GENERALITIES ON ORBIFOLD COHOMOLOGY AND TORIC DM STACKS

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Abstract. I will explain various technicalities in Gromov-Witten theory for Deligne-Mumford stacks and how to construct toric Deligne-Mumford stacks from (extended) stacky fans.

Contents

1. Orbifold Gromov-Witten theory

Let X be a smooth and separated Deligne-Mumford stack of finite type over $\mathbb{C}$.

Definition 1.1. The *inertia stack* of X is the fiber product in the diagram

$$\begin{tikzcd} IX \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

More concretely, we may think about $|X|$ as parameterizing pairs (x, g), where $x \in X$ and $g \in \mathrm{Aut}(x)$. There is another description of IX if X lives over $\mathbb{C}$.

Definition 1.2. A morphism $X \to Y$ of algebraic stacks is *representable* if for all schemes S and morphisms $S \to Y$, the fiber product $X \times_S Y$ is an algebraic space.

Remark 1.3 (Abramovich-Graber-Vistoli). Let

$$I_\mu X \coloneqq \bigsqcup_{r \geqslant 0} \mathrm{Hom}_{\mathrm{rep}}(B\mu_r, X)$$

denote the stack of representable morphisms from classifying stacks of roots of unity to X (the *cyclotomic inertia stack*). Then $I_\mu X \simeq IX$.

Recall that by the Keel-Mori theorem, X (which has finite inertia) has a coarse moduli space $|X|$, which is an algebraic space satisfying two properties:

- The morphism $\pi\colon X \to |X|$ is bijective on k-points whenever k is an algebraically closed field;
- $|X|$ is initial for morphisms from X to any algebraic space.

From now on, we will assume that $|X|$ is quasiprojective, and in particular that it is a scheme.

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1.1. **Moduli of stable maps.**