PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §2

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Proposition 1. *Let p be a prime integer. Then $\sqrt{p}$ is irrational.*

Proof 1. Suppose that $\sqrt{p}$ is rational. Then there exist integers $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ (that is, a and b have no common factors other than 1) such that $\sqrt{p} = \frac{a}{b}$. Then $p = \frac{a^2}{b^2}$, so $b^2 p = a^2$. Then p divides a^2; since p is prime, we have p divides a. Thus $a = kp$ for some $k \in \mathbb{Z}$. From this, $b^2 p = k^2 p^2$, so $b^2 = k^2 p$; thus p divides b^2, so p divides b. This contradicts our choice of a and b. □

Proof 2. Suppose that $\sqrt{p}$ is rational. Then there exist integers $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ such that $\sqrt{p} = \frac{a}{b}$. We may assume that a and b are positive. Then $\frac{a}{b}$ is a root of the polynomial $x^2 - p$. By the rational roots theorem, b divides 1 and a divides p. Then $b = 1$ and either $a = p$ or $a = 1$. But if $a = p$, we have $p^2 - p = 0$, so $p = 1$ or $p = 0$; and if $a = 1$, we have $1 - p = 0$, so $p = 1$. In either case, p is not prime, producing a contradiction. □

Exercise 1 (2.1)**.** Show that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

Proof. Our generalization above handles all of the cases $p = 3, 5, 7, 31$, so it remains to show that $\sqrt{24}$ is not rational.

Suppose that $\sqrt{24}$ is rational. Then $\sqrt{24} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and $a, b > 0$. Then $\frac{a}{b}$ is a root of the polynomial $x^2 - 24 = 0$. So b divides 1, and $a^2 = 24$. Also a divides 24, so $a \in \{1, 2, 4, 6, 8, 12, 24\}$. But none of these squared is equal to 24. □

Exercise 2 (2.4)**.** Show that $(5 - \sqrt{3})^{1/3}$ does not represent a rational number.

Proof. Suppose by way of contradiction that $\alpha = (5 - \sqrt{3})^{1/3} \in \mathbb{Q}$. If we cube a rational number, the result is rational; thus $\alpha^3 = 5 - \sqrt{3} \in \mathbb{Q}$. If we subtract 5 from a rational number, the result is rational; thus $\alpha^3 - 5 = -\sqrt{3} \in \mathbb{Q}$. If we multiply a rational number of -1, the result is rational; thus $5 - \alpha^3 = \sqrt{3} \in \mathbb{Q}$. This contradicts exercise 1. □

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Exercise 3 (2.5)**.** Show that $\alpha = [3 + \sqrt{2}]^{2/3}$ is does not represent a rational number.

Proof. We "unwind" α to find a polynomial of which it is a root. Cubing both sides gives

$$\alpha^3 = [3 + \sqrt{2}]^2 = 9 + 6\sqrt{2} + 2 = 6\sqrt{2} + 11.$$

Thus $a^3 - 11 = 6\sqrt{2}$, so $\alpha^6 - 22\alpha^3 + 121 = 72$. Then α is a root of

$$f(x) = x^6 - 22x^3 + 49.$$

Suppose that α is rational and write $\alpha = \frac{a}{b}$, where $\gcd(a, b) = 1$. By the Rational Roots Theorem, b divides 1 and a divides 49. Thus $b = 1$, so a is a root of $f(x)$; also $a \in \{1, 7, 49\}$. But none of these are roots, as can be seen by plugging in. $\square$

Exercise 4 (2.6)**.** Discuss why $4 - 7b^2$ must be rational if b is rational.

Discussion. The rational numbers are closed under the operations of addition and multiplication. That is, if $a, b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$ and $ab \in \mathbb{Q}$. Since $b \in \mathbb{Q}$, we have $b^2 \in \mathbb{Q}$. Since $-7 \in \mathbb{Q}$, we have $-7b^2 \in \mathbb{Q}$. Since $4 \in \mathbb{Q}$, we have $4 + (-7b^2) = 4 - 7b^2 \in \mathbb{Q}$. $\square$

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