CATEGORY THEORY | Practice Exam 21 | Name:
DR. PAUL L. BAILEY | Tuesday, November 19, 2019

Problem 1 (Gallian 2.25)**.** Suppose the table below is a Cayley table for a group. Fill in the blanks.

	e	a	b	c	d
e	e	____	____	____	____
a	____	b	____	____	e
b	____	c	d	e	____
c	____	d	____	a	b
d	____	____	____	s ____	____

Problem 2 (Gallian 2.34)**.** Set

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{GL}_3(\mathbb{R}) \middle| a, b, c \in \mathbb{R} \right\}.$$

Show that $H \leq \mathbf{GL}_3(\mathbb{R})$. This is called the *Heisenberg group.*

Problem 3. Let G be a group. The *center* of G is

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Show that $Z(G) \leq G$.

Problem 4. '[Gallian 3.20] Let G be a group. Let $H \leq G$ and $X \subset G$. The *centralizer of X in H* is

$$C_H(X) = \{h \in H \mid hxh^{-1} = x \text{ for all } x \in X\}.$$

Show that $C_H(X) \leq H$. Find an example where $X \leq G$ but $C_H(X)$ is not abelian.

Problem 5. Let G be a group. Let $H \leq G$ and $X \subset G$. The *normalizer of X in H* is

$$N_H(X) = \{h \in H \mid hXh^{-1} = X\}.$$

Show that $N_H(X) \leq X$.

Problem 6. Let p be the smallest positive prime integer, and let G be a group of order p^2. Show that G has a normal subgroup of order p.

Problem 7. A group of order 35 acts on a set of cardinality 6. Show that the action is not faithful.

Problem 8. Let $G = \mathbf{GL}_3(\mathbb{Z}_2)$ be the group of invertible 3×3 matrices with entries from $\mathbb{Z}_2$. Let $X = \mathbb{Z}_2^3$.

(a) Find $m = |G|$.

(b) Find $n = |X|$.

(c) Is it possible for a two-point stabilizer to act transitively on the remaining points?