LOUGHBOROUGH UNIVERSITY

Department of Mathematical Sciences

M.Sc and M.Math - Coursework Cover Sheet

Module Code: **22MAP111**

Module Name: Mathematical Modelling I

Coursework Part: One

Internal Examiner: Andrew Archer (a.j.archer@lboro.ac.uk, office SCH.1.25)

Date Set: Week 2, Semester 1

Deadline for Submission: 2:00pm, Monday, Week 7, Semester 1

MARK SCHEME: This item of coursework counts 40% towards the overall module mark. This consists of 80% for the written coursework and 20% for the oral presentation.

Of the 80% for written coursework:

- **Up to 6 marks will be awarded on the basis of your contribution as assessed at the weekly group meetings;**

  One of the Lecturers will meet with your group each week and will want to see evidence that all are contributing. When it becomes clear that some individuals are leading the group and contributing more to the work, their efforts will be rewarded by a higher mark in this category.

- **Up to 12 marks will be awarded for the written presentation, i.e., the layout, presentation of the results, good use of English, correct grammar and spelling;**

  Ask yourself the following questions about your report and if the answer is "yes" to all, then you are on the right track:

  Do the title and the abstract summarise the topic and its importance clearly?

  Does the introduction
  – put the work in context?
  – present the main conclusions?
  – give the reader a sense of the report's structure?

  Do the main statements and points of the report come across as such?

  Does the outline given by the structure of sections, subsections, and paragraphs accurately lay out the hierarchy of ideas in the report?

  Is the internal organisation of paragraphs and sections good?

  Is the writing clear?

Is the prose engaging?

Is the report free of grammatical and typographical errors?

Are figures labeled clearly and have a caption?

Do margins and spacing look good?

Is the numbering of equations clear and logical?

Are all cross-references and citations correct?

Is the mathematics typeset correctly (including punctuation)?

- **Up to 14 marks will be awarded for the content, the model itself, the amount of work done, modifications to the model etc.;**

  Does the report describe the model(s) constructed in the right depth?

  Are the models described clearly?

  Does the author exhibit a good understanding of the material?

  Are all sources used properly (don't paraphrase!) and referenced correctly?

  Are the motivations from mathematics and/or applications clear?

  Are all definitions correct?

  Is each key idea clearly explained?

  Are the logical relationships among the key ideas clear?

  Are the ideas illustrated with useful and appropriate examples, diagrams and/or figures?

  Is the level of the presentation appropriate?

- **Up to 8 marks will be awarded for the interpretation of the results, the discussion and conclusions and suggestions for further work.**

  Are all results interpreted and conclusions drawn?

  Are all the figures explained?

  Are the figures appropriately chosen?

  Are the appropriate number of equations displayed (neither too many nor too few)?

  Are the equations interpreted and explained?

  Are the different sets of results linked and the connections between things explained?

  Is there a higher level of analysis (average quantities, solutions measures, etc versus the different model parameters) and not just the basic output from the model?

Mathematical Modelling I

Competing species

Consider a lake that has two species of fish living in it. Every year fishermen are allowed to remove fish from the lake during the summer. The two species of fish compete with each other for the same food source in the lake. Make a model for how the number of each type of fish varies over time, and make a recommendation for how many of each type of fish the fishermen should take in order for the fish populations to be sustained.

Hints

A commonly employed model for the time variation of the population of a single species in a region of limited resources, is the logistic differential equation. You are asked to extend this model to consider the case of two co-existing species which compete for limited resources. Start by examining the possible equilibrium states and how these are affected by the choice of parameters. Then include the effect of the fishermen removing fish during part of the year.
The logistic equation has the form

$$\frac{dP}{dt} = r(P_M - P)P$$

where $P(t)$ is the population at time t, P_M is the maximum population that can be sustained (the carrying capacity) and r is a growth parameter. You should set up an equation for each of the two different populations $P(t)$ and $Q(t)$ bearing in mind that both populations affect the carrying capacity. Express your equations in non-dimensional form so as to reduce the number of parameters to a minimum.
Remember to consider citing relevant literature when modelling parameter values or extending the models, and think about your target audience (who are you making recommendations to?) when writing your report.

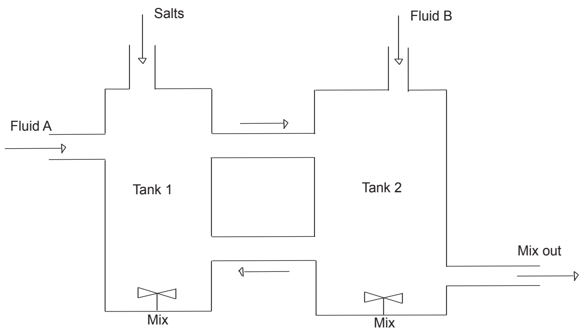
Modelling group 1

Student 1
Student 2
Student 3
Student 4

Mathematical Modelling I

Mixing chemicals

Anti-icing fluid is sprayed onto aircraft in winter in order to stop ice building up on the aeroplane. The fluid is a mixture of salts, alcohols and glycols. Assume that this fluid is made in the mixing tank illustrated below and that it is made by mixing two fluids together with the salts which are periodically added to tank 1 in the solid form. There are stirrers in both tanks that keep the fluid in each tank well mixed and there is also mixing between the two by pumping fluid back and forth between the two tanks. Fluids A and B are pumped in at a constant rate and the mixture is also pumped out of tank 2 at a constant rate. Construct a model for the system and examine how the concentration of the salt in the outflow mixture varies with time depending on how regularly the salts are added. Make a recommendation for how regularly the salts should be added to maintain a certain minimum concentration in the product.



Hints

Consider first a single tank with just fluid A and fluid B being pumped into the tank and the mixture being pumped out. In this tank if you define $y_A(t)$ as the amount of fluid A at time t in the tank, then the time evolution of this will be governed by the equation

$$\frac{\mathrm{d}y_A(t)}{\mathrm{d}t} = a_{in}(t) - a_{out}(t)$$

where $a_{in}(t)$ is the amount of A going in per unit time at time t and $a_{out}(t)$ is the amount leaving. After this you should include the salts and model periodically putting the salts in via a series of Dirac delta functions. As a final stage add the second tank and the effect of pumping the fluid back and forth between the two tanks.
Remember to consider citing relevant literature when modelling parameter values or extending the models, and think about your target audience (who are you making recommendations to?) when writing your report.

Modelling group 2

Student 1
Student 2
Student 3
Student 4

Mathematical Modelling I

Antibiotics in the bloodstream

A patient takes a course of antibiotics by ingesting a pill at discrete intervals. However, the amount in the bloodstream varies continuously with time, since the antibiotic is adsorbed continuously. You are asked to set up a mathematical model to determine the amount of antibiotic in the bloodstream as a function of time. You should then go on to modify your model to consider a two stage process in which the pill is first ingested into the stomach and from there it passes into the bloodstream over time. Derive model equations for the variation of the amount of antibiotic in the stomach and in the bloodstream as a function of time.

Hints

The ingestion of a pill at discrete intervals can be modelled mathematically by means of a series of Dirac delta functions. The equation for the amount of antibiotic in the blood at time t, $b(t)$, will take the form:

$$\frac{db(t)}{dt} = p(t) - q(t),$$

where $p(t)$ is the amount of antibiotic going into the blood at time t and $q(t)$ is the amount leaving the blood. In the one stage model $p(t)$ will be a series of Dirac delta functions. For both the one and two stage models you should obtain the solution after any number r of pills have been taken and after the course of N pills has been completed. In the two stage process model you should assume that the amount leaving the stomach enters into the bloodstream. Consider also the possibility that the rate of transfer from stomach to blood depends on the amount in the stomach and in the blood.
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Modelling group 3

Student 1
Student 2
Student 3
Student 4

Mathematical Modelling I

Heating of a factory

A factory workshop is in operation 24 hours a day. In an attempt to maintain a constant working temperature the workshop is fitted with controllable heaters which are capable of delivering heat energy at a specified rate. Set up a mathematical model of the system and examine its behaviour for different settings of the controls.

Hints

The equation for the heat energy in the factory $H(t)$ as a function of time t will take the form:

$$\frac{dH(t)}{dt} = p(t) - q(t),$$

where $p(t)$ is the amount of heat going into the factory from the heater and $q(t)$ is the amount leaving through the walls. Choose the simplest possible model in which you assume uniform temperature throughout the factory, so that the temperature in the factory $T(t) = cH(t)$, where c is a constant. Also, assume that the heat loss $q(t)$ depends on the temperature difference between the factory and the external temperature. You should take into account the variation of external temperature with time over a twenty four hour period (e.g. using a sin function). Set up the equation for the temperature as a function of time and examine the solution for different variations in external temperature. Assume that the heaters supply heat at a constant rate when switched on and examine how the behaviour depends on this rate.

Remember to consider citing relevant literature when modelling parameter values or extending the models, and think about your target audience (who are you making recommendations to?) when writing your report.

Modelling group 4

Student 1
Student 2
Student 3
Student 4

Mathematical Modelling I

Design of speed bumps

Speed bumps, or sleeping policemen, are in use on the campus roads in an attempt to keep the speed of vehicles down to 20 m.p.h. You are asked to determine the dimensions of the bumps and their spacing so as to accomplish this purpose on a straight stretch of road.

Hints

Take a simple model of the car consisting of a mass sitting on top of a solid wheel and joined to it by a spring (the suspension) in parallel with a viscous dashpot (the shock absorber). As the car passes over the speed bump there is an imposed motion of the wheel in the vertical direction and this depends on the shape of the bump and the speed with which the car travels horizontally. Write down the equation of motion of the mass and examine the solution for different shapes of bump e.g part of a parabola, portion of a sine curve, etc. Carry out calculations for different horizontal velocities with each shape. Find out from the literature what humans find uncomfortable, is it large acceleration, large displacement or large velocity?

A modification to the model would be to treat the wheel as a second mass which is in contact with the road through a spring (the tyre) and with the car mass joined to it through the spring/dashpot system. Alternatively you might consider the mass suspended between two solid wheels (one ahead of the other) through spring/dashpot combinations.

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Modelling group 5

Student 1
Student 2
Student 3
Student 4

Mathematical Modelling I

Insulating a pipe

A circular cylindrical pipe of outer radius a carries a fluid at a constant temperature. The pipe is surrounded with a cylinder of insulating material of inner radius a and outer radius b, in order to reduce the heat loss to the surroundings. You are required to make a recommendation for how thick $(b-a)$ the insulation on the pipe should be.
After this, extend the model to consider the case when the pipe is covered with two layers of different types of insulating materials. Make a recommendation for how thick these layers should be.

Hints

You may assume Fourier's law of heat conduction which states that the rate of heat flow radially out through the insulation is given by:

$$J = -kA(r)\frac{dT(r)}{dr}$$

where k is the thermal conductivity, r is the distance from the centre of the pipe, A is the area and T is the temperature.
You should also assume that the system has been in operation long enough for a steady state to be set up. Assume that the inner surface of the insulation is at the same constant temperature as the fluid and that the ambient temperature is constant and less than that of the fluid. At the outer surface of the insulation, the heat is lost according to Newton's law of cooling. Treat a as fixed and examine the variation of heat loss with thickness for different values of the conductivity of the insulating material and for different values of the heat transfer coefficient associated with Newton's law of cooling.
Remember to consider citing relevant literature when modelling parameter values or extending the models, and think about your target audience (who are you making recommendations to?) when writing your report.

Modelling group 6

Student 1
Student 2
Student 3
Student 4

Mathematical Modelling I

De-icing wing mirrors

Ice build-up (frost) on car wing mirrors can be removed by a system of heating elements built into the mirror. You are asked to construct a model to predict the time taken to melt an ice layer of specified thickness in an ambient temperature at some given value below freezing point. You may assume that the heaters produce heat energy at a known rate per unit area of the mirror surface.

Hints

The equation for the heat energy in the ice $H(t)$ as a function of time t will take the form:

$$\frac{dH(t)}{dt} = p(t) - q(t),$$

where $p(t)$ is the amount of heat going into the ice from the heater and $q(t)$ is the amount leaving, either through the fact that it is in contact with the cold air and the cold mirror, or due to the fact that when ice melts, there is a latent heat of melting. Initially assume that when the ice melts, it runs off the surface straight away. However, at a later stage you might also want to consider what is the influence of the water remaining on the ice surface and maybe also how it evaporates.
Remember to consider citing relevant literature when modelling parameter values or extending the models, and think about your target audience (who are you making recommendations to?) when writing your report.

Modelling group 7

Student 1
Student 2
Student 3
Student 4

Mathematical Modelling I

Sourdough

Sourdough can be made at home. Initially, you mix flour and water and then wait. Since flour naturally contains a variety of yeasts and bacteria, these then start to metabolise the sugars in the flour and reproduce. More flour and water needs to be added to the mix to 'refresh' it. Then, periodically, some of the culture can be taken and used instead of yeast in bread-making. You are asked to construct a model for this whole process, taking into account the regular (periodic) additions of flour and water and also the regular removal of a portion of the mixture for the bread making.

Hints

The adding and removing of yeast at discrete intervals can be modelled mathematically by means of a series of Dirac delta functions. The equation for the amount of yeast in the culture at time t, $y(t)$, will take the form:

$$\frac{dy(t)}{dt} = p(t) - q(t),$$

where $p(t)$ is the amount of yeast added at time t (both due to reproduction and due to more flour being added) and $q(t)$ is the amount removed. This model can be extended by coupling this equation to another one for the amount of flour $f(t)$ in the system at time t. Remember to consider citing relevant literature when modelling parameter values or extending the models, and think about your target audience (who are you making recommendations to?) when writing your report.

Modelling group 8

Student 1
Student 2
Student 3
Student 4