DEFORMATIONS OF SINGULARITIES

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Abstract. We will study explicitly the embedded deformations of singular affine schemes via explicit lifting of equations and relations. We prove that embedded deformations of codimension 2 Cohen-Macaulay closed subschemes are unobstructed. As a corollary, Hilbert schemes of smooth surfaces are smooth. Finally, we give an example of an obstructed deformation.

We begin by fixing some notation. Let k be a field and $R = P/I$, where $P = k[x_1, \dots, x_n]$ and $I = (f_1, \dots, f_r)$ is an ideal. Throughout this lecture, we will denote local Artinian rings with residue field k by $A, B, C, \dots$ and rings by $R, S, T, \dots$ Finally, denote $Z = \operatorname{Spec} R$.

1. Explicit criteria for flatness

We will study (embedded) deformations of singular affine schemes embedded in $\mathbb{A}^n$. The first thing we want to understand is to explicitly understand flatness of some R_A over A, where $R_A \otimes_A k = R$. We will write $R_A = P_A/I_A$, where $P_A = A[x_1, \dots, x_n] = A \otimes_k P$. Recall that over a Noetherian local ring S with residue field k, a module M is flat if and only if it is free, and this is equivalent to $\operatorname{Tor}_1^S(M, k) = 0$ by standard results in commutative algebra.

Now consider the exact sequence

$$0 \to I_A \to P_A \to R_A \to 0.$$

After tensoring with k, we have

$$0 \to \operatorname{Tor}_1(R_A, k) \to I_A \otimes_A k \to P \to R \to 0.$$

Therefore, we know that R_A is flat over A if and only if $I_A \otimes_A k = I$. We would like to understand this statement.

Consider a presentation

$$P_A^s \to P_A^r \to I_A \to 0$$

of I_A. Then we know R_A is flat over A if and only if after tensoring with k, we obtain an exact sequence

$$P^s \to P^r \to I \to 0.$$

Note that to give this presentation $P^s \to P^r \to I \to 0$ is the same as giving a complete set of relations among the generators of I.

Proposition 1.1. *Suppose that*

$$P^s \to P^r \to P \to R \to 0 \tag{1}$$

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is exact and

$$P_A^s \to P_A^r \to P_A \to R_A \to 0 \tag{2}$$

is a complex such that $P_A^r \to R_A \to R_A \to 0$ *is exact and tensoring (2) with* k *gives (1). Then* R_A *is flat over* A*.*

Proof. Note that the hypotheses are equivalent to the fact that all relations in I can be lifted to I_A. Now given $g_1', \ldots, g_r' \in P_A$ such that

$$\sum_{i=1}^{r} g_i' f_i' = 0,$$

this clearly descends to a relation in I by killing the maximal ideal of A. But now if we choose a complete set of relations for I_A, this descends to a complete set of relations in I, so we may in fact assume that (2) is exact.

In this case, there exists some L_A such that the sequence splits as

$$P_A^s \to L_A \to 0 \qquad 0 \to L_A \to P_A^r \to I_A \to 0 \qquad 0 \to I_A \to P_A \to R_A \to 0.$$

By right exactness of the tensor product, we know $P_A^s \otimes k \to L_A \otimes k \to 0$ is exact. We also know that

$$L_A \otimes k \to P_A^r \otimes k \to I_A \otimes k \to 0$$

is exact, again by right exactness. But this means that $I_A \otimes k$ is the cokernel of $P^s \to P^r$, and therefore $I_A \otimes k = I$. This means that R_A is flat. $\square$

Corollary 1.2. *Let* $R = P/I$ *and* $R_A = P_A/I_A$*, where* $I = (f_1, \ldots, f_r)$ *and* $I_A = (f_1', \ldots, f_r')$ *such that* f_i' *is a lift of* f_i*. Then* R_A *is flat over* A *if and only if every relation among the* f_i *lifts to a relation among the* f_i'*.*

Remark 1.3. This result essentially gives us that first-order embedded deformations of $\operatorname{Spec} R \subset \mathbb{A}^n$ are given by $\operatorname{Hom}(I, R)$. The first-order (not embedded) deformations of Z are given by the cokernel of

$$0 \to T_X \to T_{\mathbb{A}^n}|_X \to N_{X/\mathbb{A}^n},$$

which arises from the exact sequence

$$I/I^2 \to \Omega^1_{\mathbb{A}^n}|_X \to \Omega^1_X \to 0,$$

and this is supported on the singular points of X, so when X has isolated singularities, this is finite-dimensional.

Note that if $\operatorname{Spec} R \subset \mathbb{A}^n$ is a complete intersection, then I is generated by a regular sequence, so in particular the Koszul complex is a free resolution of R and therefore there are only trivial relations among the f_i (this means the relations are generated by $f_i f_j - f_j f_i = 0$). Clearly, because we are only considering commutative rings (after all, this is normal algebraic geometry), this means that all deformations of $\operatorname{Spec} R$ are unobstructed.

2. Hilbert schemes of smooth surfaces

We will prove that deformations of finite length closed subschemes of $\mathbb{A}^2$ are unobstructed. In particular, this will imply that the Hilbert scheme $\mathrm{Hilb}(\mathbb{A}^2, n)$ is smooth.

Let $Z \subset \mathbb{A}^2$ be a closed subscheme of dimension 0. Then because $P = k[x, y]$ has dimension 2, there exists a free resolution

$$0 \to P^s \xrightarrow{(g_{ij})} P^r \to P \to R \to 0$$

of R. In this case it is possible to understand the matrix (g_{ij}), and in fact this is the special case of a more general result. First, when we study the local behavior, we have the following result.

Theorem 2.1 (Hilbert, Burch)**.** *Let P be a regular local ring of dimension n and $R = P/I$ be a Cohen-Macaulay quotient of codimension 2. Then there exists an $(r-1) \times r$ matrix $G = (g_{ij})$ whose maximal minors $f_1, \dots, f_r$ minimally generate I, and there is a free resolution*

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \to R \to 0.$$

Proof. Note that the fact that the free resolution has this length is a corollary of the Auslander-Buchsbaum formula, which says that for a ring R and module M, we have

$$\operatorname{depth} M + \operatorname{proj.dim} M = \operatorname{depth} R$$

and the fact that depth equals dimension for Cohen-Macaulay things. Thus we have a free resolution

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(a_i)} P \to R \to 0,$$

where $a_1, \dots, a_r$ are a minimal set of generators for I. Let f_i is $(-1)^i$ times the determinant of the i-th minor of g_{ij}. We will prove that the map (f_i) is the same as the map (a_i); clearly

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \to R \to 0.$$

is a resolution. This is because at the generic point of P, we know (g_{ij}) is injective, so at least one f_i is nonzero. But then we know $\mathrm{coker}(g_{ij})$ is torsion-free (because I is torsion-free), and so it in fact must vanish by rank reasons. Thus $(a_1, \dots, a_r)$ and $(f_1, \dots, f_r)$ are isomorphic as P-modules.

At a codimension 1 point in $\operatorname{Spec} P$, note that $0 \to P^{r-1} \to P^r \xrightarrow{(a_i)} P \to B \to 0$ is split exact (because I has codimension 2). This implies that at least one f_i is a unit, and thus $(f_1, \dots, f_r)$ has codimension at least 2. But then the isomorphism $I \cong (f_1, \dots, f_r)$ is given by multiplication by some nonzero element of P which is a unit away from codimension 2. But this means it is a unit everywhere. $\square$

Considering the global picture in $\mathbb{A}^n$, we obtain the following result.

Theorem 2.2 (Hilbert, Schaps)**.** *Let $Z = \operatorname{Spec} R \subset \mathbb{A}^n$ be a Cohen-Macaulay closed subscheme of codimension 2. Then $R = P/I$ has a free resolution of the form*

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \to R \to 0$$

where the f_i are the maximal minors of the matrix (g_{ij}).

This result in fact holds over any Artinian local ring A, which we will use later.

Next, we want to understand what happens if we choose some Artinian local ring with residue field k and lift the g_{ij} to g'_{ij}, where $g'_{ij} \in P_A$.

Theorem 2.3 (Schaps)**.** *If A is a square zero extension of k, then the sequence*

$$0 \to P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(f'_i)} P_A \to R_A \to 0$$

is exact. Moreover, any lifting of R over A arises by lifting the matrix (g_{ij}).

Proof. We know that

$$L_A^\bullet \coloneqq P_A^{r-1} \to P_A^r \to P_A$$

is a complex. This is because composing the two maps amounts to evaluating determinants with a repeated column. Because P_A is free (and therefore flat), we can tensor with the exact sequence

$$0 \to \mathfrak{m}_A \to A \to k \to 0$$

to obtain an exact sequence of complexes

$$0 \to L_A^\bullet \otimes_A \mathfrak{m}_A \to L_A^\bullet \to L_A^\bullet \otimes_A k \to 0.$$

Note that

$$L_A^\bullet \otimes_A k = P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \eqqcolon L^\bullet.$$

In particular, this term is exact by Hilbert-Schaps. In addition, clearly $L_A^\bullet \otimes_A \mathfrak{m}_A = L^\bullet \otimes_k \mathfrak{m}_A$ because $A \to k$ is a square zero extension, so the complex $L_A^\bullet \otimes_A \mathfrak{m}_A$ is exact. By the long exact sequence in homology, we know that $L_A^\bullet$ is exact. Note that $L^\bullet$ extends to an exact sequence

$$0 \to P^{r-1} \to P^r \to P \to R \to 0,$$

and $L_A^\bullet$ extends to an exact sequence

$$0 \to P_A^{r-1} \to P_A^r \to P_A \to R_A \to 0.$$

However, by the homology long exact sequence, we have an exact sequence

$$0 \to R \otimes_k \mathfrak{m}_A \to R_A \to R \to 0.$$

But this implies that $R_A \otimes_A k = R$. Finally, by the local criterion for flatness, we see that R_A is flat over A.

Let $R_A = P_A/I_A$ be a lifting of R over A. Lift $f_i \in I$ to $h_i \in I_A$. By Nakayama, these generate I_A, so we obtain a free resolution

$$0 \to P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(h_i)} P_A \to R_A \to 0,$$

where g'_{ij} lift the g_{ij}. However, we already have a lift

$$0 \to P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(f'_i)} P_A \to R'_A \to 0,$$

and so we must show $R_A = R'_A$. But we know that the ideals $I_A = (h_1, \ldots, h_r)$ and $I'_A = (f'_1, \ldots, f'_r)$ are isomorphic as P_A-modules. But then if we restrict this isomorphism to $\mathbb{A}^n_A \setminus \operatorname{supp} B$, we obtain a unit in $H^0(\mathbb{A}^n_A \setminus \operatorname{supp} B, \mathcal{O}_{\mathbb{A}^n_A})$. Because functions extend over codimension 2, we have $H^0(\mathbb{A}^n_A \setminus \operatorname{supp} B, \mathcal{O}_{\mathbb{A}^n_A}) = P_A$, so this is a global unit. This gives the desired result. □

This result holds if we replace $A \to k$ with any square-zero extension of Artinian local rings $B \to A$ and P, P_A with flat things, and so we see that (embedded) deformations of codimension 2 Cohen-Macaulay subschemes of $\mathbb{A}^n$ are unobstructed. In particular, any dimension 0 closed subscheme $Z \subset \mathbb{A}^2$ is automatically Cohen-Macaulay (because it is dimension 0), so its embedded deformations are unobstructed. By some cohomological argument, the tangent space to $\mathrm{Hilb}(\mathbb{A}^2, n)$ is isomorphic to $\mathrm{Hom}(R, R)$ and has dimension $2n$, so

3. An obstructed deformation

Let $R = k[x, y, z]/(z^2, xy, xz, yz)$. Note that this scheme has an embedded point at the origin, so in particular it is **not** Cohen-Macaulay.

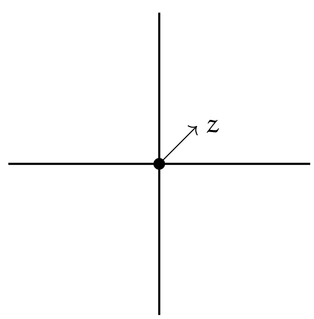

Figure 1. Drawing of Spec R

We will study embedded deformations of Spec R and see that they are obstructed. In particular, we will choose two deformations of R over $k[\varepsilon]$ that cannot be simultaneously lifted. We claim that a complete set of relations (using the ordering (xy, xz, yz, z^2) for the generators of I) is given by the matrix

$$G = \begin{pmatrix} z & -y & 0 & 0 \\ z & 0 & -x & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y \end{pmatrix}.$$

Now a first-order deformation of $\operatorname{Spec} R$ is given by lifting (xy, xz, yz, z^2) over $k[\varepsilon]$, and the first candidate is to consider $I_{\varepsilon_1} = (xy + \varepsilon_1 y, xz, yz, z^2)$. Then we note that

$$G \begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz \\ z^2 \end{pmatrix} = \varepsilon_1 \begin{pmatrix} yz \\ yz \\ 0 \\ 0 \end{pmatrix},$$

and we can lift G to kill this vector with the matrix

$$G_{\varepsilon_1} = \begin{pmatrix} z & -y & -\varepsilon_1 & 0 \\ z & 0 & -x - \varepsilon_1 & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y \end{pmatrix} = G + \begin{pmatrix} 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: G + G_1.$$

Next consider the deformation given by $I_{\varepsilon_2} = (xy, xz, yz + \varepsilon_2 z, z^2)$. We note that

$$G \begin{pmatrix} xy \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 0 \\ -xz \\ 0 \\ z^2 \end{pmatrix},$$

and we can lift G to kill this vector with the matrix

$$G_{\varepsilon_2} = \begin{pmatrix} z & -y & 0 & 0 \\ z & \varepsilon_2 & -x & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y - \varepsilon_2 \end{pmatrix} = G + \begin{pmatrix} 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: G + G_2.$$

Now we consider $I_{\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1 \varepsilon_2} = (xy + \varepsilon_1 y, xz, yz + \varepsilon_2 z, z^2)$ and attempt to lift this deformation to $k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$. Note that

$$\begin{aligned} (G + G_1 + G_2) \begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} &= \begin{pmatrix} z & -y & -\varepsilon_1 & 0 \\ z & \varepsilon_2 & -x - \varepsilon_1 & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y - \varepsilon_2 \end{pmatrix} \begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} \\ &= \varepsilon_1 \varepsilon_2 \begin{pmatrix} -z \\ -z \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

and clearly $z \notin I$, so in fact we cannot lift this deformation to $k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$. This proves obstructedness.