

*Defomation Theory Graduate Student Seminar*  
*Spring 2021*

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Lectures by Various



## **Disclaimer**

These notes were taken during the seminar using the vimtex package of the editor neovim. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at [plei@math.columbia.edu](mailto:plei@math.columbia.edu).

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## Contents

Contents • 2

- 1 Johan (Sep 24): Schlessinger's Paper • 3
- 2 Ivan and Cailan (Oct 1): Deformations of Schemes • 7
  - 2.1 DEFORMATIONS OF AFFINE SCHEMES • 7
  - 2.2 DEFORMATIONS OF SCHEMES • 10
- 3 Kevin (Oct 08): Deformations of coherent sheaves • 13
  - 3.1 TANGENT-OBSTRUCTION THEORY • 15
- 4 Patrick (Oct 15): Deformations of singularities • 17
  - 4.1 EXPLICIT CRITERIA FOR FLATNESS • 17
  - 4.2 HILBERT SCHEMES OF SMOOTH SURFACES • 18
  - 4.3 AN OBSTRUCTED DEFORMATION • 21
- 5 Avi (Oct 22): Local-global methods • 23
  - 5.1 CURVES WITH ISOLATED SINGULARITIES • 23
  - 5.2 SMOOTH CURVES WITH A FINITE GROUP ACTION • 25

## Johan (Sep 24): Schlessinger's Paper

The paper by Schlessinger is titled *Functors of Artin Rings*. Throughout this lecture,  $k$  is a field,  $\mathcal{C}$  is the category of Artinian local  $k$ -algebras  $A, B, C, \dots$  with residue field  $k$ , and  $\widehat{\mathcal{C}}$  is the category of Noetherian complete local  $k$ -algebras  $R, S, \dots$  with residue field  $k$ .

*Remark 1.0.1.* Every  $R \in \widehat{\mathcal{C}}$  is of the form  $k[[x_1, \dots, x_n]]/(f_1, \dots, f_m)$  by the Cohen structure theorem. Then  $R \in \mathcal{C}$  if and only if  $(f_1, \dots, f_m)$  contains  $(x_1, \dots, x_n)^N$  for some  $N$ .

*Remark 1.0.2.* In the paper, there is a more general setup, where  $\Lambda$  is a complete local Noetherian ring with residue field  $k$ . Then  $\mathcal{C}_\Lambda, \widehat{\mathcal{C}}$  are defined analogously, which will allow things like  $\Lambda = \mathbb{Z}_p$ .

The idea of deformation theory is to look at functors  $F: \mathcal{C} \rightarrow \text{Set}$ .

**Example 1.0.3.** Given  $R \in \widehat{\mathcal{C}}$ , we set  $h_R: \mathcal{C} \rightarrow \text{Set}$  sending  $A \mapsto \text{Hom}_{\widehat{\mathcal{C}}}(R, A)$ . This is not necessarily representable because  $R \notin \mathcal{C}$  in general, but it is pro-representable.

**Definition 1.0.4.** A functor  $F$  is *pro-representable* if  $F \simeq h_R$  for some  $R \in \widehat{\mathcal{C}}$ .

**Example 1.0.5.** Let  $M$  be a variety over  $k$  and  $m \in M(k)$ . Then define

$$\text{Def}_{M,m}(A) = \left\{ \text{Spec } A \xrightarrow{m_A} M \mid m_A|_{\text{Spec } k} = m \right\}.$$

It is easy to see that  $\text{Def}_{M,m}(A)$  is pro-representable by  $\widehat{\mathcal{O}}_{M,m}$ .

Observe that  $h_R(k) = \{*\}$  is a singleton. Also note that  $h_R(A \times_B C) = h_R(A) \times_{h_R(B)} h_R(C)$ . Here,  $A \times_B C$  is the fiber product of rings and not the tensor product.

Now consider the following conditions on  $F$ : let  $A \rightarrow B \leftarrow C$  be a diagram in  $\mathcal{C}$  and consider the morphism

$$F(A \times_B C) \xrightarrow{(*)} F(A) \times_{F(B)} F(C).$$

- (H<sub>1</sub>) The morphism  $(*)$  is surjective if  $C \twoheadrightarrow B$ ;
- (H<sub>2</sub>) The morphism  $(*)$  is bijective if  $C = k[\varepsilon] \twoheadrightarrow k = B$ ;
- (H<sub>3</sub>)  $\dim_k(t_F) < \infty$  (later, we will see that we need H<sub>2</sub> for formulate this). Here,  $t_F$  is the tangent space to  $F$ ;
- (H<sub>4</sub>) The morphism  $(*)$  is bijective if  $C \twoheadrightarrow B$ .

**Example 1.0.6.** Fix a group  $G$  and a representation  $\rho_0: G \rightarrow \mathrm{GL}_n(k)$ . Now define

$$\mathrm{Def}_{\rho_0}^{\mathrm{naive}}(A) = \{\rho: G \curvearrowright A^{\oplus n} \mid \rho \pmod{\mathfrak{m}_A} \cong \rho_0\} / \cong.$$

Better, we will define

$$\mathrm{Def}_{\rho_0}(A) = \{\rho: G \curvearrowright A^{\oplus n} \mid \rho \pmod{\mathfrak{m}_A} = \rho_0\} / \ker(\mathrm{GL}_n(A) \rightarrow \mathrm{GL}_n(k)).$$

In general these functors fail  $(H_4)$  and  $\mathrm{Def}_{\rho_0}^{\mathrm{naive}}$  even fails  $(H_2)$ .

Namely, if  $H = \mathbb{Z}$  and  $\rho_0$  is the trivial representation, then for  $\mathrm{Def}_{\rho_0}^{\mathrm{naive}}$ , we are looking at subsets of

$$\mathrm{GL}_n(A \times_B C) / \mathrm{conj} \rightarrow \mathrm{GL}_n(A) / \mathrm{conj} \times_{\mathrm{GL}_n(B) / \mathrm{conj}} \mathrm{GL}_n(C) / \mathrm{conj}.$$

This morphism is always surjective, but in general it is not injective.

For example, if  $A = k[\varepsilon_1]$ ,  $B = k$ ,  $C = k[\varepsilon_2]$ , we can look at elements of the form  $1 + \varepsilon_1 T_1 + \varepsilon_2 T_2$  and see that on the left we can only conjugate together, while on the right we can conjugate both  $T_1, T_2$  arbitrarily. Here  $A \times_B C = k[\varepsilon_1, \varepsilon_2] = k[x_1, x_2] / (x_1^2, x_1 x_2, x_2^2)$ .

**Definition 1.0.7.** A natural transformation  $t: F \rightarrow G$  of functors on  $\mathcal{C}$  is *smooth* if for all surjections  $B \twoheadrightarrow A$  the map  $F(B) \rightarrow F(A) \times_{G(A)} G(B)$  is surjective.

Note that this is equivalent to the existence of a lift in the diagram below:

$$\begin{array}{ccc} \mathrm{Spec} A & \longrightarrow & M \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec} B & \longrightarrow & N. \end{array}$$

This definition is motivated by the following example: let  $f: M \rightarrow N$  be a morphism of varieties over  $k$ . Let  $m \in M(k)$ ,  $n = f(m) \in N(k)$ . Then the following are equivalent:

1.  $\mathrm{Def}_{M,m} \rightarrow \mathrm{Def}_{N,n}$  is smooth.
2.  $f$  is smooth at  $m$ .

**Definition 1.0.8.** We say  $F$  has a *hull* if and only if  $F(k) = \{*\}$  and there exists a smooth  $t: h_R \rightarrow F$  for some  $R \in \widehat{\mathcal{C}}$  which induces an isomorphism  $t_R \cong t_F$ .

Now we will say a bit about tangent spaces.

1. When  $F(k) = \{*\}$ , then  $t_F = F(k[\varepsilon])$ .
2. If  $F$  satisfies  $(H_2)$  and  $F(k) = \{*\}$ , then  $t_F$  has a natural  $k$ -vector space structure. Here,  $H_2$  gives  $F(k[\varepsilon_1, \varepsilon_2]) \rightarrow F(k[\varepsilon]) \times F(k[\varepsilon])$  is a bijection, and then we take  $\varepsilon_1 \mapsto \varepsilon, \varepsilon_2 \mapsto \varepsilon$ , which defines addition.
3.  $t_R = \mathrm{Hom}_k(\mathfrak{m}_R / \mathfrak{m}_R^2, k) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathcal{R}, k[\varepsilon]) = h_R(k[\varepsilon]) = t_{(h_R)}$ .

**Theorem 1.0.9** (Schlessinger). Assume that  $F(k) = \{*\}$ . Then the conditions  $(H_1), (H_2), (H_3)$  hold for  $F$  if and only if  $F$  has a hull. In addition,  $(H_3)$  and  $(H_4)$  hold if and only if  $F$  is pro-representable.

*Very rough idea of proof of  $\Rightarrow$  for the hull case.* Let  $n = \dim_k(t_F)$ . Then  $(H_2)$  and  $n < \infty$  imply the following: Let  $S = k[[x_1, \dots, x_n]]$  and  $\mathfrak{m} = \mathfrak{m}_S = (x_1, \dots, x_n)$ . We can find  $\xi_1 \in F(S/\mathfrak{m}^2)$  such that

$$t_S = \text{Hom}_{\hat{\mathcal{C}}}(S, k[\varepsilon]) \xrightarrow{\xi_1} t_F$$

is an isomorphism.

Next, we will choose  $q \geq 2$  and consider pairs  $(J, \xi)$  where  $\mathfrak{m}^{q+1} \subset J \subset \mathfrak{m}^2$  and  $\xi \in F(S/J)$  such that  $\xi \mapsto \xi_1 \in F(S/\mathfrak{m}^2)$ . Say that  $(J, \xi) \leq (J', \xi')$  if  $J \subset J'$  and  $\xi \mapsto \xi'$ . Choose a minimal pair  $(J, \xi)$  for this ordering. We can choose  $J_q$  so that  $\mathfrak{m}^{q+1} + J_{q+1} = J_q$  and  $\xi_{q+1}$  maps to  $\xi_q$  for bookkeeping purposes.

Choose  $R = \lim S/J_q$ , which is a quotient of  $S$ . Set  $t: h_R \rightarrow F$  given by sending  $\varphi: R \rightarrow A$  to the following: choose  $q$  such that  $\varphi$  factors as  $R \rightarrow S/J_q \xrightarrow{\varphi_q} A$  and take  $\xi_q \mapsto F(\varphi_q)(\xi_q) \in F(A)$ .

Finally, we must show that  $t$  is smooth. Consider the diagram

$$\begin{array}{ccccc}
 S & \xrightarrow{\quad} & S/\mathfrak{m}^{q+1} & \xrightarrow{\quad} & B \\
 \downarrow & & \searrow \bar{\varphi}' & \searrow \bar{\varphi} & \downarrow \\
 R & \xrightarrow{\quad} & S/J_q & \xrightarrow{\quad} & A \\
 & & \uparrow \psi & \uparrow \text{pr}_1 & \\
 & & S/J_q \times_A B & \xrightarrow{\quad} & B
 \end{array}$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image. The actual diagram shows a complex set of maps including  $\varphi, \bar{\varphi}, \bar{\varphi}', \psi, \text{pr}_1$  and their compositions.)

with  $B \ni \tilde{\xi} \mapsto \xi \in A$  and  $S/J_q \ni \xi_q \mapsto \xi$ . First, choose  $\varphi: S \rightarrow B$  making the diagram commute. We may increase  $q$  such that  $\varphi(\mathfrak{m}^{q+1}) = 0$ , so we now have  $\bar{\varphi}: S/\mathfrak{m}^{q+1} \rightarrow B$ . Now consider the fiber product  $S/J_q \times_A B$  and  $\text{pr}_1: S/J_q \times_A B \rightarrow S/J_q$ , so we obtain  $\bar{\varphi}': S/\mathfrak{m}^{q+1} \rightarrow S/J_q \times_A B$ . By  $(H_1)$ , we obtain some  $\tilde{\xi} \in F(S/J_q \times_A B)$  mapping to  $\tilde{\xi}$  and  $\xi_q$ . We may now assume that  $B \rightarrow A$  is a small extension, which means that  $\dim_k \ker(B \rightarrow A) = 1$ , and thus  $\text{pr}_1$  is a small extension. Therefore, either  $\bar{\varphi}'$  is surjective or its image maps isomorphically via  $\text{pr}_1$  to  $S/J_q$ , so we have  $\psi$  which gives  $R \rightarrow B$  lifting our given  $r \rightarrow A$ .

The tricky part is to show that  $F(\psi)(\psi_q) = \tilde{\xi}$ , and this step is deliberately omitted.  $\square$

A generalization of this is as follows. Consider a functor  $\mathcal{F}: \mathcal{C} \rightarrow \text{Grpd}$ . We say that  $\mathcal{F}$  satisfies the *Rim-Schlessinger condition* (RS) if

$$\mathcal{F}(A \times_B C) \rightarrow \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(C)$$

is an equivalence whenever  $C \twoheadrightarrow B$ . Let  $x_0 \in \mathcal{F}(k)$  and set

$$\bar{\mathcal{F}}_{x_0}: \mathcal{C} \rightarrow \text{Set} \quad A \mapsto \{(x, \alpha) \mid x \in \mathcal{F}(A), \alpha: X_0 \rightarrow x|_k\} / \cong,$$

where  $(x, \alpha) \cong (x', \alpha')$  means that  $\varphi: x \rightarrow x'$  such that the diagram

$$\begin{array}{ccc}
 x|_k & \xrightarrow{\varphi} & x'|_k \\
 \alpha \uparrow & & \alpha' \uparrow \\
 x_0 & \xrightarrow{\text{id}} & x_0
 \end{array}$$

commutes.

**Theorem 1.0.10.** *If  $\mathcal{F}$  has (RS) then  $\overline{\mathcal{F}}_{x_0}$  has  $(H_1)$  and  $(H_2)$ . Therefore, if  $\dim t_{\overline{\mathcal{F}}_{x_0}} < \infty$  then  $\overline{\mathcal{F}}_{x_0}$  has a hull.*

In this situation,  $\overline{\mathcal{F}}_{x_0}$  has  $(H_4)$  if and only if  $\text{Aut}_A(x) \twoheadrightarrow \text{Aut}_B(x|_B)$  whenever  $A \twoheadrightarrow B$  and  $x \in \mathcal{F}_{x_0}(A)$ .

**Example 1.0.11.** Let  $\mathcal{F}(A)$  be the category of representations  $G \curvearrowright A^{\oplus n}$  with morphisms being isomorphisms of representations. This has (RS).

**Example 1.0.12.** Let  $\mathcal{F}(A)$  be the category of smooth projective families of curves of genus  $g$  over  $A$  with morphisms being isomorphisms. This has (RS).

Returning to the example of representations, it turns out that  $t_{\text{Def}_{\rho_0}} = H^1(G, M_{n \times n}(k))$ , where  $G$  acts on  $M_{n \times n}(k)$  via  $\rho_0$  by conjugation.

**Example 1.0.13.** Consider  $G = \mathbb{Z} \oplus \mathbb{Z}$  and  $\rho_0$  to be the trivial representation on  $k^{\oplus 2}$ . Then  $t_{\text{Def}_{\rho_0}} = H^1(\mathbb{Z}^2, M_2(k)) = M_2(k) \oplus M_2(k)$ . Given two matrices  $A, B$ , we have the representation

$$\begin{aligned} \mathbb{Z}^2 \rightarrow \text{GL}_2(k[\varepsilon])(1, 0) &\mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \varepsilon A \\ (0, 1) &\mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \varepsilon B. \end{aligned}$$

We get a hull  $R$  with  $h_R \rightarrow \text{Def}_{\rho_0}$ . We know that  $R$  is a some quotient of  $k[[a_{11}, \dots, a_{22}, b_{11}, \dots, b_{22}]]$  with  $\rho$  looking like

$$(1, 0) \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + A \quad (0, 1) \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + B,$$

and of course  $R$  is the quotient of the power series ring by the ideal generated by the coefficients of  $AB - BA$ .



## Ivan and Cailan (Oct 1): Deformations of Schemes

### 2.1 Deformations of affine schemes

We are looking for a Cartesian diagram of schemes

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \text{Spec } k & \longrightarrow & S \end{array}$$

where  $\pi$  is flat and surjective and  $S$  is surjective. This is called a *deformation* of  $X$  over  $S$ . For the beginning of this lecture (the part given by Ivan), we are interested in  $S = \text{Spec } A$ , where  $A \in \mathcal{C}^*$  (this category was defined in the previous lecture). This case is called a *local deformation*, and in the face where  $A$  is Artinian, it is called an *infinitesimal deformation*.

For the ring theorists, we will make the following digression. Let  $A$  be a ring and  $I \subset A$  be an ideal with  $I^2 = 0$ . Suppose that  $\bar{B}$  is an  $A/I$ -algebra,  $J$  is an  $\bar{B}$ -module, and  $h: I \rightarrow J$  is an  $A$ -module map. Then we are interested in a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & ? & \longrightarrow & \bar{B} \\ & & \uparrow h & & \uparrow & & \uparrow \\ 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I \longrightarrow 0, \end{array}$$

which we will call a deformation of  $A$ . Here are some interesting questions:

1. Is such a deformation unique?
2. If  $\bar{B}$  is flat over  $A/I$ , does that mean that  $B$  is flat over  $A$ ?

Returning to the case of schemes, we will say that two deformations  $\mathcal{X}, \mathcal{X}'$  of  $X$  over  $S$  are isomorphic if there exists an  $S$ -isomorphism  $\phi: \mathcal{X}' \rightarrow \mathcal{X}$  commuting with the inclusions of the central fibers  $X \rightarrow \mathcal{X}, \mathcal{X}'$ .

**Example 2.1.1.** The most basic example of a family is the trivial deformation

$$\begin{array}{ccc} X & \longrightarrow & X \times_k S \\ \downarrow & & \downarrow \\ \text{Spec } k & \longrightarrow & S. \end{array}$$

**Definition 2.1.2.** A scheme  $X$  is *rigid* if all deformations of  $X$  are isomorphic to the trivial deformation.

**Theorem 2.1.3.** If  $X$  is a smooth affine  $k$ -scheme and  $S = \operatorname{Spec} A$  for some local Artinian ring, then  $X$  is rigid.

**Definition 2.1.4.** A closed immersion  $i: S_0 \hookrightarrow S$  of schemes is called a *first (resp.  $n$ th) order thickening* if the ideal sheaf  $\mathcal{I} = \ker(i^b: \mathcal{O}_S \rightarrow \mathcal{O}_{S_0})$  satisfies  $\mathcal{I}^2 = 0$  (resp.  $\mathcal{I}^{n+1} = 0$ ).

**Definition 2.1.5.** A morphism  $f: X \rightarrow S$  is called *formally smooth* (resp. *unramified*, resp. *étale*) if for all first order thickenings  $i: T_0 \rightarrow T$  of affine schemes and diagrams

$$\begin{array}{ccc} T_0 & \xrightarrow{u_0} & X \\ \downarrow i & \nearrow \widetilde{u_0} & \downarrow f \\ T & \longrightarrow & S \end{array}$$

there exists a lift  $\widetilde{u_0}$  (resp. there is at most one such  $\widetilde{u_0}$ , resp. there exists a unique  $\widetilde{u_0}$ ).

**Example 2.1.6.**

1. Open immersions are formally étale. This is clear because  $T_0, T$  have the same underlying topological space.
2. Closed immersions are formally unramified. This is clear because  $X \rightarrow S$  induces an injection on  $T$ -points.
3.  $\mathbb{A}_S^n \rightarrow S$  is formally smooth. To see this, assume  $S = \operatorname{Spec} R$  is affine and then consider the corresponding lifting problem in commutative algebra.

**Proposition 2.1.7.** The classes of formally smooth (resp. étale, resp. unramified) morphisms are closed under base change, composition, and products and local on both source and target.

**Definition 2.1.8.** A  $f: X \rightarrow S$  is *smooth* if it is formally smooth and locally of finite presentation.

We will now consider differentials. Let  $X = \operatorname{Spec} A$  be an affine scheme over  $k$  and choose a  $k$ -point and consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[\varepsilon] & \longrightarrow & \operatorname{Spec} k. \end{array}$$

If  $X$  is smooth, then there exists a lift  $\operatorname{Spec} k[\varepsilon] \rightarrow X$ . But this is given by a morphism

$$\widetilde{\phi}: A \rightarrow k[\varepsilon]/\varepsilon^2 \quad a \mapsto \phi(a) + d(a)\varepsilon.$$

This motivates the following definition:

**Definition 2.1.9.** Let  $R \rightarrow A$  be a morphism of rings and  $M$  be an  $A$ -module. A *derivation*  $d: A \rightarrow M$  is an  $A$ -linear map satisfying the Leibniz rule.

**Proposition 2.1.10.** *There exists an  $A$ -module  $\Omega_{A/k}^1$  equipped with a derivation  $d: \Omega_{A/k}^1 \rightarrow \Omega_{A/k}^1$  that is universal among derivations from  $A$ . This means that all derivations  $\tilde{d}: A \rightarrow M$  factor through  $d$ , and formally, we have an identity*

$$\text{Der}_R(A, M) \simeq \text{Hom}_A(\Omega_{A/k}^1, M).$$

**Definition 2.1.11.** For an  $A$ -module  $M$  with derivation  $d: A \rightarrow M$ , define the ring  $A[M]$  as the module  $A \oplus M$  with the multiplication

$$(a, m) \cdot (a', m') = (aa', am' + a'm).$$

There is a sequence  $\phi: A \rightarrow A[M] \rightarrow A$ .

**Proposition 2.1.12.** *Let  $S \leftarrow R \rightarrow A \rightarrow B$  be a diagram of rings. Then*

1.  $\Omega_{A \otimes_R S/S}^1 \simeq \Omega_{A/R}^1 \otimes_R S$ ;
2. The sequence  $\Omega_{A/R}^1 \otimes_A B \rightarrow \Omega_{B/R}^1 \rightarrow \Omega_{B/A}^1 \rightarrow 0$  is exact.
3. If  $B = A/I$  for some ideal  $I$ , we have an exact sequence

$$I/I^2 \rightarrow \Omega_{A/R}^1 \otimes_A B \rightarrow \Omega_{B/R}^1 \rightarrow 0.$$

4. For all  $f \in A$ , we have  $\Omega_{A[f^{-1}]/R}^1 \simeq \Omega_{A/R}^1 \otimes_A A[f^{-1}]$ .

**Remark 2.1.13.** If  $J = \ker(A \otimes_R A \rightarrow A)$ , then  $\Omega_{A/R}^1 = J/J^2$ .

**Theorem 2.1.14.** *Let  $f: X \rightarrow S$  be locally of finite presentation. The following are equivalent:*

1.  $f$  is smooth;
2.  $f$  is flat with smooth fibers;
3.  $f$  is flat and has smooth geometric fibers.

We will finally return to deformation theory.

**Lemma 2.1.15.** *Let  $Z_0$  be a closed subscheme of  $Z$  determined by a nilpotent ideal sheaf  $N$ . If  $Z_0$  is affine, then so is  $Z$ .*

Proof of this result can be found in EGA, Chapter I.5.9.

*Proof of Theorem 2.1.3.* Recall that we have a diagram of the form

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \uparrow & & \uparrow \\ A & \longrightarrow & k, \end{array}$$

where  $A \rightarrow B$  is flat and  $B_0 \simeq B \otimes_A k$  is a smooth  $k$ -algebra. We need to prove that  $B_0 \simeq B \otimes_A k$ . The first step is to prove this result for first-order deformations. Suppose that  $A = k[\varepsilon]$  is a square-zero extension.

**Lemma 2.1.16.** *For a ring  $R$  with  $M, N$  flat over  $R$ , nilpotent ideal  $I \subset R$ , and  $f: M \rightarrow N$ , then if  $f \otimes_R R/I$  is an isomorphism, then so is  $f$ .*

To prove the lemma, note that the cokernel of  $f$  is preserved by  $I$ , so it must vanish. Returning to our case, we know that  $B$  is a smooth  $k[\varepsilon]$ -algebra. Now we obtain a square-zero extension  $B_0[\varepsilon]$  of  $B_0$  and a diagram

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ & & \uparrow \\ k[\varepsilon] & \xrightarrow{f} & B_0[\varepsilon] \end{array}$$

with a lift  $B \rightarrow B_0[\varepsilon]$ . But now by the lemma, we have  $B \otimes_{k[\varepsilon]} k = B_0[\varepsilon] \otimes_{k[\varepsilon]} k$ . The rest of the proof follows using an inductive argument that was verbalized but now written down.  $\square$

## 2.2 Deformations of schemes

The main theorem of this section is

**Theorem 2.2.1.** *Assume  $X$  is a smooth  $R$ -scheme. Then there is a bijection*

$$\mathrm{Def}_X^{\mathrm{sm}}(k[I]) \simeq H^1(X, T_{X/k} \otimes I).$$

*Proof.* Let  $X'$  be a smooth deformation over  $k[I]$ . Then the diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} k[I] \end{array}$$

is cartesian. Then if  $U_k = \mathrm{Spec} B_k$  is an affine cover of  $X$  and  $U'_k = \mathrm{Spec} D_k$  is an affine cover of  $X'$ , we have a  $k[I]$ -linear ring isomorphism

$$\varphi_k: k[I] \otimes_k B_k \rightarrow D_k \quad (k, i) \otimes b \mapsto s(b) + i.$$

Modulo  $I$ ,  $\varphi_k$  is the identity on  $B_k$ . Without loss of generality, we may assume that  $U_{kj} = U_k \cap U_j$  is a distinguished open for both  $U_k$  and  $U_j$ , so let  $U_{kj} = \mathrm{Spec} B_{kj}$  and  $U'_{kj} = \mathrm{Spec} D_{kj}$ . Now note that both

$$\varphi_k, \varphi_j: k[I] \otimes_k B_{kj} \rightarrow D_{kj}$$

induce the identity on  $B_{kj}$  modulo  $I$ . Now we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & D_{kj} & \longrightarrow & B_{kj} \longrightarrow 0 \\ & & \downarrow \mathrm{id} & & \downarrow \varphi_j^{-1} \varphi_k & & \downarrow \mathrm{id} \\ 0 & \longrightarrow & I & \longrightarrow & D_{kj} & \longrightarrow & B_{kj} \longrightarrow 0. \end{array}$$

**Lemma 2.2.2.** *The morphism  $g = \varphi_j^{-1} \varphi_k$  must be of the form*

$$g(i + b) = i + b + \delta(b),$$

where  $\delta: B_{kj} \rightarrow I$  is a derivation.

In particular, this means that  $\varphi_j^{-1} \circ \varphi_k(b, b') = (b, \alpha_{kj}(b) + b')$ , where  $\alpha_{kj}: B_{kj} \rightarrow I \otimes_k B_{kj}$  is a derivation.

By definition, we have

$$\begin{aligned} (T_{X/k} \otimes_k I)(B_{kj}) &= \text{Hom}_{B_{kj}}(\Omega_{B_{kj}/k}^1, B_{kj}) \otimes_k I \\ &= \text{Hom}_{B_{kj}}(\Omega_{B_{kj}/k}^1, B_{kj} \otimes_k I) \\ &= \text{Der}_k(B_{kj}, B_{kj} \otimes_k I). \end{aligned}$$

Therefore,  $\alpha_k \in H^0(B_{kj}, T_X \otimes_k I)$ . Note that

$$\varphi_\ell^{-1} \circ \varphi_j \circ \varphi_j^{-1} \circ \varphi_k^{-1} = \varphi_\ell^{-1} \circ \varphi_k,$$

which implies that

$$(b, \alpha_{j\ell}(b) + \alpha_{kj}(b) + b') = (b, \alpha_{k\ell}(b) + b')$$

and thus  $\{\alpha_{kj}\} \in Z^1(\{U_k\}, T_X \otimes_k I)$ .

If two deformations are the same, note that  $\varphi_k$  is defined using a ring section  $s_k: B_k \rightarrow D_k$  of the canonical map  $\pi_k: D_k \rightarrow B_k$ . If  $\varphi'_k$  is defined using another section  $s'_k$ , then define  $\theta_k = s'_k - s_k \in \text{Der}(B_k, I \otimes_k B_k)$ . We now compute

$$((\varphi'_j)^{-1} \circ \varphi'_k - \varphi_j^{-1} \circ \varphi_k)(b, b') = (0, \theta_k(b) - \theta_j(b)),$$

and thus the two differ by the desired coboundaries.  $\square$

We will now consider some obstructions. We are looking for a diagram of the form

$$\begin{array}{ccc} X' & \longrightarrow & X'' \\ \downarrow f & & \downarrow \\ \text{Spec } A' & \longrightarrow & \text{Spec } A''. \end{array}$$

for each pair  $(j, k)$ , we have a isomorphism  $\psi_{jk}: V'_j \rightarrow V'_k$  and a cocycle

$$c_{jkl} = \psi_{kl} \circ \psi_{jk} \circ \psi_{j\ell}^{-1}.$$

This induces  $B_{jkl} \in \text{Der}_A(D_{jkl}, J \otimes_A D_{kl}) = Z^2(U, T_{X'/A} \otimes_A J)$ .

Now we will discuss some examples.

**Theorem 2.2.3.** *Let  $C$  be a smooth projective curve,  $T = T_C$ , and  $K = \Omega_C^1$ . We have the following table:*

Table 2.1: Cohomology

	degree	$h^0$	$h^1$	$h^2$	
K	$2g - 2$	$g$	1	0	where $\varepsilon = 0$ where $g \geq 2$ , $\varepsilon = 1$ if $g = 1$ , and $\varepsilon = 3$ if $g = 0$ .
T	$2 - 2g$	$\varepsilon$	$\varepsilon + 3g - 3$	0	

For  $g \geq 2$ ,  $\deg T < 0$ , and by Riemann-Roch and Serre duality, we have  $h^1(C, T_C) = 3g - 3$ .

**Theorem 2.2.4.**  $\mathbb{P}^n$  has no infinitesimal deformations.

*Proof.* Consider the Euler sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus n+1} \rightarrow T_{\mathbb{P}^n} \rightarrow 0$$

and use the long exact sequence in cohomology. Because positive degree line bundles have no higher cohomology, we have  $H^1(T_{\mathbb{P}^n}) = 0$ .  $\square$

## Kevin (Oct 08): Deformations of coherent sheaves

There will be no mixed characteristic funny business during this lecture. Let  $X$  be a projective  $k$ -scheme (proper might be fine, but this makes certain facts more true) and  $\mathcal{F}$  be a coherent sheaf on  $X$ . Consider the deformation functor

$$D_{\mathcal{F}}: \text{Art}_k \rightarrow \text{Set} \quad A \mapsto \{\mathcal{F}_A \in \text{Coh}(X_A) \mid \mathcal{F}_A|_X \cong \mathcal{F}, \mathcal{F}_A \text{ flat over } A\}.$$

We want to study the properties of this functor, which means we will check Schlessinger's conditions:<sup>1</sup> Let  $A \rightarrow B \leftarrow C$  be a diagram in  $\mathcal{C}$  and consider the morphism

$$D(B \times_A C) \xrightarrow{r} D(B) \times_{D(A)} D(C).$$

- (H<sub>1</sub>) The morphism  $r$  is surjective if  $C \twoheadrightarrow A$ ;
- (H<sub>2</sub>) The morphism  $r$  is bijective if  $C = k[\varepsilon] \twoheadrightarrow k = A$ ;
- (H<sub>3</sub>)  $\dim_k(t_D) < \infty$  (later, we will see that we need H<sub>2</sub> to formulate this). Here,  $t_D$  is the tangent space to  $D$ ;
- (H<sub>4</sub>) The morphism  $r$  is bijective if  $C \twoheadrightarrow A$ .

Recall from Johan's lecture that (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>3</sub>) are equivalent to the existence of a hull and (H<sub>3</sub>), (H<sub>4</sub>) are equivalent to  $D$  being pro-representable.

We only need to check (H<sub>1</sub>) for small extensions, which are extensions by a  $k$ -vector space

$$0 \rightarrow I \rightarrow C \rightarrow A \rightarrow 0,$$

where  $I$  is killed by the maximal ideal of  $C$ .

**Theorem 3.0.1.** *The functor  $D_{\mathcal{F}}$  admits a hull.*

**Lemma 3.0.2.** *Let  $(A, \mathfrak{m})$  be a local Artinian ring.*

1. *If  $\mathfrak{m}M = M$ , then  $M \cong 0$ .*
2. *If  $M \rightarrow N$  induces an isomorphism  $M/\mathfrak{m}M \cong N/\mathfrak{m}N$  and  $N$  is flat over  $A$ , then  $M \cong N$ .*
3. *If  $M$  is flat, then  $M$  is free.*

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<sup>1</sup>Neither Kevin nor Johan knows why these conditions are called H

*Proof.* We know that  $m^d = 0$ , so  $m^d M = 0$ , and thus  $M = mM = m^2 M = \dots = m^d M = 0$ . Next, suppose  $M \rightarrow N$  induces an isomorphism after killing  $m$ . Then we know that the kernel and cokernel vanish because they are killed by  $m$ , so  $M \rightarrow N$  must be an isomorphism. The last part is left as an exercise.  $\square$

*Proof of theorem.* We will simply prove  $(H_1), (H_2), (H_3)$ :

1. Suppose that  $C \twoheadrightarrow A$  is a small extension and consider a pair  $(\mathcal{F}_B, \mathcal{F}_C) \in D(B) \times_{D(A)} D(C)$ . We know that we have isomorphisms  $\mathcal{F}_B|_{X_A} \cong \mathcal{F}_A, \mathcal{F}_C|_{X_A} \cong \mathcal{F}_A$ , and so we take the fiber product

$$\mathcal{F}_{B \times_A C} := \mathcal{F}_B \times_{\mathcal{F}_A} \mathcal{F}_C.$$

We only need to show that our sheaf is flat over  $B \times_A C$  because it clearly restricts to  $\mathcal{F}_B$  and  $\mathcal{F}_C$ . We can consider each sheaf as a module  $M$ , and so we know  $M_B$  is free over  $B$  by the lemma. Choose a basis  $\{e_i\}$ . Also consider the diagram

$$\begin{array}{ccc} M_B \times_{M_A} M_C & \longrightarrow & M_C \\ \downarrow & & \downarrow v \\ M_B & \xrightarrow{u} & M_A. \end{array}$$

Then  $M_A$  has  $A$ -basis  $u(e_i)$ . Because  $M_C$  surjects onto  $M_A$ , we can lift the  $u(e_i)$  to  $f_i \in C$ , and these form a  $C$ -basis for  $M_C$ . This all implies that  $M_B \times_{M_A} M_C$  is free with basis  $(e_i, f_i)$ .

2. It suffices to prove injectivity. Suppose  $\mathcal{G} \in D(B \times_k k[\varepsilon])$  maps to  $(\mathcal{F}_B, \mathcal{F}_{k[\varepsilon]}) \in D(B) \times D(k[\varepsilon])$ , and so we have morphisms

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{F}_{k[\varepsilon]} \\ \downarrow & & \downarrow \\ \mathcal{F}_B & \longrightarrow & \mathcal{F}. \end{array}$$

We will prove that this diagram is Cartesian. By the lemma, the morphism  $\mathcal{G} \rightarrow \mathcal{F}_B \times_{\mathcal{F}} \mathcal{F}_{k[\varepsilon]}$  is an isomorphism.

3. We will prove that  $T_D = \text{Ext}_X^1(\mathcal{F}, \mathcal{F})$ . We will only prove this in the case where  $\mathcal{F}$  is a vector bundle  $\mathcal{E}$  of rank  $r$ . In this case, we have  $\text{Ext}_X^1(\mathcal{E}, \mathcal{E}) = H^1(X, \text{End}(\mathcal{E}))$ . Now we will associate cocycles to deformations. To each  $\mathcal{E}_{k[\varepsilon]}$ , we will associate an open cover  $(U_j)$  and

$$h_{ij} \in \text{Aut}(\mathcal{O}_{X_{k[\varepsilon]}}^{\oplus r})(U_{ij}),$$

and we write  $g_{ij} + \varepsilon f_{ij}$ , where  $g_{ij} \in \text{Aut}_{\mathcal{O}_X^{\oplus r}}(U_{ij})$  and  $f_{ij} \in \text{End}(\mathcal{O}_X^{\oplus r})(U_{ij})$ . The cocycle condition is that

$$g_{ik} + \varepsilon f_{ik} = (g_{ij} + \varepsilon f_{ij})(g_{jk} + \varepsilon f_{jk}),$$

which is the same as

$$f_{ik} = g_{ij} f_{jk} + f_{ij} g_{jk},$$

which is exactly the Čech 1-cocycle condition. Proving that equivalent cocycles give the same deformation is easy.  $\square$

**Theorem 3.0.3.** *The condition  $(H_4)$  holds when  $\mathcal{F}$  is simple, which means that  $k \simeq \text{End}_X(\mathcal{F})$ .*



### 3.1 Tangent-obstruction theory

Suppose  $D$  is a deformation functor. Then a *tangent-obstruction theory* for  $D$  is given by finite-dimensional  $k$ -vector spaces  $(T^1, T^2)$ . Suppose we have a small extension

$$0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0.$$

Then we have another exact sequence

$$T^1 \otimes_k I \rightarrow D(B) \rightarrow D(A) \xrightarrow{\text{ob}} T^2 \otimes_k I,$$

which means that

1.  $\xi_A \in D(A)$  lifts to  $D(B)$  if and only if  $\text{ob}(\xi_A) = 0$ ;
2.  $T^1 \otimes I$  acts transitively on the fibers of  $D(B) \rightarrow D(A)$ ;
3. If  $A = k$ , then the action of  $T^1 \otimes I$  acts simply transitively on  $D(B)$ .

Note that because  $T^1$  acts simply transitively on  $D(k[\varepsilon])$ , we must have  $T^1 = D(k[\varepsilon])$ . On the other hand,  $T^2$  is not canonical.

**Theorem 3.1.1.** *The deformations  $D_{\mathcal{F}}$  admits a tangent-obstruction theory with  $T^1 = \text{Ext}_X^1(\mathcal{F}, \mathcal{F})$  and  $T^2 = \text{Ext}_X^2(\mathcal{F}, \mathcal{F})$ .*

*Proof.* We claim that if  $D$  satisfies  $(H_1)$  and  $(H_2)$ , then  $D(k[\varepsilon]) \otimes I$  naturally acts on  $D(B)$  for small extensions  $0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ . To see this, note that  $D(k[\varepsilon]) \otimes_k I = D(k[I])$ . We also note that by  $(H_2)$ ,  $D(k[I]) \times D(B) = D(k[I] \times_k B)$ . Now define  $\alpha: k[I] \times_k B \rightarrow B$  by  $\alpha(1 + i, b) = 1 + b$ , and this gives us an action of  $D(k[I]) \times D(B) = D(k[I] \times_k B) \xrightarrow{\alpha_*} D(B)$ . To prove transitivity, apply  $(H_1)$  to the diagram

$$\begin{array}{ccc} k[I] \times_k B & \xrightarrow{\alpha} & B \\ \downarrow \pi_B & & \downarrow \\ B & \longrightarrow & A. \end{array}$$

Now we will consider obstructions. We will assume again that  $\mathcal{F}$  is a rank  $r$  vector bundle, which we will call  $\mathcal{E}$ . Let

$$0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$$

be a small extension, so we will consider  $H^2(X, \text{End}(\mathcal{E}))$ . Consider an open cover  $(U_i)$  and  $g_{ij} \in \text{Aut}(\mathcal{O}_X^{\oplus r} \otimes_k A)(U_{ij})$ . We want to lift these to  $h_{ij} \in \text{Aut}(\mathcal{O}_X^{\oplus r} \otimes_k B)(U_{ij})$ . If this is possible, we have a cocycle

$$h_{ij}^{-1} h_{ij} h_{jk} \in 1 + \text{End}(\mathcal{O}_X^{\oplus r} \otimes_k I)(U_{ij}),$$

and the cocycle condition is satisfied when  $h_{ij}^{-1} h_{ij} h_{ik} = 1$ . If any other  $h'_{ij} = h_{ij} + s_{ij}$ , then we note that

$$(h'_{ij})^{-1} h'_{ij} h'_{jk} = h_{ik}^{-1} h_{ij} h_{jk} + (-s_{ik} g_{ij} g_{jk} + g_{ik}^{-1} s_{ij} g_{ij} + g_{ik}^{-1} g_{ij} s_{jk}),$$

and this gives us a class in  $H^2(X, \text{End}(\mathcal{E})) \otimes I$ . □

*Remark 3.1.2.* Let  $R$  be the hull of  $D$ , which means we have a morphism  $h_R \rightarrow D$ . Then we know  $R = k[[t_1, \dots, t_{d_1}]]/(f_1, \dots, f_{d_2})$ . We also know that  $d_1 - d_2 \leq \dim R \leq d_1$ .

**Example 3.1.3** (Good example). Let  $X$  be a smooth projective curve and  $\mathcal{E}$  be a rank  $r$  vector bundle. Then we know that

$$T^1 = H^1(X, \text{End}(\mathcal{E})), \quad T^2 = H^2(X, \text{End}(\mathcal{E})) = 0,$$

so deformations of  $\mathcal{E}$  are unobstructed. Now assume that  $\mathcal{E}$  is simple. Then  $H^0(X, \text{End}(\mathcal{E})) = k$  by definition, and we also know that  $D_\epsilon$  is pro-represented by some ring  $R$  with

$$\dim R = h^1(X, \text{End}(\mathcal{E})) = r^2(g-1) + 1$$

by Riemann-Roch.

**Example 3.1.4** (Bad example). Let  $X$  be a smooth projective variety and  $\mathcal{E}$  be a rank  $r$  vector bundle on  $X$ . Let  $\mathcal{E}_1, \mathcal{E}_2 = D_\mathcal{E}(k[\epsilon])$ . Then  $(\mathcal{E}_1, \mathcal{E}_2) \in D_\mathcal{E}(k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_1\epsilon_2, \epsilon_2^2))$ , and we would like to lift to  $D_\mathcal{E}(k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2))$ .

We will compute the obstruction explicitly. We know  $\mathcal{E}_1, \mathcal{E}_2$  give us classes  $u_1, u_2 \in H^1(X, \text{End}(\mathcal{E}))$ , and after some magical computation, the obstruction to lifting is given by

$$u_1 \smile u_2 + u_2 \smile u_1,$$

where the cup product comes from the algebra structure on  $\text{End}(\mathcal{E})$ .

Now let  $X = C_1 \times C_2$  be a product of curves. Then  $H^1(X, \mathcal{O}_X) = H^1(C_1, \mathcal{O}_{C_1}) \oplus H^1(C_2, \mathcal{O}_{C_2})$  and  $H^2(X, \mathcal{O}_X) = H^1(C_1, \mathcal{O}_{C_1}) \otimes H^1(C_2, \mathcal{O}_{C_2})$ . Suppose that  $\alpha_1 \in H^1(C_1, \mathcal{O}_{C_1})$  and  $\alpha_2 \in H^1(C_2, \mathcal{O}_{C_2})$  with nonzero cup product. Then we simply set  $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$  and

$$u_1 = \begin{pmatrix} 0 & \alpha_1 \\ 0 & 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 & 0 \\ \alpha_2 & 0 \end{pmatrix} \quad u_1 \smile u_2 + u_2 \smile u_1 = \begin{pmatrix} \alpha_1 \smile \alpha_2 & \\ & \alpha_2 \smile \alpha_1 \end{pmatrix}.$$

this gives us our obstructed deformation.

## Patrick (Oct 15): Deformations of singularities

We begin by fixing some notation. Let  $k$  be a field and  $R = P/I$ , where  $P = k[x_1, \dots, x_n]$  and  $I = (f_1, \dots, f_r)$  is an ideal. Throughout this lecture, we will denote local Artinian rings with residue field  $k$  by  $A, B, C, \dots$  and rings by  $R, S, T, \dots$ . Finally, denote  $Z = \text{Spec } R$ .

### 4.1 Explicit criteria for flatness

We will study (embedded) deformations of singular affine schemes embedded in  $\mathbb{A}^n$ . The first thing we want to understand is to explicitly understand flatness of some  $R_A$  over  $A$ , where  $R_A \otimes_A k = R$ . We will write  $R_A = P_A/I_A$ , where  $P_A = A[x_1, \dots, x_n] = A \otimes_k P$ . Recall that over a Noetherian local ring  $S$  with residue field  $k$ , a module  $M$  is flat if and only if it is free, and this is equivalent to  $\text{Tor}_1^S(M, k) = 0$  by standard results in commutative algebra.

Now consider the exact sequence

$$0 \rightarrow I_A \rightarrow P_A \rightarrow R_A \rightarrow 0.$$

After tensoring with  $k$ , we have

$$0 \rightarrow \text{Tor}_1(R_A, k) \rightarrow I_A \otimes_A k \rightarrow P \rightarrow R \rightarrow 0.$$

Therefore, we know that  $R_A$  is flat over  $A$  if and only if  $I_A \otimes_A k = I$ . We would like to understand this statement.

Consider a presentation

$$P_A^s \rightarrow P_A^r \rightarrow I_A \rightarrow 0$$

of  $I_A$ . Then we know  $R_A$  is flat over  $A$  if and only if after tensoring with  $k$ , we obtain an exact sequence

$$P^s \rightarrow P^r \rightarrow I \rightarrow 0.$$

Note that to give this presentation  $P^s \rightarrow P^r \rightarrow I \rightarrow 0$  is the same as giving a complete set of relations among the generators of  $I$ .

**Proposition 4.1.1.** *Suppose that*

$$(4.1) \quad P^s \rightarrow P^r \rightarrow P \rightarrow R \rightarrow 0$$

*is exact and*

$$(4.2) \quad P_A^s \rightarrow P_A^r \rightarrow P_A \rightarrow R_A \rightarrow 0$$

is a complex such that  $P_A^r \rightarrow R_A \rightarrow R_A \rightarrow 0$  is exact and tensoring (2) with  $k$  gives (1). Then  $R_A$  is flat over  $A$ .

*Proof.* Note that the hypotheses are equivalent to the fact that all relations in  $I$  can be lifted to  $I_A$ . Now given  $g'_1, \dots, g'_r \in P_A$  such that

$$\sum_{i=1}^r g'_i f'_i = 0,$$

this clearly descends to a relation in  $I$  by killing the maximal ideal of  $A$ . But now if we choose a complete set of relations for  $I_A$ , this descends to a complete set of relations in  $I$ , so we may in fact assume that (2) is exact.

In this case, there exists some  $L_A$  such that the sequence splits as

$$P_A^s \rightarrow L_A \rightarrow 0 \quad 0 \rightarrow L_A \rightarrow P_A^r \rightarrow I_A \rightarrow 0 \quad 0 \rightarrow I_A \rightarrow P_A \rightarrow R_A \rightarrow 0.$$

By right exactness of the tensor product, we know  $P_A^s \otimes k \rightarrow L_A \otimes k \rightarrow 0$  is exact. We also know that

$$L_A \otimes k \rightarrow P_A^r \otimes k \rightarrow I_A \otimes k \rightarrow 0$$

is exact, again by right exactness. But this means that  $I_A \otimes k$  is the cokernel of  $P^s \rightarrow P^r$ , and therefore  $I_A \otimes k = I$ . This means that  $R_A$  is flat.  $\square$

**Corollary 4.1.2.** *Let  $R = P/I$  and  $R_A = P_A/I_A$ , where  $I = (f_1, \dots, f_r)$  and  $I_A = (f'_1, \dots, f'_r)$  such that  $f'_i$  is a lift of  $f_i$ . Then  $R_A$  is flat over  $A$  if and only if every relation among the  $f_i$  lifts to a relation among the  $f'_i$ .*

*Remark 4.1.3.* This result essentially gives us that first-order embedded deformations of  $\text{Spec } R \subset \mathbb{A}^n$  are given by  $\text{Hom}(I, R)$ . The first-order (not embedded) deformations of  $Z$  are given by the cokernel of

$$0 \rightarrow T_X \rightarrow T_{\mathbb{A}^n}|_X \rightarrow N_{X/\mathbb{A}^n},$$

which arises from the exact sequence

$$I/I^2 \rightarrow \Omega_{\mathbb{A}^n}^1|_X \rightarrow \Omega_X^1 \rightarrow 0,$$

and this is supported on the singular points of  $X$ , so when  $X$  has isolated singularities, this is finite-dimensional.

Note that if  $\text{Spec } R \subset \mathbb{A}^n$  is a complete intersection, then  $I$  is generated by a regular sequence, so in particular the Koszul complex is a free resolution of  $R$  and therefore there are only trivial relations among the  $f_i$  (this means the relations are generated by  $f_i f_j - f_j f_i = 0$ ). Clearly, because we are only considering commutative rings (after all, this is normal algebraic geometry), this means that all deformations of  $\text{Spec } R$  are unobstructed.

## 4.2 Hilbert schemes of smooth surfaces

We will prove that deformations of finite length closed subschemes of  $\mathbb{A}^2$  are unobstructed. In particular, this will imply that the Hilbert scheme  $\text{Hilb}(\mathbb{A}^2, n)$  is smooth.

Let  $Z \subset \mathbb{A}^2$  be a closed subscheme of dimension 0. Then because  $P = k[x, y]$  has dimension 2, there exists a free resolution

$$0 \rightarrow P^s \xrightarrow{(g_{ij})} P^r \rightarrow P \rightarrow R \rightarrow 0$$

of  $R$ . In this case it is possible to understand the matrix  $(g_{ij})$ , and in fact this is the special case of a more general result. First, when we study the local behavior, we have the following result.

**Theorem 4.2.1** (Hilbert, Burch). *Let  $P$  be a regular local ring of dimension  $n$  and  $R = P/I$  be a Cohen-Macaulay quotient of codimension 2. Then there exists an  $(r-1) \times r$  matrix  $G = (g_{ij})$  whose maximal minors  $f_1, \dots, f_r$  minimally generate  $I$ , and there is a free resolution*

$$0 \rightarrow P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \rightarrow R \rightarrow 0.$$

*Proof.* Note that the fact that the free resolution has this length is a corollary of the Auslander-Buchsbaum formula, which says that for a ring  $R$  and module  $M$ , we have

$$\text{depth } M + \text{proj. dim } M = \text{depth } R$$

and the fact that depth equals dimension for Cohen-Macaulay things. Thus we have a free resolution

$$0 \rightarrow P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(a_i)} P \rightarrow R \rightarrow 0,$$

where  $a_1, \dots, a_r$  are a minimal set of generators for  $I$ . Let  $f_i$  is  $(-1)^i$  times the determinant of the  $i$ -th minor of  $g_{ij}$ . We will prove that the map  $(f_i)$  is the same as the map  $(a_i)$ ; clearly

$$0 \rightarrow P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \rightarrow R \rightarrow 0.$$

is a resolution. This is because at the generic point of  $P$ , we know  $(g_{ij})$  is injective, so at least one  $f_i$  is nonzero. But then we know  $\text{coker}(g_{ij})$  is torsion-free (because  $I$  is torsion-free), and so it in fact must vanish by rank reasons. Thus  $(a_1, \dots, a_r)$  and  $(f_1, \dots, f_r)$  are isomorphic as  $P$ -modules.

At a codimension 1 point in  $\text{Spec } P$ , note that  $0 \rightarrow P^{r-1} \rightarrow P^r \xrightarrow{(a_i)} P \rightarrow B \rightarrow 0$  is split exact (because  $I$  has codimension 2). This implies that at least one  $f_i$  is a unit, and thus  $(f_1, \dots, f_r)$  has codimension at least 2. But then the isomorphism  $I \cong (f_1, \dots, f_r)$  is given by multiplication by some nonzero element of  $P$  which is a unit away from codimension 2. But this means it is a unit everywhere.  $\square$

Considering the global picture in  $\mathbb{A}^n$ , we obtain the following result.

**Theorem 4.2.2** (Hilbert, Schaps). *Let  $Z = \text{Spec } R \subset \mathbb{A}^n$  be a Cohen-Macaulay closed subscheme of codimension 2. Then  $R = P/I$  has a free resolution of the form*

$$0 \rightarrow P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \rightarrow R \rightarrow 0$$

where the  $f_i$  are the maximal minors of the matrix  $(g_{ij})$ .

This result in fact holds over any Artinian local ring  $A$ , which we will use later.

Next, we want to understand what happens if we choose some Artinian local ring with residue field  $k$  and lift the  $g_{ij}$  to  $g'_{ij}$ , where  $g'_{ij} \in P_A$ .

**Theorem 4.2.3** (Schaps). *If  $A$  is a square zero extension of  $k$ , then the sequence*

$$0 \rightarrow P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(f'_i)} P_A \rightarrow R_A \rightarrow 0$$

is exact. Moreover, any lifting of  $R$  over  $A$  arises by lifting the matrix  $(g_{ij})$ .

*Proof.* We know that

$$L_A^\bullet := P_A^{r-1} \rightarrow P_A^r \rightarrow P_A$$

is a complex. This is because composing the two maps amounts to evaluating determinants with a repeated column. Because  $P_A$  is free (and therefore flat), we can tensor with the exact sequence

$$0 \rightarrow \mathfrak{m}_A \rightarrow A \rightarrow k \rightarrow 0$$

to obtain an exact sequence of complexes

$$0 \rightarrow L_A^\bullet \otimes_A \mathfrak{m}_A \rightarrow L_A^\bullet \rightarrow L_A^\bullet \otimes_A k \rightarrow 0.$$

Note that

$$L_A^\bullet \otimes_A k = P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P =: L^\bullet.$$

In particular, this term is exact by Hilbert-Schaps. In addition, clearly  $L_A^\bullet \otimes_A \mathfrak{m}_A = L^\bullet \otimes_k \mathfrak{m}_A$  because  $A \rightarrow k$  is a square zero extension, so the complex  $L_A^\bullet \otimes_A \mathfrak{m}_A$  is exact. By the long exact sequence in homology, we know that  $L_A^\bullet$  is exact. Note that  $L^\bullet$  extends to an exact sequence

$$0 \rightarrow P^{r-1} \rightarrow P^r \rightarrow P \rightarrow R \rightarrow 0,$$

and  $L_A^\bullet$  extends to an exact sequence

$$0 \rightarrow P_A^{r-1} \rightarrow P_A^r \rightarrow P_A \rightarrow R_A \rightarrow 0.$$

However, by the homology long exact sequence, we have an exact sequence

$$0 \rightarrow R \otimes_k \mathfrak{m}_A \rightarrow R_A \rightarrow R \rightarrow 0.$$

But this implies that  $R_A \otimes_A k = R$ . Finally, by the local criterion for flatness, we see that  $R_A$  is flat over  $A$ .

Let  $R_A = P_A/I_A$  be a lifting of  $R$  over  $A$ . Lift  $f_i \in I$  to  $h_i \in I_A$ . By Nakayama, these generate  $I_A$ , so we obtain a free resolution

$$0 \rightarrow P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(h_i)} P_A \rightarrow R_A \rightarrow 0,$$

where  $g'_{ij}$  lift the  $g_{ij}$ . However, we already have a lift

$$0 \rightarrow P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(f'_i)} P_A \rightarrow R'_A \rightarrow 0,$$

and so we must show  $R_A = R'_A$ . But we know that the ideals  $I_A = (h_1, \dots, h_r)$  and  $I'_A = (f'_1, \dots, f'_r)$  are isomorphic as  $P_A$ -modules. But then if we restrict this isomorphism to  $\mathbb{A}_A^n \setminus \text{supp } B$ , we obtain a unit in  $H^0(\mathbb{A}_A^n \setminus \text{supp } B, \mathcal{O}_{\mathbb{A}_A^n})$ . Because functions extend over codimension 2, we have  $H^0(\mathbb{A}_A^n \setminus \text{supp } B, \mathcal{O}_{\mathbb{A}_A^n}) = P_A$ , so this is a global unit. This gives the desired result.  $\square$

This result holds if we replace  $A \rightarrow k$  with any square-zero extension of Artinian local rings  $B \rightarrow A$  and  $P, P_A$  with flat things, and so we see that (embedded) deformations of codimension 2 Cohen-Macaulay subschemes of  $\mathbb{A}^n$  are unobstructed. In particular, any dimension 0 closed subscheme  $Z \subset \mathbb{A}^2$  is automatically Cohen-Macaulay (because it is dimension 0), so its embedded deformations are unobstructed. By some cohomological argument, the tangent space to  $\text{Hilb}(\mathbb{A}^2, n)$  is isomorphic to  $\text{Hom}(R, R)$  and has dimension  $2n$ , so

### 4.3 An obstructed deformation

Let  $R = k[x, y, z]/(z^2, xy, xz, yz)$ . Note that this scheme has an embedded point at the origin, so in particular it is **not** Cohen-Macaulay.

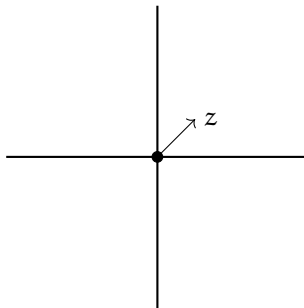


Figure 4.1: Drawing of Spec R

We will study embedded deformations of Spec R and see that they are obstructed. In particular, we will choose two deformations of R over  $k[\varepsilon]$  that cannot be simultaneously lifted. We claim that a complete set of relations (using the ordering  $(xy, xz, yz, z^2)$  for the generators of I) is given by the matrix

$$G = \begin{pmatrix} z & -y & 0 & 0 \\ z & 0 & -x & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y \end{pmatrix}.$$

Now a first-order deformation of Spec R is given by lifting  $(xy, xz, yz, z^2)$  over  $k[\varepsilon]$ , and the first candidate is to consider  $I_{\varepsilon_1} = (xy + \varepsilon_1 y, xz, yz, z^2)$ . Then we note that

$$G \begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz \\ z^2 \end{pmatrix} = \varepsilon_1 \begin{pmatrix} yz \\ yz \\ 0 \\ 0 \end{pmatrix},$$

and we can lift G to kill this vector with the matrix

$$G_{\varepsilon_1} = \begin{pmatrix} z & -y & -\varepsilon_1 & 0 \\ z & 0 & -x - \varepsilon_1 & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y \end{pmatrix} = G + \begin{pmatrix} 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: G + G_1.$$

Next consider the deformation given by  $I_{\varepsilon_2} = (xy, xz, yz + \varepsilon_2 z, z^2)$ . We note that

$$G \begin{pmatrix} xy \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 0 \\ -xz \\ 0 \\ z^2 \end{pmatrix},$$

and we can lift  $G$  to kill this vector with the matrix

$$G_{\varepsilon_2} = \begin{pmatrix} z & -y & 0 & 0 \\ z & \varepsilon_2 & -x & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y - \varepsilon_2 \end{pmatrix} = G + \begin{pmatrix} 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: G + G_2.$$

Now we consider  $I_{\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1 \varepsilon_2} = (xy + \varepsilon_1 y, xz, yz + \varepsilon_2 z, z^2)$  and attempt to lift this deformation to  $k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$ . Note that

$$\begin{aligned} (G + G_1 + G_2) \begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} &= \begin{pmatrix} z & -y & -\varepsilon_1 & 0 \\ z & \varepsilon_2 & -x - \varepsilon_1 & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y - \varepsilon_2 \end{pmatrix} \begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} \\ &= \varepsilon_1 \varepsilon_2 \begin{pmatrix} -z \\ -z \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

and clearly  $z \notin I$ , so in fact we cannot lift this deformation to  $k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$ . This proves obstructedness.



## Avi (Oct 22): Local-global methods

### 5.1 Curves with isolated singularities

Here, we will consider the same lifting problems that we have considered before. Let  $X$  be a curve over  $k$  and consider  $p: \text{Def}_X \rightarrow \mathcal{C}$  be the deformations of  $X$  as a category cofibered over  $\mathcal{C}$ .

**Definition 5.1.1.** A functor  $F: D_1 \rightarrow D_2$  is *smooth* if given  $\varphi: A' \rightarrow A$  and  $(Z, A') \rightarrow F(Y, Z)$  over  $\varphi$  over  $\varphi$  and  $(Y', A') \rightarrow (Y, A)$  over  $\varphi$  in  $D_1$ , then there exists some  $F(Y', A') \rightarrow (Z, A')$  over the identity in  $A'$ .

There is an absolute notion, for  $D_1$  to be smooth, where we set  $D_2 = \text{Def}_k$  to be the category of trivial deformations. In this case, we also called  $D_1$  *unobstructed*, and this corresponds to being unobstructed in the tangent-obstruction theory.

**Theorem 5.1.2** (Local-to-global). *Let  $X/k$  be separated of dimension at most 1 and smooth away from finitely many points. At each singularity  $p_1, \dots, p_n$ , consider the inclusions*

$$\mathcal{O}_{X,p} \rightarrow \mathcal{O}_{X,p}^h \rightarrow \hat{\mathcal{O}}_{X,p}$$

*into the henselizations and completions, respectively. Then the functors*

$$\text{Def}_X \rightarrow \prod_i \text{Def}_{\mathcal{O}_{X,p}} \xrightarrow{(2)} \prod \text{Def}_{\mathcal{O}_{X,p}^h} \xrightarrow{(3)} \prod \text{Def}_{\hat{\mathcal{O}}_{X,p}}$$

*are all smooth and (2), (3) induce isomorphisms on tangent spaces.*

This means that we only have to check unobstructedness at the completions of the local rings at each singular point.

*Proof.* First, we can reduce to the affine case.<sup>1</sup> If  $X$  is a curve, then  $X = U_1 \cup U_2$  can be covered by two affine opens such that  $U_1, U_1 \cap U_2$  are smooth. Because deformations of smooth affine schemes are unobstructed, we can essentially ignore  $U_1$ . To do this, we will prove that  $\text{Def}_X \rightarrow \text{Def}_{U_2}$  is smooth.

Because smoothness (roughly) respects products, the functor

$$\text{Def}_X \rightarrow \text{Def}_{U_1} \times_{\text{Def}_{U_1 \cap U_2}} \text{Def}_{U_2}$$

<sup>1</sup>Avi says that Johan sketched this proof and then said not to give it.

is smooth by formal reasons. Because  $U_1, U_1 \cap U_2$  are smooth, we can ignore  $\text{Def}_{U_1}, \text{Def}_{U_1 \cap U_2}$  (in fact, the arrow is an equivalence) and project down to  $\text{Def}_{U_2}$ .

Now  $U_2 = \text{Spec } P$  is affine. Then each  $p_i$  gives a maximal ideal  $\mathfrak{m}_i$ , and so we set  $J = \bigcap_i \mathfrak{m}_i$ . If we consider the completion  $\hat{P}_J$  with respect to  $J$ , this decomposes as

$$\hat{P}_J = \prod_i \hat{P}_i.$$

Applying [Lemma 91.12.5](#) in the Stacks Project, the functor

$$\text{Def}_P \rightarrow \text{Def}_{\hat{P}_J} \simeq \prod_i \text{Def}_{\hat{P}_i} = \prod_i \text{Def}_{\hat{\mathcal{O}}_{X, p_i}}$$

is smooth and an isomorphism on tangent spaces. The henselization step is similar.  $\square$

**Example 5.1.3.** Let  $X = \text{Spec } k[x, y]/(xy)$ . Then  $\hat{\mathcal{O}}_{X, 0} = k[[x, y]]/(xy)$ . We will show that all deformations look like  $k[x, y]/(xy - t)$  in some sense. Really, we will show that  $\text{Def}_X$  has a hull, which is given by  $k[[t]]$  with universal deformation

$$X_t := \text{Spec } k[[t]][x, y]/(xy - t) \rightarrow \text{Spec } k[[t]].$$

We want  $T$  such that  $\text{Hom}(T, -) \rightarrow \text{Def}_X$  is smooth and induces an isomorphism on tangent spaces. Given a morphism

$$f: k[[t]] \rightarrow A \ni f(t),$$

we obtain a deformation

$$A[[x, y]]/(xy - f(t))$$

of  $k[[x, y]]/(xy)$ . We can check that this is smooth, and so we need to check that we have an isomorphism of tangent spaces.

We want to compute  $\text{Def}_X(k[\varepsilon])$ . If  $S \in \text{Def}_X(k[\varepsilon])$ , set  $R = k[[x, y]]/(x, y)$ , and we have a diagram

$$\begin{array}{ccc} k[\varepsilon] & \longrightarrow & k \\ \downarrow & & \downarrow \\ S & \longrightarrow & R. \end{array}$$

If we consider the exact sequence

$$0 \rightarrow k \rightarrow k[\varepsilon] \rightarrow k \rightarrow 0$$

and tensor with  $S$ , by flatness we have an exact sequence

$$0 \rightarrow k \otimes_{k[\varepsilon]} S \rightarrow S \rightarrow k \otimes_{k[\varepsilon]} S \rightarrow 0.$$

But because  $k \otimes_{k[\varepsilon]} S = R$ , we have a decomposition  $S = R \oplus \varepsilon R$ . If we choose lifts  $\tilde{x}, \tilde{y}$  of  $x, y$ , the product  $\tilde{x}\tilde{y}$  lifts  $xy = 0$ , so  $\tilde{x}\tilde{y} = (f(x) + g(y))\varepsilon$ . If we choose different lifts  $\tilde{x}' = \tilde{x} + \varepsilon h, \tilde{y}' = \tilde{y} + \varepsilon h$ , we see that

$$\tilde{x}'\tilde{y}' = \tilde{x}\tilde{y} + (xj + yh)\varepsilon.$$

But now  $xj + yh \in (x, y)$ , and so if we set  $t = f(0) + g(0)$ , then we can write  $\tilde{x}\tilde{y} \in t\varepsilon + (x, y)\varepsilon$ , and so there is a canonical choice of the element of  $(x, y)\varepsilon$ , namely 0. Thus we can set

$$S_t = k[\varepsilon][[\tilde{x}, \tilde{y}]]/(\tilde{x}\tilde{y} - t\varepsilon),$$

and now we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & S_t & \longrightarrow & R \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & R & \longrightarrow & S & \longrightarrow & R \longrightarrow 0. \end{array}$$

By the five lemma,  $S_t \simeq S$ , and so we are done.

## 5.2 Smooth curves with a finite group action

In this section, we will consider lifting from characteristic  $p > 0$  to characteristic 0. This means we will have to redefine what lifting. Here, we will suppose  $k = \bar{k}$  with characteristic  $p$ . Let  $\Lambda$  be a Noetherian complete local ring with residue field  $k$ . We define  $\mathcal{C}_\Lambda$  be the category of local Artinian  $\Lambda$ -algebras with residue field  $k$ .<sup>2</sup>

Forgetting about the requirement that  $k$  is algebraically closed, if  $k = \mathbb{F}_p$ , then we can set  $\Lambda = \mathbb{Z}_p$ . Let  $W = W(k)$  be the ring of Witt vectors in  $k$ . This is apparently some universal Noetherian local ring lifting  $k$ , and for example,  $W(\mathbb{F}_p) = \mathbb{Z}_p$ , and  $W(\bar{\mathbb{F}}_p) = \mathcal{O}_{\check{\mathbb{Q}}_p}$ . We will consider some extension  $\mathcal{O}/W$  of rings and write  $K$  for the fraction field of  $\mathcal{O}$ .

We are interested in deforming smooth proper curves  $X/k$  with the action of some finite group  $G$ , conveniently denoted as a pair  $(X, G)$ .

**Definition 5.2.1.** The pair  $(X, G)$  *lifts to characteristic 0* if there exists  $\mathcal{O}$  and  $\mathcal{X}/\mathcal{O}$  a projective smooth curve with an action of  $G$  such that  $(\mathcal{X} \times_0 k, G) \simeq (X, G)$ .

Given the action of  $G$  on  $X$ , we can construct the quotient  $X/G$ . To remember the data of the action, we remember the ramified Galois cover  $\pi: X \rightarrow X/G$ .<sup>3</sup>

Consider the action of  $\mathrm{PGL}_2(\mathbb{F}_q)$  on  $\mathbb{P}_k^1$ . This includes in  $\mathbb{P}_\mathcal{O}^1$ , which has an action of  $\mathrm{PGL}_2(\mathcal{O}) \subset \mathrm{PGL}_2(K)$ . We want an embedding  $\mathrm{PGL}_2(\mathbb{F}_q) \subseteq \mathrm{PGL}_2(K)$ , but generally this is not possible. For example, set  $q = 9$  and consider the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} a+1 & a \\ -a & 1-a \end{pmatrix},$$

which generate a  $(\mathbb{Z}/3)^3$ . But then all finite subgroups of  $\mathrm{PGL}_2(K)$  are cyclic, dihedral,  $A_4$ ,  $A_5$ , or  $S_4$ , so this action clearly cannot lift.

**Proposition 5.2.2.** Suppose  $(X, G)$  is such that for all  $x \in X$ , the stabilizer  $G_x$  has order  $|G_x|$  prime to  $p$ . Then  $(X, G)$  lifts to characteristic 0.

Note that this is some local statement, so we need some kind of local-to-global method. Recall that  $\hat{\mathcal{O}}_{X,x} = k[[t]]$ , and this has an action of  $G_x$ . Thus we define a local action as some action of a finite group  $G$  on  $k[[t]]$ , and this lifts if there exists  $\mathcal{O}$  with an action of  $G$  on  $\mathcal{O}[[t]]$  specializing to the original action on  $k[[t]]$ .

**Theorem 5.2.3.** For  $(X, G)$ , suppose that the local action of  $G_x$  on  $\hat{\mathcal{O}}_{X,x}$  lifts to characteristic 0. Then  $(X, G)$  lifts to characteristic 0.<sup>4</sup>

<sup>2</sup>Apparently this is interesting in infinite combinatorics number theory.

<sup>3</sup>This is useful if you want to look at some of the omitted proofs.

<sup>4</sup>Avi was lured in to a paper with French title and French body but English abstract, and in the end did not read this paper.

It is a fact that if  $G$  acts faithfully on  $k[[t]]$ , then  $G = P \rtimes C$ , where  $P$  is  $p$ -Sylow and  $C$  is cyclic.

*Proof of proposition.* We will simply prove that if  $|G_x|$  is prime to  $p$ , then the local action lifts. Then we know that  $G_x = G$  is cyclic of order  $n$  with generator  $\sigma$ . If we write

$$\sigma^i t = \sum_{m \geq 0} a_{m,i} t^m,$$

then of course if  $a_{0,i} \neq 0$ , then  $\sigma^i t$  is a unit and hence  $t$  is a unit, so  $a_{0,i} = 0$ . Our goal is to replace  $t$  by some generator with a more explicit action of  $\sigma$ . Let  $V$  be the vector space spanned by  $\sigma^i t$ . Then on our designated basis, we have

$$\sigma = \begin{pmatrix} 1 & & & & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

This has eigenvalues the  $n$ -th roots of 1, so we choose  $z \in V$  such that  $\sigma z = \zeta_n z$  and replace  $k[[z]] = k[[t]]$ . But now the action of  $\mu_n$  clearly lifts to characteristic 0, so we are done.  $\square$

We have essentially proved that local actions of cyclic groups of order prime to  $p$  lift to characteristic 0. Here is a mild generalization:

**Conjecture 5.2.4** (Oort<sup>5</sup>). *The local action on  $k[[t]]$  by the action of any cyclic  $G$  lifts to characteristic 0.*

This is now a theorem due to (among others?) Obus, Wewers, and Pop. Because of this, if the stabilizers  $G_x$  are all cyclic, then  $(X, G)$  lifts. However, this is not a necessary condition for lifting. If we consider an action of  $(\mathbb{Z}/2)^2$  on  $\mathbb{P}_k^1$ , this does lift (because  $D_4 = (\mathbb{Z}/2)^2$ ).

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<sup>5</sup>Fun fact: Oort was Johan's advisor.