RATIONAL CURVES ON CALABI-YAU HYPERSURFACES

PATRICK LEI

Abstract. I will discuss sweeping out of Calabi-Yau hypersurfaces by abelian varieties as well as rational curves on such hypersurfaces. We will focus in particular on the case of quantic threefolds and discuss Clemens' conjecture.

1. Introduction

First, we recall some basic definitions.

Definition 1.1. Let X be a smooth (or maybe with nice singularities prescribed by the MMP) variety. Then X is *Calabi-Yau* if $\omega_X = \mathcal{O}_X$.

Example 1.2. By the adjunction formula, a degree $n+2$ hypersurface in $\mathbb{P}^{n+1}$ is Calabi-Yau. For some low dimensional examples, this means that elliptic curves, quartic surfaces, and quintic threefolds are Calabi-Yau.

Example 1.3. Recall that if $f\colon Y \to X$ is a double cover branched along a divisor D, we have

$$K_Y = f^*\left(K_X + \frac{1}{2}D\right).$$

This implies that the double cover of $\mathbb{P}^2$ branched along a sextic is Calabi-Yau as is the double cover of $\mathbb{P}^3$ branched along an octic surface.

2. Rational curves on Calabi-Yau hypersurfaces

We will now consider rational curves on Calabi-Yau hypersurfaces.

Theorem 2.1 (Clemens)**.** *Let X be a general Calabi-Yau hypersurface of dimension $n \geqslant 2$. Then X contains rational curves of arbitrarily large degree.*

For a quintic threefold, note that if we have $\mathbb{P}^1 \simeq C \subset X$, then the exact sequence

$$0 \to T_C \to T_{\mathbb{P}^3}|_C \to N_{C/X} \to 0$$

tells us that $\deg N_{C/X} = -2$. Hirzebruch-Riemann-Roch for a smooth curve C of genus g says that

$$\chi(C, \mathcal{E}) = \operatorname{rk}\mathcal{E}(1-g) + c_1(\mathcal{E}),$$

and this gives us $\chi(N_{C/X}) = 0$ in our case. Generically, we expect (by upper semicontinuity of cohomology) that $h^0(N_{C/X}) = h^1(N_{C/X}) = 0$, and thus we expect

$$N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

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Actually, for any smooth rational curve $\mathbb{P}^1 \hookrightarrow X$ in a general quintic threefold, that is exactly the normal bundle by a result of Bin Wang in 2015. By basic deformation theory, we see that such C are infinitesimally rigid. In fact, we can strengthen the previous theorem:

Theorem 2.2. *Let X be a quintic threefold. then X has infinitely many infinitesimally rigid rational curves.*

On the flip side, we expect that there are not too many rational curves on a quintic threefold.

Conjecture 2.3 (Clemens)**.** *Let X be a general quintic threefold. Recall that by the Lefschetz hyperplane theorem, $H_2(X) = \mathbb{Z}$ is generated by the class of a line. Then for any $d > 0$, X contains only finitely many rational curves of degree d.*

This conjecture is true in degrees $d \leqslant 11$ by a result of Cotterill in 2012. It also inspired the development of enumerative geometry, mirror symmetry, and many other ideas, but we will not discuss them here.

Remark 2.4. The conjecture is crucially supported by the fact that deformations of $C \simeq \mathbb{P}^1$ inside X are unobstructed. On the other hand, if Q is a quartic surface and $C \subset Q$ is a rational curve, then $N_{C/Q} = \mathcal{O}(-2)$, and because $h^1(N_{C/X}) = 1$, we see that deformations of C in Q are obstructed.

Remark 2.5. The Clemens conjecture is false if we let X be a double cover of $\mathbb{P}^3$ ramified along an octic surface S. Consider lines ℓ such that $\ell \cap S = 2p_1 + 2p_2 + 2p_3 + p_4 + p_5$ as cycles. There is a 1-dimensional family of such lines. If $\pi \colon X \to \mathbb{P}^3$ is the double cover, then $\pi^{-1}(\ell)$ is ramified over ℓ at 2 points and is therefore rational. But this means we have a 1-parameter family of rational curves in X all of the same degree.

The remark is related to the following result:

Proposition 2.6. *Let X be a Calabi-Yau threefold and suppose that X contains a smooth rational curve $C \simeq \mathbb{P}^1$ with normal bundle*

$$N_{C/X} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1).$$

Assume there exists a rational curve $C' \subset X$, a neighborhood $U \supset C \cup C'$, and an involution $i \colon U \to U$ such that

(1) The fixed locus of i is a smooth hypersurface of U that meets C transversally;

(2) $i(C) = C' \neq C$.

Then,

(1) If i has at least one fixed point on C, then X contains a one-parameter family of rational curves.

(2) If i has two fixed points $p, p' \in C$ and the tangent directions to C' at p, p' are distinct in $\mathbb{P}(N_{C/X}) = C \times \mathbb{P}^1$, then X is swept out by a two-parameter family of elliptic curves.

To obtain this result, we need to choose a special octic surface. Let ℓ be a line in $\mathbb{P}^3$ and note that the map

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(8)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$$

is surjective (this is an exercise in Hartshorne). Now if we choose $f \in H^0(\mathbb{P}^1, \mathcal{O}(8))$ to vanish at three points with degrees $2, 2, 4$, then by a Bertini-type argument, the general octic surface S living above f is smooth. Now note that $\pi^{-1}(\ell)$ splits into two components because the local equation looks like

$$y^2 = x^4(x+1)^2(x-1)^2,$$

and therefore the two components look like $y = \pm x^2(x+1)(x-2)$. But now if we take C, C' to be the two components and i to exchange the two sheets, we see that both assumptions of the proposition are satisfied. We have already exhibited a one-parameter family of rational curves above, and later we will produce a two-parameter family of elliptic curves.

3. Sweeping out of hypersurfaces

In this section we will discuss sweeping out of hypersurfaces by abelian varieties. This will roughly mean that a variety is covered by a generically finite morphism from a family of abelian varieties.

Definition 3.1. Let S be a quasiprojective variety and $f\colon \mathcal{Y} \to S$ be a smooth projective morphism of relative dimension r. Also let X be a variety of dimension n. Then X is *rationally swept out* by members of $\mathcal{Y}$ if there exists a quasiprojective variety B of dimension $n-r$, a morphism $B \to S$, and a dominant rational map $\mathcal{Y}_B \dashrightarrow X$.

For example, we can consider when S is a moduli space of varieties and $\mathcal{Y}$ is the universal family. Also, the definition of sweeping out is equivalent to the property that the union of the images of generically finite rational maps $Y_s \dashrightarrow X$ contains a Zariski open set of X.

The main result that Voisin proves about sweeping out of hypersurfaces by varieties is the following:

Theorem 3.2 (Voisin). *Let $1 \leqslant r \leqslant n$ and $\gamma = \lceil \frac{r-1}{2} \rceil$. Let S have dimension C and $\mathcal{Y} \to S$ be a family of r-dimensional smooth projective varieties. Finally fix a positive integer d. Then, if the two inequalities*

(1) $(d+1)r \geqslant 2n + C + 2$;

(2) $(\gamma+1)d \geqslant 2n - r + 1 + C$

*hold, the very general hypersurface of degree d in $\mathbb{P}^{n+1}$ is **not** swept out by members of the family $\mathcal{Y} \to S$.*

This result is a consequence of a Hodge-theoretic result. First, let $U \subset |\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ be the open set parameterizing smooth hypersurfaces. Let $\rho\colon \mathcal{M} \to U$ be a morphism with $\mathcal{M}$ smooth and quasiprojective such that the corank of ρ is constant and equal to C and that the image of ρ is stable under GL_{n+2}. Let $\mathcal{X}_U$ be the universal smooth hypersurface and $j\colon \mathcal{X}_{\mathcal{M}} \hookrightarrow \mathcal{M} \times \mathbb{P}^{n+1}$ be the natural closed immersion.

Theorem 3.3 (Nori)**.**

(1) Assume that $(d+1)r \geqslant 2n + C + 2$*. Then the restriction*
$$j^*\colon F^n H^{2n-r}(\mathcal{M} \times \mathbb{P}^{n+1}, \mathbb{C}) \to F^n H^{2n-r}(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$$
is surjective.

(2) If $(\gamma+1)d \geqslant 2n + 1 - r + C$*, then the restriction*
$$j^*\colon H^{2n-r-i}(\mathcal{M} \times \mathbb{P}^{n+1}, \mathbb{C}) \to H^{2n-r-i}(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$$
is surjective for any $i \geqslant 1$*.*

The proof of this is of course Hodge-theoretic and uses spectral sequences, and it is omitted here. Finally, we state a conjecture about the sweeping out of hypersurfaces by abelian varieties.

Proof of Voisin assuming Nori. Suppose that there is a B of dimension $n - r$, maps $\rho\colon B \to S$ and $m\colon B \to U$, and a dominant $\phi\colon \mathcal{Y}_B \dashrightarrow \mathcal{X}_B$. But then if $f\colon \mathcal{Y}_B \to B$ and $\pi\colon \mathcal{X}_U \to U$ are the families, we have $\pi \circ \phi = m \circ f$.

Now by shrinking B, we may assume that all B_s for $s \in S$ are smooth and that the corank of $m_s \coloneqq m|_{B_S}$ is constant and $\leqslant C$. Now if $\mathcal{Y}_s \coloneqq f^{-1}(\mathcal{B}_s)$, write $\phi_s \coloneqq \phi|_{\mathcal{Y}_s}$. Therefore we have a rational map
$$\phi_s\colon \mathcal{Y}_s = Y_s \times \mathcal{B}_s \dashrightarrow \mathcal{X}_U \times_U \mathcal{B}_s \eqqcolon \mathcal{X}_s.$$
But then the graph $\Gamma_{\phi_s} \subset Y_s \times \mathcal{B}_s \times_{\mathcal{B}_s} \mathcal{X}_s = Y_s \times \mathcal{X}_s$. This gives a cohomology class $\gamma_s \in H^{2n}(Y_s \times \mathcal{X}_s, \mathbb{Q})$. Now Nori's theorem implies the class $\gamma_{s,r} \in H^r(Y_s, \mathbb{Q}) \otimes H^{2n-r}(\mathcal{X}_s, \mathbb{Q})$ vanishes in
$$H^r(Y_s, \mathbb{Q})_{tr} \otimes (H^{2n-r}(\mathcal{X}_s, \mathbb{Q})/H^{2n-r}(\mathbb{P}^{n+1} \times \mathcal{B}_s, \mathbb{Q})).$$

Finally, returning to $\Gamma_\phi \subset \mathcal{Y}_B \times_U \mathcal{X}$, which has codimension n. If we fix $s \in S$, the class $[\Gamma_\phi] \in H^{2n}(\mathcal{Y}_B \times_U \mathcal{X}, \mathbb{Q})$ restricts to γ_s. If we fix $u \in U$ and let $B_u \coloneqq m^{-1}(u)$. Then ϕ restricts to
$$\phi_u\colon \mathcal{Y}_u \eqqcolon \mathcal{Y} \times_S B_u \dashrightarrow X_u,$$
which is dominant and generically finite on fibers. But now, a Hodge-theoretic argument tells us that
$$\phi_u^*\colon H^n(X_u, \mathbb{Q})_{tr} \to H^n(\mathcal{Y}_u, \mathbb{Q})$$
is injective and is in fact nonzero in $\mathrm{Hom}(H^n(X_u, \mathbb{Q})_{tr}, H^{n-r}(B_u, R^r f_* \mathbb{Q}_{tr}))$. But now $[\Gamma_\phi]$ is nontrivial in $H^{2n}(\mathcal{X}_B, R^r f_* \mathbb{Q}_{tr})/H^{2n}(\mathcal{Y}_B \times \mathbb{P}^{n+1}, \mathbb{Q})$, and so by a spectral sequence argument, $\gamma_{s,r}$ is nonzero, which is a contradiction. □

Conjecture 3.4 (Lang)**.** *Let* X *be not of general type. Then the union of the images of non-constant rational maps* $\phi\colon A \dashrightarrow X$ *from an abelian variety* A *is* X*.*

Equivalently, this becomes:

Conjecture 3.5. *Let* X *be a variety of Kodaira dimension* $0 \leqslant \kappa(X) < \dim X$ *(in particular,* X *is not of general type). Then* X *is rationally swept out by abelian varieties of dimension* $r \geqslant 1$*.*

Note that this conjecture is true when X is a double cover of $\mathbb{P}^3$ branched along an octic surface S. Consider the two-parameter family of lines $\ell \subset \mathbb{P}^3$ such that

$$\ell \cap S = 2p_1 + 2p_2 + p_3 + p_4 + p_5 + p_6.$$

Then the preimage of ℓ in X is branched over $\ell \simeq \mathbb{P}^1$ at 4 points, and therefore its normalization is an elliptic curve. But this implies that X is swept out by a two-parameter family of elliptic curves.

4. Sweeping out of Calabi-Yau hypersurfaces by abelian varieties

Theorem 4.1. *Let X be a very general Calabi-Yau hypersurface in $\mathbb{P}^{n+1}$ (that is, of degree $d = n+2$). Then X is **not** swept out by r-dimensional abelian varieties for any $r \geqslant 2$.*

Proof. This boils down to checking some inequalities. First, recall that

$$\dim \mathcal{A}_r = \frac{r(r+1)}{2}.$$

Then we need to check the two inequalities. The first is

$$(n+3)r \geqslant 2n + \frac{r(r+1)}{2} + 2,$$

which is clearly true for $r \geqslant 2$ because

$$r\left(n + 3 - \frac{r+1}{2}\right) \geqslant 2(n+1),$$

which holds because $n + 3 - \frac{r+1}{2} + r \geqslant n+3$ and both $r \geqslant 2$ and $n + 3 - \frac{r+1}{2} \geqslant 3$. The second inequality is

$$(\gamma+1)(n+2) \geqslant 2n - r + 1 + \frac{r(r+1)}{2},$$

and this holds for $r \geqslant 2$ by high school algebra (of the kind that I cannot bother to figure out). □

Corollary 4.2. *If Lang's conjecture is true for a very general Calabi-Yau hypersurface X, then X is swept out by elliptic curves.*

This will imply that X has a uniruled divisor, but first we need the following lemma:

Lemma 4.3. *Let X be a very general Calabi-Yau hypersurface of dimension $\dim X \geqslant 2$. Then X is not swept out by an isotrivial family of elliptic curves.*

Lemma 4.4. *Suppose a variety X is swept out by a non-isotrivial family of elliptic curves. Then X has a uniruled divisor.*

Proof. By assumption, we have a diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, dashed, "\phi"] \arrow[d, "\pi"] & X \\ B, & \end{tikzcd}$$

where $\mathcal{K} \to B$ is a family of elliptic curves. Now this is given by $B \to \mathcal{M}_{1,1}$, so we can choose a smooth projective model $B' \supset B$ and a smooth projective model $\mathcal{K}' \supset \mathcal{K}$. Now we can replace ϕ with an honest morphism

$$\begin{tikzcd} \mathcal{K}' \arrow[r, "\phi"] \arrow[d, "\pi"] & X \\ B'. & \end{tikzcd}$$

But now the j-invariant map

$$j \colon B' \xrightarrow{\mathcal{K}'} \overline{\mathcal{M}}_{1,1} \to \mathbb{P}^1$$

(composing the defining map of $\mathcal{K}'$ with the coarse moduli space) is surjective. But now for $t \in \mathbb{P}^1$, consider the divisor

$$\mathcal{K}'_t \coloneqq (j \circ \pi)^{-1}(t).$$

For a generic $t \in \mathbb{P}^1$, this is sent to a divisor of X, and in particular for any t, the image $\phi(\mathcal{K}'_t)$ contains a divisor of X. But now if we take $t = \infty$, noting that an elliptic curve must degenerate to a union of rational curves, we see that any component of $\mathcal{K}'_\infty$ is uniruled, so we are done. $\square$

Corollary 4.5. *Clemens' conjecture and Lang's conjecture contradict each other.*