COMPLEX ANALYSIS
TOPIC XVII: SERIES (DRAFT)

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1. Series

It is convenient at this point to allow our sequences to start at index 0. Let it be so.

Definition 1. Let (a_n) be a sequence of complex numbers. The n^{th} *partial sum* of this sequence is

$$s_n = \sum_{i=0}^{\infty} a_n$$

The *series* associated with (a_n) is the sequence (s_n) of partial sums of (a_n). We say the the series *converges* if the sequence (s_n) converges.

The notation $\sum a_n$ is used to denote this series. The notation $\sum_{n\in\mathbb{N}} a_n$ or $\sum_{n=0}^{\infty}$ may be used to denote the series, or to denote the limit of the sequence of partial sums:

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n} a_n.$$

A *series over* $\mathbb{C}$ is a series $\sum a_n$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$.

Proposition 1. (Linearity of Series)
Let $\sum a_n$ *and* $\sum b_n$ *be convergent series, and let* $c \in \mathbb{C}$. *Then*

(a) $\displaystyle\sum_{n=0}^{\infty}(a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$;

(b) $\displaystyle\sum_{n=0}^{\infty} ca_n = c\sum_{n=0}^{\infty} a_n$.

Note that if $\sum a_n$ is a series over $\mathbb{C}$, then each a_n may be written as $a_n = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Thus

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x_n + i\sum_{n=0}^{\infty} y_n.$$

If any two of these converge, then so does the third.

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Proposition 2. (Geometric Series Test)
A geometric series *is a series of the form* $\sum_{i=0}^{\infty} r^n$*, where* $r \in \mathbb{C}$*. This series converges if and only if* $|r| < 1$*.*

Proof. We have seen the identity

$$\frac{1 - r^{n+1}}{1 - r} = \sum_{i=0}^{n} r^n.$$

If $|r| < 1$, then $\lim_{n \to \infty} r_{n+1} = 0$, so taking the limit of both sides of our identity gives

$$\sum_{i=0}^{\infty} r^n = \frac{1}{1 - r}.$$

If $|r| \geq 1$, this diverges. $\square$

Proposition 3. (n^{th} Term Test)
Let $\sum a_n$ *be a convergent series. Then* $\lim a_n = 0$*.*

Proof. Let $s_n = \sum_{i=0}^{n} a_n$. Then (s_n) is a convergent sequence. Shifted the sequence cannot change it's limit, so (s_{n-1}) converges to the same value. That is, $\lim s_n = \lim s_{n-1}$, which by the arithmetic properties of limits implies that $\lim(s_n - s_{n-1}) = 0$. But $s_n - s_{n-1} = a_n$, so $\lim a_n = 0$. $\square$

Example 1. (Harmonic Series)
The *harmonic series* is $\sum_{n=1}^{\infty} \frac{1}{n}$. One way to see this is to write

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \\
&\geq + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \infty
\end{aligned}$$

This series diverges, even though $\lim \frac{1}{n} = 0$. Thus the converse of the n^{th} term test is not true.

Example 2. (Telescoping Series)
Recall the *triangular numbers*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Leibnitz was challenged by Huygens to find the sum of their reciprocals. First factor out a 2 from all the terms $\frac{2}{n(n+1)}$; then compute

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \left[\frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}\right] \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots \\
&= 1 - \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{4}\right) - \dots \\
&= 1.
\end{aligned}$$

Thus the sum of the reciprocals of the triangular numbers is 2.

Jacob Bernoulli, who knew that the harmonic series $\sum \frac{1}{n}$ diverges, then realized that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2.$$

Euler was able to compute the value to which the sum of the reciprocals of the square natural numbers converges.

Proposition 4. (Comparison Test)
Let $\sum c_n$ be a convergent series of real numbers, and let $\sum d_n$ be a divergent series of real numbers.

(a) *If $0 \le a_n \le c_n$ for all $n \in \mathbb{N}$, then $\sum a_n$ converges.*
(b) *If $0 \le d_n \le b_n$ for all $n \in \mathbb{N}$, then $\sum b_n$ diverges.*

Proposition 5. (Alternating Series Test)
Let (a_n) be a decreasing sequence of nonnegative real numbers which converges to zero. Then $\sum (-1)^n a_n$ converges.

Reason. Note that $0 \le s_2 \le s_4 \le s_6 \le \cdots \le a_1$. Thus (s_{2n}) is a bounded monotone sequence, and so it converges, say to s. Then $\lim s_{2n+1} = \lim s_{2n} + \lim a_{2n+1} = s + 0 = s$. □

Proposition 6. (Ratio Test)
Let (a_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then $\sum a_n$ converges if $L < 1$ and $\sum a_n$ diverges if $L > 1$.

Reason. Suppose $0 < L < 1$. Select r such that $0 < L < r < 1$. Let N be so large that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad \text{for} \quad n \geq N.$$

Then $|a_{n+1}| < r|a_n|$, for $n \geq N$.

In particular, $|a_{N+1}| < r|a_N|$, $|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$, and in general, $|a_{N+k}| < r^k|a_N|$. Now

$$\sum_{k=1}^{\infty} |a_n| < \sum_{k=1}^{\infty} |a_N| r^k,$$

which converges. □

Proposition 7. (Root Test)
Let (a_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \sqrt{n} a_n = L.$$

Then $\sum a_n$ converges if $L < 1$ and $\sum a_n$ diverges if $L > 1$.

Definition 2. Let $\sum a_n$ be a series over $\mathbb{C}$. We say that $\sum a_n$ is *absolutely convergent*, or *converges absolutely*, if $\sum |a_n|$ converges.

Proposition 8. *If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.*

Proof. But we can prove it by doing the following.

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} |x_n + iy_n| \\
&= \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \\
&\geq \sum_{n=1}^{\infty} \sqrt{x_n^2 + 0} \\
&= \sum_{n=1}^{\infty} |x_n|
\end{aligned}$$

We can do the same to show $|y_n|$ converges. Since the theorem holds for series over $\mathbb{R}$, we see that $\sum x_n$ and $\sum y_n$ converge. Thus $\sum a_n$ converges. □

2. Power Series

Definition 3. A *power series* is a series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $z_0 \in \mathbb{C}$, and $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$. We call z_0 the *center* of the power series, and we call the numbers a_n the *coefficients* of the series.

We view z as a variable. The power series $f(z)$ either converges or diverges, dependent on z. We view $f(z)$ as a function, whose domain is the set of all $z \in \mathbb{C}$ such that $f(z)$ converges.

We may use the ratio test to compute the radius of convergence. If all of the coefficients are nonzero, we easily get a precise formula.

Proposition 9. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, with $a_n \neq 0$ for all $n \in \mathbb{N}$. Then f converges inside a disk of radius R, and diverges outside this disk, where*

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

assuming that this limit exists.

Proof. We let $p \Leftrightarrow^* x < y$ mean that p is true if $x < y$ and false if $x > y$, but we are unsure of p if $x = y$. The ratio test tells us that

$$\begin{aligned} f(z) \ \texttt{converges} \ &\Leftrightarrow^* \lim_{n\to\infty} \left| \frac{a_{n+1}(z - z_0)^{n+1}}{a_n (z - z_0)^n} \right| < 1 \\ &\Leftrightarrow^* \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |z - z_0| < 1 \\ &\Leftrightarrow^* |z - z_0| < \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} \\ &\Leftrightarrow^* |z - z_0| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \end{aligned}$$

□

With a little extra real analysis, we get the precise statement involving the ratio test and the limit superior:

$$\limsup \sqrt[n]{|a_n|} = \frac{1}{R}.$$

The next theorem implies that a power series is differentiable, and that its derivative is also a power series, so that the power series is actually infinitely differentiable.

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converge in an open disk $B_r(z_0)$. Then f is differentiable in this disk, and f has a primitive F in this disk, given by*

$$f'(z) = \frac{d}{dz} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{d}{dz} a_n (z - z_0)^n = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

and

$$F(z) = \int \Big(\sum_{n=0}^{\infty} a_n (z - z_0)^n \, dz \Big) = \sum_{n=0}^{\infty} \int \Big(a_n (z - z_0)^n \Big) \, dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} + C.$$

Definition 4. Let $D \subset \mathbb{C}$. A *sequence of functions* (f_n) on D consists of one function $f_n : D \to \mathbb{C}$ for each nonnegative integer n.

We say that (f_n) *converges pointwise at* $z \in D$ if the sequence $(f_n(z))$ converges. We say that (f_n) converges pointwise on D if it converges pointwise at z for every $z \in D$.

Suppose that (f_n) converges pointwise on D. We say that the function $f : D \to \mathbb{C}$ is the *limit* of (f_n) if $(f_n(z))$ converges to $f(z)$ for all $z \in D$.

We say that (f_n) *converges uniformly on* D if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \epsilon$.

In the definition of pointwise convergence, for a given ϵ, the N that works is specific to the given z. Different z's may require a different N's. Uniform convergence means that the same N will work for all $z \in D$.

Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series, and set $f_n(z) = \sum_{i=0}^{n} a_i(z-z_0)^i$. In this way, we may view $f(z)$ as the limit of a sequence of polynomial functions.

Recall that if $r > 0$, the disk of radius r about z_0 is denoted

$$B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

Theorem 2. *Let $D \subset \mathbb{C}$ be open and let $f : D \to \mathbb{C}$ be continuously differentiable on D. Let $z_0 \in D$. Let $r, R \in \mathbb{R}$ with $0 < r < R$. If $B_R(z_0) \subset D$, then f has a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^k,$$

which converges absolutely and uniformly in $B_r(z_0)$. Furthermore, the coefficients are given by

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{k+1}} dw,$$

where C is a positively oriented circle of radius r centered at z_0.

Example 3. This is a phenomenon which applies to complex differentiability and not real differentiability. For example, take $g(x) = 2|x|$ and integrate to get

$$G(x) = \begin{cases} -x^2 & \text{if } x < 0; \\ x^2 & \text{if} x \geq 0. \end{cases}$$

This is a function which is continuously differentiable at $x_0 = 0$, but whose derivative is not differentiable.

Example 4. More extreme (subtle?) cases also exist for the reals. Let

$$f(x) = \frac{1}{e^{x^2}}.$$

This function is infinitely differentiable, but each of the derivatives evaluated at 0 produce 0. Thus the Taylor series is constantly zero, but clearly the function is not. Thus is it not "analytic" in the sense that it equals its Taylor expansion. This function is not complex differentiable.

3. The Identity Theorem

Let f be an analytic at a point w. Then f has a power series expansion at w. If f is not constantly zero, there exists a nonnegative integer m such that

$$f(z) = (z - w)^m g(z),$$

where g is analytic at w and $g(w) \neq 0$. We call m the *multiplicity* of w as a zero of f. Of course, if $m = 0$, this means $f(w) \neq 0$. Since g is continuous at w and nonvanishing at w, there exists a neighborhood of w such that g is nonvanishing in that neighborhood.

Theorem 3. (Identity Theorem) *Let $D \subset \mathbb{C}$ be an open connected set and let $f : D \to \mathbb{C}$ be analytic. Let (z_n) be an injective sequence in D which converges to $w \in D$. Suppose that $f(z_n) = 0$ for all $n \in \mathbb{N}$. Then $f(z) = 0$ for all $n \in \mathbb{N}$.*

Proof. Suppose that f is not constantly zero. Write $f(z) = (z-w)^m g(z)$, where g is analytic at w and $g(w) \neq 0$. Since g is continuous at w, there exists a neighborhood U of w such that $g(z) \neq 0$ for all $z \in U$. Let $N \in \mathbb{N}$ be so large that $n \geq N$ implies $z_n \in U$. Then $f(z_n) = 0$, so $(z_n - w)^m g(z_n) = 0$, and since $z_n \neq w$, we have $g(z_n) = 0$. This contradiction proves the theorem. $\square$

We have these alternate forms of the identity theorem.

Corollary 1. *Let f and g be analytic functions in a domain D which agree on a convergent sequence. Then $f = g$.*

Corollary 2. *Let f be an analytic function which is constant on an open subset of its domain. Then f is constant.*

Corollary 3. *Let f and g be analytic functions which agree on an open set. Then $f = g$.*

4. Singularities

Much of this section is derived from *Complex Variables* by Stephen S. Fischer.

Definition 5. Let f be a complex valued function of a complex variable, and let $z_0 \in \mathbb{C}$. We say that f has an *isolated singularity* at z_0 if f is defined and analytic is a deleted neighborhood of z_0, but is undefined at z_0.

There are three possibilities:

(1) f is bounded in a deleted neighborhood of z_0;
(2) $\lim_{z \to z_0} f(z) = \infty$;
(3) neither **(1)** or **(2)**.

4.1. Removable Singularities.

Case 1: Suppose f is bounded in a deleted neighborhood of z_0.

Let $\epsilon > 0$ and $M > 0$ such that $|f(z)| < M$ for $0 < z - z_0 < M$. Consider the function

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \neq z_0; \\ 0 \quad \text{if } z = z_0. \end{cases}$$

Then g is differentiable at 0, since

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} f(z) = 0.$$

Thus g is analytic for $|z - z_0| < \epsilon$, so g equals its Taylor expansion

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots .$$

But $b_0 = g(z_0) = 0$ and $b_1 = g(z_0) = 0$, so

$$g(z) = b_2(z - z_0)^2 + b_3(z - z_0)^3 + \cdots ,$$

whence

$$f(z) = b_2 + b_3(z - z_0) + \cdots$$

for $0 < |z - z_0|$.

Set $f(z_0) = b_2$. Then f is analytic for $|z - z_0| < \epsilon$.

Note that $b_2 = \lim_{z \to z_0} f(z)$, which we could have used as a definition; however, the approach above is more thorough. We have shown that the simple condition of boundedness implies not only continuity, but also analyticity.

We call z_0 a *removable singularity* of f.

4.2. **Poles.** *Case 2:* Suppose that $\lim_{z \to z_0} f(z) = \infty$.

Let r be so small that $|z - z_0| < r$ implies $|f(z)| > 1$. Then, $g(z) = \dfrac{1}{f(z)}$ is analytic on the punctured disk $\{z \in \mathbb{C} \mid 0 \leq |z - z_0| < r\}$ and is bounded there with $0 < |g(z)| < 1$. Thus 0 is a removable singularity of $g(z)$, and it is easy to see that we may extend g by setting $g(z_0) = 0$. Let m be the order of the zero of g at z_0; then $g(z) = (z - z_0)^m h(z)$ with $h(z)$ analytic on this disk and $h(z_0) \neq 0$. Since g does not vanish on the punctured disk, neither does h. Thus $H(z) = \dfrac{1}{h(z)}$ is analytic for $|z - z_0| < r$. Now

$$f(z) = \frac{H(z)}{(z - z_0)^m},$$

where $H(z)$ is analytic on the disk and $H(z_0) \neq 0$.

We call z_0 a *pole of order m* of f.

4.3. **Essential Singularities.** *Case 3:* Suppose that f is not bounded near z_0, yet $f(z)$ does not diverge to ∞ as $z \to z_0$. In this case we say that z_0 is an *essential singularity* of f.

5. Laurent Series

Definition 6. A *Laurent series* is an expression of the form

$$\sum_{n \in \mathbb{Z}} a_n (z - z_0)^n,$$

where $z_0 \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$.

We say that such a series *converges at z* if the two series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{a_{(-n)}}{(z - z_0)^n}$$

both converge.

Let $f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$. As above, there are three possibilities:

(1) $a_n = 0$ for all $n < 0$;
(2) $a_n = 0$ for all $n < -k$ for some positive integer k;
(3) neither **(1)** or **(2)**.

In the first case, we have a regular power series, which produces a function which is analytic at z_0, and all analytic functions have such a power series.

In the second case, f has a pole at z_0 of order k for some positive integer k. In this case, there exists a function $g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ which is analytic at z_0 such that

$$f(z) = \frac{1}{(z - z_0)^k} g(z) = \frac{1}{(z - z_0)^k} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=-k}^{\infty} a_{n+k} (z - z_0)^n.$$

In the third case, f is not analytic at z_0 and f does not have a pole at z_0; so, f has an essential singularity at z_0.

Example 5. Find the Laurent series of $f(z) = \dfrac{e^z}{(z-2)^3}$ at $z_0 = 2$.

Solution. Since $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$, we have

$$e^z = e^2 \cdot e^{(z-2)} = e^2 \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^n.$$

Thus

$$f(z) = \frac{e^z}{(z-2)^3} = \sum_{n=-3}^{\infty} \frac{e^2}{(n+3)!}(z-2)^n.$$

□

Example 6. Find the Laurent series of $f(z) = \dfrac{z}{z^2 - 7z + 10}$ at the pole $z_0 = 2$.

Solution. First we note that $f(z) = \dfrac{z}{(z-2)(z-5)}$. Let $g(z) = \dfrac{z}{z-5}$; then $f(z) = \dfrac{g(z)}{z-2}$, where $g(z)$ is analytic at $z_0 = 2$. We compute the power series of g at z_0 in the normal way:

$g(z) = \dfrac{z}{z-5}$; $g'(z) = -5(z-5)^{-2}$; $g''(z) = 10(z-5)^{-3}$; $g'''(z) = -5{\cdot}2{\cdot}3(z-5)^{-4}$; $\ldots$

$$g^{(n)} = \frac{(-1)^n \cdot 5 \cdot n!}{(z-5)^{n+1}}.$$

Plug in $z = 2$ to get

$$g^{(n)}(2) = \frac{(-1)^n \cdot 5 \cdot n!}{(-3)^{n+1}} = -\frac{5n!}{3^{n+1}}.$$

Divide by $n!$ to obtain the coefficients of the power series:

$$a_n = \frac{g^{(n)}(2)}{n!} = -\frac{5}{3^{n+1}}.$$

Thus

$$g(z) = \sum_{n=0}^{\infty} -\frac{5}{3^{n+1}}(z-2)^n, \quad \text{so} \quad f(z) = \frac{g(z)}{z-2} = \sum_{n=-1}^{\infty} -\frac{5}{3^{n+2}}(z-2)^n.$$

□

6. Existence and Domain of Laurent Series

From the examples above, it is clear that Laurent series exist for poles. It is unclear that they exist for essential singularities, and we have not stated what to expect from their domains.

Thus let us begin with a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, which is analytic at z_0 with radius of convergence $R > 0$; then $f(z)$ converges if $|z| < R.$, Plug in z^{-1} to obtain a Laurent series, and see that $f(z^{-1})$ converges if $|z^{-1}| < R$, that is, if $|z| > \frac{1}{R}$. If the original function $f(z)$ is entire, then $f(z^{-1})$ converges nowhere. Substitute $z - z_0$ with z above to center around some other point.

Thus to understand the domain of a Laurent series $f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$, we break it into two parts:

$$f(z) = f_1(z) + f_2(z),$$

where

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad f_1(z) = \sum_{n=1}^{\infty} \frac{a_{(-n)}}{(z - z_0)^{(-n)}}.$$

Now $f_1(\frac{1}{z} + z_0)$ is a power series, with radius of convergence r_1, and $f_2(z + z_0)$ is a power series with radius of converge R_2. Let $R_1 = \frac{1}{r_1}$. Then f converges in an annulus

$$\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}.$$

7. The Residue Theorem

Let f be analytic in a punctured disk $U = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$. Then f has a Laurent series $f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$.

Let C be a positively oriented circle centered at z_0 of radius s with $0 < s < r$. We see that $a(z-z_0)^n$ has a primitive for all $n \in \mathbb{Z}$, except $n = -1$. Thus

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C \Big(\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n\Big)\,dz \\
&= \dots + \int_C \frac{a_{-2}}{(z-z_0)^2}\,dz + \int_C \frac{a_{-1}}{z-z_0}\,dz + \int_C a_0\,dz + \int_C a_1(z-z_0)\,dz = \dots \\
&= \dots + 0 + \int_C \frac{a_{-1}}{z-z_0}\,dz + 0 + 0 + \dots \\
&= \int_C \frac{a_{-1}}{z-z_0}\,dz \\
&= 2\pi i a_{-1}.
\end{aligned}$$

The *residue* of f at z_0 is

$$\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz.$$

Since a simply closed curve about a region containing isolated points is homotopic to a series of small circles around the isolated points, linked by line segments, we obtain the Residue Theorem.

Theorem 4. (Residue Theorem)
Let D be a simply connected open set and let $z_1, \dots, z_n$ be distinct points in D. Let $U = D \setminus \{z_1, \dots, z_n\}$, and let f be analytic on U.

Let γ be a positively oriented simple closed path in U. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \operatorname{Res}(f, z_k).$$

More generally, if γ is any closed path in U, we have

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} n(\gamma, z_k)\operatorname{Res}(f, z_k),$$

where

$$n(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-w)}$$

denotes the winding number of γ about w.

8. The Argument Principle

Recall that a function $f : D \to \mathbb{C}$ is called *meromorphic* if f is analytic on D, except possibly at a finite number of isolated points, where the function has poles. Some of the details of what follows were obtained from Wikipedia.

Theorem 5. (Argument Principle) *Let f be meromorphic on a simply connected open set D. Let γ be a positively oriented simple closed path in D, along which f has no zeros or poles. Let Z and P be the number of zeros and poles, respectively, of f inside γ, counted with multiplicity. Then*

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(Z - P).$$

More generally, let γ be any closed path in D, along which f has no zeros or poles. Let $z_1, \ldots, z_k$ be the zeros of f inside γ, and let $p_1, \ldots, p_l$ be the poles of f inside of γ; these are repeated for multiple zeros or poles. Then

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\Big(\sum_{i=1}^{k} n(\gamma, z_i) - \sum_{j=1}^{l} n(\gamma, p_j)\Big).$$

Proof. We find the residues of the integrand at zeros of f, and then at poles of f.

Let w be a zero of f. We can write $f(z) = (z-w)^m g(z)$, where m is the multiplicity of the zero, and $g(w) \neq 0$. We get

$$f'(z) = m(z-w)^{m-1}g(z) + (z-w)^m g'(z),$$

whence

$$\frac{f'(z)}{f(z)} = \frac{m}{z-w} + \frac{g'(z)}{g(z)}.$$

Since $g(w) \neq 0$, $\dfrac{g'(z)}{g(z)}$ is analytic at w, so the residue of $\dfrac{f'(z)}{f(z)}$ at w is m.

Let w be a pole of f. Then $f(z) = \dfrac{g(z)}{(z-w)^m}$, where m is the order of the pole, and $g(w) \neq 0$. Then

$$f'(z) = \frac{-m}{(z-w)^{m+1}}g(z) + \frac{g'(z)}{(z-w)^m},$$

whence

$$\frac{f'(z)}{f(z)} = \frac{m}{z-w} + \frac{g'(z)}{g(z)}.$$

Since $g(w) \neq 0$, $\dfrac{g'(z)}{g(z)}$ is analytic at w, so the residue of $\dfrac{f'(z)}{f(z)}$ at w is $-m$.

Adding these residues gives the result, via the Residue Theorem. $\square$

To interpret these results, we use the substitution $w = f(z)$ to obtain

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_{f\circ\gamma} \frac{1}{w}\,dw.$$

We see that the latter quantity is $2\pi i$ times the winding number around zero of the image of γ under f. So, this integral measure the total change in argument of $f(z)$ as z travels around γ.

9. Local Mapping Theorem

Theorem 6. (Local Mapping Theorem) *Let f be analytic at z_0, with $f(z_0) = w_0$. Suppose $f(z) - w_0$ has a zero of multiplicity n at z_0. Then there exists an open neighborhood V of w_0 such that for every $v \in V \setminus \{w_0\}$, the equation $f(z) = v$ has exactly n distinct solutions.*

Proof. Note that f' is also analytic at z_0; thus, by the Identity Theorem, it is possible to choose $\delta > 0$ such that f is defined and analytic on the disk $|z - z_0| < \delta$, and such that $f(z) \neq w_0$ and $f'(z) \neq 0$ for z in the punctured disk $0 < |z - z_0| < \delta$.

Let γ parameterize the circle $|z - z_0| = \delta$. Let $\Gamma = f \circ \gamma$ be the image of γ under f. Let ϵ be so small that the open disk given by $|w - w_0| < \epsilon$ is enclosed by Γ. Let V denote the open disk given by $|w - w_0| < \epsilon$. Let $v \in V$.

By the Argument Principle, the number of solutions (counted with multiplicity) of $f(z) = v$ is

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - v}\, dz = \int_\Gamma \frac{1}{w - v}\, dw = n(\Gamma, v).$$

This is independent of which v in V we select. For $v = w_0$, this integral equals n, so it equals n for all $v \in V$. In passing, we reiterate that n is the winding number of Γ around every point in V.

Now suppose that $v \neq w_0$, and that $f(u) = v$. The function $f(z) - v$ vanishes to order 1 at $z = u$, since $f'(u) \neq 0$. Thus the solutions of $f(z) = v$ are distinct. □

Definition 7. Let $D \subset \mathbb{C}$ be an open subset and let g be analytic on D. We say that g is *locally injective* on D if for every $w \in D$ there exists a neighborhood W of w such that, for every $a, b \in W$,

$$f(a) = f(b) \Rightarrow a = b.$$

Corollary 4. *Let $D \subset \mathbb{C}$ be an open subset and let f be analytic on D. If $f'(z)$ does not vanish on D, then f is locally injective on D.*

Proof. Let $z_0 \in D$ and $f(z_0) = w_0$. Then f is n to 1 in a neighborhood of z_0, where n is the order of vanishing of $f(z) - w_0$. Since $f'(z_0) \neq 0$, this order is 1, as can be seen from the power series expansion of f about z_0. □

Corollary 5. (Open Mapping Theorem) *Let $D \subset \mathbb{C}$ be an open set and let $f : D \to \mathbb{C}$ be analytic. Then f is an open map.*

Proof. Recall that this means that f maps open sets to open sets. Thus let $U \subset D$ be open and let $W = f(U)$. Let $w \in W$ and let $u \in U$ such that $f(u) = w$. By Theorem 6, there exists an open neighborhood V of w such that f maps onto V. Then $V \cap W$ is an open neighborhood of w contained in W. Thus w is an interior point of W, and W is open. □

10. Rational Functions

Let D be an open subset of $\mathbb{C}$ which is connected and simply connected. A function f is *meromorphic* on D if f is analytic on D, except for isolated singularities which are poles. We view f as a function from D to $\mathbb{C}_\infty$.

We can extend this definitions to include ∞ in the domain by letting $D \in \mathbb{C}_\infty$, and defining $f(\infty) = \lim_{z\to\infty} f(z)$. Now we may state that $f(z)$ is analytic at $z = \infty$ if $f(1/z)$ is analytic at zero.

For simplicity, we will focus on the case where f is defined on the entire Riemann sphere. The poles of f are isolated by the Identity Theorem, and since the Riemann sphere is compact, there are finitely many of them. In fact, if $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is nonconstant, the Identity Theorem implies that the fiber over every point in finite, for otherwise, we would have an infinite subset of the sphere, which would necessarily have an accumulation point, implying that f is constant.

Proposition 10. *Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a nonconstant meromorphic function. Then*

- **(a)** *f is surjective;*
- **(b)** *the cardinality of every fiber of f is finite, and the equation $f(z) = w$ has the same number of solutions, when counted with multiplicity, for every $w \in \mathbb{C}_\infty$;*
- **(c)** *f is a rational function.*

Proof. Much of this bit is copied from my college work. It is here just to show how quickly it can go when good definitions have already been made.

Suppose f is nonconstant. The image of f is open by the Open Mapping Theorem, and it is also closed because the domain is compact and the image is Hausdorff. The image of f is both open and closed, so it is a component of the codomain, and since $\mathbb{C}_\infty$ is connected, the image must be all of $\mathbb{C}_\infty$; thus f is surjective.

Let $w \in \mathbb{C}_\infty$ and let $F = f^{-1}(w)$; the Identity Theorem implies that F is a discrete subset of $\mathbb{C}_\infty$, so F is finite because $\mathbb{C}_\infty$ is compact.

Define a function $n : \mathbb{C}_\infty \to \mathbb{Z}$ by $n(z)$ is the number of points in the fiber of f over $f(z)$, counted with multiplicity. that is, $n(z)$ is the cardinality of the fiber of f in which z resides, counted with multiplicity. By the Local Mapping Theorem, n is a continuous map into a discrete set, and since $\mathbb{C}_\infty$ is connected, it is constant. In particular, the number of zeros of f equals the number of poles of f, when counted with multiplicity.

Let f have zeros $a_1, \ldots, a_n$ and poles $b_1, \ldots, b_n$. Set

$$g(x) = \frac{\Pi_{i=1}^n (x - a_i)}{\Pi_{j=1}^n (x - b_j)}.$$

Then f/g is a function without zeros or poles, which is constant by the Open Mapping Theorem. Thus $f = ag$ for some $a \in \mathbb{C}$, and f is a rational function. $\square$

11. Branch Points

Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a nonconstant meromorphic function. By the previous proposition, f is a rational function. The *degree* of f, denoted $\deg(f)$, is the cardinality of the "generic fiber"; that is, it is the number of solutions to $f(z) = w$, when counted with multiplicity; this number does not depend on w. If $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are polynomials without common factors, then

$$\deg(f) = \max\{\deg g, \deg h\}.$$

Since f is a rational function, it is differentiable, and its derivative is also meromorphic, so by the Identity theorem there are only finitely many points where the derivative vanishes. These points in the domain are called the *ramification points* of the cover. Their images are called the *branch points* of the cover.

The *ramification index* of a point $z_0 \in \mathrm{dom}(f)$, denoted $\mathrm{ri}_f(z_0)$, is the multiplicity of z_0 as a zero of the function $f(z) - f(z_0)$. The Local Mapping Theorem tells us that f is n-to-1 in a neighborhood of z_0, where n is the ramification index of z_0. This ramification index is 1 plus the order of vanishing of the derivative at the ramification point.

The sum of the ramification in a fiber is the degree of the function. Specifically, let $w \in \mathbb{C}_\infty$ be in the range of f, and let $F = f^{-1}(w)$ be the fiber over w. Then

$$\sum_{z\in F} \mathrm{ri}_f(z) = \deg(f).$$

To find the ramification points, take the derivative and see where it vanishes. The corresponding ramification index is one more than the order of vanishing of the derivative.

We now wish to clarify the role of ∞ in this discussion. It is relatively clear what is meant to say ∞ is a branch point; it simply means that one of the points in the fiber over ∞ is ramified. We thus wish to outline what is meant by the order of vanishing of the derivative at ∞.

To investigate this, let's back up a little, and begin with $g(z) = \dfrac{az+b}{cz+d}$. Then $g'(z) = \dfrac{ad-bc}{(cz+d)^2}$; if $ad - bc = 0$, the $g'(z) = 0$ so g is constant; otherwise, g is a Möbius transformation, that is, g is a degree one rational function with nonvanishing derivative, and g is bijective. Assume this is the case.

Let f be a meromorphic function and let $h = f \circ g$. By the chain rule, $h'(z) = f'(g(z))g'(z)$, so h' vanishes at z_0 if and only if f' vanishes at $g(z_0)$. Apply this in the case $g(z) = \dfrac{1}{z}$ and $z_0 = \infty$; we see that $h'(z) = f(1/z)$ vanishes at 0 if and only if f is ramified at infinity.

We point out that, if $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a polynomial of degree n, then ∞ is ramified to index n. The easiest way to see this is to consider the fact that the only point in the fiber over ∞ is ∞, so

$$n = \deg(f) = \sum_{z\in f^{-1}(\infty)} \mathrm{ri}_f(z) = \mathrm{ri}_f(\infty).$$

Example 7. Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the given rational function. Analyze the function as follows.

- Find the degree of f.
- Find the ramification points of f.
- Find the ramification index of each ramification point of f.
- Find the branch points of f.

(a) Analyze the polynomial function

$$f : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad f(z) = z^2 + 16.$$

The degree of f is 2. Take the derivative and find that $f'(z) = 2z$, so if $f'(z) = 0$, we have $z = 0$. Thus the only finite ramification point is $z = 0$. Since this is a two-sheeted cover, the only possible ramification index for a ramification point is 2; so f is ramified to index 2 at 0 and ∞. The branch points are $f(0) = 16$ and $f(\infty) = \infty$.

(b) Analyze the polynomial function

$$f : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad f(z) = z^3 - 3z^2 - 45z.$$

The degree is 3. We have $f'(z) = 3z^2 - 6z - 45 = 0$ implies $z^2 - 2z - 15 = 0$, so $z = -3$ or $z = 5$. In each case, the ramification index is 2. Also, since this is a polynomial, it is ramified to index 3 at infinity. The finite branch points are $f(-3) = -81$ and $f(5) = -175$. We can find the fiber over the branch points by solving the polynomial equations $f(z) + 81 = 0$ and $f(z) + 175 = 0$. Since we already know that -3 and 5 are double zeros of these respective equations, we can use polynomial division (via synthetic division) to find that $f^{-1}(-81) = \{-3, 9\}$ and $f^{-1}(-175) = \{5, -7\}$.

(c) Analyze the polynomial function

$$f : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad f(z) = z^4 - 4z^3 + 2z^2 - 12z + 1.$$

The degree is four. We have $f'(z) = 4z^3 - 12z^2 + 4z - 12 = 0$ implies $z^3 - 3z^2 + z - 3 = 0$, so $(z^2 + 1)(z - 3) = 0$, so $z = \pm i$ or $z = 3$. Each of these is ramified to index 2. The branch points are $f(i) = -8i$, $f(-i) = 8i$, and $f(3) = -44$. Infinity is ramified over infinity to index four.

(d) Analyze the rational function

$$f : \mathbb{C}_\infty \to \mathbb{C}_\infty \quad \text{given by} \quad f(z) = z + \frac{1}{z}.$$

Rewrite f as $f(z) = \dfrac{z^2 + 1}{z}$. Then we see that the degree is two. The derivative is $f'(z) = \dfrac{z^2 - 1}{z^2}$, so $f'(z) = 0$ implies $z = \pm 1$. The branch points are $f(1) = 2$ and $f(-1) = -2$. Also, ∞ is unramified, since the fiber over infinity consists of 0 and ∞.

Example 8. Find a rational function f of degree 3 with integer coefficients which is ramified to index 2 at $z = 0$, 2 at $z = 1$, and 3 at $z = \infty$, such that $f(1) = 1$. Find the branch points of your function. What is the fiber over ∞?

Solution. Consider the derivative $z(z-1) = z^2 - z$. Integrate to get $\frac{1}{3}z^3 - \frac{1}{2}z^2 + C$. Multiplying by a constant will not change the ramification points, so let $f(z) = 2z^3 - 3z^2 + C$. Set $C = 2$ so that $f(1) = 1$. The branch points are $f(0) = 2$ and $f(1) = 1$. The only point over infinity is infinity, since this is a polynomial. $\square$

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