Defomation Theory Graduate Student Seminar Spring 2021

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Lectures by Various

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Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Seminar Website: `https://www.math.columbia.edu/~dejong/seminar.html`

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1

Johan (Sep 24): Schlessinger's Paper

The paper by Schlessinger is titled *Functors of Artin Rings*. Throughout this lecture, k is a field, $\mathcal{C}$ is the category of Artinian local k-algebras $A, B, C, \ldots$ with residue field k, and $\widehat{\mathcal{C}}$ is the category of Noetherian complete local k-algebras $R, S, \ldots$ with residue field k.

Remark 1.0.1. Every $R \in \widehat{\mathcal{C}}$ is of the form $k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_m)$ by the Cohen structure theorem. Then $R \in \mathcal{C}$ if and only if and only if $(f_1, \ldots, f_m)$ contains $(x_1, \ldots, x_n)^N$ for some N.

Remark 1.0.2. In the paper, there is a more general setup, where Λ is a complete local Noetherian ring with residue field k. Then $\mathcal{C}_\Lambda, \widehat{\mathcal{C}}$ are defined analogously, which will allow things like $\Lambda = \mathbb{Z}_p$.

The idea of deformation theory is to look at functors $F \colon \mathcal{C} \to \mathrm{Set}$.

Example 1.0.3. Given $R \in \widehat{\mathcal{C}}$, we set $h_R \colon \mathcal{C} \to \mathrm{Set}$ sending $A \mapsto \mathrm{Hom}_{\widehat{\mathcal{C}}}(R, A)$. This is not necessarily representable because $R \notin \mathcal{C}$ in general, but it is pro-representable.

Definition 1.0.4. A functor F is *pro-representable* if $F \simeq h_R$ for some $R \in \widehat{\mathcal{C}}$.

Example 1.0.5. Let M be a variety over k and $m \in M(k)$. Then define

$$\mathrm{Def}_{M,m}(A) = \left\{ \mathrm{Spec}\, A \xrightarrow{m_A} M \mid m_A|_{\mathrm{Spec}\, k} = m \right\}.$$

It is easy to see that $\mathrm{Def}_{M,m}(A)$ is pro-representable by $\widehat{\mathcal{O}}_{M,m}$.

Observe that $h_R(k) = \{*\}$ is a singleton. Also note that $h_R(A \times_B C) = h_R(A) \times_{h_R(B)} h_R(C)$. Here, $A \times_B C$ is the fiber product of rings and not the tensor product.

Now consider the following conditions on F: let $A \to B \leftarrow C$ be a diagram in $\mathcal{C}$ and consider the morphism

$$F(A \times_B C) \xrightarrow{(*)} F(A) \times_{F(B)} F(C).$$

- (H_1) The morphism $(*)$ is surjective if $C \twoheadrightarrow B$;
- (H_2) The morphism $(*)$ is bijective if $C = k[\varepsilon] \twoheadrightarrow k = B$;
- (H_3) $\dim_k(t_F) < \infty$ (later, we will see that we need H_2 for formulate this). Here, t_F is the tangent space to F;
- (H_4) The morphism $(*)$ is bijective if $C \twoheadrightarrow B$.

Example 1.0.6. Fix a group G and a representation $\rho_0\colon G \to GL_n(k)$. Now define

$$\mathrm{Def}^{\mathrm{naive}}_{\rho_0}(A) = \left\{\rho\colon G \curvearrowright A^{\oplus n} \mid \rho \pmod{m_A} \cong \rho_0\right\}/\cong.$$

Better, we will define

$$\mathrm{Def}_{\rho_0}(A) = \left\{\rho\colon G \curvearrowright A^{\oplus n} \mid \rho \pmod{m_A} = \rho_0\right\}/\ker(GL_n(A) \to GL_n(k)).$$

In general these functors fail (H_4) and $\mathrm{Def}^{\mathrm{naive}}_{\rho_0}$ even fails (H_2).

Namely, if $H = \mathbb{Z}$ and ρ_0 is the trivial representation, then for $\mathrm{Def}^{\mathrm{naive}}_{\rho_0}$, we are looking at subsets of

$$GL_n(A \times_B C)/\mathrm{conj} \to GL_n(A)/\mathrm{conj} \times_{GL_n(B)/\mathrm{conj}} GL_n(C)/\mathrm{conj}.$$

This morphism is always surjective, but in general it is not injective.

For example, if $A = k[\varepsilon_1], B = k, C = k[\varepsilon_2]$, we can look at elements of the form $1 + \varepsilon_1 T_1 + \varepsilon_2 T_2$ and see that on the left we can only conjugate together, while on the right we can conjugate both T_1, T_2 arbitrarily. Here $A \times_B C = k[\varepsilon_1, \varepsilon_2] = k[x_1, x_2]/(x_1^2, x_1x_2, x_2^2)$.

Definition 1.0.7. A natural transformation $t\colon F \to G$ of functors on $\mathcal{C}$ is *smooth* if for all surjections $B \twoheadrightarrow A$ the map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

Note that this is equivalent to the existence of a lift in the diagram below:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d, hook] & M \arrow[d, "f"] \\
\operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & N.
\end{tikzcd}$$

This definition is motivated by the following example: let $f\colon M \to N$ be a morphism of varieties over k. Let $m \in M(k), n = f(m) \in N(k)$. Then the following are equivalent:

1. $\mathrm{Def}_{M,m} \to \mathrm{Def}_{N,n}$ is smooth.
2. f is smooth at m.

Definition 1.0.8. We say F has a *hull* if and only if $F(k) = \{*\}$ and there exists a smooth $t\colon h_R \to F$ for some $R \in \widehat{\mathcal{C}}$ which induces an isomorphism $t_R \cong t_F$.

Now we will say a bit about tangent spaces.

1. When $F(k) = \{*\}$, then $t_F = F(k[\varepsilon])$.
2. If F satisfies (H_2) and $F(k) = \{*\}$, then t_F has a natural k-vector space structure. Here, H_2 gives $F(k[\varepsilon_1, \varepsilon_2]) \to F(k[\varepsilon]) \times F(k[\varepsilon])$ is a bijection, and then we take $\varepsilon_1 \mapsto \varepsilon, \varepsilon_2 \mapsto \varepsilon$, which defines addition.
3. $t_R = \mathrm{Hom}_k(m_R/m_R^2, k) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(R, k[\varepsilon]) = h_R(k[\varepsilon]) = t_{(h_R)}$.

Theorem 1.0.9 (Schlessinger)**.** *Asssume that $F(k) = \{*\}$. Then the conditions $(H_1), (H_2), (H_3)$ hold for F if and only if F has a hull. In addition, (H_3) and (H_4) hold if and only if F is pro-representable.*

Very rough idea of proof of $\Rightarrow$ for the hull case. Let $n = \dim_k(t_F)$. Then (H_2) and $n < \infty$ imply the following: Let $S = k[[x_1, \dots, x_n]]$ and $\mathfrak{m} = \mathfrak{m}_S = (x_1, \dots, x_n)$. We can find $\xi_1 \in F(S/\mathfrak{m}^2)$ such that

$$t_S = \mathrm{Hom}_{\widehat{\mathcal{C}}}(S, k[\varepsilon]) \xrightarrow{\xi_1} t_F$$

is an isomorphism.

Next, we will choose $q \geqslant 2$ and consider pairs (J, ξ) where $\mathfrak{m}^{q+1} \subset J \subset \mathfrak{m}^2$ and $\xi \in F(S/J)$ such that $\xi \mapsto \xi_1 \in F(S/\mathfrak{m}^2)$. Say that $(J, \xi) \leqslant (J', \xi')$ if $J \subset J'$ and $\xi \mapsto \xi'$. Choose a minimal pair (J, ξ) for this ordering. We can choose J_q so that $\mathfrak{m}^{q+1} + J_{q+1} = J_q$ and ξ_{q+1} maps to ξ_q for bookkeeping purposes.

Choose $R = \lim S/J_q$, which is a quotient of S. Set $t\colon h_R \to F$ given by sending $\varphi\colon R \to A$ to the following: choose q such that φ factors as $R \to S/J_q \xrightarrow{\varphi_q} A$ and take $\xi_q \mapsto F(\varphi_q)(\xi_q) \in F(A)$.

Finally, we must show that t is smooth. Consider the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[dd] \arrow[drrr, dashed, bend left, "\varphi"] & S/\mathfrak{m}^{q+1} \arrow[dr, dashed, "\overline{\varphi}'"'] \arrow[drr, dashed, "\overline{\varphi}"] & & \\
& & S/J_q \times_A B \arrow[r] \arrow[dl, "pr_1"] & B \arrow[d, two heads] \\
R \arrow[r] & S/J_q \arrow[rr] \arrow[ur, dashed, bend left, "\psi"] & & A
\end{tikzcd}$$

with $B \ni \widetilde{\xi} \mapsto \xi \in A$ and $S/J_q \ni \xi_q \mapsto \xi$. First, choose $\varphi\colon S \to B$ making the diagram commute. We may increase q such that $\varphi(\mathfrak{m}^{q+1}) = 0$, so we now have $\overline{\varphi}\colon S/\mathfrak{m}^{q+1} \to B$. Now consider the fiber product $S/J_q \times_A B$ and $pr_1\colon S/J_q \times_A B \to S/J_q$, so we obtain $\overline{\varphi}'\colon S/\mathfrak{m}^{q+1} \to S/J_q \times_A B$. By (H_1), we obtain some $\widetilde{\widetilde{\xi}} \in F(S/J_q \times_A B)$ mapping to $\widetilde{\xi}$ and ξ_q. We may now assume that $B \to A$ is a small extension, which means that $\dim_k \ker(B \to A) = 1$, and thus pr_1 is a small extension. Therefore, either $\overline{\varphi}'$ is surjective or its image maps isomorphically via pr_1 to S/J_q., so we have ψ which gives $R \to B$ lifting our given $r \to A$.

The tricky part is to show that $F(\psi)(\psi_q) = \widetilde{\widetilde{\xi}}$, and this step is deliberately omitted. $\square$

A generalization of this is as follows. Consider a functor $\mathcal{F}\colon \mathcal{C} \to \mathsf{Grpd}$. We say that $\mathcal{F}$ satisfies the *Rim-Schlessinger condition* (RS) if

$$\mathcal{F}(A \times_B C) \to \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(C)$$

is an equivalence whenever $C \twoheadrightarrow B$. Let $x_0 \in \mathcal{F}(k)$ and set

$$\overline{\mathcal{F}}_{x_0}\colon \mathcal{C} \to \mathsf{Set} \qquad A \mapsto \{(x, \alpha) \mid x \in \mathcal{F}(A), \alpha\colon X_0 \to x|_k\}/\cong,$$

where $(x, \alpha) \cong (x', \alpha')$ means that $\varphi\colon x \to x'$ such that the diagram

$$\begin{tikzcd}
x|_k \arrow[r, "\varphi"] & x'|_k \\
x_0 \arrow[u, "\alpha"] \arrow[r, "\mathrm{id}"] & x_0 \arrow[u, "\alpha'"']
\end{tikzcd}$$

commutes.

Theorem 1.0.10. *If $\mathcal{F}$ has (RS) then $\overline{\mathcal{F}}_{x_0}$ has (H_1) and (H_2). Therefore, if $\dim t_{\overline{\mathcal{F}}_{x_0}} < \infty$ then $\overline{\mathcal{F}}_{x_0}$ has a hull.*

In this situation, $\overline{\mathcal{F}}_{x_0}$ has (H_4) if and only if $\mathrm{Aut}_A(x) \twoheadrightarrow \mathrm{Aut}_B(x|_B)$ whenever $A \twoheadrightarrow B$ and $x \in \mathcal{F}_{x_0}(A)$.

Example 1.0.11. Let $\mathcal{F}(A)$ be the category of representations $G \curvearrowright A^{\oplus n}$ with morphisms being isomorphisms of representations. This has (RS).

Example 1.0.12. Let $\mathcal{F}(A)$ be the category of smooth projective families of curves of genus g over A with morphisms being isomorphisms. This has (RS).

Returning to the example of representations, it turns out that $t_{\mathrm{Def}_{\rho_0}} = H^1(G, M_{n\times n}(k))$, where G acts on $M_{n\times n}(k)$ via ρ_0 by conjugation.

Example 1.0.13. Consider $G = \mathbb{Z} \oplus \mathbb{Z}$ and ρ_0 to be the trivial representation on $k^{\oplus 2}$. Then $t_{\mathrm{Def}_{\rho_0}} = H^1(\mathbb{Z}^2, M_2(k)) = M_2(k) \oplus M_2(k)$. Given two matrices A, B, we have the representation

$$\begin{aligned} \mathbb{Z}^2 \to GL_2(k[\varepsilon])(1,0) &\mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \varepsilon A \\ (0,1) &\mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \varepsilon B. \end{aligned}$$

We get a hull R with $h_R \to \mathrm{Def}_{\rho_0}$. We know that R is a some quotient of $k[[a_{11}, \ldots, a_{22}, b_{11}, \ldots, b_{22}]]$ with ρ looking like

$$(1,0) \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + A \qquad (0,1) \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + B,$$

and of course R is the quotient of the power series ring by the ideal generated by the coefficients of $AB - BA$.

2

Ivan and Cailan (Oct 1): Deformations of Schemes

2.1 Deformations of affine schemes

We are looking for a Cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[r] & S \end{tikzcd}$$

where π is flat and surjective and S is surjective. This is called an *deformation* of X over S. For the beginning of this lecture (the part given by Ivan), we are interested in $S = \operatorname{Spec} A$, where $A \in \mathcal{C}^*$ (this category was defined in the previous lecture). This case is called a *local deformation*, and in the face where A is Artinian, it is called an *infinitesimal deformation*.

For the ring theorists, we will make the following digression. Let A be a ring and $I \subset A$ be an ideal with $I^2 = 0$. Suppose that $\overline{B}$ is an A/I-algebra, J is an $\overline{B}$-module, and $h\colon I \to J$ is an A-module map. Then we are interested in a diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & ? \arrow[r] & \overline{B} & \\ 0 \arrow[r] & I \arrow[r] \arrow[u, "h"] & A \arrow[r] \arrow[u] & A/I \arrow[r] \arrow[u] & 0, \end{tikzcd}$$

which we will call a deformation of A. Here are some interesting questions:

1. Is such a deformation unique?
2. If $\overline{B}$ is flat over A/I, does that mean that B is flat over A?

Returning to the case of schemes, we will say that two deformations $\mathcal{X}, \mathcal{X}'$ of X over S are isomorphic if there exists an S-isomorphism $\phi\colon \mathcal{X}' \to \mathcal{X}$ commuting with the inclusions of the central fibers $X \to \mathcal{X}, \mathcal{X}'$.

Example 2.1.1. The most basic example of a family is the trivial deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times_k S \arrow[d] \\ \operatorname{Spec} k \arrow[r] & S. \end{tikzcd}$$

Definition 2.1.2. A scheme X is *rigid* if all deformations of X are isomorphic to the trivial deformation.

Theorem 2.1.3. *If X is a smooth affine k-scheme and $S = \operatorname{Spec} A$ for some local Artinian ring, then X is rigid.*

Definition 2.1.4. A closed immersion $i\colon S_0 \hookrightarrow S$ of schemes is called a *first (resp. nth) order thickening* if the ideal sheaf $\mathcal{I} = \ker(i^\flat\colon \mathcal{O}_S \to \mathcal{O}_{S_0})$ satisfies $I^2 = 0$ (resp. $I^{n+1} = 0$).

Definition 2.1.5. A morphism $f\colon X \to S$ is called *formally smooth* (resp. unfamified, resp. étale) if for all first order thickenings $i\colon T_0 \to T$ of affine schemes and diagrams

$$\begin{tikzcd} T_0 \arrow[r, "u_0"] \arrow[d, "i"] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "\widetilde{u_0}"] & S \end{tikzcd}$$

there exists a lift $\widetilde{u_0}$ (resp. there is at most one such $\widetilde{u_0}$, resp. there exists a unique $\widetilde{u}_0$).

Example 2.1.6.

1. Open immersions are formally étale. This is cleaer because T_0, T have the same underlying topological space.

2. Closed immersions are formally unramified. This is clear because $X \to S$ induces an injection on T-points.

3. $\mathbb{A}^n_S \to S$ is formally smooth. To see this, assume $S = \operatorname{Spec} R$ is affine and then consider the corresponding lifting problem in commutative algebra.

Proposition 2.1.7. *The classes of formally smooth (resp. étale, resp. unramified) morphisms are closed under base change, composition, and products and local on both source and target.*

Definition 2.1.8. A $f\colon X \to S$ is *smooth* if it is formally smooth and locally of finite presentation.

We will now consider differentials. Let $X = \operatorname{Spec} A$ be an affine scheme over k and choose a k-point and consider the diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k[\varepsilon] \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

If X is smooth, then there exists a lift $\operatorname{Spec} k[\varepsilon] \to X$. But this is given by a morphism

$$\widetilde{\phi}\colon A \to k[\varepsilon]/\varepsilon^2 \qquad a \mapsto \phi(a) + d(a)\varepsilon.$$

This motivates the following definition:

Definition 2.1.9. Let $R \to A$ be a morphism of rings and M be an A-module. A *derivation* $d\colon A \to M$ is an A-linear map satisfying the Leibniz rule.

Proposition 2.1.10. *There exists an* A*-module* $\Omega^1_{A/k}$ *equipped with a derivation* $d\colon \Omega^1_{A/k}$ *that is universal among derivations from* A*. This means that all derivations* $\tilde{d}\colon A \to M$ *factor through* d*, and formally, we have an identity*

$$\mathrm{Der}_R(A, M) \simeq \mathrm{Hom}_A(\Omega^1_{A/k}, M).$$

Definition 2.1.11. For an A-module M with derivation $d\colon A \to M$, define the ring $A[M]$ as the module $A \oplus M$ with the multiplication

$$(a, m) \cdot (a', m') = (aa', am' + a'm).$$

There is a sequence $\phi\colon A \to A[M] \to A$.

Proposition 2.1.12. *Let* $S \leftarrow R \to A \to B$ *be a diagram of rings. Then*

1. $\Omega^1_{A \otimes S/S} \simeq \Omega^1_{A/R} \otimes_R S$;
2. *The sequence* $\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$ *is exact.*
3. *If* $B = A/I$ *for some ideal* I*, we have an exact sequence*

$$I/I^2 \to \Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to 0.$$

4. *For all* $f \in A$*, we havae* $\Omega^1_{A[f^{-1}]/R} \simeq \Omega^1_{A/R} \otimes_A A[f^{-1}]$.

Remark 2.1.13. If $J = \ker(A \otimes_R A \to A)$, then $\Omega^1_{A/R} = J/J^2$.

Theorem 2.1.14. *Let* $f\colon X \to S$ *be locally of finite presentation. The following are equivalent:*

1. f *is smooth;*
2. f *is flat with smooth fibers;*
3. f *is flat and has smooth geometric fibers.*

We will finally return to deformation theory.

Lemma 2.1.15. *Let* Z_0 *be a closed subscheme of* Z *determined by a nilpotent ideal sheaf* N*. If* Z_0 *is affine, then so is* Z*.*

Proof of this result can be found in EGA, Chapter I.5.9.

Proof of Theorem 2.1.3. Recall that we have a diagram of the form

$$\begin{tikzcd} B \arrow[r] & B_0 \\ A \arrow[u] \arrow[r] & k, \arrow[u] \end{tikzcd}$$

where $A \to B$ is flat and $B_0 \simeq B \otimes_A k$ is a smooth k-algebra. We need to prove that $B_0 \simeq B \otimes_A k$. The first step is to prove this result for first-order deformations. Suppose that $A = k[\varepsilon]$ is a square-zero extension.

Lemma 2.1.16. *For a ring* R *with* M, N *flat over* R*, nilpotent ideal* $I \subset R$*, and* $f\colon M \to N$*, then if* $f \otimes_R R/I$ *is an isomorphism, then so is* f*.*

To prove the lemma, note that the cokernel of f is preserved by I, so it must vanish. Returning to our case, we know that B is a smooth $k[\varepsilon]$-algebra. Now we obtain a square-zero extension $B_0[\varepsilon]$ of B_0 and a diagram

$$\begin{tikzcd} B \arrow[r] & B_0 \\ k[\varepsilon] \arrow[r, "f"] & B_0[\varepsilon] \arrow[u] \end{tikzcd}$$

with a lift $B \to B_0[\varepsilon]$. But now by the lemma, we have $B \otimes_{k[\varepsilon]} k = B_0[\varepsilon] \otimes_{k[\varepsilon]} k$. The rest of the proof follows using an inductive argument that was verbalized but now written down. $\square$

2.2 Deformations of schemes

The main theorem of this section is

Theorem 2.2.1. *Assume X is a smooth R-scheme. Then there is a bijection*

$$\mathrm{Def}_X^{\mathrm{sm}}(k[I]) \simeq H^1(X, T_{X/k} \otimes I).$$

Proof. Let $\mathcal{X}'$ be a smooth deformation over $k[I]$. Then the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[I] \end{tikzcd}$$

is cartesian. Then if $U_k = \operatorname{Spec} B_k$ is an affine cover of X and $U'_k = \operatorname{Spec} D_k$ is an affine cover of X', we have a $k[I]$-linear ring isomorphism

$$\varphi_k \colon k[I] \otimes_k B_k \to D_k \qquad (k, i) \otimes b \mapsto s(b) + i.$$

Modulo I, φ_k is the identity on B_k. Without loss of generality, we may assume that $U_{kj} = U_k \cap U_j$ is a distinguished open for both U_k and U_j, so let $U_{kj} = \operatorname{Spec} B_{kj}$ and $U'_{kj} = \operatorname{Spec} D_{kj}$. Now note that both

$$\varphi_k, \varphi_j \colon k[I] \otimes_k B_{kj} \to D_{kj}$$

induce the identity on B_{kj} modulo I. Now we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{id}"] & D_{kj} \arrow[r] \arrow[d, "\varphi_j^{-1}\varphi_k"] & B_{kj} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & I \arrow[r] & D_{kj} \arrow[r] & B_{kj} \arrow[r] & 0. \end{tikzcd}$$

Lemma 2.2.2. *The morphism $g = \varphi_j^{-1}\varphi_k$ must be of the form*

$$g(i + b) = i + b + \delta(b),$$

where $\delta \colon B_{kj} \to I$ is a derivation.

In particular, this means that $\varphi_j^{-1} \circ \varphi_k(b, b') = (b, \alpha_{kj}(b) + b')$, where $\alpha_{kj}\colon B_{kj} \to I \otimes_k B_{kj}$ is a derivation.

By definition, we have

$$\begin{aligned}(T_{X/k} \otimes_k I)(B_{kj}) &= \mathrm{Hom}_{B_{kj}}(\Omega^1_{B_{kj}/k}, B_{kj}) \otimes_k I \\ &= \mathrm{Hom}_{B_{kj}}(\Omega^1_{B_{kj}/k}, B_{kj} \otimes_k I) \\ &= \mathrm{Der}_k(B_{kj}, B_{kj} \otimes_k I).\end{aligned}$$

Therefore, $\alpha_k \in H^0(B_{kj}, T_X \otimes_k I)$. Note that

$$\varphi_\ell^{-1} \circ \varphi_j \circ \varphi_j^{-1} \circ \varphi_k^{-1} = \varphi_\ell^{-1} \circ \varphi_k,$$

which implies that

$$(b, \alpha_{j\ell}(b) + \alpha_{kj}(b) + b') = (b, \alpha_{k\ell}(b) + b')$$

and thus $\{\alpha_{kj}\} \in Z^1(\{U_k\}, T_X \otimes_k I)$.

If two deformations are the same, note that φ_k is defined using a ring section $s_k\colon B_k \to D_k$ of the canonical map $\pi_k\colon D_k \to B_k$. If φ'_k is defined using another section s'_k, then define $\theta_k = s'_k - s_k \in \mathrm{Der}(B_k, I \otimes_k B_k)$. We now compute

$$((\varphi'_j)^{-1} \circ \varphi'_k - \varphi_j^{-1} \circ \varphi_k)(b, b') = (0, \theta_k(b) - \theta_j(b)),$$

and thus the two differ by the desired coboundaries. $\square$

We will now consider some obstructions. We are looking for a diagram of the form

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & X'' \arrow[d] \\ \mathrm{Spec}\, A' \arrow[r] & \mathrm{Spec}\, A''. \end{tikzcd}$$

for each pair (j, k), we have a isomorphism $\psi_{jk}\colon V'_j \to V'_k$ and a cocycle

$$c_{jk\ell} = \psi_{k\ell} \circ \psi_{jk} \circ \psi_{j\ell}^{-1}.$$

This induces $B_{jk\ell} \in \mathrm{Der}_A(D_{jk\ell}, J \otimes_A D_{k\ell}) = Z^2(U, T_{X'/A} \otimes_A J)$.

Now we will discuss some examples.

Theorem 2.2.3. *Let C be a smooth projective curve, $T = T_C$, and $K = \Omega^1_C$. We have the following table:*

Table 2.1: Cohomology

	degree	h^0	h^1	h^2
K	$2g-2$	g	1	0
T	$2-2g$	ε	$\varepsilon + 3g - 3$	0

where $\varepsilon = 0$ where $g \geqslant 2$, $\varepsilon = 1$ if $g = 1$, and $\varepsilon = 3$ if $g = 0$.

For $g \geqslant 2$, $\deg T < 0$, and by Riemann-Roch and Serre duality, we have $h^1(C, T_C) = 3g - 3$.

Theorem 2.2.4. *$\mathbb{P}^n$ has no infinitesimal deformations.*

Proof. Consider the Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

and use the long exact sequence in cohomology. Because positive degree line bundles have no higher cohomology, we have $H^1(T_{\mathbb{P}^n}) = 0$. $\square$

3

Kevin (Oct 08): Deformations of coherent sheaves

There will be no mixed characteristic funny business during this lecture. Let X be a projective k-scheme (proper might be fine, but this makes certain facts more true) and $\mathcal{F}$ be a coherent sheaf on X. Consider the deformation functor

$$D_{\mathcal{F}} \colon \mathsf{Art}_k \to \mathsf{Set} \qquad A \mapsto \{\mathcal{F}_A \in \mathsf{Coh}(X_A) \mid \mathcal{F}_A|_X \cong \mathcal{F}, \mathcal{F}_A \text{ flat over } A\}.$$

We want to study the properties of this functor, which means we will check Schlessinger's conditions:[1] Let $A \to B \leftarrow C$ be a diagram in $\mathcal{C}$ and consider the morphism

$$D(B \times_A C) \xrightarrow{r} D(B) \times_{D(A)} D(C).$$

- (H_1) The morphism r is surjective if $C \twoheadrightarrow A$;
- (H_2) The morphism r is bijective if $C = k[\varepsilon] \twoheadrightarrow k = A$;
- (H_3) $\dim_k(t_D) < \infty$ (later, we will see that we need H_2 for formulate this). Here, t_D is the tangent space to D;
- (H_4) The morphism r is bijective if $C \twoheadrightarrow A$.

Recall from Johan's lecture that $(H_1), (H_2), (H_3)$ are equivalent to the existence of a hull and $(H_3), (H_4)$ are equivalent to D being pro-representable.

We only need to check (H_1) for small extensions, which are extensions by a k-vector spaace

$$0 \to I \to C \to A \to 0,$$

where I is killed by the maximal ideal of C.

Theorem 3.0.1. *The functor $D_{\mathcal{F}}$ admits a hull.*

Lemma 3.0.2. *Let $(A, \mathfrak{m})$ be a local Artinian ring.*

1. *If $\mathfrak{m}M = M$, then $M \cong 0$.*
2. *If $M \to N$ induces an isomorphism $M/\mathfrak{m}M \cong N/\mathfrak{m}N$ and N is flat over A, then $M \cong N$.*
3. *If M is flat, then M is free.*

[1] Neither Kevin nor Johan knows why these conditions are called H

Proof. We know that $\mathfrak{m}^d = 0$, so $\mathfrak{m}^d M = 0$, and thus $M = \mathfrak{m}M = \mathfrak{m}^2 M = \cdots = \mathfrak{m}^d M = 0$. Next, suppose $M \to N$ induces an isomorphism after killing $\mathfrak{m}$. Then we know that the kernel and cokernel vanish because they are killed by $\mathfrak{m}$, so $M \to N$ must be an isomorphism. The last part is left as an exercise. □

Proof of theorem. We will simply prove $(H_1), (H_2), (H_3)$:

1. Suppose that $C \twoheadrightarrow A$ is a small extension and consider a pair $(\mathcal{F}_B, \mathcal{F}_C) \in D(B) \times_{D(A)} D(C)$. We know that we have isomorphisms $\mathcal{F}_B|_{X_A} \cong \mathcal{F}_A, \mathcal{F}_c|_{X_A} \cong \mathcal{F}_A$, and so we take the fiber product
$$\mathcal{F}_{B \times_A C} \coloneqq \mathcal{F}_B \times_{F_A} \mathcal{F}_C.$$
We only need to show that our sheaf is flat over $B \times_A C$ because it clearly restricts to $\mathcal{F}_B$ and $\mathcal{F}_C$. We can consider each sheaf as a module M, and so we know M_B is free over B by the lemma. Choose a basis $\{e_i\}$. Also consider the diagram
$$\begin{tikzcd} M_B \times_{M_A} M_C \arrow[r] \arrow[d] & M_C \arrow[d, "v"] \\ M_B \arrow[r, "u"] & M_A. \end{tikzcd}$$
Then M_A has A-basis $u(e_i)$. Because M_C surjects onto M_A, we can lift the $u(e_i)$ to $f_i \in C$, and these form a C-basis for M_C. This all implies that $M_B \times_{M_A} M_C$ is free with basis (e_i, f_i).

2. It suffices to prove injectivity. Suppose $\mathcal{G} \in D(B \times_k k[\varepsilon])$ maps to $(\mathcal{F}_B, \mathcal{F}_{k[\varepsilon]}) \in D(B) \times D(k[\varepsilon])$, and so we have morphisms
$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{F}_{k[\varepsilon]} \arrow[d] \\ \mathcal{F}_B \arrow[r] & \mathcal{F}. \end{tikzcd}$$
We will prove that this diagram is Cartesian. By the lemma, the morphism $\mathcal{G} \to \mathcal{F}_B \times_{\mathcal{F}} \mathcal{F}_{k[\varepsilon]}$ is an isomorphism.

3. We will prove that $T_D = \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F})$. We will only prove this in the case where $\mathcal{F}$ is a vector bundle $\mathcal{E}$ of rank r. In this case, we have $\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E}) = H^1(X, \mathrm{End}(\mathcal{E}))$. Now we will associate cocycles to deformations. To each $\mathcal{E}_{k[\varepsilon]}$, we will associate an open cover (U_j) and
$$h_{ij} \in \mathrm{Aut}(\mathcal{O}_{X_{k[\varepsilon]}}^{\oplus r})(U_{ij}),$$
and we write $g_{ij} + \varepsilon f_{ij}$, where $g_{ij} \in \mathrm{Aut}_{\mathcal{O}_X^{\oplus r}}(U_{ij})$ and $f_{ij} \in \mathrm{End}(\mathcal{O}_X^{\oplus r})(U_{ij})$. The cocycle condition is that
$$g_{ik} + \varepsilon f_{ik} = (g_{ij} + \varepsilon f_{ij})(g_{jk} + \varepsilon f_{jk}),$$
which is the same as
$$f_{ik} = g_{ij} f_{jk} + f_{ij} g_{jk},$$
which is exactly the Čech 1-cocycle condition. Proving that equivalent cocycles give the same deformation is easy. □

Theorem 3.0.3. *The condition (H_4) holds when $\mathcal{F}$ is simple, which means that $k \simeq \mathrm{End}_X(\mathcal{F})$.*

3.1 Tangent-obstruction theory

Suppose D is a deformation functor. Then a *tangent-obstruction theory* for D is given by finite-dimensional k-vector spaces (T^1, T^2). Suppose we have a small extension

$$0 \to I \to B \to A \to 0.$$

Then we have another exact sequence

$$T^1 \otimes_k I \to D(B) \to D(A) \xrightarrow{\text{ob}} T^2 \otimes_k I,$$

which means that

1. $\xi_A \in D(A)$ lifts to $D(B)$ if and only if $\text{ob}(\xi_A) = 0$;
2. $T^1 \otimes I$ acts transitively on the fibers of $D(B) \to D(A)$;
3. If $A = k$, then the action of $T^1 \otimes I$ acts simply transitively on $D(B)$.

Note that because T^1 acts simply transitively on $D(k[\varepsilon])$, we must have $T^1 = D(k[\varepsilon])$. On the other hand, T^2 is not canonical.

Theorem 3.1.1. *The deformations* $D_{\mathcal{F}}$ *admits a tangent-obstruction theory with* $T^1 = \text{Ext}^1_X(\mathcal{F}, \mathcal{F})$ *and* $T^2 = \text{Ext}^2_X(\mathcal{F}, \mathcal{F})$.

Proof. We claim that if D satisfies (H_1) and (H_2), then $D(k[\varepsilon]) \otimes I$ naturally acts on $D(B)$ for small extensions $0 \to I \to B \to A \to 0$. To see this, note that $D(k[\varepsilon]) \otimes_k I = D(k[I])$. We also note that by (H_2), $D(k[I]) \times D(B) = D(k[I] \times_k B)$. Now define $\alpha\colon k[I] \times_k B \to B$ by $\alpha(1 + i, b) = 1 + b$, and this gives us an action of $D(k[I]) \times D(B) = D(k[I] \times_k B) \xrightarrow{\alpha_*} D(B)$. To prove transitivity, apply (H_1) to the diagram

$$\begin{tikzcd} k[I] \times_k B \arrow[r, "\alpha"] \arrow[d, "\pi_B"] & B \arrow[d] \\ B \arrow[r] & A. \end{tikzcd}$$

Now we will consider obstructions. We will assume again that $\mathcal{F}$ is a rank r vector bundle, which we will call $\mathcal{E}$. Let

$$0 \to I \to B \to A \to 0$$

be a small extension, so we will consider $H^2(X, \text{End}(\mathcal{E}))$. Consider an open cover (U_i) and $g_{ij} \in \text{Aut}(\mathcal{O}_X^{\oplus r} \otimes_k A)(U_{ij})$. We want to lift these to $h_{ij} \in \text{Aut}(\mathcal{O}_X^{\oplus r} \otimes_k B)(U_{ij})$. If this is possible, we have a cocycle

$$h_{ij}^{-1} h_{ij} h_{jk} \in 1 + \text{End}(\mathcal{O}_X^{\oplus r} \otimes_k I)(U_{ij}),$$

and the cocycle condition is satisfied when $h_{ij}^{-1} h_{ij} h_{ik} = 1$. If any other $h'_{ij} = h_{ij} + s_{ij}$, then we note that

$$(h'_{ij})^{-1} h'_{ij} h'_{jk} = h_{ik}^{-1} h_{ij} h_{jk} + (-s_{ik} g_{ij} g_{jk} + g_{ik}^{-1} s_{ij} g_{ij} + g_{ik}^{-1} g_{ij} s_{jk}),$$

and this gives us a class in $H^2(X, \text{End}(\mathcal{E})) \otimes I$. $\square$

Remark 3.1.2. Let R be the hull of D, which means we have a morphism $h_R \to D$. Then we know $R = k[[t_1, \dots, t_{d_1 1}]]/(f_1, \dots, f_{d_2})$. We also know that $d_1 - d_2 \leqslant \dim R \leqslant d_1$.

Example 3.1.3 (Good example)**.** Let X be a smooth projective curve and $\mathcal{E}$ be a rank r vector bundle. Then we know that

$$T^1 = H^1(X, \mathrm{End}(\mathcal{E})), \qquad T^2 = H^2(X, \mathrm{End}(\mathcal{E})) = 0,$$

so deformations of $\mathcal{E}$ are unobstructed. Now assume that $\mathcal{E}$ is simple. Then $H^0(X, \mathrm{End}(\mathcal{E})) = k$ by definition, and we also know that D_ε is pro-represented by some ring R with

$$\dim R = h^1(X, \mathrm{End}(\mathcal{E})) = r^2(g-1)+1$$

by Riemann-Roch.

Example 3.1.4 (Bad example)**.** Let X be a smoorh projective variety and $\mathcal{E}$ be a rank r vector bundle on X. Let $\mathcal{E}_1, \mathcal{E}_2 = D_{\mathcal{E}}(k[\varepsilon])$. Then $(\mathcal{E}_1, \mathcal{E}_2) \in D_{\mathcal{E}}(k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_1\varepsilon_2, \varepsilon_2^2))$, and we would like to lift to $D_{\mathcal{E}}(k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2))$.

We will compute the obstruction explicitly. We know $\mathcal{E}_1, \mathcal{E}_2$ give us classes $u_1, u_2 \in H^1(X, \mathrm{End}(\mathcal{E}))$, and after some magical computation, the obstruction to lifting is given by

$$u_1 \smile u_2 + u_2 \smile u_1,$$

where the cup product comes from the algebra structure on $\mathrm{End}(\mathcal{E})$.

Now let $X = C_1 \times C_2$ be a product of curves. Then $H^1(X, \mathcal{O}_X) = H^1(C, \mathcal{O}_{C_1}) \oplus H^1(C_2, \mathcal{O}_{C_2})$ and $H^2(X, \mathcal{O}_X) = H^1(C, \mathcal{O}_{C_1}) \otimes_j H^2(C_2, \mathcal{O}_{C_2})$. Suppose that $\alpha_1 \in H^1(C_1, \mathcal{O}_{C_1})$ and $\alpha_2 \in H^1(C_2, \mathcal{O}_{C_2})$ with nonzero cup product. Then we simply set $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$ and

$$u_1 = \begin{pmatrix} 0 & \alpha_1 \\ 0 & 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 & 0 \\ \alpha_2 & 0 \end{pmatrix} \qquad u_1 \smile u_2 + u_2 \smile u_1 = \begin{pmatrix} \alpha_1 \smile \alpha_2 & \\ & \alpha_2 \smile \alpha_1 \end{pmatrix}.$$

this gives us our obstructed deformation.

4

Patrick (Oct 15): Deformations of singularities

We begin by fixing some notation. Let k be a field and $R = P/I$, where $P = k[x_1, \dots, x_n]$ and $I = (f_1, \dots, f_r)$ is an ideal. Throughout this lecture, we will denote local Artinian rings with residue field k by $A, B, C, \dots$ and rings by $R, S, T, \dots$ Finally, denote $Z = \operatorname{Spec} R$.

4.1 Explicit criteria for flatness

We will study (embedded) deformations of singular affine schemes embedded in $\mathbb{A}^n$. The first thing we want to understand is to explicitly understand flatness of some R_A over A, where $R_A \otimes_A k = R$. We will write $R_A = P_A/I_A$, where $P_A = A[x_1, \dots, x_n] = A \otimes_k P$. Recall that over a Noetherian local ring S with residue field k, a module M is flat if and only if it is free, and this is equivalent to $\operatorname{Tor}_1^S(M, k) = 0$ by standard results in commutative algebra.

Now consider the exact sequence

$$0 \to I_A \to P_A \to R_A \to 0.$$

After tensoring with k, we have

$$0 \to \operatorname{Tor}_1(R_A, k) \to I_A \otimes_A k \to P \to R \to 0.$$

Therefore, we know that R_A is flat over A if and only if $I_A \otimes_A k = I$. We would like to understand this statement.

Consider a presentation

$$P_A^s \to P_A^r \to I_A \to 0$$

of I_A. Then we know R_A is flat over A if and only if after tensoring with k, we obtain an exact sequence

$$P^s \to P^r \to I \to 0.$$

Note that to give this presentation $P^s \to P^r \to I \to 0$ is the same as giving a complete set of relations among the generators of I.

Proposition 4.1.1. *Suppose that*

$$P^s \to P^r \to P \to R \to 0 \tag{4.1}$$

is exact and

$$P_A^s \to P_A^r \to P_A \to R_A \to 0 \tag{4.2}$$

is a complex such that $P_A^r \to R_A \to R_A \to 0$ is exact and tensoring (2) with k gives (1). Then R_A is flat over A.

Proof. Note that the hypotheses are equivalent to the fact that all relations in I can be lifted to I_A. Now given $g_1', \dots, g_r' \in P_A$ such that

$$\sum_{i=1}^{r} g_i' f_i' = 0,$$

this clearly descends to a relation in I by killing the maximal ideal of A. But now if we choose a complete set of relations for I_A, this descends to a complete set of relations in I, so we may in fact assume that (2) is exact.

In this case, there exists some L_A such that the sequence splits as

$$P_A^s \to L_A \to 0 \qquad 0 \to L_A \to P_A^r \to I_A \to 0 \qquad 0 \to I_A \to P_A \to R_A \to 0.$$

By right exactness of the tensor product, we know $P_A^s \otimes k \to L_A \otimes k \to 0$ is exact. We also know that

$$L_A \otimes k \to P_A^r \otimes k \to I_A \otimes k \to 0$$

is exact, again by right exactness. But this means that $I_A \otimes k$ is the cokernel of $P^s \to P^r$, and therefore $I_A \otimes k = I$. This means that R_A is flat. $\square$

Corollary 4.1.2. *Let $R = P/I$ and $R_A = P_A/I_A$, where $I = (f_1, \dots, f_r)$ and $I_A = (f_1', \dots, f_r')$ such that f_i' is a lift of f_i. Then R_A is flat over A if and only if every relation among the f_i lifts to a relation among the f_i'.*

Remark 4.1.3. This result essentially gives us that first-order embedded deformations of Spec $R \subset \mathbb{A}^n$ are given by $\mathrm{Hom}(I, R)$. The first-order (not embedded) deformations of Z are given by the cokernel of

$$0 \to T_X \to T_{\mathbb{A}^n}|_X \to N_{X/\mathbb{A}^n},$$

which arises from the exact sequence

$$I/I^2 \to \Omega^1_{\mathbb{A}^n}|_X \to \Omega^1_X \to 0,$$

and this is supported on the singular points of X, so when X has isolated singularities, this is finite-dimensional.

Note that if Spec $R \subset \mathbb{A}^n$ is a complete intersection, then I is generated by a regular sequence, so in particular the Koszul complex is a free resolution of R and therefore there are only trivial relations among the f_i (this means the relations are generated by $f_i f_j - f_j f_i = 0$). Clearly, because we are only considering commutative rings (after all, this is normal algebraic geometry), this means that all deformations of Spec R are unobstructed.

4.2 Hilbert schemes of smooth surfaces

We will prove that deformations of finite length closed subschemes of $\mathbb{A}^2$ are unobstructed. In particular, this will imply that the Hilbert scheme $\mathrm{Hilb}(\mathbb{A}^2, n)$ is smooth.

Let $Z \subset \mathbb{A}^2$ be a closed subscheme of dimension 0. Then because $P = k[x, y]$ has dimension 2, there exists a free resolution

$$0 \to P^s \xrightarrow{(g_{ij})} P^r \to P \to R \to 0$$

of R. In this case it is possible to understand the matrix (g_{ij}), and in fact this is the special case of a more general result. First, when we study the local behavior, we have the following result.

Theorem 4.2.1 (Hilbert, Burch). *Let P be a regular local ring of dimension n and $R = P/I$ be a Cohen-Macaulay quotient of codimension 2. Then there exists an $(r-1) \times r$ matrix $G = (g_{ij})$ whose maximal minors $f_1, \dots, f_r$ minimally generate I, and there is a free resolution*

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \to R \to 0.$$

Proof. Note that the fact that the free resolution has this length is a corollary of the Auslander-Buchsbaum formula, which says that for a ring R and module M, we have

$$\operatorname{depth} M + \operatorname{proj.dim} M = \operatorname{depth} R$$

and the fact that depth equals dimension for Cohen-Macaulay things. Thus we have a free resolution

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(a_i)} P \to R \to 0,$$

where $a_1, \dots, a_r$ are a minimal set of generators for I. Let f_i is $(-1)^i$ times the determinant of the i-th minor of g_{ij}. We will prove that the map (f_i) is the same as the map (a_i); clearly

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \to R \to 0.$$

is a resolution. This is because at the generic point of P, we know (g_{ij}) is injective, so at least one f_i is nonzero. But then we know coker(g_{ij}) is torsion-free (because I is torsion-free), and so it in fact must vanish by rank reasons. Thus $(a_1, \dots, a_r)$ and $(f_1, \dots, f_r)$ are isomorphic as P-modules.

At a codimension 1 point in $\operatorname{Spec} P$, note that $0 \to P^{r-1} \to P^r \xrightarrow{(a_i)} P \to B \to 0$ is split exact (because I has codimension 2). This implies that at least one f_i is a unit, and thus $(f_1, \dots, f_r)$ has codimension at least 2. But then the isomorphism $I \cong (f_1, \dots, f_r)$ is given by multiplication by some nonzero element of P which is a unit away from codimension 2. But this means it is a unit everywhere. □

Considering the global picture in $\mathbb{A}^n$, we obtain the following result.

Theorem 4.2.2 (Hilbert, Schaps). *Let $Z = \operatorname{Spec} R \subset \mathbb{A}^n$ be a Cohen-Macaulay closed subscheme of codimension 2. Then $R = P/I$ has a free resolution of the form*

$$0 \to P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P \to R \to 0$$

where the f_i are the maximal minors of the matrix (g_{ij}).

This result in fact holds over any Artinian local ring A, which we will use later.

Next, we want to understand what happens if we choose some Artinian local ring with residue field k and lift the g_{ij} to g'_{ij}, where $g'_{ij} \in P_A$.

Theorem 4.2.3 (Schaps). *If A is a square zero extension of k, then the sequence*

$$0 \to P_A^{r-1} \xrightarrow{(g'_{ij})} P_A^r \xrightarrow{(f'_i)} P_A \to R_A \to 0$$

is exact. Moreover, any lifting of R over A arises by lifting the matrix (g_{ij}).

Proof. We know that

$$L_A^\bullet \coloneqq P_A^{r-1} \to P_A^r \to P_A$$

is a complex. This is because composing the two maps amounts to evaluating determinants with a repeated column. Because P_A is free (and therefore flat), we can tensor with the exact sequence

$$0 \to \mathfrak{m}_A \to A \to k \to 0$$

to obtain an exact sequence of complexes

$$0 \to L^\bullet_A \otimes_A \mathfrak{m}_A \to L^\bullet_A \to L^\bullet_A \otimes_A k \to 0.$$

Note that

$$L^\bullet_A \otimes_A k = P^{r-1} \xrightarrow{(g_{ij})} P^r \xrightarrow{(f_i)} P =: L^\bullet.$$

In particular, this term is exact by Hilbert-Schaps. In addition, clearly $L^\bullet_A \otimes_A \mathfrak{m}_A = L^\bullet \otimes_k \mathfrak{m}_A$ because $A \to k$ is a square zero extension, so the complex $L^\bullet_A \otimes_A \mathfrak{m}_A$ is exact. By the long exact sequence in homology, we know that $L^\bullet_A$ is exact. Note that $L^\bullet$ extends to an exact sequence

$$0 \to P^{r-1} \to P^r \to P \to R \to 0,$$

and $L^\bullet_A$ extends to an exact sequence

$$0 \to P^{r-1}_A \to P^r_A \to P_A \to R_A \to 0.$$

However, by the homology long exact sequence, we have an exact sequence

$$0 \to R \otimes_k \mathfrak{m}_A \to R_A \to R \to 0.$$

But this implies that $R_A \otimes_A k = R$. Finally, by the local criterion for flatness, we see that R_A is flat over A.

Let $R_A = P_A/I_A$ be a lifting of R over A. Lift $f_i \in I$ to $h_i \in I_A$. By Nakayama, these generate I_A, so we obtain a free resolution

$$0 \to P^{r-1}_A \xrightarrow{(g'_{ij})} P^r_A \xrightarrow{(h_i)} P_A \to R_A \to 0,$$

where g'_{ij} lift the g_{ij}. However, we already have a lift

$$0 \to P^{r-1}_A \xrightarrow{(g'_{ij})} P^r_A \xrightarrow{(f'_i)} P_A \to R'_A \to 0,$$

and so we must show $R_A = R'_A$. But we know that the ideals $I_A = (h_1, \dots, h_r)$ and $I'_A = (f'_1, \dots, f'_r)$ are isomorphic as P_A-modules. But then if we restrict this isomorphism to $\mathbb{A}^n_A \setminus \operatorname{supp} B$, we obtain a unit in $H^0(\mathbb{A}^n_A \setminus \operatorname{supp} B, \mathcal{O}_{\mathbb{A}^n_A})$. Because functions extend over codimension 2, we have $H^0(\mathbb{A}^n_A \setminus \operatorname{supp} B, \mathcal{O}_{\mathbb{A}^n_A}) = P_A$, so this is a global unit. This gives the desired result. □

This result holds if we replace $A \to k$ with any square-zero extension of Artinian local rings $B \to A$ and P, P_A with flat things, and so we see that (embedded) deformations of codimension 2 Cohen-Macaulay subschemes of $\mathbb{A}^n$ are unobstructed. In particular, any dimension 0 closed subscheme $Z \subset \mathbb{A}^2$ is automatically Cohen-Macaulay (because it is dimension 0), so its embedded deformations are unobstructed. By some cohomological argument, the tangent space to $\operatorname{Hilb}(\mathbb{A}^2, n)$ is isomorphic to $\operatorname{Hom}(R, R)$ and has dimension $2n$, so

4.3 An obstructed deformation

Let $R = k[x, y, z]/(z^2, xy, xz, yz)$. Note that this scheme has an embedded point at the origin, so in particular it is **not** Cohen-Macaulay.

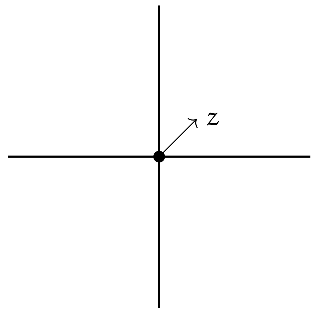

Figure 4.1: Drawing of $\operatorname{Spec} R$

We will study embedded deformations of $\operatorname{Spec} R$ and see that they are obstructed. In particular, we will choose two deformations of R over $k[\varepsilon]$ that cannot be simultaneously lifted. We claim that a complete set of relations (using the ordering (xy, xz, yz, z^2) for the generators of I) is given by the matrix

$$G = \begin{pmatrix} z & -y & 0 & 0 \\ z & 0 & -x & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y \end{pmatrix}.$$

Now a first-order deformation of $\operatorname{Spec} R$ is given by lifting (xy, xz, yz, z^2) over $k[\varepsilon]$, and the first candidate is to consider $I_{\varepsilon_1} = (xy + \varepsilon_1 y, xz, yz, z^2)$. Then we note that

$$G \begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz \\ z^2 \end{pmatrix} = \varepsilon_1 \begin{pmatrix} yz \\ yz \\ 0 \\ 0 \end{pmatrix},$$

and we can lift G to kill this vector with the matrix

$$G_{\varepsilon_1} = \begin{pmatrix} z & -y & -\varepsilon_1 & 0 \\ z & 0 & -x - \varepsilon_1 & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y \end{pmatrix} = G + \begin{pmatrix} 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & -\varepsilon_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: G + G_1.$$

Next consider the deformation given by $I_{\varepsilon_2} = (xy, xz, yz + \varepsilon_2 z, z^2)$. We note that

$$G \begin{pmatrix} xy \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} = \varepsilon_2 \begin{pmatrix} 0 \\ -xz \\ 0 \\ z^2 \end{pmatrix},$$

and we can lift G to kill this vector with the matrix

$$G_{\varepsilon_2} = \begin{pmatrix} z & -y & 0 & 0 \\ z & \varepsilon_2 & -x & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y-\varepsilon_2 \end{pmatrix} = G + \begin{pmatrix} 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: G + G_2.$$

Now we consider $I_{\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\varepsilon_2} = (xy + \varepsilon_1 y, xz, yz + \varepsilon_2 z, z^2)$ and attempt to lift this deformation to $k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$. Note that

$$\begin{aligned}(G + G_1 + G_2)\begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} &= \begin{pmatrix} z & -y & -\varepsilon_1 & 0 \\ z & \varepsilon_2 & -x-\varepsilon_1 & 0 \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y-\varepsilon_2 \end{pmatrix}\begin{pmatrix} xy + \varepsilon_1 y \\ xz \\ yz + \varepsilon_2 z \\ z^2 \end{pmatrix} \\ &= \varepsilon_1\varepsilon_2 \begin{pmatrix} -z \\ -z \\ 0 \\ 0 \end{pmatrix},\end{aligned}$$

and clearly $z \notin I$, so in fact we cannot lift this deformation to $k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_2^2)$. This proves obstructedness.