IMA Journal of Numerical Analysis (2020) **40**, 2871–2897
doi:10.1093/imanum/drz048
Advance Access publication on 27 January 2020

A lowest-order staggered DG method for the coupled Stokes–Darcy problem

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[Received on 28 April 2019; revised on 26 August 2019]

In this paper we propose a locally conservative, lowest-order staggered discontinuous Galerkin method for the coupled Stokes–Darcy problem on general quadrilateral and polygonal meshes. This model is composed of Stokes flow in the fluid region and Darcy flow in the porous media region, coupling together through mass conservation, balance of normal forces and the Beavers–Joseph–Saffman condition. Stability of the proposed method is proved. A new regularization operator is constructed to show the discrete trace inequality. Optimal convergence estimates for all the approximations covering low regularity are achieved. Numerical experiments are given to illustrate the performances of the proposed method. The numerical results indicate that the proposed method can be flexibly applied to rough grids such as the trapezoidal grid and h-perturbation grid.

Keywords: lowest-order SDG method; Stokes equations; Darcy's law; Beavers–Joseph–Saffman condition; regularization operator; low regularity.

1. Introduction

The staggered discontinuous Galerkin (SDG) method pioneered by Chung & Engquist (2006, 2009) has been successfully applied to a wide range of partial differential equations arising from practical applications; see, for example, Chung *et al.* (2018) and the references therein. SDG methods possess many distinctive features such as local and global conservations, optimal convergence and feasibility of the local postprocessing scheme. In addition the staggered continuity property naturally gives rise to a flux condition across the interface, whereas an artificial numerical flux needs to be introduced in other DG methods. However, due to the special construction of the basis functions, SDG methods result in more degrees of freedom compared to discontinuous Galerkin methods and other conforming Galerkin methods. It will be highly appreciated if one can reduce the number of degrees of freedom and at the same time maintain the desirable properties of SDG methods. Quadrilateral grids can represent highly irregular geometries with relatively few degrees of freedom, and practically the quadrilateral grids are not always orthogonal. Therefore, the goal of this paper is to develop an SDG method for the coupled Stokes–Darcy model that can be flexibly applied to quadrilateral and polygonal grids of arbitrary shapes and have minimal dimension, which can facilitate SDG methods for more general applications.

The coupled Stokes–Darcy model has extensive applications such as petroleum engineering, hydrology and environmental sciences. There have been a large number of works devoted to developing suitable numerical methods for this coupled model. The commonly used technique to develop finite element formulations for the coupled Stokes–Darcy problem arises from appropriate combinations of stable elements for the fluid flow and for the porous medium flow, which can be traced back to Discacciati *et al.* (2002); Layton *et al.* (2003). In Discacciati *et al.* (2002) continuous finite element methods are employed for both regions. In Layton *et al.* (2003) a combination of a continuous finite element method for the Stokes region and a mixed finite element method for the Darcy region is studied. In Rivière & Yotov (2005) a combination of the DG method for the Stokes region and a mixed finite element method for the Darcy region is investigated. Then in Rivière (2005) a locally conservative DG method is proposed for the coupled Stokes–Darcy problem. In a more recent work (Gatica *et al.*, 2011) a fully-mixed finite element method for the coupled Stokes–Darcy problem is developed. On the other hand, among other methods, mortar finite element methods, conforming finite volume element methods, discontinuous finite volume element methods, stabilized methods and weak Galerkin methods (Burman & Hansbo, 2007; Masud, 2008; Girault *et al.*, 2014; Chen *et al.*, 2016; Li *et al.*, 2018b,c,d) have been developed to solve the coupled Stokes–Darcy problem.

In this paper our purpose is to develop lowest-order SDG methods on general quadrilateral and polygonal meshes for the coupled Stokes–Darcy model. The primal mixed formulation is considered in both domains. Namely, there are three variables in the fluid region: stress, velocity and pressure and there are two variables in the porous media region: velocity and pressure. By doing so, on the one hand further unknowns of physical interest are introduced, and on the other hand, the same family of finite element spaces for the stress in the fluid region and velocity in the porous media region, for the velocity in the fluid region and pressure in the porous media region are utilized. Then for a given initial partition for both domains each quadrilateral or polygonal mesh is divided into the union of triangles by connecting the center point to the nodal vertices. We consider piecewise constant functions for all the approximations, and the continuity of the functions is staggered on the interelement boundaries. In other words, on each interelement boundary, if one of the functions is continuous then the other can be discontinuous. By this construction the resulting SDG finite element spaces are locally continuous and of minimal dimension. Based on the staggered triangulation we define the SDG formulations in both domains, where the interface condition is imposed strongly without resort to Lagrange multipliers. The local conservation property of the proposed method stems from the use of staggered variables. Since the finite element functions are composed of piecewise constant functions the computation is quite efficient. In addition the meshes from the two regions may be nonmatching on the interface. Another feature of our method is that if the quadrilateral is reduced to a square, then the mass matrices for the stress in the fluid region and the velocity in the porous medium region become just a diagonal matrix.

The existing works for the coupled model only establish the error analysis with full regularity. Problems with low regularity are particularly interesting in practical applications. Therefore, we attempt to develop *a priori* error analysis covering low regularity. A new regularization operator is proposed to show the discrete trace inequality. We employ the duality argument to show the L^2 error estimates for the velocity in the fluid region and for the pressure in the porous medium region. Since piecewise constant functions are employed to approximate the velocity in the fluid region and pressure in the porous media region, the interpolation error estimates in L^2 norm for the corresponding velocity and pressure are only optimal in rate, not in regularity. Due to the reduced convergence rate with respect to the regularity it is also important to treat the terms present in the duality argument properly in order to achieve optimal convergence rates. Allowing rough grids is highly appreciated from a practical point of view; to verify the flexibility of the proposed method, we test a trapezoidal grid and h perturbation of smooth grids. The results confirm optimal convergence for all the approximations on rough grids. To the best of our knowledge this is the first result on L^2 error analysis of velocity in the fluid region and pressure in the porous medium region covering low regularity for the coupled Stokes–Darcy problem.

The rest of the paper is organized as follows. In the next section we briefly introduce the coupled Stokes–Darcy model and derive the corresponding lowest-order SDG method. Then in Section 3 the stability of the discrete scheme is proved and *a priori* error estimates are presented for all the

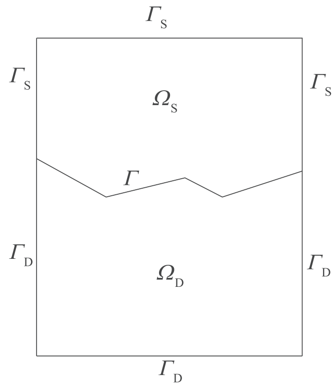

FIG. 1. Coupled domain with interface Γ.

approximations. In Section 4 numerical experiments are carried out to confirm the theoretical results. Finally, a conclusion is given at the end of the paper.

2. SDG method

In this section we will develop a lowest-order SDG method for the coupled Stokes–Darcy problem. Some basic notation is introduced first, then a weak formulation is given by employing integration by parts and proper treatments of the interface conditions. We emphasize that the partitions of subdomains are not necessarily matching across the interface.

2.1 *Preliminaries*

We consider the coupled Stokes–Darcy model in a bounded domain $\Omega \subset \mathbb{R}^2$, consisting of a fluid region Ω_S and a porous medium region Ω_D, with interface $\Gamma = \partial\Omega_S \cap \partial\Omega_D$. Define $\Gamma_i = \partial\Omega_i \backslash \Gamma$ for $i = S, D$. Moreover, $\boldsymbol{n}_i$ denotes the outward unit normal vector to $\partial\Omega_i$, and $\boldsymbol{n}_{12}$ denotes the unit normal vector of Γ pointing from Ω_S to Ω_D and $\boldsymbol{t}$ denotes the corresponding unit tangential vector. Fig. 1 gives a schematic representation of the geometry.

In Ω_S the fluid flow is assumed to be governed by the Stokes equations

$$-\nabla \cdot \boldsymbol{\sigma}_S + \nabla p_S = \boldsymbol{f}_S \qquad \text{in } \Omega_S, \tag{2.1}$$

$$\boldsymbol{\sigma}_S = \nu \nabla \boldsymbol{u}_S \qquad \text{in } \Omega_S, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{u}_S = 0 \qquad \text{in } \Omega_S, \tag{2.3}$$

where ν is the kinematic viscosity, assumed to be a positive constant, $\boldsymbol{u}_S$ is the fluid velocity, p_S is the kinematic pressure and $\boldsymbol{f}_S$ denotes a general body force term that includes gravitational acceleration.

In Ω_D the flow is governed by Darcy's law:

$$-\nabla \cdot \boldsymbol{u}_D = f_D \qquad \text{in } \Omega_D, \tag{2.4}$$

$$\boldsymbol{u}_D = K \nabla p_D \quad \text{in } \Omega_D, \tag{2.5}$$

where K is a symmetric and positive definite tensor that represents the rock permeability. For brevity we assume that $K = kI$, where I is the identity matrix and k is a positive constant. Also $\boldsymbol{u}_D$ is the specific discharge rate in the porous medium and f_D is a sink/source term.

For simplicity we assume that $\boldsymbol{u}_S$ and p_D satisfy a homogeneous Dirichlet boundary condition on Γ_i, i.e., $\boldsymbol{u}_S = 0$ on Γ_S and $p_D = 0$ on Γ_D.

On the interface we prescribe the following interface conditions:

$$\boldsymbol{u}_S \cdot \boldsymbol{n}_{12} = -\boldsymbol{u}_D \cdot \boldsymbol{n}_{12} \qquad \text{on } \Gamma, \tag{2.6a}$$

$$p_S - \nu \boldsymbol{n}_{12} \frac{\partial \boldsymbol{u}_S}{\partial \boldsymbol{n}_{12}} = p_D \qquad \text{on } \Gamma, \tag{2.6b}$$

$$-\nu \boldsymbol{t} \frac{\partial \boldsymbol{u}_S}{\partial \boldsymbol{n}_{12}} = G \boldsymbol{u}_S \cdot \boldsymbol{t} \qquad \text{on } \Gamma. \tag{2.6c}$$

Condition (2.6a) represents continuity of the fluid velocity's normal components, (2.6b) represents the balance of forces acting across the interface and (2.6c) is the Beaver–Joseph–Saffman condition (Beaver & Joseph, 1967). The constant $G > 0$ is given and is usually obtained from experimental data.

Let $D \subset \mathbb{R}^2$; the $L^2(D)$ inner product (or duality pairing) and norm are denoted by $(\cdot, \cdot)_D$ and $\|\cdot\|_{0,D}$, respectively, for scalar-, vector- and tensor-valued functions. For example, for tensor-valued functions $A, B : D \to \mathbb{R}^{2\times 2}$,

$$(A, B)_D = \sum_{i,j=1}^{2} \int_D A_{ij}(x) B_{ij}(x)\, \mathrm{d}x = \int_D A : B \;\, \mathrm{d}x.$$

For a connected open subset of the edge $e \in \mathbb{R}$ we write $\langle \cdot, \cdot \rangle_e$ and $\|\cdot\|_{0,e}$ for the $L^2(e)$ inner product (or duality pairing) and norm, respectively; for scalar-valued functions λ, μ and vector-valued functions $\boldsymbol{\psi} = (\psi_1, \psi_2), \boldsymbol{\phi} = (\phi_1, \phi_2)$,

$$\langle \lambda, \mu \rangle_e := \int_e \lambda\, \mu \;\, \mathrm{d}x, \quad \langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle_e := \int_e \sum_{i=1}^{2} \psi_i \phi_i \;\, \mathrm{d}s.$$

The Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$ are defined in the usual way for $\Omega = \Omega_S$ or $\Omega = \Omega_D$ with the usual norm and seminorm $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. In the sequel we use C to denote a generic positive constant independent of the mesh size, which may have a different value at different occurrences.

Following Zhao & Park (2018), Zhao *et al.* (2019), we first let $\mathcal{T}_{u_i}$ $(i = S, D)$ be the initial partition of the domain Ω_i into nonoverlapping quadrilateral or polygonal elements. We require that $\mathcal{T}_{u_i}$ be aligned with Γ. We also let $\mathcal{F}_{u,i}$ be the set of all edges in the initial partition $\mathcal{T}_{u_i}$ and $\mathcal{F}^0_{u,i} \subset \mathcal{F}_{u,i}$ be the subset of all interior edges. The union of all the nodal points in the initial partition $\mathcal{T}_{u_i}$ is denoted by $\mathcal{P}_i$, and $\mathcal{P} = \mathcal{P}_S \cup \mathcal{P}_D$. For each primal element E in the initial partition $\mathcal{T}_{u_i}$, we select an interior point ν and create new edges by connecting ν to the vertices of the primal element. This process will divide E into the union of triangles, where the triangle is denoted by τ, and the union of these triangles is renamed $S(\nu)$. The union of all $\nu \in \mathcal{T}_{u_i}$ is denoted by $\mathcal{N}_i$, and $\mathcal{N} = \mathcal{N}_S \cup \mathcal{N}_D$. We remark that the $S(\nu)$ are the

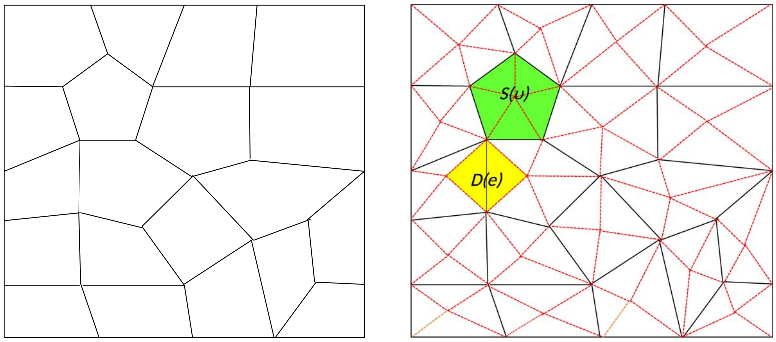

FIG. 2. Schematic of the primal mesh $S(\nu)$, the dual mesh $D(e)$ and the primal simplexes.

quadrilateral or polygonal meshes in the initial partition. Moreover, we will use $\mathcal{F}_{p,i}$ to denote the set of all new edges generated by this subdivision process and use $\mathcal{T}_{h_i}$ to denote the resulting triangulation, on which our basis functions are defined. Furthermore, $\mathcal{T}_{h_i}$ is assumed to satisfy a local quasi-uniform assumption in the sense that for any pair of elements τ and τ' in $\mathcal{T}_{h_i}$ that share an edge, there exists a constant κ independent of h_τ and $h_{\tau'}$ such that $\kappa^{-1} \leq h_\tau/h_{\tau'} \leq \kappa$. In addition we define $\mathcal{F}_i := \mathcal{F}_{u,i} \cup \mathcal{F}_{p,i}$ and $\mathcal{F}_i^0 := \mathcal{F}_{u,i}^0 \cup \mathcal{F}_{p,i}$. For each triangle $\tau \in \mathcal{T}_{h_i}$ we let h_τ be the diameter of τ and $h_i = \max\{h_\tau, \tau \in \mathcal{T}_{h_i}\}$, and we define $h = \max\{h_S, h_D\}$. Also, we let h_e denote the length of edge $e \in \mathcal{F}_i$. This construction is illustrated in Fig. 2, where black solid lines are edges in $\mathcal{F}_{u,i}$ and red dotted lines are edges in $\mathcal{F}_{p,i}$.

For each interior edge $e \in \mathcal{F}_{u,i}^0$ we use $D(e)$ to denote the union of the two triangles in $\mathcal{T}_{h_i}$ sharing edge e, and for each boundary edge $e \in \mathcal{F}_{u,i} \backslash \mathcal{F}_{u,i}^0$, we use $D(e)$ to denote the triangle in $\mathcal{T}_{h_i}$ having edge e; see Fig. 2. Moreover, the union of all $D(e)$ for $e \in \mathcal{F}_{u,i}$ is denoted by $\mathcal{D}_i$.

For each edge e we define a unit normal vector $\boldsymbol{n}_e$ as follows: if $e \in \mathcal{F}_i \setminus \mathcal{F}_i^0$ then $\boldsymbol{n}_e$ is the unit normal vector of e pointing towards the outside of Ω_i. If $e \in \mathcal{F}_i^0$, an interior edge, we then fix $\boldsymbol{n}_e$ as one of the two possible unit normal vectors on e. When there is no ambiguity we use $\boldsymbol{n}$ instead of $\boldsymbol{n}_e$ to simplify the notation.

We also make the following mesh regularity assumptions (cf. Beirão da Veiga *et al.*, 2013; Cangiani *et al.*, 2017).

ASSUMPTION 2.1 We assume the existence of a constant $\rho > 0$ such that

(1) for every element $E \in \mathcal{T}_{u_i}$ $(i = S, D)$ and every edge $e \in \partial E$ it satisfies $h_e \geq \rho h_E$, where h_E denotes the diameter of E;

(2) each element E in $\mathcal{T}_{u_i}$ is star shaped with respect to a ball of radius $\geq \rho h_E$.

We remark that the above assumptions ensure that the triangulation $\mathcal{T}_{h_i}$ is shape regular.

For a scalar or vector function v belonging to a broken Sobolev space its jump on $e \in \mathcal{F}_{p,i}$ is defined as

$$[\![v]\!] = v_1 - v_2,$$

where $v_i = v\,|_{\tau_i}$ and $\tau_1,\ \tau_2$ are the two triangles in $\mathcal{T}_{h_i}$ sharing edge e.

Let $r \geq 0$ be the order of approximation. For every $\tau \in \mathcal{T}_{h_i}$ and $e \in \mathcal{F}_i$ we define $P_r(\tau)$ and $P_r(e)$ as the spaces of polynomials of degree less than or equal to r on τ and e, respectively. Next we will introduce the SDG spaces. First, we define the following locally $H^1(\Omega)$-conforming SDG space U_{h_i} for $i = S, D$,

$$U_{h_i} := \{v : v\,|_{D(e)} \in P_0(D(e))\ \forall\, D(e) \in \mathcal{D}_i;\, v\,|_{\Gamma_i} = 0\}.$$

The discrete H^1 norm for the space U_{h_i} is given by

$$\|v\|_{Z_i}^2 = \sum_{e\in\mathcal{F}_{p,i}} h_e^{-1}\|[\![v]\!]\|_{0,e}^2.$$

Here we also define the following seminorm for $\boldsymbol{v} = (v_1, v_2) \in [U_{h_S}]^2$:

$$\|\boldsymbol{v}\|_h^2 = \|v_1\|_{Z_S}^2 + \|v_2\|_{Z_S}^2.$$

We next define the locally $H(\text{div}, \Omega)$-conforming SDG space $\boldsymbol{V}_{h_i}$,

$$\boldsymbol{V}_{h_i} := \{\boldsymbol{\tau} : \boldsymbol{\tau}\,|_{\tau} \in [P_0(\tau)]^2\ \forall\, \tau \in \mathcal{T}_{h_i};\, [\![\boldsymbol{\tau}\cdot\boldsymbol{n}]\!]\,|_e = 0\ \forall\, e \in \mathcal{F}_{p,i}\},$$

which is equipped with

$$\|\boldsymbol{\tau}\|_{X_i'}^2 = \|\boldsymbol{\tau}\|_{0,\Omega_i}^2 + \sum_{e\in\mathcal{F}_{p,i}} h_e\|\boldsymbol{\tau}\cdot\boldsymbol{n}\|_{0,e}^2.$$

Scaling arguments imply that there exists a positive constant C such that

$$C\|\boldsymbol{\tau}\|_{X_i'} \leq \|\boldsymbol{\tau}\|_{0,\Omega_i} \leq \|\boldsymbol{\tau}\|_{X_i'} \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{V}_{h_i}. \tag{2.7}$$

Finally, locally $H^1(\Omega)$-conforming finite element space for pressure is defined as

$$P_h := \left\{q : q\,|_{S(\nu)} \in P_0(S(\nu))\ \forall\, \nu \in \mathcal{N}_S\right\}$$

with norm

$$\|q\|_P^2 = \|q\|_{0,\Omega_S}^2 + \sum_{e\in\mathcal{F}_{p,S}} h_e\|q\|_{0,e}^2.$$

Norm equivalence yields

$$C\|q\|_P \leq \|q\|_{0,\Omega_S} \leq \|q\|_P \quad \forall\, q \in P_h. \tag{2.8}$$

2.2 *Weak formulation*

In this subsection we attempt to derive the weak formulation for the coupled system. To this end we define

$$\boldsymbol{X} = \left\{\boldsymbol{v} \in [H^1(\Omega_S)]^2;\ \boldsymbol{v} = 0 \text{ on } \Gamma_S\right\},$$

$$Y = \left\{v \in H^1(\Omega_D);\ v = 0 \text{ on } \Gamma_D\right\}.$$

Taking the scalar product of (2.1) with $\boldsymbol{v}_S \in \boldsymbol{X}$, and applying Green's formula yields

$$(\boldsymbol{\sigma}_S, \nabla \boldsymbol{v}_S)_{\Omega_S} - \langle \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}_S \rangle_\Gamma - (p_S, \nabla \cdot \boldsymbol{v}_S)_{\Omega_S} + \langle p_S, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12} \rangle_\Gamma = (\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S}. \tag{2.9}$$

Next take the scalar product of (2.4) with $q \in Y$ over Ω_D, apply Green's formula and use the orientation of $\boldsymbol{n}_{12}$:

$$(\boldsymbol{u}_D, \nabla q)_{\Omega_D} + \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_{12}, q \rangle_\Gamma = (f_D, q)_{\Omega_D}. \tag{2.10}$$

Then by adding (2.9) and (2.10) we have

$$\begin{aligned}(\boldsymbol{\sigma}_S, \nabla \boldsymbol{v}_S)_{\Omega_S} - \langle \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}_S \rangle_\Gamma - (p_S, \nabla \cdot \boldsymbol{v}_S)_{\Omega_S} + \langle p_S, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12} \rangle_\Gamma + (\boldsymbol{u}_D, \nabla q)_{\Omega_D} + \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_{12}, q \rangle_\Gamma \\ = (\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S} + (f_D, q)_{\Omega_D}.\end{aligned}$$

Simple algebraic calculation yields

$$\langle \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}_S \rangle_\Gamma = \langle \boldsymbol{n}_{12} \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12} \rangle_\Gamma + \langle \boldsymbol{n}_{12} \boldsymbol{\sigma}_S \boldsymbol{t}, \boldsymbol{v}_S \cdot \boldsymbol{t} \rangle_\Gamma. \tag{2.11}$$

This with (2.6b) and (2.6c) implies

$$\langle (p_S I - \boldsymbol{\sigma}_S) \boldsymbol{n}_{12}, \boldsymbol{v}_S \rangle_\Gamma = \langle p_D, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12} \rangle_\Gamma + G \langle \boldsymbol{u}_S \cdot \boldsymbol{t}, \boldsymbol{v}_S \cdot \boldsymbol{t} \rangle_\Gamma. \tag{2.12}$$

On the other hand (2.6a) reveals that

$$\langle \boldsymbol{u}_D \cdot \boldsymbol{n}_{12}, q \rangle_\Gamma = -\langle \boldsymbol{u}_S \cdot \boldsymbol{n}_{12}, q \rangle_\Gamma.$$

Collecting the above results we propose the following variational formulation: find $(\boldsymbol{\sigma}_S, \boldsymbol{u}_S, p_S) \in [L^2(\Omega_S)]^{2\times 2} \times \boldsymbol{X} \times L^2(\Omega_S)$ and $(\boldsymbol{u}_D, p_D) \in [L^2(\Omega_D)]^2 \times Y$ such that

$$\begin{aligned}&(\boldsymbol{\sigma}_S, \nabla \boldsymbol{v}_S)_{\Omega_S} + \langle p_D, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12} \rangle_\Gamma + G \langle \boldsymbol{u}_S \cdot \boldsymbol{t}, \boldsymbol{v}_S \cdot \boldsymbol{t} \rangle_\Gamma - (p_S, \nabla \cdot \boldsymbol{v}_S)_{\Omega_S} \\ &\qquad + (\boldsymbol{u}_D, \nabla q_D)_{\Omega_D} - \langle \boldsymbol{u}_S \cdot \boldsymbol{n}_{12}, q_D \rangle_\Gamma = (f_D, q_D)_{\Omega_D} + (\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S}, \\ &K^{-1}(\boldsymbol{u}_D, \boldsymbol{v}_D)_{\Omega_D} - (\nabla p_D, \boldsymbol{v}_D)_{\Omega_D} + \nu^{-1}(\boldsymbol{\sigma}_S, \boldsymbol{\tau}_S)_{\Omega_S} - (\nabla \boldsymbol{u}_S, \boldsymbol{\tau}_S)_{\Omega_S} = 0, \\ &(\nabla \cdot \boldsymbol{u}_S, q_S)_{\Omega_S} = 0 \\ &\quad \forall (\boldsymbol{\tau}_S, \boldsymbol{v}_S, q_S) \in [L^2(\Omega_S)]^{2\times 2} \times \boldsymbol{X} \times L^2(\Omega_S), \quad \forall (\boldsymbol{v}_D, q_D) \in [L^2(\Omega_D)]^2 \times Y.\end{aligned} \tag{2.13}$$

Lemma 2.2 Any solution $(\boldsymbol{\sigma}_S, \boldsymbol{u}_S, p_S) \in [L^2(\Omega_S)]^{2\times 2} \times \boldsymbol{X} \times L^2(\Omega_S)$ and $(\boldsymbol{u}_D, p_D) \in [L^2(\Omega_D)]^2 \times Y$ of the coupled problem (2.4)–(2.6c) is also a solution to the variational problem (2.13). Conversely, any solution of problem (2.13) satisfies (2.4)–(2.6c).

Proof. We first choose $\boldsymbol{v}_S \in C_0^\infty(\Omega_S)^2$ and $q_D = 0$, then $\boldsymbol{v}_S = 0$ and $q_D \in C_0^\infty(\Omega_D)$ in the first equation of (2.13). Then in the sense of distributions on Ω_S and Ω_D we obtain

$$-\nabla \cdot \boldsymbol{\sigma}_S + \nabla p_S = \boldsymbol{f}_S, \tag{2.14}$$
$$-\nabla \cdot \boldsymbol{u}_D = f_D. \tag{2.15}$$

Similarly, in the sense of distributions we get

$$\boldsymbol{\sigma}_S = \nu \nabla \boldsymbol{u}_S,$$
$$\boldsymbol{u}_D = K \nabla p_D.$$

Then we multiply (2.14) by $\boldsymbol{v} \in \boldsymbol{X}$, (2.15) by $q \in Y$; it follows from integration by parts that

$$\begin{aligned}&(\boldsymbol{\sigma}_S, \nabla \boldsymbol{v})_{\Omega_S} - \langle \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}\rangle_\Gamma - (p_S, \nabla \cdot \boldsymbol{v})_{\Omega_S} + \langle p_S, \boldsymbol{v} \cdot \boldsymbol{n}_{12}\rangle_\Gamma + (\boldsymbol{u}_D, \nabla q)_{\Omega_D} + \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_{12}, q\rangle_\Gamma \\ &\quad = (\boldsymbol{f}_S, \boldsymbol{v})_{\Omega_S} + (f_D, q)_{\Omega_D}.\end{aligned}$$

Comparing with (2.13) yields

$$-\langle \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}\rangle_\Gamma + \langle p_S, \boldsymbol{v}\cdot \boldsymbol{n}_{12}\rangle_\Gamma + \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_{12}, q\rangle_\Gamma = \langle p_D, \boldsymbol{v}\cdot \boldsymbol{n}_{12}\rangle_\Gamma + G\langle \boldsymbol{u}_S \cdot \boldsymbol{t}, \boldsymbol{v}\cdot \boldsymbol{t}\rangle_\Gamma - \langle \boldsymbol{u}_S \cdot \boldsymbol{n}_{12}, q\rangle_\Gamma. \tag{2.16}$$

Choosing $\boldsymbol{v} = 0$ gives

$$\langle \boldsymbol{u}_D \cdot \boldsymbol{n}_{12}, q\rangle_\Gamma = -\langle \boldsymbol{u}_S \cdot \boldsymbol{n}_{12}, q\rangle_\Gamma.$$

Thus, (2.16) is reduced to

$$-\langle \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}\rangle_\Gamma + \langle p_S, \boldsymbol{v} \cdot \boldsymbol{n}_{12}\rangle_\Gamma = \langle p_D, \boldsymbol{v} \cdot \boldsymbol{n}_{12}\rangle_\Gamma + G\langle \boldsymbol{u}_S \cdot \boldsymbol{t}, \boldsymbol{v} \cdot \boldsymbol{t}\rangle_\Gamma,$$

which together with (2.11) implies (2.6b) and (2.6c). □

2.3 *Discrete formulation*

Following the framework given in Chung & Engquist (2009), multiplying (2.1) by $\boldsymbol{v}_S \in [U_{h_S}]^2$, (2.2) by $\boldsymbol{\tau}_S \in [\boldsymbol{V}_{h_S}]^2$, (2.3) by $q_S \in P_h$ and integrating by parts, we get

$$\begin{aligned}&-\sum_{e\in\mathcal{F}_{p,S}} \langle \boldsymbol{\sigma}_S \boldsymbol{n}_e, [\![\boldsymbol{v}_S]\!]\rangle_e - \langle \boldsymbol{\sigma}_S \boldsymbol{n}_{12}, \boldsymbol{v}_S\rangle_\Gamma + \sum_{e\in\mathcal{F}_{p,S}} \langle p_S, [\![\boldsymbol{v}_S \cdot \boldsymbol{n}_e]\!]\rangle_e + \langle p_S, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12}\rangle_\Gamma = (\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S},\\ &\nu^{-1}(\boldsymbol{\sigma}_S, \boldsymbol{\tau}_S)_{\Omega_S} = \sum_{e\in\mathcal{F}_{u,S}^0} \langle [\![\boldsymbol{\tau}_S \boldsymbol{n}_e]\!], \boldsymbol{u}_S\rangle_e + \langle \boldsymbol{\tau}_S \boldsymbol{n}_{12}, \boldsymbol{u}_S\rangle_\Gamma,\\ &\sum_{e\in\mathcal{F}_{u,S}^0} \langle \boldsymbol{u}_S \cdot \boldsymbol{n}_e, [\![q_S]\!]\rangle_e + \langle \boldsymbol{u}_S \cdot \boldsymbol{n}_{12}, q_S\rangle_\Gamma = 0.\end{aligned} \tag{2.17}$$

Then multiplying (2.4) by $q_D \in U_{h_D}$, (2.5) by $\boldsymbol{v}_D \in \boldsymbol{V}_{h_D}$ and integrating by parts, we obtain

$$\begin{aligned} -\sum_{e\in\mathcal{F}_{p,D}} \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_e, [\![q_D]\!]\rangle_e + \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_{12}, q_D\rangle_\Gamma &= (f_D, q_D)_{\Omega_D}, \\ K^{-1}(\boldsymbol{u}_D, \boldsymbol{v}_D)_{\Omega_D} &= \sum_{e\in\mathcal{F}^0_{u,D}} \langle p_D, [\![\boldsymbol{v}_D \cdot \boldsymbol{n}_e]\!]\rangle_e + \langle p_D, \boldsymbol{v}_D \cdot \boldsymbol{n}_D\rangle_\Gamma. \end{aligned} \tag{2.18}$$

By adding the first equations of (2.17) and (2.18), and applying the interface condition (2.6a) and (2.12) we have

$$\begin{aligned} &-\sum_{e\in\mathcal{F}_{p,S}} \langle \boldsymbol{\sigma}_S \boldsymbol{n}_e, [\![\boldsymbol{v}_S]\!]\rangle_e + \sum_{e\in\mathcal{F}_{p,S}} \langle p_S, [\![\boldsymbol{v}_S \cdot \boldsymbol{n}_e]\!]\rangle_e - \sum_{e\in\mathcal{F}_{p,D}} \langle \boldsymbol{u}_D \cdot \boldsymbol{n}_e, [\![q_D]\!]\rangle_e + \langle p_D, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12}\rangle_\Gamma \\ &\quad + G\langle \boldsymbol{u}_S \cdot \boldsymbol{t}, \boldsymbol{v}_S \cdot \boldsymbol{t}\rangle_\Gamma - \langle \boldsymbol{u}_S \cdot \boldsymbol{n}_{12}, q_D\rangle_\Gamma = (f_D, q_D)_{\Omega_D} + (\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S}. \end{aligned}$$

We can now introduce the following bilinear forms:

$$\begin{aligned} a_S(\boldsymbol{\tau}_S, \boldsymbol{v}_S) &= -\sum_{e\in\mathcal{F}_{p,S}} \langle \boldsymbol{\tau}_S \boldsymbol{n}_e, [\![\boldsymbol{v}_S]\!]\rangle_e, \\ a^*_S(\boldsymbol{v}_S, \boldsymbol{\tau}_S) &= \sum_{e\in\mathcal{F}^0_{u,S}} \langle \boldsymbol{v}_S, [\![\boldsymbol{\tau}_S \boldsymbol{n}_e]\!]\rangle_e + \langle \boldsymbol{v}_S, \boldsymbol{\tau}_S \boldsymbol{n}_{12}\rangle_\Gamma, \\ b_S(\boldsymbol{v}_S, q_S) &= -\sum_{e\in\mathcal{F}^0_{u,S}} \langle \boldsymbol{v}_S \cdot \boldsymbol{n}_e, [\![q_S]\!]\rangle_e - \langle \boldsymbol{v}_S \cdot \boldsymbol{n}_{12}, q_S\rangle_\Gamma, \\ b^*_S(q_S, \boldsymbol{v}_S) &= \sum_{e\in\mathcal{F}_{p,S}} \langle q_S, [\![\boldsymbol{v}_S \cdot \boldsymbol{n}_e]\!]\rangle_e, \\ a_D(\boldsymbol{v}_D, q_D) &= -\sum_{e\in\mathcal{F}_{p,D}} \langle \boldsymbol{v}_D \cdot \boldsymbol{n}_e, [\![q_D]\!]\rangle_e, \\ a^*_D(q_D, \boldsymbol{v}_D) &= \sum_{e\in\mathcal{F}^0_{u,D}} \langle q_D, [\![\boldsymbol{v}_D \cdot \boldsymbol{n}_e]\!]\rangle_e + \langle q_D, \boldsymbol{v}_D \cdot \boldsymbol{n}_D\rangle_\Gamma. \end{aligned}$$

Then we arrive at the following discrete formulation by combining the above results: find $(\boldsymbol{\sigma}_{S,h}, \boldsymbol{u}_{S,h}, p_{S,h}) \in [\boldsymbol{V}_{h_S}]^2 \times [U_{h_S}]^2 \times P_h$ and $(\boldsymbol{u}_{D,h}, p_{D,h}) \in \boldsymbol{V}_{h_D} \times U_{h_D}$ such that

$$\begin{aligned} &a_S(\boldsymbol{\sigma}_{S,h}, \boldsymbol{v}_S) + b^*_S(p_{S,h}, \boldsymbol{v}_S) + a_D(\boldsymbol{u}_{D,h}, q_D) + \langle p_{D,h}, \boldsymbol{v}_S \cdot \boldsymbol{n}_{12}\rangle_\Gamma + G\langle \boldsymbol{u}_{S,h} \cdot \boldsymbol{t}, \boldsymbol{v}_S \cdot \boldsymbol{t}\rangle_\Gamma \\ &\quad - \langle \boldsymbol{u}_{S,h} \cdot \boldsymbol{n}_{12}, q_D\rangle_\Gamma = (f_D, q_D)_{\Omega_D} + (\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S}, \\ &K^{-1}(\boldsymbol{u}_{D,h}, \boldsymbol{v}_D)_{\Omega_D} - a^*_D(p_{D,h}, \boldsymbol{v}_D) + \nu^{-1}(\boldsymbol{\sigma}_{S,h}, \boldsymbol{\tau}_S)_{\Omega_S} - a^*_S(\boldsymbol{u}_{S,h}, \boldsymbol{\tau}_S) = 0, \\ &b_S(\boldsymbol{u}_{S,h}, q_S) = 0 \\ &\forall\, (\boldsymbol{\tau}_S, \boldsymbol{v}_S, q_S) \in [\boldsymbol{V}_{h_S}]^2 \times [U_{h_S}]^2 \times P_h, \quad \forall\, (\boldsymbol{v}_D, q_D) \in \boldsymbol{V}_{h_D} \times U_{h_D}. \end{aligned} \tag{2.19}$$

The bilinear forms introduced above possess some desirable properties; see e.g., Kim *et al.* (2013). We list some of the properties that will be used throughout this paper. First, we have the following inf-sup condition

$$\inf_{q\in P_h\backslash\{0\}} \sup_{\boldsymbol{v}\in[U_{h_S}]^2\backslash\{0\}} \frac{b_S(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_h\|q\|_{0,\Omega_S}} \geq C. \tag{2.20}$$

Second, the following adjoint properties hold:

$$\begin{aligned} a_S(\boldsymbol{w}_S,\boldsymbol{v}) &= a_S^*(\boldsymbol{v},\boldsymbol{w}_S) \quad \forall\,(\boldsymbol{w}_S,\boldsymbol{v})\in[\boldsymbol{V}_{h_S}]^2\times[U_{h_S}]^2,\\ a_D(\boldsymbol{v}_D,q) &= a_D^*(q,\boldsymbol{v}_D) \quad \forall\,(\boldsymbol{v}_D,q)\in\boldsymbol{V}_{h_D}\times U_{h_D},\\ b_S(\boldsymbol{v}_S,q_S) &= b_S^*(q_S,\boldsymbol{v}_S) \quad \forall\,(\boldsymbol{v}_S,q_S)\in[U_{h_S}]^2\times P_h. \end{aligned} \tag{2.21}$$

Then we define three interpolation operators $I_{h_i}: H^1(\Omega_i)\to U_{h_i}$, $\pi_h: H^\alpha(\Omega_S)\to P_h$ and $J_{h_i}: [H^\alpha(\Omega_i)]^2\to\boldsymbol{V}_{h_i}$, $1/2<\alpha\leq 1$, which are given explicitly as

$$\begin{aligned} I_{h_i}v\,|_{D(e)} &= \frac{1}{h_e}\langle v,1\rangle_e \quad \forall\,e\in\mathcal{F}_{u,i},\\ \pi_h q\,|_{S(\nu)} &= \frac{1}{|S(\nu)|}(q,1)_{S(\nu)} \quad \forall\,\nu\in\mathcal{N}_{\mathcal{S}},\\ J_{h_i}\boldsymbol{w}\cdot\boldsymbol{n}_e\,|_e &= \frac{1}{h_e}\langle\boldsymbol{w}\cdot\boldsymbol{n}_e,1\rangle_e \quad \forall\,e\in\mathcal{F}_{p,i}, \end{aligned} \tag{2.22}$$

where $|S(\nu)|$ denotes the area of $S(\nu)$.

By the definitions of the given operators we infer that

$$a_D^*(I_{h_D}p_D-p_D,\boldsymbol{q})=0 \quad \forall\,\boldsymbol{q}\in\boldsymbol{V}_{h_D}, \tag{2.23}$$

$$a_D(J_{h_D}\boldsymbol{u}_D-\boldsymbol{u}_D,v)=0 \quad \forall\,v\in U_{h_D} \tag{2.24}$$

and

$$a_S^*(I_{h_S}\boldsymbol{u}_S-\boldsymbol{u}_S,\boldsymbol{\tau})=0 \quad \forall\,\boldsymbol{\tau}\in[\boldsymbol{V}_{h_S}]^2, \tag{2.25}$$

$$a_S(J_{h_S}\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_S,\boldsymbol{v})=0 \quad \forall\,\boldsymbol{v}\in[U_{h_S}]^2, \tag{2.26}$$

where $I_{h_S}\boldsymbol{u}_S=(I_{h_S}u_S^1,I_{h_S}u_S^2)$ for $\boldsymbol{u}_S=(u_S^1,u_S^2)$ and $J_{h_S}\boldsymbol{\sigma}_S=(J_{h_S}\boldsymbol{\sigma}_S^1,J_{h_S}\boldsymbol{\sigma}_S^2)$ for $\boldsymbol{\sigma}_S=(\boldsymbol{\sigma}_S^1,\boldsymbol{\sigma}_S^2)$.

Finally, we recall some important interpolation error estimates (cf. Ciarlet, 1978), which are useful for the subsequent analysis:

$$\begin{aligned}
\|q-\pi_h q\|_P &\le Ch^\alpha \|q\|_{\alpha,\Omega_S} \quad \forall\, q \in H^\alpha(\Omega_S),\ 1/2<\alpha\le 1,\\
\|v-I_{h_i}v\|_{0,\Omega_i} &\le Ch_i\|\nabla v\|_{0,\Omega_i} \quad \forall\, v\in H^1(\Omega_i),\\
\|\boldsymbol{\tau}-J_{h_i}\boldsymbol{\tau}\|_{0,\Omega_i} &\le Ch_i^\alpha\|\boldsymbol{\tau}\|_{\alpha,\Omega_i} \quad \forall\, \boldsymbol{\tau}\in[H^\alpha(\Omega_i)]^2,\ 1/2<\alpha\le 1.
\end{aligned} \tag{2.27}$$

3. Error analysis

This section is devoted to the construction of error estimates for various approximations. A new regularization operator is established to show the discrete trace inequality. Furthermore, the convergence estimates covering the low regularity assumption, i.e., $1/2 < \alpha \le 1$, are achieved.

3.1 *Discrete trace inequality*

In this subsection we attempt to show the discrete trace inequality as stated in Lemma 3.1, which can be obtained using regularization. We define the regularization operator $R(z_h)$ for $z_h \in U_{h_D}$, which consists of piecewise linear polynomials and is continuous over all the nodes interior to $\mathcal{T}_{h_D}$. Since z_h is partially continuous we cannot define $R(z_h)$ by direct applications of the approach used in Girault & Wheeler (2008). In fact we combine the strategies used in Girault & Wheeler (2008) and Brenner (2003), and the details are given below. To define a piecewise linear function it suffices to specify its values at the nodal points of $\mathcal{T}_{h_D}$. If $m \in \Gamma_D$ then $R(z_h)(m) = 0$. For the nodal point m of $\mathcal{T}_{h_D}$ that lies on the edges belonging to $\mathcal{F}^0_{u,D} \cup \Gamma$ we define

$$R(z_h)(m) := z_h(e_m),$$

where $e_m \in \mathcal{F}^0_{u,D} \cup \Gamma$ is a fixed edge that contains m.

For the nodal point ν interior to each $S(\nu)$ belonging to $\mathcal{N}_S$ let nl be the number of nodal points of $S(\nu)$ and let $m_1, \ldots, m_{nl}$ be the nodal points of $S(\nu)$; we define

$$R(z_h)(\nu) := \frac{1}{nl}\sum_{i=1}^{nl} R(z_h)(m_i).$$

We remark that the resulting $R(z_h) \in H^1(\Omega_D)$ and vanishes on Γ_D.

LEMMA 3.1 The following estimate holds for $z_h \in U_{h_D}$:

$$\|z_h\|^2_{0,\Gamma} \le C \sum_{e\in\mathcal{F}_{p,D}} h_e^{-1}\|[\![z_h]\!]\|^2_{0,e}.$$

Proof. First, it follows from the triangle inequality that

$$\|z_h\|_{0,\Gamma} \le \|z_h - R(z_h)\|_{0,\Gamma} + \|R(z_h)\|_{0,\Gamma}. \tag{3.1}$$

Since $R(z_h) \in H^1(\Omega_D)$ and $R(z_h) = 0$ on Γ_D then we have

$$\|R(z_h)\|_{1,\Omega_D} \le C\|\nabla R(z_h)\|_{0,\Omega_D}.$$

An application of the trace theorem yields

$$\|R(z_h)\|_{0,\Gamma} \le C\|R(z_h)\|_{1,\Omega_D}.$$

Inserting the above inequalities into (3.1) implies

$$\begin{aligned} \|z_h\|_{0,\Gamma} &\le \|z_h - R(z_h)\|_{0,\Gamma} + \|R(z_h)\|_{0,\Gamma} \\ &\le C(\|z_h - R(z_h)\|_{0,\Gamma} + \|\nabla R(z_h)\|_{0,\Omega_D}). \end{aligned} \tag{3.2}$$

Fix an edge $e \in \Gamma$ and let a_1, a_2 be the two endpoints of e; since $R(z_h)$ is a linear function and z_h is a constant function over each triangle we obtain

$$\begin{aligned} \|z_h - R(z_h)\|_{0,e}^2 &\le \frac{h_e}{2}\Big((z_h(e) - R(z_h)(a_1))^2 + (z_h(e) - R(z_h)(a_2))^2\Big) \\ &\le \frac{h_e}{2}\Big((z_h(e) - z_h(e_{a_1}))^2 + (z_h(e) - z_h(e_{a_2}))^2\Big), \end{aligned}$$

where $e_{a_i} \in \mathcal{F}_{u,D}^0 \cup \Gamma, i = 1, 2$ are the edges containing a_i.

Since $e_{a_i}, i = 1, 2$ is the edge containing a_i as a vertex, there exists a sequence of $J \ge 1$ edges, $e_1 = e, e_2, \ldots, e_J = e_{a_i}$ such that $D(e_k)$ and $D(e_{k+1})$ are adjacent. In addition the integer J is bounded by a fixed constant that is independent of h. Thus, we can write

$$z_h(e) - z_h(e_{a_i}) = (z_h(e) - z_h(e_1)) + (z_h(e_1) - z_h(e_2)) + \cdots + (z_h(e_{J-1}) - z_h(e_{a_i})),$$

where $e_1 = e,\ e_1$ and e_2 belong to $D(e_1)$ and $D(e_2)$ such that $D(e_1)$ and $D(e_2)$ are adjacent etc., and the function $z_h(e_k)$ is constant over $D(e_k)$, which implies

$$|z_h(e_{k-1}) - z_h(e_k)| = h_e^{-1/2}\|[\![z_h]\!]\|_{0,e} \quad e \in \partial D(e_{k-1}) \cap \partial D(e_k).$$

Thus

$$|z_h(e) - z_h(e_{a_i})| \le C\sum_{k=1}^{J-1} h_{e_k}^{-1/2}\|[\![z_h]\!]\|_{0,\partial D(e_k)\cap\partial D(e_{k+1})}.$$

It is easy to see that

$$\|z_h - R(z_h)\|_{0,e}^2 \le Ch_e\left(\sum_{k=1}^{J-1} h_{e_k}^{-1/2}\|[\![z_h]\!]\|_{0,\partial D(e_k)\cap\partial D(e_{k+1})}\right)^2.$$

Summing over all the edges in Γ yields

$$\|z_h - R(z_h)\|_{0,\Gamma}^2 \le C \sum_{e\in\mathcal{F}_p} h_e^{-1} \|[\![z_h]\!]\|_{0,e}^2. \tag{3.3}$$

It remains to estimate the second term of (3.2): fix $\tau \in \mathcal{T}_{h_D}$, let $a_i, i = 1, 2, 3$ be the three nodal points of τ, then scaling argument implies

$$\|\nabla R(z_h)\|_{0,\tau}^2 \le C\Big(|R(z_h)(a_1) - R(z_h)(a_2)|^2 + |R(z_h)(a_2) - R(z_h)(a_3)|^2 + |R(z_h)(a_3) - R(z_h)(a_1)|^2\Big).$$

By the definition of $R(z_h)(a_i)$ the same argument implies

$$\|\nabla R(z_h)\|_{0,\Omega_D}^2 \le C \sum_{e\in\mathcal{F}_{p,D}} h_e^{-1} \|[\![z_h]\!]\|_{0,e}^2. \tag{3.4}$$

The desired estimate follows by combining (3.2)–(3.4).

Remark 3.2 Proceeding similarly we can show that for $\boldsymbol{v}_h \in [U_{h_S}]^2$,

$$\|\boldsymbol{v}_h\|_{0,\Gamma}^2 \le C \sum_{e\in\mathcal{F}_{p,S}} h_e^{-1} \|[\![\boldsymbol{v}_h]\!]\|_{0,e}^2.$$

□

3.2 *Convergence estimates*

In this subsection we will prove the convergence of the proposed SDG method for all the approximations. To begin we state the following lemma (cf. Kim *et al.*, 2013), which is useful for the subsequent analysis.

Lemma 3.3 Let $(\boldsymbol{\phi}, \boldsymbol{\tau}) \in [U_{h_S}]^2 \times [\boldsymbol{V}_{h_S}]^2$ and $(\xi, \boldsymbol{v}) \in U_{h_D} \times \boldsymbol{V}_{h_D}$ such that

$$\begin{aligned} \nu^{-1}(\boldsymbol{\tau}, \boldsymbol{\varphi})_{\Omega_S} &= a_S^*(\boldsymbol{\phi}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in [\boldsymbol{V}_{h_S}]^2,\\ K^{-1}(\boldsymbol{v}, \boldsymbol{q})_{\Omega_D} &= a_D^*(\xi, \boldsymbol{q}) \quad \forall \boldsymbol{q} \in \boldsymbol{V}_{h_D}. \end{aligned}$$

Then we have

$$\begin{aligned} \|\boldsymbol{\tau}\|_{0,\Omega_S} &\le C\|\boldsymbol{\phi}\|_h,\\ \|\boldsymbol{\phi}\|_h &\le C\|\boldsymbol{\tau}\|_{0,\Omega_S},\\ \|\boldsymbol{v}\|_{0,\Omega_D} &\le C\|\xi\|_{Z_D},\\ \|\xi\|_{Z_D} &\le C\|\boldsymbol{v}\|_{0,\Omega_D}. \end{aligned}$$

LEMMA 3.4 The discrete scheme (2.19) has a unique solution. Moreover, it is stable in the sense that

$$\begin{aligned}&\|\boldsymbol{u}_{S,h}\|_{0,\Omega_S}+\|p_{S,h}\|_{0,\Omega_S}+\|\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}+\|\boldsymbol{u}_{D,h}\|_{0,\Omega_D}+\|p_{D,h}\|_{0,\Omega_D}\leq C(\|\boldsymbol{f}_S\|_{0,\Omega_S}+\|f_D\|_{0,\Omega_D}),\\&\|\boldsymbol{u}_{S,h}\|_h+\|p_{D,h}\|_{Z_D}\leq C(\|\boldsymbol{f}_S\|_{0,\Omega_S}+\|f_D\|_{0,\Omega_D}).\end{aligned}\tag{3.5}$$

Proof. Note that if (3.5) holds then the existence and uniqueness of the discrete solution follow immediately. All we need to show are the bounds (3.5). From Lemma 3.3 and the second equation of (2.19) we have

$$\begin{aligned}&\|\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}\leq C\|\boldsymbol{u}_{S,h}\|_h,\quad \|\boldsymbol{u}_{S,h}\|_h\leq C\|\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S},\\&\|\boldsymbol{u}_{D,h}\|_{0,\Omega_D}\leq C\|p_{D,h}\|_{Z_D},\quad \|p_{D,h}\|_{Z_D}\leq C\|\boldsymbol{u}_{D,h}\|_{0,\Omega_D}.\end{aligned}\tag{3.6}$$

We choose $(\boldsymbol{v}_D,q_D)=(\boldsymbol{u}_{D,h},p_{D,h})$ and $(\boldsymbol{\tau}_S,\boldsymbol{v}_S,q_S)=(\boldsymbol{\sigma}_{S,h},\boldsymbol{u}_{S,h},p_{S,h})$ in (2.19). Adding the resulting equations yields

$$G\|\boldsymbol{u}_{S,h}\|_{0,\Gamma}^2+K^{-1}\|\boldsymbol{u}_{D,h}\|_{0,\Omega_D}^2+\nu^{-1}\|\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}^2=(\boldsymbol{f}_S,\boldsymbol{u}_{S,h})+(f_D,p_{D,h}).$$

Now it follows from (3.6) and the discrete Poincaré inequality (cf. Brenner, 2003) that

$$\begin{aligned}\|\boldsymbol{u}_{S,h}\|_h^2+\|p_{D,h}\|_{Z_D}^2&\leq C(\|\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}^2+\|\boldsymbol{u}_{D,h}\|_{0,\Omega_D}^2)\\&\leq C(\|\boldsymbol{f}_S\|_{0,\Omega_S}+\|f_D\|_{0,\Omega_D})(\|\boldsymbol{u}_{S,h}\|_{0,\Omega_S}+\|p_{D,h}\|_{0,\Omega_D})\\&\leq C(\|\boldsymbol{f}_S\|_{0,\Omega_S}+\|f_D\|_{0,\Omega_D})(\|\boldsymbol{u}_{S,h}\|_h+\|p_{D,h}\|_{Z_D}),\end{aligned}$$

which yields

$$\|\boldsymbol{u}_{S,h}\|_h+\|p_{D,h}\|_{Z_D}\leq C(\|\boldsymbol{f}_S\|_{0,\Omega_S}+\|f_D\|_{0,\Omega_D}).\tag{3.7}$$

A further application of the discrete Poincaré inequality implies

$$\|\boldsymbol{u}_{S,h}\|_{0,\Omega_S}+\|p_{D,h}\|_{0,\Omega_D}\leq C(\|\boldsymbol{u}_{S,h}\|_h+\|p_{D,h}\|_{Z_D})\leq C(\|\boldsymbol{f}_S\|_{0,\Omega_S}+\|f_D\|_{0,\Omega_D}).$$

By (3.6) and (3.7) we obtain

$$\|\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}+\|\boldsymbol{u}_{D,h}\|_{0,\Omega_D}\leq C(\|\boldsymbol{u}_{S,h}\|_h+\|p_{D,h}\|_{Z_D})\leq C(\|\boldsymbol{f}_S\|_{0,\Omega_S}+\|f_D\|_{0,\Omega_D}).$$

Finally, (2.7), the first equation of (2.19), the inf-sup condition (2.20), the discrete adjoint property (2.21), Lemma 3.1, Remark 3.2, (3.7) and the Cauchy–Schwarz inequality imply

$$
\begin{aligned}
\|p_{S,h}\|_{0,\Omega_S} &\le C \sup_{\boldsymbol{v}\in[U_{h_S}]^2} \frac{b_S(\boldsymbol{v}, p_{S,h})}{\|\boldsymbol{v}\|_h}\\
&= C \sup_{\boldsymbol{v}\in[U_{h_S}]^2} \frac{(\boldsymbol{f}_S, \boldsymbol{v})_{\Omega_S} - a_S(\boldsymbol{\sigma}_{S,h}, \boldsymbol{v}) - \langle p_{D,h}, \boldsymbol{v}\cdot\boldsymbol{n}_{12}\rangle_\Gamma - G\langle \boldsymbol{u}_{S,h}\cdot\boldsymbol{t}, \boldsymbol{v}\cdot\boldsymbol{t}\rangle_\Gamma}{\|\boldsymbol{v}\|_h}\\
&\le C(\|\boldsymbol{f}_S\|_{0,\Omega_S} + \|\boldsymbol{\sigma}_{S,h}\|_{X_S'} + \|p_{D,h}\|_{0,\Gamma} + \|\boldsymbol{u}_{S,h}\|_{0,\Gamma})\\
&\le C(\|\boldsymbol{f}_S\|_{0,\Omega_S} + \|f_D\|_{0,\Omega_D}).
\end{aligned}
$$

Combining the above estimates the proof is complete. □

The rest of this section is devoted to establishing the convergence of the proposed scheme. To begin with we note that $\boldsymbol{\sigma}_S, \boldsymbol{u}_S, p_S, \boldsymbol{u}_D, p_D$ satisfy the following equation:

$$
\begin{aligned}
&a_S(\boldsymbol{\sigma}_S, \boldsymbol{v}_S) + b_S^*(p_S, \boldsymbol{v}_S) + a_D(\boldsymbol{u}_D, q_D) + \langle p_D, \boldsymbol{v}_S\cdot\boldsymbol{n}_{12}\rangle_\Gamma + G\langle \boldsymbol{u}_S\cdot\boldsymbol{t}, \boldsymbol{v}_S\cdot\boldsymbol{t}\rangle_\Gamma\\
&\quad - \langle \boldsymbol{u}_S\cdot\boldsymbol{n}_{12}, q_D\rangle_\Gamma = (f_D, q_D)_{\Omega_D} + (\boldsymbol{f}_S, \boldsymbol{v}_S)_{\Omega_S} \quad \forall\, \boldsymbol{v}_S \in [U_{h_S}]^2,\ q_D \in U_{h_D}.
\end{aligned}
\tag{3.8}
$$

Subtracting (2.19) from (3.8) we get the following error equation:

$$
\begin{aligned}
&a_S(\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}, \boldsymbol{v}_S) + b_S^*(p_S - p_{S,h}, \boldsymbol{v}_S) + a_D(\boldsymbol{u}_D - \boldsymbol{u}_{D,h}, q_D) + \langle p_D - p_{D,h}, \boldsymbol{v}_S\cdot\boldsymbol{n}_{12}\rangle_\Gamma\\
&\quad + G\langle (\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot\boldsymbol{t}, \boldsymbol{v}_S\cdot\boldsymbol{t}\rangle_\Gamma - \langle (\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot\boldsymbol{n}_{12}, q_D\rangle_\Gamma = 0 \quad \forall\, \boldsymbol{v}_S \in [U_{h_S}]^2, q_D \in U_{h_D}.
\end{aligned}
\tag{3.9}
$$

We are now ready to state the next theorem.

THEOREM 3.5 Let $(\boldsymbol{\sigma}_S, \boldsymbol{u}_S, p_S) \in [H^\alpha(\Omega_S)]^{2\times 2} \times [H^{1+\alpha}(\Omega_S)]^2 \times H^\alpha(\Omega_S)$ and $(\boldsymbol{u}_D, p_D) \in [H^\alpha(\Omega_D)]^2 \times H^{1+\alpha}(\Omega_D)$ be the weak solution of (2.13), and let $(\boldsymbol{\sigma}_{S,h}, \boldsymbol{u}_{D,h}, p_{S,h})$ be the numerical solution of (2.19). Then for $\frac{1}{2} < \alpha \le 1$,

$$
\|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S} + \|\boldsymbol{u}_D - \boldsymbol{u}_{D,h}\|_{0,\Omega_D} \le Ch^\alpha(\|p_D\|_{1+\alpha,\Omega_D} + \|\boldsymbol{u}_S\|_{1+\alpha,\Omega_S} + \|p_S\|_{\alpha,\Omega_S}),
\tag{3.10}
$$

$$
\|p_S - p_{S,h}\|_{0,\Omega_S} \le Ch^\alpha(\|p_D\|_{1+\alpha,\Omega_D} + \|\boldsymbol{u}_S\|_{1+\alpha,\Omega_S} + \|p_S\|_{\alpha,\Omega_S}).
\tag{3.11}
$$

Proof. We construct $\tilde{\boldsymbol{\sigma}}_S \in [\boldsymbol{V}_{h_S}]^2, \tilde{\boldsymbol{u}}_D \in \boldsymbol{V}_{h_D}$ by

$$
\nu^{-1}(\tilde{\boldsymbol{\sigma}}_S, \boldsymbol{\tau}_S)_{\Omega_S} = a_S^*(I_{h_S}\boldsymbol{u}_S, \boldsymbol{\tau}_S) \quad \forall\, \boldsymbol{\tau}_S \in [\boldsymbol{V}_{h_S}]^2,
\tag{3.12}
$$

$$
K^{-1}(\tilde{\boldsymbol{u}}_D, \boldsymbol{v}_D)_{\Omega_D} = a_D^*(I_{h_D}p_D, \boldsymbol{v}_D) \quad \forall\, \boldsymbol{v}_D \in \boldsymbol{V}_{h_D}.
\tag{3.13}
$$

Then the definition of I_{h_i}, (2.24), (2.26), (3.9) and the fact that $b_S(I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}, \tilde{p}) = 0\ \forall \tilde{p} \in P_h$ imply

$$
\begin{aligned}
&a_S(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, I_{h_D}p_D - p_{D,h})\\
&\quad = a_S(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_D, I_{h_D}p_D - p_{D,h})\\
&\qquad + b_S^*(p_{S,h} - p_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) - \langle p_D - p_{D,h}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{n}_{12}\rangle_\Gamma\\
&\qquad + \langle (\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{n}_{12}, I_{h_D}p_D - p_{D,h}\rangle_\Gamma - G\langle (\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{t}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{t}\rangle_\Gamma\\
&\quad = a_S(\tilde{\boldsymbol{\sigma}}_S - J_{h_S}\boldsymbol{\sigma}_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + a_D(\tilde{\boldsymbol{u}}_D - J_{h_D}\boldsymbol{u}_D, I_{h_D}p_D - p_{D,h})\\
&\qquad + b_S^*(\tilde{p} - p_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) - \langle I_{h_D}p_D - p_{D,h}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{n}_{12}\rangle_\Gamma\\
&\qquad + \langle (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{n}_{12}, I_{h_D}p_D - p_{D,h}\rangle_\Gamma - G\langle (\boldsymbol{u}_S - I_{h_S}\boldsymbol{u}_S) \cdot \boldsymbol{t}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{t}\rangle_\Gamma\\
&\qquad - G\langle (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{t}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{t}\rangle_\Gamma.
\end{aligned}
$$

For $\boldsymbol{u}_S = (u_S^1, u_S^2)$ we have from (2.22),

$$\int_e (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_S) \cdot \boldsymbol{t}\ \ \mathrm{d}s = 0 \quad \forall\, e \in \Gamma.$$

Therefore, we have from the Cauchy–Schwarz inequality, (2.7) and (2.8),

$$
\begin{aligned}
&a_S(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, I_{h_D}p_D - p_{D,h})\\
&\quad = a_S(\tilde{\boldsymbol{\sigma}}_S - J_{h_S}\boldsymbol{\sigma}_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + a_D(\tilde{\boldsymbol{u}}_D - J_{h_D}\boldsymbol{u}_D, I_{h_D}p_D - p_{D,h})\\
&\qquad + b_S^*(\tilde{p} - p_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) - G\langle (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{t}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) \cdot \boldsymbol{t}\rangle_\Gamma\\
&\quad \le C(\|\tilde{\boldsymbol{\sigma}}_S - J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S}\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_h + \|\tilde{\boldsymbol{u}}_D - J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}\|I_{h_D}p_D - p_{D,h}\|_{Z_D}\\
&\qquad + \|p_S - \tilde{p}\|_P\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_h).
\end{aligned}
\tag{3.14}
$$

On the other hand (2.19), (3.12) and (3.13) imply

$$
\begin{aligned}
\nu^{-1}(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, \boldsymbol{\tau}_S)_{\Omega_S} &= a_S^*(I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}, \boldsymbol{\tau}_S) \quad \forall\, \boldsymbol{\tau}_S \in [\boldsymbol{V}_{h_S}]^2,\\
K^{-1}(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, \boldsymbol{v}_D)_{\Omega_D} &= a_D^*(I_{h_D}p_D - p_{D,h}, \boldsymbol{v}_D) \quad \forall\, \boldsymbol{v}_D \in \boldsymbol{V}_{h_D}.
\end{aligned}
\tag{3.15}
$$

Then employing the adjoint property (2.21) and Lemma 3.3 we have

$$
\begin{aligned}
&a_S(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, I_{h_D}p_D - p_{D,h})\\
&\quad = \nu^{-1}\|\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}^2 + K^{-1}\|\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}\|_{0,\Omega_D}^2\\
&\quad \ge C(\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_h^2 + \|I_{h_D}p_D - p_{D,h}\|_{Z_D}^2).
\end{aligned}
\tag{3.16}
$$

Therefore, (3.14) and (3.16) imply

$$\|I_{h_S}\boldsymbol{u}_S-\boldsymbol{u}_{S,h}\|_h+\|I_{h_D}p_D-p_{D,h}\|_{Z_D}\le C(\|\tilde{\boldsymbol{\sigma}}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S}+\|\tilde{\boldsymbol{u}}_D-J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}+\|p_S-\tilde{p}\|_P). \quad (3.17)$$

It follows from the discrete adjoint property (2.21), (2.24), (2.25), (3.12) and (3.13),

$$\begin{aligned}\nu^{-1}(\tilde{\boldsymbol{\sigma}}_S,\boldsymbol{\tau}_S)&=a_S^*(I_{h_S}\boldsymbol{u}_S,\boldsymbol{\tau}_S)=a_S^*(\boldsymbol{u}_S,\boldsymbol{\tau}_S)=\nu^{-1}(\boldsymbol{\sigma}_S,\boldsymbol{\tau}_S),\\ K^{-1}(\tilde{\boldsymbol{u}}_D,\boldsymbol{v}_D)&=a_D^*(J_{h_D}\boldsymbol{u}_D,\boldsymbol{v}_D)=a_D^*(\boldsymbol{u}_D,\boldsymbol{v}_D)=K^{-1}(\boldsymbol{u}_D,\boldsymbol{v}_D),\end{aligned}$$

which means that $\tilde{\boldsymbol{\sigma}}_S$ is the L^2 projection of $\boldsymbol{\sigma}_S$ onto $[\boldsymbol{V}_{h_S}]^2$ and $\tilde{\boldsymbol{u}}_D$ is the L^2 projection of $\boldsymbol{u}_D$ onto $\boldsymbol{V}_{h_D}$.

Thus

$$\begin{aligned}\|\tilde{\boldsymbol{\sigma}}_S-\boldsymbol{\sigma}_S\|_{0,\Omega_S}&\le\|\boldsymbol{\sigma}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S},\\ \|\tilde{\boldsymbol{u}}_D-\boldsymbol{u}_D\|_{0,\Omega_D}&\le\|\boldsymbol{u}_D-J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}.\end{aligned} \quad (3.18)$$

This with the triangle inequality implies

$$\begin{aligned}\|\tilde{\boldsymbol{\sigma}}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S}&\le\|\tilde{\boldsymbol{\sigma}}_S-\boldsymbol{\sigma}_S\|_{0,\Omega_S}+\|\boldsymbol{\sigma}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S}\le 2\|\boldsymbol{\sigma}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S},\\ \|\tilde{\boldsymbol{u}}_D-J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}&\le\|\tilde{\boldsymbol{u}}_D-\boldsymbol{u}_D\|_{0,\Omega_D}+\|\boldsymbol{u}_D-J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}\le 2\|\boldsymbol{u}_D-J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}.\end{aligned}$$

Then we have from (3.17),

$$\|I_{h_S}\boldsymbol{u}_S-\boldsymbol{u}_{S,h}\|_h+\|I_{h_D}p_D-p_{D,h}\|_{Z_D}\le C(\|\boldsymbol{\sigma}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S}+\|\boldsymbol{u}_D-J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}+\|p_S-\tilde{p}\|_P). \quad (3.19)$$

An appeal to Lemma 3.3 and (3.15) yields

$$\|\tilde{\boldsymbol{\sigma}}_S-\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}+\|\tilde{\boldsymbol{u}}_D-\boldsymbol{u}_{D,h}\|_{0,\Omega_D}\le C(\|I_{h_S}\boldsymbol{u}_S-\boldsymbol{u}_{S,h}\|_h+\|I_{h_D}p_D-p_{D,h}\|_{Z_D}), \quad (3.20)$$

which together with the triangle inequality, (3.18) and (3.19) implies

$$\begin{aligned}&\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}+\|\boldsymbol{u}_D-\boldsymbol{u}_{D,h}\|_{0,\Omega_D}\\ &\quad\le\|\boldsymbol{\sigma}_S-\tilde{\boldsymbol{\sigma}}_S\|_{0,\Omega_S}+\|\tilde{\boldsymbol{\sigma}}_S-\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}+\|\boldsymbol{u}_D-\tilde{\boldsymbol{u}}_D\|_{0,\Omega_D}+\|\tilde{\boldsymbol{u}}_D-\boldsymbol{u}_{D,h}\|_{0,\Omega_D}\\ &\quad\le C(\|\boldsymbol{\sigma}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S}+\|\boldsymbol{u}_D-J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}+\|p_S-\tilde{p}\|_{0,\Omega_S}).\end{aligned}$$

Estimate (3.10) follows from (2.27).

It remains to show estimate (3.11). The inf-sup condition (2.20) implies

$$\|p_{S,h}-\tilde{p}\|_{0,\Omega_D}\le C\sup_{\boldsymbol{v}\in[U_{h_S}]^2\setminus\{0\}}\frac{b_S(\boldsymbol{v},p_{S,h}-\tilde{p})}{\|\boldsymbol{v}\|_h}. \quad (3.21)$$

We have from (2.26) and (3.9),

$$\begin{aligned} b_S^*(p_{S,h}-\tilde{p},\boldsymbol{v}) &= b_S^*(p_S-\tilde{p},\boldsymbol{v}) + a_S(\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h},\boldsymbol{v}) + \langle p_D-p_{D,h},\boldsymbol{v}\cdot\boldsymbol{n}_{12}\rangle_\Gamma + G\langle(\boldsymbol{u}_S-\boldsymbol{u}_{S,h})\cdot\boldsymbol{t},\boldsymbol{v}\cdot\boldsymbol{t}\rangle_\Gamma \\ &= b_S^*(p_S-\tilde{p},\boldsymbol{v}) + a_S(J_{h_S}\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h},\boldsymbol{v}) + \langle I_{h_D}p_D-p_{D,h},\boldsymbol{v}\cdot\boldsymbol{n}_{12}\rangle_\Gamma \\ &\quad + G\langle(I_{h_S}\boldsymbol{u}_S-\boldsymbol{u}_{S,h})\cdot\boldsymbol{t},\boldsymbol{v}\cdot\boldsymbol{t}\rangle_\Gamma. \end{aligned}$$

Then the Cauchy–Schwarz inequality, (2.7), Lemma 3.1, Remark 3.2, (3.19) and (3.21) reveal that

$$\begin{aligned} \|p_{S,h}-\tilde{p}\|_0 &\le C(\|p_S-\tilde{p}\|_P + \|J_{h_S}\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S} + \|I_{h_D}p_D-p_{D,h}\|_{Z_D} + \|I_{h_S}\boldsymbol{u}_S-\boldsymbol{u}_{S,h}\|_h) \\ &\le C(\|p_S-\tilde{p}\|_P + \|\boldsymbol{\sigma}_S-J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{u}_D-I_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D}). \end{aligned}$$

Set $\tilde{p}=\pi_h p_S$; then the proof is complete using (2.27). □

Finally, we state the following theorem.

THEOREM 3.6 Let $(\boldsymbol{\sigma}_S,\boldsymbol{u}_S,p_S) \in [H^\alpha(\Omega_S)]^{2\times 2}\times[H^{1+\alpha}(\Omega_S)]^2\times H^\alpha(\Omega_S), \boldsymbol{u}_S|_E\in H^2(E), p_S|_E\in H^1(E), E\in\mathcal{T}_{u_S}$ and $(\boldsymbol{u}_D,p_D)\in[H^\alpha(\Omega_D)]^2\times H^{1+\alpha}(\Omega_D),\ p_D|_E\in H^2(E), E\in\mathcal{T}_{u_D}$ be the weak solution of (2.13), and let $(\boldsymbol{u}_{S,h},p_{D,h})$ be the numerical solution of (2.19). Then for $\frac{1}{2}<\alpha\le 1$,

$$\begin{aligned} \|\boldsymbol{u}_S-\boldsymbol{u}_{S,h}\|_{0,\Omega_S} + \|p_D-p_{D,h}\|_{0,\Omega_D} \le Ch\Bigg(&\|\boldsymbol{u}_S\|_{1+\alpha,\Omega_S} + \|p_D\|_{1+\alpha,\Omega_D} + \|p_S\|_{\alpha,\Omega_S} \\ &+ \left(\sum_{E\in\mathcal{T}_{u_S}}\|\boldsymbol{u}_S\|_{2,E}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{T}_{u_D}}\|p_D\|_{2,E}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{T}_{u_S}}|p_S|_{1,E}^2\right)^{1/2}\Bigg) \end{aligned}$$

Proof. We assume that the auxiliary problem

$$\nabla\cdot\boldsymbol{\xi}_S + \nabla r_S = I_{h_S}\boldsymbol{u}_S-\boldsymbol{u}_{S,h}, \tag{3.22a}$$

$$\boldsymbol{\xi}_S = -\nu\nabla\boldsymbol{w}_S, \tag{3.22b}$$

$$\nabla\cdot\boldsymbol{w}_S = 0, \tag{3.22c}$$

$$\nabla\cdot\boldsymbol{\zeta}_D = J_{h_D}p_D-p_{D,h}, \tag{3.22d}$$

$$\boldsymbol{\zeta}_D = -K\nabla\varphi_D, \tag{3.22e}$$

with interface condition

$$\begin{aligned} \boldsymbol{w}_S\cdot\boldsymbol{n}_{12} &= \boldsymbol{\xi}_D\cdot\boldsymbol{n}_{12}, \\ r_S-\nu\boldsymbol{n}_{12}\frac{\partial\boldsymbol{w}_S}{\partial\boldsymbol{n}_{12}} &= -\varphi_D, \\ -\nu\boldsymbol{t}\frac{\partial\boldsymbol{w}_S}{\partial\boldsymbol{n}_{12}} &= G\boldsymbol{w}_S\cdot\boldsymbol{t}, \end{aligned} \tag{3.23}$$

satisfies the regularity assumption

$$\|\boldsymbol{w}_S\|_{1+\alpha,\Omega_S} + \|r_S\|_{\alpha,\Omega_S} + \|\varphi_D\|_{1+\alpha,\Omega_D} \leq C(\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_{0,\Omega_S} + \|I_{h_D}\phi_D - \phi_{D,h}\|_{0,\Omega_D}). \tag{3.24}$$

Let $\tilde{\boldsymbol{\sigma}}_S$ and $\tilde{\boldsymbol{u}}_D$ be defined by (3.12) and (3.13). Multiplying (3.22a) by $I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}$, (3.22b) by $\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}$, (3.22c) by $\pi_h p_S - p_{S,h}$, (3.22d) by $I_{h_D}p_D - p_{D,h}$ and (3.22e) by $\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}$ we obtain

$$\begin{aligned}
&\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_{0,\Omega_S}^2 + \|I_{h_D}p_D - p_{D,h}\|_{0,\Omega_D}^2 \\
&\quad = (\nabla\cdot\boldsymbol{\xi}_S + \nabla r_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h})_{\Omega_S} + \nu^{-1}(\boldsymbol{\xi}_S, \tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h})_{\Omega_S} + (\nabla(\boldsymbol{w}_S - I_{h_S}\boldsymbol{w}_S), \tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h})_{\Omega_S} \\
&\qquad - (\nabla\cdot(\boldsymbol{w}_S - I_{h_S}\boldsymbol{w}_S), \pi_h p_S - p_{S,h})_{\Omega_S} + (\nabla\cdot\boldsymbol{\zeta}_D, I_{h_D}p_D - p_{D,h})_{\Omega_D} \\
&\qquad + K^{-1}(\boldsymbol{\zeta}_D, \tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h})_{\Omega_D} + (\nabla(\varphi_D - I_{h_D}\varphi_D), \tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h})_{\Omega_D}.
\end{aligned}$$

It follows from integration by parts and (3.23)

$$\begin{aligned}
&\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_{0,\Omega_S}^2 + \|I_{h_D}p_D - p_{D,h}\|_{0,\Omega_D}^2 \\
&\quad = -a_S(\boldsymbol{\xi}_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + b_S^*(r_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + \nu^{-1}(\boldsymbol{\xi}_S, \tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h})_{\Omega_S} \\
&\qquad + a_S(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, I_{h_S}\boldsymbol{w}_S) + b_S^*(\pi_h p_S - p_{S,h}, I_{h_S}\boldsymbol{w}_S) - a_D(\boldsymbol{\zeta}_D, I_{h_D}p_D - p_{D,h}) \\
&\qquad + K^{-1}(\boldsymbol{\zeta}_D, \tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h})_{\Omega_D} + a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, I_{h_D}\varphi_D) - \langle\varphi_D, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot\boldsymbol{n}\rangle_\Gamma \\
&\qquad + G\langle\boldsymbol{w}_S\cdot\boldsymbol{t}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot\boldsymbol{t}\rangle_\Gamma + \langle\boldsymbol{w}_S\cdot\boldsymbol{n}_{12}, I_{h_D}p_D - p_{D,h}\rangle_\Gamma.
\end{aligned}$$

The above equation can be rewritten as

$$\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_{0,\Omega_S}^2 + \|I_{h_D}p_D - p_{D,h}\|_{0,\Omega_D}^2 = \sum_{i=1}^{5} R_i, \tag{3.25}$$

where

$$\begin{aligned}
R_1 &= a_S(\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}, I_{h_S}\boldsymbol{w}_S) + b_S^*(p_S - p_{S,h}, I_{h_S}\boldsymbol{w}_S) + a_D(\boldsymbol{u}_D - \boldsymbol{u}_{D,h}, I_{h_D}\varphi_D), \\
R_2 &= -\langle\varphi_D, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot\boldsymbol{n}_{12}\rangle_\Gamma + G\langle\boldsymbol{w}_S\cdot\boldsymbol{t}, (I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot\boldsymbol{t}\rangle_\Gamma + \langle\boldsymbol{w}_S\cdot\boldsymbol{n}_{12}, I_{h_D}p_D - p_{D,h}\rangle_\Gamma, \\
R_3 &= \nu^{-1}(\boldsymbol{\xi}_S, \tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h})_{\Omega_S} - a_S(\boldsymbol{\xi}_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}) + K^{-1}(\boldsymbol{\zeta}_D, \tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h})_{\Omega_D} \\
&\quad - a_D(\boldsymbol{\zeta}_D, I_{h_D}p_D - p_{D,h}), \\
R_4 &= b_S^*(r_S - \pi_h r_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}), \\
R_5 &= a_S(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_S, I_{h_S}\boldsymbol{w}_S) + b_S^*(\pi_h p_S - p_S, I_{h_S}\boldsymbol{w}_S) + a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_D, I_{h_D}\varphi_D).
\end{aligned}$$

By the error equation (3.9) we have

$$
\begin{aligned}
R_1 = &-\langle p_D - p_{D,h}, I_{h_S}\boldsymbol{w}_S \cdot \boldsymbol{n}_{12}\rangle_\Gamma - G\langle(\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot \boldsymbol{t}, I_{h_S}\boldsymbol{w}_S \cdot \boldsymbol{t}\rangle_\Gamma \\
&+ \langle(\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\cdot \boldsymbol{n}_{12}, I_{h_D}\varphi_D\rangle_\Gamma = -R_2.
\end{aligned}
\tag{3.26}
$$

An application of (3.15) yields

$$
\begin{aligned}
\nu^{-1}(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, J_{h_S}\boldsymbol{\xi}_S)_{\Omega_S} &= a_S^*(I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}, J_{h_S}\boldsymbol{\xi}_S) = a_S(J_{h_S}\boldsymbol{\xi}_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}),\\
K^{-1}(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, J_{h_D}\boldsymbol{\zeta}_D)_{\Omega_D} &= a_D^*(I_{h_D}p_D - p_{D,h}, J_{h_D}\boldsymbol{\zeta}_D) = a_D(J_{h_D}\boldsymbol{\zeta}_D, I_{h_D}p_D - p_{D,h}).
\end{aligned}
$$

In view of (2.24), (2.26), (2.27), (3.19) and (3.20) we obtain

$$
\begin{aligned}
R_3 &= \nu^{-1}(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, \boldsymbol{\xi}_S - J_{h_S}\boldsymbol{\xi}_S)_{\Omega_S} - a_S(\boldsymbol{\xi}_S - J_{h_S}\boldsymbol{\xi}_S, I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h})\\
&\quad + K^{-1}(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, \boldsymbol{\zeta}_D - J_{h_D}\boldsymbol{\zeta}_D)_{\Omega_D} - a_D(\boldsymbol{\zeta}_D - J_{h_D}\boldsymbol{\zeta}_D, I_{h_D}p_D - p_{D,h})\\
&= \nu^{-1}(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}, \boldsymbol{\xi}_S - J_{h_S}\boldsymbol{\xi}_S)_{\Omega_S} + K^{-1}(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}, \boldsymbol{\zeta}_D - J_{h_D}\boldsymbol{\zeta}_D)_{\Omega_D}\\
&\le C(h^\alpha\|\boldsymbol{\xi}_S\|_{\alpha,\Omega_S}\|\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S} + h^\alpha\|\boldsymbol{\zeta}_D\|_{\alpha,\Omega_D}\|\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_{D,h}\|_{0,\Omega_D})\\
&\le Ch^{2\alpha}(\|\boldsymbol{\sigma}_S\|_{\alpha,\Omega_S} + \|\boldsymbol{u}_D\|_{\alpha,\Omega_D} + \|p_S\|_{\alpha,\Omega_S})(\|\boldsymbol{\xi}_S\|_{\alpha,\Omega_S} + \|\boldsymbol{\zeta}_D\|_{\alpha,\Omega_D}).
\end{aligned}
\tag{3.27}
$$

By (2.27) and (3.19) we see that

$$
\begin{aligned}
R_4 &\le Ch^\alpha\|r_S\|_{\alpha,\Omega_S}\|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_h\\
&\le Ch^\alpha(\|\boldsymbol{\sigma}_S - J_{h_S}\boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{u}_D - J_{h_D}\boldsymbol{u}_D\|_{0,\Omega_D} + \|p_S - \tilde{p}\|_P)\\
&\le Ch^{2\alpha}\|r_S\|_{\alpha,\Omega_S}(\|\boldsymbol{\sigma}_S\|_{\alpha,\Omega_S} + \|\boldsymbol{u}_D\|_{\alpha,\Omega_D} + \|p_S\|_{\alpha,\Omega_S}).
\end{aligned}
\tag{3.28}
$$

The following three terms can be bounded using the Cauchy–Schwarz inequality and standard interpolation error estimates (cf. Chung & Engquist, 2006);

$$\begin{aligned} a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_D, I_{h_D}\varphi_D) &= a_D(\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_D, I_{h_D}\varphi_D - \varphi_D) \\ &\leq \sum_{e\in\mathcal{F}_{p,D}} \|\tilde{\boldsymbol{u}}_D - \boldsymbol{u}_D\|_{0,e} \|I_{h_D}\varphi_D - \varphi_D\|_{0,e} \\ &\leq Ch \left(\sum_{E\in\mathcal{T}_{u_D}} \|\boldsymbol{u}_D\|_{1,E}^2\right)^{1/2} \|\varphi_D\|_{\alpha+1,\Omega_D}. \end{aligned}$$

Similarly, we have

$$a_S(\tilde{\boldsymbol{\sigma}}_S - \boldsymbol{\sigma}_S, I_{h_S}\boldsymbol{w}_S) \leq Ch \left(\sum_{E\in\mathcal{T}_{u_S}} \|\boldsymbol{\sigma}_S\|_{1,E}^2\right)^{1/2} \|\boldsymbol{w}_S\|_{1+\alpha,\Omega_S}$$

and

$$\begin{aligned} b_S^*(\pi_h p_S - p_S, I_{h_S}\boldsymbol{w}_S) &= b_S^*(\pi_h p_S - p_S, I_{h_S}\boldsymbol{w}_S - \boldsymbol{w}_S) \\ &\leq C \sum_{e\in\mathcal{F}_{p,S}} \|\pi_h p_S - p_S\|_{0,e} \|(I_{h_S}\boldsymbol{w}_S - \boldsymbol{w}_S)\cdot\boldsymbol{n}_e\|_{0,e} \\ &\leq Ch \left(\sum_{E\in\mathcal{T}_{u_S}} \|\nabla p_S\|_{0,E}^2\right)^{1/2} \|\boldsymbol{w}_S\|_1. \end{aligned}$$

Therefore,

$$\begin{aligned} R_5 \leq C\Bigg(& h^{2\alpha}(\|\boldsymbol{\sigma}_S\|_{\alpha,\Omega_S} + \|\boldsymbol{u}_D\|_{\alpha,\Omega_D})(\|\boldsymbol{w}_S\|_{1+\alpha,\Omega_S} + \|\varphi_D\|_{1+\alpha,\Omega_D}) \\ & + h\left(\left(\sum_{E\in\mathcal{T}_{u_S}} \|\boldsymbol{\sigma}_S\|_{1,E}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{T}_{u_D}} \|\boldsymbol{u}_D\|_{1,E}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{T}_{u_S}} \|\nabla p\|_{0,E}^2\right)^{1/2}\right) \\ & (\|\boldsymbol{w}_S\|_{1+\alpha,\Omega_S} + \|\varphi_D\|_{1+\alpha,\Omega_D})\Bigg). \end{aligned} \tag{3.29}$$

Combining the regularity estimate (3.24) and (3.25)–(3.29) we can get

$$\begin{aligned} \|I_{h_S}\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_{0,\Omega_S} + \|I_{h_D}p_D - p_{D,h}\|_{0,\Omega_D} \leq C(& h^{2\alpha}(\|\boldsymbol{\sigma}_S\|_{\alpha,\Omega_S} + \|\boldsymbol{u}_D\|_{\alpha,\Omega_D} + \|p_S\|_{\alpha,\Omega_S}) \\ & + h\left(\left(\sum_{E\in\mathcal{T}_{u_S}} \|\boldsymbol{\sigma}_S\|_{1,E}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{T}_{u_D}} \|\boldsymbol{u}_D\|_{1,E}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{T}_{u_S}} \|\nabla p_S\|_{0,E}^2\right)^{1/2}\right). \end{aligned}$$

Finally, it follows from the triangle inequality that

$$
\begin{aligned}
&\|\boldsymbol{u}_S - \boldsymbol{u}_{S,h}\|_{0,\Omega_S} + \|p_D - p_{D,h}\|_{0,\Omega_D} \\
&\quad \le \|\boldsymbol{u}_S - I_{h_S}\boldsymbol{u}_S\|_{0,\Omega_S} + \|\boldsymbol{u}_{S,h} - I_{h_S}\boldsymbol{u}_S\|_{0,\Omega_S} + \|p_D - I_{h_D}p_D\|_{0,\Omega_D} + \|p_{D,h} - I_{h_D}p_D\|_{0,\Omega_D} \\
&\quad \le Ch\left(\|\boldsymbol{u}_S\|_{1+\alpha,\Omega_S} + \|p_D\|_{1+\alpha,\Omega_D} + \|p_S\|_{\alpha,\Omega_S} + \left(\sum_{E\in\mathcal{T}_{u_S}} \|\boldsymbol{u}_S\|_{2,E}^2\right)^{1/2}\right. \\
&\qquad \left. + \left(\sum_{E\in\mathcal{T}_{u_D}} \|p_D\|_{2,E}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{T}_{u_S}} |p_S|_{1,E}^2\right)^{1/2}\right).
\end{aligned}
$$

□

4. Numerical experiments

In this section we examine the lowest-order SDG method for the coupled Stokes–Darcy problem by testing its convergence and flexibility on rough grids. Both the smooth solution and the solution with low regularity are chosen to test the convergence. The trapezoidal grid and h-perturbation grid are employed to test the flexibility of the proposed method.

As is well known it is difficult to find an analytical solution that satisfies the interface condition (2.6). To overcome this issue we employ the trick of generalizing the equations to include a nonhomogeneous term as proposed in Arbogast & Brunson (2007). That is, we replace (2.6b) and (2.6c) by

$$
\begin{aligned}
p_S - \nu \boldsymbol{n}_{12}\frac{\partial \boldsymbol{u}_S}{\partial \boldsymbol{n}_{12}} &= p_D + g_1 \qquad \text{on } \Gamma, \\
-\nu \boldsymbol{t}\frac{\partial \boldsymbol{u}_S}{\partial \boldsymbol{n}_{12}} &= G\boldsymbol{u}_S\cdot\boldsymbol{t} + g_2 \qquad \text{on } \Gamma.
\end{aligned} \tag{4.1}
$$

The variation formulation (2.13) has only a small change: the first equation of (2.13) now includes the two terms $-\langle g_1, \boldsymbol{v}_S\cdot\boldsymbol{n}_{12}\rangle_\Gamma - \langle g_2, \boldsymbol{v}_S\cdot\boldsymbol{t}\rangle_\Gamma$ on the right-hand side. A similar change can be made for the discrete formulation (2.19).

EXAMPLE 4.1 This example is taken from Li *et al.* (2018a). Let the computation domain be $\Omega = [0,1]\times[0,2]$, where $\Omega_S = [0,1]\times[1,2]$, $\Omega_D = [0,1]\times[0,1]$ and $\Gamma = [0,1]\times\{1\}$. The exact solution is given by

$$
\begin{aligned}
p_D &= -\frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right)y, \\
\boldsymbol{u}_S &= \left[-\cos^2\left(\frac{\pi y}{2}\right)\sin\left(\frac{\pi x}{2}\right), \frac{1}{4}\cos\left(\frac{\pi x}{2}\right)(\sin(\pi y) + \pi y)\right]^{\mathrm{T}}, \\
p_S &= -\frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right)\left(y - 2\cos^2\left(\frac{\pi y}{2}\right)\right),
\end{aligned}
$$

where we set the corresponding parameters to be $G = 1, \nu = 1, k = 1$.

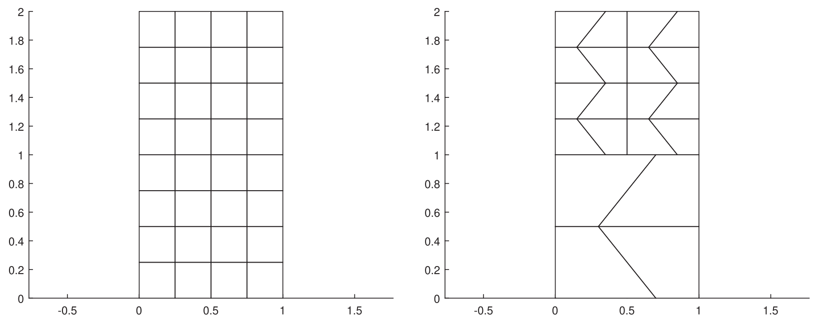

FIG. 3. Mesh of 4×4 square on the left and trapezoids on the right.

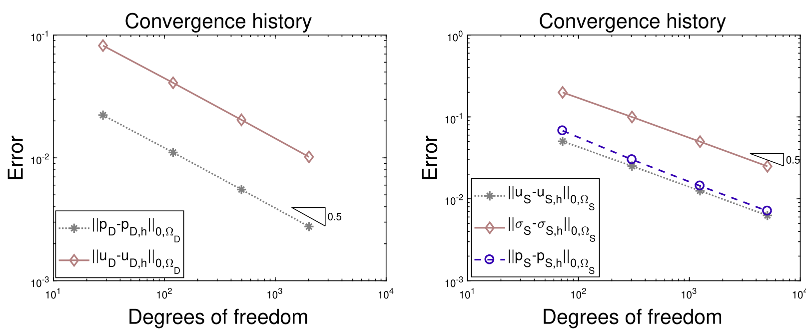

FIG. 4. Convergence history for the square mesh.

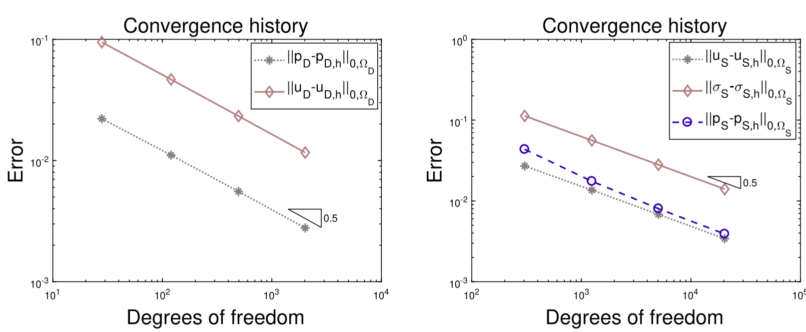

FIG. 5. Convergence history for the trapezoidal mesh.

We remark that the interface condition (2.6) is satisfied. The finite element solutions are computed by adopting two different sequences of meshes. The first is a uniform mesh of n^2 square elements in each $\Omega_i, i = S, D$ and the second is a mesh of trapezoids of base h in the vertical direction and parallel horizontal edges of sizes $0.6h$ and $1.4h$, as proposed in Arnold *et al.* (2005) and shown in Fig. 3.

The convergence histories for the two different sequences of meshes are displayed in Figs 4 and 5, and first-order convergence is obtained for all the approximations on both meshes. We conclude that the proposed method can be flexibly applied to rough grids.

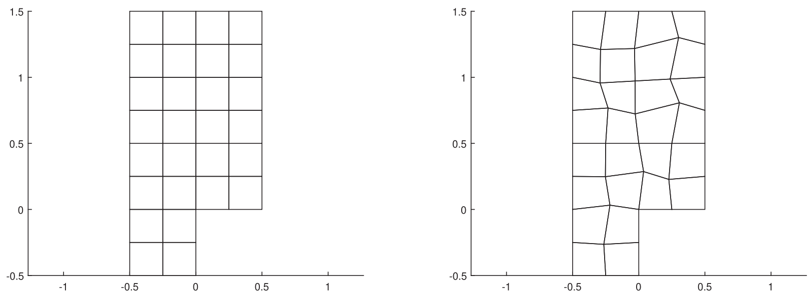

FIG. 6. Initial mesh.

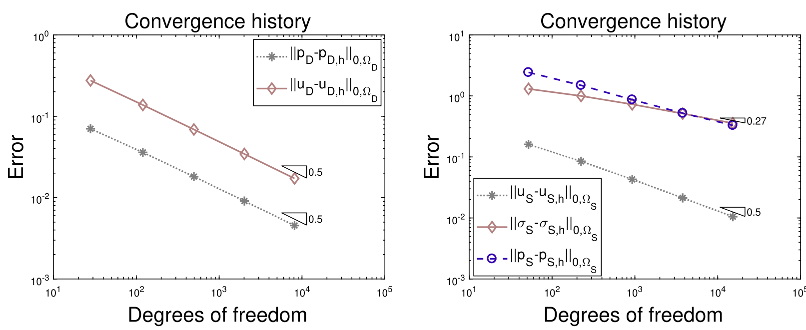

FIG. 7. Convergence history for the square mesh.

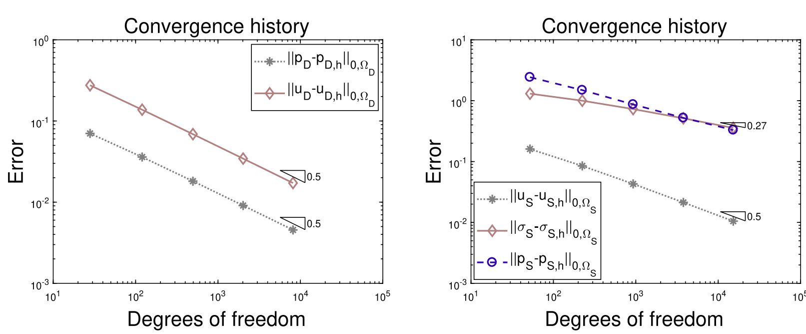

FIG. 8. Convergence history for the h-perturbation grids.

EXAMPLE 4.2 In this example we take $\Omega_S = (-0.5, 0.5)^2 \backslash [0, 0.5] \times [-0.5, 0]$, $\Omega_D = (-0.5, 0.5) \times (0.5, 1.5)$. The exact solution for the Stokes part in the polar coordinate system is given as (cf. Houston *et al.*, 2009)

$$\boldsymbol{u}_S(\rho, \phi) = \begin{pmatrix} \rho^\lambda((1+\lambda)\sin(\phi)\psi(\phi) + \cos(\phi)\psi'(\phi)) \\ \rho^\lambda(-(1+\lambda)\cos(\phi)\psi(\phi) + \sin(\phi)\psi'(\phi)) \end{pmatrix},$$

$$p_S(\rho, \phi) = -\rho^{\lambda-1}((1+\lambda)^2\psi'(\phi) + \psi'''(\phi))/(1-\lambda),$$

where

$$\psi(\phi) = \sin((1+\lambda)\phi)\cos(\lambda\omega)/(1+\lambda) - \cos((1+\lambda)\phi)$$
$$- \sin((1-\lambda)\phi)\cos(\lambda\omega)/(1-\lambda) + \cos((1-\lambda)\phi),$$

$\lambda \approx 0.544483736782\text{46}$ and $\omega = 3\pi/2$, $(\boldsymbol{u}_S, p_S) \in [H^{1+\lambda}(\Omega)]^2 \times H^{\lambda}(\Omega)$. For the Darcy part we choose the same solution as the Stokes part. In addition we set $G = 1, \nu = 1, k = 1$.

With this choice of the analytical solution the interface condition (2.6) is not satisfied, so (4.1) will be used. We notice that the Stokes region inhibits singularity and the Darcy region is smooth.

The finite element solutions are computed by employing two different sequences of meshes. The first is a uniform mesh of square elements in each $\Omega_i, i = S, D$ and the second is the h-perturbation grids as proposed in Aavatsmark *et al.* (2007), Mishev (2002) and shown in Fig. 6. The convergence histories for the errors measured in different norms on both meshes are reported in Figs. 7 and 8. The optimal convergence rates match the theoretical results. This example once again highlights that the proposed method is a viable option for computing rough solutions. We remark that in the preasymptotic region the convergence order of $\|p_S - p_{S,h}\|_{0,\Omega_S}$ is slightly higher than 0.27 as shown in Figs. 7 and 8.

5. Conclusion

We have developed a lowest-order SDG method for the coupled Stokes–Darcy model on general quadrilateral and polygonal meshes. Unlike previous formulations based on mixed finite element methods (see Gatica *et al.* 2009 and Gatica *et al.* 2011), this approach employs piecewise constant functions for all the approximations, which renders the proposed method computationally efficient. The stability of the discrete scheme is presented. In addition the optimal convergence rates covering low regularity for all the approximations are achieved. The main ingredients involved here are the new regularization operator. The numerical results indicate that the proposed method can be flexibly applied to rough grids and optimal convergence rates matching the theoretical results can be obtained. Note that the method proposed in this paper can also be extended to high-order polynomial cases. Our future work will extend the current approach to the coupling of flows governed by the Stokes and Darcy–Forchheimer equations (Girault & Wheeler, 2008; Kim & Park, 1999; Park, 2005).

Funding

National Research Foundation of Korea (2015R1A5A1009350 to E.-J.P.).

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