Definition 1. Let X be a set. A *partition* on X is a collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of X satisfying

(P0) $C \in \mathcal{C} \Rightarrow C \neq \varnothing$

(P1) $C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cap C_2 = \varnothing \vee C_1 = C_2$

(P2) $\cup \mathcal{C} = X$

Problem 1. Let $X = \{a, b, c\}$. Write all partitions of X.

Solution. There are five distinct partitions.

- $\{\{a, b, c\}\}$ The trivial partition
- $\{\{a, b\}, \{c\}\}$
- $\{\{a, c\}, \{b\}\}$
- $\{\{a\}, \{b, c\}\}$
- $\{\{a\}, \{b\}, \{c\}\}$ The discrete partition

□

Definition 2. Let $f : X \to Y$. The *partition of X induced by f* is the collection of subsets of X given by

$$\mathcal{C} = \{f^{-1}(y) \mid y \in Y\}.$$

Problem 2. Let $f : \mathbb{Z} \to \mathbb{R}$ be given by $f(n) = \sin\left(\dfrac{\pi}{6}n\right)$. Describe the partition of $\mathbb{Z}$ induced by f.

Solution. The range of the function is

$$f(\mathbb{Z}) = \left\{0, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1\right\}.$$

We introduce some notation for this problem. Let

$$B_r = \{n \in \mathbb{Z} \mid n = 12k + r \text{ for some } k \in \mathbb{Z}\}.$$

Thus, the partition of $\mathbb{Z}$ consists of five blocks:

- $f^{-1}(0)$ $= B_0 \cup B_6$
- $f^{-1}(1/2)$ $= B_1 \cup B_5$
- $f^{-1}(\sqrt{3}/2)$ $= B_2 \cup B_4$
- $f^{-1}(1)$ $= B_3$
- $f^{-1}(-1/2)$ $= B_7 \cup B_{11}$
- $f^{-1}(\sqrt{3}/2)$ $= B_8 \cup B_{10}$
- $f^{-1}(-1)$ $= B_9$

□

Definition 3. A *partitioned set* is a pair $(X, \mathcal{C})$, where X is a set, and $\mathcal{C}$ is a partition of X.

We wish to study "partition preserving functions" from one partitioned set to another. Of course, the first step is to write down what this means (I can think of at least two potentially significant definitions).

Problem 3. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be partitioned sets, and let $f : X \to Y$. Define what it means to say that f is *partition-preserving* with respect to $\mathcal{C}$ and $\mathcal{D}$.

Solution. We give six different potential definitions.

- for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $f(C) \subset D$.

- for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $f(C) = D$.

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- for every $D \in \mathcal{D}$ there exists a unique $C \in \mathcal{C}$ such that $f(C) \subset D$.

- for every $D \in \mathcal{D}$ there exists a unique $C \in \mathcal{C}$ such that $f(C) = D$.

What is the difference between these definitions? Do any of these imply that f is injective? surjective? bijective? Are any of these situations impossible? ☐

We will use the following definition.

Definition 4. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be partitioned sets, and let $f : X \to Y$. We say that f is *partition-preserving* with respect to $\mathcal{C}$ and $\mathcal{D}$ if for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $f(C) \subset D$.

Problem 4. Let
$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \quad \text{and} \quad Y = \{a, b, c, d, e, f, g, h, i, j\}.$$
Let
$$\mathcal{C} = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7, 8\}\} \quad \text{and} \quad \mathcal{D} = \{\{a\}, \{b, c\}, \{d, e, f\}, \{g, h, i, j\}\}.$$
Consider the partitioned sets $(X, \mathcal{C})$ and $(Y, \mathcal{D})$.

(a) Define a partition-preserving function $X \to Y$, or, demonstrate that your definition does not allow such a function.

(b) Define a partition-preserving function $Y \to X$, or, demonstrate that your definition does not allow such a function.

Solution. The first step is the carefully read the problem and realize that it contains a typo. The next step is to fix the typo. Only after that may we form a solution.

We define
$$\mathcal{C} = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7, 8, 9\}\} \quad \text{and} \quad \mathcal{D} = \{\{a\}, \{b, c\}, \{d, e, f\}, \{g, h, i, j\}\}.$$

(a) $\{1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d, 5 \mapsto e, 6 \mapsto g, 7 \mapsto h, 8 \mapsto i, 9 \mapsto j\}$.

(b) $\{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4, e \mapsto 5, f \mapsto 5, g \mapsto 6, h \mapsto 7, i \mapsto 8, j \mapsto 9\}$.

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