COMPLEX ANALYSIS
TOPIC XVIII: COMPACT RIEMANN SURFACES
DRAFT (MORE TO COME!)

PAUL L. BAILEY

ABSTRACT. Our goal for the last month of the course is to introduce the concept of compact Riemann surfaces, and glimpse their role in the theory of meromorphic functions. This is an extremely advanced topic for high school students, so we focus on understanding the definitions. We begin by reviewing what we have already seen about topological spaces, then define connectedness and compactness sufficiently for our purposes. We then move on to locally Euclidean spaces, manifolds, Riemann surfaces, and genera. Finally, we discover the Riemann-Hurwitz formula for discovering the genus of a ramified cover.

1. TOPOLOGY

Definition 1. A *topological space* is a set X together with a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ such that

- **(T1)** $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
- **(T2)** $\mathcal{U} \subset \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$;
- **(T3)** $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ finite $\Rightarrow \cap\mathcal{U} \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.

A subset $A \subset X$ is called *open* if $A \in \mathcal{T}$, and is called *closed* if $X \setminus A \in \mathcal{T}$.

The set of real numbers and the set of complex numbers are topological spaces, with the definitions of open sets we have already given. Also, $\mathbb{R}^n$ is a topological space, where the open sets are unions of open balls. We outline this next.

Definition 2. Let $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ be points in $\mathbb{R}^n$. The *distance* from p to q is

$$d(p, q) = \sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2}.$$

Let $r \in \mathbb{R}$, $r > 0$. The *ball of radius r about p* is

$$B_r(p) = \{q \in \mathbb{R}^n \mid |p - q| < r\}.$$

Let $U \subset \mathbb{R}^n$. We say that U is *open* if for every $u \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(u) \subset U$. The collection of open subsets of $\mathbb{R}^n$ is a topology on $\mathbb{R}^n$, making $\mathbb{R}^n$ a topological space.

For the purposes of topology, we view $\mathbb{C}$ as $\mathbb{R}^2$, with the extra structure of complex multiplication.

Date: April 23, 2018.

2. Subspaces

Any subset of a topological space is naturally a topological space, with the subspace topology.

Definition 3. Let X be a topological space and let $A \subset X$. A subset $W \subset A$ is called *relatively open* if there exists a set $U \subset X$ which is open in X such that $W = A \cap U$. The set of relatively open subsets of A forms a topology on A, called the *subspace topology.*

Example 1. Let $I = [0, 1]$. This is a subspace of $\mathbb{R}$. Let $U = (-0.5, 0.5)$; this is an open set in $\mathbb{R}$. Thus the set $W = U \cap I = [0, 0.5)$ is relatively open in I. If we view I as a topological space, then W is an open set in I.

Next, we define some standard topological spaces; each is endowed with the subspace topology inherited from the appropriate version of $\mathbb{R}^n$. We may think of these as building blocks to form new topological spaces.

The *open n-ball* is

$$B^n = \{q \in \mathbb{R}^n \mid d(0, q) < 1\}.$$

The *closed n-ball* is

$$D^n = \{q \in \mathbb{R}^n \mid d(0, q) \leq 1\}.$$

The *n-sphere* is

$$S^n = \{q \in \mathbb{R}^{n+1} \mid d(0, q) = 1\}.$$

So, $B^1 = (-1, 1)$ is an open interval and $D^1 = [-1, 1]$ is a closed interval. Also, S^1 is a circle, but D^2 is a closed disk.

3. Classification of Points

The definitions of neighborhood, deleted neighborhood, closure point, interior point, boundary point, accumulation point, isolated point, all carry over from our previous discussions virtually unchanged into this more general context, as do the concepts of the closure, interior, and boundary of a set. We review this now.

Definition 4. Let X be a topological space, and let $p \in X$. A *neighborhood* of p is a set which contains an open set which contains p. A *deleted neighborhood* of p is a set of the form $N \smallsetminus \{p\}$, where N is a neighborhood of p.

Let $A \subset X$. We say that p is a *closure point* of A if every neighborhood of p intersects A. We say that p is a *interior point* of A if there exists a neighborhood of p which is contained in A. We say that p is a *boundary point* of A if every neighborhood of p intersects A and $X \smallsetminus A$. We say that p is an *accumulation point* of A if every deleted neighborhood of p intersects A. We say that p is an *isolated point* of A if there exists a neighborhood U of p such that $A \cap U = \{p\}$.

The *closure* of A, denoted $\overline{A}$, is the set of closure points of A. The *interior* of A, denoted A°, is the set of interior points of A. The *boundary* of A, denoted ∂A, is the set of boundary points of A.

We say that A is *discrete* if every point in A is isolated.

Let $B \subset A$. We say that B is *dense* in A if $\overline{B} = A$.

4. Bases and Subbases

Let X be any set; this set admits many different collections of subsets which satisfy the axioms of a topology If we have a collection of different topologies on X, we attempt to make a new topology on X by declaring that a given subset is open if and only if if is a member of each of the topologies in the collection. It is relatively easy to show that what we obtain in this manner is again a topology on X. We state this as a proposition.

Proposition 1. *Let X be a set. The intersection of topologies on X is a topology on X.*

We use this to produce the easiest definition of generated topologies; we wish to define the topology generated by a collection of subsets of X to be the coarsest (that is, with the fewest open sets) topology on X such that each of the sets in our collection is open.

Definition 5. Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. The *topology generated by* $\mathcal{C}$ is the intersection of all topologies on X which contain $\mathcal{C}$.

The topology generated by $\mathcal{C}$ may to constructed in stages as follows. First, take the collection of all finite intersections of all the sets in $\mathcal{C}$. Then, take the collection of all possible unions of sets obtained in above. There are definitions for these things.

Definition 6. Let X be a set and let $\mathcal{T}$ be a topology on X.

A *subbasis* for $\mathcal{T}$ is a collection of subsets of X which generated the topology $\mathcal{T}$.

A *basis* for $\mathcal{T}$ is a collection of subsets of X such that the collection of all possible unions of these subsets is $\mathcal{T}$.

For example, the collection of open interval of finite length is a basis for the standard topology on $\mathbb{R}$; also, the collection of all open disks in the complex plane is a basis for the standard topology on $\mathbb{C}$.

5. Discrete and Trivial Topologies

Definition 7. Let X be a set.

The *power set* of X, denoted $\mathcal{P}(X)$, is the collection of all subsets of X.

The *discrete topology* on X is the topology in which every subset of X is open; that is, $\mathcal{T} = \mathcal{P}(X)$. Thus the discrete topology is the topology generated by the collection of singleton sets.

The *trivial topology* on X is the topology in which the only open sets are the empty set and the whole space; that is, $\mathcal{T} = \{\varnothing, X\}$. Thus the trivial topology is the topology generated by the empty collection.

6. Continuity

Continuity and convergence may now be defined on any topological space.

Definition 8. Let X and Y be topological spaces, and let $f : X \to Y$. We say that f is *continuous at* $x \in X$ if, for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. We say that f is continuous if it is continuous at every point in the domain.

Proposition 2. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if the preimage of every open set in Y is open in X.*

Proof. We prove both directions of the implication.

($\Rightarrow$) Suppose that f is continuous at every point in X. Let $V \subset Y$ be open and let $U = f^{-1}(V)$; we wish to show that U is open in X.

For every $x \in U$, V is a neighborhood of $f(x)$, so there exists an open neighborhood U_x of x such that $f(U_x) \subset V$. But then $U_x \subset U$, and U is the union of such sets; thus U is open, and f is continuous. Suppose that f is continuous, and let $x_0 \in X$. Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V.

($\Leftarrow$) Conversely, suppose that the preimage of every open set in Y is open in X, and let $x_0 \in X$. We wish to show that f is continuous at x_0.

Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V. □

Definition 9. Let X and Y be topological spaces and let $f : X \to Y$. We say that f is *open* if the image of every open set in X is open in Y. We say that f is *bicontinuous* if f is open and continuous.

Example 2. Continuous functions are characterized by having at least enough of open sets in the domain, and open maps are characterized by having a at least enough open sets in the range.

Let X be any set. Let $X_{\mathtt{T}}$ and $X_{\mathtt{D}}$ denote the topological space X together with the trivial or discrete topology, respectively. Let $f(x) = x$ be the identity map on X.

Then $f : X_{\mathtt{T}} \to X_{\mathtt{D}}$ is open but not continuous, because the range has more open sets than the domain. On the other hand, $f : X_{\mathtt{D}} \to X_{\mathtt{T}}$ is continuous, but not open, because the domain has more open sets than the range.

Definition 10. Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective continuous function $f : X \to Y$ whose inverse is also continuous. We say that X and Y are *homeomorphic* if there exists a homeomorphism between them.

A homeomorphism between topological spaces preserves all of the features of the domain which can be described exclusively using open sets; we may call such features "topological". Because of this, we view two topological spaces as equivalent, or essentially the same, if they are homeomorphic. However, a space may have additional structure beyond its topology, which is not preserved by homeomorphism.

7. Product Topology

Let X and Y be topological spaces; the set $X \times Y$ is the set of all ordered pairs of elements from X and Y:

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

We wish to put the "most natural" topology we can on $X \times Y$. Certainly, whatever we choose in this regard should conform with what we already experience with subsets of $\mathbb{R}^n$.

For example, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the standard cartesian plane. let $I = [0, 1] \subset \mathbb{R}$ be the closed unit interval. Then $I \times I$ is a square, and its topology should be that which it inherits as a subspace of $\mathbb{R}^2$. Let's list some more examples.

- $I \times I$ is a square;
- $I \times S^1$ is a cylinder;
- $S^1 \times S^1$ is a torus (the surface of a donut);
- $S^1 \times D^2$ is a solid torus (the entire donut).

Mathematicians think of these things in terms of mappings. The most primitive useful mappings on $X \times Y$ are the projections.

Let $p_X : X \times Y \to X$ be given by $(x, y) \mapsto x$ and $p_Y : X \times Y \to Y$ be given by $(x, y) \mapsto y$. These are called *projections* onto X and Y, respectively.

We wish to define the topology on $X \times Y$ to be the coarsest topology on $X \times Y$ such that the projection maps are continuous. What is required is that the preimages of open sets are open. So, for p_X to be open, we require that if U is open in X, the $p_X^{-1}(U) = U \times Y$ is open in $X \times Y$. A similar statement may be made regarding p_Y. Thus, a subbasis for the topology we seek is the collection of sets of the form $U \times Y$ and $X \times V$, and a basis for the topology is the collection of sets of the form $U \times V$, where U is open in X and V is open in Y.

Definition 11. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by sets of the form $U \times Y$ and $X \times V$, where U is open in X and V is open in Y.

8. Quotient Topology

Let X and Y be topological spaces. We know that a function $f : X \to Y$ is continuous if and only if the preimage of an open set in Y is open in X. So, in some sense, Y has at least as many open sets as it needs to the map to be continuous, and possibly more. On the other hand, we say the f is an open map if the image of an open set in X is open in Y. Here, we see that X has at least as many and potentially more open sets as it needs for f to be open. If f is bicontinuous, the number of open sets in X and Y is "just right" for f. Goldilocks would be proud.

Definition 12. Let X and Y be topological spaces, and let $f : X \to Y$ be a surjective function. We say that Y has the *quotient topology* with respect to f if

$$V \subset Y \text{ is open} \quad \Leftrightarrow \quad V = f(U) \text{ for some open } U \subset X.$$

the open sets is Y are exactly the images of the open sets in Y.

Now suppose that X is a topological space, Y is any set, and $f : X \to Y$ is a surjective function. We may *define* a topology on Y by declaring a subset of Y to be open if and only if its preimage in X is open.

One way the occurs is by creating an equivalence relation on X. That is, we partition X is disjoint subsets which cover X. Two elements from X are considered equivalent if the belong to the same set in the partition; these sets are called *equivalence classes*. Then, we view the set of equivalence classes as a set in its own right. These are the details.

Definition 13. Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. We say that $\mathcal{C}$ is a *partition* of X if

(P1) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$, then $C_1 \cap C_2 = \varnothing$;
(P2) $\cap \mathcal{C} = X$.

The members of $\mathcal{C}$ are called *blocks*.

Let $\mathcal{C}$ be a partition of X. Let $x_1, x_2 \in X$, we say that x_1 is *equivalent* to x_2, and write $x_1 \equiv x_2$, if $x_1, x_2 \in C$ for some $C \in \mathcal{C}$.

Let $a \in X$. The *equivalence class* of a is the set

$$\overline{a} = \{x \in X \mid a \equiv x\}.$$

The *quotient* of X by $\mathcal{C}$ is

$$\overline{X} = \{\overline{x} \mid x \in X\}.$$

Thus $\overline{X}$ is the set of equivalence class. There is a natural function from X to $\overline{X}$, given by sending a point x to the block that it is in:

$$\beta : X \to \overline{X} \quad \text{given by } \beta(x) = \overline{x}.$$

If X is a topological space, we then impose the quotient topology on $\overline{X}$.

9. Convergence

Definition 14. Let X be a topological space and let (x_n) be a sequence in X. We say that (x_n) *converges* to $L \in X$ if, for every neighborhood V of L there exists $N \in \mathbb{N}$ such that $x_n \in V$ whenever $n \geq N$.

Example 3. Let $I = (0, \infty)$. The function $\exp : \mathbb{R} \to I$ given by $\exp(x) = e^x$ is a homeomorphism, so it preserves all topological properties. For example, if x_0 is an boundary point of $A \subset \mathbb{R}$, then $f(x_0)$ is a boundary point of $f(A)$. If a sequence (x_n) converges to $L \in \mathbb{R}$, then the sequence $f(x_n)$ converges to $f(L) \in I$.

It may seem odd that the limit of a sequence is not necessary unique in every space. Next we give a condition that will ensure that the limit of a sequence is unique.

10. Hausdorff Spaces

It may seem odd that the limit of a sequence is not necessary unique in every space.

Example 4. Consider the "bug-eyed" line segment, constructed at follows. Let $X = (0, 1] \cup \{a, b\}$. Declare a subset U of X to be open if their exists an open set V in $\mathbb{R}$ such that

$$U = \begin{cases} X \cap (V \cup \{a, b\}) & \text{if } 0 \in V \ ; \\ X \cap V & \text{if } 0 \notin V \ . \end{cases}$$

Then the sequence (x_n), where $x_n = \dfrac{1}{n}$, converges to both a and b. The problem with this space is that the points a and b cannot be "separated".

We may construct this space using the quotient topology: Let $X = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1] \text{ and } y \in \{1, 2\}\}$. Define a relation on X by

$$(x_1, y_1) \equiv (x_2, y_2) \quad \Leftrightarrow \quad x_1 = x_2 \text{ and } x_1 \neq 0\}.$$

Then $\overline{X}$, with the quotient topology, it the "bug-eyed" line segment.

Definition 15. Let X be a topological space. We say that X is *Hausdorff* if for every distinct $x_1, x_2 \in X$ their exists neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \varnothing$.

Proposition 3. *Let X be a Hausdorff space and let (x_n) be a sequence in X which converges to L_1 and to L_2. Then $L_1 = L_2$.*

Proof. Suppose not. Since X is Hausdorff, there exists neighborhoods of U_1 of L_1 and U_2 of L_2 such that $U_1 \cap U_2 = \varnothing$. Since (x_n) converges to L_1, there exist $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U_1$. But then, for $n \geq N$, $x_n \notin U_2$. This contradicts that (x_n) converges to L_2. $\square$

11. Connectedness

A space is connected if it has only one “piece”. We state this formally as follows.

Definition 16. Let X be a topological space.

A *separation* of X is a pair of nonempty open sets $U, V \subset X$ such that $U \cap V = \varnothing$ and $U \cup V = X$.

We say that X is *connected* if there does not exist a separation of X.

A *component* of X is a maximal connected subset; that is, it is a connected subset which is not properly contained in a connected subset.

If we speak of a subset of X being connected, we mean that it is connected as topological space with the subspace topology. It is clear that a set is connected if and only if it has exactly one component.

We give some examples.

- A nonempty subset of $\mathbb{R}$ is connected if and only if it is a singleton or an interval.
- S^n is connected unless $n = 0$ (in which case S^0 is a set containing two points.
- No finite subset of $\mathbb{R}^n$ is connected.
- Consider the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. Then $f^{-1}(B_1(0))$ is connected, but $f^{-1}(B_1(2)$ has two components.

Proposition 4. *Let $f : X \to Y$ be continuous, and let $A \subset X$. If A is connected, then $f(A)$ is connected.*

Proof. We use a proof by contrapositive; assume that $f(A)$ is not connected. Then there exist disjoint open sets V_1 and V_2 with $f(A) \subset V_1 \cup V_2$, with $f(A) \cap V_1 \neq \varnothing$ and $f(a) \cap V_2 \neq \varnothing$. Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$; since f is continuous, U_1 and U_2 are open. Since V_1 and V_2 are disjoint, so are U_1 and U_2. Moreover, $U_1 \cap A$ and $U_2 \cap A$ are nonempty. Finally, $A \subset U_1 \cap U_2$. $\square$

Definition 17. Let X be a topological space. We say that X is *path-connected* if for every $x_1, x_2 \in X$, there exists a continuous function $\gamma : I \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

Proposition 5. *If X is path-connected, then X is connected.*

Proof. Suppose that X is not connected, and let $U_1, U_2 \subset X$ be disjoint nonempty open sets which cover X. Let $x_1 \in U_1$ and $x_2 \in U_2$. If X is path-connected, there exists a continuous function $\gamma : I \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. The image of γ is a subset of X which we denote by $\gamma(I)$. Since γ is continuous and I is connected, then $\gamma(I)$ is connected. But $\gamma I \cup U_1$ and $\gamma(I) \cup U_2$ is a separation of $\gamma(I)$, so $\gamma(I)$ cannot be connected; this contradiction implies that X is not path-connected. $\square$

12. Compactness

A space is compact if it is not *too* big, and if it doesn't have any "holes". This may be stated in multiple ways, which are equivalent for "well-behaved" spaces.

Definition 18. Let X be a topological space.

A *cover* of X is a collection of subsets of X whose union is X.

An *open cover* of X is a cover consisting of open sets.

A *finite cover* of X is a cover consisting of finitely many sets.

A *subcover* of a cover is a subset of the cover whose union is X.

We say that X is *compact* if every open cover has a finite subcover.

If we speak of a subset of X being compact, we mean that it is compact as topological space with the subspace topology.

Note that in the phrase "every open cover has a finite subcover", the word finite is describing the collection which is the subcover, but the word open is describing the sets in the cover.

We give some examples.

- Open balls are not compact.
- Closed balls are compact.
- A punctured disk is not compact.
- The entire real line is not compact.

Proposition 6. *A compact subset of a Hausdorff space is closed.*

Proposition 7. *Let $f : X \to Y$ be continuous, and let $A \subset X$. If A is compact, then $f(A)$ is compact.*

Proof. Consider an open cover of $f(A)$. The collection of preimages of the sets in the cover form an open cover of A. Since A is compact, a finite subset of these cover A. The collection of images of these sets form a finite subcover of the original cover of $f(A)$. $\square$

Theorem 1. (Heine-Borel Theorem) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

We have the following alternate variations of the definition of compactness, which are equivalent to the standard definition in most cases in which we are interested.

Definition 19. Let X be a topological space.

We say that X is *sequentially compact* if every sequence in X has a cluster point in X.

We say that X is *limit point compact* if every infinite subset of X has an accumulation point in X.

Theorem 2. (Bolzano-Weierstrauss Theorem) *A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.*

Corollary 1. *A subset of $\mathbb{R}^n$ is compact if and only if it is sequentially compact.*

13. Topological Manifolds

Definition 20. Let X be a topological space. We say that X is *locally Euclidean* if, for every point $x \in X$, there exists a neighborhood of x which is homeomorphic to B^n for some n.

A *topological manifold* is a locally Euclidean Hausdorff space.

If X is locally Euclidean and connected, then the dimension n is constant throughout the entire space, and is called the *dimension* of the manifold.

In order to compute with locally Euclidean spaces, we become more specific about these local homeomorphisms.

Definition 21. Let X be a topological space.

A *chart* on X is a bijective bicontinuous function $\psi : U \to B^n$, where $U \subset X$ is an open subset of X.

Let $\psi_1 : U_1 \to B^n$ and $\psi_2 : U_2 \to B^n$ be charts on X. The *transition function* given by these charts is

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2).$$

We say that ψ_1 and ψ_2 are *compatible* if the function

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$$

is a homeomorphism.

A collection of charts on X is said to *cover* X if every point on X is in the domain of one of the charts in the collection. An *atlas* on X is a collection of charts on X which cover X, such that every pair of charts in the collection are compatible.

We say that two atlases are *compatible* if their union is an atlas.

For a topological manifold, any two atlases are compatible. However, if we wish to put additional structure on our manifold, this may no longer be the case. For example, doing calculus on manifolds requires what is known as a differentiable manifold.

Definition 22. Let X be a topological manifold.

We say that two charts on X are *differentiably compatible* if the corresponding transition function is differentiable. A *differentiable atlas* is an atlas consisting of differentiably compatible charts.

The set of all differentiable atlases on X is partially ordered by inclusion. A *differentiable structure* on X is a maximal atlas with respect to this partial order.

A *differentiable manifold* is a topological space X together with a differentiable structure on X.

14. Riemann Surfaces

A Riemann surface is a topological 2-manifold, together with a complex structure. Put another way, a Riemann surface is a topology space such that each point has a neighborhood homoeomorphic to the open unit disk in the complex plane, such that the transition functions are analytic. We given the details now.

Definition 23. Let X be a topological space, and let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

A *complex chart* on X is a bijective bicontinuous function $\psi : U \to \Delta$, where $U \subset X$ is an open subset of X.

Let $\psi_1 : U_1 \to \Delta$ and $\psi_2 : U_2 \to \Delta$ be charts on X. The *transition function* given by these charts is

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2).$$

We say that ψ_1 and ψ_2 are *compatible* if the function

$$\psi_2 \circ \psi_1^{-1} : \psi(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$$

is analytic.

A collection of charts on X is said to *cover* X if every point on X is in the domain of one of the charts in the collection. An *atlas* on X is a collection of charts on X which cover X, such that every pair of charts in the collection are compatible.

We say that two atlases are *compatible* if their union is an atlas. Compatibility is an equivalence relation on the set of all atlases, and the union of compatible atlas is again an atlas. Thus the union of all atlases in an equivalence class is the maximal atlas in the class. A *complex structure* on X is a maximal atlas.

A *Riemann surface* is a topological space X together with a complex structure on X.

Department of Mathematics, BASIS Scottsdale
E-mail address: `paul.bailey@basised.com`