Topology and Geometry of Singular Spaces
Math 797D

Notes by Patrick Lei, Spring 2020

Lectures by Paul Gunnells

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Disclaimer

These notes were taken during lecture using the `vimtex` package of the editor `neovim`. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style (omit lengthy computations, use category theory) and that of the instructor. If you find any errors, please contact me at `plei@umass.edu`.

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1.1 Course Description

Singular spaces arise naturally in many contexts, including algebraic geometry and representation theory. Any singular space has a decomposition into manifolds (a "stratification"), and so the study of them is a mixture of topology, geometry, and combinatorics. The goal of this course is to present the basics of stratified spaces and to illustrate the general theory with some important examples. Potential topics: general material, including stratifications, Whitney conditions, local structures, intersection homology; isolated singularities of complex hypersurfaces (the Milnor fibration and related topics), including connections to knot theory; and compactifications of locally symmetric spaces.

1.2 Organization

This class has a webpage: `www.math.umass.edu/~gunnells/singspc/singspc.html`. Grading will be based on an expository paper of around 10 pages. More information is on the website. He says he does this because most grad students don't do enough writing until the thesis.[1] Occasionally Paul will suggest exercises that we can think about.

1.3 Overview

Obviously this courses is about singular spaces, in particular stratified spaces. Recall that a nice space is a smooth manifold, which is locally Euclidean. An example of this is a torus. However, not all useful spaces are manifolds.

Example 1.1. Consider the nodal cubic given by collapsing a circle on the torus to a point. Algebraically, this is given by $y^2 = x^3 + x^2$. This is not a manifold, but it has a decomposition (stratification) into manifolds $(S^1 \times S^1 \setminus S^1, \mathrm{pt})$.

Example 1.2. Affine algebraic varieties have a natural stratification (take the smooth locus, then repeat on the singular locus). For example, for a hypersurface $(f = 0)$, the singular locus is given by the vanishing of all partial derivatives of f.

For a simple example, consider $f = x^3 - y^2$. This is singular at the origin. If we consider $S^3 \cap V_f$, we obtain an S^1. Viewing $S^3 = \mathbb{R}^3 \cup \{\infty\}$, our S^1 is a trefoil knot.[2] The stratification into manifolds is obvious.

[1] Hey, it's good to get this in my undergrad.

[2] Andreas spoke for everyone and asked for justification, and Paul said he said the word overview, so he can say whatever he wants without explaining anything.

Example 1.3. Consider a manifold M and topological group G acting on M. Sometimes the space M/G is a manifold (for example when $M = S^n, G = \mathbb{Z}/2\mathbb{Z}$, $M/G = \mathbb{P}^n(\mathbb{R})$). However, typically the quotient is not a manifold. Sometimes the space is not even Hausdorff.[3] If we suppose that the action is proper (this is automatically satisfied if G is compact $((S^1)^k, SO(n))$), then M/G will be a stratified space. If G is finite, then M/G is a global orbifold.

Definition 1.4. Let G be a Lie group and K a maximal compact subgroup. For example, consider $G = SL_n(\mathbb{R})$ and $K = SO(n)$. Alternatively, consider $G = SL_n(\mathbb{C})$ and $K = SU(n)$. The space $X = G/K$ is a contractible smooth manifold diffeomorphic to Euclidean space. This is a *global symmetric space*. Now let $\Gamma \subset G$ be a discrete group (for example $SL_n(\mathbb{Z})$). Another example is $SL_n(\mathbb{Z}[i]) \subset SL_n(\mathbb{C})$. Now Γ has a natural action on G/K. Thus a *locally symmetric space* is $\Gamma \backslash G/K$.

Example 1.5. Consider $G = SL_2(\mathbb{R})$ and $K = SO(2)$. Then $X = G/K \simeq \mathscr{H}$, the upper half plane. Now $\Gamma = SL_2(\mathbb{Z})$ acts by Möbius transformations. Then $\Gamma \backslash \mathscr{H} \simeq \mathbb{C}$.

In general, locally symmetric spaces are orbifolds. However, they are not compact, so we want to build good compactifications for these spaces. For this, we have a well-developed theory that leads to very interesting singular spaces. In the previous example, we may consider principal congruence subgroups $\Gamma(N)$, and it turns out $\Gamma(N)\backslash \mathscr{H}$ is a curve of some genus minus finitely many points.

It should be apparent that this subject is all over the place. For this reason, Paul finds this subject very interesting. Resources can be found on the course webpage.

Definition 1.6. Let M be a smooth manifold and Z be a subset of M. Then a *stratification* of Z is a filtration by subsets $Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z$ such that:

1. $S_i = Z_i \setminus Z_{i-1}$ is a locally closed regular submanifold of M of dimension i (possibly disconnected or even empty).[4] The connected components of the S_i are *strata*.

2. If S, T are strata and $\overline{S} \cap T \neq \emptyset$, then $T \subset \overline{S}$. This gives a poset structure on the set of strata.

3. The stratification is locally finite. Equivalently, any point is contained in the closure of finitely many strata.

Next lecture we will see the Whitney conditions, which guarantee that the geometry is locally homogeneous.

[3] "I don't know why you'd want to see an example of this. You must be a bad person"

[4] A special case of this is when S_{n-1} is empty, a pseudomanifold.

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2.1 Basic Examples

Last time we defined what a stratification is. Here is an example that is not very exciting:

Example 2.1. A manifold with boundary has a very simple stratification. More generally, a manifold with corners has a stratification given by the codimension of the corner. Note that topologically, a manifold with corners is the same thing as a manifold with boundary. A tetrahedron is an example of a manifold with corners, but an octahedron is not.

Example 2.2. Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let Z be a finite collection of affine subspaces. This defines a stratification of V. Note this works for any locally finite connection of subspaces, e.g. a periodization of Z by some lattice.

Example 2.3. Let X be a smooth manifold and consider X^n. Then for some $I \subset [n]$ with $|I| \geq 2$, we can consider the set $\Delta_I = \{x_j = x_k \text{ iff } j, k \in I\}$. We will consider the space

$$Z = \bigcup_{|I| \geq 2} \Delta_I.$$

This is related to configuration spaces. Locally this looks like a subspace arrangement.

Example 2.4. Let G be a finite group acting on X. X has a stratification given by the various stabilizers. For example, we can obtain the previous example by considering the action of S_n on X^n.

Example 2.5. Let $M = Gr(k, n)$. For example, $Gr(1, n) = \mathbb{P}^{n-1}$. Now choose a fixed flag in $\mathbb{R}^n$. We will obtain a stratification by looking at intersections with the flag. Considering $Gr(2, 4)$, we projectivize our setup and find six possible configurations. We can form a poset by taking limiting configurations, and the closures of strata are called Schubert varieties. The strata are called Schubert cells.

There is another construction (the matroid stratification) that we can perform on the Grassmannian. Fix a basis $e_1, \dots, e_n$. Then consider the $n!$ fixed flags that come from this basis. Now take all possible intersections of all Schubert cells. This is not a stratification because the axiom of the frontier fails.

Exercise 2.6. Figure out what the Schubert varieties of $Gr(2, 4)$ look like.

Example 2.7. Let Z be a complex projective variety. Z has a natural stratification given by iteratively taking the singular locus. However, this is not always the best stratification. Consider the Whitney cusp given by $(x^3 + z^2x^2 - y^2 = 0)$. The singular locus is the z-axis. The generic cross-section looks like a nodal curve, but the the cross-section at $z = 0$ is is cuspidal curve.

2.2 Some Geometry

Recall that two regular submanifolds M, N intersect transversely inside X if $T_pM + T_pN \subset T_pX$ has maximal possible dimension. We ran out of time, so next time we will define normal slices and links.

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3.1 Normal Slices and Links

Let $M \supset Z$ with $\mathscr{S}$ the set of strata. Then let $p \in Z$ be contained in stratum S. Choose a small disk $N \ni p$ and transverse to all strata containing p with $\dim N + \dim S = \dim M$.

Definition 3.1. THe *normal slice* to p is $N \cap Z$ and the *link* is defined as $\partial N \cap Z$.

Example 3.2. On the Whitney cusp, the normal slice of a generic singular point is an X-shape and the normal slice of the origin is a V-shape. The links are four points and two points, respectively.

Example 3.3. Let p be contained in a codimension k corner. Then it is not difficult to see that $L(p) = \Delta^k$, the $(k-1)$-simplex.

Exercise 3.4. Stratify $\mathbb{C}^n$ by the coordinate subspaces. Compute the links. As a warm up, try this with $\mathbb{R}^n$, or if you feel ambitious, do this with V^n for any real vector space.

3.2 The Whitney Conditions

3.2.1 Condition A

Let $Z = \bigcup S_\alpha$ be a stratification. Then

1. Suppose $S_\alpha \subset \overline{S_\beta}$.
2. In addition, suppose that we have a sequence $\{x_i\} \subset S_\beta$ and $y \in S_\alpha$.

Then if $T_{x_i} S_\beta \to P$, we have $T_y S_\alpha \subset P$. Here the limiting process of spaces takes place in the Grassmannian.[1]

3.2.2 Condition B

Suppose that conditions 1 and 2 hold has above. Additionally suppose that for $y \in S_\alpha$, $\{x_i\} \subset S_\beta$, $\{y_i\} \subset S_\alpha$ with $x_i \to y$, $y_i \to y$, consider the secant lines ℓ_i through x_i, y_i. Then if $\ell_i \to \ell$, we have $\ell \subset P$.

Example 3.5. On the Whitney cusp, this condition fails at the origin.

Exercise 3.6. Prove that condition B implies condition A.

[1]Note that Whitney originally came up with condition A, then realized it was strong enough. He then came up with condition B, which implies condition A. For some reason, he presented them in this order and now every presentation of the subject does this.

Definition 3.7. Let $M \supset Z$ with $\mathfrak{S}$ the set of strata. This is a *Whitney stratification* if it is a stratification and satisfies Whitney's condition B.

Example 3.8. If we make the origin a stratum on the Whitney cusp, then we obtain a Whitney stratification.

Definition 3.9. Suppose Z is Whitney stratified. Then

1. The homeomorphism types of links and normal slices are uniquely determined as soon as the disk N is sufficiently small.

2. The normal slice is homeomorphic to the cone on the link.

3. Locally near p, Z looks like $S \times cL(p)$ (local triviality along the strata). More precisely, given $S \subset Z$, there exists a closed neighborhood $T_S \subset Z$ and a locally trivial fibration $T_S \to S$ such that $f^{-1}(P) \simeq cL_p(S)$.

Proof of this theorem can be found in Mather's notes.

Definition 3.10. A map $f : X \to Y$ of spaces is a *fibration* if it satisfies the path lifting property: If $x \in f^{-1}(y)$ and we have a closed loop in Y based at y, it can be lifted to a path in X based at x.

Definition 3.11. A fibration is *locally trivial* if there exists a covering $\{U\}$ such that $f^{-1}(U)$ is homeomorphic to a product $U \times F$. Here F is called the *fiber*.

Example 3.12. Any fiber bundle is an example of a locally trivial fibration. If F is equipped with the action of a topological group G, the fiber bundle is determined by a covering $\{U\}$ and a G-valued cocycle. Let $F = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\gamma \in SL_2(\mathbb{Z})$. Then let $M = T^2 \times I/(t,0) \sim (\gamma t, 1)$. This gives a torus fibration M over the circle.

Exercise 3.13. Compute the homology of M.

Theorem 3.14. *Let X be a real or complex semianalytic semianalytic variety. Then X admits a filtration by semianalytic varieties whose connected components form the strata of a Whitney stratification.*[2]

A recent proof of this was given by Kaloshin.

[2]Most reasonable spaces are like this. The Cantor set is not like this, or the cone on the Cantor set, but is that really what you want to be doing?

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February 4

4.1 Whitney, Continued

Last time we discussed the Whitney conditions and their consequences. Here is an example of a nontrivial link bundle.

Example 4.1. Recall the construction in Example 3.12. Then take the matrix

$$\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

This is a torus bundle $T^2 \to M \to S^1$ over the circle, but is not trivial. Now we will construct three singular spaces:

- Set $\overline{X} = M \times [0, 1)$. This is a manifold with boundary.
- In $\partial\overline{X} = M$, collapse the fibers of M over S^1. This is a stratified space with a circle glued onto a four dimensional manifold. Call this $\widehat{X}$.
- Now collapse all of the boundary S^1 to a single point. Call this X^*.

Now consider the boundary strata and the link of a point.

	Boundary Stratum	Link of point
$\overline{X}$	M	pt
$\widehat{X}$	S^1	T^2
X^*	pt	M

Remark 4.2. These singularities appear in the study of Hilbert modular surfaces.

4.2 Thom's Presentation of Stratified Spaces

Our goal here is to explain how to assemble a stratified space from simpler ones (manifolds with corners) by gluing them together. The main reference is a paper in the Liverpool Singularities Symposium. This will allow us to compute links.

Consider a torus with an open slice taken out, so it has boundary two circles. Then if we glue the two boundary components together and collapse them, we obtain the nodal cubic. Here, we use

Theorem 4.3 (Collar Neighborhood Theorem). *Let $\partial X = Y$. Then there exists a neighborhood of Y in X that locally looks like $Y \supset U \times [0, \varepsilon)$.*

Now we generalize this to manifolds with corners.[1] Suppose M_n is a manifold with corners. Then let $M_{i,n}$ be a codimension 1 corner where $0 \le i \le n-1$. Now let $U_{i,n} \subset M$ be a neighborhood of $M_{i,n}$. Note that we have maps

$$M_{i,n} \xleftarrow{\pi_i} U_{i,n} \xrightarrow{\rho_i} [0, \infty).$$

Choosing small ε, we set $U_{i,n}^{\varepsilon} := \rho_i^{-1}([0, \varepsilon))$. If ε is sufficiently small, we can ensure that:

1. $U_{i,n}^{\varepsilon} \xrightarrow{\pi_i \times \rho_i} M_{i,n} \times [0, \varepsilon)$ is an isomorphism;
2. On $U_{i,n}^{\varepsilon} \cap U_{j,n}^{\varepsilon}$, we have $\pi_i \pi_j = \pi_j \pi_i$;
3. We also have $\rho_j \pi_i = \rho_j$;
4. If $S \subset \{0, \dots, n-1\}$, set
$$U_{S,n}^{\varepsilon} = \bigcap_{i \in S} U_{i,n}^{\varepsilon}.$$

   Then $M_{S,n}$ is a corner of codimension $|S|$, and similarly to the above, we have $U_{S,n}^{\varepsilon} \simeq M_{S,n} \times [0, \varepsilon)^{|S|}$.

Given all of this data, we can recover the manifold with corners. We can see the links of corners. If p is contained in a corner, then we can recover the link by

$$L(p) = \left\{ x \mid \Pi(x) = p, \sum \rho_i(x) = \frac{\varepsilon}{2} \right\}.$$

4.2.1 Thom Presentation

Let M_i be a manifold with corners for $i = 0, \dots, n$. For $j < i$ let M_{ji} be a codimension 1 boundary component of M_i. Define M_S similarly for $S \subset \{0, \dots, n\}$ with largest element i. Now suppose we have $M_{SS'}$ where every element of S' is larger than every element of S. Then we have

- A proper fibration $\pi : M_{SS'} \to M_S$;
- An inclusion $M_{SS'} \hookrightarrow M_{S'}$;

Both the inclusions and fibrations satisfy the expected compatibility relations.

[1] "I hope everyone doesn't mind these randomly generated abbreviations. This isn't English class so we can do whatever we want."

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February 6

5.1 Thom, Continued

Last time we bagan the Thom presentation of stratified spaces. Recall that we have manifolds with boundary M_i with M_S boundary components for S with largest element i. Also recall we have proper fibrations deleting on the right and inclusions deleting on the left. Also assume that the fibrations and inclusions satisfy the expected compatibility relations. Now we will construct a singular space from this as

$$\left(\bigsqcup M_S^{(\alpha)}\right) / \sim,$$

where $\sim$ is determined by the fibrations.

Theorem 5.1. *Any Whitney stratified space has a Thom presentation.*

We discussed the Whitney cusp, a picture of which is included. Now the strata are images of open strata in the manifolds with corners. We now describe how to compute links. Suppose $p \in S_i$. Consider all sets $\{i\} \neq S \ni i$ such that i is the smallest element. Then for each S let $M_S^p \subset M_S$ be the inverse image of p in M_S. Then on this collection, we obtain an induced Thom presentation. The link is the associated singular space.

Example 5.2. For the Whitney cusp, the link at a generic singular point is four points. At the origin, the link is three circles.

Exercise 5.3. Do the same for $\mathbb{C}^2$ with the coordinate axes, do $\mathbb{C}^n, \mathbb{R}^n$, and then do the Whitney cusp over $\mathbb{C}$.

Example 5.4. Recall the space $\overline{X}$ in Example 4.1. This has the Thom presentation M_0, M_{01}, M_1 where $M_0 = M_{01} = Y$, the torus bundle over S^1. Therefore the link at a boundary component is a point.

For $\widehat{X}$, the presentation is now $M_0 = S^1$. Then the fibration is $\pi : Y \to S^1$, so the link at a point is the torus.

Finally, for X^*, the presentation now has $M_0 = \text{pt}$. Clearly here the link is now all of Y.

We will give a geometric explation of why the links work this way. The idea is that the link parameterizes the directions we can move from the boundary component into the space.

5.2 Thin Singularities

A modification of this scheme works for some singularities. The idea is that we know the links of the corners in a manifold with corners. A codimension k corner has link Δ^{k-1}. We will try to use this in a nontrivial way. The resulting links will look like $\Delta \times \{\text{spaces}\}/\sim$ on proper faces of Δ.

Example 5.5. For $\mathbb{R}^2$ stratified by the coordinate axes, our thin scheme is given by taking each of the quadrants. We can recover the link from this presentation.

Next time we will define the thin presentation.

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Last time we began an overview of thin singularities. Recall that this is a variation of the Thom presentation that uses knowledge of manifolds with corners.

6.1 Thin Singularities in Detail

We will consider manifolds with corners $M_{S,T}$, where S, T are disjoint subsets of $\{0, \dots, n-1\}$. Here, S tracks the depth of the corner and T tracks the codimension of this piece in the whole picture.

- Clearly the top (N) dimensional manifold with corners is $M_{\emptyset,\emptyset}$;
- $M_{a,\emptyset}$ is a codimension 1 corner in $M_{\emptyset,\emptyset}$;
- Similarly, $M_{T,\emptyset}$ is a codimension $|T|$ corner in $M_{\emptyset,\emptyset}$;
- $M_{\emptyset,a}$ is a $(N-1)$-dimensional manifold with corners;
- $M_{S,T}$ is a codimension $|S|$ corner in a manifold with corners of dimension $N - |T|$.

6.1.1 Fibrations For $a \notin S \cup T$ with $S \cap T = \emptyset$, then we have a fibration $M_{S\cup\{a\},T} \to M_{S,T\cup\{a\}}$. In addition, we assume that all natural commutativity relations hold. In particular, if S, T, A are pairwise disjoint, we should obtain a unique fibration $M_{S\cup A,T} \to M_{S,T\cup A}$.

6.1.2 Fullness Note that this is not some sort of alternative healing thing.[1]. Recall that the fiber product of two maps is the limit $X \times_Z Y$ of

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Here, consider the diagram

$$\begin{tikzcd} M_{Sab,T} \arrow[r] \arrow[d] & M_{Sb,Ta} \arrow[d] \\ M_{Sa,Tb} \arrow[r] & M_{S,Tab} \end{tikzcd}$$

[1]It does have to do with fiber, so maybe it does

We will require that the map $M_{Sab,T} \to M_{Sb,Ta} \times_{M_{S,Tab}} M_{Sa,Tb}$ is surjective. In addition, we will require this for all such diagrams.

Now we can assemble our singular space.

Definition 6.1. Set

$$X = \bigsqcup M_{A,B} / \sim$$

where the equivalences come from the fibrations.

Example 6.2. We can do this for the plane stratified by the coordinate axes, where the closure of each quadrant is an $M_{\emptyset,\emptyset}$.

6.2 Constructing Links

Let $p \in M_{\emptyset,T}$. We will consider the image of p in X. Then we consider the full inverse image of p in $M_{T,\emptyset} \to M_{\emptyset,T}$. Here, the link of p is the quotient

$$\Delta^{|T|-1} \times M^p_{T,\emptyset} / \sim$$

where on the face where the coordinates corresponding to $A \subset T$ are nonzero, we collapse the fibers of $M^p_{T,\emptyset} \to M^p_{A,B}$.

Remark 6.3. Note that if the third coordinate is nonzero, that means you are moving towards the point $(0, 0, 1)$ because you are moving away from the opposite edge where the third coordinate is zero.[2]

Example 6.4. In the plane stratified by the coordinate axes, the link of a generic point on a coordinate axis is just a point. The link of the origin is a circle.

Example 6.5. Consider the product of the first quadrant and the torus. Then we can collapse to obtain two cylinders, and then further collapse to a point. Then we have a four-dimensional stratum, two two-dimensional strata, and a point. We want to compute the link of S_0. Note that the map from $M_{01,\emptyset} \to M_{1,0}$ is collapsing vertically, so we have a solid torus and two circles. Paul claims that the link is S^3 and that this example is really $\mathbb{C}^2$ stratified by the coordinate axes. In addition, S^3 meets the two coordinate axes in circles, which form a Hopf link.

Remark 6.6. S^3 is the union of two solid tori. To see this, note that

$$\begin{aligned} S^3 &= \partial B^4 \\ &= \partial(B^2 \times B^2) \\ &= \partial B^2 \times B^2 \cup B^2 \times \partial B^2 \\ &= S^1 \times B^2 \cup B^2 \times S^1. \end{aligned}$$

Remark 6.7. In $\mathbb{R}^2$, the link of the point is S^1 which contains two S^0, which are linked in some sense.[3]

[2]This is a geometric thing, so it's best thought of in the privacy of your own home.

[3]Nobody will ever write a PHD thesis about this, so don't get too excited.

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7.1 Toric Varieties

Recall Example 6.5 from last time. We can generalize this example in the following way: Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n. We will assume that P is *simple*, which means that each vertex meets n facets.

Example 7.1. Both the cube and the tetrahedron are simple polytopes. However, each vertex of the octahedron meets 4 facets, so it is not simple.

Note that for a simple polytope P, each vertex looks like a corner in $\mathbb{R}^n$. In particular, any face of codimension k is determined by k facets containing it.

We can consider the space $X = P \times T^n / \sim$. We will have to collapse certain things on the boundary of P. For each facet $F \subset P$, choose a primitive normal vector $v_F \in \mathbb{Z}^n$ that points inward. Then each v_F determines a subgroup $T_F \subset \mathbb{T}^n$. Similarly, any face determines a subgroup, which is the direct sum of the subgroups determined by the facets containing it.

Now for each $p \in P$, let $F(p)$ be the face that contains it in its relative interior. Now we can define the equivalence relation on $P \times T^n$ by $(p, g) \sim (q, h)$ if $p = q$ and $gh^{-1} \in T_{F(p)}$. This gives us a presentation of a stratified space with thin singularities, which happen to be *projective toric varieties* attached to simple polytopes.

Example 7.2. If $P = I = [0, 1]$, then $X = S^2$. If $P = I^n$, then $X = (S^2)^n$. Also, if P is the simplex, then $X = \mathbb{P}^2_{\mathbb{C}}$.

Remark 7.3. Note that toric varieties are compactification of the (algebraic) torus $\mathbb{C}^\times$.[1] The reason that this is the same as our presentation is because $\mathbb{C}^\times \simeq S^1 \times \mathbb{R}$.

Proposition 7.4. *X is a manifold if and only if for each vertex $p \in P$, the associated normal vectors $v_1, \dots, v_n$ corresponding to facets $F_1, \dots, F_n$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

Remark 7.5. This condition guarantees that locally at each vertex we get the picture from Example 6.5. This also answers a question of Tetsuya about whether the space is uniquely determined by the homeomorphism type of the polytope.

Example 7.6. Consider the triangle with vertices $(0, 0), (1, 0), (1, 3)$. Then at the origin, X the two normals do not generate $\mathbb{Z}^2$, so X is singular at the origin. Then the link at this point is a quotient of S^3 called a *lens space*.[2]

[1] This is different from the topological torus that we can buy at Dunkin Donuts.
[2] In algebraic geometry, we call these cyclic quotient singularities.

7.2 Intersection Homology

Note that algebraic topology is not a prerequisite for this course, so students are not assumed to know what homology is.[3]. The idea is to attach a collection of vector spaces that are topological invariants. This type of construction is ubiquitous throughout mathematics.

7.2.1 Homology Crash Course

We will define groups $H_i(X, G)$, where G is typically $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and where $0 \leq i \leq \dim X$. We expect this to be an invariant under homeomorphism.[4]

Example 7.7. $H_i(S^1, \mathbb{Z}) = \mathbb{Z}$ for $i = 0, 1$. Similarly, $H_i(S^n) = \mathbb{Z}$ for $i = 0, n$ and is zero otherwise.

Example 7.8. $H_i(T^n) = \mathbb{Z}^{\binom{n}{i}}$ for all i.

Example 7.9. Let $L(p, q)$ be the lens space corresponding to the cyclic quotient singularity determined by $\mathbb{Z}/p\mathbb{Z}$ acting by character $(1, q)$. Then $H_k(L(p, q), \mathbb{Z}) = \mathbb{Z}$ for $k = 0, 3$ and $H_1(L(p, q), \mathbb{Z}) = \mathbb{Z}/\alpha\mathbb{Z}$.

The general formula[5] for homology is as follows:

1. From X, produce a chain complex of abelian groups with boundary map ∂.
2. Compute the kernel $Z_i(X) = \ker \partial_i$.
3. Compute the image $B_i(X) = \operatorname{Im} \partial_{i+1}$.
4. Compute the homology $H_i(X, \mathbb{Z}) = Z_i / B_i$.

[3] Although the note taker believes everyone does know what homology is
[4] The best we can actually do is homotopy type.
[5] Implementation details are left for the lowlife programmers.

8

February 20

Last time we discussed homology. There are many ways to define homology, but we will use a combinatorial approach in this course.

Pros	Cons
Geometric meaning easy to understand	Depends on extra data
Easy to compute	Not clearly an invariant

The idea is to decompose our space X by simplices in a reasonable way. For example, the tetrahedron i s a model of the sphere. Unfortunately, not all spaces (or even topological manifolds) can be triangulated.[1] Fortunately, smooth manifolds and Whitney stratified spaces can be triangulated. We can require that the stratification is compatible with the stratification.

Remark 8.1. There is a class of triangularizable spaces between **Top** and **Diff**, namely the category **PL** of piecewise-linear manifolds.

8.1 Piecewise Linear Manifolds

Definition 8.2. An *abstract simplicial complex* Δ is a family of nonempty finite subsets of a set S such that:

1. If $A \in \Delta$ and $\emptyset \neq B \subset A$, then $B \in \Delta$;
2. Any $B \in \Delta$ is only contained in finitely many $A \in \Delta$.

The subsets are called *faces* and the dimension of face A is defined to be $|A| - 1$.

Definition 8.3. An abstract simplicial complex Δ is *pure* if all facets have the same dimension.

We now define the *geometric realization* of a simplicial complex. Suppose $A \in \Delta$ has dimension n. Then $F(A)$ is the standard n-simplex on $\mathbb{R}^n$. If we choose a total ordering on S, then $B \subset A$ can be identified with a face $F(B) \subset F(A)$. Then we can define the geometric realization to be the colimit

$$|\Delta| = \varinjlim_{A \in \Delta} F(A).$$

Definition 8.4. A *triangulation* of a space X is an abstract simplicial complex Δ and a homeomorphism $|\Delta| \to X$.

[1] They aren't too bad. Maybe you wouldn't go home with them, but you would have a drink with them.

Definition 8.5. An abstract simplicial complex Δ' is a *subdivision* of Δ if $|\Delta'| \simeq |\Delta|$ and every simplex of Δ' lines in some simplex of Δ.

Definition 8.6. A *morphism* $f : \Delta \to \Delta'$ of simplicial complexes is a map $f : |\Delta| \to \Delta'$ such that $f|_A$ maps linearly onto a simplex of Δ'.

Definition 8.7. A *piecewise-linear map* $f : \Delta_1 \to \Delta_2$ is a map f such that there exist subdivisions $\Delta_1' \to \Delta_2'$ such that $f : \Delta_1' \to \Delta_2'$ is a morphism.

Definition 8.8. A *combinatorial n-manifold* is a triangulated manifold such that links of p-simplices are PL-isomorphic to the boundary of an $(n-p)$-simplex.

Definition 8.9. A *PL-manifold* is a manifold where the transition functions are PL.

Remark 8.10. Piecewise linear implies combinatorial, so PL-manifolds have triangulations.

8.2 Defining Homology

Given a simplicial complex Δ, we can make a chain complex $C_*(\Delta)$. We consider a total ordering on the set S and we consider symbols $[v_0, \dots, v_n]$, where permuting $v_0, \dots, v_n$ changes the sign of the symbol. Now we define

$$C_n(\Delta) = \oplus_{A \in \Delta_n} \mathbb{Z}[v_0, \dots, v_n]$$

and the boundary map $\partial : C_n(\Delta) \to C_{n-1}(\Delta)$ by

$$[v_0, \dots, v_n] \mapsto \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n].$$

For example, we can see that $\partial([v_0, v_1]) = [v_1] - [v_0]$.

Exercise 8.11. Check that $\partial^2 = 0$ (just check what happens when you delete two vertices in the opposite order).

Remark 8.12. The purpose of the sign in the definition of δ is to respect the orientation of the simplices.[2]

Example 8.13. Consider the circle S^1 represented as a triangle with vertices $0, 1, 2$. Then we see that

$$C_1(\Delta) = \mathbb{Z}[01, 12, 02], C_0(\Delta) = \mathbb{Z}[0, 1, 2].$$

Then the boundary ∂ is defined by the matrix

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then we see that $H_1(S^1) \simeq \mathbb{Z}$ and $H^0(S^1) \simeq \mathbb{Z}$.

Remark 8.14. In order to compute the homology, we need to use Smith normal form, which can be found in musty old British algebra textboks.

[2]How to keep track of sign: work in characteristic 2.

8.2.1 A Variation There is a variation which goes by various names (local coefficients, twisted coefficients, local systems), where we have a nontrivial action on the coefficients G of the homology.

Definition 8.15. A *local system* $\widetilde{G}$ is given by a representation $\pi_1(X, x) \to \operatorname{Aut} G$.

There is a construction due to Steenrod of $H_*(X, \widetilde{G})$ which is not seen in a first topology course but is very important. Combinatorially, for each face B of a simplex A, choose a point x_B in the relative interior of the geometric realization. Then over each x_B we place a copy G_B of G. Then for each face, we choose isomorphisms $G_A \to G_B$. Then we require all natural compatibility relations to hold.

Remark 8.16. Local systems are intermediate between ordinary homology and sheaf cohomology.

9

February 25

9.1 Homology with Local Systems

Last time we began our discussion of homology with coefficient systems. Today we will define this using simplicial homology.

Let Δ be a triangulation of X. Over each simplex $\sigma \in \Delta$, choose a point x_σ in the relative interior of σ. Now attach a copy G_σ of G at each x_σ. If $\tau < \sigma$, choose an isomorphism $h_{\tau\sigma} : G_\sigma \to G_\tau$. Then we require all compositions to commute.

For a reference, see Steenrod's paper or Whitehead's big book on homotopy theory.

We now need to create a homology theory out of this. Define the chains $C_n(X, \widetilde{G})$ to be G-linear combinations of symbols $[x_0, \dots, x_n]$ with boundary map

$$\partial(g[x_0, \dots, x_n]) = \sum_{i=0}^{n} (-1)^i h_i(g)[x_0, \dots, \hat{x}_i, \dots, x_n],$$

where h_i is the isomorphism $h_{\sigma'\sigma}$ obtained by deleting x_i. Then we define $H_*(X, \widetilde{G})$ as the homology of the chain complex.

Example 9.1. Let $X = S^1$ and $G = \mathbb{Q}^2$. Here the local system will be given by $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then we can represent S^1 as a triangle with the x_σ of the edges being their midpoints. We choose all maps to be the identity except for one, which is γ.

The 0-chain is $C_0(X, \widetilde{G}) \simeq \mathbb{Q}^6$ and $C_1(X, \widetilde{G}) \simeq \mathbb{Q}^6$. Now we need to understand the boundary map. Now the boundary is given by

$$\partial = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

We can check the rank of ∂, which is 5, so $H_0 \simeq H_1 \simeq \mathbb{Q}$.

Remark 9.2. If we replace γ is the identity matrix, then the rank of ∂ is now 4, so $H_0 \simeq H_1 \simeq \mathbb{Q}^2$. If we set $\gamma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then $H_1 \simeq H_1 \simeq 0$.

Remark 9.3. Recall our manifold M_γ given by a T^2-bundle over S^1. Our question is: Given a fiber bundle $F \hookrightarrow E \to B$, can we compute $H_*(E)$ in terms of $H_*(B), H_*(F)$? Recall that if $E = B \times F$, then $H_*(E) = H_*(B) \otimes H_*(F)$. In general, there is the Leray-Serre spectral sequence. Under favorable circumstances, the E^2 page will determine $H_*(E)$. For Example 9.1, we computed $H_p(S^1, \widetilde{H_1(T^2)})$. Then the homology of M_γ is determined by taking direct sums along the diagonals of the E^2 page.[1]

9.2 Some Technical Remarks

Remark 9.4. There is a way to eliminate dependence on the triangulation, namely PL geometric chains. Write $C_*^\Delta(X)$ for the chain complex attached to the triangulation. Suppose that Δ' is another triangulation and that Δ, Δ' have a common refinement Δ''. Then we can take the direct limit of this system and call the elements PL geometric chains. This makes sense for any PL manifold.

Remark 9.5. Simplicial homology is not the only way to work. It is the easiest way to work in the elementary treatment, but there are others:

Regular Cell Complexes The closure of a cell must be homeomorphic to $(B, \partial B)$, and ∂B must be a union of cells. The boundary map is slightly more complicated.

CW Complexes These are much more efficient and need very few cells, but writing down the boundary is much more difficult.

Singular Homology Here, we work with maps $\Delta^n \to X$.

Remark 9.6. Homology is a functor $H_* : \mathbf{HoTop} \to \mathbf{Ab}$. There is a similar construction for local systems as long as the local systems are related.

Theorem 9.7 (Poincare Duality). *Let X be a compact oriented manifold. Then there exists a perfect pairing*

$$H_i(X) \otimes H_{n-i}(X) \to \mathbb{Q}.$$

This implies that $H_i(X) \simeq H_{n-i}(X)^*$. The pairing is called the intersection form *on X.*[2]

[1] Paul was always mystified by these kinds of computations as a grad student, and he wants us to see that it's not so bad.

[2] The modern way of thinking about this uses cohomology.

10

February 27

Recall Poincare duality from last time. The classical picture is using dual cell decompositions. Let Δ be a triangulation of X and let Δ' be the barycentric subdivision. Then we can obtain the dual cell decomposition by taking unions of simplices that are transverse to the original $\sigma \in \Delta$. Note that this is a regular cell complex Z, but not a simplicial complex. If we compute the homology $H_*(Z_*)$, we obtain the same result as if we compute $H_*(\Delta_*)$.

Remark 10.1. This fails if we take integral coefficients. For example, if we take the lens space $\Lambda(a, b)$. This has homology groups $\mathbb{Z}, \mathbb{Z}/a\mathbb{Z}, 0, \mathbb{Z}$, so we see Poincare duality does not hold. This is an example of a *rational homology sphere*.

10.1 Cohomology

Cohomology is the dual theory to homology. We will consider cochain complexes with maps $d : C^i \to C^{i+1}$. Here d is defined as the adjoint of ∂ with respect to the pairing $C^i \otimes C_i \to \mathbb{Z}$ given by[1]

$$\langle d\xi, \eta \rangle = \langle \xi, \partial\eta \rangle.$$

Then recall that cohomology has a cup product structure, which is dual to the intersection product and turns cohomology into a ring.

If X is a compact oriented manifold, then we can phrase everything in terms of the cup product. The reason cohomology is easier to define a product for is because the construction is local.

10.2 Intersection Homology

We want a homology theory for singular spaces that has as many nice properties of usual homology as possible. The first problem is that Poincare duality fails for ordinary homology.

Example 10.2. Consider $X = \{xy = 0\} \subset \mathbb{P}^2$. Then X is a union of two $\mathbb{P}^1$. Then we can see that $H_0(X) = \mathbb{Q}$, but $H_2(X) = \mathbb{Q}^2$.

Example 10.3. Now consider the suspension ΣT^2. Recall that $H_*(T^2) = \mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}$. It is a good exercise to determine $H_*(\Sigma T^2)$. However, note that the suspensions of two independent cycles do not intersect in a cycle, so we cannot define an intersection product.

Now let α be a cycle on the torus and $W \in H_2$ be the suspension of the other cycle β on T^2. Then α, W meet transversely and intersect in one point. Then we see that α is a boundary, so it has no dual.

[1] This is just Stoke's Theorem.

Remark 10.4. Returning to Example 10.2, if we do not allow chains to pass through the intersection point, then we recover Poincare duality. Thus the main idea of intersection homology is to introduce data to control how chains meet the singular locus. More generally, we will impose conditions on how chains meet all strata.

Doing this will give the opportunity to both eliminate and create homology. This new data is called *perversity*.[2] This will allow us to pair cycles created with different perversities and recover Poincare duality.

[2] "I suppose it was a quainter time when this was invented."

11

March 3

Note: I was away on this day, notes were provided by Arthur Wang.

Definition 11.1. A *pseudomanifold* is a stratified space with a singular locus of codimension at least 2. Alternatively, a pseudomanifold has a triangulation where every $(n-1)$-simplex is a face of exactly two n-simplices.

Recall that a compact oridnted manifolds have a fundamental class, which generates the top homology. This comes from the sum of all the n-simplices.

Example 11.2. Every complex projective variety is a pseudomanifold.

Note we want the stratification to be compatible with the PL structure. Recall that in a Whitney stratification, we have a neighborhood of p that looks like $B_i \times c^o(L(p))$. We need this to be compatible with the PL structure. In particular, we need a filtered space $V = V_n \supset \cdots \supset V_i = \text{pt}$ and a PL i-ball B_i such that we have a map $V \times B_i \to X$ that is a PL homeomorphism onto a neighborhood of $p \in X$. Fortunately, this can always be arranged for a Whitney stratified space.

To define intersection homology, we will use PL geometric chains. We work with chains $\xi \in C^{\Delta}(X)$ for some triangulation in the direct system.

Definition 11.3. The *support* $|\xi|$ of a geometric chain ξ js the union of the closures of the simplices where the coefficient of ξ is nonzero.

Definition 11.4. A *perversity* $\overline{p}$ is a sequence $\overline{p} = (o_2, p_3, \ldots)$ such that

1. $p_2 = 0$;
2. $p_k \leq p_{k+1} \leq p_k + 1$.

Example 11.5. If $X = \Sigma T^2$, then $\overline{p} = (p_2, p_3)$, so there are two perversities: $(0,0)$ and $(0,1)$.

Remark 11.6. We always have two distinguished perversities:

$$\bar{t} = (0, 1, 2, 3, \ldots) \tag{11.1}$$

$$\overline{0} = (0, 0, \ldots). \tag{11.2}$$

These are called the *top* and *zero* perversities.

There are two other interesting perversities:

$$\overline{m} = (0, 0, 1, 1, 2, 2, \dots) \tag{11.3}$$

$$\overline{n} = (0, 1, 1, 2, 2, \dots). \tag{11.4}$$

Note that $\overline{m} + \overline{n} = \overline{t}$.

Definition 11.7. If $\overline{p} + \overline{q} = \overline{t}$, then they are *complementary*.

For a complex projective variety, all strata have even dimension, so we can define $\overline{p}(2n) = n - 1$. This is called *middle perversity* and is most common in the literature.

The condition of slow growth is necessary to prove the topological invariance of intersection homology. In particular, the result is independent of the stratification.

Example 11.8. Consider a homogeneous variety X (like a Grassmannian or flag variety). This has a stratification into Schubert varieties. We get many interesting constructions from this stratification going beyond the topology of X, which leads to perverse sheaves.

Definition 11.9. A subspace $Y \subset X$ is called $(\overline{p}, i)$-allowable if

1. $\dim Y \leq i$;
2. $\dim(Y \cap X_{n-k}) \leq i - k + p_k$ for all $k \geq 2$.

Definition 11.10. The intersection chain complex consists of chains ξ such that both $\xi, \partial\xi$ are allowable.

The intersection homology is defined exactly as expected.

Remark 11.11. The second condition in allowability controls how much a chain can meet the singular locus and corresponds to a deviation from transversality.

Example 11.12. Consider the two lines $xy = 0$ in $\mathbb{P}^2$. This has intersection homology $\mathbb{Q}^2, 0, \mathbb{Q}^2$, which satisfies Poincare duality. On the other hand, the ordinary homology is $\mathbb{Q}, 0, \mathbb{Q}^2$.

Exercise 11.13. Do the same for the nodal cubic.

12

March 5

Today we compute some examples of perversity. Last time, we considered the example $xy = 0$ in $\mathbb{P}^2$, and we saw that the perversity is identically 0.

Example 12.1. Consider the nodal cubic. Then we see that the perversity is the zero vector, so there are no allowable 0-chains and 1-chains that meet the singular point, but there is an allowable 2-chain. Note that each ordinary homology group is $\mathbb{Q}$. Then we can see that the intersection homology changes in dimension 1, where it is now zero.

12.1 Normal Spaces and IH

Let X be a stratified pseudomanifold with singular set Σ. Then we say X is *normal* if each point has a fundamental system of neighborhoods U such that $U \setminus \Sigma$ is connected. In algebraic geometry, a space is normal if the local ring at each point is integrally closed.

If a space is normal, then IH with the top perversity recovers the usual homology. In general, IH of a space with top perversity will recover the homology of its normalization.In addition, if X is normal, IH with the zero perversity recovers the cohomology of X. In general, IH with zero perversity is the cohomology of the normalization.

Example 12.2. Consider $X = \Sigma T^2$. Then the possible perversities are the zero perversity and the top perversity. First we note that the ordinary homology groups are $H_0 = \mathbb{Q}, H_1 = 0, H_2 = \mathbb{Q}^2, H_3 = \mathbb{Q}$. This matches the intersection homology with the top perversity.

i	$i-k$	top	zero
0	$0-3$	-2	-3
1	$1-3$	-1	-2
	$0-3$	-2	-3
2	$2-3$	0	-1
	$1-3$	-1	-2
3	$3-3$	1	0
	$2-3$	0	-1

Table 12.1: Consitions for Allowable Chains.

Now we can compute the intersection homology with the zero perversity, which is $\mathbb{Q}, \mathbb{Q}^2, 0, \mathbb{Q}$. Then Poincare Duality[1] says there is a pairing

$$IH_i^{\overline{p}}(X) \otimes IH_{n-1}^{\overline{q}} \to IH_0^{\overline{t}}(X).$$

Our goal is to explain local properties of intersection homology groups. This is a key point in the passage from IH1 (geometric) to IH2 (sheafy).[2]

Example 12.3. Consider an isolated point singularity. Then locally we obtain the open cone on the link L of the singularity, so we obtain $c^o L = X$. When can a class survive? Only somewhere in between $p_n \in \{0, \dots, n-2\}$.

Exercise: Find answer.

[1]Goresky and MacPherson had a giant notebook with many examples, and they

[2]The story goes that MacPherson was at a party in France with Deligne, where this definition was written down.

13

March 10

Note: I was not here on this day; notes were provided by Arthur Wang.

Consider the open cone on a link of an isolated singularity. Recall that this is the same as a neighborhood of the singularity. **Note**: There was a probably incorrect calculation done, that is not reproduced here.

We will consider two generalizations:

1. Suppose X has only isolated singularities. Then some discussion shows that
$$IH_i^{\overline{p}}(X) = \begin{cases} H_i(X - X_0) & i < n - p_n - 1 \\ H_i(X) & i > n - p_n - 1 \\ Im(H_i(X - X_0) \to H_i(X)) & i = n - p_n - 1 \end{cases}.$$
2. Suppose L is an $(n-1)$-dimensional stratified space. Then clearly $cL = X$ is a stratified n dimensional space with strata the cone point and the cones on the strata of L. Then we can compute
$$IH_i^{\overline{p}}(cL) = \begin{cases} IH_{i-1}^{\overline{p}}(L) & i \geq n - p_n \\ 0 & i < n - p_n \end{cases}.$$

13.1 Local Coeffieicents for IH

Let $X = X_n \supset X_{n-2} \supset \cdots$. To specify a local system on X for IH, we only need to do it on $X \setminus \Sigma$. For example, if $X \setminus \Sigma$ is connected, then we only need a representation of $\pi_1(X \setminus \Sigma)$. Note this is not true for regular homology (recall how regular homology with local systems worked).

Note that $IC_i^{\overline{p}}(X) \subset C_i(X)$. Then $(\overline{p}, i)$ allowability means that we only need the data for a local system in $X \setminus \Sigma$. Suppose Δ is an i-simplex in $|\xi|$ and $\partial\Delta$ is an $(i-1)$-simplex in $|\partial\xi|$. Then we require
$$\dim \Delta \cap X_{n-k} \leq i - k + p_k$$
$$\dim \partial\Delta \cap X_{n-k} \leq i - 1 - k + p_k.$$
How large can these dimensions be? With the top perversity, then we have
$$\dim \Delta \cap X_{n-k} \leq i - 2$$
$$\dim \partial\Delta \cap X_{n-k} \leq i - 3.$$

Our goal is now to present Deligne's definition, but we must beware of cheating. The key to the definition are local intersection homology groups. These have very spevific structure involving the links of strata.

13.2 Local Homology

We present some more facts about regular homology. Let $X \supset Y$ and suppose X is triangulated so that Y is compatibly triangulated. Then $C_i(X) \supset C_i(Y)$. We can make a quotient $C_i(X,Y) := C_i(X)/C_i(Y)$. Then we obtain a boundary map making $C_*(X,Y)$ into a complex. Then $H_*(X,Y)$ is defined to be the homology of that complex.

Note that cycles in $C_i(X,Y)$ are chains in X with boundaries in $C_{i-1}(Y)$. Also if $Y = \varnothing$, then $H_*(X,\varnothing) = H_*(X)$.

Example 13.1. Let $X = S^2$ and Y be the two poles. Note that we have a new class joining the two poles. Thus $H_i(X,Y) \neq 0$. We can compute things using the long exact sequence of a pair:

$$\cdots \to H_i(Y) \to H_i(X) \to H_i(X,Y) \to H_{i-1}(Y) \to H_{i-1}(X) \to \cdots.$$

Definition 13.2. The *local homology group* at $x \in X$ is $H_*(X, X \setminus \{x\})$.

14

March 24

Because of the coronavirus, we moved lectures online. Paul says he practiced, so hopefully things go well. All lectures will go on YouTube and a link will be sent later today. In an email, Paul asked that we do not share the link, so the link will not be in the notes.

14.1 Cleaning up Loose Ends

Last time we computed the intersection homology on the link of an isolated singularities. Paul took information from different sources and there was conflicting information. After discussing this, we will go back to local systems in IH.

Recall that for compact manifolds there is only one homology theory that is really useful to think about. For noncompact manifolds, there are two different useful theories: usual homology, which is compactly supported, and $H_*^{BM}(M)$, which is called Borel-Moore homology or homology with closed support. Here we allow infinitely supported chains. For noncompact manifolds, these usually give different answers.

Example 14.1. Recall that the usual homology of $\mathbb{R}^n$ is $\mathbb{Q}$ is degree 0 and 0 everywhere else. On the other hand, the Borel-Moore homology of $\mathbb{R}^n$ is $\mathbb{Q}$ is degree n and 0 everywhere else.

Remark 14.2. Recall that $H_i(X) \simeq H_i(X \times \mathbb{R})$. On the other hand, $H_i^{BM}(X) \simeq H_{i+1}^{BM}(X \times \mathbb{R})$. Thus Borel-Moore homology does not introduce new information, but is not truly homotopy invariant.

Let L be a compact $(n-1)$-dimensional compact manifold and $X = cL$, the open cone on L. Then we have $X = X_n$ and X_0 is the cone point. Then, the usual intersection homology of X is

$$IH_k^{\overline{p}}(X) = \begin{cases} H_k(L) & k \leq \ell - 2 \\ 0 & k \geq \ell - 1 \end{cases}.$$

Here, $p_n = n - \ell$.

In the Borel-Moore situation, we can compute

$$\widetilde{IH}_k^{\overline{p}}(X) = \begin{cases} 0 & k \leq \ell - 1 \\ H_{k-1}(L) & k \geq \ell \end{cases}.$$

Remark 14.3. Nobody else uses this tilde notation. Please do not use it ever when you talk to other people.

Remark 14.4. What happens when L is itself a stratified space and not a manifold? Then we should have the same story but replacing normal homology with intersection homology. We have

$$\widetilde{IH}_k^{\overline{p}}(X) = \begin{cases} 0 & k < n - p_n \\ IH_{k-1}^{\overline{p}}(L) & k \geq n - p_n \end{cases}.$$

There is a similar statement for the usual intersection homology, which is left as an exercise.

14.2 Local Homology

Recall the definition of relative homology $H_*(X, Y)$. If $Y = X \setminus \{x\}$, this only makes sense in $\dim X = n$, and we have $H_n(X, X \setminus \{x\}) = H_n(B^n, B_n \setminus \{0\})$ by the excision theorem. By the long exact sequence of a pair, this becomes $H_{n-1}(B^n \setminus \{0\}) \simeq \mathbb{Z}$.

We can now consider relative intersection homology $\widetilde{IH}_*^{\overline{p}}(X, X \setminus \{x\})$. Suppose $x \in S_k$. We should still have some form of excision and we expect some information about the link of x. Recall that a neighborhood of $x \in S_k$ looks like $\mathbb{R}^k \times cL$. We have

$$\widetilde{IH}_i^{\overline{p}}(X, X \setminus \{x\}) = \begin{cases} 0 & i < n - p_{n-k} \\ IH_{i-k-1}^{\overline{p}} & i \geq p_{n-k}. \end{cases}.$$

A remark of this is in IH2, Borel seminar volume. Our goal is to motivate Deligne's construction of IH via sheaf theory. Note this will involve a lot of lying as Paul will not tell us what a sheaf is, what a theory is, or motivate anything.

Remark 14.5. Sheaves are nice because there are many canonical operations we can do with them without really thinking about what is going on geometrically.

Our goal is to construct a complex computing IH. We begin with the nonsingular part $X \setminus \Sigma$. Then set $S_i = X_i \setminus X_{i-1}$ and $U_i = X - X_{n-i}$.

We have a complex

$$0 \leftarrow 0 \leftarrow \mathbb{Q} \leftarrow 0 \leftarrow 0$$

attached to the open set U_2. We denote this by $P(\mathbb{Q})_2^\bullet$. Then we grow this to U_3 by

$$P(\mathbb{Q})_3^\bullet := \tau_{\leq p(2)} R(i_2)_*(P(\mathbb{Q})_2^\bullet).$$

15

March 26

15.1 Sheafy Stuff (with caveats)

Let $X = X_n \supseteq X_{n-2} \supseteq \cdots \subseteq X_0$ be a stratified space and $S_i = X_i \setminus X_{i-1}$ be the strata. Consider the open sets $U_i = X \setminus X_{n-i}$. Recall that we defined $P(\mathbb{Q})_2^\bullet$ to be the complex with $\mathbb{Q}$ at degree 0 and is 0 everywhere. Then we define $P(\mathbb{Q})_3^\bullet$ to be

$$P(\mathbb{Q})_3^\bullet = \tau_{\leq p(2)} R(i_2)_* P(\mathbb{Q})_2^\bullet.$$

Here

- $(i_2)_*$ is the pushforward from U_2 to U_3 along the inclusion.[1]
- R means to take the derived version of the pushforward.[2] What this means is to replace $P(\mathbb{Q})_2^\bullet$ with an injective resolution and then apply $(i_2)_*$.
- The $\tau_{\leq p(2)}$ truncates the complex, killing anything in degree above $p(2)$ and replacing the object in degree $p(2)$ with $\ker d_{p(2)}$.

Then we turn this into an inductive process:

$$P(\mathbb{Q})_{k+1}^\bullet = \tau_{\leq p(k)} R(i_k)_* R(\mathbb{Q})_k^\bullet.$$

Once we finish, we have a complex that computes intersection homology after taking the derived functor of global sections of the complex.[3] In fact, the two complexes are quasi-isomorphic. This is different from the complex we obtained from the other definition, but it computes the same homology.

Remarks 15.1.

1. We could start with a local system instead of $\mathbb{Q}$.
2. This definition makes sense in a broader setting because it uses definitions that make sense whenever we have the six functors formalism.
3. A good reference for this is the IH2 paper, where they explain what they did and how it relates to what came before.

[1] Note that there are various operations associated to a morphism of schemes $f : X \to Y$: $f_*, f^*, f_!, f^!$.
[2] "Even though it looks like there are three things happening, there are only two."
[3] Up to degree shifts and multiplying the degree by -1.

15.2 Isolated Singularities of Complex Hypersurfaces

Now we make a shift in the course to discussing examples and applications of this theory. The first example, isolated singularities, does not rely on intersection homology. The standard reference for this is the book by Milnor.

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ and suppose that $\nabla f = 0$ only at the origin. Assume $f(0) = 0$. Then $V(f)$ is a hypersurface in $\mathbb{C}^n$ which has an isolated singularity at the origin.

Example 15.2. Take $n = 2$ and $f \in \mathbb{C}[x, y]$. The link of 0 in $\mathbb{C}^2$ is S^3. The link of the singular point is simply $V(f) \cap S^3$. This has real dimension 1, so L is a link in S^3. If the hypersurface is smooth, then the link is the usual circle in S^3.

If $V(f)$ is singular, then the link can have arbitrarily many connected components and can be arbitrarily complicated. For example, if $f = xy$, then the link is the Hopf link. If f is a product of independent linear forms, then L is a collection of circles. Each circle is standardly embedded, but the components are typically linked in a complicated way. This depends on the actual linear forms.

Example 15.3. Let $f = x^3 - y^2$. Let (a, b) be a fixed real solution. Then for all angles θ, $(ae^{2i\theta}, be^{3i\theta})$ is also a solution. Every solution of this form, so we can see what L looks like. First note that all points sit on a torus $|x| = a, |y| = b$. We can see that we obtain a line of slope $3/2$ in the torus. Thus L is a trefoil knot embedded in the torus.

More generally, if we have $f = x^p - y^q$, then we get the (q, p) torus knot if $(p, q) = 1$.

Question 15.4. *What kinds of knots and links can we obtain can arise as links of isolated singularities of hypersurfaces in $\mathbb{C}^2$? Do those that arise have special properties? Classify the ones that arise.*

Question 15.5. *How do we compute the link given f?*

We will discuss these question in more depth later.