FGA Explained Learning Seminar
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Lectures by Various

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Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Seminar Website: `https://www.math.columbia.edu/~calebji/fga.html`

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Caleb (Oct 16): Representable Functors and Grothendieck Topologies

1.1 Representable Functors

We will always denote categories by C.

Definition 1.1.1. Given an object $x \in C$, define the functor $h_X \colon C^{\mathrm{op}} \to \mathsf{Set}$ by $h_x = \mathrm{Hom}(-, X)$.

Any morphism $f \colon x \to y$ induces a natural transformation $h_f \colon h_x \to h_y$. By the Yoneda lemma, this correspondence is bijective.

Lemma 1.1.2 (Yoneda Lemma)**.** *Let $x \in C$ and $F \colon C^{\mathrm{op}} \to \mathsf{Set}$ be a functor. Then* $\mathrm{Hom}(h_x, F) \simeq F(x)$*.*

Proof. Let $\theta \colon h_x \to F$. This gives a map $\theta_x \colon h_x(x) \to F(x)$, and we can consider $\mathrm{id} \to \theta_x(\mathrm{id})$. Now given $t \in F(x)$, we need $h_x(U) \to F(U)$. Given $U \to x$, then we have a map $F(x) \to F(U)$ and then $t \mapsto F_f(t)$. We can check that these are inverses. □

Definition 1.1.3. A functor $F \colon C^{\mathrm{op}} \to \mathsf{Set}$ is *representable* if it is naturally isomorphic to h_x for some x.

Definition 1.1.4. If F is a presheaf, a *universal object* for F is a pair (X, ξ) such that $\xi \in FX$ and for any (U, σ) where $\sigma \in FU$, there exists a unique $f \colon U \to X$ such that $F_f(\xi) = \sigma$.

Note that representability is equivalent to having a universal object.

Example 1.1.5.

1. For the first example, consider $C = \mathsf{Sch}/R$ for some ring R. Then if $F = \Gamma(\mathcal{O})$, then clearly this is isomorphic to $h_{\mathbb{A}^1}$ and the universal object is $(\mathbb{A}^1, x)$.

2. Let $F(X) = \{\mathcal{L}, s_0, \dots, s_n\}$ where $\mathcal{L}$ is a line bundle and $s_0, \dots, s_n$ generate $\mathcal{L}$, then $(\mathbb{P}^n, x_0, \dots, x_n)$ is a universal object.

1.2 Grothendieck Topologies

According to Wikipedia, this is supposed to be a pun on "Riemann surface." We want to generalize the idea of a topology because the Zariski topology is awful. Instead of open sets, we will consider suitable maps (coverings).

Definition 1.2.1. A *Grothendieck topology* on a category C is a specification of *coverings* $\{U_i \to U\}$ of U for each $U \in C$. Here are the axioms for coverings:

1. If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering.
2. If $\{U_i \to U\}$ is a covering, for all $V \to U$, the fiber products $\{U_i \times_U V \to V\}$ and form a covering of V.
3. If $\{U_i \to U\}$ is a covering and $\left\{V_{ij} \to U_i\right\}$ are coverings, then $\left\{V_{ij} \to U\right\}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

Example 1.2.2. Here are some topological examples. Let X be a topological space.

1. The site of X is the poset category of open subsets of X. The fiber product is just the intersection, and a covering is a normal open covering.
2. (Global classical topology) Let $C = \mathsf{Top}$. Here, the coverings are sets of open embeddings such that the union of the images covers the whole space.
3. (Global étale topology) Here, $C = \mathsf{Top}$ and the coverings are now local homeomorphisms.

Returning to schemes, we have several examples of Grothendieck topologies.

1. (Global Zariski Topology). Let $C = \mathsf{Sch}$. The coverings are jointly surjective open embeddings.
2. (Big étale site over S) The objects are schemes over S and the morphisms are S-morphisms that are étale and locally of finite presentation.
3. (Small étale site) This the same as the big étale site, but with the added requirement that $U \to S$ is also étale.
4. (fppf topology) This stands for the French *fidèlement plat et présentation finie*. The morphisms are $U_i \to U$ flat and locally of finite presentation. A covering is a set of jointly surjective morphisms such that the map $\bigsqcup U_i \to U$ is faithfully flat and of finite presentation. Note that flat and locally of finite presentation implies open.
5. (fpqc topology) This stands for the French *Fidèlement plat et quasi-compacte*. An *fpqc* morphism is a morphism $X \to Y$ that is faithfully flat and one of the following equivalent conditions:
   a) Every quasicompact open subset of Y is the image of a quasicompact open subset of X.
   b) There exists an affine open cover $\{V_i\}$ of Y such that V_i is the image of a quasicompact open subset of X.
   c) Given $x \in X$, there exists a neighborhood $U \ni x$ such that $f(U)$ is open in Y and $U \to f(U)$ is quasicompact.
   d) Given $x \in X$, there exists a quasicompact open neighborhood $U \ni x$ such that $f(U)$ is open and affine in Y.

   The fpqc topology is given by maps $\{U_i \to U\}$ such that $\bigsqcup U_i \to U$ is an fpqc morphism.

To check that this is a topology, we have to do a lot of work. However, we will list some properties of fpqc morphisms and coverings.

Proposition 1.2.3. *1. The composition of fpqc morphisms is fpqc.*

2. *Given $f\colon X \to Y$, if $f^{-1}(V_i) \to V_i$ is fpqc, then f is fpqc.*

3. *Open and faithfully flat implies fpqc. Moreover, faithfully flat and locally of finite presentation implies fpqc. This means that fppf implies fpqc.*

4. *Base change preserves fpqc morphsisms.*

5. *All fpqc morphsism are submersive. Thus $f^{-1}(V)$ is open if and only if V is open.*

Note that Zariski is coarser than étale is coarser than fppf is coarser than fpqc.

1.3 Sheaves on Sites

Recall that a presheaf on a space is a functor $X_{\mathrm{cl}}^{\mathrm{op}} \to \mathsf{Set}$. Similarly, if C is a site, then a presheaf is a functor $C^{\mathrm{op}} \to \mathsf{Set}$.

Definition 1.3.1. A presheaf on a site C is a *sheaf* if

1. Given a covering $\{U_i \to U\}$ and $a, b \in FU$ such that $p_i^* a = p_i^* b$, then $a = b$.

2. Given a covering $\{U_i \to U\}$ and $a_i \in FU_i$ such that $p_i^* a_j = p_j^* a_i$ (in the fiber product) for all i, j, there exists a unique $a \in FU$ such that $p_i^* a = a_i$.

An alternative definition of a sheaf is that $FU \to \prod FU_i \rightrightarrows F(U_i \times_U U_j)$ is an equalizer.

Theorem 1.3.2 (Grothendieck). *A representable functor on Sch/S is a sheaf in the fpqc topology.*

This means that given any fpqc cover $\{U_i \to U\}$, then applying h_X, if we have $f_i\colon U_i \to X$ that glue on $U_i \times_X U_j \to X$, then the sheaf condition says we can glue to a unique $f\colon U \to X$. In the Zariski topology, this is trivial. This also means that the fpqc topology is *subcanonical*, which means that h_X are all sheaves.

We will prove this result by reducing to the category of all schemes. Note that the topology on Sch/S comes from the topology on Sch. Then we can show that if C is subcanonical, then C/S is subcanonical. Then we use the following lemma.

Lemma 1.3.3. *Let S be a scheme and $F\colon \mathsf{Sch}/S^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. If F is a Zariski sheaf if $V \to U$ is a faithfully flat morphism of affine S-schemes, then $FU \to FV \rightrightarrows F(V \times_U V)$ is an equalizer, then F is an fpqc sheaf.*

Proof. Given $\{U_i \to U\}$ an fpqc covering, let $V = \coprod U_i$. Then consider the diagram

$$\begin{tikzcd}
FU \arrow[r] \arrow[d] & FV \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(V \times_U V) \arrow[d] \\
FU \arrow[r] & \prod FU_i \arrow[r, shift left] \arrow[r, shift right] & F(U_i \times_U U_j),
\end{tikzcd}$$

the columns are bijective, so it suffices to check this for single coverings.

Now if $\{U_i \to U\}$ are finite and all affine and the second assumption holds, we have the diagram

$$\begin{tikzcd}
FU \arrow[r] \arrow[d] & FV \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(V \times_U V) \\
\prod_i F(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_a FU_{ia} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_{ab} F(V_{ia} \times_U V_{ib}) \\
\prod_{ij} F(U_i \cap U_j) \arrow[r] & \prod_{ij} \prod_{ab} F(U_{ia} \cap U_{jb}).
\end{tikzcd}$$

Then the middle row is an equalizer. □

Proof of Theorem 1.3.2. If X, U, V are affine, then we know that $\mathrm{Hom}(R, -)$ is left exact, so the result follows from commutative algebra. Now it suffices to check the general case for single covers. If $X = \bigcup X_i$ is a union of affines, then separatedness follows by restricting to the X_i and using the affine case.

Please read the rest of this yourself. □