

# Effects and Coeffects in Call-By-Push-Value

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Effect and coeffect tracking integrate many types of compile-time analysis, such as cost, liveness, or dataflow, directly into a language's type system. In this paper, we investigate the addition of effect and coeffect tracking to the type system of call-by-push-value (CBPV), a computational model useful in compilation for its isolation of effects and for its ability to cleanly express both call-by-name and call-by-value computations. Our main result is *effect-and-coeffect soundness*, which asserts that the type system accurately bounds the effects that the program may trigger during execution and accurately tracks the demands that the program may make on its environment. This result holds for two different dynamic semantics: a general one and one that is specialized for reasoning about resource usage. In particular, the second semantics discards the evaluation of unused values and pure computations, while ensuring that effectful computations are always evaluated, even if their results are not required. Our results have been mechanized using the Coq proof assistant.

CCS Concepts: • **Theory of computation** → **Type theory**.

Additional Key Words and Phrases: Types, CBPV, Effects, Coeffects

## 1 INTRODUCTION

Computations interact with the world in which they run. Sometimes the computation does something the world can observe, known as an *effect* [Lucassen and Gifford 1988], and sometimes computations demand something that the world must provide, known as a *coeffect* [Brunel et al. 2014; Orchard et al. 2022; Petricek et al. 2014]. For example, running a computation might take time (a clock ticking is an effect) and might require resources (using input parameters is a coeffect).

Some programming languages track various of effects and coeffects statically. Frank [Convent et al. 2020], Koka [Leijen 2014], and the Verse functional logic language [Verse development team 2023] do this for effects such as state, exceptions, divergence, and failure; Linear Haskell [Bernardy et al. 2017] does this for a resource management coeffect, while Agda and Idris 2 [Brady 2021] do this for a relevancy coeffect. The Effekt language [Brachthäuser et al. 2022] both tracks effects statically and uses a limited form of coeffect tracking to ensure that effect handlers are well-scoped. Finally, the Granule language [Orchard et al. 2019] uses monads and comonads graded by abstract structures to track a variety of effects and coeffects in a flexible and expressive system.

We would like to update the type systems of existing languages with effect and coeffect tracking by annotating their existing type systems: rather than constrain effects and coeffects to monadic and comonadic structures, we would like our extended type system to describe the effects and coeffects of arbitrary computations. Therefore, to study this problem in a comprehensive way, we address it in the context of Call-By-Push-Value (CBPV) [Levy 2001], a calculus that makes the ambient computational monad and comonad explicit. In this setting, we can annotate these existing structures directly, instead of adding new features to the language for effect and coeffect tracking.

CBPV's ambient computational monad is similar to that found in Moggi's computational lambda calculus [Moggi 1989] and fine-grained CBV [Levy et al. 2003]. Like these systems, CBPV separates

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inert “values” from executable “computations”, and includes “return” and “let” constructs to sequence evaluation in the ambient monad. However, CBPV also provides access to its ambient comonad through the addition of thunks, which temporarily suspend computations and treat them as values.

CBPV is a low-level language and is appropriate for use as a compiler intermediate representation [Garbuzov et al. 2018; Rizkallah et al. 2018]. Its distinction between values and computations allows CBPV to work with strict and nonstrict language features explicitly, enabling it to model both call-by-value and call-by-name languages with the same facility. Adding effects and coeffects to CBPV would enrich this intermediate representation to support program optimizations; for example, to justify dead code elimination for pure code whose coeffect annotations mark it as unused.

The ability of CBPV to model both CBV and CBN also lets us observe how evaluation order changes the way a program alters and makes demands on the world. Levy characterizes the difference between values and computations with the slogan: “a value *is*, a computation *does*.” [Levy 2001] Our interpretation of this slogan is that only computations may contain effectful subcomponents—values must be pure throughout. Conversely, coeffects describe the demands a program makes on its inputs, which are always values in CBPV.

CBPV uses separate types for values and computations. Values have *positive* types (for which we use the metavariable  $A$ ), while computations have *negative* types (for which we use  $B$ ). These two forms are connected via an adjunction: the thunk type  $U\ B$  suspends a computation as an inert value, and the type of return  $F\ A$  creates a fine-grained structure similar to monadic bind that threads values through computations. Due to the structure of the adjunction, the combination  $U\ (F\ A)$  forms a monad and the combination  $F\ (U\ B)$  forms a comonad [Levy 2003].

The duality between values and computations gives CBPV its power, and it is reflected in the structures we use to statically track effects and coeffects. For effects, we add effect information  $\phi$  to the thunk type  $U_\phi\ B$ , recording the latent effect of suspended computations. Similarly, to track coeffects, we add coeffect information  $q$  to the returner type  $F_q\ A$ , describing the demands subsequent computation is allowed to make on the returned value. With this augmentation, we will show that the types  $U_\phi\ (F\ A)$  and  $F_q\ (U\ B)$  can encode the graded monads and comonads associated with effect and coeffect tracking.

Following this duality, this paper begins with two mirrored halves and then combines them. The first part (Section 2) extends CBPV with effect tracking and shows how we can recover the graded monad by grading the thunk type with latent effects. The second part (Section 3) extends CBPV with coeffect tracking and recovers a graded comonad by grading the returner type with latent coeffects; we also discuss modifications to the system for resource tracking with coeffects (Section 4). Finally, we combine the two systems and explore their interaction (Section 5). This paper is best read in color: effects  $\phi$  appear in red and coeffects  $q$  in blue. Without these colorful annotations, the type system and semantics are the standard rules of CBPV.

Along the way, we prove the following results about our extensions.

- We prove *effect soundness* for our effect-annotated extension of CBPV, demonstrating that the type-and-effect system accurately bounds what happens at runtime. To do so, we define an environment-based big-step operational semantics for CBPV instrumented to precisely track effects during evaluation, and we use a logical relation to prove our soundness theorem. (Section 2.3)
- We prove that the standard translations from call-by-value (CBV) and call-by-name (CBN) lambda calculi to CBPV are *type-and-effect preserving*. Starting with a well-typed CBV or (monadic) CBN program, we can produce a well-typed CBPV program with the same effects as the source program. (Section 2.4)

- We prove *coeffect soundness* for our coeffect-annotated extension of CBPV, demonstrating that the type-and-coeffect system accurately bounds the demands a program may make on its inputs. We do so using an environment-based big-step operational semantics for CBPV, where the environment has been instrumented to track coeffects during evaluation. (Section 3.1)
- While our coeffect-tracking operational semantics is generic with respect to coeffects, we observe that its behavior has some strange implications when reasoning about resource usage. Therefore, we modify the rules of our semantics so that it does not demand resources for discarded values, providing a better model of how the program uses its inputs. (Section 4)
- We prove that the standard translations from both CBN and CBV to CBPV are *type-and-coeffect* preserving for this updated coeffect system. Starting with a well-typed CBN or CBV program, we can produce a well-typed CBPV program with the same coeffects. (Section 4.1)
- We combine the ‘tick’ effect and resource tracking coeffect together into the same CBPV type system and prove combined versions of the results from each: *type-and-effect-and-coeffect soundness* and *type-and-effect-and-coeffect* preservation of the standard translations from CBV and CBN. Finally, we also prove a result relating our resource semantics to the more general one. (Section 5)

We are not the first to extend CBPV with effect tracking and our type system is most similar to Kammar and Plotkin [2012] and Forster et al. [2017]. However, all other definitions and results of this paper are novel. In particular, we have found little work that explores the interaction between CBPV and coeffects. Furthermore, while we are able to use the standard translations to interpret CBV and CBN in CBPV, designing the effect and coeffect systems so that these translations “just work” is a contribution of this paper. Our approach to reasoning about effect-and-coeffect soundness is also new—we employ a novel environment-based big-step semantics for CBPV that leads to simple proofs. For simplicity, our effect system only tracks clock effects and our coeffect system only counts variable usages. As a result, we do not explore more sophisticated interactions between effects and coeffects, such as local and global state [Nanevski 2003], or between information flow and nondeterminism and between usage analysis and errors [Gaborardi et al. 2016].

The results of this paper have been completely formalized in Coq and are available online.<sup>1</sup> For space reasons, some definitions have been elided. Full details are available in an extended version of this paper [Torczon et al. 2024].

## 2 CALL-BY-PUSH-VALUE (CBPV) AND EFFECT TRACKING

In this section, we extend the type system of CBPV with effect tracking. Our modifications to the base system, which are limited to reasoning about effect annotations  $\phi$ , are marked in red.

CBPV syntactically separates terms into *values*  $V$ , inhabiting positive types  $A$ , and *computations*  $M$ , inhabiting negative types  $B$ , as shown by the following grammar.

|                          |     |       |  |
|--------------------------|-----|-------|--|
| <i>value types</i>       | $A$ | $::=$ | $\text{unit} \mid \mathbf{U}_\phi B \mid A_1 \times A_2$   |
| <i>computation types</i> | $B$ | $::=$ | $A \rightarrow B \mid \mathbf{F} A \mid B_1 \& B_2$  |
| <i>values</i>            | $V$ | $::=$ | $x \mid () \mid \{M\} \mid (V_1, V_2)$   |
| <i>computations</i>      | $M$ | $::=$ | $\lambda x. M \mid M V \mid V! \mid \text{case } V \text{ of } (x_1, x_2) \rightarrow N$<br>$\mid \langle M_1, M_2 \rangle \mid M.1 \mid M.2 \mid \text{return } V \mid x \leftarrow M \text{ in } N \mid \text{tick}$ |

Values in CBPV mostly correspond to the values found in a call-by-value typed functional language, such as unit and positive products of values. CBPV values additionally include suspended

<sup>1</sup> <https://github.com/plclub/cbpv-effects-coeffects/>

computations, called *thunks* and written  $\{M\}$ . (Variables always represent values, so they are always declared with value types in the context.)

Computations in CBPV include abstractions  $(\lambda x.M)$ , applications  $(M V)$ , elimination (*forcing*) of thunks  $(V!)$ , and positive product elimination (**case**  $V$  **of**  $(x_1, x_2) \rightarrow N$ ). In addition to positive products, CBPV also includes negative products, of type  $B_1 \& B_2$ . These are introduced with a pair of computations  $\langle M_1, M_2 \rangle$  and eliminated by projecting either the first or second component, *i.e.*  $M.1$  or  $M.2$ .

Values can be threaded through computations via the “return” and “letin” constructs. In the latter, written  $x \leftarrow M$  in  $N$ , the first subcomputation must evaluate to the form **return**  $V$ , and the second computation can then reference  $V$ .

One of the advantages of CBPV is that this bind-like method of threading values through computations makes it readily extensible with effectful language features. Levy [2001, 2006, 2022] demonstrates how to add nontermination, nondeterminism, errors, I/O, state, and control effects to CBPV. In each case, Levy extends the language with new computations and modifies the operational semantics to account for the new features.

For simplicity, we describe a single effect in this paper, the **tick** computation. This effect advances a virtual clock in the operational semantics, simulating the cost of the program.

## 2.1 CBPV: Type-and-effect system

Our **type-and-effect system** for CBPV is shown in Figure 1. Under some typing context  $\Gamma$ , this system assigns a value type to values ( $\Gamma \vdash_{\text{eff}} V : A$ ) and both a computation type and effect to computations ( $\Gamma \vdash_{\text{eff}} M :^{\phi} B$ ), where  $\phi$  is an upper bound on the effects that could occur during evaluation of  $M$ . The judgement for values does not need an effect annotation because values are pure. The latent effect of a suspended computation is recorded in the thunk type  $U_{\phi} B$  using rule **EFF-THUNK**.

Following Katsumata [2014], our system models effects using an arbitrary *preordered monoid*. This gives us an identity element  $\varepsilon$ , an associative combining operation  $\phi_1 \cdot \phi_2$ , and a preorder relation  $\leq_{\text{eff}}$  that respects the operation. We also include a primitive effect **Tick** produced by the **tick** computation. However, the only parts of the system that are specific to this effect are the rules for **tick**, which is our *only* effectful computation. All other rules are presented generically and are adaptable to other effects and effectful computations (e.g. a **Read** effect produced by a **read** computation).

Concretely, we could use the natural number monoid with the usual ordering, 0 as the identity element, and addition as the combining operation to have our type system perform a cost analysis. Using 1 as our model of the **Tick** effect, the system would statically bound the number of **ticks** that are evaluated. For example, the type system would tell us that the computation  $\langle \text{tick}, y \leftarrow \text{tick in tick} \rangle$  advances the clock at most twice. If the first component of the pair is projected, the type system overapproximates the effect produced during execution. Note that to track other behaviors with our type system, we need only change our preordered monoid accordingly (e.g. we could track possible effects with the power set monoid ordered by set inclusion).

Rules **EFF-RET** and **EFF-LETIN** motivate the choice of a monoid structure. Returning a value has no effect, so the effect of **return**  $V$  should always be  $\varepsilon$ . Rule **EFF-LETIN** must combine effects, because  $x \leftarrow M$  in  $N$  is the only computation in our system with two subcomputations, both of which may be effectful. Finally, because return and letin satisfy identity and associativity properties as the building blocks of the CBPV monad, we need these same properties in our effect structure.

Rule **EFF-SUB** allows for imprecision in the type system. That is, an effect annotation  $\phi$  on the type of a program indicates that the program will have *at most*  $\phi$  as its effect; it may have less.

|   |   |   |  |
|---|---|---|--|
| $\Gamma \vdash_{\text{eff}} V : A$  |   | <i>(value effect typing)</i>  |  |
| $\frac{\text{EFF-VAR} \quad x : A \in \Gamma}{\Gamma \vdash_{\text{eff}} x : A}$  | $\frac{\text{EFF-THUNK} \quad \Gamma \vdash_{\text{eff}} M : \phi B}{\Gamma \vdash_{\text{eff}} \{M\} : \mathbf{U}_{\phi} B}$                                     | $\frac{\text{EFF-UNIT}}{\Gamma \vdash_{\text{eff}} () : \mathbf{unit}}$   | $\frac{\text{EFF-PAIR} \quad \Gamma \vdash_{\text{eff}} V_1 : A_1 \quad \Gamma \vdash_{\text{eff}} V_2 : A_2}{\Gamma \vdash_{\text{eff}} (V_1, V_2) : A_1 \times A_2}$ |
| $\Gamma \vdash_{\text{eff}} M : \phi B$   |   | <i>(computation effect typing)</i>  |  |
| $\frac{\text{EFF-ABS} \quad \Gamma, x : A \vdash_{\text{eff}} M : \phi B}{\Gamma \vdash_{\text{eff}} \lambda x. M : \phi A \rightarrow B}$  | $\frac{\text{EFF-APP} \quad \Gamma \vdash_{\text{eff}} M : \phi A \rightarrow B \quad \Gamma \vdash_{\text{eff}} V : A}{\Gamma \vdash_{\text{eff}} M V : \phi B}$ | $\frac{\text{EFF-FORCE} \quad \Gamma \vdash_{\text{eff}} V : \mathbf{U}_{\phi} B}{\Gamma \vdash_{\text{eff}} V! : \phi B}$  | $\frac{\text{EFF-RET} \quad \Gamma \vdash_{\text{eff}} V : A}{\Gamma \vdash_{\text{eff}} \mathbf{return} V : \epsilon F A}$  |
| $\frac{\text{EFF-LETIN} \quad \Gamma \vdash_{\text{eff}} M : \phi_1 F A \quad \Gamma, x : A \vdash_{\text{eff}} N : \phi_2 B}{\Gamma \vdash_{\text{eff}} x \leftarrow M \mathbf{in} N : \phi_1 \cdot \phi_2 B}$ |   | $\frac{\text{EFF-SPLIT} \quad \Gamma \vdash_{\text{eff}} V : A_1 \times A_2 \quad \Gamma, x_1 : A_1, x_2 : A_2 \vdash_{\text{eff}} N : \phi B}{\Gamma \vdash_{\text{eff}} \mathbf{case} V \mathbf{of} (x_1, x_2) \rightarrow N : \phi B}$ |  |
| $\frac{\text{EFF-CPAIR} \quad \Gamma \vdash_{\text{eff}} M_1 : \phi B_1 \quad \Gamma \vdash_{\text{eff}} M_2 : \phi B_2}{\Gamma \vdash_{\text{eff}} \langle M_1, M_2 \rangle : \phi B_1 \& B_2}$                |   | $\frac{\text{EFF-FST} \quad \Gamma \vdash_{\text{eff}} M : \phi B_1 \& B_2}{\Gamma \vdash_{\text{eff}} M.1 : \phi B_1}$   | $\frac{\text{EFF-SND} \quad \Gamma \vdash_{\text{eff}} M : \phi B_1 \& B_2}{\Gamma \vdash_{\text{eff}} M.2 : \phi B_2}$  |
| $\frac{\text{EFF-TICK}}{\Gamma \vdash_{\text{eff}} \mathbf{tick} : \mathbf{Tick} F \mathbf{unit}}$  |   | $\frac{\text{EFF-SUB} \quad \Gamma \vdash_{\text{eff}} M : \phi_1 B \quad \phi_1 \leq_{\text{eff}} \phi_2}{\Gamma \vdash_{\text{eff}} M : \phi_2 B}$  |  |

Fig. 1. CBPV typing and effect tracking

If the type system determines that the computation will complete within 5 ticks, it is also sound, but less precise, for it to say that it will complete within 7 ticks. Choosing the discrete ordering (equality) forces the type system to track effects precisely. (Note that to allow the discrete ordering, we do not assume  $\epsilon \leq_{\text{eff}} \phi$  from the effect structure.)

This imprecision allows more programs to type check. In a program with branching, different branches may have different effects. For example, in rule **EFF-CPAIR**, only one side of a computational pair will ever be evaluated. However, for soundness, both computations must be typed with the same effect (which may be an overapproximation due to subeffecting).

Unlike in effect systems for the  $\lambda$ -calculus, the latent effects of function bodies are not recorded in function types. Instead, they are propagated to the conclusion of rule **EFF-ABS**. This makes sense because abstractions are not values in CBPV. From an operational sense, they are computations that pop the argument off the stack before continuing [Levy 2001].

## 2.2 Instrumented operational semantics and effect soundness

We next define a big-step, *environment-based* operational semantics for CBPV. Here, an **environment**,  $\rho$ , is a mapping from variables to **closed values**,  $W$ , and can be thought of as a sequence of delayed

$$\boxed{\rho \vdash V \Downarrow W} \quad \text{(Value closing)}$$

$$\begin{array}{c}
\text{EVAL-VAL-VAR} \\
\frac{x \mapsto W \in \rho}{\rho \vdash x \Downarrow W}
\end{array}
\quad
\begin{array}{c}
\text{EVAL-VAL-UNIT} \\
\frac{}{\rho \vdash () \Downarrow ()}
\end{array}
\quad
\begin{array}{c}
\text{EVAL-VAL-THUNK} \\
\frac{}{\rho \vdash \{M\} \Downarrow \mathbf{clo}(\rho, \{M\})}
\end{array}
\quad
\begin{array}{c}
\text{EVAL-VAL-VPAR} \\
\frac{\rho \vdash V_1 \Downarrow W_1 \quad \rho \vdash V_2 \Downarrow W_2}{\rho \vdash (V_1, V_2) \Downarrow (W_1, W_2)}
\end{array}$$

Fig. 2. Operational semantics of CBPV with effect tracking

$$\boxed{\rho \vdash_{\text{eff}} M \Downarrow T \# \phi} \quad \text{(Computation rules)}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-APP-ABS} \\
\frac{\rho \vdash_{\text{eff}} M \Downarrow \mathbf{clo}(\rho', \lambda x. M') \# \phi_1 \quad \rho \vdash V \Downarrow W \quad \rho', x \mapsto W \vdash_{\text{eff}} M' \Downarrow T \# \phi_2}{\rho \vdash_{\text{eff}} M V \Downarrow T \# \phi_1 \cdot \phi_2}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-ABS} \\
\frac{}{\rho \vdash_{\text{eff}} \lambda x. M \Downarrow \mathbf{clo}(\rho, \lambda x. M) \# \epsilon}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-FORCE-THUNK} \\
\frac{\rho \vdash V \Downarrow \mathbf{clo}(\rho', \{M\}) \quad \rho' \vdash_{\text{eff}} M \Downarrow T \# \phi}{\rho \vdash_{\text{eff}} V! \Downarrow T \# \phi}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-RETURN} \\
\frac{}{\rho \vdash_{\text{eff}} \mathbf{return} V \Downarrow \mathbf{return} W \# \epsilon}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-LETIN-RET} \\
\frac{\rho \vdash_{\text{eff}} M \Downarrow \mathbf{return} W \# \phi_1 \quad \rho, x \mapsto W \vdash_{\text{eff}} N \Downarrow T \# \phi_2}{\rho \vdash_{\text{eff}} x \leftarrow M \mathbf{in} N \Downarrow T \# \phi_1 \cdot \phi_2}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-SPLIT} \\
\frac{\rho \vdash V \Downarrow (W_1, W_2) \quad \rho, x_1 \mapsto W_1, x_2 \mapsto W_2 \vdash_{\text{eff}} N \Downarrow T \# \phi}{\rho \vdash_{\text{eff}} \mathbf{case} V \mathbf{of} (x_1, x_2) \rightarrow N \Downarrow T \# \phi}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-CPAIR} \\
\frac{}{\rho \vdash_{\text{eff}} \langle M_1, M_2 \rangle \Downarrow \mathbf{clo}(\rho, \langle M_1, M_2 \rangle) \# \epsilon}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-FST} \\
\frac{\rho \vdash_{\text{eff}} M \Downarrow \mathbf{clo}(\rho', \langle M_1, M_2 \rangle) \# \phi_1 \quad \rho' \vdash_{\text{eff}} M_1 \Downarrow T \# \phi_2}{\rho \vdash_{\text{eff}} M.1 \Downarrow T \# \phi_1 \cdot \phi_2}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-EFF-COMP-SND} \\
\frac{\rho \vdash_{\text{eff}} M \Downarrow \mathbf{clo}(\rho', \langle M_1, M_2 \rangle) \# \phi_1 \quad \rho' \vdash_{\text{eff}} M_2 \Downarrow T \# \phi_2}{\rho \vdash_{\text{eff}} M.2 \Downarrow T \# \phi_1 \cdot \phi_2}
\end{array}$$

Fig. 3. Operational semantics of CBPV with effect tracking

substitutions. Closed values include closures, *i.e.* suspended computations paired with closing environments, as well as unit and positive products.

$$\begin{array}{ll}
\text{environments } \rho & ::= \emptyset \mid \rho, x \mapsto W \\
\text{closed values } W & ::= () \mid \mathbf{clo}(\rho, \{M\}) \mid (W_1, W_2)
\end{array}$$

This semantics is new but straightforward. Past presentations of CBPV define its operational behavior using small-step, big-step, or stack-based semantics, but all the ones we have found use immediate substitution [Levy 2022]. We choose an environment-based big-step semantics for two reasons. First, the big-step structure corresponds closely to the structure of the type system; there is

only one rule of the operational semantics for each rule of the type system. Together with the use of environments, this semantics eliminates the need for substitution lemmas, leading to a remarkably straightforward soundness proof (Section 2.3). Second, the environment lets us track the demands that computations make on their inputs in our coeffect soundness proof (Section 3.1). For example, with resource usage, we can include annotations in the environment that count how many times the program accesses each variable during computation, mirroring the annotations in the context in the type system. A substitution-based semantics does not support this instrumentation.

Figure 3 shows the definition of the operational semantics. This semantics consists of two relations. The **first relation**, written  $\rho \vdash V \Downarrow W$ , uses the provided environment  $\rho$  to “evaluate” a value  $V$  to a closed value  $W$ . This operation is essentially a substitution operation in that it replaces each variable found in the value with its definition in the environment.

The **second relation**, written  $\rho \vdash_{\text{eff}} M \Downarrow T \# \phi$ , shows how computations evaluate to **closed terminal computations**,  $T$ . Closed terminals are computations that cannot step any further, such as returned (closed) values and suspended abstractions and pairs. The effect annotation  $\phi$  on this relation counts the number of ticks that occur during evaluation of  $M$ . While suspended abstractions and pairs resemble closures, they are not first class. Instead, they provide convenient notation to describe how the environment is propagated through evaluation.

$$\text{closed terminals } T ::= \text{return } W \mid \text{clo}(\rho, \lambda x.M) \mid \text{clo}(\rho, \langle M_1, M_2 \rangle)$$

The operational semantics of the **tick** computation is trivial—it merely produces a unit value and a single **Tick** effect. Other computations either produce no effect (as in rule **EVAL-EFF-COMP-ABS**) or combine the effects of their subcomponents (as in rule **EVAL-EFF-COMP-APP-ABS**). As in the type-and-effect system, the only rule that is specific to the **Tick** effect is the rule for **tick**. All other effects in these rules are parameterized over the input monoid.

While the type system allows for imprecision, the operational semantics precisely tracks the effects of computation.

### 2.3 Type and effect soundness

We state our effect soundness theorem as follows: closed, well-typed computations of type  $\mathbf{F} A$  return closed values and produce effects that are bounded by the type system.

**THEOREM 2.1 (EFFECT SOUNDNESS).** *If  $\emptyset \vdash_{\text{eff}} M :^\phi \mathbf{F} A$  then  $\emptyset \vdash_{\text{eff}} M \Downarrow \text{return } W \# \phi_1$  where  $\phi_1 \leq_{\text{eff}} \phi$ .*

The proof is simple and based on the following logical relation, consisting of three functions defined mutually over the structure of types: closed values  $\mathcal{W}[A]$ , closed terminal computations  $\mathcal{T}[B]^\phi$ , and computations tupled with environments  $\mathcal{M}[B]^\phi$ .

**Definition 2.2 (CBPV with Effects: Logical Relation).**

$$\begin{aligned} \mathcal{W}[\mathbf{U}_\phi B] &= \{ \text{clo}(\rho, \{M\}) \mid (\rho, M) \in \mathcal{M}[B]^\phi \} \\ \mathcal{W}[\mathbf{unit}] &= \{ () \} \\ \mathcal{W}[A_1 \times A_2] &= \{ (W_1, W_2) \mid W_1 \in \mathcal{W}[A_1] \text{ and } W_2 \in \mathcal{W}[A_2] \} \\ \mathcal{T}[\mathbf{F} A]^\phi &= \{ \text{return } W \mid W \in \mathcal{W}[A] \text{ and } \phi \equiv \varepsilon \} \\ \mathcal{T}[A \rightarrow B]^\phi &= \{ \text{clo}(\rho, \lambda x.M) \mid \text{for all } W \in \mathcal{W}[A], ((\rho, x \mapsto W), M) \in \mathcal{M}[B]^\phi \} \\ \mathcal{T}[B_1 \& B_2]^\phi &= \{ \text{clo}(\rho, \langle M_1, M_2 \rangle) \mid (\rho, M_1) \in \mathcal{M}[B_1]^\phi \text{ and } (\rho, M_2) \in \mathcal{M}[B_2]^\phi \} \\ \mathcal{M}[B]^\phi &= \{ (\rho, M) \mid \rho \vdash_{\text{eff}} M \Downarrow T \# \phi_1 \text{ and } T \in \mathcal{T}[B]^\phi \text{ and } \phi_1 \cdot \phi_2 \leq_{\text{eff}} \phi \} \end{aligned}$$



We use this relation to define semantic typing for environments, values, and computations.

*Definition 2.3 (CBPV with Effects: Semantic Typing).*

$$\begin{aligned} \Gamma \models \rho &= x : A \in \Gamma \text{ implies } x \mapsto W \in \rho \text{ and } W \in \mathcal{W}[A] \\ \Gamma \models_{\text{eff}} V : A &= \Gamma \models \rho \text{ implies } \rho \vdash V \Downarrow W \text{ and } W \in \mathcal{W}[A] \\ \Gamma \models_{\text{eff}} M :^{\phi} B &= \Gamma \models \rho \text{ implies } (\rho, M) \in \mathcal{M}[B]^{\phi} \end{aligned}$$

Using these definitions, we can prove semantic typing lemmas corresponding to each of the syntactic typing rules shown in Figure 1. With these lemmas, we show the fundamental lemma as a straightforward induction.

LEMMA 2.4 (FUNDAMENTAL LEMMA: EFFECT SOUNDNESS). (1) If  $\Gamma \vdash_{\text{eff}} V : A$  then  $\Gamma \models_{\text{eff}} V : A$ .  
 (2) If  $\Gamma \vdash_{\text{eff}} M :^{\phi} B$  then  $\Gamma \models_{\text{eff}} M :^{\phi} B$ .

The effect soundness theorem (2.1) follows from the second claus of this lemma, instantiating  $\Gamma$  with the empty context and  $B$  with  $\text{F } A$ . Unfolding the definition of  $\emptyset \models_{\text{eff}} M :^{\phi} \text{F } A$  and gives us some  $\phi_1$  and  $\phi_2$  such that  $\emptyset \vdash_{\text{eff}} M \Downarrow T \# \phi_1$  and  $T \in \mathcal{T}[\text{F } A]^{\phi_2}$  and  $\phi_1 \cdot \phi_2 \leq_{\text{eff}} \phi$ . Further unfolding definitions means that  $T$  must be **return**  $W$ ,  $\phi_2$  must be  $\varepsilon$  and thus  $\phi_1 \leq_{\text{eff}} \phi$ .

## 2.4 Type-and-effect preserving translations

Levy [2006] provides translations from call-by-value (CBV) and call-by-name (CBN)  $\lambda$ -calculi to CBPV and shows that those translations preserve types, denotational semantics, and (substitution-based) big-step operational semantics. We show here that those translations also preserve effects.

For the CBV translation, we start with a  $\lambda$ -calculus that has a simple type-and-effect system, loosely based on Lucassen and Gifford [1988]. However, as few CBN languages directly include effects, for the CBN translation we start with a simply-typed  $\lambda$ -calculus that encapsulates effects using a *graded monad*. Furthermore, we show that we can use this same monad with the CBV translation because effects are encapsulated.

2.4.1 *CBV type-and-effect system.* The simple CBV language with effect tracking in this subsection features the same **tick** term and **Tick** effect as before, along with the usual forms of the  $\lambda$ -calculus.<sup>2</sup>

$$\begin{array}{c} \text{LAM-EFF-VAR} \\ \frac{x : \tau \in \Gamma}{\Gamma \vdash_{\text{eff}} x :^{\varepsilon} \tau} \end{array} \quad \begin{array}{c} \text{LAM-EFF-ABS} \\ \frac{\Gamma, x : \tau_1 \vdash_{\text{eff}} e :^{\phi} \tau_2}{\Gamma \vdash_{\text{eff}} \lambda x. e :^{\varepsilon} \tau_1 \xrightarrow{\phi} \tau_2} \end{array} \quad \begin{array}{c} \text{LAM-EFF-APP} \\ \frac{\Gamma \vdash_{\text{eff}} e_1 :^{\phi_1} \tau_1 \xrightarrow{\phi_3} \tau_2 \quad \Gamma \vdash_{\text{eff}} e_2 :^{\phi_2} \tau_1}{\Gamma \vdash_{\text{eff}} e_1 e_2 :^{\phi_1 \cdot \phi_2 \cdot \phi_3} \tau_2} \end{array} \quad \begin{array}{c} \text{LAM-EFF-TICK} \\ \frac{}{\Gamma \vdash_{\text{eff}} \text{tick} :^{\text{Tick}} \text{unit}} \end{array}$$

Function types, written  $\tau_1 \xrightarrow{\phi} \tau_2$ , are annotated with *latent effects*, which occur when the function is called. In the application rule rule LAM-EFF-APP, this latent effect is combined with  $\phi_1$ , the effects that occur when evaluating the function  $e_1$  to a  $\lambda$  expression, and  $\phi_2$ , the effects that occur when evaluating the argument to a value.

The CBV type and term translations follow directly from Levy [2022]. Besides adding a case for the **tick** expression, the only change that we make is moving the latent effect from the function type to the thunk type. All other cases are exactly as in prior work. Because of this, we only show the translation for function types and the **tick** expression.

$$\begin{array}{ll} \text{Type translation} & \text{Term translation} \\ \llbracket \tau_1 \xrightarrow{\phi} \tau_2 \rrbracket_{\text{v}} = \mathbf{U}_{\phi} (\llbracket \tau_1 \rrbracket_{\text{v}} \rightarrow \mathbf{F} \llbracket \tau_2 \rrbracket_{\text{v}}) & \llbracket \text{tick} \rrbracket_{\text{v}} = \text{tick} \end{array}$$

<sup>2</sup> In this part, the treatment for unit and products is straightforward. Therefore, we elide the typing rules for those features from this section. These rules are available in the extended version.



This translation preserves types and effects from the source language.

LEMMA 2.5 (CBV TRANSLATION IS TYPE CORRECT). *If  $\Gamma \vdash_{\text{eff}} e : \phi \tau$  then  $\llbracket \Gamma \rrbracket_v \vdash_{\text{eff}} \llbracket e \rrbracket_v : \phi \text{ F } \llbracket \tau \rrbracket_v$ .*

This result is easy to prove, reassuring us that our effect system design is correct: we can use CBPV to encode the well-studied type-and-effect systems developed over the past 40 years.

**2.4.2 Graded monads.** CBPV is designed to serve as a convenient translation target for both CBV and CBN languages. However, in CBN languages, effects are usually tracked using parametric effect monads, also known as *graded monads* [Katsumata 2014; Orchard and Petricek 2014; Smirnov 2008; Wadler and Thiemann 2003].<sup>3</sup> Therefore, here we translate a *CBN language with graded monads* to CBPV. Our source language for this translation is the simply-typed  $\lambda$ -calculus with unit and products, together with a graded monadic type  $\mathbf{T}_\phi \tau$ , the monadic operations **return** and **bind**, and the **tick** operation, with a monadic type. To account for imprecision, we include an explicit type coercion for the graded monad.

$$\begin{array}{c}
\text{LAM-MON-RETURN} \\
\frac{\Gamma \vdash_{\text{mon}} e : \tau}{\Gamma \vdash_{\text{mon}} \mathbf{return} e : \mathbf{T}_\epsilon \tau} \\
\\
\text{LAM-MON-TICK} \\
\frac{}{\Gamma \vdash_{\text{mon}} \mathbf{tick} : \mathbf{T}_{\text{Tick}} \mathbf{unit}} \\
\\
\text{LAM-MON-BIND} \\
\frac{\Gamma \vdash_{\text{mon}} e_1 : \mathbf{T}_{\phi_1} \tau_1 \quad \Gamma, x : \tau_1 \vdash_{\text{mon}} e_2 : \mathbf{T}_{\phi_2} \tau_2}{\Gamma \vdash_{\text{mon}} \mathbf{bind} x = e_1 \text{ in } e_2 : \mathbf{T}_{\phi_1 \cdot \phi_2} \tau_2} \\
\\
\text{LAM-MON-COERCE} \\
\frac{\Gamma \vdash_{\text{mon}} e : \mathbf{T}_{\phi_1} \tau \quad \phi_1 \leq_{\text{eff}} \phi_2}{\Gamma \vdash_{\text{mon}} \mathbf{coerce} e : \mathbf{T}_{\phi_2} \tau}
\end{array}$$

Below, we extend Levy’s *translation of the CBN  $\lambda$ -calculus* to include the graded monad. The translation of the core language is as in prior work and all effects are isolated to the monadic type, so we only show the monadic portion in the figure.

*Type translation*

$$\llbracket \mathbf{T}_\phi \tau \rrbracket_N = \mathbf{F} (\mathbf{U}_\phi \text{ F } (\mathbf{U}_\epsilon \llbracket \tau \rrbracket_N))$$

*Term translation*

$$\begin{aligned}
\llbracket \mathbf{return} e \rrbracket_N &= \mathbf{return} \{ \mathbf{return} \{ \llbracket e \rrbracket_N \} \} \\
\llbracket \mathbf{bind} x = e_1 \text{ in } e_2 \rrbracket_N &= \mathbf{return} \{ x \leftarrow (y \leftarrow \llbracket e_1 \rrbracket_N \text{ in } y!) \text{ in } z \leftarrow \llbracket e_2 \rrbracket_N \text{ in } z! \} \\
\llbracket \mathbf{coerce} e \rrbracket_N &= \mathbf{return} \{ x \leftarrow \llbracket e \rrbracket_N \text{ in } x! \} \\
\llbracket \mathbf{tick} \rrbracket_N &= \mathbf{return} \{ x \leftarrow \mathbf{tick} \text{ in } \mathbf{return} \{ \mathbf{return} x \} \}
\end{aligned}$$

This translation preserves types (with embedded effects) from the source language. Note that, because the monadic type marks effectful code, the translation produces CBPV computations that can be checked with the “pure” effect  $\epsilon$ .

LEMMA 2.6 (CBN TRANSLATION IS TYPE CORRECT). *If  $\Gamma \vdash_{\text{mon}} e : \tau$  then  $\llbracket \Gamma \rrbracket_N \vdash_{\text{eff}} \llbracket e \rrbracket_N : \epsilon \llbracket \tau \rrbracket_N$ .*

One difficulty of this translation is that the monadic type in the CBPV adjunction is  $\mathbf{U} \mathbf{F}$ . This type is a value type, and the standard CBN translation produces terms with computation types. Therefore, to use  $\mathbf{U} \mathbf{F}$  as the monad in our CBN translation, we need to bracket it: on the outside by  $\mathbf{F}$  to form a computation type, and then on the inside by  $\mathbf{U}$  to construct the value type that the monad expects. This bracketing produces an awkward translation of the monadic operations with doubled thunking. This awkwardness is due to the presence of the monad in the source language; it is a separate structure from the ambient monad of the computation language.

<sup>3</sup> Instead of graded monads, we could also consider a translation from call-by-name language that does not encapsulate effects, such as the one defined by McDermott and Mycroft [2018].

|   |   |  |
|---|---|--|
| $\gamma \cdot \Gamma \vdash_{\text{coeff}} V : A$   |   | <i>(value coeffect typing)</i>   |
| $\frac{\text{COEFF-VAR}}{\bar{0} \cdot \Gamma_1, x : \bar{1} A, \bar{0} \cdot \Gamma_2 \vdash_{\text{coeff}} x : A}$  | $\frac{\text{COEFF-THUNK} \quad \gamma \cdot \Gamma \vdash_{\text{coeff}} M : B}{\gamma \cdot \Gamma \vdash_{\text{coeff}} \{M\} : \mathbf{U} B}$   | $\frac{\text{COEFF-UNIT}}{\bar{0} \cdot \Gamma \vdash_{\text{coeff}} () : \mathbf{unit}}$  |
| $\frac{\text{COEFF-PAIR} \quad \gamma_1 \cdot \Gamma \vdash_{\text{coeff}} V_1 : A_1 \quad \gamma_2 \cdot \Gamma \vdash_{\text{coeff}} V_2 : A_2}{\gamma_1 + \gamma_2 \cdot \Gamma \vdash_{\text{coeff}} (V_1, V_2) : A_1 \times A_2}$  | $\frac{\text{COEFF-VSUB} \quad \gamma' \cdot \Gamma \vdash_{\text{coeff}} V : A \quad \gamma \leq_{\text{co}} \gamma'}{\gamma \cdot \Gamma \vdash_{\text{coeff}} V : A}$  |  |
| $\gamma \cdot \Gamma \vdash_{\text{coeff}} M : B$   |   | <i>(computation coeffect typing)</i>   |
| $\frac{\text{COEFF-ABS} \quad \gamma \cdot \Gamma, x : \bar{q} A \vdash_{\text{coeff}} M : B \quad q' \leq_{\text{co}} q}{\gamma \cdot \Gamma \vdash_{\text{coeff}} \lambda x^{\bar{q}}. M : A^{q'} \rightarrow B}$   | $\frac{\text{COEFF-APP} \quad \gamma_1 \cdot \Gamma \vdash_{\text{coeff}} M : A^q \rightarrow B \quad \gamma_2 \cdot \Gamma \vdash_{\text{coeff}} V : A}{\gamma_1 + (q \cdot \gamma_2) \cdot \Gamma \vdash_{\text{coeff}} M V : B}$ | $\frac{\text{COEFF-FORCE} \quad \gamma \cdot \Gamma \vdash_{\text{coeff}} V : \mathbf{U} B}{\gamma \cdot \Gamma \vdash_{\text{coeff}} V! : B}$                           |
| $\frac{\text{COEFF-SPLIT} \quad \gamma_1 \cdot \Gamma \vdash_{\text{coeff}} V : A_1 \times A_2 \quad \gamma_2 \cdot \Gamma, x_1 : \bar{q} A_1, x_2 : \bar{q} A_2 \vdash_{\text{coeff}} N : B}{(q \cdot \gamma_1) + \gamma_2 \cdot \Gamma \vdash_{\text{coeff}} \mathbf{case}_q V \text{ of } (x_1, x_2) \rightarrow N : B}$ | $\frac{\text{COEFF-RET} \quad \gamma \cdot \Gamma \vdash_{\text{coeff}} V : A}{q \cdot \gamma \cdot \Gamma \vdash_{\text{coeff}} \mathbf{return}_q V : \mathbf{F}_q A}$   |  |
| $\frac{\text{COEFF-LETIN} \quad \gamma_1 \cdot \Gamma \vdash_{\text{coeff}} M : \mathbf{F}_{q_1} A \quad \gamma_2 \cdot \Gamma, x : \bar{q_1 \cdot q_2} A \vdash_{\text{coeff}} N : B}{(q_2 \cdot \gamma_1) + \gamma_2 \cdot \Gamma \vdash_{\text{coeff}} x \leftarrow^{q_2} M \text{ in } N : B}$                          | $\frac{\text{COEFF-CPAIR} \quad \gamma \cdot \Gamma \vdash_{\text{coeff}} M_1 : B_1 \quad \gamma \cdot \Gamma \vdash_{\text{coeff}} M_2 : B_2}{\gamma \cdot \Gamma \vdash_{\text{coeff}} \langle M_1, M_2 \rangle : B_1 \& B_2}$    |  |
| $\frac{\text{COEFF-FST} \quad \gamma \cdot \Gamma \vdash_{\text{coeff}} M : B_1 \& B_2}{\gamma \cdot \Gamma \vdash_{\text{coeff}} M.1 : B_1}$   | $\frac{\text{COEFF-SND} \quad \gamma \cdot \Gamma \vdash_{\text{coeff}} M : B_1 \& B_2}{\gamma \cdot \Gamma \vdash_{\text{coeff}} M.2 : B_2}$   | $\frac{\text{COEFF-CSUB} \quad \gamma' \cdot \Gamma \vdash_{\text{coeff}} M : B \quad \gamma \leq_{\text{co}} \gamma'}{\gamma \cdot \Gamma \vdash_{\text{coeff}} M : B}$ |

Fig. 4. CBPV type system with coeffect tracking

### 3 CBPV AND COEFFECTS (VERSION 1: GENERAL SEMANTICS)

Next, we construct a parallel extension of CBPV with *coeffects*. Figure 4 lists the *typing rules*, with coeffect annotations in blue. Coeffect systems are designed for reasoning about inputs, so we annotate variables at their binding sites and in the context.

Coeffects annotations consist of *grades*  $q$  taken from a *preordered semiring*. This structure provides an addition operation  $q_1 + q_2$ , an additive identity element  $0$ , a multiplication operation  $q_1 \cdot q_2$ , a multiplicative identity  $1$ , and a reflexive and transitive binary relation  $\leq_{\text{co}}$  that respects addition and multiplication. (The preorder does not have to be the one defined by the addition operation.) The need for a semiring rather than a monoid arises from the fact that any value may be bound to a variable that may then be used multiple times, requiring a notion of coeffect multiplication.

Similarly to the previous section, our type system in this section is general across coeffects and can be specialized via the choice of semiring and preorder. For concreteness, we discuss it in terms of a running example, resource usage, an application that we will explore more thoroughly in the next section. In this case, grades bound the *uses* of variables, as in bounded linear logic, and

come from the natural number semiring with the usual addition and multiplication operators. The additive and multiplicative identity elements of this semiring mark  $0$  and at most  $1$  (affine) use of a variable respectively, and the addition and multiplication semiring operations calculate the total number of times each variable is used in the program.

As in many systems for bounded linear logic,  $q_1 \leq_{co} q_2$  indicates that  $q_1$  is *less precise* or *less restrictive* than  $q_2$ . When counting variable usage, this has the opposite order from the usual one—we have  $3 \leq_{co} 2$  because allowing at most 3 uses is less restrictive than at most 2. With other coeffects, such as security levels, this ordering has a more intuitive interpretation: a higher grade corresponds to a higher security level, which is more restrictive than a low security level.

Like the effect system with subeffecting, this type system includes a rule for *subcoeffecting*: if a judgment holds with some annotation  $q_2$  on a variable in the context, then it is also derivable with any  $q_1 \leq_{co} q_2$ . For example, we can weaken a judgment that a computation makes zero ( $0$ ) uses of some variable to observe at most one use (affine) or any other number. This corresponds to the usual weakening lemma from typed  $\lambda$ -calculus.

Again, as in the effect section, including a preorder with the semiring allows for imprecision, needed when analyzing branching computations. For example, if one branch requires  $1$  use of a variable  $x$ , but the other branch requires  $0$  uses, the system will record that the program must have the resources to use  $x$  at least once, because  $1 \leq_{co} 0$ . This is dual to the preorder's role in the effect system—if one branch ticks once and the other does not tick, then the system will record at most one tick. In both cases, replacing the ordering with the discrete preorder means that the type system must be precise and would reject both of these examples.

The type system uses a *grade vector*  $\gamma$ , a comma-separated list of grades, to represent the annotations for the variables in a typing context. When combined with a typing context  $\Gamma$ , written  $\gamma \cdot \Gamma$ , the grade vector must have the same length as  $\Gamma$ . We extend a combined grade vector and typing context simultaneously with the notation  $\gamma \cdot \Gamma, x :^q A$ , equivalent to  $(\gamma, q) \cdot (\Gamma, x : A)$ .

The grade vector written  $\bar{0}$  contains only zeros and is used where its length can be inferred from context. Grade vectors of the same length can be added together pointwise, written  $\gamma_1 + \gamma_2$ , and compared pointwise, written  $\gamma_1 \leq_{co} \gamma_2$ . Grade vectors can also be pointwise scaled, written  $q \cdot \gamma$ .

The basis of this system is rule **COEFF-VAR**. When introducing a variable  $x$ , the context must grade  $x$  with  $1$ , indicating that it is used once. No other variables in the context should affect the typing judgement, so they must have grade  $0$ . Similarly, the unit value  $()$  can make no demands on the environment, so rule **COEFF-UNIT** requires that all variables in the typing context be graded  $0$ .

In rule **COEFF-THUNK** and rule **COEFF-FORCE**, there is a single subterm that makes exactly the same demands on its environment as the term in the conclusion, so we use the same grade vector in the conclusion and the premise.

In other rules, the term in the conclusion has multiple subterms, so we combine the demands made by each. In rule **COEFF-PAIR**, the subterms both get evaluated and do not directly interact, so we combine their grade vectors via simple pointwise addition. Conversely, with negative products, the two subterms must use the same resources, so we use the same grade vector in each premise and the conclusion. Intuitively, this is because we can only ever project out one subterm from a computation pair (see rule **COEFF-FST** and rule **COEFF-SND**), so the projected term will make all the same demands on the environment as the pair.

In rule **COEFF-ABS**, we know from the premise that  $M$  will require a grade of  $q$  on  $x$ , so we store that grade as an annotation on  $x$  in the term syntax. For flexibility, we allow the annotation in the type,  $q'$ , to be a less precise approximation of  $q$ . (This expressiveness is useful for the translation results in the next section. Note that subcoeffecting is not sufficient as it can not allow the annotation

on the  $\lambda$  to differ from the annotation on the function type.) Both the premise and the conclusion make the same demands on the variables in  $\Gamma$ , so  $\gamma$  is otherwise the same in both.

In some rules, we must combine the grade vectors of subterms using both scaling and addition. For example, in rule **COEFF-APP**,  $\gamma_1$  denotes the demands the operator  $M$  makes on the environment, and  $\gamma_2$  denotes the demands the argument  $V$  makes.  $M$  has type  $A^q \rightarrow B$ , indicating that when it is reduced to some terminal  $\lambda x^{q'}.M'$ , then  $M'$  will require  $x$  to have a grade of  $q'$ , where  $q \leq_{co} q'$ . This means we must scale  $\gamma_1$  by  $q$  before adding it to  $\gamma_2$  to calculate the total demands that  $M V$  makes on its environment.

In the effect system, we annotate the type  $U_\phi B$  with the effect of the suspended computation. In the coeffect system, we dually annotate the returner type  $F_q A$ . In our resource usage example, the  $q$  indicates that we require enough resources from the environment to produce  $q$  copies of a value. For example, **return**<sub>3</sub>  $V$  indicates that we require the resources to create 3 copies of  $V$ . Therefore, rule **COEFF-RET** scales the demands needed to create  $V$  by  $q$ .

In rule **COEFF-LETIN**,  $M$  has returner type  $F_{q_1} A$ , and its result value has been scaled by  $q_1$ . However, the expression includes another scaling annotation  $q_2$ , that allows duplication of the computation  $M$  itself. If  $\gamma_1$  denotes the demands  $M$  makes on its environment,  $q_1 \cdot \gamma_2$  denotes the grade  $N$  requires  $x$  to have, and  $\gamma_2$  denotes the demands  $N$  makes from the rest of the environment, then we need  $q_2 \cdot \gamma_1 + \gamma_2$  to type the entire term.

The scaling annotations in **return** <sub>$q$</sub>   $V$  and  $x \leftarrow^q M$  in  $N$  increase the expressiveness of the language and are required for the translation of a CBV  $\lambda$ -calculus to CBPV described in Section 4.1.2. Because CBV is strict, when translating an application, we must use a let binding to evaluate the translated argument before applying the translated function to it. However, the function may require a particular grade  $q$  on its argument, so we must be able to scale this computation. Similarly, to translate the graded CBV comonadic type, we need to be able to duplicate values.

The two subsumption rules **COEFF-VSUB** and **COEFF-CSUB** allow for subcoeffecting.

### 3.1 General instrumented operational semantics and coeffect soundness

Next, we develop an **instrumented operational semantics**, shown in Figure 5, that tracks coeffects using an environment  $\rho$ , which maps variables to closed values, and a grade vector  $\gamma$  of equal length, which implicitly maps variables to their coeffects. As before, we extend both a grade vector and corresponding environment simultaneously with the notation  $\gamma \cdot \rho$ ,  $x \mapsto^q W$ , equivalent to  $(\gamma, q) \cdot (\rho, x \mapsto W)$ .

We also use  $W$  as a metavariable for *closed* values and  $T$  as a metavariable for *closed terminal* computations. However, closed terminals include coeffects here. They have the form **return** <sub>$q$</sub>   $W$ , **clo**( $\gamma \cdot \rho, \lambda x^q.M$ ), or **clo**( $\gamma \cdot \rho, \langle M_1, M_2 \rangle$ ), where **clo**( $\gamma \cdot \rho, M$ ) denotes the *closure* of  $M$  under  $\gamma \cdot \rho$ . The grade vector in the closure indicates the demands on the variables used by  $M$ .

Unlike our instrumented operational semantics for effects, which calculates the exact effect of a computation, this semantics cannot track coeffects with precision. For example, suppose we have a term  $\lambda x^1.M$  where  $M$  is a computation that both branches on its argument and uses it in at exactly one branch, such as **case**<sub>1</sub>  $x$  **of** **inl**  $x_1 \rightarrow$  **return**  $x$ ; **inr**  $x_2 \rightarrow$  **return** **inr**  $()$ .<sup>4</sup> What should this step to? If provided with an argument of the form **inl**  $y$ , it should step to **clo**( $x \mapsto^1$  **inl**  $y, \lambda x^1.M$ ). If provided with an argument of the form **inr**  $y$ , it should step to **clo**( $x \mapsto^0$  **inr**  $y, \lambda x^1.M$ ). But, if this term is the entire program, it is not clear what it should step to. In general, depending on the argument, the body of a function  $\lambda x^q.M$  may require a different exact grade on  $x$ ; all we know from the typing judgement is that  $q$  must be a bound on that usage. We cannot write a precise rule

<sup>4</sup> The system in our extended version and Coq development includes sums.

|  |  |
|--|--|
| $\gamma \cdot \rho \vdash_{\text{coeff}} V \Downarrow W$   | <i>(Value rules)</i>   |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-VAL-VSUB</div> $\frac{\gamma_1 \cdot \rho \vdash_{\text{coeff}} V \Downarrow W \quad \gamma_2 \leq_{\text{co}} \gamma_1}{\gamma_2 \cdot \rho \vdash_{\text{coeff}} V \Downarrow W}$   | <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-VAL-VAR</div> $\frac{}{\bar{0}_1 \cdot \rho_1, x \mapsto^1 W, \bar{0}_2 \cdot \rho_2 \vdash_{\text{coeff}} x \Downarrow W}$   |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-VAL-UNIT</div> $\frac{}{\bar{0} \cdot \rho \vdash_{\text{coeff}} () \Downarrow ()}$   |  |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-VAL-THUNK</div> $\frac{}{\gamma \cdot \rho \vdash_{\text{coeff}} \{M\} \Downarrow \mathbf{clo}(\gamma \cdot \rho, \{M\})}$  | <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-VAL-VPAIR</div> $\frac{\gamma_1 \cdot \rho \vdash_{\text{coeff}} V_1 \Downarrow W_1 \quad \gamma_2 \cdot \rho \vdash_{\text{coeff}} V_2 \Downarrow W_2}{\gamma_1 + \gamma_2 \cdot \rho \vdash_{\text{coeff}} (V_1, V_2) \Downarrow (W_1, W_2)}$   |
| $\gamma \cdot \rho \vdash_{\text{coeff}} M \Downarrow T$   | <i>(Computation rules)</i>   |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-ABS</div> $\frac{q' \leq_{\text{co}} q}{\gamma \cdot \rho \vdash_{\text{coeff}} \lambda x^q. M \Downarrow \mathbf{clo}(\gamma \cdot \rho, \lambda x^q. M)}$  | <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-CPAIR</div> $\frac{}{\gamma \cdot \rho \vdash_{\text{coeff}} \langle M_1, M_2 \rangle \Downarrow \mathbf{clo}(\gamma \cdot \rho, \langle M_1, M_2 \rangle)}$   |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-APP-ABS</div> $\frac{\gamma_1 \cdot \rho \vdash_{\text{coeff}} M \Downarrow \mathbf{clo}(\gamma' \cdot \rho', \lambda x^q. M') \quad \gamma_2 \cdot \rho \vdash_{\text{coeff}} V \Downarrow W \quad \gamma' \cdot \rho', x \mapsto^q W \vdash_{\text{coeff}} M' \Downarrow T \quad \gamma \equiv \gamma_1 + q \cdot \gamma_2}{\gamma \cdot \rho \vdash_{\text{coeff}} M V \Downarrow T}$ | <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-SPLIT</div> $\frac{\gamma_1 \cdot \rho \vdash_{\text{coeff}} V \Downarrow (W_1, W_2) \quad \gamma_2 \cdot \rho, x_1 \mapsto^q W_1, x_2 \mapsto^q W_2 \vdash_{\text{coeff}} N \Downarrow T \quad \gamma \equiv q \cdot \gamma_1 + \gamma_2}{\gamma \cdot \rho \vdash_{\text{coeff}} \mathbf{case}_q V \text{ of } (x_1, x_2) \rightarrow N \Downarrow T}$ |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-RETURN</div> $\frac{\gamma \cdot \rho \vdash_{\text{coeff}} V \Downarrow W}{q \cdot \gamma \cdot \rho \vdash_{\text{coeff}} \mathbf{return}_q V \Downarrow \mathbf{return}_q W}$   | <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-FORCE-THUNK</div> $\frac{\gamma \cdot \rho \vdash_{\text{coeff}} V \Downarrow \mathbf{clo}(\gamma' \cdot \rho', \{M\}) \quad \gamma' \cdot \rho' \vdash_{\text{coeff}} M \Downarrow T}{\gamma \cdot \rho \vdash_{\text{coeff}} V! \Downarrow T}$   |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-LETIN-RET</div> $\frac{\gamma_1 \cdot \rho \vdash_{\text{coeff}} M \Downarrow \mathbf{return}_{q_1} W \quad \gamma_2 \cdot \rho, x \mapsto^{q_1 \cdot q_2} W \vdash_{\text{coeff}} N \Downarrow T}{q_2 \cdot \gamma_1 + \gamma_2 \cdot \rho \vdash_{\text{coeff}} x \leftarrow^{q_2} M \text{ in } N \Downarrow T}$  | <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-FST</div> $\frac{\gamma \cdot \rho \vdash_{\text{coeff}} M \Downarrow \mathbf{clo}(\gamma' \cdot \rho', \langle M_1, M_2 \rangle) \quad \gamma' \cdot \rho' \vdash_{\text{coeff}} M_1 \Downarrow T}{\gamma \cdot \rho \vdash_{\text{coeff}} M.1 \Downarrow T}$   |
| <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-SND</div> $\frac{\gamma \cdot \rho \vdash_{\text{coeff}} M \Downarrow \mathbf{clo}(\gamma' \cdot \rho', \langle M_1, M_2 \rangle) \quad \gamma' \cdot \rho' \vdash_{\text{coeff}} M_2 \Downarrow T}{\gamma \cdot \rho \vdash_{\text{coeff}} M.2 \Downarrow T}$   | <div style="text-align: center; margin-bottom: 10px;">EVAL-COEFF-COMP-CSUB</div> $\frac{\gamma' \cdot \rho \vdash_{\text{coeff}} M \Downarrow T \quad \gamma \leq_{\text{co}} \gamma'}{\gamma \cdot \rho \vdash_{\text{coeff}} M \Downarrow T}$  |

Fig. 5. Instrumented operational semantics

for evaluating abstractions to their closed terminal forms, because we do not have access to the argument yet when doing that evaluation.

Therefore, as in the typing rules, the operational semantics also includes rules for subcoeffecting, rules **EVAL-COEFF-VAL-VSUB** and **EVAL-COEFF-COMP-CSUB**. These rules say that if we can step a term with grades given by  $\gamma$  attached to the environment, then we can step it with  $\gamma'$  for any  $\gamma' \leq_{\text{co}} \gamma$ , i.e., any less precise accounting.

As in the semantics for CBPV without coefficients, we define “evaluation” of values using the given environment (see Figure 5). These rules mirror the typing rules: rule **EVAL-COEFF-VAL-VAR** requires the evaluating variable to have **1** as its corresponding grade and all other variables to have **0**; rule **EVAL-COEFF-VAL-UNIT** requires that every variable be graded with **0**; rule **EVAL-COEFF-VAL-THUNK** simply includes the grade vector in the closure along with the environment, and rule **EVAL-COEFF-VAL-VPAIR** sums the grades needed to evaluate subterms to their closures.

Figure 5 also shows computations. Rules **EVAL-COEFF-COMP-ABS**, **EVAL-COEFF-COMP-FORCE-THUNK**, **EVAL-COEFF-COMP-CPAIR**, **EVAL-COEFF-COMP-FST**, and **EVAL-COEFF-COMP-SND** are largely the same as before, just with the inclusion of grade vectors along with environments.

In rule **EVAL-COEFF-COMP-RETURN**, we scale the grade needed to evaluate the subterm to its closure by  $q$ . In the elimination rules **EVAL-COEFF-COMP-APP-ABS**, **EVAL-COEFF-COMP-LETIN-RET**, and **EVAL-COEFF-COMP-SPLIT**, if we are eliminating a value  $V$  and binding it to a variable  $x$  with a grade  $q$  for use in some computation  $M$ , we must scale the grade vector needed to evaluate  $V$  by  $q$  before adding it to the grade vector needed to continue with  $M$ , as in the typing rules.

We prove a coefficient soundness theorem stating that if a term is well-typed with some grade vector  $\gamma$ , then given  $\gamma$  and some environment  $\rho$  that provides values of the correct type for all free variables, it can evaluate to a terminal. Because both values and computations make demands on their inputs, we state this property for both. We formalize the requirement on  $\rho$  as  $\Gamma \models \rho$  in our logical relation below, and this theorem follows immediately from the fundamental lemma.

**THEOREM 3.1 (COEFFICIENT SOUNDNESS).** *Let  $\Gamma$  be a context and  $\rho$  an environment mapping all variables in the domain of  $\Gamma$  to closed values of the expected type. Then:*

- (1) *If  $\gamma \cdot \Gamma \vdash_{\text{coeff}} V : A$  then  $\gamma \cdot \rho \vdash_{\text{coeff}} V \Downarrow W$  for some closed value  $W$ .*
- (2) *If  $\gamma \cdot \Gamma \vdash_{\text{coeff}} M : B$  then  $\gamma \cdot \rho \vdash_{\text{coeff}} M \Downarrow T$  for some closed terminal computation  $T$ .*

The proof of coefficient soundness is similar to the proof of effect soundness, and requires a similar logical relation.

**Definition 3.2 (CBPV with General Coefficients: Logical Relation).**

$$\begin{aligned}
\mathcal{W}[\![U B]\!] &= \{ \text{clo}(\gamma \cdot \rho, \{M\}) \mid (\gamma \cdot \rho, M) \in \mathcal{M}[\![B]\!] \} \\
\mathcal{W}[\![\text{unit}]\!] &= \{ () \} \\
\mathcal{W}[\![A_1 \times A_2]\!] &= \{ (W_1, W_2) \mid W_1 \in \mathcal{W}[\![A_1]\!] \text{ and } W_2 \in \mathcal{W}[\![A_2]\!] \} \\
\mathcal{T}[\![F_q A]\!] &= \{ \text{return}_q W \mid W \in \mathcal{W}[\![A]\!] \} \\
\mathcal{T}[\![A^q \rightarrow B]\!] &= \{ \text{clo}(\gamma \cdot \rho, \lambda x^q. M) \mid \text{for all } W \in \mathcal{W}[\![A]\!], ((\gamma \cdot \rho, x \mapsto^q W), M) \in \mathcal{M}[\![B]\!] \} \\
\mathcal{T}[\![B_1 \& B_2]\!] &= \{ \text{clo}(\gamma \cdot \rho, \langle M_1, M_2 \rangle) \mid (\gamma \cdot \rho, M_1) \in \mathcal{M}[\![B_1]\!] \text{ and } (\gamma \cdot \rho, M_2) \in \mathcal{M}[\![B_2]\!] \}
\end{aligned}$$

*Closures*

$$\begin{aligned}
\mathcal{V}[\![A]\!] &= \{ (\gamma \cdot \rho, V) \mid \gamma \cdot \rho \vdash_{\text{coeff}} V \Downarrow W \text{ and } W \in \mathcal{W}[\![A]\!] \} \\
\mathcal{M}[\![B]\!] &= \{ (\gamma \cdot \rho, M) \mid \gamma \cdot \rho \vdash_{\text{coeff}} M \Downarrow T \text{ and } T \in \mathcal{T}[\![B]\!] \}
\end{aligned}$$

**Definition 3.3 (CBPV with General Coefficients: Semantic Typing).**

$$\begin{aligned}
\Gamma \models \rho &= x : A \in \Gamma \text{ implies exists } W, x \mapsto W \in \rho \text{ and } W \in \mathcal{W}[\![A]\!] \\
\gamma \cdot \Gamma \models_{\text{coeff}} V : A &= \text{for all } \rho, \Gamma \models \rho \text{ implies exists } W, \gamma \cdot \rho \vdash_{\text{coeff}} V \Downarrow W \text{ and } W \in \mathcal{W}[\![A]\!] \\
\gamma \cdot \Gamma \models_{\text{coeff}} M : B &= \text{for all } \rho, \Gamma \models \rho \text{ implies } (\gamma \cdot \rho, M) \in \mathcal{M}[\![B]\!]
\end{aligned}$$

We can now state the fundamental lemma, which derives the soundness theorem as a corollary.

**THEOREM 3.4 (FUNDAMENTAL LEMMA: COEFFICIENT SOUNDNESS).** *For all  $\gamma, \Gamma$ , if  $\gamma \cdot \Gamma \vdash_{\text{coeff}} V : A$  then  $\gamma \cdot \Gamma \models_{\text{coeff}} V : A$ , and for all  $\gamma, \Gamma$ , if  $\gamma \cdot \Gamma \vdash_{\text{coeff}} M : B$  then  $\gamma \cdot \Gamma \models_{\text{coeff}} M : B$ .*

$$\boxed{\gamma \cdot \Gamma \vdash_{lin} M : B} \quad (Modified \text{ typing rule})$$

$$\begin{array}{c}
\text{LIN-LETIN} \\
\frac{\gamma' = q_2 \parallel 1 \quad \gamma_1 \cdot \Gamma \vdash_{lin} M : F_{q_1} A \quad \gamma_2 \cdot \Gamma, x : \gamma_1 \cdot \gamma' A \vdash_{lin} N : B}{(\gamma' \cdot \gamma_1) + \gamma_2 \cdot \Gamma \vdash_{lin} x \leftarrow^{q_2} M \text{ in } N : B}
\end{array}$$

$$\boxed{\gamma \cdot \rho \vdash_{lin} M \Downarrow T} \quad (New \text{ and modified computation rules})$$

$$\begin{array}{c}
\text{EVAL-LIN-COMP-APP-ABS-ZERO} \\
\frac{\gamma \cdot \rho \vdash_{lin} M \Downarrow \text{clo}(\gamma' \cdot \rho', \lambda x^0. M') \quad (\gamma' \cdot \rho'), (x \mapsto^0 \frac{1}{2}) \vdash_{lin} M' \Downarrow T}{\gamma \cdot \rho \vdash_{lin} M V \Downarrow T}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-LIN-COMP-RET-ZERO} \\
\frac{}{\bar{0} \cdot \rho \vdash_{lin} \text{return}_0 V \Downarrow \text{return}_0 \frac{1}{2}}
\end{array}
\quad
\begin{array}{c}
\text{EVAL-LIN-COMP-SPLIT-ZERO} \\
\frac{\gamma \cdot \rho, x_1 \mapsto^0 \frac{1}{2}, x_2 \mapsto^0 \frac{1}{2} \vdash_{lin} N \Downarrow T}{\gamma \cdot \rho \vdash_{lin} \text{case}_0 V \text{ of } (x_1, x_2) \rightarrow N \Downarrow T}
\end{array}$$

$$\begin{array}{c}
\text{EVAL-LIN-COMP-LETIN-RET} \\
\frac{\gamma' = q_2 \parallel 1 \quad \gamma_1 \cdot \rho \vdash_{lin} M \Downarrow \text{return}_{q_1} W \quad \gamma_2 \cdot \rho, x \mapsto \gamma_1 \cdot \gamma' W \vdash_{lin} N \Downarrow T}{\gamma' \cdot \gamma_1 + \gamma_2 \cdot \rho \vdash_{lin} x \leftarrow^{q_2} M \text{ in } N \Downarrow T}
\end{array}$$

Fig. 6. Typing rules and instrumented operational semantics for resource tracking

### 3.2 A Strange Semantics?

The operational semantics and soundness proof in this section are generic and work for any instantiation of the coeffect semiring. However, this semantics has strange implications for the resource usage coeffect. Here, the soundness theorem should say that if  $\gamma \cdot \rho \vdash_{coeff} M \Downarrow T$ , then the evaluation of  $M$  used its variables at most the number of times indicated by  $\gamma$ . If  $\gamma$  says that a variable  $x$  has grade  $0$ , then there should never be a use of rule **EVAL-COEFF-VAL-VAR** with the variable  $x$ .

But, on closer examination of the operational semantics, this is not exactly what this soundness theorem implies. Consider the following example:

$$x :^0 U (F \text{ unit}) \vdash_{coeff} z_1 \leftarrow^0 x! \text{ in } \text{return}_1 () : F_1 \text{ unit}$$

$x!$  does not contribute to the final result, and the resources used in its evaluation are accordingly multiplied by  $0$  when we calculate the grade for  $x$  in the context. However, our semantics evaluates  $x$  once here using rule **EVAL-COEFF-VAL-VAR**, violating the principle we described above.

More generally, we encounter this issue with any rule in the operational semantics that scales resources based on some annotation in the terms. For example, in rule **EVAL-COEFF-COMP-APP-ABS**, the resources used by the evaluation of the argument  $\gamma_2$  are scaled by  $q$ , the grade on the function argument. The total resources of the application  $\gamma$  must equal this scaled vector plus  $\gamma_1$ , the resources used to evaluate the function – i.e., we must have  $\gamma \equiv \gamma_1 + q \cdot \gamma_2$ . What if  $q$  is  $0$ ? The resources needed to compute the argument are then not accounted for in  $\gamma$ . This suggests that we should not evaluate the argument at all in this case, so we need to adjust our operational semantics.

## 4 CBPV AND COEFFECTS (VERSION 2: RESOURCE TRACKING)

In this section, we discuss how, with a few additional axioms, we can modify our instrumented operational semantics and type system to produce a better model for resource tracking. Our goal is



to ensure that we never evaluate values and computations without including their resource usage in the final count. The modifications that we discuss here are summarized in Figure 6. We use the judgements  $\gamma \cdot \Gamma \vdash_{lin} M : B$  and  $\gamma \cdot \Gamma \vdash_{lin} V : A$  to refer to the [modified typing rules](#) of this section and  $\gamma \cdot \rho \vdash_{lin} M \Downarrow T$  to refer to the [modified operational semantics](#), highlighting the connection between resource usage coeffects and bounded linear logic.

First, we axiomatize that the semiring is nontrivial. If  $1 = 0$ , resource tracking via grades is meaningless, and our general semantics degenerates to standard CBPV. Second, we require that if  $0 \leq_{co} q_1 + q_2$ , then  $q_1 = 0$  and  $q_2 = 0$ . If either subterm in a value pair requires nonzero resources, we should not be able to evaluate the pair with no resources. Finally, for similar reasons, we require that there be no nonzero zero divisors in the semiring, i.e., if  $0 = q_1 \cdot q_2$ , then  $q_1 = 0$  or  $q_2 = 0$ .

In this system, the  $0$  grade denotes that the corresponding variable is inaccessible, i.e., used  $0$  times, so anywhere we eliminate a value and bind it to an inaccessible variable (or return a value with grade  $0$ ), we require special treatment. Rules [EVAL-COEFF-COMP-APP-ABS](#), [EVAL-COEFF-COMP-RETURN](#), and [EVAL-COEFF-COMP-SPLIT](#) all have this property, so we modify these rules to require that the relevant grade be nonzero. We also add new rules that apply when the grade is zero. These rules, shown in Figure 6, discard the unused value  $V$  without evaluating it and use a new, untyped, closed value  $\zeta$  in place of the result of evaluating  $V$ . Because values are pure, discarding an unused value does not alter any effects of the program.

However, rule [EVAL-COEFF-COMP-LETIN-RET](#) requires special consideration. Unlike in the rules above, which discard values, this rule discards a *computation* – but because that computation could be effectful, this could change the semantics in unintended ways. Following related work [Dal Lago and Gavazzo 2022; Gavazzo 2018], we reconcile this by adding a notion of  $q \parallel 1$ , which is equivalent to  $q$  when  $q$  is nonzero and  $1$  otherwise. We continue to allow the syntax of the term itself to contain any  $q_2$ , but the rest of the typing rule refers to  $q_2 \parallel 1$  instead. (All other typing rules stay the same.) The evaluation rule, rule [EVAL-LIN-COMP-LETIN-RET](#), follows the same pattern (see Figure 6). Note that this modified evaluation rule introduces a new source of imprecision: we may consume resources to evaluate code without ever using its result, making our final resource accounting more of an overapproximation.

With these modifications, we update our logical relation with a special case for zero resources below. For brevity we show only the changes.

*Definition 4.1 (CBPV with Resource Coeffects: Logical Relation).*

*Closed graded values*

$$\begin{aligned} \mathcal{W}_0[A] &= \{ \zeta \} \\ \mathcal{W}_q[A] &= \mathcal{W}[A] \text{ when } q \neq 0 \end{aligned}$$

*Closed terminals*

$$\begin{aligned} \mathcal{T}[F_q A] &= \{ \text{return}_q W \mid W \in \mathcal{W}_q[A] \} \\ \mathcal{T}[A^q \rightarrow B] &= \{ \text{clo}(\gamma \cdot \rho, \lambda x^q. M) \mid \text{forall } W \in \mathcal{W}_q[A], \\ &\quad ((\gamma \cdot \rho, x \mapsto^q W), M) \in \mathcal{M}[B] \} \end{aligned}$$

Furthermore, we update our semantic typing relation for environments to also include a special case for zero. (The remaining definitions do not change other than to use the resource accounting operational semantics. In particular,  $\mathcal{V}[A]$  still requires the resulting closed value to be in  $\mathcal{W}[A]$ .)

*Definition 4.2 (CBPV with Resource Coeffects: Semantic Typing).*

$$\gamma \cdot \Gamma \vDash \rho \quad = \quad x :^q A \in \gamma \cdot \Gamma \text{ and } q \neq 0 \text{ implies } x \mapsto W \in \rho \text{ and } W \in \mathcal{W}[A]$$

With these updates, we again prove the fundamental lemma.

$$\begin{array}{c}
\text{CBNCOEFF-APP} \\
\frac{\gamma_1 \cdot \Gamma \vdash_{\text{cbncoeff}} e_1 : \tau_1^q \rightarrow \tau_2 \quad \gamma_2 \cdot \Gamma \vdash_{\text{cbncoeff}} e_2 : \tau_1}{\gamma_1 + q \cdot \gamma_2 \cdot \Gamma \vdash_{\text{cbncoeff}} e_1 e_2 : \tau_2} \\
\text{CBNCOEFF-UNBOX} \\
\frac{\gamma_1 \cdot \Gamma \vdash_{\text{cbncoeff}} e_1 : \Box_{q_1} \tau \quad \gamma_2 \cdot \Gamma, x : q_1 \cdot q' \vdash_{\text{cbncoeff}} e_2 : \tau'}{q' \cdot \gamma_1 + \gamma_2 \cdot \Gamma \vdash_{\text{cbncoeff}} \text{unbox}_{q_2} x = e_1 \text{ in } e_2 : \tau'}
\end{array}
\quad
\begin{array}{c}
\text{CBNCOEFF-BOX} \\
\frac{\gamma_1 \cdot \Gamma \vdash_{\text{cbncoeff}} e : \tau}{q \cdot \gamma_1 \cdot \Gamma \vdash_{\text{cbncoeff}} \text{box}_q e : \Box_q \tau}
\end{array}$$

Fig. 7. CBN with coeffect tracking

$$\begin{array}{c}
\text{CBVCOEFF-APP} \\
\frac{q' = q \parallel 1 \quad \gamma_1 \cdot \Gamma \vdash_{\text{cbvcoeff}} e_1 : \tau_1^{q'} \rightarrow \tau_2 \quad \gamma_2 \cdot \Gamma \vdash_{\text{cbvcoeff}} e_2 : \tau_1}{\gamma_1 + q' \cdot \gamma_2 \cdot \Gamma \vdash_{\text{cbvcoeff}} e_1^q e_2 : \tau_2} \\
\text{CBVCOEFF-BOX} \\
\frac{q' = q \parallel 1 \quad \gamma_1 \cdot \Gamma \vdash_{\text{cbvcoeff}} e : \tau}{q' \cdot \gamma_1 \cdot \Gamma \vdash_{\text{cbvcoeff}} \text{box}_q e : \Box_{q'} \tau} \\
\text{CBVCOEFF-UNBOX} \\
\frac{q' = q_2 \parallel 1 \quad \gamma_1 \cdot \Gamma \vdash_{\text{cbvcoeff}} e_1 : \Box_{q_1} \tau \quad \gamma_2 \cdot \Gamma, x : q_1 \cdot q' \vdash_{\text{cbvcoeff}} e_2 : \tau'}{q' \cdot \gamma_1 + \gamma_2 \cdot \Gamma \vdash_{\text{cbvcoeff}} \text{unbox}_{q_2} x = e_1 \text{ in } e_2 : \tau'}
\end{array}$$

Fig. 8. CBV with coeffect tracking

**THEOREM 4.3 (FUNDAMENTAL LEMMA: RESOURCE SOUNDNESS).** *For all  $\gamma, \Gamma$ , if  $\gamma \cdot \Gamma \vdash_{\text{lin}} V : A$ , then  $\gamma \cdot \Gamma \models_{\text{lin}} V : A$ , and for all  $\gamma, \Gamma$ , if  $\gamma \cdot \Gamma \vdash_{\text{lin}} M : B$ , then  $\gamma \cdot \Gamma \models_{\text{lin}} M : B$ .*

As in the previous section, if we unfold the definitions above, this gives us exactly the [soundness theorem](#) we would like.

For example, suppose we type check some computation  $M$  in the context of an inaccessible variable  $x$ . Instantiating the theorem above with this context assures us that evaluation succeeds even when variables are mapped to  $\frac{1}{2}$  in the environment.

**COROLLARY 4.4 (INACCESSIBLE VARIABLE EXAMPLE).** *For all  $M$  and  $B$ , if  $x : \frac{0}{2} A \vdash_{\text{lin}} M : B$ , then there exists some  $T$ , such that  $x \mapsto \frac{0}{2} \vdash_{\text{lin}} M \Downarrow T$ .*

Because the operational semantics does not include any rules for evaluating  $\frac{1}{2}$ , we can conclude that 0-marked variables are never used by the operational semantics. Furthermore, there are no assumptions about the structure of  $\frac{1}{2}$  values, so we can discard them during computation.

#### 4.1 Translation soundness

As with effects, we explore the translation of coeffect-aware CBN and CBV  $\lambda$ -calculi to CBPV. As in our CBPV extension with coeffects, the source type systems are parameterized by a pre-ordered semiring structure of coeffects and combine the typing context with  $\gamma$ , a vector of coeffect annotations that describe the demands on each variable.

The type-and-coeffect system that we consider as the starting point of our CBN translation is adapted from the simple type system of [Choudhury et al. \[2021\]](#) and is similar to the system developed by [Abel and Bernardy \[2020\]](#). The differences between this source language and the related work are minor. The design of our CBV language is inspired by [Dal Lago and Gavazzo \[2022\]](#). To make the comparison clear, we present it as a standard CBV lambda calculus instead of fine-grained CBV. Other changes to the language include the introduction of subcoeffecting, allowing functions to take  $q$  copies of their argument instead of one (and annotating applications

with  $q$ ), and replacing  $q \wedge 1$  with  $q \parallel 1$  to force the evaluation of subterms. (We choose  $q \parallel 1$  over  $q \wedge 1$  to avoid requiring the existence of  $q \wedge 1$  as an axiom of the semiring. The difference is minor.)

The rules for the CBN version of the system appear in Figure 7; the rules for the CBV version are in Figure 8. Most of the rules parallel those of the corresponding terms in CBPV; for brevity, then, we show only the rules for application, boxing and unboxing here. The two languages differ in the application rule. In CBV, we annotate applications with the number of times the function uses its argument. Because the argument will always be evaluated once in CBV, if  $q$  is zero, we force it to be one.

These latter two terms introduce and eliminate the modal type  $\Box_q \tau$ . The introduction form requires grade  $q$  on its argument. When we unbox the argument, the second subterm has access to it with grade  $q_1 \cdot q_2$ . The  $q_1$  comes from when the box was created, and the  $q_2$  comes from the unboxing term, as in let bindings in CBPV. In CBV, we use **letin** in both translations, so we include  $q \parallel 1$  in both rules in an analogous way to its use in rule **COEFF-LETIN**. In CBN, we use **letin** in the translation of **unbox** but not **box**, so we can drop  $q \parallel 1$  from the typing rule for **box**. This imprecision makes sense in the source languages for the same reason it makes sense in CBPV: because we are combining effects and coeffects, we sometimes need to evaluate subterms for their effects even if the results of those subterms are never used.

**4.1.1 Call-by-name translation.** We first consider a call-by-name translation to CBPV. For brevity, we show just the translation of function and box types on the left below and the translation of applications and the **box** and **unbox** terms on the right.

$$\begin{array}{lll} \llbracket \tau_1^q \rightarrow \tau_2 \rrbracket_N & = (U \llbracket \tau_1 \rrbracket_N)^q \rightarrow \llbracket \tau_2 \rrbracket_N & \llbracket e_1 e_2 \rrbracket_N = \llbracket e_1 \rrbracket_N \{ \llbracket e_2 \rrbracket_N \} \\ \llbracket \Box_q \tau \rrbracket_N & = F_q (U \llbracket \tau \rrbracket_N) & \llbracket \mathbf{box}_q e \rrbracket_N = \mathbf{return}_q \{ \llbracket e \rrbracket_N \} \\ & & \llbracket \mathbf{unbox}_q x = e_1 \text{ in } e_2 \rrbracket_N = x \leftarrow^q \llbracket e_1 \rrbracket_N \text{ in } \llbracket e_2 \rrbracket_N \end{array}$$

In this translation, the coeffect on the  $\lambda$ -calculus function type translates directly to the coeffect on the CBPV function type. Furthermore, the modal type  $\Box_q \tau$  is a graded comonad, so it can be translated to the comonad in CBPV, adding the grade to the returner type.

The CBN translation of  $\lambda$  terms is as usual. However, the translation of the box introduction and elimination forms follows from the definition of the CBPV comonadic type. To create a box, we return the thunked translation of the expression. To eliminate a box, we use **letin** to move the thunk to the environment.

**4.1.2 Call-by-value translation.** Next, we define a corresponding CBV translation to CBPV. For brevity, we again show only the translation of function and graded modal types and of applications and the **box** and **unbox** terms.

$$\begin{array}{ll} \llbracket \tau_1^q \rightarrow \tau_2 \rrbracket_v & = U (\llbracket \tau_1 \rrbracket_v^q \rightarrow F_1 \llbracket \tau_2 \rrbracket_v) \\ \llbracket \Box_q \tau \rrbracket_v & = U (F_q \llbracket \tau \rrbracket_v) \\ \llbracket e_1^q e_2 \rrbracket_v & = x \leftarrow^1 \llbracket e_1 \rrbracket_v \text{ in } y \leftarrow^q \llbracket e_2 \rrbracket_v \text{ in } x! y \\ \llbracket \mathbf{box}_q e \rrbracket_v & = x \leftarrow^q \llbracket e \rrbracket_v \text{ in } \mathbf{return}_1 \{ \mathbf{return}_q \parallel 1 x \} \\ \llbracket \mathbf{unbox}_q x = e_1 \text{ in } e_2 \rrbracket_v & = y \leftarrow^q \llbracket e_1 \rrbracket_v \text{ in } x \leftarrow^q y! \text{ in } \llbracket e_2 \rrbracket_v \end{array}$$

As above, we propagate the coeffect from the  $\lambda$ -calculus function type directly to the CBPV function type. Similarly, we propagate the grade in the modal type to the inner returner type and let binding in CBPV.

For applications, we use let bindings to access the translations of the function and the argument. The argument is not thunked in translation, so it is strict, but the function is thunked in translation,

so we must force it before applying it. **box** is also strict in CBV, so its translation first evaluates its argument. The rest of the translation follows its type definition. In CBPV, the computation  $x \leftarrow^1 M$  in **return**<sub>1</sub>  $x$  is equivalent to  $M$ , but the computation  $x \leftarrow^q M$  in **return**<sub>q</sub>  $x$  corresponds to duplicating  $M$   $q$  times in a resource usage coeffect. This propagation of the grade is exactly the feature that we need to translate the **box** term. Like the CBN translation of the modal type, in the CBV translation, the comonadic type is difficult to access. In this translation, **box** must include an extra thunk that is forced in the translation of the **unbox** term, giving us access to the comonadic type **F U**. We must also use the annotation capability of **letin** (twice) to mirror the annotation in the source language. The correctness proofs for both the **CBN** and **CBV** translations follow from the corresponding proofs of the combined system (taking the trivial effect).

## 5 COMBINED EFFECTS AND COEFFECTS

Next, we present a system that tracks both effects and coeffects, by combining the effect system of Section 2 with the resource usage system of Section 4, and adding one new rule.

*Definition 5.1 (Combined type system).* The judgements  $\gamma \cdot \Gamma \vdash_{full} V : A$  and  $\gamma \cdot \Gamma \vdash_{full} M :^\phi B$  refer to the CBPV type system with effect annotations from Figure 1 and coeffect annotations (resource tracking version) from Figure 6.<sup>5</sup>

This type system is a straightforward combination of the systems presented earlier. For example, the typing rule **FULL-LETIN** combines rule **EFF-LETIN** with rule **LIN-LETIN** and includes both the grade vector  $q'_2 \cdot \gamma_1 + \gamma_2$  and the effect  $\phi_1 \cdot \phi_2$  for the computation.

$$\frac{\text{FULL-LETIN} \quad q'_2 = q_2 \parallel 1 \quad \gamma_1 \cdot \Gamma \vdash_{full} M_1 :^{\phi_1} F_{q_1} A \quad \gamma_2 \cdot \Gamma, x :^{q_1 \cdot q'_2} A \vdash_{full} M_2 :^{\phi_2} B}{(q'_2 \cdot \gamma_1) + \gamma_2 \cdot \Gamma \vdash_{full} x \leftarrow^{q_2} M_1 \text{ in } M_2 :^{\phi_1 \cdot \phi_2} B}$$

Similarly, we augment our instrumented operational semantics to track both effects and coeffects.

*Definition 5.2 (Combined Resource Semantics).* The judgements  $\gamma \cdot \rho \vdash_{full} V \Downarrow W$  and  $\gamma \cdot \rho \vdash_{full} M \Downarrow T \# \phi$  refer to the CBPV operational semantics with effect annotations from Figures 2 and 3 and coeffect annotations from Figure 5, with updates for resource tracking from Figure 6.

For example, the **letin** evaluation rule computes the instrumented grade vector and effect and requires that the computation  $M$  be evaluated at least once, as in rule **Eval-LIN-COMP-LETIN-RET**.

$$\frac{\text{EVAL-FULL-COMP-LETIN} \quad q'_2 = q_2 \parallel 1 \quad \gamma_1 \cdot \rho \vdash_{full} M \Downarrow \text{return}_{q_1} W \# \phi_1 \quad \gamma_2 \cdot \rho, x \mapsto^{q_1 \cdot q'_2} W \vdash_{full} N \Downarrow T \# \phi_2}{q'_2 \cdot \gamma_1 + \gamma_2 \cdot \rho \vdash_{full} x \leftarrow^{q_2} M \text{ in } N \Downarrow T \# \phi_1 \cdot \phi_2}$$

We can use this operational semantics to show both effect and coeffect soundness of the combined type system. However, before we do so, we make one more extension to the language.

*Skipping unused discardable computations.* In Section 4, we developed several “zero” rules for discarding unused *values*. But, unused *computations* could not be discarded, because they may have effects. However, in this system, we can identify unused, pure computations, and add a new syntactic form, written  $x \leftarrow^0_\epsilon M$  in  $N$ , indicating that they can be discarded. The typing rule (below

<sup>5</sup> For reasons of space, we do not include the entire combined system here. The full rules of this system are available in the extended version of this paper and in the Coq development.

left) requires that  $M$  be effect free and its result unused in  $N$ .

$$\begin{array}{c}
 \text{FULL-LETIN-ZERO} \\
 \frac{\gamma_1 \cdot \Gamma \vdash_{full} M :^{\varepsilon} F_q A \quad \gamma_2 \cdot \Gamma, x :^0 A \vdash_{full} N :^{\phi} B}{\gamma_2 \cdot \Gamma \vdash_{full} x \leftarrow_{\varepsilon}^0 M \text{ in } N :^{\phi} B}
 \end{array}
 \quad
 \begin{array}{c}
 \text{EVAL-FULL-COMP-LETIN-ZERO} \\
 \frac{\gamma \cdot \rho, x \mapsto^0 \bot \vdash_{full} N \Downarrow T \# \phi}{\gamma \cdot \rho \vdash_{full} x \leftarrow_{\varepsilon}^0 M \text{ in } N \Downarrow T \# \phi}
 \end{array}$$

Furthermore, the operational semantics of this new expression form (above right) does not evaluate  $M$ . Instead it uses the junk value  $\bot$  for the result of this computation.

To see this rule in action, consider the CBV translation of an expression  $y_2 (y_1 x)$ , where  $y_2$  is a constant function and  $y_1$  is pure. In this case, the type system can observe that  $x$  does not contribute to the final result when the application of  $y_1$  to  $x$  is marked as discardable.

$$x :^0 A, y_1 :^0 U_{\varepsilon} (A^1 \rightarrow F A), y_2 :^1 U_{\phi} (A^0 \rightarrow B) \vdash_{full} z \leftarrow_{\varepsilon}^0 y_1! x \text{ in } y_2! z :^{\phi} B$$

*Soundness proof for discardable computations.* We next show that discarding unused values and unused pure computations does not change the evaluation behavior of computations. To do so, we need the following properties that state that  $\varepsilon$  is the minimum element of the effect preorder.<sup>6</sup>

*Definition 5.3 (Min identity).* (1) For all  $\phi, \varepsilon \leq_{eff} \phi$  (2) For all  $\phi, \phi \leq_{eff} \varepsilon$  implies  $\phi = \varepsilon$ .

To prove that discarding is sound, we establish a relation between our combined resource semantics and one that does not discard terms.

*Definition 5.4 (Combined nondiscarding semantics).* The judgement  $\gamma \cdot \rho \vdash_{gen} M \Downarrow T \# \phi$  refers to the operational semantics that is the combination of CBPV with effect annotations from Figure 3 and coeffect annotations from Figure 5, with the modified rule **FULL-LETIN** that always evaluates computations. This semantics does not include rules that discard values or computations and uses rule **EVAL-FULL-COMP-LETIN** to evaluate the new **letin** expression.

Our simulation lemma states that for closed boolean-valued computations<sup>7</sup>, evaluating with either the nondiscarding semantics (Definition 5.4) or with the resource semantics (Definition 5.2) produces the same result and the same effect.

**LEMMA 5.5 (RESOURCE SIMULATION).** *If  $\emptyset \cdot \emptyset \vdash_{full} M :^{\phi} F_1$  (**unit + unit**) then either*

- (1)  $\emptyset \cdot \emptyset \vdash_{gen} M \Downarrow \text{return}_1(\text{inl } ()) \# \phi_1$  and  $\emptyset \cdot \emptyset \vdash_{full} M \Downarrow \text{return}_1(\text{inl } ()) \# \phi_1$  or
- (2)  $\emptyset \cdot \emptyset \vdash_{gen} M \Downarrow \text{return}_1(\text{inr } ()) \# \phi_1$  and  $\emptyset \cdot \emptyset \vdash_{full} M \Downarrow \text{return}_1(\text{inr } ()) \# \phi_1$ .

This simulation lemma is a corollary of a much more general result—the fundamental lemma for a binary logical relation between computations that are evaluated with the two different semantics. This relation, shown below, is mutually defined with relations between closed values and closed terminals (not shown, but available in the extended version and in the Coq development).

$$\begin{aligned}
 \mathcal{M}[[B]]^{\phi} = \{ & (\gamma \cdot \rho_1, M_1, \gamma \cdot \rho_2, M_2) \mid \gamma \cdot \rho_1 \vdash_{gen} M_1 \Downarrow T_1 \# \phi_1 \text{ and } \gamma \cdot \rho_2 \vdash_{full} M_2 \Downarrow T_2 \# \phi_1 \\
 & \text{and } (T_1, T_1) \in \mathcal{T}[[B]]^{\phi_2} \text{ and } (T_1, T_2) \in \mathcal{T}[[B]]^{\phi_2} \text{ and } \phi_1 \cdot \phi_2 \leq_{eff} \phi \}
 \end{aligned}$$

Using this relation, we define a binary version of the semantic typing relation. Two environments  $\rho_1$  and  $\rho_2$  are related when the closed values in the first environment are related to themselves, and, if the usage is nonzero, the closed value in the second environment is related to the first. The first condition ensures that we know something about closed values in the first relation even when the corresponding value in the second relation has been discarded in the resource semantics.

<sup>6</sup> These properties hold for **tick** effects, but we have not used them before now. <sup>7</sup> The system in our extended version and Coq development includes sums, necessary to implement the boolean type.

*Definition 5.6 (Semantic double typing).*

$$\begin{aligned}
\gamma \cdot \Gamma \models \rho_1 \sim \rho_2 &= x :^q A \in \gamma \cdot \Gamma \text{ implies } x \mapsto W_1 \in \rho_1 \text{ and } (W_1, W_1) \in \mathcal{W}[A] \text{ and} \\
&\quad (q = 0 \text{ or } ((q \neq 0) \text{ and } x \mapsto W_2 \in \rho_2 \text{ and } (W_1, W_2) \in \mathcal{W}[A])) \\
\gamma \cdot \Gamma \models_{full} V_1 \sim V_2 : A &= \text{forall } \rho_1, \rho_2, \gamma \cdot \Gamma \models \rho_1 \sim \rho_2 \\
&\quad \text{implies } \gamma \cdot \rho_1 \vdash_{full} V_1 \Downarrow W_1 \text{ and } \gamma \cdot \rho_2 \vdash_{full} V_1 \Downarrow W_1 \\
&\quad \text{and } (W_1, W_1) \in \mathcal{W}[A] \text{ and } (W_1, W_2) \in \mathcal{W}[A] \\
\gamma \cdot \Gamma \models_{full} M_1 \approx M_2 :^\phi B &= \text{forall } \rho_1, \rho_2, \gamma \cdot \Gamma \models \rho_1 \sim \rho_2 \text{ implies } (\gamma \cdot \rho_1, M_1, \gamma \cdot \rho_2, M_2) \in \mathcal{M}[B]^\phi
\end{aligned}$$

The fundamental theorem shows that this binary relation is reflexive.

**THEOREM 5.7 (FUNDAMENTAL LEMMA: SIMULATION).**

- (1) For all  $\gamma, \Gamma$ , if  $\gamma \cdot \Gamma \vdash_{full} V : A$ , then  $\gamma \cdot \Gamma \models_{full} V \sim V : A$ , and
- (2) for all  $\gamma, \Gamma$ , if  $\gamma \cdot \Gamma \vdash_{full} M :^\phi B$ , then  $\gamma \cdot \Gamma \models_{full} M \approx M :^\phi B$ .

This fundamental lemma combines and generalizes prior results of this paper. In particular, it shows the effect-and-coeffect soundness of the combined type system with respect to both the nondiscarding and resource accounting semantics—the effects and coeffects of the evaluation are bounded by the type system. For clarity, we also separately show effect-and-coeffect soundness of the combined type system in the Coq development.

*CBN and CBV translations.* Finally, we have defined CBN and CBV with combined effects and coeffects and have proved the soundness of translations to the combined CPBV type system.

**THEOREM 5.8 (CBN AND CBV TRANSLATION CORRECTNESS).**

- (1) For all  $\gamma, \Gamma, e, \tau$ , if  $\gamma \cdot \Gamma \vdash_{cbncoeff} e : \tau$ , then  $\gamma \cdot \llbracket \Gamma \rrbracket_N \vdash_{full} \llbracket e \rrbracket_N :^\epsilon \llbracket \tau \rrbracket_N$ , and
- (2) For all  $\gamma, \Gamma, e, \tau$ , if  $\gamma \cdot \Gamma \vdash_{cbvcoeff} e : \tau$ , then  $\gamma \cdot \llbracket \Gamma \rrbracket_v \vdash_{full} \llbracket e \rrbracket_v :^\epsilon \mathbf{F}_1 \llbracket \tau \rrbracket_v$ .

Like 2.5, these proofs follow by simple induction, so we omit them here; however, they can be found in the Coq development.

## 6 RELATED WORK

Call-by-push-value (CBPV) was originally developed by Levy [2001]. Forster et al. [2019] mechanized proofs of its metatheoretic properties and translation soundness and inspired our mechanized proofs. Current applications of CBPV include modeling compiler intermediate languages [New 2019; Rizkallah et al. 2018], understanding the role that polarity plays in bidirectional typing [Dunfield and Krishnaswami 2021] and subtyping [Lakhani et al. 2022], and incorporating effects into dependent type theories [Pédrot and Tabareau 2019; Pédrot et al. 2019].

*CBPV and effects.* Call-by-value languages with effect tracking go back to FX [Lucassen and Gifford 1988]. Wadler and Thiemann [2003] showed the connection between graded monads and effects by translating the effect system of Talpin and Jouvelot [1994] to a language that isolates effects using graded monads. Our monadic effect language is inspired by this paper, generalized following Katsumata [2014]. In this paper, our translation is the reverse of Wadler and Thiemann, mapping a language with graded monads to an effect-style extension of CBPV. Like us, Rajani et al. [2021] use a logical relation to show the soundness of their monadic cost analysis.

Although CBPV has often been used to model the semantics of effects, its type system has only rarely been extended with effect tracking. The type system that we present in Section 2 is most similar to MAM (multi-adjunctive metalanguage) from Forster et al. [2017], which builds on Kammar and Plotkin [2012] and Kammar et al. [2013]. Forster et al. use MAM to compare the relative expressiveness of effect handlers, monadic reflection and delimited control. As in our system, MAM annotates the thunk type with an effect annotation.



**Wuttke [2021]** defines a cost-annotated version of CBPV by annotating the thunk type in CBPV with a bound  $[a < I]$  that limits the number of times that thunks can be forced. This work includes both call-by-value and call-by-name translations from cost-annotated PCF terms to cost-annotated CBPV. For expressiveness, the system includes subtyping and indexed types.

Some extensions of CBPV annotate effects on  $F A$  instead of  $U B$ . These systems isolate effects so that they need not be tracked by the typing judgement. Extended Call-by-Push-Value (ECBPV) [McDermott and Mycroft 2019] adds call-by-need evaluation to CBPV and layers an effect system to augment equational reasoning. This system uses an operation  $\langle \phi \rangle B$  to extend the effect annotation to other computation types, combining effects in returner types and pushing effects to the result type of functions and inside with-products. Rioux and Zdancewic [2020] tracks divergence. In this system, the sequencing operation requires that the annotation on the returner type be less than or equal to any annotation on the result of the continuation.

*Coeffects.* Type systems that track coeffects were introduced by Brunel et al. [2014]; Ghica and Smith [2014]; Petricek et al. [2014] and developed by Abel and Bernardy [2020]; Orchard and et al. [2022]; Orchard et al. [2019]. Early applications were for bounded linearity; but these systems have also been used for tracking information flow in differential privacy [Reed and Pierce 2010], dynamic binding [Nanevski 2003] and have also been applied for resource usage in Haskell [Bernardy et al. 2017] and irrelevance in dependently-typed languages [Abel et al. 2023; Atkey 2018; Choudhury et al. 2021]. Petricek et al. [2014] give a number of additional examples, including dataflow (the number of past values needed in a stream processing language) and data liveness (whether references to a variable are still needed).

As in our work, all prior semantics that “count” uses of variables are imprecise and allow execution to waste resources. Abel et al. [2023] and Choudhury et al. [2021] use a heap-based operational semantics to show coeffect soundness for a language with a small-step, call-by-name semantics, but do not consider the interactions with effects. Bianchini et al. [2023] proves resource soundness for a fine-grained call-by-value language using a big-step semantics. Their language includes a nontermination effect through recursive functions and recursive types. Their soundness proof is based on a heap-based semantics, which must simultaneously evaluate  $q$  copies of an expression. In contrast, because our environment-based semantics can separate the resource usage of a subexpression from the rest of the computation, our semantics can use multiplication instead of multi-usage. For consistency with effects, several rules of their type system require that the number of copies of the produced value to be nonzero, similar to our use of  $q \parallel 1$ .

Dal Lago and Gavazzo [2022] also explore the addition of effects and coeffects to a fine-grained call-by-value language. They also force the `letin` term to count the coeffects of the computation at least once, through the use of  $q \wedge 1$ . (This rule is derived from Gavazzo [2018].) Unlike this work, they give a denotational semantics based on a monadic evaluation function and do not track resource usage. Their main result is a definition of a program relation in the presence of effects and coeffects. Their approach is to refine a standard logical relation with *relators* and *corelators* that capture the interaction of effects and coeffects with the language semantics. This approach is more general than ours, which is tied to a specific effect and coeffect.

*CBPV and linearity.* Our extension of CBPV with coeffect typing is novel and inspired by the duality with effects. The most related systems are those involving linearity in the context of low-level or compiler intermediate languages. Schöpp [2015] develops a low-level language, similar to CBPV, that includes linear operations in its type system. The *enriched effect calculus* [Egger et al. 2009, 2012] extends a type theory for computational effects, with primitives from linear logic. Ahmed et al. [2007] augment a variant of typed assembly language with linear types.



*Interactions between effects and coeffects.* Several systems describe interactions between effects and coeffects. [Nanevski \[2003\]](#) uses comonads to guard the usage of local state (dynamic binding) and monads to guard the usage of global state. In each case the type system tracks the set of locations can be safely read and updated.

[Gaborardi et al. \[2016\]](#) present a combined calculus featuring effects and coeffects. Unlike this work, their lambda calculus isolates effects and coeffects using graded monadic and comonadic modal types. A key feature of their system are “graded distributive laws”, that permit interactions between the monad and comonad. The exact interactions are mediated by operations determined by the particular effects and coeffects being modeled. For example, we could distribute a term of type  $\Box_3 T_{2_{\text{eff}}} \tau$  into a term of type  $T_{6_{\text{eff}}} \Box_3 \tau$ . That is, it could turn 3 copies of a monad which ticks twice and returns a term of type  $\tau$  into a monad which ticks 6 times then returns 3 copies of the term.

In future work, we hope to add distributivity. Such distributive laws would not change the structure of the computation as described by the returner and thunk type constructors, but rather redistribute grades on types in the form  $F_{q_1} U_{\phi} F_{q_2} A$  or  $U_{\phi_1} F_q U_{\phi_2} B$ . However, we have yet to determine what sorts of rearrangement are sound in this context.

## 7 CONCLUSION AND FUTURE WORK

In this paper we have annotated the ambient monad and comonad of CBPV to statically track effects and coeffects. We have presented these extensions separately to provide a gentle introduction, before developing a combined calculus that tracks both simultaneously. We have identified semantic subtleties in resource tracking and have developed an alternative semantics that better describes our understanding of this coeffect. We have proven soundness for all versions of our type system, identifying the required assumptions of the effect and coeffect algebras. To make sure that our designs are expressive, we have shown the standard translations from call-by-value and call-by-name lambda calculi into call-by-push-value preserve tick and resource tracking with our system. By exploring both effects and coeffects together, we were also able to observe similarities between these dual notions, and, more importantly, identify their differences.

However, this work is only the starting point for investigation in this space. The natural next step is to go beyond a single effect (tick) and single coeffect (resource usage) to develop a more general structure for extensions of CBPV, perhaps based on algebraic effects [\[Plotkin and Pretnar 2008\]](#) or effect signatures [\[Katsumata 2014\]](#). This structure would allow us to verify that our rules stay general in the presence of other effects, such as nontermination and state, or other coeffects, such as information-flow tracking and differential privacy.

We can also extend this work by adding language features that interact with effect and coeffect tracking, such as polymorphism, indexed or dependent types, and quantification over effects and coeffects. Subtyping would capture the idea that the type  $U_{\phi_1} B$  is a subtype of  $U_{\phi_2} B$  when  $\phi_1 \leq_{\text{eff}} \phi_2$ , and that the type  $F_{q_1} A$  is a subtype of  $F_{q_2} A$  when  $q_2 \leq_{\text{co}} q_1$ . Finally, we would like to explore the practical concerns of this system in more depth, focusing on how users or compilers might make effective use of the statically tracked information.

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