Lie Groups and Representations
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Disclaimer

These notes were taken during lecture using the `vimtex` package of the editor `neovim`. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the instructor. If you find any errors, please contact me at `plei@math.columbia.edu`.

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Basic Notions

Definition 1.0.1. A *Lie group* is a group that is also a manifold. Here, manifold could mean a smooth manifold, complex manifold, or many other options.

Example 1.0.2. Locally, every Lie group looks like $(\mathbb{R}^n, +)$. An example of a complex Lie group is $\mathbb{C}^n$.

Remark 1.0.3. If $\mathbb{F}$ is a field with topology, then the additive or multiplicative group of $\mathbb{F}$ are topological groups, but usually not Lie groups.

Example 1.0.4. Let p be prime and consider the field $\mathbb{Q}_p$ of p-adic numbers. This has a topology, but is not locally isomorphic to a vector space. Here, the base of neighborhoods of 0 is formed by the fractional ideals $p^n\mathbb{Z}_p$, whereas neighborhoods of 0 in $\mathbb{R}^n$ are not subgroups.

Remark 1.0.5. It is possible to develop analysis for the p-adics and consider p-adic Lie groups.

More generally, one can define a class of "manifolds" by postulating local models and the corresponding algebras of functions. For real manifolds, the local model is $\mathbb{R}^n$ and the algebra is $C^\infty(\mathbb{R}^n)$. Then a map is C^∞ if and only if it pulls smooth functions back to smooth functions. To an inclusion $U'' \subset U$, we will associate a "restriction" $C^\infty(U) \to C^\infty(U'')$. This is known as a *(pre)sheaf* of algebras on M. To be a sheaf means that given the restrictions

$$C^\infty(U) \to \prod_i C^\infty(V_i) \rightrightarrows \prod_{i<j} C^\infty(V_i \cap V_j),$$

the first restriction is injective and its image is

$$\{f_i \in C^\infty(V_i) \mid \mathrm{res}_1 f_i = \mathrm{res}_2 f_i\}.$$

To define complex manifolds, we consider the sheaf of holomorphic functions.

For some of the "other options," we may consider algebraic varieties over a field $\mathbb{F}$, where the algebras of functions are reduced commutative algebras of the form $\mathbb{F}[x_1, \ldots, x_N]/I$. If we give up the idea of being reduced, we obtain schemes over $\mathbb{F}$. Algebraic varieties are both more flexible (singularities are allowed) and more rigid (any piece of the map determines the whole thing) than smooth manifolds.

Example 1.0.6. Consider the group $G = SL(n, \mathbb{F})$. G is a Lie group of dimension $n^2 - 1$ for $\mathbb{R}$ and $\mathbb{C}$, and in general, $SL(n, \mathbb{F})$ is the group of $\mathbb{F}$-points in the *algebraic group* $SL(n)$ defined by the equation $\det = 1$.

In algebraic geometry, any variety contains an open set of smooth points. Because any group is a homogeneous space, all points must have the same properties, so they must all be smooth.

Definition 1.0.7. A Lie group G *acts* on a manifold M if there is a map of manifolds $G \times M \to M$ such that $1 \cdot m = m$ and $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$.

For all $m \in M$ we can define the orbit and the stabilizer. If all of M forms one orbit, we say M is *homogeneous* or that the action is *transitive*.

Example 1.0.8. Let $M = G$. Then the action can be given by

Left $g \cdot h = gh$;

Right $g \cdot h = hg^{-1}$;

Adjoint $g \cdot h = ghg^{-1}$;

1.1 Examples of Lie Groups

For now, for real Lie groups, the notion of manifold we will use is that of a smooth manifold.

Example 1.1.1. The additive group $\mathbb{R}^n$ is a Lie group. In one dimension, the only connected manifolds are $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z} = SO(2, \mathbb{R})$.

Example 1.1.2. Recall the classification of two-dimensional manifolds. First, any Lie group is orientable, so we will consider only the orientable manifolds. These are classified by their genus, and only the torus $S^1 \times S^1$ is a Lie group. To see this, note that Lie groups have a trivial tangent bundle. A global frame for TG is given by translating a basis of $T_1 G$ by G. Therefore, we can write $TG = G \times T_1 G = G \times \mathfrak{g}$.

Remark 1.1.3. By the Hopf index theorem, the self-intersection of M inside TM is $\chi(M)$, so the index of a vector field on Σ_g is $2 - 2g$.

We will now consider some 3-manifolds. In particular, S^3 is the Lie group $SU(2)$. Note that unitary transformations of $\mathbb{C}^2$ send S^3 to itself. Therefore, we can send one basis element anywhere, but if we insist that the determinant is 1, the second basis vector is sent to a unique target. Thus $SU(2)$ acts transitively on S^3 with trivial stabilizers. Alternatively, we can write

$$SU(2) = \left\{ \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \right\}.$$

All connected real Lie groups, as manifolds, have the form

$$G = (\text{maximal compact subgroup}) \times \mathbb{R}^r.$$

The maximal compact subgroup is a product of S^1 and simple nonabelian Lie groups. The simple nonabelian Lie groups are all built from $SU(2)$ in some sense.

Corollary 1.1.4. *The only finite-dimensional division algebras over* $\mathbb{R}$ *are* $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Note that $SO(4, \mathbb{R})$ acts on S^3 by rotations. Therefore we have a map $SU(2) \times SU(2) \to SO(4, \mathbb{R})$ with kernel $(-1, -1)$, so we have an exact sequence

$$1 \to (\pm 1) \to SU(2) \times SU(2) \to SO(4, \mathbb{R}) \to 1.$$

Heuristically, this means that left translation and right translation are as different as can possibly be.

Now we have seen groups like $GL(n, \mathbb{R}), SL(n, R), U(n), SU(n), SO(n, \mathbb{R})$. In fact, the groups $SU(n), SO(n)$, the symplectic groups, and a few exceptional Lie groups, make up all simple compact nonabelian Lie groups (up to discrete centers).

1.2 Lie Group Actions on Manifolds

Recall the notions of action, orbit, stabilizer, etc. Then we have a map

$$G \times \{m\} \to \operatorname{orbit}(m) \subset M$$

that is equivariant, so the differential of this map has constant rank r. Locally, the map looks like $\mathbb{R}^{n-r} \times \mathbb{R}^r \to \mathbb{R}^r$. Locally, the orbit looks like $\mathbb{R}^r$ and the stabilizer looks like $\mathbb{R}^{\dim G - r}$.

Theorem 1.2.1. *The stabilizer of any $m \in M$ is a submanifold of G. By definition, it can be upgraded to a Lie subgroup of G. In addition, there is a neighborhood U of $1 \in G$ such that $U \cdot m$ is a submanifold in M. If G is compact, then $G \cdot m$ is a submanifold.*

Remark 1.2.2.

1. $G \cdot m$ need not be a submanifold of M. The classical example is when $\mathbb{R}$ acts on M by flow along a vector field. For example, if $M = \mathbb{R}^2/\mathbb{Z}^2$, we can choose the vector field to be a constant vector field with irrational slope. The orbits are dense in M. In fact, we obtain a group homomorphism $\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ with dense image.

2. For algebraic actions, orbits are much nicer because for $f : X \to Y$ algebraic, $f(X)$ contains a dense open subset of its closure and in fact is open in its closure (consider $\mathbb{C}^*$ acting on $\mathbb{C}$).

Let G be a Lie group acting on a manifold M. We will denote the stabilizer of $x \in M$ as G_x and the orbit of x as $G \cdot x$. Note the orbit is not necessarily a submanifold., but is locally a submanifold. One such example is when $\mathbb{R}$ acts on M by time evolution according to some ODE. This type of behavior is studied in the field of dynamical systems.

Example 1.2.3. A homomorphism $G \xrightarrow{\varphi} H$ is a special case of an action. If G acts by left or right multiplication, we see that $G_{1_H} = \ker \varphi$ is a Lie subgroup of G and $\operatorname{Im} \varphi = G \cdot 1_H$ may or may not be a Lie subgroup. We will call these "virtual Lie subgroups."

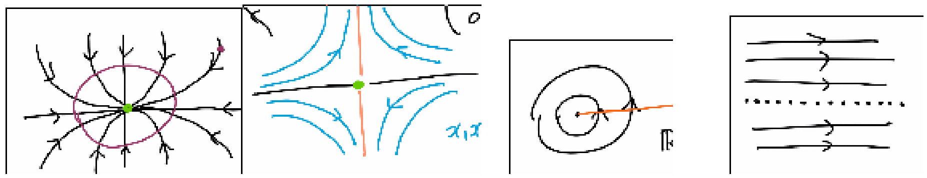
Now we will discuss the space of orbits M/G. Right now, this is still a set, but we can canonically make it a topological space with the quotient topology.

Remark 1.2.4. This quotient space is very rarely Hausdorff, so in particular it is almost never a manifold. For example, if we consider $\mathbb{R}^\times$ acting on $\mathbb{R}$, we see that the quotient space $\mathbb{R}/\mathbb{R}^\times$ consists of two points, where the point 0 is closed and the point $\mathbb{R}^\times$ is a generic point.

Example 1.2.5. Now consider actions $\mathbb{R} \xrightarrow{\varphi} GL(2,\mathbb{R})$ acting on $\mathbb{R}^2$. Then there are several cases:

$$\varphi(t) = \begin{cases} \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix} & ab > 0 \\ \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix} & ab < 0 \\ \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix} & \text{complex conjugate} \\ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \end{cases}.$$

In these cases, the orbits look like this:



(a) (b) (c) (d)

Figure 1.1: Orbits in various cases.

In the third case, the orbit space is $\mathbb{R}_{\geq 0}$. In the final case, even though the orbits are closed, the space is still not Hausdorff. In summary, M/G may be non-Hausdorff in a very complicated way.

Note that M/G has a natural sheaf of functions. If $U \subset M/G$ is open, then $\pi^{-1}(U)$ is open in M, where $\pi : M \to M/G$ is the projection. We will declare the functions on U to be the G-invariants. In the best case scenario, when U is sufficiently small, we have $\pi^{-1}(U) = U \times G$ where the action is contained entirely in the second factor. Therefore functions on U defined in the new sense are the same as normal functions on U.

However, there is no reason to expect this kind of behavior. We may get interesting behavior even for quotients by a finite group. For example, consider $M = \mathbb{R}^2$ and let $G = \{\pm 1\}$. Then M/G is simply the closure of the upper half-plane with the negative and positive real axes glued together, so we obtain a cone with total angle π. We see that $\mathbb{R}^2/\{\pm 1\}$ is a manifold near every point except 0. At 0, we study functions of the form $f(x_1, x_2)$ such that $f(-x_1, -x_2) = f(x_1, x_2)$, which are functions of $u = x_1^2, v = x_2^2, w = x_1 x_2$. It is easy to see that these satisfy the equation $uv = w^2$.

Similarly, if we consider $\mathbb{Z}/m\mathbb{Z}$ acting on $\mathbb{R}^2$ by roots of unity, then we will obtain invariants $(u, v, w) = (x_1^m, x_2^m, x_1 x_2)$ satisfying $uv = w^m$. This is known as the A_{m-1} surface singularity.

Remark 1.2.6. Note that $\mathbb{R}^2/\{\pm 1\}$ is very different from $\mathbb{C}/\{\pm 1\}$ because we take the second quotient in the category of complex manifolds. Here, $\mathbb{C}/\{\pm 1\} \simeq \mathbb{C}$ and the projection is a double cover branched at 0. There is a similar result for $\mathbb{C}/\zeta_m$.

More generally, we have the following result: a finite subgroup $G \subset GL(n, \mathbb{C})$ is generated by complex reflections if and only if $\mathbb{C}^n/G \simeq \mathbb{C}^n$. In general, $\mathbb{C}^n/(\text{finite subgroup of } GL(n, \mathbb{C}))$ is singular.

Example 1.2.7. Consider the permutation group $S_n \subset GL(n, \mathbb{C})$. Each transposition (ij) is a reflection in the hyperplane $x_i = x_j$. Therefore the coordinates on $\mathbb{C}^n/S_n$ are the elementary symmetric functions

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

for $k = 1, \ldots, n$. This is an important notion in representation theory because we can consider the map

$$GL(n, \mathbb{F})/\text{conjugation} \xrightarrow{\text{eigenvalues}} \overline{\mathbb{F}}^n/S_n.$$

In general, S_n can be replaced by the Weyl group.

In summary, M/G may have complicated topology in singular. Complexity is good in math, but it is also good to have the simple cases. The best possible case is when the action is *proper*, i.e. that the map $G \times M \to M \times M, (g, x) \mapsto (gx, x)$ is proper (ensures the quotient is Hausdorff), and *free*, i.e. that there are no stabilizers.

Theorem 1.2.8. *Let $G \times M \to M$ be a free and proper action of a Lie group G on a manifold M. Then M/G is a manifold and the projection $M \xrightarrow{\pi} M/G$ is a locally trivial fibration with fiber G.*

Example 1.2.9. Here are some examples of a free and proper action:

1. The action of G on $H \supset G$. Then $(g,h) \mapsto (gh,h)$ is an embedding and is in particular proper. Therefore H/G is a manifold.

2. Any free action of a compact Lie group.

Proposition 1.2.10. *The map $\pi : M \to M/G$ is open.*

Proof. Note that $\pi^{-1}(\pi(V)) = \bigcup_{g \in G} g \cdot V$ is open. □

Note that the quotient by a group is a special case of a quotient by an equivalence relation.

Proposition 1.2.11. *Suppose $\pi : M \to Y$ is open and a quotient by a closed equivalent relation. Then Y is Hausdorff.*

Proof. Suppose that x, x' be such that $\pi(x,x') = (y,y')$ such that $(x,x') \notin R$. Then there exists a neighborhood of (x,x') not intersecting R, so there exist $U, U' \ni x, x'$ such that $U \times U' \cap R = \emptyset$. These project to disjoint opens, so Y is Hausdorff. □

In our situation, R is the image of $G \times M \to M \times M$. By definition, the action is proper if this map is proper.

Proposition 1.2.12. *Suppose $f : X \to Y$ is proper. Then the image of a closed set is closed.*

Proof. It suffices to prove $f(X)$ is closed. Suppose $f(x_i) \to y_\infty$. Then $x_i \in f^{-1}(\{f(x_i, y_\infty)\})$, which is compact. Thus we can find a subsequence $x_{k_i} \to x_\infty \in X$, so because f is continuous, $f(x_\infty) = y_\infty$. □

In particular, in a Hausdorff space, all points are closed. In terms of our group action, this means all orbits are closed.

Proof of Theorem 1.2.8. Fix a point $x \in M$ and look at the neighborhood of $\pi(x)$. Consider the differential $\mathfrak{g} \oplus T_xM \to T_xM$ of the action map $G \times M \to M$. Because this map has maximal rank everywhere, if we choose coordinates ξ along $G.x$ and η along M and ξ' for G, the differential is simply $(\xi', \xi, \eta) \mapsto (\xi' + \xi, \eta)$.

Choose a submanifold S transverse to the orbit, we can consider the action $G \times S \to M$. We see that this map is a local diffeomorphism, so we need $\pi^{-1}(\pi(S)) = G \times S$. This is not obvious and requires properness. One can imagine that there exists g such that $gS \cap S \neq 0$ for any S.

Choose a sequence S_n that shrinks to x. Then we consider the set $\{g \mid gS_n \cap S_n \neq \emptyset\} \setminus \{1\}$. Then we can consider

$$\{g \mid g\overline{S}_n \cap \overline{S}_n \neq \emptyset\}$$

with a fixed neighborhood of 1 removed. This is compact by properness. If g lies in the intersection of all such sets, it must stabilize x, which is impossible. □

Remark 1.2.13. For algebraic actions, it is possible that $G.x$ is free but for all open U, $\mathrm{Stab}(x') \neq \{e\}$ for all $x' \in U$. For an example, $SL(2, \mathbb{C})$ acts on cubic polynomials in x_1, x_2. The generic polynomial has three roots and has stabilizer μ_3, but $x_1^2 x_2$ has trivial stabilizer.

Theorem 1.2.14. *Suppose the action of G on M is proper (but possibly not free). Then the normal bundle of $G.x$ is a vector space with an action of G_x, so there exists a neighborhood of $G.x$ isomorphic to $G \times N_x/G_x$. This is a vector bundle over the orbit with an action of G (it is precisely the associated bundle).*

Now note that the stabilizer of G_x is compact because the action is proper. We would like to find a G_x-invariant slice at x. To do this, we will need to discuss metrics. This is a smooth nondegenerate positive-definite quadratic form on each fiber. Then we can define the length of a curve by

$$\int \sqrt{\left\|\dot{x}(t)\right\|^2}\mathrm{d}t.$$

There exist curves that minimize length locally, and these are called *geodesics*.

Then there is a map $T_xM \to M$ given by following a vector v along the geodesic in the direction of v for time 1. This is a local diffeomorphism and is closely related to the exponential for Lie groups. Later in the lecture, we will prove that if G acts on M and G is compact, then M has a G-invariant Riemannian metric.

In particular, there is a G_x-invariant Riemannian metric on M. Then we can write $T_xM = T_xG \cdot x \oplus (T_xG \cdot x)^\perp$. Identifying $(T_xG \cdot x)^\perp$ with the normal bundle $\nu_{G\cdot x}$, then the slice is simply $S := \exp(\nu_{G\cdot x})$.

Now consider the action $q : G \times S \to M$. If $h \in G_x$, then $q(gh, s) = q(g, hs)$ by definition. Therefore, we can write

$$q : G \times S/G_x \to M,$$

where G_x acts by $(g, x) \mapsto (gh^{-1}, hs)$. This is G-equivariant and locally an isomorphism.

Theorem 1.2.15. *This is an isomorphism of $G \times N_x/G_x \to M$ is a neighborhood of the orbit of x. Note that we can scale any $\mathbb{R}^n$ to the unit ball.*

Corollary 1.2.16. *The neighborhood of the orbit of x in M/G looks like N_x/G_x.*

Corollary 1.2.17. *The stabilizer of any nearby point is conjugate to a subgroup of G_x.*

Remark 1.2.18. Sometimes the quotient $G \times Y/H$ by the action $(g, y) \mapsto (gh^{-1}, hy)$ is denoted by $G \times_H Y$.

Corollary 1.2.19. *The manifold M has a G-invariant metric.*

Proof. Let $h \in G_x$. Then for a vector $v \in T_{gx}M$, we can associate $g^{-1}v$ and $h^{-1}g^{-1}v$ to v, but these must have the same length because they differ by an element of the stabilizer. This gives an invariant metric in a neighborhood of the orbit. Finally, we can sum the local metrics over a partition of unity to obtain a global invariant metric. □

Theorem 1.2.20. *Let H be a compact Lie group acting on a manifold M. Then M has an H-invariant metric.*

Proof. Choose some Riemannian metric $\|-\|_0$ on M. Then choose a Haar measure on H and define

$$\|v\|^2 := \int_H \mathrm{d}h \|h \cdot v\|_0^2.$$

To show invariance, note that

$$\|g \cdot v\|^2 = \int_H \mathrm{d}h \|g \cdot h \cdot v\|_0^2 = \int_H \mathrm{d}\left(g^{-1}h'\right) \|h' \cdot v\|_0^2 = \int_H \mathrm{d}h' \|h' \cdot v\|_0^2.$$

□

More generally, suppose H acts on an affine linear space by affine transforations and suppose there is a closed convex H-invariant subset B. Then there exists an H-fixed point (by the same integration argument).

Now we will show existence of the Haar measure. Recall that $TH \simeq T_1H \times H$ by left translation. Then we may choose an $\mathfrak{h}$-valued 1-form $g^{-1}\mathrm{d}g$ on H, and this gives a finite volume form if the group is compact.

2

Classification of Lie Groups

2.1 Topology of Lie Groups

Recall that if G acts on M properly and freely then $\pi : M \to M/G$ is a locally trivial fibration with fiber G and M/G is a manifold. Now suppose H is a Lie subgroup of G. Then H acts freely and properly on G. Therefore we have a map $G \to G/H$ and G/H is a manifold. Our goal is to use this fibration to understand the geometry of its ingredients.

Example 2.1.1. Consider $G = SU(2) \simeq S^3$ and let H be the set of diagonal matrices in G. Then to compute G/H, note that G acts on $\mathbb{C}^2$ and hence on $\mathbb{CP}^1$. The stabilizer of a point is H, and thus $G \to G/H$ is the *Hopf fibration* $S^1 \to S^3 \to S^2$. An illustration of the Hopf fibration is below:

Figure 2.1: The Hopf fibration

Any two fibers are linked as the Hopf link, so this is not a globally trivial fibration.

Example 2.1.2. Consider $G = SU(2)$ acting on itself by conjugation. This fixes $1 \in G$, so it acts by conjugation on $\mathfrak{g} = T_1 G = \{\xi \in M_2(\mathbb{C}) \mid \xi + \xi^\dagger = 0, \operatorname{tr} \xi = 0\}$. Because the norm of the matrix is preserved, we have a map $SU(2) \to SO(3, \mathbb{R})$ with kernel the center of $SU(2)$, which is just $\{\pm 1\}$. By dimension arguments, the map is surjective, and thus we have realized $SO(3, \mathbb{R}) \simeq \mathbb{RP}^3$.

We can discuss various topological invariants of Lie groups, in particular their homotopy, homology, and cohomology groups. We will begin with $\pi_0(X)$, the set of connected components. If G is a Lie group, then $\pi_0(G)$ is a **group** isomorphic to G/G_0, where G_0 is the connected component of the identity. It is clear that $\pi_0(G/H) = \pi_0(G)/\operatorname{Im}\pi_0(H)$ under the natural map $\pi_0(H) \to \pi_0(G)$.

Example 2.1.3. Let $G = SU(2)$ and let $H = Z(SU(2))$. Then any path connecting ± 1 in G descends to $G/H \simeq \mathbb{RP}^3$, so the kernel of $\pi_0(H) \to \pi_0(G)$ is exactly the image of the transport $\pi_1(G/H) \to \pi_0(H)$.

Recall that we have a long exact sequence

$$\cdots \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1$$

arising from the fibration.

Theorem 2.1.4. *Let G be a topological group. Then $\pi_1(G)$ is abelian.*

Proof. The homotopy $[0,1] \times [0,1] \to G, (t,s) \mapsto \gamma_1(t)\gamma_2(s)$ exhibits a homotopy between $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$ for two loops γ_1, γ_2 based at the identity. $\square$

Remark 2.1.5. $\pi_1(\mathbb{R}^2 \setminus \pm 1)$ is not abelian. In fact it is equal to $\mathbb{Z} * \mathbb{Z} = F_2$.

Let $\widetilde{X} \xrightarrow{\pi} X$ be a covering space. Then we have a map $\pi_1(X) \to \widetilde{x} \to X$ given by transport, and if $\pi_1(\widetilde{X}) = 1$, then $\widetilde{X}$ is the universal cover.

Proposition 2.1.6. *If G is a Lie group then so is its universal cover $\widetilde{G}$ by multiplication $\gamma_1(t)\gamma_2(t)$.*

Corollary 2.1.7. *If G is a connected Lie group, then there exists a unique simply connected Lie group $\widetilde{G}$ such that $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$ is an exact sequence.*

Example 2.1.8. The map $SU(2) \to SO(3, \mathbb{R})$ is a universal covering. An even more basic example is the cover $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$.

Proposition 2.1.9. *Suppose G is a connected Lie group and $\Gamma \subset G$ is a discrete normal subgroup. Then $\Gamma \subset Z(G)$.*

Proof. Let $\gamma \in \Gamma$ and consider the map $G \ni g \mapsto g\gamma g^{-1} \in \Gamma$. Because G is connected, the image is a point, which must be γ because we can choose $g = 1$. $\square$

In summary, any connected Lie group G has the form $\widetilde{G}/\Gamma$, where $\widetilde{G}$ is simply connected and Γ is a discrete subgroup of the center.

Corollary 2.1.10. *If G is abelian, then G is of the form $\mathbb{R}^n/\Lambda$, where Λ is some discrete subgroup, and this means that $G \simeq \mathbb{R}^k \times (S^1)^{n-k}$.*

Remark 2.1.11. This allows us to prove the fundamental theorem of algebra. If $\mathbb{F}/\mathbb{R}$ is a field extension, then $(\mathbb{F}^*)_0$ is abelian and connected. Then for $d = 1$, this is $\mathbb{R}$, for $d = 2$ this is $S^1 \times \mathbb{R}$, and for $d \geq 3$ this is $S^{d-1} \times \mathbb{R}$, which is impossible.

Here is a very important ideal in Lie theory: Consider a Lie group G with identity 1. Then if we consider $\mathfrak{g} = T_1 G$, we can reconstruct a lot of information about G. An obvious limitation of this approach is that some Lie groups are locally isomorphic.

Example 2.1.12. Recall that $SU(2)$ is a double cover of $SO(3)$, so they are locally isomorphic. In particular, any group G is locally isomorphic to G/Γ, where $\Gamma \subset Z(G)$ is discrete.

Our strategy for dealing with this is to determine the universal cover, which is equivalent to determining $\pi_1(G)$. We will see later that simply-connected Lie groups are determined by the infinitesimal data.

Previously, we discussed the long exact sequence arising from a fibration. To make this precise, we need to define $\pi_n(X, *)$. But this is simply the group $[S^n, X]^0$ of homotopy classes of based maps from S^0 to X. For $n > 1$, we can see that π_n is commutative by the following picture (or the fact that S^n is a double suspension).

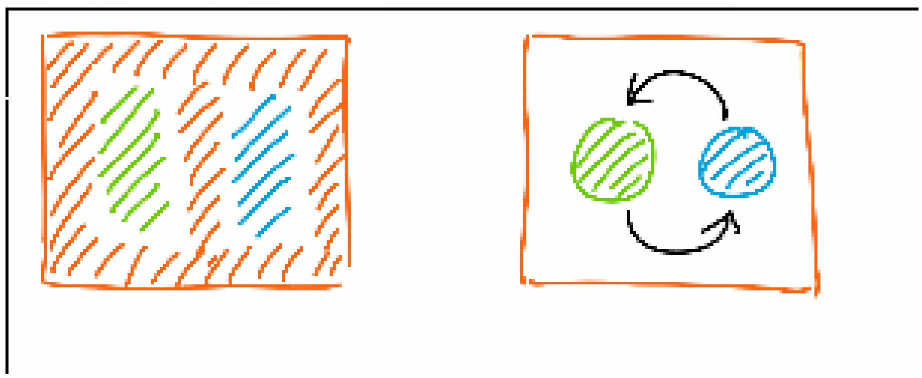


Figure 2.2: Proof that π_2 is abelian by picture

Very often, G/H is a sphere, with homotopy groups

$$\pi_k(S^n) = \begin{cases} 0 & k < n \\ \mathbb{Z} & k = n \\ ??? & k \gg n \end{cases}.$$

Not much is known about the higher homotopy groups of spheres, and computing them is a central problem in modern algebraic topology. Fortunately, it is easier to compute the homotopy groups of Lie groups.

2.2 Lie Algebras

We will discuss reconstruction of simply-connected Lie groups from their local data. This can be phrased as an equivalence of categories. On one side, we have the category of simply-connected Lie groups, and on the other side, we have the category of Lie algebras. We will define a functor Lie from Lie groups to Lie algebras that is an equivalence of categories.

We will define $\mathsf{Lie}(G) = T_1(G)$ plus some extra data, and for any $f \colon G \to G'$, we wil define $\mathsf{Lie}(f) = \mathrm{d}f \colon T_1 G \to T_1 G'$.

Example 2.2.1. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a Lie group homomorphism. Then we can differentiate to see that

$$\left.\frac{\mathrm{d}}{\mathrm{d}y}\right|_{y=0} f(x+y) = f'(x) = f'(0).$$

This means that the morphism is determined by an ODE, and so it must be linear. Then $f'(0)$ is precisely the map between Lie algebras.

In general for $f\colon G \to G'$, then we can differentiate with respect to g_2 at $g_2 = 1$ to obtain a system of first order ODEs. Then solvability of these ODEs is a condition on $\mathrm{d}f$ that is equivalent to being a homomorphism of Lie algebras.

Definition 2.2.2. A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear operation

$$[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

such that $[x,x] = 0$ that satisfies the Jacobi identity:

$$[z,[x,y]] + [y,[z,x]] + [x,[y,z]] = 0.$$

We will construct the Lie bracket out of multiplication in G. If we simply differentiate the multiplication m, the differential

$$\mathrm{d}m\colon T_1 G \oplus T_1 G \to T_1 G$$

is simply the addition. This is a linear map, but it tells us that all Lie groups are the same locally.

Therefore, we need to consider higher order terms in the Taylor expansion of m. This is

$$m(\xi,\eta) = 0 + (\xi + \eta) + \text{quadratic terms} + \cdots$$

and the quadratic term is a bilinear form $B(\xi,\eta)$ with no quadratic terms in ξ or η. This is **not** independent of the coordinates because if we choose $\xi' = \xi + Q(\xi), \eta' = \eta + Q(\eta)$, then the multiplication becomes

$$m(\xi',\eta') = \xi + \eta + B(\xi,\eta) + Q(\xi+\eta) + \cdots$$

In barticular, we have $B' = B + Q(\xi+\eta) - Q(\xi) - Q(\eta)$, which does not necessarily vanish. However, it is symmetric, so we can define the *Lie bracket*

$$[\xi,\eta] = B(\xi,\eta) - B(\eta,\xi).$$

> *In mathematics, there is a high road and a low road. The low road is to write everything in coordinates, which is sort of what we did here. It's good to be able to take the low road, for example when we need to compute with a computer.* *A. Okounkov*

Remark 2.2.3. There are many other definitions of the Lie bracket.

1. We can consider the commutator in G. If we assume $G \subset GL(n,\mathbb{F})$, then we have $[\xi,\eta] = \xi\eta - \eta\xi$. The differential of the commutator vanishes, but the second differential is precisely the bracket.

We need to prove that the commutator satisfies the Jacobi identity, which can be restated as

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

Now define $\mathrm{ad}_x(-) = [x,-]$. Then the Jacobi identity says that ad_x is a derivation of the Lie bracket.

Note that if V is a vector space with $\star\colon V \otimes V \to V$, then $\mathrm{Aut}(\star) \subset GL(V)$ is an algebraic subgroup. If we apply the Lie functor, we see that

$$\mathsf{Lie}(\mathrm{Aut}(\star)) = \{D \in \mathfrak{gl}(V) \mid D(y \star z) = D(y) \star z + y \star D(z)\}.$$

Consider the adjoint action of G on itself. This fixes $h = 1$, so it gives a representation of G on $\mathfrak{g} = T_1 G$. If G is a matrix group, then this is really a product $\xi \mapsto g\xi g^{-1}$. If we write this out in coordinates, we see that the adjoint representation Ad of G takes values in $\mathrm{Aut}([-,-])$ and thus when we differentiate. If we take the derivative $\mathrm{ad} = \mathrm{d}_1\mathrm{Ad}$, we obtain a map

$$g \to \mathrm{Der}([-,-]) \subset \mathfrak{gl}(\mathfrak{g}).$$

We simply need to check that $\mathrm{ad}_x = [x,-]$. But this is obvious because the left hand side comes from differentiating ghg^{-1} and the right hand side comes from differentiating $ghg^{-1}h^{-1}$.

Remark 2.2.4. This gives yet another definition of the Lie bracket.

2.3 Correspondence between Lie groups and Lie algebras

Now let M be a smooth manifold with an action of a Lie group G. Then $C^\infty(M)$ is an algebra that is acted on by G. Then we know that $\mathrm{Der}(C^\infty(M))$ is simply the space of vector fields. Then we can write $f(x) = f(x_0) + \mathrm{d}f + \mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the ideal of functions that vanish at x_0. For any derivation, $D(\mathfrak{m}_x^2)\big|_{x=x_0} = 0$. This defines a tangeng vector at any $x_0 \in M$. In particular, $\mathfrak{g}$ defines C^∞ vector fields on M. Then we know that derivations form a Lie algebra.

For example, consider the action of G on itself by left translation. Then we have a map $\mathfrak{g} \to H^0(G, TG)$. In addition, it is easy to see that this gives right-invariant vector fields on G. On the other hand, right-invariant vector fields are determined by their value at 1, so we have an isomorphism $H^0(G, TG)^G \simeq \mathfrak{g}$ of vector spaces. In fact, this can be upgraded to an isomorphism of Lie algebras.

Returning to the main point, we want to prove

Theorem 2.3.1. *The functor* $\mathsf{Lie}\colon \{1\text{-}connected\ Lie\ groups\} \to \{Lie\ algebras\}$ *is an equivalence of categories.*

Theorem 2.3.2. *Let G_1, G_2 be connected Lie groups with $\mathsf{Lie}(G_i) = \mathfrak{g}_i$. Then any homomorphism $f\colon G_1 \to G_2$ is uniquely determined by $\mathrm{d}f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$.*

Proof. We know that $f(g_1 g_2) = f(g_1)f(g_2)$. Then if we differentiate with respect to $g_1 = 1 + \xi$, then

$$\frac{\mathrm{d}}{\mathrm{d}\xi} f(g) = f(\xi) \cdot f(g).$$

This gives us a system of first order ODE. By connectedness, there is a unique solution with prescribed initial condition. $\square$

Later, we will see that the mixed partials are equal (or the curvature vanishes) if and only if $\mathrm{d}f$ is a Lie algebra homomorphism.

Remark 2.3.3. There is no homomorphism of Lie groups from $G_1\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R} = G_2$ corresponding to the identity homomorphism between the Lie algebras.

These ODE that we obtain from differentiating the multiplication naturally lead to the concepts of connections and curvature. Suppose we have a locally trivial fiber bundle over a base B with fibers F. Then the idea of a connection is to be able to lift paths downstairs to paths upstairs respecting concatenations of paths.

Fix a Lie group G acting on the fiber F. Then we say the *structure group* is contained in G if all transition functions may be chosed to be in G. For example, a vector bundle is a locally trivial bundle with structure group $GL(n)$. We can use the same transition functions to glue copies of G, and we obtain a principal bundle $\mathcal{P}$. Then the old bundle can be obtained using the associated bundle construction $\mathfrak{P} \times_G F$ and a connection on a principal G-bundle induces a connection on any associated bundle.

In our case, we are talking about the trivial G-bundle over H. Then sections are maps from H to G. In coordinates, these are lifts that are invariant under the action of G on the right. This means we can consider the value at $1 \in G$, which means we have a map $\alpha \colon T_b B \to \mathfrak{g}$. Thus a connection can be thought of as a right-invariant Lie algebra-valued 1-form. However, this is dependent on the trivialization. If we change the section by a function $g(b)$, we know a section is contant if

$$\frac{\mathrm{d}}{\mathrm{d}\xi} - \alpha(\xi) = 0.$$

However, if we conjugate by g, then we need to differentiate $\mathrm{d}\big(g^{-1}\big) = -g^{-1}\,\mathrm{d}g\, g^{-1}$, and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi} - \widetilde{\alpha}(\xi) = 0,$$

where $\widetilde{\alpha} = g\alpha g^{-1} + \mathrm{d}g \cdot g^{-1}$.

Next, when does the transport along the path depend only on the endpoints? We can consider

1. Small changes, i.e. homotopies with fixed endpoints. In this case, the connection is *flat*.
2. Paths up to homotopy, i.e. $\pi_1(B)$.

Proposition 2.3.4. *A connection is flat if and only if its curvature is identically zero.*

The curvature is a certain 2-form that measures the difference between two solutions to an ODE. If we transport along $\xi_1\xi_2\xi_1^{-1}\xi_2^{-1}$, then we obtain the commutator

$$\left[\frac{\partial}{\partial\xi_2} - \alpha_2, \frac{\partial}{\partial\xi_1} - \alpha_1\right] = -\left(\frac{\partial}{\partial\xi_2}\alpha_1 - \frac{\partial}{\partial\xi_1}\alpha_2 + [\alpha_2, \alpha_1]\right).$$

Thus if the connection is flat, then the curvature vanishes. In the other direction, suppose we have two homotopic paths. Then if we break down the square $[0,1]^2$ into squares of size ε, then each square changes the result by $\varepsilon^2 \cdot \text{curvature} + O(\varepsilon^3)$, and so if we make ε small enough, the change vanishes.

Returning to our original problem, suppose we have a map $f \colon H \to G$. Then $\mathrm{d}f : \mathfrak{h} \to \mathfrak{g}$ determines a connection on $G \times_H H$. On H we have the canonical 1-form $\mathrm{d}h \cdot h^{-1}$ If $\mathrm{d}f$ is a Lie algebra homomorphism, then

$$[\mathrm{d}f\,(\xi_1), \mathrm{d}f\,(\xi_2)] = \mathrm{d}f\,([\xi_1, \xi_2])$$

and thus the curvature of the connection α induced from $\mathrm{d}h \cdot h^{-1}$ is the differential of the curvature of $\mathrm{d}h \cdot h^{-1}$, which is identically zero.

Remark 2.3.5. All of this can be expressed in elementary terms. First, we have $\frac{\partial}{\partial\xi_i} g = \alpha_i g$. Then we have

$$0 = \frac{\partial^2 g}{\partial\xi_i\partial\xi_j} - \frac{\partial^2 g}{\partial\xi_j\partial\xi_i} = \left(\frac{\partial\alpha_j}{\partial\xi_i} - \frac{\partial\alpha_i}{\partial\xi_j} + [\alpha_i, \alpha_j]\right) g.$$

Thus we have proved the following theorem:

Theorem 2.3.6. *If H is a simply connected Lie group and $\varphi \colon \mathsf{Lie}\, H \to \mathsf{Lie}\, G$ is a Lie algebra homomorphism, then there a unique map $f \colon H \to G$ such that $\mathrm{d}f\,\big|_1 = \varphi$.*

For example, if we want to prove that $\log(xy) = \log x + \log y$, we write

$$\log(xy) = \int_1^x \frac{\mathrm{d}t}{t} + \int_x^{xy} \frac{\mathrm{d}t}{t}$$

and note that the second term in the sum equals $\int_1^y \frac{\mathrm{d}t}{t}$.

Recall the differential equations that we constructed for a homomorphism of Lie groups from the Lie algebra. These equations are right-invariant in both the source and the target and imply that f is a homomorphism.

Example 2.3.7. Here is a silly example: Note the isomorphism $(\mathbb{R}_{>0}, \times) \to (\mathbb{R}, +)$. Then $f(xy) = f(x) + f(y)$ and so we have

$$y \frac{\mathrm{d}}{\mathrm{d}y} f = c$$

for some constant c. In the other direction, we have $\varphi(x+y) = \varphi(x)\varphi(y)$, so

$$\frac{\mathrm{d}}{\mathrm{d}y} \varphi = c \cdot \varphi$$

for some constant c. The solution to the second equation is clearly the exponential function, and the solution to the first is

$$f(y) = \int_1^y c \frac{\mathrm{d}t}{t} =: c \log y.$$

Invariance implies that log is a homomorphism.

Theorem 2.3.8 (Lie)**.** *For any simply connected Lie group G, the map*

$$\mathrm{Hom}_{\textit{Lie Groups}}(H, G) \xrightarrow{\mathrm{d}} \mathrm{Hom}_{\textit{Lie Algebras}}(\mathsf{Lie}(H), \mathsf{Lie}(G))$$

is an isomorphism.

Here are some applications:

1. Any connected abelian Lie group G of dimension n is of the form $\mathbb{R}^n/\Gamma$, where $\Gamma \cong \mathbb{Z}^k$. Thus $G = (S^1)^k \times \mathbb{R}^{n-k}$.

2. Not all Lie groups are matrix Lie groups. However, every Lie algebra is a matrix Lie algebra in characteristic 0. In the category of Lie algebras, we can always lift the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ to the center of $\mathfrak{g}$. However, when we consider the adjoint representation of G as a Lie group, we cannot lift to the center.

   For example, consider $SL(2,\mathbb{R})$. This has $\pi_1(SL(2,\mathbb{R})) = \mathbb{Z}$. Then $G = \widetilde{SL(2,\mathbb{R})}$ has $\mathbb{Z}$ in the center, and so any map

   $$f\colon G \to GL(N, \mathbb{C})$$

   corresponds to a map of Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(N,\mathbb{C})$. This gives a map $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(N,\mathbb{C})$, which then lifts to a map $SL(2,\mathbb{C}) \to GL(N,\mathbb{C})$. In particular, all linear representations of G factor through $SL(2,\mathbb{C})$ are are thus trivial on the center.

Definition 2.3.9. Suppose G is a real Lie group with Lie algebra $\mathfrak{g}$. Then if we take the complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, this gives a map $G \to G_{\mathbb{C}}$ for $G_{\mathbb{C}}$ the complex Lie group corresponding to $\mathfrak{g}_{\mathbb{C}}$. We say that $G_{\mathbb{C}}$ is a *complexification* of G and that G is a *real form* of $G_{\mathbb{C}}$.

Remark 2.3.10. Note that this relation is generally many-to-one. In particular, both $SL(n, \mathbb{R})$ and $SU(n)$ are real forms of $SL(n, \mathbb{C})$.

Now let G be a Lie group and $\xi \in \mathfrak{g}$. Then $\mathbb{R} \ni t \mapsto t\xi \in \mathfrak{g}$ is a homomorphism of Lie algebras. Thus there exists a unique map

$$\mathbb{R} \ni t \mapsto \exp(t\xi) \in G$$

and this is the matrix exponential for matrix groups. Also, $\frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi) = \xi \exp(t\xi)$. In addition, if $[\xi, \eta] = 0$, then we can exponentiate $\exp(t\xi + s\eta) = \exp(t\xi)\exp(s\eta)$ in either order.

Theorem 2.3.11 (Lie)**.** *For any Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$, there exists a unique simply-connected Lie group G with Lie algebra $\mathfrak{g}$.*

This gives a correspondence between our linear local data and nonlinear global data. We can construct manifolds either as:

1. A quotient of something simpler. If G is a simply-connected Lie group, then if we choose a point $x \in \mathfrak{g}$, then we can consider smooth paths g from 1 to x. Then $\dot{g}(t)g^{-1}(t) = \xi(t)$ the tangent vector at time t, and for a smooth homotopy between paths, If the curvature vanishes, then we have

$$\frac{\partial}{\partial s}\xi - \frac{\partial}{\partial t}\eta = [\xi, \eta].$$

   Then we can write G as the paths in the Lie algebra modulo solutions to the equation. Fortunately, the analysis reduces to first-order deformations.

2. As a "submanifold" of something simpler. Write $\mathfrak{g} \hookrightarrow \mathfrak{gl}(n, \mathbb{R})$. Then we have a map $G \to GL(n, \mathbb{R})$ with some kernel Γ. Thus G is the universal cover of $G/\Gamma \subset GL(n, \mathbb{R})$. Therefore, at least locally, every element of the Lie group is a matrix.

   A problem with this approach is that G/Γ need not be a submanifold. If we have the map $\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ with dense image, we obtain a foliation, so individual leaves are (locally) submanifolds, but we do not globally obtain a submanifold. In particular, if $H \to G$ is an injective Lie algebra homomorphism, we have a foliation with leaves corresponding to the cosets of H in G. In particular, we have the field of tangent planes $f(\mathsf{Lie}\, H) \cdot g$. Thus the cosets may be reconstructed either as leaves of the foliation or as integral manifolds for this field of tangent $(k = \dim H)$-planes (a section of a bundle of Grassmannians over M).

Note that an *integral manifold* for a field of tangent k-planes is a k-dimensional manifold $L \xrightarrow{\iota} M$ which is locally a submanifold such that T_xL is precisely the value of the field of k-planes at every point $x \in L$. The idea to construct this in our situation is to start by finding an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(N, \mathbb{C})$. The main obstacle to this plan is that a field of k-planes may not have any integral manifolds for $k > 1$.

A connection is a special case of a field of k-planes. Here, a connection on a locally trivial bundle Π gives a field of tangent planes that are transverse to the fibers of Π. Then the existence of integral manifolds is equivalence to flatness. There is a classical criterion for the existence of integral manifolds.

Theorem 2.3.12 (Frobenius)**.** *A field V of tangent k-planes in M has integral manifolds if and only if for all $m \in M$ the set of vector fields tangent to V forms a Lie subalgebra of $\Gamma(M, TM)$.*

Proof. Suppose we have integral manifolds $f_1 = c_1, \cdots, f_{n-k} = c_{n-k}$. Then a vector field v is tangent to the integral manifolds if and only if $\frac{\mathrm{d}}{\mathrm{d}v} f_i = 0$, which implies that $\left[\frac{\mathrm{d}}{\mathrm{d}v_1}, \frac{\mathrm{d}}{\mathrm{d}v_2}\right] f_i = 0$.

Conversely, suppose that $\mathcal{V}$ is our field of k-planes. Then denote $\Gamma_{\mathcal{V}}(M)$ to be the set of vector fields tangent to $\mathcal{V}$. Then we have this following picture:

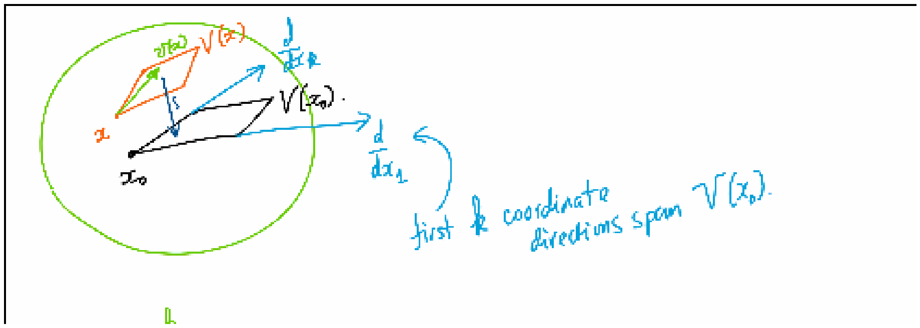


Figure 2.3: Projection of tangent planes

Thus we have

$$v = \sum_{i=1}^{k} c_i(x) \frac{\mathrm{d}}{\mathrm{d}x_i} + \sum_{i=k+1}^{n} c_j(x) \frac{\mathrm{d}}{\mathrm{d}x_i}.$$

Here the first c_i are arbitrary and the last c_j are uniquely determined. In particular, we have a basis of the form

$$v_i = \frac{\mathrm{d}}{\mathrm{d}x_i} + \sum_{j>k} c_j(x) \frac{\mathrm{d}}{\mathrm{d}x_j}.$$

Then we see that

$$[v_i, v_j] = 0 + \sum_{j>k} c_j \frac{\mathrm{d}}{\mathrm{d}x_j}$$

and so $[v_i, v_j] = 0$. This means that locally, the connection has curvature zero, and thus there are integral manifolds. $\square$

Returning to our case, consider $GL(N, \mathbb{C})$ and consider the tangent field $\mathfrak{g}g$, where $\mathfrak{g} \subset \mathfrak{gl}(N, \mathbb{C})$ is a Lie subalgebra. Then

$$\Gamma_{\mathcal{V}} = C^{\infty}(G \otimes)\{\xi g \mid \xi \in \mathfrak{g}\}.$$

The right-hand factor is already closed under the commutator. By the Leibniz rule, we see that $\Gamma_{\mathcal{V}}$ is also closed under $[-, -]$ and hence has integral manifolds.

Choosing the integral manifold that contains $1 \in GL(N, \mathbb{C})$, this is a subgroup of $GL(N, \mathbb{C})$. However, it is not a Lie subgroup in general. For any integral manifold L and $h \in GL(N, \mathbb{C})$, then Lh is also an integral manifold because the field was right invariant. If $h \in G$, then $Gh^{-1} = G$ because Gh^{-1} is integral and contains 1, so for all $g_1, g_2 \in G$, then $g_1 g_2^{-1} \in G$. Hence G is a connected Lie group. If G is not 1-connected, then we can take the universal cover. Thus we have proved

Theorem 2.3.13 (Lie)**.** *For any Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there exists a unique simply-connected Lie group G with* $\mathsf{Lie}(G) = \mathfrak{g}$.

Remark 2.3.14. When is $G \subset GL(N, \mathbb{C})$ algebraic? Not every Lie algebra over $\mathbb{C}$ is a Lie algebra of an algebraic group. Most differential equations with algebraic coefficients do not have algebraic solutions.

For example, the irrational winding of the torus corresponds to $\left\{ \frac{\log z}{\log w} = \text{const} \right\} \subset \mathbb{C}^* \times \mathbb{C}^*$.

3

Representations

Let G be a group. Then a *representation* of G over a field $\mathbb{F}$ is a map $G \to GL(n, \mathbb{F})$. Here, we will take $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and strongly prefer $\mathbb{F} = \mathbb{C}$. We will call a representation of G a G-module. These form an abelian category, which is not something we will dwell on too much. A map between two representations is an "intertwining operator" which is something that makes

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ \downarrow g & & \downarrow g \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

commute. Then both $\ker f, \operatorname{Im} f$ are submodules. If V has a nontrivial submodule, then it is *reducible*. Otherwise, we call it irreducible. If we have an exact sequence

$$0 \to V_1 \to V \to V_2 \to 0,$$

then V_1 is a submodule of V and V_2 is a quotient.

Definition 3.0.1. A representation V is called *semisimple* (or completely reducible), if $V = \bigoplus V_i$, where V_i is irreducible.

If V_1, V_2 are representations, then G acts on $V_1 \otimes V_2$ by $\pi_1(g) \otimes \pi_2(g)$. This comes from the map $\Delta\colon G \to G \times G$.

We have a notion of *characters* that send a representation V to the conjugation-invariant function $\chi_V(g) = \operatorname{tr}_V g$. Then it is easy to see that $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$ and that $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$. This is a semiring homomorphism, so we can form the *representation ring* Rep_G. This is the K-group of the category Mod_G. Also, note that if

$$0 \to V_1 \to V \to V_2 \to 0$$

is exact, then we see that $\chi_V = \chi_{V_1} + \chi_{V_2}$. Thus we can impose the relation $[V] = [V_1] + [V_2]$.

At first sight, it seems that taking the character loses a lot of information. However, if we have all of the traces, this means we can compute all of the eigenvalues. Thus, we can reconstruct V up to conjugation from its character.

Next, if V is a G-module, then G also acts on V^* by $(g\ell)(v) = \ell(g^{-1}v)$. Also, $(V^*)^* = V$ as a G-module.

3.1 Finite Dimensional Representations

Consider the simplest groups we know: $SU(2), SL(2,\mathbb{R}), SL(2,\mathbb{C})$. First, all of these groups have the same finite-dimensional representations given by

$$\begin{tikzcd}
SU(2) \arrow[dr] \arrow[drr] & & \\
& SL(2,\mathbb{C}) \arrow[r] & GL(V) \\
SL(2,\mathbb{R}) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

and in the other direction, we can complexity $\mathfrak{su}(2)$ to $\mathfrak{sl}(2,\mathbb{C})$, and then $SL(2,\mathbb{C})$ is simply-connected. However, note that $GL(1,\mathbb{C})$ is not simply-connected, so we cannot use the same argument for $U(1)$. Also, representations of S^1 and $GL(1,\mathbb{C})$ are semisimple, but $\mathbb{R}$ has the representation

$$z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

which is a nontrivial representation in two dimensions that is not semisimple.

Second, we note that all representations are semisimple because $SU(2)$ is compact. To see this, note that $SU(2)$ is compact and thus every representation has an invariant Hermitian form (by averaging). Then for $V_1 \subset V$, we can write $V = V_1 \oplus V_1^{\perp}$.

Then all irreducible representations of $SL(2,\mathbb{C})$ can be described as symmetric powers $S^k\mathbb{C}^2 \cong \mathbb{C}[(C^2)^*]_k$. This has basis $v_1^{k_1}v_2^{k_2}$, where $k_1 + k_2 = k$. Here, the maximal torus acts by

$$\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \mapsto \begin{pmatrix} z^k & & & \\ & z^{k-2} & & \\ & & \ddots & \\ & & & z^{-k} \end{pmatrix}.$$

We will find this structure in the representation of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. We know $\mathfrak{g}$ has a basis

$$h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we see that

$$h \mapsto \begin{pmatrix} k & & & \\ & k-2 & & \\ & & \dots & \\ & & & -k \end{pmatrix}$$

and $e = v_1 \frac{\partial}{\partial v_2}, f = v_2 \frac{\partial}{\partial v_1}$. Also, note that $[h,e] = 2e, [h,f] = -2f, [e,f] = h$. Then e shifts the weight by $+2$ and f by -2.

Lemma 3.1.1. *If v is an eigenvector of h with eigenvalue λ, then $h(ev) = (\lambda+2)ev$ and $h(fv) = (\lambda-2)fv$.*

Proof. Note that $h(ev) = [h,e]v + ehv = 2ev + e\lambda v = (\lambda+2)ev$. A similar argument gives the result for f. $\square$

Classification of irreps of $\mathfrak{sl}(2)$. Let V be irreducible and v be an eigenvector of h with eigenvalue λ. If $ev \neq 0$, then replace v by ev. The eigenvalue cannot grow forever, so eventually we reach v such that $hv = \lambda v$ and $ev = 0$. This is called the highest weight vector.

Now we will show that V is the span of $v, fv, f^2v, \ldots$. This is clearly invariant under h, f by construction. Then note that

$$ef^m v = [e, f^m]v + f^m ev = [e, f^m]v.$$

Because $[e, f^m]v$ is a combination of h, f by the commutation relations, we see that the span of $v, fv, \ldots$ is a subrepresentation, so it must be everything. Then because $h \cdot f^m v = (\lambda - 2m) f^m v$, then there exists a minimal m such that $f^m v = 0$. This implies that $\lambda = m - 1$, but to show this, consider $ef^m v$, which is a multiple of $f^{m-1}v$.

The rest of this proof is left as an exercise. □

Remark 3.1.2. Representations of $GL(n, \mathbb{C})$ correspond to integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

3.2 Harmonic Analysis on Compact Groups

Let G be a compact Lie group and consider representations $G \to GL(V)$. We would like to do harmonic analysis on G. Our prototype will be $G = \mathbb{R}/\mathbb{Z}$. If $\mathrm{d}x$ is the Haar measure, then we can write

$$L^2(G, \mathrm{d}x) = \widehat{\bigoplus_{n \in \mathbb{Z}} \mathbb{C}} \cdot e^{2\pi i n x}$$

as a **Hilbert Space**. Recall that a Hilbert space is a complete inner product space, and the inner product on L^2 is

$$(f, g) = \int_X f\overline{g}\, \mathrm{d}x\,.$$

We can write $\|f\|^2 = (f, f)$. Then recall that $x \mapsto e^{2\pi i n x}$ are precisely the irreducible representations of G. Here, $\widehat{\bigoplus}$ is the closure of the algebraic direct sum. Next, for any $f(x) \in L^2$, we can take the Fourier transform

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x},$$

where $\widehat{f}(n) = (f, e^{2\pi i n x})$. We will generalize this to an arbitrary compact group.

Our main issue is that for non-abelian groups, not all irreducible representations have dimension 1. However, we will have a correspondence

$$\begin{pmatrix} \text{matrix elements of} \\ \text{irreducible representations} \end{pmatrix} \longrightarrow L^2(G, \mathrm{d}\mu),$$

where μ is the Haar measure. These matrix elements of irreducible representations are in fact analytic. When we pass to $G_{\mathbb{C}}$, they become holomorphic.

Then if V is a complex representation of G, then its matrix elements are a map $V^* \otimes V \to C^\infty(G)$, where

$$(\ell \otimes v)(g) = \ell(gv).$$

These satisfy the following orthogonality relation which compares the inner product in $L^2(G)$ and on $\bigoplus V_i^* \otimes V_i$.

Proposition 3.2.1. *Every irreducible representation V has a unique G-invariant Hermitian inner product up to multiple.*

Proof. To show existence, take any Hermitian inner product and average over the group G. Then if $(-,-), \langle -,-\rangle$ are different invariant inner products, we can write

$$\langle v_1, v_2\rangle = (Bv_1, v_2)$$

for some Hermitian matrix B. But then this commutes with G, so by Schur's Lemma, B is a constant. $\square$

Lemma 3.2.2 (Schur)**.** *Let V_1, V_2 be irreducible G-modules. Then*

$$\mathrm{Hom}_G(V_1, V_2) = \begin{cases} 0 & v_1 \not\simeq V_2 \\ \mathbb{C} & V_1 \simeq V_2 \end{cases}.$$

Proof. If $f \in \mathrm{Hom}_G(V_1, V_2)$, then $\ker f \subset V_1$ and $\mathrm{Im}\, f \subset V_2$. Thus either the kernel is everything and the image is zero, or the image is everything and the kernel is zero, so either f is zero or an isomorphism.

Now assume $V_1 \simeq V_2$ and consider $\mathrm{Hom}_G(V, V)$. Then $\mathrm{Hom}_G(V, V)$ is a division algebra over $\mathbb{C}$. But over $\mathbb{C}$, this must be $\mathbb{C}$ because for λ an eigenvalue, then $f - \lambda$ has nontrivial kernel and thus must be the zero map. $\square$

Now for the irrep V, take the unique inner product $(-,-)$. This gives $V^* \otimes V$ a canonical inner product. To write this concretely, write $V^* \otimes V = \mathrm{End}\, V$, and then write $(A, B) = \mathrm{tr}\, AB^\dagger$. For matrices $E_{ij}, E_{k\ell}$, we have

$$(E_{ij}, E_{k\ell}) = \delta_{ik}\delta_{j\ell}.$$

Theorem 3.2.3 (Orthogonality)**.** *Let $V_1, V_2, \ldots$ be a collection of ditinct irreducible representations. Then consider the space $\bigoplus_i V_i^* \otimes V_i$ with inner product $(A, B) = \frac{1}{\dim V_i}\,\mathrm{tr}\, AB^\dagger$. Then the embedding*

$$\bigoplus_i V_i^* \otimes V_i \xrightarrow{\text{matrix elements}} L^2(G)$$

is an isometry.

Proof. Denote $g \xrightarrow{\pi_i} GL(V_i)$. Then take an arbitrary $f\colon V_i \to V_j$ and make it G-invariant by averaging

$$\overline{f} = \int_G \mathrm{d}g\, \pi_j(G) f \pi_i(g)^{-1}.$$

By Schur, we see that

$$\overline{f} = \begin{cases} 0 & i \neq j \\ \frac{\mathrm{tr}\, f}{\dim V_i} & i = j \end{cases}.$$

Because the inner product on each V_i is G-invariant, then $G \to U(V_i) \subset GL(V_i)$. Therefore, for all $f \in \mathrm{Hom}(V_i, V_j)$,

$$\int_G \mathrm{d}g\, \pi_j(g) f \pi_i(g)^* = \begin{cases} 0 & i \neq j \\ \frac{\mathrm{tr}\, f}{\dim V_i} & i = j \end{cases}.$$

This equality of matrices is equivalent to the desired result. $\square$

Theorem 3.2.4 (Peter-Weyl)**.** *Let G be compact. Then if μ is the Haar measure, we have*

$$L^2(G, \mathrm{d}\mu) = \widehat{\bigoplus_{\text{irreps } V}} V^* \otimes V$$

as modules over $G \times G$.

The key statement is that $\bigoplus V^* \otimes V$ is dense in $L^2(G)$. For the second part, note that $G \times G$ acts on $L^2(G)$ by

$$[(g_1, g_2) \cdot f](h) = f(g_1^{-1} h g_2).$$

Then $\varphi_{\ell v}(h) = \ell(h \cdot v)$, and under (g_1, g_2), this becomes $\ell(g_1^{-1} h g_2 v) = \varphi_{g_1 \ell, g_2 v}(h)$, so $\bigoplus V_i^* \otimes V_i \to L^2(G)$ is a map of $G \times G$ modules.

Conversely, the subspace of $L^2(G)$ that transforms in V under the right regular action of G is $V^* \otimes V$. Indeed, suppose $f_k(h)$ are such that

$$f_k(hg) = \sum_j \pi_{k\ell}(g) f_\ell(h).$$

In particular, if we take $h = 1$, we get $f_k \in V^* \otimes V$.

Here are some reformulations of Peter-Weyl:

1. The unitary representation
$$\left(\bigoplus V^* \boxtimes V\right)^\perp$$
cannot have finite dimensional submodules, so we have a matrix element of an infinite-dimensional unitary representation of G. Thus Peter-Wel is equivalent to every irreducible representation of G being finite-dimensional.

2. Every compact group has a faithful finite-dimensional representation $G \hookrightarrow GL(n, \mathbb{C})$ for some n. To see this, suppose $G \subset GL(V)$. Then $G \subset U(V)$. We can now consider the algebra generated by matrix elements g_{ij}, where $g_{ij} g_{k\ell}$ is a matrix element of $V \otimes V$. Then we see that
$$\bigoplus_k (V^{\otimes k})^* \otimes V^{\otimes k} \longrightarrow \begin{pmatrix} \text{matrix elements of} \\ V \otimes \cdots \otimes V \end{pmatrix} \subset L^2(G).$$
This is an algebra of complex valued functions stable under complex conjugation, which separates points of G. By Stone-Weierstrass, this is dense in $C(G)$ and thus in $L^2(G)$.

Remark 3.2.5. Let V be a faithful finite-dimensional representation of a compact group G. Then any irreducible representation of G is contained in the decomposition of $V^k \otimes (V^*)^\ell$. However, for $U(n)$, the defining representation is not in the decomposition of $(\mathbb{C}^n)^{\otimes k}$.

Remark 3.2.6. This proves Peter-Weyl for all compact groups because they are all matrix groups. Also, we showed that $L^2(G)$ is separable, which means that compact groups have only countable many irreps.

Continuing the proof of the second item, we have an exact sequence

$$1 \to G_N \to G \to GL\left(\bigoplus_{i \le N} V_i\right).$$

Then $G_1 \supset G_2 \supset \cdots$ has to eventually stabilize and write G_∞ for the colimit. Then if $G_\infty = 1$, we are done. Otherwise, we have a contradiction because all functions take the same value on G_∞-cosets.

Now we will continue our proof of Peter-Weyl. Our strategy is to break $L^2(G)$ into finite-dimensional G-invariant pieces. We can consider the $G \times G$ invariant metric on G and consider the corresponding Laplace operator and its eigenspaces. Note that $\mathfrak{g}$ has a positive-definite invariant metric and an invariant tensor in $S^2\mathfrak{g}^* \to S^2\mathfrak{g}$. Then if ξ_i is an orthonormal basis of $\mathfrak{g}$, then $_i$ is a first-order differential operator of G, and we define the Laplacian to be

$$\Delta = \sum \xi_i^2,$$

which is also called the "Casimir[1] element." Because this is invariant, it acts by a scalar in V, which is the eigenvalue of the Laplacian in $V^* \boxtimes V$.

Instead of doing this, we will use integral operators because they are easier to work with. We have an action of G on $L^2(G)$ on the right, which yields $f(h) \mapsto f(hg)$, so we will smear out our operators following the philosophy of functional analysis. Then we will have

$$f(h) \mapsto \int_G c(g) f(hg)\, \mathrm{d}g,$$

where c is an arbitrary function. The key point will be to show that this operator is **compact** and self-adjoint if $c(g^{-1}) = \overline{c(g)}$. Here a compact operator A is compact if the closure of the image of the unit ball is compact. Here are some basic properties:

1. Compact operators form a two-sided closed ideal in all bounded operators.
2. A is compact if and only if there exist a sequence A_n of finite-rank operators such that $\|A - A_n\| \to 0$. Each A_n is given by choosing finitely many dimensions and projecting there.
3. If A is compact and self-adjoint, then $\mathcal{H} = \bigoplus \mathcal{H}_{\lambda_i}$, where λ_i are real eigenvalues, $\lambda_i \to 0$ as $i \to \infty$, and $\dim \mathcal{H}_{\lambda_i} < \infty$.
4. If A is compact and general, then

   $$A = \sum \lambda_i (\psi_n, -)\varphi_n,$$

   where $(\psi_n, \psi_m) = (\varphi_n, \varphi_m) = \delta_{nm}$.

The main point is that integral operators are typically compact. Modulo this, we have proved Peter-Weyl.

Now we need to show that $\left(\widehat{\bigoplus} V^* \boxtimes V\right)^{\perp} = 0$. To do this, we will use spectral decomposition for operators that come from the right action of G. These are compact and self-adjoint operators. Recall that for Hilbert space $\mathcal{H}$ and compact and self-adjoint operator K, then we can write

$$\mathcal{H} = \bigoplus_{\lambda} \mathcal{H}^{\lambda},$$

where $\lambda \neq 0$ and $\dim \mathcal{H}^{\lambda} < \infty$. Here, we will write

$$[K(f)](g) = \int_G f(gh) c(h)\, \mathrm{d}h$$

where $c(h)$ is a kernel function. Then recall that if $[M(i,j)]$ is a matrix, then we have

$$M \cdot v(i) = \sum_j M(i,j) v(j).$$

[1] Andrei had no idea how this name came to be, so we looked at Wikipedia in real time and found that he was a Dutch physicist. We still have no idea why the name was given.

Given two spaces X, Y, then for some kernel function $K \in L^2(X \times Y)$, we can define an integral operator

$$[K(f)](x) = \int_Y \mathrm{d}y\, K(x,y) f(y).$$

Now we have the inequality

$$\begin{aligned}
\|K(f)\|^2_{L^2(X)} &= \int_X |K(f)|^2 \,\mathrm{d}x \\
&\leq \int_X \mathrm{d}x \left[\left(\int_Y \mathrm{d}y\, |f(y)|^2 \right) \left(\int_Y \mathrm{d}y\, |K(x,y)|^2 \right) \right] \\
&\leq \|f\|^2_{L^2(Y)} \|K\|^2_{L^2(X\times Y)},
\end{aligned}$$

and thus K defines a bounded operator on X.

Remark 3.2.7. Recall that if V is a finite-dimensional vector space, we have

$$V^* \otimes V \xrightarrow{\simeq} \mathrm{End}(V).$$

More generally, for finite dimensional vector spaces, we have

$$V_1^* \otimes V_2 \xrightarrow{\simeq} \mathrm{Hom}(V_1, V_2).$$

If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, then we have a map

$$\mathcal{H}_1^* \widehat{\otimes} \mathcal{H}_2 \to B(\mathcal{H}_1, \mathcal{H}_2),\ v_1 \otimes v_2 \mapsto (-, v_1) v_2.$$

This is not surjective and goes into an ideal of Hilbert-Schmidt operators, which are those that have finite L^2 norm under the norm $\sum |M_{ij}|^2$. Then we can write

$$K(x,y) = \sum_{ij} \varphi_i(x) \psi_j(y)$$

and see that $\|K\|^2 = \sum |K_{ij}|^2$.

Remark 3.2.8. Consider the two kernels $X_1 \xrightarrow{K_{21}(x_2,x_1)} X_2 \xrightarrow{K_{32}(x_3,x_2)} X_3$. Then the composition has kernel

$$\int_{X_2} \mathrm{d}x_2\, K_{32}(x_3, x_2) K_{21}(x_2, x_1).$$

This is analogous to matrix multiplication.

Remark 3.2.9. Consider a measure space $(X, \mathrm{d}x)$. Then the assignment

$$(X, \mathrm{d}x) \longrightarrow \text{Functions}$$

is a functor. We have the pullback as usually defined, but we also have pushforwards defined by integration with respect to $\mathrm{d}x$. Here, analytic issues with this integration process are ignored. This generalizes to the structure

$$\begin{tikzcd}
 & F(X_1 \times X_2) \arrow[dr, "(p_2)_*"] & \\
F(X_1) \arrow[ur, "p_1^*"] & & F(X_2)
\end{tikzcd}$$

where we have $Kf = (p_2)_*(p_1^*(f) \otimes K)$. In the context of sheaves on algebraic varieties, this gives a Fourier-Mukai transform.

We now return to our compact operator K. We know that $\sum K_{ij}\psi_i(x)\varphi_j(y)$ converges. Then we can write

$$K \cdot f = \sum_{i,j} K_{ij}\psi_i(x) \int_Y f(y)\varpi_j(y)\, \mathrm{d}y\,.$$

Thus K is a limit in the operator norm of operators of finite rank and thus K is compact. The particular operator we want is

$$f \mapsto \int_G f(gh)c(h)\,\mathrm{d}h = \int_G f(h)c(g^{-1}h)\,\mathrm{d}h\,.$$

This is self-adjoint if $c(g^{-1}) = \overline{c(g)}$. However, we don't need to worry about this because if K is compact and commutes with left translation then K^*K is compact and self-adjoint. Now we use the fact that because the sum of the nonzero eigenspaces of operators like K^*K are dense in $L^2(G)$ and thus the image of operators of this form are dense. Thus if $c(g) \to \delta(e)$, then

$$\int f(gh)c(h)\,\mathrm{d}h \to f(g),$$

where δ is the Dirac delta distribution. Thus any function is in the closure of the image. Here, convergence here is convergence in the weak sense in the space of ditributions $C^\infty(G)^\vee$. This concludes the proof of Peter-Weyl.

Remark 3.2.10. Recall that we have the inclusion

$$\bigoplus_V V^* \boxtimes V \subset \widehat{\bigoplus_V} V^* \boxtimes V = L^2(G).$$

Then note that the finite direct sum is an **algebra**. Here, we simply note that matrix elements of V_1 times matrix elements of V_2 are matrix elements of $V_1 \otimes V_2$. This is finitely generated (by matrix elements of any faithful representation and its dual). Therefore, the space of functions has a finitely generated dense subset that is an algebra.

In our Lie group $G \subset G_{\mathbb{C}}$ with Lie algebra $\mathsf{Lie}(G) \otimes \mathbb{C}$, recall that $G_{\mathbb{C}}$ is a complex Lie group and is thus analytic. In fact, we will see that $G_{\mathbb{C}}$ is an affine algebraic group, and thus $\bigoplus_V V^* \boxtimes V$ is the algebra of functions on $G_{\mathbb{C}}$. This tells us that every compact Lie group is the real form of a complex algebraic group.

3.3 Representation Theory of Unitary Groups

We have been discussing the Peter-Weyl theorem, and now we will apply this to study the representation theory of compact Lie groups, and in particular, the most important such group $U(n)$. Recalling that

$$L^2(G) = \widehat{\bigoplus_{V \text{ irrep}}} \operatorname{End}(V),$$

there is a distinguished element of each factor: the identity 1_V. This is invariant under $G \subset G \times G$ and corresponds to the character $\operatorname{tr}_V g \in C^\infty(G)$ of V. Recalling that the metric on $\operatorname{End}(V)$ was

$$(A_1, A_2) = \frac{1}{\dim V} \operatorname{tr} A_1 A_2^\dagger,$$

we see that characters of irreducible representations are orthonormal. Taking invariants, we now see

$$L^2(G/\text{conjugation}) = \widehat{\bigoplus_{\text{irreps } V}} \mathbb{C} \cdot \chi_V(g).$$

For $U(n)$, we will describe this space of functions and the lattice $\bigoplus_V \mathbb{Z} \cdot \chi_V(g)$ explicitly. Then we will find an orthonormal basis of this lattice. We will use the basic fact that $O(n, \mathbb{Z})$ is generated by S_n and ± 1.

Now recall that by the spectral theorem, all unitary matrices can be diagonalized. Thus we have

$$U(n)/\mathrm{conj} = \left\{ \begin{pmatrix} t_1 & & \\ & \dots & \\ & & t_n \end{pmatrix} \right\} / S_n.$$

This set of diagonal matrices is usually denoted by T and is isomorphic to $U(1)^n$. This is a maximal torus. In general this S_n is replaced by the Weyl group. To draw an explicit picture in the case of $SU(2)$, we have

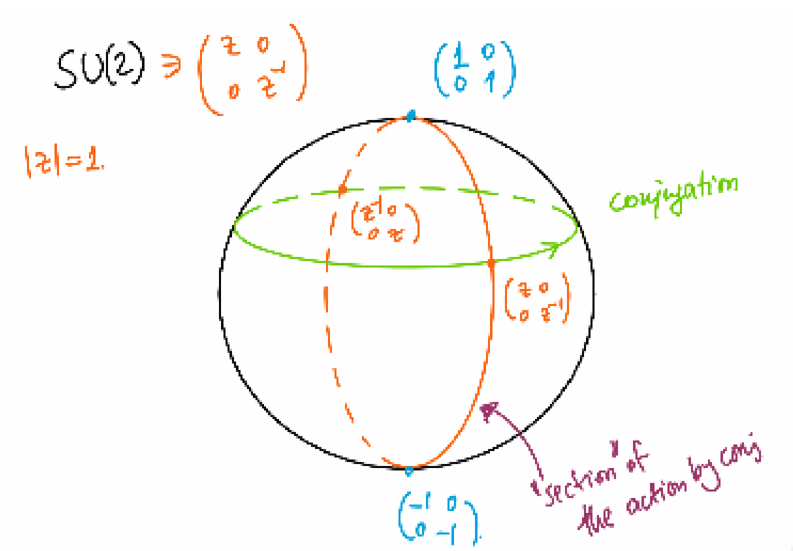


Figure 3.1: Conjugation action of $SU(2)$ on itself

and then we see that

$$N(T) = \left\{ g \mid gTg^{-1} \subset T \right\} = \left\{ (*.*), \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$$

and then the Weyl group is

$$N(T)/T = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

For $U(n)$, $N(T)$ is the monomial matrices (or rook placements) and $N(T)/T$ is the set of permutations, or S_n.

Returning to $SU(2)$, we then see that all orbits are parameterized by the eigenvalue $z = e^{i\varphi}$, and each orbit is an S^2 with radius $\sin\varphi$. Thus we have

$$L^2(U(n))^\Delta = L^2(T, \text{interesting measure})^{S_n}.$$

This measure is some constant multiple of $\sin^2 \varphi \, d\varphi$, which is the same as

$$\frac{1}{|W|}(1 - z^2)(1 - \bar{z}^2)\frac{dz}{2\pi i z}.$$

Among all functions on the torus, we may consider the functions given by the coordinates, so we then have a lattice $\mathbb{Z}[t_i^{\pm}]^{S_n}$. Then we note that this contains $\chi_V(g)$ for all V because as a representation of T, we can split V into 1-dimensional representations $t^\mu = \prod t_i^{\mu_i}$, where $\mu_i \in \mathbb{Z}$. These μ are called the *weights* and the number of times a weight μ appears is the *multiplicity*. Now all unitary matrices are diagonalizable, and thus it suffices to compute characters on T. Therefore we have

$$\chi_V(t) = \sum_\mu \mathrm{mult}_V(\mu) \cdot t^\mu.$$

We see each multiplicity is a W-invariant and nonnegative. For $SU(3)$, we have $\mu \in \mathbb{Z}^3/\mathbb{Z}(1,1,1)$. In general, the set $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ is a fundamental domain for S_n.

Theorem 3.3.1. *For any dominant μ, there exists V^μ such that*

$$\chi_{V^\mu}(t) = t^\mu + \textit{lower order terms}$$

with respect to a certain ordering on monomials.

Example 3.3.2. For $SU(2)$, we have

$$\chi_{S^m\mathbb{C}^2}\left(\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix}\right) = z^m + z^{m-2} + \cdots + z^{-m}.$$

Then if we consider the adjoint action of $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ on $\mathfrak{gl}(n, \mathbb{C})$, and the weights for this action are called *roots*. Differentiating this, we can consider roots on $\mathsf{Lie}(T)$ and the pairing

$$\left\langle \begin{pmatrix} \xi_1 & & \\ & \cdots & \\ & & \xi_n \end{pmatrix}, \delta_i - \delta_j \right\rangle = \xi_i - \xi_j.$$

Then the weight lattice contains the root lattice, which contains the cone spanned by positive roots ($i < j$).

Theorem 3.3.3. *For a dominant weight $\mu = (\mu_1 \geq \cdots \geq \mu_n) \in \mathbb{Z}^n/S_n$, there exists V^μ such that*

$$\chi_{V^\mu}(t) = t^\mu + \sum_{\eta \text{ negative}} * t^{\mu+\eta}.$$

Here, we write $\mu > \nu$ if $\mu - \nu$ is in the cone spanned by the positive roots $(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0 \ldots, 0)$. In particular, this means that $t^\mu > t^\nu$ if and only if $t^{\mu-\nu} < 1$, which is the same thing as $t^{\mu-\nu} \to 0$.

Proof. We will first consider a certain infinite-dimensional representation of $\mathfrak{gl}(n) \supset \mathfrak{u}(n) = \mathsf{Lie}(U(n))$. It has to have a vector v_μ such that $t \cdot v_\mu = t^\mu v_\mu$. Then for $i < j$, we know that $E_{ii} v_\mu = \mu_i v_\mu$ and $E_{ij} v_\mu = 0$ because this is a vector of weight strictly greater than μ. Now we define $M(\mu)$ to be the free module generated by these relations.

Note that a "free module" for $\mathfrak{gl}(n)$ behaves like the universal enveloping algebra[2] $\mathcal{U}\mathfrak{gl}(\mathfrak{n})$, which is simply $C\langle E_{ij}\rangle / (xy - yx = [x, y])$ by the Poincaré-Birkhoff-Witt theorem. As a linear space, these are monomials in E_{ij} ordered arbitrarily. In this point of view, we now have

$$M(\mu) = \mathcal{U}\mathfrak{gl}(n) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C} \cdot v_\mu,$$

[2] This is also the algebra of right-invariant differential operators on $GL(n)$.

where $\mathfrak{b}$ is the Lie algebra of upper-triangular matrices and $\mathbb{C}v_\mu$ is a one-dimensional representation of $\mathfrak{b}$ given by $E_{ii}v_\mu = \mu_i v_\mu$ and $E_{ij}v_\mu = 0$ for $i < j$. As a linear space, $M(\mu)$ is spanned by things like $E_{53}E_{74}\cdots v_\mu$, which are arbitrary products of E_{ij} for $i > 1$. Then if $i \le j$, we can write

$$E_{ij} \cdot E_{53}E_{74}E_{21}v_\mu = E_{53}E_{74}E_{21}E_{ij}v_\mu$$

using the commutation relations, so for this mildly noncommutative algebra, we have a Gröbner basis.

Thus the Verma module $M(\mu)$ has the form $t^\mu + \sum_{v<\mu} m(\mu-\nu)t^\nu$, where $m(\mu-\nu)$ is the number of ways to write $\mu-\nu$ as a sum of positive roots. As a generating function, this becomes

$$\begin{aligned}\chi_{M(\mu)}(t) &= \frac{t^\mu}{\prod_{\alpha>0}(1-t^{-\alpha})} \\ &= \frac{t^\mu}{\prod_{i>j}(1-t_i/t_j)}.\end{aligned}$$

Now consider the following remarks:

1. We can define $M(\mu)$ for any $\mu \in \mathbb{C}^n$ by $E_{ii}v_\mu = \mu_i v_\mu$. For generic μ, this is irreducible.
2. For positive dominant weights μ, this is always reducible. If $M' \subset M(\mu)$ is a submodule, then the weights of M' contain μ if and only if $M' = M$. Thus $M(\mu)$ contains a maximal proper submodule M', and we call $L(\mu) = M(\mu)/M'$ to be the irreducible module with *highest weight* μ. Then we can write $\chi_{(L(\mu))}(t) = t^\mu +$ lower order terms.

We now need to show that $\dim L(\mu) < \infty$ by relating it to the group. The most direct way is through matrix elements. Write $\varphi(g)$ to be the coefficient of v_μ in $g \cdot v_\mu$. For diagonal matrices $t \in T$, we have $\varphi(t) = t^\mu$. For the upper-triangular matrices, we have $\varphi(gu_+) = \varphi(g)$. Then for any lower-triangular u_-, we have $\varphi(u_-) = 0$. Finally, because generic matrices have a Gauss decomposition, the set U_-TU_+ is dense in $GL(n)$. Thus the function is determined uniquely by the values we already have.

Now because the minor of the $i \times i$ submatrix given by the first i rows and first i columns is invariant under U_i and U_+, we see that

$$\varphi = \Delta_i^{\mu_1-\mu_2}\Delta_2^{\mu_2-\mu_3}\cdots \in (V^\mu)^* \boxtimes V^\mu \subset L^2(U(n)).$$

Considering the span under the right regular representation, we obtain V^μ. This span is finite dimensional because it is contained in the space of polynomials. $\square$

Corollary 3.3.4. *1. We have the identity*

$$\bigoplus_V \mathbb{Z}\chi_V(t) = \mathbb{Z}[t_1^\pm, \ldots, t_n^\pm]^{S_n}.$$

2. *We can compute $\chi_V^\mu(t)$ by Gram-Schmidt because only finitely many weights are smaller. This will follow from orthogonality of $\chi_{V^\mu}, \chi_{V^\nu}$ whenever $\nu < \mu$.*
3. *There is a formula for this inner product, and orthogonality can be done explicitly in one step.*

Now we want to prove the *Weyl integration formula*. Let f be a conjugation-invariant function on $G = U(n)$. Then we have

$$\int_G f(g)\,\mathrm{d}g = \frac{1}{n!}\int_T f(t)\prod_{i<j}|t_i - t_j|^2\,\mathrm{d}t\,,$$

where $\mathrm{d}t = \prod_k \frac{\mathrm{d}t_k}{2\pi i t_k}$ is the Haar measure on T.

Remark 3.3.5. For $SU(2)$, recall that for $t = e^{i\varphi}$, we have $\sin^2\varphi \propto \left|t - t^{-1}\right|^2$.

Consider the map $G/T \times T \to G$ given by $(g, t) \mapsto gtg^{-1}$. This is an $n!$-to-1 map, and so we can write

$$\int_G f(g)\,\mathrm{d}g = \frac{1}{n!}\int_{G/T\times T} f(t)\,\mathrm{d}t\,\mathrm{d}g/t \cdot \mathcal{J}(t),$$

where $\mathcal{J}$ is some Jacobian. Because the Haar measure is invariant under conjugation, this Jacobian is independent of g. Now computing at the point $g = 1$, we need to compute

$$\begin{aligned}(1+\delta x)(t+t\delta t)(1+\delta x)^{-1} &= 1 + (t\delta t + \delta x t - t\delta x) + \cdots \\ &= 1 + t(\delta t + t^{-1}\delta x t - \delta x) + \cdots .\end{aligned}$$

The linear term is $\delta t + (\mathrm{Ad}(t^{-1}) - 1)\delta x$, so we need to compute $\det\bigl(\mathrm{Ad}(t^{-1}) - 1\bigr)$, but this operator has eigenvalues $t^{-\alpha} - 1$, where α is a root. Therefore, we have

$$\mathcal{J} = \prod_{\alpha}(t^{-\alpha} - 1) = \prod_{\alpha>0}(t^{\alpha} - 1)(t^{-\alpha} - 1) = \prod_{\alpha>0}|t^{\alpha} - 1|^2.$$