Problem 1. Find the vector and parametric equations of the line in $\mathbb{R}^3$ through the point $(-1, 2, -3)$ which is perpendicular to the plane with general equation $x + 2y - 4z = 8$.

Solution. Since the line is perpendicular to the plane, we may use a normal vector for the plane as a direction vector for the line. Let $\vec{v} = \langle 1, 2, -4\rangle$; this is a normal vector for the plane, and so is a direction vector for the line. Let $P_0 = (-1, 2, -3)$; this is a point on the line. Thus the vector equation for the line is

$$\vec{r}(t) = P_0 + t\vec{v} = \langle -1 + t, 2 + 2t, -3 - 4t\rangle$$

The parametric equations are

$$\begin{aligned} x &= -1 + t; \\ y &= 2 + 2t; \\ z &= -3 - 4t. \end{aligned}$$

□

Problem 2. Find the cosine of the angle between the planes in $\mathbb{R}^3$ with general equations $2x - 3y + z = 3$ and $x + 2y + z = -3$.

Solution. The angle between the planes is the same as the angle between their normal vectors. The normal vectors are $\vec{v} = \langle 2, -3, 1\rangle$ and $\vec{w} = \langle 1, 2, 1\rangle$. Now if θ is the angle between the planes,

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|} = \frac{2 - 6 + 1}{\sqrt{4 + 9 + 1}\sqrt{1 + 4 + 1}} = \frac{-3}{\sqrt{84}} = \frac{-3}{2\sqrt{21}} = -\frac{\sqrt{21}}{14}.$$

□

Problem 3. Find the vector form of the equation of the line in $\mathbb{R}^3$ which is the intersection of the planes with general equations $2x - 3y + z = 3$ and $x + 2y + z = -3$.

Solution. We attempt to find the point where this line intersects the xz-plane, so we set $y = 0$ and get $2x + z = 3$ and $x + z = -3$. Subtracting gives $x = 6$, so $z = -9$. We double check that the point $Q = (6, 0, -9)$ is in fact on the line.

Setting $y = 1$ give $2x + z = 6$ and $x + z = -5$. Subtracting these gives $x = 11$. Now $z = -16$, so the point $R = (11, 1, -16)$ is also on the line.

A direction vector for the line is $\vec{v} = R - Q = \langle 5, 1, -25\rangle$, so the vector equation is

$$\vec{r}(t) = Q + t\vec{v} = \langle 6 + 5t, t, -9 - 25t\rangle.$$

□

Problem 4. Find the general form of the equation of the plane in $\mathbb{R}^3$ which is the set of all points in $\mathbb{R}^3$ which are equidistant between the points $Q(1, 3, 2)$ and $R(2, 0, 1)$.

Solution. The plane is perpendicular to the line which passes through Q and R, so the direction vector for that line serves as a normal vector for the plane. Moreover, the midpoint between Q and R is a point on the plane.

Let $\vec{n} = R - Q = \langle 1, -3, -1\rangle$; this is the normal vector. Let $P_0 = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$; this is the midpoint. Let $P = (x, y, z)$. The normal equation for the plane is

$$(P - P_0)\cdot\vec{n} = 0.$$

Now $P_0 \cdot \vec{n} = \frac{3}{2} - \frac{9}{2} - \frac{3}{2} = -\frac{9}{2}$, so the general equation of the plane is

$$x - 3y - z = -\frac{9}{2}.$$

□

Problem 5. Find the cosine of the angle between the plane in $\mathbb{R}^3$ with general equation $5x - 2y + z = 3$ and the line with vector equation $\vec{r}(t) = \langle 2 - 4t, -3 + 3t, 5 + t \rangle$.

Solution. The angle between the plane and the line is the angle between the normal vector of the plane and the direction vector of the line. A normal vector for the plane is $\vec{n} = \langle 5, -2, 3 \rangle$, and a direction vector for the line is $\vec{v} = \langle -4, 3, 1 \rangle$. Thus, the cosine of the angle between them is

$$\cos\theta = \frac{\vec{n} \cdot \vec{v}}{|\vec{n}||\vec{v}|} = \frac{-20 - 6 + 3}{\sqrt{25 + 4 + 9}\sqrt{16 + 9 + 1}} = \frac{-23}{\sqrt{38}\sqrt{26}} = -\frac{23}{2\sqrt{247}}.$$

$\square$

Problem 6. Find the point on the plane in $\mathbb{R}^3$ with general equation $x + 2y + 3z = 6$ which is closest to the point $(5, -2, 8)$.

Solution. Let $Q = (5, -2, 8)$. The plan is to create the line through Q which is perpendicular to the plane, and then find the point of intersection of the line and the plane.

The normal vector of the plane is $\vec{v} = \langle 1, 2, 3 \rangle$. We use this as a direction vector for the line, and get

$$\vec{r}(t) = Q + t\vec{v} = \langle 5 + t, -2 + 2t, 8 + 3t \rangle.$$

Plug these coordinates into the plane to get $(5+t)+2(-2+2t)+3(8+3t) = 6$, and solve to t to get $t = -\frac{19}{14}$. Then the point of intersection is

$$\vec{r}\Big(-\frac{19}{14}\Big) = \Big(5 - \frac{19}{14}, -2 - 2 \cdot \frac{19}{14}, 8 - 3 \cdot \frac{19}{14}\Big) = \Big(\frac{51}{14}, -\frac{66}{14}, \frac{46}{14}\Big).$$

$\square$