Name:

Abstract Algebra (Math 3063)
Midterm Exam II - Take Home

Professor Paul Bailey
Friday, April 3, 2009

Due Friday, April 10, 2009.
Please write neatly, in complete sentences. Attach this sheet to your solutions.

Problem 1. Let p be a positive odd integer.

(a) How many p-cycles are in A_p?

(b) How many distinct cyclic subgroups of order p are in A_p?

Problem 2. Let p be a positive prime integer and define

$$\phi : \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{by } \phi(a) = a^p.$$

(a) Show that ϕ is bijective.

(b) Show that $\phi(ab) = \phi(a)\phi(b)$.

(c) Show that $\phi(a + b) = \phi(a) + \phi(b)$ (hint: use the binomial theorem).

Problem 3. Let G be a group and let $H = \{h \in G \mid h = g^2 \text{ for some } g \in G\}$. Suppose that $H \leq G$.

(a) Show that $H \triangleleft G$.

(b) Show that G/H is abelian.

Problem 4. Consider the groups $\mathbb{R}$ under addition and $\mathbf{GL}_2(\mathbb{R})$ under matrix multiplication. Let

$$M = \left\{ A \in \mathbf{SL}_2(\mathbb{R}) \mid A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\}.$$

(a) Show that $\phi : \mathbb{R} \to \mathbf{GL}_2(\mathbb{R})$ given by $\phi(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ is a group homomorphism.

(b) Show that $\phi(\mathbb{R}) = M$.

(c) Conclude that $M \leq \mathbf{SL}_2(\mathbb{R})$ and that $M \cong \mathbb{R}/2\pi\mathbb{Z}$.

Problem 5. Let $X \subset \mathbb{R}^2$ be a subset of the cartesian plane. If $\vec{v}, \vec{w} \in X$, the distance between $\vec{v}$ and $\vec{w}$ is denoted $d(\vec{v}, \vec{w})$. An *isometry* of X is a function $f : X \to X$ which preserves the distance between any to points, so that

$$d(\vec{v}, \vec{w}) = d(f(\vec{v}), f(\vec{w})).$$

Let

$$\mathrm{Iso}(X) = \{f : X \to X \mid f \text{ is an isometry}\}.$$

This is a group under composition.

For example, if X is a square, the isometries of X are rotations and reflections, and $\mathrm{Iso}(X) \cong D_4$; that is, the group of isometries of X is isomorphic to the dihedral group on 4 points.

Describe $\mathrm{Iso}(X)$ (number of elements, elements and their orders, how elements interact, interesting subgroups, etc.) in each of these cases

(a) $X = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ (a parabola)

(b) $X = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{9} + \frac{y^2}{4} = 1\}$ (an ellipse)

(c) $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (a circle)

(d) $X = \{(x, y) \in \mathbb{R}^2 \mid y = \tan(x)\}$