Enumerative invariants and birational geometry
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Lectures by Various

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Disclaimer

These notes were taken during the lectures using `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. Also, notation may differe between lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

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Seminar Website: https://math.columbia.edu/~plei/s24-birat.html

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Preliminaries

1.1 Givental formalism (Patrick, Feb 01)

1.1.1 Introduction Let X be a smooth projective variety. Then for any $g, n \in \mathbb{Z}_{\geqslant 0}, \beta \in H_2(X, \mathbb{Z})$, there exists a moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ (Givental's notation is $X_{g,n,\beta}$) of *stable maps* $f\colon C \to X$ from genus-g, n-marked prestable curves to X with $f_*[C] = \beta$. It is well-known that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in A_\delta(\overline{\mathcal{M}}_{g,n}(X, \beta)), \qquad \delta = \int_\beta c_1(X) + (\dim X - 3)(1-g) + 3.$$

In addition, there is a universal curve and sections

$$\mathcal{C} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} \overline{\mathcal{M}}_{g,n}(X, \beta).$$

In this setup, there are tautological classes

$$\psi_i \coloneqq c_1(\sigma_i^* \omega_\pi) \in H^2(\overline{\mathcal{M}}_{g,n}(X, \beta)).$$

This allows us to define individual Gromov-Witten invariants by

$$\left\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \phi_i \cdot \psi_i^{a_i}.$$

These invariants satisfy various relations. The first is the *string equation:*

$$\left\langle \tau_0(1) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n+1,\beta} = \sum_{i=1}^n \left\langle \tau_{a_i - 1}(\phi_i) \prod_{j \neq i} \tau_{a_j}(\phi_j) \right\rangle^X_{g,n,\beta}.$$

The next is the *dilaton equation:*

$$\left\langle \tau_1(1) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n+1,\beta} = (2g - 2 + n) \left\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n,\beta}.$$

Finally, we have the *divisor equation* when one insertion is a divisor $D \in H^2(X)$:

$$\left\langle \tau_0(D)\tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n+1,\beta} = \left(\int_\beta D\right)\cdot\left\langle \tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n,\beta} + \sum_{i=1}^n \left\langle \tau_{a_i-1}(\phi_i\cdot D)\prod_{j\neq i}\tau_{a_j}(\phi_j)\right\rangle^X_{g,n,\beta}.$$

It is often useful to package Gromov-Witten invariants into various generating series.

Definition 1.1.1. The *quantum cohomology* $QH^*(X)$ of X is defined by the formula

$$(a\star_t b, c) := \sum_{\beta,n}\frac{Q^\beta}{n!}\langle a,b,c,t,\ldots,t\rangle^X_{0,3+n,\beta}$$

for any $t \in H^*(X)$. This is a commutative and associative product.

The *small quantum cohomology* is obtained by setting $t = 0$ and the ordinary cohomology is obtained by further setting $Q = 0$.

Remark 1.1.2. Convergence of the formula does not hold in general, so quantum cohomology needs to be treated as a formal object.

Definition 1.1.3. Let ϕ_i be a basis of $H^*(X)$ and ϕ^i be the dual basis. Then the J-*function* of X is the cohomology-valued function

$$J_X(t,z) := z + t + \sum_i\sum_{n,\beta}\frac{Q^\beta}{n!}\left\langle\frac{\phi_i}{z-\psi},t,\ldots,t\right\rangle^X_{0,n+1,\beta}\phi^i.$$

Definition 1.1.4. The *genus*-0 *GW potential* of X is the (formal) function

$$\mathcal{F}^X(t(z)) = \sum_{\beta,n}\frac{Q^\beta}{n!}\langle t(\psi),\ldots,t(\psi)\rangle^X_{0,n,\beta}.$$

The associativity of the quantum product is equivalent to the PDE

$$\sum_{e,f}\mathcal{F}^X_{abe}\eta^{ef}\mathcal{F}_{cdf} = \sum_{e,f}\mathcal{F}^X_{ade}\eta^{ef}\mathcal{F}^X_{bcf}$$

for any a,b,c,d, which are known as the *WDVV equations*. Here, we choose coordinates on $H^*(X)$ and set $z = 0$ (only consider primary insertions). In addition, set η_{ef} to be the components of the Poincaré pairing and let η^{ef} be the inverse matrix.

1.1.2 Frobenius manifolds

A Frobenius manifold can be thought of as a formalization of the WDVV equations.

Definition 1.1.5. A *Frobenius manifold* is a complex manifold M with a flat symmetric bilinear form $\langle -,-\rangle$ (meaning that the Levi-Civita connection has zero curvature) on TM and a holomorphic system of (commutative, associative) products $\star_t$ on T_tM satisfying:

1. The unit vector field $\mathbf{1}$ is flat: $\nabla\mathbf{1} = 0$;

2. For any t and $a, b, c \in T_tM$, $\langle a \star_t b, c\rangle = \langle a, b \star_t c\rangle$;
3. If $c(u, v, w) := \langle u \star_t v, w\rangle$, then the tensor $(\nabla_z c)(u, v, w)$ is symmetric in $u, v, w, z \in T_tM$.

If there exists a vector field E such that $\nabla\nabla E = 0$ and complex number d such that:

1. $\nabla\nabla E = 0$;
2. $\mathcal{L}_E(u \star v) - \mathcal{L}_E u \star v - u \star \mathcal{L}_E v = u \star v$ for all vector fields u, v;
3. $\mathcal{L}_E\langle u, v\rangle - \langle \mathcal{L}_E u, v\rangle - \langle u, \mathcal{L}_E v\rangle = (2 - d)\langle u, v\rangle$ for all vector fields u, v,

then E is called an *Euler vector field* and the Frobenius manifold M is called *conformal*.

Example 1.1.6. Let X be a smooth projective variety. Then we can give $H^*(X)$ the structure of a Frobenius algebra with the Poincaré pairing and the quantum product. Note that the quantum product does not converge in general, so we must treat this as a formal object. The Euler vector field is given by

$$E_X = c_1(X) + \sum_i \left(1 - \frac{\deg \phi_i}{2}\right) t^i \phi_i,$$

where a general element of $H^*(X)$ is given by $t = \sum_i t^i \phi_i$. We will also impose that $\phi_1 = \mathbf{1}$. There is another very important structure, the *quantum connection*, which is given by the formula

$$\nabla_{t^i} := \partial_{t^i} + \frac{1}{z}\phi_i \star_t$$
$$\nabla_{z\frac{d}{dz}} := z\frac{d}{dz} - \frac{1}{z}E_X \star_t + \mu_X.$$

Here, μ_X is the *grading operator*, defined for pure degree classes $\phi \in H^*(X)$ by

$$\mu_X(\phi) = \frac{\deg \phi - \dim X}{2}\phi.$$

Finally, in the direction of the Novikov variables, we have

$$\nabla_{\xi Q \partial_Q} = \xi Q\, \partial_Q + \frac{1}{z}\xi \star_t .$$

Remark 1.1.7. For a general conformal Frobenius manifold $(H, (-,-), \star, E)$, there is still a *deformed flat connection* or *Dubrovin connection* given by

$$\nabla_{t^i} := \frac{\partial}{\partial t^i} + \frac{1}{z}\phi_i \star$$
$$\nabla_{z\frac{d}{dz}} := z\frac{d}{dz} - \frac{1}{z}E \star .$$

Definition 1.1.8. The *quantum* D-*module* of X is the module $H^*(X)[z][\![Q, t]\!]$ with the quantum connection defined above.

Remark 1.1.9. It is important to note that the quantum connection has a fundamental solution matrix $S^X(t, z)$ given by

$$S_X(t, z)\phi = \phi + \sum_i \sum_{n,\beta} \frac{Q^\beta}{n!}\phi^i \left\langle \frac{\phi_i}{z - \psi}, \phi, t, \ldots, t \right\rangle^X_{0,n+2,\beta}.$$

It satisfies the important equation

$$S^*_X(t, -z)S(t, z) = 1.$$

Using this formalism, the J-function is given by $S^*_X(t, z)\mathbf{1} = z^{-1}J_X(t, z)$.

1.1.3 Givental formalism The Givental formalism is a geometric way to package enumerative (CohFT) invariants cleanly. We begin by defining the symplectic space

$$\mathcal{H} \coloneqq H^*(X, \Lambda)(\!(z^{-1})\!)$$

with the symplectic form

$$\Omega(f, g) \coloneqq \mathrm{Res}_{z=0}(f(-z)g(z)).$$

This has a polarization by Lagrangian subspaces

$$\mathcal{H}_+ \coloneqq H^*(X, \Lambda)[z], \qquad \mathcal{H}_- \coloneqq z^{-1}H^*(X, \Lambda)[\![z^{-1}]\!]$$

giving $\mathcal{H} \cong T^*\mathcal{H}_+$ as symplectic vector spaces. Choose Darboux coordinates $\underline{p}, \underline{q}$ on $\mathcal{H}$. For example, there is a choice in Coates's thesis which gives a general element of $\mathcal{H}$ as

$$\sum_{k\geqslant 0} \sum_{i} q_k^i \phi_i z^k + \sum_{\ell\geqslant 0} \sum_{j} p_\ell^j \phi^j (-z)^{-\ell-1}.$$

Taking the *dilaton shift*

$$q(z) = t(z) - z = -z + t_0 + t_1 z + t_2 z^2 + \cdots,$$

we can now think of $\mathcal{F}^X$ has a formal function on $\mathcal{H}_+$ near $q = -z$. This convention is called the *dilaton shift*.

Before we continue, we need to recast the string and dilaton equations in terms of $\mathcal{F}^X$. Write $t_x = \sum t_k^i \phi_i$. Then the string equation becomes

$$\partial_0^1 \mathcal{F}(t) = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty} \sum_{j} t_{n+1}^j \, \partial_n^j \mathcal{F}(t)$$

and the dilaton equation becomes

$$\partial_1^1 \mathcal{F}(t) = \sum_{n=0}^{\infty} t_n^j \, \partial_n^j \mathcal{F}(t) - 2\mathcal{F}(t).$$

There are also an infinite series of topological recursion relations

$$\partial_{k+1}^i \, \partial_\ell^j \, \partial_m^k \mathcal{F}(t) = \sum_{a,b} \partial_k^i \, \partial_0^a \mathcal{F}(t) \eta^{ab} \, \partial_0^b \, \partial_\ell^j \, \partial_m^k \mathcal{F}(t).$$

We can make sense of these three relations for any (formal) function $\mathcal{F}$ on $\mathcal{H}_+$.

Now let

$$\mathcal{L} = \left\{ (\underline{p}, \underline{q}) \in \mathcal{H} \mid \underline{p} = d_{\underline{q}} \mathcal{F} \right\}$$

be the graph of $d\mathcal{F}$. This is a formal germ at $q = -z$ of a Lagrangian section of the cotangent bundle $T^*\mathcal{H}_+$ and is therefore a formal germ of a Lagrangian submanifold in $\mathcal{H}$.

Theorem 1.1.10. *The function $\mathcal{F}$ satisfies the string equation, dilaton equation, and topological recursion relations if and only if $\mathcal{L}$ is a Lagrangian cone with vertex at the origin $q = 0$ such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along zL.*

Because of this theorem, $\mathcal{L}$ is known as the *Lagrangian cone*. It can be recovered from the J-function by the following procedure. First consider $\mathcal{L} \cap (-z + z\mathcal{H}_-)$. Via the projection to $-z + H$ along $\mathcal{H}_-$, this can be considered as the graph of the J-function. Next, we consider the derivatives $\frac{\partial J}{\partial t^i}$, which form a basis of $L \cap z\mathcal{H}_-$, which is a complement to zL in L. Then we know that

$$z\frac{\partial J}{\partial t^i} \in zL \subset \mathcal{L},$$

so

$$z\frac{\partial^2 J}{\partial t^i\,\partial t^j} \in L \cap z\mathcal{H}_-.$$

Writing these in terms of the first derivatives $\frac{\partial J}{\partial t^i}$ and using the fact that J is a solution of the quantum connection, so we recover the Frobenius structure of quantum cohomology.

We will now express some classical results in this formalism. Let X be a toric variety with toric divisors $D_1, \ldots, D_N$ such that $D_1, \ldots, D_k$ form a basis of $H^2(X)$ and Picard rank k. Then define the I-*function*

$$I_X = ze^{\sum_{j=1}^{k} t_i D_i} \sum_{\beta} Q^{\beta} \frac{\prod_{j=1}^{N} \prod_{m=-\infty}^{0} (D_j + mz)}{\prod_{j=1}^{N} \prod_{m=-\infty}^{\langle D_j, \beta\rangle} (D_j + mz)}.$$

Theorem 1.1.11 (Mirror theorem)**.** *The formal functions* I_X *and* J_X *coincide up to some change of variables, which if* $c_1(X)$ *is semi-positive is given by components of the* I*-function.*

Theorem 1.1.12 (Mirror theorem in this formalism)**.** *For any* t, *we have*

$$I_X(t, z) \in \mathcal{L}.$$

Another direction in Gromov-Witten theory is the Virasoro constraints. In the original formulation, these involved very complicated explicit differential operators, but in the Givental formalism, there is a very compact formulation.

Define $\ell^{-1} = z^{-1}$ and

$$\ell_0 = z\frac{d}{dz} + \frac{1}{2} + \mu + \frac{c_1(X) \cup -}{z}.$$

Then define

$$\ell_n = \ell_0(z\ell_0)^n.$$

Theorem 1.1.13 (Genus-0 Virasoro constraints)**.** *Suppose the vector field on* $\mathcal{H}$ *defined by* ℓ_0 *is tangent to* $\mathcal{L}$*. Then the same is true for the vector fields defined by* ℓ_n *for any* $n \geqslant 1$*.*

Proof. Let L be a tangent space to $\mathcal{L}$. Then if $f \in zL \subset \mathcal{L}$, the assumption gives us $\ell_0 f \in L$. But then $z\ell_0 f \in zL$, so $\ell_0 z\ell_0 f = \ell_1 f \in L$. Continuing, we obtain $\ell_n f \in L$ for all n. □

Next week, we will learn that the Quantum Riemann-Roch theorem can be stated in this formalism. Let $\mathcal{L}^{tw}$ be the twisted Lagrangian cone (where the twisted theory will be defined next week).

Theorem 1.1.14 (Quantum Riemann-Roch)**.** *For some explicit linear symplectic transformaiton* Δ*, we have* $\mathcal{L}^{tw} = \Delta\mathcal{L}$*.*

1.1.4 Quantization In the last part of the talk, we will briefly discuss the quantization formalism, which encodes the higher-genus theory. In Darboux coordinates p_a, q_b, we will quantize symplectic transformations by the standard rules

$$\widehat{q_a q_b} = \frac{q_a q_b}{\hbar}, \qquad \widehat{q_a p_b} = q_a \frac{\partial}{\partial q_b}, \qquad \widehat{p_a p_b} = \hbar \frac{\partial^2}{\partial q_a \, \partial q_b}.$$

This determines a differential operator acting on functions on $\mathcal{H}_+$.

We also need the genus-g potential

$$\mathcal{F}_g^X \coloneqq \sum_{\beta, n} \frac{Q^\beta}{n!} \langle t(\psi), \dots, t(\psi) \rangle_{g,n,\beta}^X$$

and the *total descendent potential*

$$\mathcal{D} \coloneqq \exp\left(\sum_{g \geqslant 0} \hbar^{g-1} \mathcal{F}_g^X \right).$$

In this formalism, the Virasoro conjecture can be expressed as follows. Let $L_n = \widehat{\ell}_n + c_n$, where c_n is a carefully chosen constant.

Conjecture 1.1.15 (Virasoro conjecture)**.** *If* $L_{-1}\mathcal{D} = L_0 \mathcal{D} = 0$, *then* $L_n \mathcal{D} = 0$ *for all* $n \geqslant 1$.

In this formalism, the higher-genus version of the Quantum Riemann-Roch theorem takes the very simple form

Theorem 1.1.16 (Quantum Riemann-Roch)**.** *Let* $\mathcal{D}^{\mathrm{tw}}$ *be the twisted descendent potential. Then*

$$\mathcal{D}^{\mathrm{tw}} = \widehat{\Delta} \mathcal{D}.$$

1.2 Quantum Riemann-Roch (Shaoyun, Feb 08)

We will state and prove the Quantum Riemann-Roch theorem in genus 0, following Coates-Givental.

1.2.1 Twisted Gromov-Witten invariants Again, let X be a smooth projective variety. Let E be a vector bundle on X. We should note that

$$\overline{\mathcal{M}}_{0,n+1}(X, \beta) \xrightarrow{\pi} \overline{\mathcal{M}}_{0,n}(X, \beta)$$

is the universal curve, and the universal morphism is simply ev_{n+1}. We will consider the sheaf

$$E_{0,n,\beta} \coloneqq R\pi_* \, \mathrm{ev}_{n+1}^* \, E \in K^0(\overline{\mathcal{M}}_{0,n}(X, \beta)).$$

We need to check that this is a well-defined K-theory class. Choose an ample line bundle $L \to X$. By definition, for $N \gg 1$, the cohomology

$$H^i(X, E \otimes L^N) = 0$$

whenever $i \geqslant 1$. This gives us an exact sequence

$$0 \to \ker(=: A) \to H^0(X, E \otimes L^N) \otimes L^{-N} (=: B) \to E \to 0.$$

For any stable map $f\colon \Sigma \to X$ of positive degree, we obtain a long exact sequence

$$0 \to H^0(\Sigma, f^*E) \to H^1(\Sigma, f^*A) \to H^1(\Sigma, f^*B) \to H^1(\Sigma, f^*E) \to 0,$$

so we obtain

$$R^0\pi_* \operatorname{ev}_{n+1}^* E - R^1\pi_* \operatorname{ev}_{n+1}^* E = R^1\pi_* \operatorname{ev}_{n+1}^* B - R^1\pi_* \operatorname{ev}_{n+1}^* A.$$

This expresses $E_{0,n,\beta}$ as a difference of vector bundles.

We will now introduce a *universal characteristic class*

$$\mathbf{c}(-) = \exp\left(\sum_{k=0}^{\infty} s_k \operatorname{ch}_k(-)\right),$$

where $s_0, s_1, s_2, \ldots$ are formal variables and ch_k is the k-th Chern character

$$\frac{x_1^k}{k!} + \cdots + \frac{x_r^k}{k!},$$

where x_i are the Chern roots.

Example 1.2.1. Let $E \to X$ be a vector bundle and equip it with the fiberwise $\mathbb{C}^*$-action by scaling. Let λ be the equivariant parameter and ρ_i be the Chern roots. Then

$$e(E) = \sum_i (\lambda + \rho_i).$$

We then rewrite

$$\begin{aligned} \prod(\lambda + \rho_i) &= \exp\left(\sum_i \left(\log\lambda - \sum_k \frac{(-\rho_i)^k}{k\lambda^k}\right)\right) \\ &= \exp\left(\operatorname{ch}_0(E)\log\lambda + \sum_{k>0} \frac{(-1)^{k-1}(k-1)!}{\lambda^k} \operatorname{ch}_k(E)\right), \end{aligned}$$

so for the (equivariant Euler class), we obtain

$$\begin{aligned} s_0 &= \log\lambda \\ s_k &= \frac{(-1)^{k-1}(k-1)!}{\lambda^k}, \qquad k > 0. \end{aligned}$$

We are now ready to define the $(E, \mathbf{c})$-twisted Gromov-Witten invariants.

Definition 1.2.2. Define the *twisted Gromov-Witten invariants* by

$$\left\langle \alpha_1\psi_1^{k_1}, \ldots, \alpha_n\psi_n^{k_n} \right\rangle_{0,n,\beta}^{X,(E,\mathbf{c})} := \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \operatorname{ev}_i^*(\alpha_i)\psi_i^{k_i} \cup \mathbf{c}(E_{0,n,\beta})$$

for $\alpha_i \in H^*(X)$ and $k_i \in \mathbb{Z}_{\geqslant 0}$.

We will now construct the Lagrangian cone for the twisted theory. Let R be the coefficient ring containing $s_0, s_1, \ldots$ and define

$$\mathcal{H}_X^{\mathrm{tw}} := H^*(X) \otimes R(\!(z^{-1})\!)[\![Q]\!].$$

We also introduce the *twisted Poincaré pairing*

$$(a,b)_{(E,\mathbf{c})} = \int_X a \cup b \cup \mathbf{c}(E).$$

The symplectic structure is defined by

$$\Omega_{tw}(f,g) = \mathrm{Res}_{z=0}(f(-z)g(z))_{(E,\mathbf{c})}.$$

There is a polarization

$$\mathcal{H}_X^{tw} = \mathcal{H}_+^{tw} \oplus \mathcal{H}_-^{tw}$$

with

$$\mathcal{H}_+^{tw} := H^*(X) \otimes R[z][\![Q]\!]$$
$$\mathcal{H}_-^{tw} := H^*(X) \otimes R[\![z]\!][\![Q]\!].$$

Finally, we have the *twisted genus-0 descendent potential*

$$\mathcal{F}^0_{X,tw}(t) := \sum_{\beta,n} \frac{Q^\beta}{n!} \langle t, \dots, t \rangle_{0,n,\beta}^{X,(E,\mathbf{c})}.$$

Identifying $\mathcal{H}_X^{tw}$ with $T^*\mathcal{H}_+^{tw}$, we obtain the twisted Lagrangian cone $\mathcal{L}_X^{tw}$ as the graph of $d\mathcal{F}^0_{X,tw}$. Denote the untwisted Lagrangian cone as $\mathcal{L}_X$.

Theorem 1.2.3. *We have*

$$\mathcal{L}_X^{tw} = \Delta \mathcal{L}_X,$$

where

$$\Delta = \exp\left(\sum_{m \geqslant 0} \sum_{\ell \geqslant 0} s_{2m-1+\ell} \frac{B_{2m}}{(2m)!} \mathrm{ch}_\ell(E) z^{2m-1} \right).$$

Here, the Bernoulli numbers B_{2m} are defined by

$$\frac{t}{1-e^{-t}} = \frac{t}{2} + \sum_{m \geqslant 0} \frac{B_{2m}}{(2m!)} t^{2m}.$$

1.2.2 Proof of Theorem 1.2.3

The idea is to use the Grothendieck-Riemann-Roch theorem.

Proposition 1.2.4. *We can write*

$$[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir} \cap \mathrm{ch}_k(E_{0,n,\beta}) = \pi_*\left(\sum_{\substack{r+\ell=k+1 \\ r,\ell \geqslant 0}} \frac{B_r}{r!} \mathrm{ch}_\ell(\mathrm{ev}_{n+1}^* E) \Psi(r) \right),$$

where

$$\begin{aligned} \Psi(r) = {} & \psi_{n+1}^r \cap [\overline{\mathcal{M}}_{0,n+1}(X,\beta)]^{vir} \\ & - \sum_{i=1}^n (\sigma_i)_*(\psi_i^{n-1} \cap [\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir}) \\ & + \frac{1}{2} j_*\left(\sum_{\substack{a+b=r-2 \\ a,b \geqslant 0}} (-1)^a \psi_+^a \psi_i^b \cap [\widetilde{Z}_{0,n+1,\beta}]^{vir} \right). \end{aligned}$$

Here, $Z_{0,n+1,\beta}$ is formed by the nodes of π, $\widetilde{Z}_{0,n+1,\beta}$ is a double cover of $Z_{0,n+1,\beta}$ formed by a choice of branch of the nodes, ψ_+ and ψ_- are the ψ-classes at the two branches of the nodes, and

$$j\colon \widetilde{Z}_{0,n+1,\beta} \to Z_{0,n+1,\beta} \to \overline{\mathcal{M}}_{0,n+1}(X,\beta)$$

is the "inclusion."

Proof. We will first assume that $\overline{M}_{0,n+1}(X,\beta)$, $\overline{\mathcal{M}}_{0,n}(X,\beta)$, and $Z_{0,n+1,\beta}$ are all smooth and that $\pi(Z_{0,n+1,\beta})$ is a normal crossings divisor. In general, we need a Cartesian diagram

$$\begin{tikzcd}
 & \mathrm{ev}_{n+1}^* E \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
\overline{\mathcal{M}}_{0,n+1}(X,\beta) \arrow[rr] \arrow[dd] & & \mathcal{C} \arrow[dd] & \\
 & Z_{0,n+1,\beta} \arrow[ul, hook] & & Z \arrow[ul, hook] \\
\overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[rr] & & \mathcal{M}. &
\end{tikzcd}$$

Continuing in the ideal situation, we apply Grothendieck-Riemann-Roch[1] to obtain

$$\begin{aligned}
\mathrm{ch}(E_{0,n,\beta}) &= \mathrm{ch}(R\pi_* \mathrm{ev}_{n+1}^* E) \\
&= \pi_*(\mathrm{ch}(\mathrm{ev}_{n+1}^* E) \cdot \mathrm{td}^\vee \Omega_\pi),
\end{aligned}$$

where $\mathrm{td}^\vee$ is the dual Todd class, defined by $\frac{-x}{1-e^{tx}}$, and Ω_π is the sheaf of relative differentials.

We then have two short exact sequences

$$0 \to \Omega_\pi \to \omega_\pi \to \mathcal{O}_{Z_{0,n+1,\beta}} \to 0$$

and

$$0 \to \omega_\pi \to L_{n+1} \to \bigoplus_{i=1}^{n} \mathcal{O}_{D_i} \to 0,$$

where D_i is the divisor where the marked points $i, n+1$ collide and their component has exactly three special points. Now we obtain

$$\Omega_\pi = L_{n+1} - \sum_{i=1}^{n} \mathcal{O}_{D_i} - \mathcal{O}_{Z_{0,n+1,\beta}}$$

in K-theory. Using the facts that $c_1(L_{n+1}) = \psi_{n+1}$, $D_i \cap D_j = \emptyset$ for $i \neq j$, and $D_i \cap Z_{0,n+1,\beta} = \emptyset$, we see that L_{n+1} is trivial when restricted to D_i and $Z_{0,n+1,\beta}$. Now we apply the dual Todd class.

Lemma 1.2.5. *If $x_1 \cup x_2 = 0$, then*

$$(\mathrm{td}^\vee(x_1) - 1)(\mathrm{td}^\vee(x_2) - 1) = 0.$$

[1] We need to be careful about directly applying Grothendieck-Riemann-Roch in the stacky setting (and in general we are only quasi-smooth).

Using the lemma, we obtain

$$\begin{aligned}\mathrm{td}^\vee(\Omega_\pi) &= \mathrm{td}^\vee(L_{n+1})\prod_{i=1}^{n}\mathrm{td}^\vee(-\mathcal{O}_{D_i})\,\mathrm{td}^\vee(\mathcal{O}_{Z_{0,n+1,\beta}})^{-1}\\ &= 1+(\mathrm{td}^\vee(L_{n+1})-1)+\sum_{i=1}^{n}\left(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{D_i})}-1\right)+\left(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{Z_{n+1,\beta}})}-1\right).\end{aligned}$$

The first term in the statement comes from the dual Todd class of L_{n+1}, the second comes from

$$0\to\mathcal{O}(-D_i)\to\mathcal{O}\to\mathcal{O}_{D_i}\to 0$$

and the relation between $\mathcal{O}(-D_i)$ and L_i, and the last term can be found in Appendix A of Coates-Givental. □

To obtain the Quantum Riemann-Roch theorem, we use the previous proposition and manipulate the generating function. If E is convex and $Y\subset X$ is a complete intersection defined by E, then $\mathcal{L}_X^{\mathrm{tw}}$ is closely related to $\mathcal{L}_Y$, so we are able to study the Gromov-Witten theory of Y using this.

1.3 Shift operators (Melissa, Feb 15)

Let X be a semiprojective smooth variety. This means that X is projective over its affinization. Also assume that X has an action by $T=(\mathbb{C}^\times)^m$ such that all T-weights in $H^0(X,\mathcal{O})$ are contained in a strictly convex cone in $\mathrm{Hom}(T,\mathbb{C}^\times)_{\mathbb{R}}$ and $H^0(X,\mathcal{O})^T=\mathbb{C}$. All such X imply that

(a) The fixed locus X^T is projective;

(b) The T-variety X is equivariantly formal. This means that $H_T^*(X)$ is a free module over $H_T^*(\mathrm{pt})=\mathbb{Q}[\lambda]:=\mathbb{Q}[\lambda_1,\dots,\lambda_m]$ and there is a non-canonical isomorphism

$$H_T^*(X)\cong H^*(X)\otimes H_T^*(\mathrm{pt})$$

as $H_T^*(\mathrm{pt})$-modules.

(c) The evaluation maps $\mathrm{ev}_i\colon X_{0,n,d}\to X$ are proper.

Using (b), we may choose a basis $\{\phi_i\}_{i=0}^N$ of $H_T^*(X)$ over $H_T^*(\mathrm{pt})$. Let τ^i be the dual coordinates.

1.3.1 Equivariant big quantum cohomology Let $(-,-)$ be the T-equivariant Poincaré pairing, which in general takes values in $\mathbb{Q}(\lambda)$. Then the T-equivariant big quantum product is defined by

$$\begin{aligned}(\phi_i\star_\tau\phi_j,\phi_k) &= \langle\!\langle\phi_i,\phi_j,\phi_k\rangle\!\rangle_{0,3}^{X,T}\\ &=\sum_{d,n}\frac{Q^d}{n!}\langle\phi_i,\phi_b,\phi_j,\tau,\dots,\tau\rangle_{0,n+3,d}^{X,T}.\end{aligned}$$

This can also be defined using the evaluation maps

$$(\mathrm{ev}_i)_*\colon H_T^*(X_{0,n+3,d})\to H_T^{*-2(c_1(X)\cdot d+n)}(X)$$

as

$$\phi_i\star_\tau\phi_j=\sum_{d,n}\frac{Q^d}{n!}(\mathrm{ev}_3)_*\left(\mathrm{ev}_1^*(\phi_i)\,\mathrm{ev}_2^*(\phi_j)\prod_{i=4}^{n+3}\mathrm{ev}_i^*(\tau)\cap[X_{0,n+3,d}]^{\mathrm{vir}}\right)\in H_T^*(X)[\![Q]\!][\![\tau_0,\dots,\tau_n]\!].$$

1.3.2 Quantum connection We will define

$$\nabla_i \colon H_T^*(X)[z][\![Q]\!][\![\tau]\!] \to z^{-1}H_T^*(X)[z][\![Q]\!][\![\tau^0, \dots, \tau^N]\!]$$

by setting

$$\nabla_i = \frac{\partial}{\partial \tau^i} + \frac{1}{z}(\phi_i \star).$$

We can view z as the loop variable by setting $\widehat{T} = T \times \mathbb{C}^\times$. If the extra copy of $\mathbb{C}^\times$ acts trivially on X, then

$$H_{\widehat{T}}^*(X) = H_T^*(X)[z].$$

This has a fundamental solution

$$M(\tau) \colon H_{\widehat{T}}^*(X)[\![Q, \tau]\!] \to H_{\widehat{T}}^*(X)_{\mathrm{loc}}[\![Q, \tau]\!]$$

where

$$H_{\widehat{T}}^*(X)_{\mathrm{loc}} \coloneqq H_{\widehat{T}}^*(X) \otimes_{\mathbb{Q}[\lambda, z]} \mathbb{Q}(\lambda(z)).$$

This satisfies the differential equation

$$z\frac{\partial}{\partial \tau^i} M(\tau) = M(\tau)(\phi_i \star),$$

which is equivalent to

$$\frac{\partial}{\partial \tau^i} \circ M(\tau) = M(\tau) \circ \nabla_i.$$

The solution has the form

$$(M(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j) + \left\langle\!\!\left\langle \phi_i, \frac{\phi_j}{z - \psi} \right\rangle\!\!\right\rangle_{0,2}^{X,T}.$$

1.3.3 Shift operators Let $k \colon \mathbb{C}^\times \to T$ be a cocharacter of T. Then define a $\widehat{T}$-action ρ_k on X by

$$\rho_k(t, x)x = tu^k \cdot x$$

for $t \in T, u \in \mathbb{C}^\times, x \in X$. Under the group automorphism

$$\phi_k \colon \widehat{T} \to \widehat{T} \qquad \phi_k(t, u) = (tu^{-k}, u),$$

the identity map $(X, \rho_0) \to (X, \rho_k)$ is $\widehat{T}$-equivariant, so we obtain isomorphisms

$$\Phi_k \colon H_{\widehat{T}, \rho_0}^*(X) \to H_{\widehat{T}, \rho_k}^*(X).$$

Now define the bundle

$$E_k = (X \times (\mathbb{C}^2 \setminus 0))/\mathbb{C}^\times,$$

where $\mathbb{C}^\times$ acts by

$$s \cdot (x, v_1, v_2) = (s^k x, s^{-1}v_1, s^{-1}v_2).$$

This is an X-bundle over $\mathbb{P}^1$ with an action on $\widehat{T}$ by

$$(t, u) \cdot [x, (v_1, v_2)] = [t \cdot x, (v_1, uv_2)].$$

Setting $0 = [1, 0]$ and $\infty = [0, 1]$, we see that $\widehat{T}$ acts on X_0 by ρ_0 and X_∞ by ρ_k.

Definition 1.3.1. A cocharacter $k\colon \mathbb{C}^\times \to T$ is *seminegative* if all weights of $H^0(X, \mathcal{O})$ are nonpositive with respect to k and is *negative* if all nonzero weights of $H^0(X, \mathcal{O})$ are negative.

Lemma 1.3.2. *If k is seminegative, then E_k is semiprojective.*

Now let $\pi\colon E_k \to \mathbb{P}^1$ be the projection. We now consider *section classes*, which are those effective classes in $H_2(E_k, \mathbb{Z})$ satisfying $\pi_* d = [\mathbb{P}^1]$. For the $\mathbb{C}^\times$-action on X given by k, there is a unique fixed component $F_{\min}$ whose normal weights are all positive (one way to see this is to consider the moment map of the corresponding circle action). Therefore, there is a minimal section class $\sigma_{\min}$ corresponding to $F_{\min}$.

Lemma 1.3.3. *Given $\tau \in H^*_T(X)$, there exists $\widehat{\tau} \in H^*_{\widehat{T}}(E_k)$ such that $\widehat{\tau}|_{X_0} = \tau$ and $\widehat{\tau}|_{X_\infty} = \Phi_k(\tau)$.*

Lemma 1.3.4. *If k is seminegative, then*

$$\mathrm{Eff}(E_k)^{\mathrm{sec}} = \sigma_{\min} + \mathrm{Eff}(X).$$

Definition 1.3.5. Let $k\colon \mathbb{C}^\times \to T$ be seminegative. Given $\tau \in H^*_T(X)$, we define the *shift operator*

$$\widetilde{\mathbb{S}}_k\colon H^*_{\widehat{T},\rho_0}(X)[\![Q]\!] \to H^*_{\widehat{T},\rho_k}(X)[\![Q]\!]$$

by the formula

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha, \beta) = \sum_{\widehat{d} \in \mathrm{Eff}(E_k)^{\mathrm{sec}}} \frac{Q^{\widehat{d}-\sigma_{\min}}}{n!} \langle (\iota_0)_*\alpha, (\iota_\infty)_*\beta, \widehat{\tau}, \dots, \widehat{\tau} \rangle^{E_k, \widehat{T}}_{0,n+2,\widehat{d}},$$

where $\alpha \in H^*_{\hat{t},\rho_0}(X)$ and $\beta \in H^*_{\widehat{T},\rho_k}(X)$. We also define

$$\mathbb{S}_k(\tau) = \Phi_k^{-1} \circ \widehat{\mathbb{S}}_k(\tau).$$

Theorem 1.3.6. *We have the formula*

$$M(\tau) \circ \mathbb{S}_k(\tau) = \mathcal{S}_k \circ M(\tau),$$

where $\mathcal{S}_k$ is defined via the commutative diagram

$$\begin{tikzcd} H^*_{\widehat{T}}(X)_{\mathrm{loc}} \arrow[r, "\mathcal{S}_k"] \arrow[d] & H^*_{\widehat{T}}(X)_{\mathrm{loc}} \arrow[d, "\iota^*"] \\ H^*_{\widehat{T}}(X^T)_{\mathrm{loc}} \arrow[r, "\bigoplus_i \Delta_i(k) e^{-2k\delta_\lambda}"] & H^*_{\widehat{T}}(X^T)_{\mathrm{loc}}. \end{tikzcd}$$

Here, we define

$$\Delta_i(k) = Q^{\sigma_i - \sigma_{\min}} \prod_\alpha \prod_{j=1}^{\mathrm{rk}\, N_{i,\alpha}} \frac{\prod_{c=-\infty}^{0} (\rho_{i,\alpha,j} + \alpha + cz)}{\prod_{c=-\infty}^{-\alpha\cdot k} (\rho_{i,\alpha,j} + \alpha + cz)} \in H^*_{\widehat{T}}(F_i)_{\mathrm{loc}}[\![Q]\!],$$

where

$$N_i = N_{F_i/X} = \bigoplus_\alpha N_{i,\alpha}$$

is the normal bundle of F_i in X and $\rho_{i,\alpha,j}$ are its Chern roots.

The idea of the proof is to decompose

$$E^{\widehat{T}}_{k,0,n+2,\widehat{d}} = \bigsqcup_{i} \bigsqcup_{I_1 \cup I_2 = [n+2]} \bigsqcup_{d_0 + d_\infty + \widehat{\sigma} = \widehat{d}} (X_0)^T_{0,I_1 \sqcup p, d_0} \times_{F_i} (X_\infty)^T_{0, I_2 \sqcup q, d_\infty}.$$

Using the exact sequence

$$0 \to \mathrm{Aut}(C,x) \to \mathrm{Def}(f) \to T^1 \to \mathrm{Def}(C,x) \to \mathrm{Obs}(f) \to T^2 \to 0,$$

we obtain the explicit formulae

$$\begin{aligned} \mathrm{Aut}(C,x)^m &= \mathrm{Aut}(C_0,x_0)^m + \mathrm{Aut}(C_\infty,x_\infty)^m \\ \mathrm{Def}(C,x)^m &= \mathrm{Def}(C_0,x_0)^m \oplus \mathrm{Def}(C_0,x_0)^m \oplus T_p C_0 \otimes T_p \mathbb{P}^1 \oplus T_q C_\infty \otimes T_q \mathbb{P}^1. \end{aligned}$$

This gives the virtual normal bundle, and using virtual localization, we obtain

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha, \beta) = (\widetilde{\mathbb{S}}_k M(\tau,z)\alpha, M'(\tau', -z)\beta),$$

where

$$M'(\tau',z) = \Phi_k \circ M(\tau,z) \circ \Phi_k^{-1}.$$

Using the unitarity property of M, we obtain the desired result.