - b_n) + (a_n - b_{n-1}) + \cdots + (a_2 - b_1) + (a_1 - a) \geq 0,$$

which implies that $\ell(J) \geq \sum_{k=1}^{n} \ell(I_k)$.

Next, suppose that $\mathcal{I} = \{I_k \mid k \in \mathbb{N}\}$ is an infinite countable collection of intervals. Then for every partial sum, $\sum_{k=1}^{n} \ell(I_k) \leq \ell(J)$. The sequence of partial sums is a bounded nondecreasing sequence, so it converges; thus

$$\sum_{k \in \mathbb{N}} \ell(I_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \ell(I_k) \leq \ell(J).$$

$\square$

Definition 3. Let $G \subset \mathbb{R}$ be open. Define the *length* of G, denoted $\ell(G)$, to be the sum of the length of the disjoint components of G.

Let $F \subset \mathbb{R}$ be bounded and closed, and let $J = \operatorname{sci}(F)$. Define the *length* of F, denoted $\ell(F)$, to be $\ell(J) - \ell(J \setminus F)$.

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