

# GOOD MODULI SPACES, POSITIVITY AND RATIONALITY

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ABSTRACT. This course will be an introduction to the theory of good moduli spaces (in the sense of Alper). We will generalize filtrations from a vector space to an algebraic variety, define theta-reductivity, S-completeness, and unpuncturedness of inertia. Examples will include good moduli spaces for Deligne-Mumford stable curves, and K-semistable Fanos, with the focus on the cases where there is a positive tautological line bundle and the good moduli space is particularly simple, i.e., rational.

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## 1. MODULI OF RATIONAL CURVES

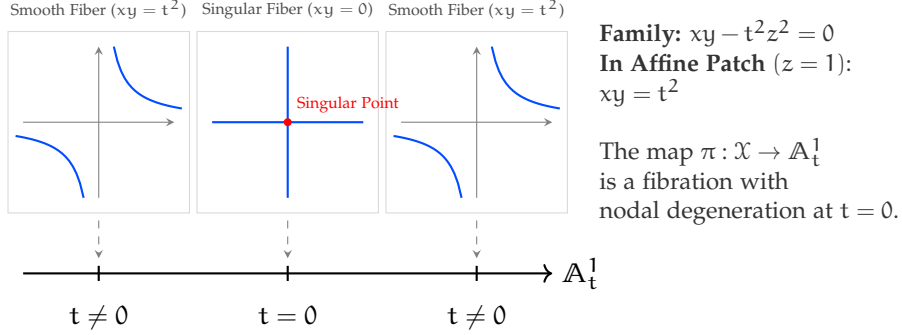
We will begin by interpolating between  $\overline{M}_{0,n}$  (which is a scheme) and the stack  $(\mathbb{P}^1)^n // \mathrm{PGL}(2)$  (which has a good moduli space by definition) via Hassett spaces  $\overline{M}_{0,\Lambda}$ . There is a single smooth rational curve, which is  $\mathbb{P}^1 = \{x = 0\} \subseteq \mathbb{P}_{x,y,z}^2$ .

Viewing the curve inside  $\mathbb{P}^2$ , we may (see Figure 1) consider the family

$$\begin{array}{ccc} \{xy - t^2z^2 = 0\} & \xlongequal{\quad} & \mathfrak{X} \hookrightarrow \mathbb{P}^2 \\ & & \downarrow \\ & & \mathbb{A}_t^1 \end{array}$$

This morphism is a (flat) isotrivial degeneration, and will be the first example of a test configuration. What this means is that there is an action of  $G_m$  on  $\mathfrak{X}$  by the formula

$$\lambda([x, y, z], t) = ([\lambda x, \lambda y, z], \lambda t)$$

FIGURE 1. The family  $\{xy - t^2 z^2 = 0\}$  over  $\mathbb{A}^1$ 

such that  $\mathcal{X} \rightarrow \mathbb{A}^1$  is equivariant. More generally, any tree of rational curves joined by nodes is also a rational curve (as in it has genus 0).

*Remark 1.1.* A **reduced** rational curve is a “tree” of  $\mathbb{P}^1$ ’s meeting in  $m$ -fold rational points, where each singularity is locally analytically isomorphic to the union of the coordinate axes in  $\mathbb{A}^m$ .

In fact, the definition of a rational curve tells about what kinds of singularities it has, and then there is a classification of curve singularities. Another viewpoint is that if we degenerate a rational normal curve in  $\mathbb{P}^m$  to a cone over a hyperplane section, we get a union of  $m$  lines meeting at a point in exactly the way described above.

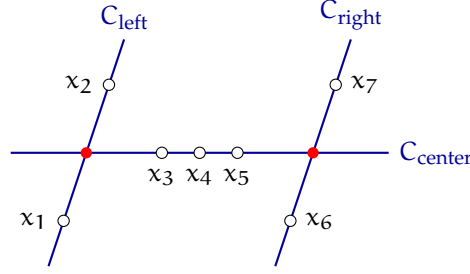
We will now consider the functor  $\tilde{\mathcal{U}}_{0,n}$  of  $n$ -pointed smooth rational curves. In particular, a family  $\mathcal{X} \rightarrow B$  must have  $n$  sections  $\sigma_1, \dots, \sigma_n: B \rightarrow \mathcal{X}$ , which are not required to be disjoint. We may now consider the larger functor  $\tilde{\mathcal{M}}_{0,n}$  of  $n$ -pointed reduced rational curves with sections  $\sigma_i$  avoiding the singularities of the fibers. Unfortunately, this is extremely non-separated, so we will consider stability conditions.

Define the functor  $\overline{\mathcal{M}}_{0,n}$  of  $n$ -pointed rational curves with at worst nodal singularities such that all sections  $\sigma_i(B)$  are disjoint and that the relative dualizing sheaf

$$\omega_{\mathcal{X}/B}(\sigma_1 + \dots + \sigma_n)$$

is relatively ample over  $B$ . One reason we consider this (and that it makes sense) is that the  $m$ -fold rational point is Gorenstein if and only if  $m \leq 2$ . This stability condition was introduced by Deligne and Mumford. In fact, the condition on the relative dualizing sheaf is equivalent to enforcing that the automorphism group of each fiber is finite, but is easier to modify for obvious reasons. From either perspective, it is clear that we must have a tree of  $\mathbb{P}^1$ ’s with at least three special points on each component. One example can be seen in Figure 2.

One way to turn a family with an  $m$ -fold rational point into a nodal one is to blow up the singular point. The remaining conditions are non-disjoint sections, which can be separated by finitely many blowups. Finally, the unstable



Configuration in  $\overline{M}_{0,7}$   
 Dual Graph: Tree (Genus 0)  
 Stability: Each component has  $\geq 3$   
 special points (nodes + marks).

FIGURE 2. An example of a stable curve of genus 0 with marked points

components can be contracted by the basic theory of surfaces (they have self-intersection either  $-1$  or  $-2$ ). In particular, we have sketched a proof that  $\overline{M}_{0,n}$  is universally closed. In fact, it is also separated, but we will omit the proof here.

Suppose we have a family  $\pi: \mathcal{X} \rightarrow B$  in  $\widetilde{M}_{0,n}$ . Then we have  $n+1$  line bundles on  $B$  where  $n$  of them are given by

$$\sigma_i^* \omega_{\mathcal{X}/B} = \sigma_i^* (N_{\sigma_i(B)/\mathcal{X}}^\vee)$$

(this is fine even in the non-Gorenstein case because the sections avoid the singularities) and the last is given by

$$\delta = \pi_*(c_2(\Omega_{\mathcal{X}/B}^1)),$$

which counts singularities. For example, if  $B$  is a smooth projective curve, then

$$\deg \delta = \sum_m m \cdot (\#\{m\text{-fold singularities}\}).$$

**Theorem 1.2.** *The line bundle*

$$\psi - \delta = \left[ \sum_{i=1}^n \psi_i \right] - \delta$$

*is ample on  $\overline{M}_{0,n}$ .*

*Remark 1.3.* Projectivity of  $\overline{M}_{0,n}$  can be proved in more synthetic ways, for example via the explicit blowup construction of Keel.

**1.1. Hassett moduli spaces.** There is a modification of this, which is the functor  $\overline{M}_{0,A}$ , where

$$A = (\alpha_1, \dots, \alpha_n) \in (0, 1] \cap \mathbb{Q}.$$

Here, we consider  $n$ -pointed rational curves which are at worst nodal, but where the sections  $\sigma_{i_1}, \dots, \sigma_{i_k}$  can meet if and only if

$$\alpha_{i_1} + \dots + \alpha_{i_k} \leq 1$$

and the condition on the dualizing sheaf is that  $\omega_{\mathcal{X}/B}(\sum_i \alpha_i \sigma_i)$  is ample. Projectivity of this is proven using similar methods to  $\overline{M}_{0,n}$ .

More specifically, the functor is given by

$$\overline{M}_{0,A}(T) = \left\{ \begin{array}{c} \mathcal{C} \\ \pi \downarrow \uparrow \sigma_1, \dots, \sigma_n \\ T \end{array} \middle| \begin{array}{l} \text{fibers at worst nodal} \\ \omega_{\mathcal{C}/T}(\sum \alpha_i \sigma_i) \text{ is } \pi\text{-ample} \\ \text{sections with sum of weights} > 1 \text{ cannot meet} \end{array} \right\}.$$

This functor is actually representable by a scheme because the three conditions imply that there are no automorphisms.

**Example 1.4.** If  $A = (1, \dots, 1)$  or more generally if the sum of any two weights is strictly greater than 1, then  $\overline{M}_{0,A} = \overline{M}_{0,n}$ .

**Example 1.5.** If  $A = (1, \varepsilon, \dots, \varepsilon)$  such that  $1 - (n-1)\varepsilon > 2$  and  $\varepsilon \leq \frac{1}{n-2}$ , then the combinatorics imply that all marked points lie on the same component. If we call the heavy marked point  $\infty$ , then all of the light markings lie in  $\mathbb{A}^1$ . By translation invariance, we may assume that

$$x_1 + \dots + x_{n-1} = 0,$$

and therefore we obtain

$$[x_1 : \dots : x_{n-1}] \in \mathbb{C}^{n-2} \setminus \{0\} / \mathbb{C}^\times \simeq \mathbb{P}^{n-3}.$$

*Remark 1.6.* There exists a morphism  $\overline{M}_{0,n} \rightarrow \overline{M}_{0,A}$  given (when the base is a curve) by changing the weights and contracting all of the  $(-1)$ -curves in the fibers where the total weight of the sections has weight at most 1.

**Example 1.7.** If we set  $A = (\varepsilon, \dots, \varepsilon)$  such that  $\varepsilon \approx \frac{2}{n} + \varepsilon'$ , then the curve must be a  $\mathbb{P}^1$  with  $n$  points such that we cannot have  $\lceil \frac{2}{n} \rceil$  colliding. More concretely, if  $n = 2m + 1$  and  $\varepsilon = \frac{1}{m}$ , then we have a  $\mathbb{P}^1$  with  $2m + 1$  marked points such that at most  $m$  can collide. In particular, we have

$$\begin{aligned} \overline{M}_{0,A} &= \frac{(\mathbb{P}^1)^{2m+1} \setminus \{(m+1)\text{-fold diagonals}\}}{\mathrm{PGL}(2)} \\ &= ((\mathbb{P}^1)^{2m+1})^{\mathrm{ss}} // \mathrm{PGL}(2), \end{aligned}$$

where the linearization is the tensor product of the  $\mathcal{O}(1)$  on each factor. For an example when  $m = 7$ , see Figure 3.

**1.2. Semistable objects.** Now we will define a new moduli space

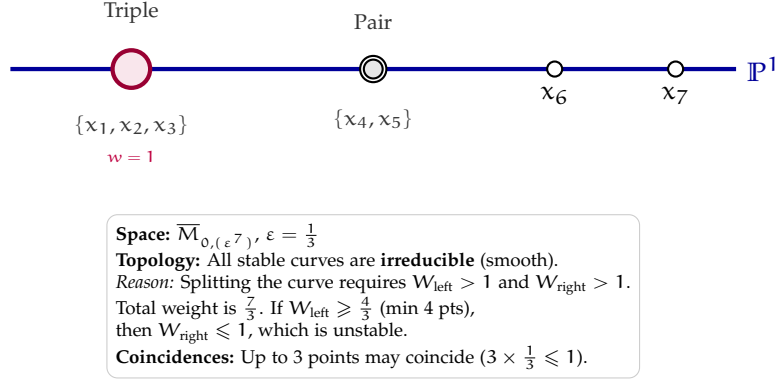
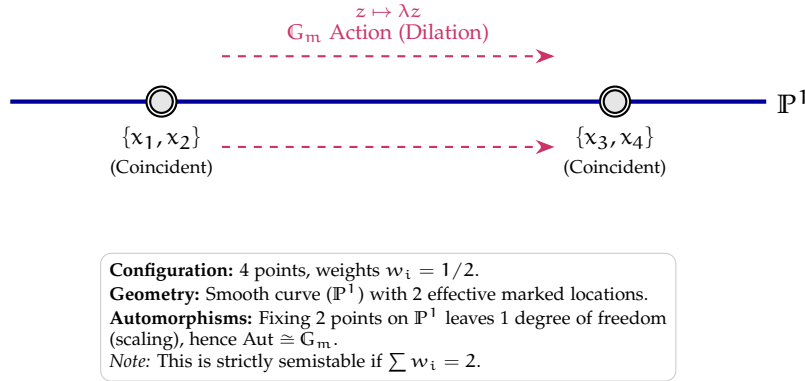
$$\overline{M}_{0,A}^*(T) = \left\{ \begin{array}{c} \mathcal{C} \\ \pi \downarrow \uparrow \sigma_1, \dots, \sigma_n \\ T \end{array} \middle| \begin{array}{l} \text{fibers at worst nodal} \\ \omega_{\mathcal{C}/T}(\sum \alpha_i \sigma_i) \text{ is } \pi\text{-nef} \\ \text{sections with sum of weights} > 1 \text{ cannot meet} \end{array} \right\}$$

by relaxing the ampleness condition. This can happen if we have sections whose weights sum to exactly 1.

**Example 1.8.** If  $n = 2m$  and  $A = (\frac{1}{m}, \dots, \frac{1}{m})$ , then we have

$$\overline{M}_{0,A}^* = (\mathbb{P}^1)^{2m} // \mathrm{PGL}(2).$$

This is not a scheme and does not even have a coarse moduli space because the curve where the first  $m$  markings collide and the last  $m$  markings collide has an entire  $G_m$  of automorphisms. For an example when  $n = 4$ , see Figure 4.

FIGURE 3. An object in  $\overline{M}_{0,(\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10})}$ FIGURE 4. A strictly semistable object in  $\overline{M}_{0,\frac{1}{2}}^*$ .

The Hassett moduli spaces have a wall-and chamber-structure, where for example, if  $A = (\alpha, \dots, \alpha)$  is  $S_n$ -symmetric, then  $\overline{M}_{0,A}$  is constant for  $\alpha \in (\frac{1}{k+1}, \frac{1}{k})$ , but on the walls we only have  $\overline{M}_{0,A}^*$ . The main observation is that if we write  $\overline{M}_{0,\alpha}$  for  $\overline{M}_{0,(\alpha, \dots, \alpha)}$ , then there are open immersions

$$\overline{M}_{0,\frac{1}{k}-\varepsilon} \subset \overline{M}_{0,\frac{1}{k}}^* \supset \overline{M}_{0,\frac{1}{k}+\varepsilon}.$$

If we consider the complement  $\overline{M}_{0,\frac{1}{k}}^* \setminus \overline{M}_{0,\frac{1}{k}-\varepsilon}$ , then it consists of curves which contain an irreducible component which is a tree in the dual graph and contains  $k$  marked points, so this part of the moduli space has a factor which describes the rest of the curve and one which is  $[\mathbb{A}^{k-2}/G_m]$ . More precisely, we have

$$\overline{M}_{0,\frac{1}{k}}^* \setminus \overline{M}_{0,\frac{1}{k}-\varepsilon} = \overline{M}_{0,(1,\frac{1}{k}, \dots, \frac{1}{k})}^* \times \overline{M}_{0,(1,\frac{1}{k}, \dots, \frac{1}{k})}^*$$

This actually allows us to construct a test configuration, where if we have  $\mathbb{P}^1 \times \mathbb{A}^1$  with a bunch of sections, we can blow up a point on the central fiber and then contract the central  $(-1)$ -curve with all of the sections.

**Exercise 1.9.** Figure out what  $\overline{\mathcal{M}}_{0, \frac{1}{k}}^* \setminus \overline{\mathcal{M}}_{0, \frac{1}{k} + \varepsilon}$  is.

For now, we will denote this picture by  $\mathcal{M}^- \subset \mathcal{M} \supset \mathcal{M}^+$ . Then in fact we actually have

$$\mathcal{M} \setminus \mathcal{M}^+ = \overline{\mathcal{M}}_{0, (1, \frac{1}{k}, \dots, \frac{1}{k})} \times [\mathbb{C}/\mathbb{C}^\times],$$

where the second factor comes from the place where  $k$  of the sections collide. In this picture, we can degenerate out a new rational tail which has only the  $k$  colliding sections (which has a  $\mathbb{C}^\times$ -action), which is another example of a test configuration, as in Figure 5.

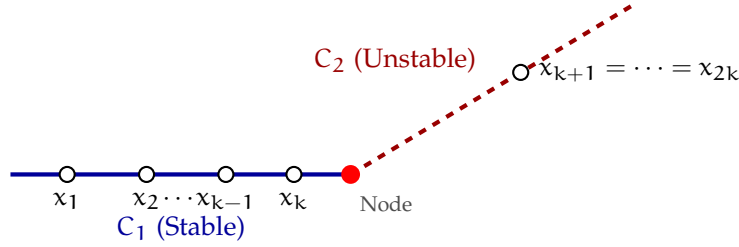


FIGURE 5. A curve that we bubble out

Later, when we discuss good moduli spaces, there is a diagram

$$\begin{array}{ccccc} \mathcal{M}^+ & \longleftrightarrow & \mathcal{M} & \longleftrightarrow & \mathcal{M}^- \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}^+ & \longrightarrow & \mathcal{M} & \xleftarrow{\sim} & \mathcal{M}^-, \end{array}$$

where the rightward arrow in the bottom row is a divisorial contraction.

In fact, there is a series of divisorial contractions

$$\overline{\mathcal{M}}_{0,n} \rightarrow \cdots \rightarrow \overline{\mathcal{M}}_{0,\mathcal{A}} \rightarrow \cdots \rightarrow (\mathbb{P}^1)^n // \mathrm{PGL}(2)$$

and another series of divisorial contractions

$$\overline{\mathcal{M}}_{0,n} \rightarrow \cdots \rightarrow \overline{\mathcal{M}}_{0,\mathcal{A}} \rightarrow \cdots \rightarrow \mathbb{P}^{n-3}.$$

**1.3. Divisors on  $\overline{\mathcal{M}}_{0,n}$ .** There are boundary divisors  $\Delta_{I,J}$  where  $I \sqcup J = \{1, \dots, n\}$  where the curve degenerates into two components, one with the markings from  $I$  and the other with the markings from  $J$ . It is clear that

$$\Delta_{I,J} \simeq \overline{\mathcal{M}}_{0,|I|+1} \times \overline{\mathcal{M}}_{0,|J|+1}.$$

We will denote their union by  $\Delta$  and call it the total boundary.

It is easy to see that  $\overline{\mathcal{M}}_{0,n} \setminus \Delta = \mathcal{M}_{0,n}$  consists only of smooth curves and is given by

$$\mathcal{M}_{0,n} = (\mathbb{C}^\times \setminus 1)^{n-3} \setminus \text{all diagonals}.$$

This has no Picard group, so the entire Picard group of  $\overline{\mathcal{M}}_{0,n}$  is generated by the boundary divisors. For example,  $\overline{\mathcal{M}}_{0,3}$  is a point,  $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$ , and  $\overline{\mathcal{M}}_{0,5}$  is  $\mathbb{P}^1 \times \mathbb{P}^1$  blown up at three points.

**Exercise 1.10.** Compute the Picard rank of  $\overline{\mathcal{M}}_{0,n}$ .

If we consider a 1-dimensional base  $B$ , then basic the theory of surfaces tells us that  $\psi_{i|_B} = -\sigma_i^2$ , and if we blow down all rational tails in  $\mathcal{C}$  to form a new  $\mathbb{P}^1$ -bundle  $\mathcal{D} \rightarrow B$  which has a section  $\sigma_1$  with self-intersection  $-r$  and a number of other sections  $\tilde{\sigma}_i$  with self-intersection  $+r$ . If we now blow up the other collision points, then we see that

$$(\psi_1 + \psi_i)|_B = \sum_{\substack{I \in I \\ i \in J}} \Delta_{I \sqcup J} \cdot B.$$

In particular, because  $\overline{M}_{0,n}$  is smooth, we have derived Keel's relation, which is that for all  $i \neq j$ , then

$$\psi_i + \psi_j \sim \sum_{\substack{I \in I \\ j \in J}} \Delta_{I \sqcup J}.$$

These in fact generate all relations.

We now consider the morphism  $f_{n+1}: \overline{M}_{0,n+1} \rightarrow \overline{M}_{0,n}$  given by deleting the last marked point and stabilizing. This is in fact the universal family over  $\overline{M}_{0,n}$ , so in particular it has  $n$  disjoint sections  $\sigma_1, \dots, \sigma_n$ . From this perspective, we can construct  $\overline{M}_{0,5}$  by blowing up the points on  $\mathbb{P}^1 \times \mathbb{P}^1$  where the diagonal intersects the three horizontal sections corresponding to  $0, 1$ , and  $\infty$ . If we blow down the four  $(-1)$ -curves corresponding to the diagonal section and the three fibers that we blew up, the family becomes the pencil of conics passing through four points on  $\mathbb{P}^2$  and the reducible fibers are in fact the three degenerate members (see Figure 6). In particular, we see that  $\text{Pic}(\overline{M}_{0,5}) = \mathbb{Z}^5$ .

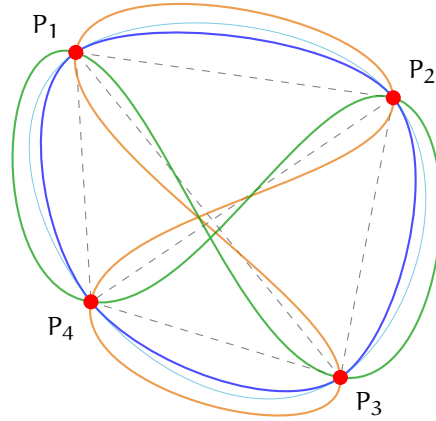


FIGURE 6. Pencil of conics through four points in general position on  $\mathbb{P}^2$ . The three degenerate conics correspond to pairs of lines through these points (dashed).

Another way to see this is to consider the sequence

$$\overline{M}_{0,n} \simeq \overline{M}_{0,(\frac{1}{2}, \dots, \frac{1}{2})} \rightarrow \overline{M}_{0,(\frac{1}{3}, \dots, \frac{1}{3})} \rightarrow \dots \rightarrow (\mathbb{P}^1)^n // \text{PGL}(2)$$

and use this to compute the Picard rank, which is

$$\rho(\overline{M}_{0,n}) = 2^{n-1} - \binom{n}{2} - 1.$$

A more precise description is that

$$\text{Pic}(\overline{M}_{0,n}) = \mathbb{Z}\langle \Delta_{I \sqcup J}, \psi_1, \dots, \psi_n \rangle / \text{Keel}.$$

The blowup construction (really the one that starts with  $\mathbb{P}^{n-3}$ ) proves that the Picard group also has no torsion.

One way to find a basis for the Picard group is the following. We will call  $\Delta_{I \sqcup J}$  *non-adjacent* if  $I$  is not an arc when we arrange  $1, \dots, n$  in a circle.

**Proposition 1.11.** *Non-adjacent boundary divisors form a basis of  $\text{Pic}(\overline{M}_{g,n})$ .*

*Proof.* We use Keel's relations to rewrite every divisor as a linear combination of non-adjacent ones, and then we check that the number of non-adjacent ones is exactly the Picard rank.

For example, if we have  $I = \{1, 2\}$ , we will use the relations for  $\psi_1 + \psi_2$ ,  $\psi_3 + \psi_n$  (with a plus sign),  $\psi_1 + \psi_n$ , and  $\psi_2 + \psi_3$  (with a minus sign), then the left hand side vanishes while the only adjacent boundary divisor remaining is the one with  $I = \{1, 2\}$ . The general case of this is left as an exercise, as is counting the non-adjacent boundary divisors.  $\square$

We have discussed the fact that  $\overline{M}_{0,n}$  is a projective variety. We know it is proper, but we would like to produce an ample line bundle. One candidate is

$$\psi - \Delta := \psi_1 + \dots + \psi_n - \sum \Delta_{I \sqcup J}.$$

The first step to proving that this is ample is to sum all of Keel's relations, which gives us

$$(n-1)\psi = |I||J|\Delta_{I \sqcup J}.$$

Because  $|I||J| \geq 2(n-2) > n-1$  (here  $n \geq 4$ ), we see that

$$\psi - \Delta = \sum c_{I \sqcup J} \Delta_{I \sqcup J},$$

where all coefficients are positive.

We will now use the ampleness criterion and show that  $(\psi - \Delta) \cdot B > 0$  for any irreducible proper curve  $B \hookrightarrow \overline{M}_{0,n}$ . If  $B$  intersects the interior nontrivially, then by definition it has to intersect the boundary with positive degree. The other case is when  $B \hookrightarrow \Delta_{I \sqcup J}$  lives inside one of the boundary divisors.

In this case, we have

$$B \hookrightarrow \Delta_{I \sqcup J} \simeq \overline{M}_{0,k+1} \times \overline{M}_{0,n-k+1}.$$

Restricting  $\psi - \Delta$ , we obtain

$$(\psi - \Delta)|_{\Delta_{I \sqcup J}} \simeq (\psi - \Delta)|_{\overline{M}_{0,k+1}} \boxtimes (\psi - \Delta)|_{\overline{M}_{0,n-k+1}},$$

so we induct using the fact that on  $\overline{M}_{0,4} \simeq \mathbb{P}^1$ ,  $\psi - \Delta = \mathcal{O}(1)$ .



## 2. GEOMETRIC INVARIANT THEORY

**2.1. Groupoids and coarse moduli.** We will now recall the Yoneda embedding, which is the fully faithful functor

$$\mathrm{Sch}_S \rightarrow \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{Set}), \quad X \mapsto h_X = \mathrm{Mor}_{\mathrm{Sch}_S}(-, X).$$

**Definition 2.1.** A scheme  $G$  is a *group scheme* if the functor  $h_G$  factors through the embedding  $\mathrm{Grp} \hookrightarrow \mathrm{Set}$ .

**Example 2.2.**

- (1) There is the group scheme  $G_m = S \times \mathbb{Z}[x, x^{-1}]$  which sends a scheme  $T$  to  $H^0(T, \mathcal{O}_T)^\times$ .
- (2) There is the group scheme  $G_a = S \times \mathbb{Z}[x]$  which sends a scheme  $T$  to  $H^0(T, \mathcal{O}_T)$  with addition.

**Definition 2.3.** A scheme  $G$  is a *group scheme* if there exist morphisms

$$\mu: G \times_S G \rightarrow G, \quad \beta: G \rightarrow G, \quad e: S \rightarrow G$$

satisfying the usual group axioms:

- (1) The diagram

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{1 \times \mu} & G \times G \\ \downarrow \mu \times 1 & & \downarrow \mu \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

commutes (associativity);

- (2) The diagram

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G \times G & \xrightarrow[\beta \times 1]{1 \times \beta} & G \times G \\ \downarrow & & & & \downarrow \mu \\ S & \xrightarrow{e} & & & G \end{array}$$

commutes (inverse axiom).

- (3) The diagram

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & \text{---} & \searrow & \\ G & \xrightarrow{\quad} & G \times_S S & \xrightarrow{1 \times e} & G \times_S G \xrightarrow{\mu} G \\ & \searrow & & \nearrow & \\ & & S \times_S G & \xrightarrow{e \times 1} & \end{array}$$

commutes (identity axiom).

**Example 2.4.** If  $k$  is algebraically closed, then a (Zariski)-closed subgroup of  $\mathrm{GL}_n(k)$  is a group scheme over  $k$ , and is known as a *linear algebraic group*.

**Definition 2.5.** An *algebraic group* over  $k$  is a smooth group scheme over  $k$ , and if it is affine, we call it *linear*.

**Definition 2.6.** An *action* of an algebraic group  $G$  on a scheme  $X$  is a morphism  $\sigma: G \times X \rightarrow X$  such that the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mu \times 1} & G \times X \\ \downarrow 1 \times \sigma & & \downarrow \sigma \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

commutes and the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\simeq} & S \times X & \xrightarrow{e \times 1} & G \times X & \xrightarrow{\sigma} & X \\ & & & & \searrow & \nearrow & \\ & & & & & 1_X & \end{array}$$

commutes.

On  $k$ -points over an algebraically closed field, this says that  $g(g'(x)) = (gg')(x)$  and that  $e \cdot x = x$  which are the usual group action axioms.

**Construction 2.7** (Action groupoid). Suppose that  $G$  acts on  $X$ . Define  $R := G \times X$ ,  $s := \text{pr}_2: R \rightarrow X$  (source morphism), and  $t := \sigma: R \rightarrow X$  (target morphism). Then set  $j := (s, t): R \rightarrow X \times X$ . There is a morphism

$$c: R \times_{(s,t)} R \simeq G \times G \times X \xrightarrow{\mu \times 1} G \times X \cong R$$

giving composition. Finally there is the morphism

$$e: X \simeq S \times X \xrightarrow{e \times 1} G \times X \simeq R$$

and the reflexivity morphism

$$i: R \xrightarrow{\beta \times 1_X} R \simeq G \times X.$$

This construction gives a groupoid in schemes called the *action groupoid* associated to the action of  $G$  on  $X$ .

More generally, if we replace  $G \times X$  by an arbitrary  $R$ , we obtain the general notion of a *groupoid* in the category of schemes. Our goal will be to prove the following theorems.

**Theorem 2.8.** Let  $j: R \rightarrow X \times X$  be a flat groupoid such that the stabilizer  $j^{-1}(\Delta_X) \rightarrow \Delta_X \simeq X$  is finite. Then there exists an algebraic space “quotient”  $X/R$  and a quotient morphism  $q: X \rightarrow X/R$  which is a uniformly categorical and geometric quotient. Moreover, if  $j$  is finite, then  $X/R$  is separated.

**Definition 2.9.** Let  $\sigma: G \times X \rightarrow X$  be a group action. Then  $\phi: X \rightarrow Y$  is a *geometric quotient* if the square in the diagram

$$\begin{array}{ccc} T \times X & \xrightarrow{\sigma} & X \\ \downarrow \text{pr}_2 & & \downarrow \phi \\ X & \xrightarrow{\phi} & Y \end{array} \quad \begin{array}{c} \searrow \psi \\ \downarrow \exists! \\ \downarrow \psi \end{array}$$

commutes and the dashed arrow is unique for any commutative diagram where the  $\psi$  exists. The categorical quotient is *uniform* if it is preserved by flat base change and *universal* if it is preserved by arbitrary base change.

Naively, we may form a quotient pre-sheaf

$$X/R: \text{Sch}_S^{\text{op}} \rightarrow \text{Set}, \quad T \mapsto X(T)/\sim,$$

where  $x \sim x'$  if there exists  $g \in G(T)$  such that  $g \cdot x = x'$ . However, this is not a sheaf in general (and is definitely not a scheme), so we need to improve our constructions.

**2.2. Good quotients.** Let  $S = \text{Spec } k$  for an algebraically closed field  $k$  and  $G$  be an algebraic group over  $k$ . Suppose that  $X$  is a scheme with an action

$$\sigma: G \times X \rightarrow X$$

of  $G$ . Now suppose we have another action

$$\sigma': G \times X' \rightarrow X'$$

on another scheme  $X'$ .

**Definition 2.10.** A morphism  $f: X \rightarrow X'$  is *G-invariant* if the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ 1 \times f \downarrow & & \downarrow f \\ G \times X' & \xrightarrow{\sigma'} & X' \end{array}$$

commutes.

In the finite-type case, this reduces to the equation

$$g \circ g(x) = f(g \circ x)$$

for all  $k$ -points  $g \in G(k)$  and  $x \in X(k)$ .

**Definition 2.11.** If  $G$  acts trivially on  $Y$ , then a  $G$ -equivariant  $f: X \rightarrow Y$  is *G-invariant*.

In particular,  $f$  has to collapse all orbits of  $G$  to points.

**Definition 2.12.** A morphism  $\phi: X \rightarrow Y$  is a *categorical quotient* of  $X$  by  $G$  if it is  $G$ -invariant and for any other  $G$ -invariant morphism  $\psi$ , there exists a unique  $g: Y \rightarrow Y'$  such that  $\psi = g \circ \phi$ , or in other words the dashed arrow in

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & \searrow \psi & \downarrow g \\ & & Y' \end{array}$$

exists and is unique.

**Exercise 2.13.** If  $X$  has property  $\mathcal{P}$  being one of normal, irreducible, reduced, connected, or locally factorial, and if  $\phi: X \rightarrow Y$  is a categorical quotient, then  $Y$  also has  $\mathcal{P}$ .

*Remark 2.14.* If  $G$  acts on  $X$  and  $U \subset X$  is a  $G$ -invariant open subset, then the group  $G(k)$  acts on  $\mathcal{O}_X(U)$ , which is a  $k$ -algebra.

**Definition 2.15.** Suppose that  $G$  acts on  $X$ . A morphism  $\pi: X \rightarrow Y$  is a *good quotient* if

- (1) It is  $G$ -invariant;
- (2)  $\pi$  is affine and surjective;
- (3) If  $U \subset Y$  is open, the morphism of rings  $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\pi^{-1}(U))^G$  is an isomorphism;
- (4) If  $W_1$  and  $W_2$  are disjoint  $G$ -invariant closed subsets in  $X$ , then  $\pi(W_1) \cap \pi(W_2) = \emptyset$  and the two images are closed.

**Proposition 2.16.** Every good quotient is a categorical quotient.

*Proof.* Suppose that  $\pi: X \rightarrow Y$  is a good quotient and  $\phi: X \rightarrow Z$  is  $G$ -invariant. We will construct a morphism  $g: Y \rightarrow Z$  filling in the dashed arrow in

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & \searrow \phi & \downarrow g \\ & & Y'. \end{array}$$

Let  $\{V_i\}$  be a cover of  $Z$  by affine open subschemes.

The first claim we will prove is that there exists an open  $U_i \subset Y$  such that  $\pi^{-1}(U_i) = \phi^{-1}(V_i)$ . If  $x \in V_i$  is a closed point and we set  $W = Z \setminus V_i$ , then  $\phi^{-1}(x)$  and  $\phi^{-1}(W)$  are disjoint  $G$ -invariant closed subsets of  $X$ . This is true for all  $x \in V_i$ , so we see that  $\phi^{-1}(V_i) = \pi^{-1}(U_i)$  for some  $U_i \subset X$ . Because  $Y \setminus U_i = \pi(\phi^{-1}(W))$  is closed,  $U_i$  must be open.

We now treat  $U_i$  as a subset of  $Y$ . We need to construct a morphism  $g_i: U_i \rightarrow V_i = \text{Spec } A_i$ . There is a map

$$A_i \rightarrow \mathcal{O}_X(\phi^{-1}(V_i))^G = \mathcal{O}_X(\pi^{-1}(U_i))^G \simeq \mathcal{O}_Y(U_i),$$

which is the desired morphism. These morphisms glue, so we are done.  $\square$

We will denote good quotients by  $X // G$ .

**Lemma 2.17.** If  $\pi: X \rightarrow X // G$  is a good quotient, then for all closed  $y \in X // G$ , the fiber  $\pi^{-1}(y)$  contains a unique closed  $G$ -orbit.

*Proof.* Consider the orbit  $G \cdot x \subset \pi^{-1}(y)$  of the smallest dimension. Then  $\overline{G \cdot x} \setminus G \cdot x$  is a union of orbits, so it must be empty.  $\square$

**Corollary 2.18.** The set  $X // G(k)$  is in bijection with the set of closed orbits in  $X$ .

**Definition 2.19.** A good quotient is *geometric* if all orbits of closed points are closed. Equivalently,  $X // G(k)$  is in bijection with the set of  $G(k)$ -orbits in  $X(k)$ . A third equivalent definition is that  $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$  for all  $x_1, x_2 \in X(k)$  in disjoint orbits.

**Lemma 2.20.** Suppose that  $f: X \rightarrow Y$  is finite and  $G$ -equivariant. Then  $G \cdot x$  is closed if and only if  $G \cdot f(x)$  is closed.

*Proof.* Note that equivariance implies that  $G \cdot f(x) = f(G \cdot x)$ . If  $G \cdot x$  is closed, then so is  $G \cdot f(x)$  by finiteness. If  $G \cdot x$  is not closed and there exists  $x' \in \overline{G \cdot x} \setminus G \cdot x$  but  $G \cdot f(x)$  is closed, then  $\dim G \cdot x' < \dim G \cdot x$ . The sequence of containments

$$G \cdot f(x') = f(G \cdot x') \subset f(\overline{G \cdot x}) = \overline{G \cdot f(x)} = G \cdot f(x)$$

implies that  $G \cdot f(x') = G \cdot f(x)$ , contradicting the dimension inequality.  $\square$

**Definition 2.21.** Let  $X$  be a  $G$ -scheme. A  $G$ -invariant subset  $U \subseteq X$  is *saturated* if for all  $x \in U$  and  $y \in X$  such that  $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$ , then  $y \in U$ .

**Example 2.22.** Suppose we have  $f_1, \dots, f_n \in \mathcal{O}_X(X)^G$ . Then the open set

$$D(f_1, \dots, f_n) = D(f_1) \cap \dots \cap D(f_n)$$

is saturated.

**Exercise 2.23.** Suppose that  $\pi: X \rightarrow X // G$  is a good quotient. Then a  $G$ -invariant subset  $U \subseteq X$  is saturated if and only if  $U = \pi^{-1}(\pi(U))$ .

One major statement we want to prove is the following.

**Theorem 2.24** (Affine GIT). *Suppose that  $G$  is a linearly reductive group acting on an affine scheme  $X = \text{Spec } R$ . Then*

$$\pi: X \rightarrow Y = \text{Spec } R^G$$

*is a universal good quotient.*

**Definition 2.25.** A group scheme  $G$  is *linearly reductive* if every finite-dimensional rational representation of  $G$  is completely reducible.

*Remark 2.26.* In characteristic zero, a group  $G$  is reductive (equivalently, linearly reductive), if and only if  $BG = [\bullet/G] \rightarrow \bullet$  is a good moduli space.

**Definition 2.27.** Let  $V$  be a finite-dimensional  $G$ -representation. Consider the vector space

$$V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}.$$

By complete reducibility, there is a direct sum decomposition  $V = V^G \oplus V'$ , so we define a map

$$\rho: V \rightarrow V^G$$

by projecting onto the first factor. This is clearly  $G$ -equivariant, surjective, and idempotent, and is called the *Reynolds operator*.

By Schur's lemma, the diagram

$$\begin{array}{ccc} V & \xrightarrow{\psi} & W \\ \downarrow \rho & & \downarrow \rho \\ V^G & \longrightarrow & W^G \end{array}$$

commutes.

By gluing Reynolds operators on finite-dimensional  $G$ -representations together, we obtain a Reynolds operator on infinite-dimensional rational representations, and in particular on  $k$ -algebras  $R$  as

$$\rho: R \rightarrow R^G.$$

*Remark 2.28.* Note that  $R^G$  is a  $k$ -subalgebra of  $R$ .

Let  $G$  be a linear algebraic group. Then if  $G = \text{Spec } S$  (note here that  $S$  is a Hopf algebra), an action

$$G \times \text{Spec } R \rightarrow \text{Spec } R$$

of  $G$  on an affine scheme  $\text{Spec } R$  corresponds to a coaction

$$R \rightarrow R \otimes_k S.$$

Composing this with  $g^*: S \rightarrow k$  corresponding to  $g \in G(k)$ , we obtain an action

$$g^*: R \rightarrow R.$$

This defines a  $k$ -linear action

$$G(k) \times R \rightarrow R \quad (g, f) \mapsto g^*(f).$$

**Definition 2.29.**  $G(k)$  acts *rationaly* on  $R$  if every element of  $R$  lies in a finite-dimensional  $G(k)$ -invariant subspace of  $R$ .

**Lemma 2.30.** The action  $G(k) \times R \rightarrow R$  coming from the action of  $G$  on  $\text{Spec } R$  is rational.

*Idea of proof.* For any  $r \in R$ , consider the coaction

$$r \in R \mapsto \sum_i r_i \otimes s_i \in R \otimes_k S.$$

Then  $r$  lies in the span of the  $r_i$ , which is finite-dimensional and  $G(k)$ -invariant.  $\square$

This clarifies what we meant by gluing Reynolds operators together. By complete reducibility, we can write  $R = R^G \oplus R'$ .

**Claim 2.31.**  $R'$  is an  $R^G$ -module and the direct sum decomposition holds at the level of  $R^G$ -modules.

*Proof.* We need to prove that  $R^G R' \subseteq R'$ . Choose a finite-dimensional subrepresentation  $V \subseteq R'$  and  $r \in R'$ . We want to prove that  $rV \subseteq R'$ . Now consider the sequence

$$V \xrightarrow{r} rV \xrightarrow{\rho} (rV)^G \subseteq R^G.$$

Because  $V$  is a direct sum of nontrivial representations and  $R^G$  is a trivial representation, the composition is zero by Schur's lemma.  $\square$

Now consider the Cartesian diagram

$$\begin{array}{ccc} \text{Spec } T \otimes_{R^G} R & \longrightarrow & \text{Spec } R \\ \downarrow & & \downarrow \\ \text{Spec } T & \longrightarrow & \text{Spec } R^G. \end{array}$$

**Corollary 2.32.** For all  $R^G$ -algebras  $T$ , we have

- (1) A canonical isomorphism  $(T \otimes_{R^G} R)^G \cong T$ .
- (2) Suppose that  $\{I_i\}$  are  $G$ -invariant ideals of  $R$ . Then

$$\left( \sum I_i \right) \cap R^G = \sum (I_i \cap R^G) = \sum I_i^G.$$

*Proof.*

- (1) Because  $R \simeq R^G \oplus R'$  as  $R^G$ -modules, we have

$$T \otimes_{R^G} R \simeq T \oplus (T \otimes_{R^G} R').$$

Applying the Reynolds operator, we see that

$$(T \otimes_{R^G} R)^G = \rho(T) \oplus (T \otimes_{R^G} \rho(R')) = T.$$

- (2) We use the fact that  $\rho(\sum I_i) = \sum \rho(I_i)$ .  $\square$

*Proof of Theorem 2.24.* The morphism is clearly  $G$ -invariant and affine. We will begin with surjectivity. Choose  $T = k$ , so we are looking at a  $k$ -point of  $\text{Spec } R^G$ . We only need to prove that  $k \otimes_{R^G} R$  is nonzero, but we already know that  $(k \otimes_{R^G} R)^G = k$  is nonzero. In particular, this means that every fiber is nonempty.

Now let  $\text{Spec } T$  be an open affine in  $\text{Spec } R^G$ . Now we see that

$$\mathcal{O}_Y(T) = T = (T \otimes_{R^G} R)^G = \pi_* \mathcal{O}_X(T),$$

and this holds for all open affines, so it extends to an isomorphism

$$\mathcal{O}_Y \simeq (\pi_* \mathcal{O}_X)^G$$

of sheaves.

Now let  $W_i = V(I_i)$  be disjoint  $G$ -invariant closed subsets of  $X$ . Then  $I_1$  and  $I_2$  are  $G$ -invariant ideals, and in particular

$$\overline{\pi(W_i)} = V(I_i \cap R^G).$$

This implies that

$$\overline{\pi(W_1)} \cap \overline{\pi(W_2)} = \overline{\pi(W_1 \cap W_2)},$$

so the closures are disjoint. Finally, if we choose  $W_2 = \pi^{-1}(\text{pt})$ , we see that  $\pi(W_1)$  is closed.  $\square$

Now if we have a  $G$ -invariant ideal  $I \subseteq R$ , the sequence

$$0 \rightarrow I^G \rightarrow R^G \rightarrow R^G/I^G \rightarrow 0$$

is also exact. In particular, if we have a  $G$ -invariant closed subscheme  $Z \hookrightarrow X$ , then  $Z \rightarrow \pi(Z)$  is also a good quotient and the diagram

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \pi(Z) & \hookrightarrow & Y \end{array}$$

commutes.

**Lemma 2.33.** *If  $X = \text{Spec } R$  is Noetherian, then so is  $Y = X // G = \text{Spec } R^G$ .*

*Proof.* If  $W \subseteq Y$  is closed, then  $\pi(\pi^{-1}(W)) = W$ , so ascending chains of ideals downstairs correspond to ones upstairs. Here, we use the fact that

$$(R/RJ)^G = ((R^G/J) \otimes_{R^G} R)^G = R^G/J$$

and thus  $(RJ)^G = J$  for any ideal  $J \subseteq R^G$ .  $\square$

**Proposition 2.34.** *If  $R$  is a finitely-generated  $k$ -algebra, then so is  $R^G$ . Equivalently, if  $X$  is finite type, then so is  $X // G$ .*

*Proof.* We first consider the case that

$$R = \bigoplus_{d \geq 0} R_d$$

is a graded ring with  $R_0 = k$  and  $G$  preserves the grading. In this case,  $R^G \subseteq R$  is also graded and is Noetherian, and is thus finitely generated.

In the second case,  $G$  acts rationally on  $R$ . Suppose that  $R$  is generated by  $x_1, \dots, x_n$ . Then the generators are contained in some finite-dimensional  $G$ -subrepresentation  $V \subseteq R$ , so conclude using the diagram

$$\begin{array}{ccc} A = \text{Sym}^* V & \longrightarrow & R \\ \downarrow & & \downarrow \\ A^G & \longrightarrow & R^G \end{array}$$

to show that  $R^G$  is finitely-generated.  $\square$

**2.3. Numerical criteria for stability.** Now suppose that  $\sigma: G \times X \rightarrow X$  is a group action. Consider the morphism

$$\psi = (\sigma, \text{pr}_2): G \times X \rightarrow X \times X \quad \psi(g, x) = (gx, x).$$

If  $j_x: \text{Spec } k \rightarrow X$  is the inclusion of a point, then consider the Cartesian diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\psi} & X \times X \\ 1_G \times j_x \uparrow & & \uparrow \\ G \times \text{Spec } k & \xrightarrow{\psi_x} & X \times \text{Spec } k. \end{array}$$

This defines a morphism  $\psi_x: G \rightarrow X$  and  $G \cdot x = \text{Im}(\psi_x)$  is the orbit of  $x$ .

**Definition 2.35.** The action  $\sigma$  is called:

- (1) *closed* if all maps  $\psi_x$  are closed;
- (2) *separated* if  $\psi$  itself is closed;
- (3) *proper* if  $\psi$  is proper;
- (4) *free* if  $\psi$  is a closed embedding.

If we consider the Cartesian diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\psi} & X \times X \\ \uparrow & & \uparrow \Delta \\ I_X & \longrightarrow & X, \end{array}$$

then in fact the fibers of  $I_X \rightarrow X$  are the stabilizers of points in  $X$ .

**Lemma 2.36.** *If  $\sigma$  is proper, then all  $\psi_x$  are proper. This further implies that all orbits are closed and all stabilizers are proper.*

**Remark 2.37.** There are examples where the converse is false, even when all orbits are closed and stabilizers are trivial.

*Proof.* Use the valuative criterion of properness.  $\square$

**Definition 2.38 (1-PS).** A 1-parameter subgroup of an algebraic group  $G$  is a morphism

$$\lambda: G_m = \text{Spec } k[t, t^{-1}] \rightarrow G.$$



**Example 2.39.** If  $G = \mathrm{GL}_n$ , then one example of a 1-parameter subgroup is the set of matrices of the form

$$\begin{pmatrix} t^{a_1} & & \\ & \cdots & \\ & & t^{a_n} \end{pmatrix}$$

in some basis.

**Proposition 2.40.** *An action of a **reductive** group on a finite type and separated<sup>1</sup>  $X$  over  $k$  is proper if and only if for every nontrivial 1-parameter subgroup*

$$\lambda: \mathbb{G}_m \rightarrow G,$$

*the induced action of  $\mathbb{G}_m$  on  $X$  is proper.*

**Theorem 2.41** (Iwahori). *For a 1-PS  $\lambda: \mathbb{G}_m \rightarrow G$  and a DVR  $R$  with fraction field  $K$ , denote by  $\langle \lambda \rangle_K$  the  $K$ -point*

$$\mathrm{Spec} K \rightarrow \mathrm{Spec} k[t, t^{-1}] \rightarrow G$$

*induced by sending  $t$  to a uniformizer  $\pi$  of  $R$ . Then*

$$G(K) = \bigcup_{\lambda} G(R) \langle \lambda \rangle_K G(R)$$

*is a union of double cosets.*

**Theorem 2.42.** *If  $X = \mathrm{Spec} R$  is finite type and  $G$  is linearly reductive, then the action is proper if and only if all stabilizers are finite.*

*Remark 2.43.* Note that finite stabilizers imply that all orbits are closed.

**Corollary 2.44.** *If  $X // G = X/G$  is a geometric good quotient by a reductive group, then the action is proper.*

*Remark 2.45.* If  $\pi: X \rightarrow Y$  is  $G$ -invariant and surjective then the following are equivalent:

- (1)  $\mathrm{Im}(\psi) = X \times_Y X \subset X \times X$ ;
- (2) All fibers of  $\pi$  are orbits.

In particular, if this holds and  $Y$  is separated, then the diagram

$$\begin{array}{ccc} X \times_Y X & \longrightarrow & X \times X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

is Cartesian, so  $\psi$  is separated.

*Proof of Proposition 2.40.* We apply the valuative criterion. Let  $R$  be a DVR with fraction field  $K$ . Consider the diagram

$$\begin{array}{ccc} \mathrm{Spec} K & \xrightarrow{(\phi, \zeta)} & G \times X \\ \downarrow & & \downarrow \\ \mathrm{Spec} R & \longrightarrow & X \times X \end{array}$$

---

<sup>1</sup>Note that in Mumford's book, all schemes are assumed to be schemes and non-separated schemes are referred to as preschemes.

Then we know that

$$\psi((\phi, \zeta)) = (\phi\zeta, \zeta) \in X(R) \times X(R)$$

by assumption. Also consider the Iwahori decomposition

$$\phi = \psi_1 \langle \lambda \rangle \psi_2$$

for  $\psi_i \in G(R)$ . Now consider a 1-PS  $\lambda: \mathbb{G}_m \rightarrow G$ . Then we compute

$$\begin{aligned} \psi(\langle \lambda \rangle, \psi_2 \zeta) &= (\langle \lambda \rangle \psi_2 \zeta, \psi_2 \zeta) \\ &= (\psi_1^{-1} \phi \zeta, \psi_2 \zeta) \in X(R) \times X(R). \end{aligned}$$

Therefore the morphism  $\mathbb{G}_m \times X \rightarrow X \times X$  is proper, so  $\langle \lambda \rangle \in \mathbb{G}_m(R) \subset G(R)$ .  $\square$

**Proposition 2.46.** *Suppose that  $\mathbb{G}_m$  acts on  $X$  such that the action is induced from an action of  $\mathbb{G}_m$  on  $\mathbb{P}_k^n$  and a locally closed  $\mathbb{G}_m$ -equivariant embedding  $X \hookrightarrow \mathbb{P}_k^n$ . Then the action is proper if and only if all stabilizers are finite.*

*Proof of Theorem 2.42.* If a good quotient exists and all orbits are closed with finite stabilizers, then  $X // G = X/G = \text{Spec } R^G$  is geometric. Therefore,  $\sigma$  is separated, which implies that all 1-PS actions are proper, and thus that  $\sigma$  is proper.  $\square$

*Proof of Proposition 2.46.*  $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$  acts on  $\mathbb{P}^n = \text{Proj } k[x_0, \dots, x_n]$  by the  $x_i \mapsto t^{\alpha_i} x_i$  for  $\alpha_i \in \mathbb{Z}$ . Choose  $R = k[[t]]$  and  $K = k((t))$ . Now the calculation with the diagram

$$\begin{array}{ccc} \mathbb{G}_m \times X & \xleftarrow{(\phi, \zeta)} & \text{Spec } K \\ \downarrow & & \downarrow \\ X \times X & \xleftarrow{\quad} & \text{Spec } R \end{array}$$

becomes very explicit. We assume that  $\phi\zeta, \zeta \in X(R) \subset \mathbb{P}^n(R)$ . Write  $\zeta = [a_0 : \dots : a_n] \in \mathbb{P}^n(R)$  for  $a_i \in R$ . Then we see that

$$\phi\zeta = [A^{\alpha_i} a_i] \in \mathbb{P}^n(R),$$

where  $\phi = A \in K^\times = \mathbb{G}_m(K)$ . Because  $(\phi\zeta, \phi) \in \text{Im}(\psi)$ , so is  $(\phi\zeta(0), \zeta(0))$  by separatedness. Therefore,  $(\phi\zeta)(0) = \phi_0(\zeta(0))$  for some  $\phi_0 \in \mathbb{G}_m(k) = k^\times$ . Now there exists some  $\tilde{\phi} \in \mathbb{G}_m(R)$  such that  $\tilde{\phi}(0) = \phi_0$ . Replace  $\zeta \in X(R)$  by  $\tilde{\zeta} = \tilde{\phi}^{-1} \zeta \in X(R)$ . Therefore,

$$\phi\tilde{\zeta}(0) = \tilde{\zeta}(0)$$

and thus  $(\phi\tilde{\zeta}, \tilde{\zeta}) \in X(R) \times X(R)$  satisfies

$$(\phi\tilde{\zeta}(0), \tilde{\zeta}(0)) \in \Delta_X \subset X(k) \times X(k).$$

Now write

$$\phi\tilde{\zeta} = [A^{\alpha_0} \tilde{a}_0 : \dots : A^{\alpha_n} \tilde{a}_n] \in \mathbb{P}_R^n.$$

Clearly, we have  $\tilde{a}_i(0) = A^{\alpha_i} \tilde{a}_i(0)$ , so if  $\tilde{a}_i \neq 0$  and  $\alpha_i \neq 0$ , then  $A$  is a unit. But the point

$$[\tilde{a}_0 : \dots : \tilde{a}_n] \in \mathbb{P}_k^n$$

has finite stabilizer under the action with weights  $\alpha_0, \dots, \alpha_n$ , so the desired  $i$  exists.  $\square$

**Example 2.47** (Non-proper action, Mumford). Let  $G = \mathrm{SL}(2)$  and

$$X = \left\{ (F_1, F_4) \mid F_1 \neq 0, F_4 = Q^2, Q \text{ quadric}, \det(Q) = 1 \right\} \subset \mathbb{C}^2 \times \mathrm{Sym}^4(\mathbb{C}^2).$$

The action will be induced from the natural action of  $\mathrm{SL}(2)$  on  $\mathbb{C}^2$ .

**Exercise 2.48.** There is a map

$$\pi: X \rightarrow \mathbb{A}^1 \quad \pi(\alpha X + \beta Y, Q^2) = Q(-\beta, \alpha) \in \mathbb{C}$$

which is  $\mathrm{SL}(2)$ -invariant with  $\pi^{-1}(\pi(x)) = \mathrm{SL}(2) \cdot x$ . Moreover, all stabilizers are trivial.

There is a locus

$$Y = \left\{ \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}, (\lambda x + y, x^2 y^2) \right\} \hookrightarrow \mathrm{SL}(2) \times X,$$

which is closed. Then we see that

$$\begin{aligned} \Psi(Y) &= (\sigma, \mathrm{pr}_2)(Y) \\ &= \left\{ (-\lambda x + y, x^2 y^2), (\lambda x + y, x^2 y^2) \right\} \subset X \times X, \end{aligned}$$

which is not closed. Therefore, the orbit space map cannot be any kind of good quotient.

This example is morally the same the example of the action of  $G_m \rtimes \mathbb{Z}_2$  on  $\mathbb{A}^2 \setminus 0$ , where  $G_m$  acts with weights 1 and  $-1$  and  $\mathbb{Z}_2$  acts by swapping the coordinates. This has a map to  $\mathbb{A}^1$  which separates orbits, but this is still not a good quotient because even though the diagonal  $\Delta$  of  $(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^2 \setminus 0)$  is closed,  $\Psi(\Delta)$  is not closed.

In the language of stacks, in fact there is an isomorphism

$$[X/\mathrm{SL}(2)] \simeq [(\mathbb{A}^1 \text{ with doubled origin})/\mathbb{Z}_2],$$

where  $\mathbb{Z}_2$  acts by swapping the two origins. This stack does not have a good moduli space which is a scheme, but it does have a good moduli space which is a non-separated algebraic space by Keel-Mori.

**Example 2.49** (Hironaka). There is a proper but non-projective threefold  $X$  with a free action of  $G_m$  which is not proper, and the quotient is not a scheme but is an algebraic space.

**2.4. Linearizations and GIT.** Let  $X$  be finite type and separated over an algebraically closed field  $k$ ,  $\mathcal{L}$  be a line bundle on  $X$ , and  $\sigma: G \times X \rightarrow X$  be an action.

**Definition 2.50.** A  $G$ -linearization of  $\mathcal{L}$  is an action

$$\Sigma: G \times \mathbb{L} \rightarrow \mathbb{L}$$

on the total space of  $\mathcal{L}$  such that the diagram

$$\begin{array}{ccc} G \times \mathbb{L} & \xrightarrow{\Sigma} & \mathbb{L} \\ \downarrow 1_G \times \pi & & \downarrow \pi \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

commutes and the induced map

$$G \times \mathbb{L} \rightarrow \sigma^* \mathbb{L} = (G \times X) \times_X \mathbb{L}$$

is a morphism of line bundles over  $G \times X$ .

**Example 2.51.** If  $G$  acts on  $V$  rationally by some morphism  $G \rightarrow GL(V)$ , then the induced action on  $\mathbb{P}(V)$  comes with a linearization of  $\mathcal{O}_{\mathbb{P}(V)}(1)$ , which comes from the fact that the total space of  $\mathcal{O}_{\mathbb{P}(V)}(1)$  is the blowup of  $V$  at the origin, and the action on  $V$  lifts to an action on the blowup (since we blew up a fixed point).

**Definition 2.52.** Let  $X$  be finite type over an algebraically closed field  $k$ ,  $G$  act on  $X$ , and  $\mathcal{L}$  be a  $G$ -linearized line bundle on  $X$ . A point  $x \in X(k)$  is *semistable* if there exists  $s \in H^0(X, \mathcal{L}^n)$  for some positive integer  $n$  such that

- (1)  $s$  is  $G$ -invariant under the natural action of  $G$  on  $H^0(X, \mathcal{L}^n)$  induced by the linearization;
- (2)  $X_s = X \setminus \mathbb{V}(s)$  is affine;
- (3)  $x \in X_s$ .

Furthermore,  $x$  is *stable* if the action on  $X_s$  is closed.

*Remarks 2.53.*

- (1) We see that the *semistable locus*  $X^{ss} \subset X$  is open.
- (2) If  $\mathcal{L}$  is ample, then  $X_s$  is automatically affine.