PRINCIPLES OF ANALYSIS
SET THEORY SYNOPSIS

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Abstract. It is nearly impossible to fully grasp abstract mathematics without fluent control over the basic concepts of set theory. This document rapidly lists some of what you should know. It is hoped that much of this is review; feel free to bring any questions you may have to the attention of the class.

1. Sets

Set and *element* are undefined terms, except to the extent that we know the relationship between them is *containment*; elements are contained in sets.

If two symbols a and b represent the same element, we write $a = b$. If the symbols a and b represent different elements, we write $a \neq b$. If an element a is a contained in a set A, this relation is written $a \in A$. If a is not in A, this fact is denoted $a \notin A$. We assume that the statements $a \in A$ and $a = b$ are always either true or false, although we may not know which.

Two sets are considered equal when they contain the same elements:

$$A = B \Leftrightarrow [x \in A \Leftrightarrow x \in B].$$

2. Subsets

Let A and B be sets. We say that B is a *subset* of A and write $A \subset B$ if $x \in B \Rightarrow x \in A$.

It is clear that $A = B$ if and only if $A \subset B$ and $B \subset A$.

A set with no elements is called an *empty set*. Since two sets are equal if and only if they contain the same elements, there is only one empty set, and it is denoted $\varnothing$. The empty set is a subset of any other set. A set is *nonempty* if it is not equal to the empty set. Two sets A, B are called *disjoint* if $A \cap B = \varnothing$.

If X is any set and $p(x)$ is a proposition whose truth or falsehood depends on each element $x \in X$, we may construct a new set consisting of all of the elements of X for which the proposition is true; this set is denoted:

$$\{x \in X \mid p(x)\}.$$

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3. Set Operations

Let X be a set and let $A, B \subset X$.

The *intersection* of A and B is denoted by $A \cap B$ and is defined to be the set containing all of the elements of X that are in both A and B:

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}.$$

The *union* of A and B is denoted by $A \cup B$ and is defined to be the set containing all of the elements of X that are in either A or B:

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}.$$

We note here that there is no concept of "multiplicity" of an element in a set; that is, if x is in both A and B, then x occurs only once in $A \cup B$.

The *complement* of A with respect to B is denoted $A \smallsetminus B$ and is defined to be the set containing all of the elements of A which are not in B:

$$A \smallsetminus B = \{x \in X \mid x \in A \text{ and } x \notin B\}.$$

The *symmetric difference* of A and B is denoted $A \triangle B$ and is defined to be the set containing all of the elements X which are in either A or B but not both:

$$A \triangle B = \{x \in X \mid x \in A \cup B \text{ and } x \notin A \cap B\}.$$

Proposition 1. *Let X be a set and let $A, B, C \subset X$. Then*

- $A = A \cup A = A \cap A$;
- $\varnothing \cap A = \varnothing$;
- $\varnothing \cup A = A$;
- $A \subset B \Leftrightarrow A \cap B = A$;
- $A \subset B \Leftrightarrow A \cup B = B$;
- $A \cap B = B \cap A$;
- $A \cup B = B \cup A$;
- $(A \cap B) \cap C = A \cap (B \cap C)$;
- $(A \cup B) \cup C = A \cup (B \cup C)$;
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
- $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$;
- $A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$;
- $A \subset B \Rightarrow A \cup (B \smallsetminus A) = B$;
- $A \subset B \Rightarrow A \cap (B \smallsetminus A) = \varnothing$;
- $A \smallsetminus (B \smallsetminus C) = (A \smallsetminus B) \cup (A \cap B \cap C)$;
- $(A \smallsetminus B) \smallsetminus C = A \smallsetminus (B \cup C)$;
- $A \triangle B = (A \cup B) \smallsetminus (A \cap B)$;
- $A \triangle B = (A \smallsetminus B) \cup (B \smallsetminus A)$.

4. Cartesian Product of Two Sets

If a and b are elements, we can construct a new element

$$(a, b) = \{\{a\}, \{a, b\}\},$$

called an *ordered pair*. Ordered pairs obey the "defining property":

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

If (a, b) is an ordered pair, then a is called the *first coordinate* and b is called the *second coordinate*.

Let A and B be sets. The *cartesian product* of A and B is denoted $A \times B$ and is defined to be the set of ordered pairs whose first coordinate is in A and whose second coordinate is in B:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Proposition 2. *Let X be a set and let $A, B, C \subset X$. Then*

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

5. Numbers

The following notation for sets of numbers is standard:

$$\begin{aligned}
\text{Natural Numbers:} \quad & \mathbb{N} = \{1, 2, 3, \dots\} \\
\text{Integers:} \quad & \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \\
\text{Rational Numbers:} \quad & \mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\} \\
\text{Real Numbers:} \quad & \mathbb{R} = \{\text{Cuts of } \mathbb{Q}\} \\
\text{Complex Numbers:} \quad & \mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}
\end{aligned}$$

We view $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. We note that in treatments prepared for algebra, $0 \in \mathbb{N}$, whereas in analysis, it is typical to begin the natural numbers at 1.

The following notation gives subsets of the real numbers, called *intervals*:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ (closed)
- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ (open)
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$ (closed)
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$ (open)
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$ (closed)
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$ (open)

6. Functions

Let A and B be sets. A *function* from A to B is a subset $f \subset A \times B$ such that

$$\forall a \in A \exists ! b \in B \mid (a,b) \in f;$$

in words: "for every element a in the set A there exists a unique element b in the set B such that the ordered pair (a,b) is in the set f."

If f is such a subset of $A \times B$, we indicate this fact by writing $f : A \to B$. If $a \in A$, the unique element of B such that $(a,b) \in f$ is denoted $f(a)$. Functions obey the "defining property":

- for every $a \in A$ there exists $b \in B$ such that $f(a) = b$;
- if $f(a) = b$ and $f(a) = c$, then $b = c$.

Let $f : A \to B$. The *domain* of f is A, and the *codomain* of f is B. The domain of f is denoted $\mathrm{dom}(f)$:

$$\mathrm{dom}(f) = A.$$

If $C \subset A$, the *image* of C is $f(C) = \{b \in B \mid f(c) = b \text{ for some } c \in C\}$. The image of the function $f : A \to B$ is the image of its domain A, and may be denoted $\mathrm{img}(f)$:

$$\mathrm{img}(f) = f(A).$$

If $D \subset B$, the *preimage* of D is $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$.

The word *range* is used by some authors to mean the codomain of f, and by others to mean the image of f; thus we avoid this word.

We say that f is *injective* (or *one to one*) if for every $a_1, a_2 \in A$ we have

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

We say that f is *surjective* (or *onto*) if

$$\forall b \in B \; \exists a \in A \mid f(a) = b;$$

in words, "for every $b \in B$ there exists $a \in A$ such that $f(a) = b$." A function is surjective if and only if its image is equal to its codomain.

We say that f is *bijective* if it is both injective and surjective.

If A is a set, define the *identity function* on A to be the function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$ for all $a \in A$. This function is bijective.

If $f : A \to B$ and $g : B \to C$, define the *composition* of f and g to be the function $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$.

We say that f is *invertible* if there exists a function $f^{-1} : B \to A$, called the *inverse* of f, such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Proposition 3. *A function is invertible if and only if it is bijective.*

If f is injective, we define the *inverse* of f to be a function $f^{-1} : f(A) \to A$ by $f^{-1}(y) = x$, where $f(x) = y$. Since an invertible function is bijective, it is injective, and this definition of inverse agrees with our previous one in this case.

If $f : A \to B$ is a function and $C \subset A$, we define a function $f \restriction_C : C \to B$, called the *restriction* of f to C, by $f \restriction_C (c) = f(c)$. If f is injective, then so is $f \restriction_C$.

7. Cardinality

We say that two sets have the same *cardinality* if and only if there is a bijective function between them.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers and for $n \in \mathbb{N}$ let

$$H_n = \{m \in \mathbb{N} \mid m < n\}.$$

A set X is called *finite* if there exists a surjective function from H_n to X for some $n \in \mathbb{N}$. If there exists a bijective function $H_n \to X$, we say that the cardinality of X is n, and write $|X| = N$.

A set X is called *infinite* if there exists an injective function $\mathbb{N} \to X$.

Proposition 4. *A set is infinite if and only if it is not finite.*

Proposition 5. *Let A be a finite set and let $f : A \to A$ be a function. Then f is injective if and only if f is surjective.*

Proposition 6. *Let A and B be finite sets. Then $|A \times B| = |A| \cdot |B|$.*

8. Collections

A *collection* is a set whose elements are themselves sets.

Let X be a set. The collection of all subsets of X is called the *power set* of X and is denoted $\mathcal{P}(X)$.

Let $\mathcal{C}$ be a collection of subsets of X; then $\mathcal{C} \subset \mathcal{P}(X)$. Define the *intersection* and *union* of the collection by

- $\cap\mathcal{C} = \{a \in A \mid a \in C \text{ for all } C \in \mathcal{C}\}$
- $\cup\mathcal{C} = \{a \in A \mid a \in C \text{ for some } C \in \mathcal{C}\}$

If $\mathcal{C}$ contains two subsets of X, this definition concurs with our previous definition for the union of two sets.

Let A and B sets. The collection of all functions from A to B is denoted $\mathcal{F}(A, B)$, and is a subset of $\mathcal{P}(A \times B)$.

Proposition 7. *Let X be a set of cardinality $n \in \mathbb{N}$ and let $T = \{0, 1\}$. Then*

(a) $|\mathcal{P}(X)| = |\mathcal{F}(X, T)|$;
(b) $|\mathcal{P}(X)| = 2^n$.

9. Families

Let I and X be sets. A *family* of subsets of X indexed by I is the image of an injective function $A : I \to \mathcal{P}(X)$. For each $\alpha \in I$, the set $A(\alpha)$ may be denoted by A_α. The family itself may denoted by $\{A_\alpha \subset X \mid \alpha \in I\}$

Let $\{A_\alpha \subset X \mid \alpha \in I\}$ be a family of subsets of a set X. The *intersection* and *union* of the family is defined by

- $\cap_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for all } \alpha \in I\}$;
- $\cup_{\alpha \in I} A_\alpha = \{x \in X \mid x \in A_\alpha \text{ for some } \alpha \in I\}$.

If $I = \{1, \dots, n\}$, we may write

- $\cap_{i=1}^n A_i = \{x \in X \mid x \in A_i \text{ for all } i \in I\}$;
- $\cup_{i=1}^n A_i = \{x \in X \mid x \in A_i \text{ for some } i \in I\}$.

If $I = \mathbb{N}$, we may write

- $\cap_{i=1}^\infty A_i = \{x \in X \mid x \in A_i \text{ for all } i \in I\}$;
- $\cup_{i=1}^\infty A_i = \{x \in X \mid x \in A_i \text{ for some } i \in I\}$.

Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$ be a collection of subsets of X. Then $\mathcal{C}$ is a family of subsets of X, indexed by itself via the identity function. Our definitions of intersection and union of a family of subsets concur with our definitions of intersections and union of a collection of subsets under this correspondence.

Proposition 8. DeMorgan's Laws
Let X be a set and let $\{A_\alpha \mid \alpha \in I\}$ be a family of subsets of X indexed by I. Then

(a) $X \smallsetminus \left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I}(X \smallsetminus A_\alpha)$;
(b) $X \smallsetminus \left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I}(X \smallsetminus A_\alpha)$.

10. Cartesian Product of a Family

Let X be a set and let $\mathcal{C} = \{A_\alpha \subset X \mid \alpha \in I\}$ be a family of subsets of X. Let $Y = \cup_{\alpha \in I} A_\alpha$.

The *cartesian product* of $\mathcal{C}$ is denoted by $\times\mathcal{C}$ or by $\times_{\alpha \in I} A_\alpha$ and is defined to be the collection of all functions from Y into the union of the family such that each element of α is mapped to an element of A_α:

$$\times_{\alpha \in I} A_a = \{f \in \mathcal{F}(I, Y) \mid f(\alpha) \in A_\alpha\}.$$

We needed to define the cartesian product of two sets in order to define function, which in turn we have used to define the cartesian product of more than two sets. These definitions concur according to the following proposition.

Proposition 9. *Let X be a set and let $A_1, A_2 \subset X$. Let $I = \{1, 2\}$ and let $Y = A_1 \cup A_2$. For any $x_1, x_2 \in X$, define a function $f_{x_1,x_2} : I \to X$ by $f(1) = x_1$ and $f(2) = x_2$. Define a function*

$$\phi : A_1 \times A_2 \to \{f \in \mathcal{F}(I, Y) \mid f(\alpha) \in A_\alpha\} \quad \text{by} \quad \phi(x_1, x_2) = f_{x_1,x_2}.$$

Then ϕ is a bijection.

The *axiom of choice* states that the cartesian product of a family of nonempty sets is nonempty. This axiom is independent of the other axioms of set theory, and is not accepted by all mathematicians. We will accept it.

11. Relations

A *relation* on a set A is a subset of $R \subset A \times A$. If $(a, b) \in R$, we may indicate this by writing aRb; that is $(a, b) \in R \Leftrightarrow aRb$.

A relation is called *reflexive* if aRa for every $a \in A$.

A relation is called *symmetric* if $aRb \Leftrightarrow bRa$ for every $a, b \in A$.

A relation is called *antisymmetric* if aRb and bRa implies $a = b$.

A relation is called *transitive* if aRb and bRc implies aRc.

A relation is called *definite* if either aRb or bRa for every $a, b \in A$.

A *partial order* on A is a relation which is reflexive, antisymmetric, and transitive. A *total order* on A is a definite partial order.

An *equivalence relation* on A is a relation which is reflexive, symmetric, and transitive.

If $\equiv$ is an equivalence relation on A and $a \in A$, the *equivalence class* of a is the set

$$[a]_{\equiv} = \{b \in A \mid a \equiv b\}.$$

A collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A is said to be *pairwise disjoint* if every every distinct pair of members of $\mathcal{C}$ are disjoint. A collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A is said to *cover* A if $\cup\mathcal{C} = A$.

A *partition* of A is a collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A such that

- $\varnothing \notin \mathcal{C}$;
- $\cup\mathcal{C} = A$;
- $A, B \in \mathcal{C} \Rightarrow A \cap B = \varnothing$ or $A = B$.

That is, a partition of A is a pairwise disjoint collection of nonempty subsets of A which covers A.

If $\equiv$ is an equivalence relation A, then the collection of equivalence classes under $\equiv$ is a partition of A. If $\mathcal{C}$ is a partition of A, we may define an equivalence relation $\equiv$ on A by $a \equiv b$ if and only if they are in the same subset of the partition.

If $f : A \to X$ is a surjective function, then the relation $\equiv$ on A defined by $a \equiv b \Leftrightarrow f(a) = f(b)$ is an equivalence relation which partitions A into blocks of elements which are sent to the same place by f. There is a natural bijective function from the partition into X given by sending each block to the appropriate element in X.

12. Binary Operations

A *binary operation* on a set A is a function

$$*: A \times A \to A.$$

We write $a * b$ instead of $*(a, b)$.

Let $*$ be a binary operation on A.

We say that $*$ is *commutative* if $a * b = b * a$ for every $a, b \in A$.

We say that $*$ is *associative* if $a * (b * c) = (a * b) * c$ for every $a, b, c \in A$.

We say that $e \in A$ is an *identity* for $*$ if $a * e = e * a$ for every $a \in A$. Identity elements are necessarily unique, when they exist.

We say that b is an *inverse* for a if $a * b = b * a = e$ for every $a, b \in A$, where e is an identity for $*$. Inverses are necessarily unique, when they exist.

Additive notation is in force when the binary operation is denoted by a plus sign:

$$+ : A \times A \to A.$$

In this case, the following conventions are followed:

- $+$ has an identity and inverses, and is associative and commutative;
- 0 denotes the identity element;
- the inverse of a is denoted by $-a$;
- na means $a + \cdots + a$ (n times), where $n \in \mathbb{N}$, and $na = 0$ if $n = 0$.

Multiplicative notation is in force when the binary operation is denoted by a dot:

$$\cdot : A \times A \to A.$$

In this case, the dot is usually suppressed, and the operation is denoted by juxtiposition. Also, the following conventions are followed:

- $\cdot$ has an identity, and is associative;
- 1 denotes the identity element;
- the inverse of a is denoted by a^{-1}, if it exists;
- a^n means $a \cdots a$ (n times), where $n \in \mathbb{N}$, and $a^n = 1$ if $n = 0$.

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