TOPOLOGICAL CYCLIC HOMOLOGY

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Abstract. These are notes taken live while watching the YouTube lectures about topological cyclic homology by Thomas Nikolaus and Achim Krause. Note that we use homological grading throughout because the lecturers are homotopy theorists.

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Date: Spring 2025.

1. Motivation: trace methods

One motivation to study topological cyclic homology is its relation to crystalline cohomology and syntomic cohomology in arithmetic geometry. However, this is very involved, so we will not discuss it in depth. We will instead discuss the original motivation, which are trace methods in algebraic K-theory.

Let R be a ring. There is an invariant, called *algebraic K-theory*, which produces groups $K_*(R)$ for all $* \geq 0$. These are very important, but are almost impossible to compute or to understand structurally. For example, even $K_*(\mathbb{Z})$ is not very well-understood. They are known in all odd degrees, conjectured to vanish in degrees divisible by 4 (this is equivalent to the Kummer-Vandiver conjecture in number theory).

If you are a topologist, then one motivation is to study whether a retract X of a finite CW complex is itself a finite CW complex. The obstruction to this lies in

$$K_*(\mathbb{Z}[\pi_1(X)]).$$

Another motivation is that by work of Whitehead and others (the s-cobordism theorem), the obstruction for a cobordism to be trivial lies in an algebraic K-theory group. There is now a higher version of this theorem which relates diffeomorphism groups to algebraic K-theory. A third motivation comes from the Baum-Connes conjecture, which has other applications in geometric topology.

For other applications, the group K_0 was first invented by Grothendieck to give a statement of the Grothendieck-Riemann-Roch theorem, and algebraic K-theory is also related to special values of L-functions.

Definition 1.1. For a ring R, the group $K_0(R)$ is the Grothendieck group of isomorphism classes of finitely generated projective R-modules under the operation of direct sum.

Higher K-groups are defined as homotopy groups of the space obtained by group completing the category of finitely-generated projective R-modules. This can be made precise using higher algebra and actually produces a spectrum, and then we can take its homotopy groups. We can see why this is so complicated.

Example 1.2. It is easy to see that $K_0(\text{field}) = \mathbb{Z}$, but of course the higher K-groups are very complicated. Quillen computed them for finite fields, but if we leave this case this becomes extremely difficult.

1.1. Approximation of algebraic K-theory.

We will attempt to approximate algebraic K-theory using more algebraic invariants. There is a *Dennis trace map* to Hochschild homology. There are refinements of this which fit into a diagram

$$\begin{tikzcd}
K_*(R) \arrow[dr, "\text{cyclotomic trace}"] & & \\
& \mathsf{TC}_*(R) \arrow[r] \arrow[d] & \mathsf{HC}^-_*(R) \arrow[d] \\
& \mathsf{THH}_*(R) \arrow[r] & \mathsf{HH}_*(R)
\end{tikzcd}$$

Slogan 1.3. *The cyclotomic trace is often close to an isomorphism.*

There are also relative K-groups $K_*(R, I)$ for any ideal $I \subseteq R$, which is simply the homotopy fiber of $K(R) \to K(R/I)$. There are also relative TC groups, which are defined in the same way. We can also define groups with coefficients like $K_*(R, \mathbb{Z}_p)$ and $\mathsf{TC}_*(R, \mathbb{Z}_p)$.

Theorem 1.4 (Dundas-Goodwillie-McCarthy, Clausen-Matthew-Morrow)**.**

(1) If $I \subseteq R$ is a nilpotent ideal, then

$$\mathrm{cyctr}\colon K_*(R, I) \to \mathsf{TC}_*(R, I)$$

is an isomorphism.

(2) If R is commutative and I-complete, the same conclusion holds with p-adic coefficients. In other words, we have

$$K_*(R, I, \mathbb{Z}_p) \cong \mathsf{TC}_*(R, I, \mathbb{Z}_p).$$

(3) If R is p-complete, then

$$\mathsf{TC}_*(R, \mathbb{Z}_p) \cong K_*^{\acute{e}t}(R, \mathbb{Z}_p).$$

1.2. **Why the trace map is a trace.** Let k be a field, R be a k-algebra, and P be a finitely generated projective right R-module. Then we have

$$\mathrm{Hom}_R(P, P) \cong P \otimes_R \mathrm{Hom}_R(P, R).$$

We would attempt to go to R by an evaluation map

$$P \otimes_R \mathrm{Hom}_R(P, R) \xrightarrow{\mathrm{ev}} R,$$

but this doesn't actually land in R. Instead, we see that

$$\begin{aligned} x \otimes r\varphi &\mapsto r \cdot \varphi(x) \\ xr \otimes \varphi &\mapsto \varphi(xr) = \varphi(x) \cdot r. \end{aligned}$$

The best we can do is to consider the quotient

$$R/[R, R]$$

as an abelian groups, so in conclusion we have

$$\mathrm{tr}\colon \mathrm{Hom}_R(P, P) \to R/[R, R].$$

Note that an R-R-bimodule is equivalently an $R \otimes_k R^{\mathrm{op}}$-module. Then, we in fact have

$$R/[R, R] \cong R \otimes_{R \otimes_k R^{\mathrm{op}}} R.$$

Deriving this expression, we will see later that

$$\mathsf{HH}(R, k) \cong R \otimes^{\mathbb{L}}_{R \otimes_k R^{\mathrm{op}}} R.$$

The Dennis trace will refine this trace in the sense that on K_0, it will send a projective module P to the trace of the identity endomorphism.

2. Infinity-categories

Infinity-categories are a simultaneous generalization of both ordinary categories and of the homotopy theory of spaces which form a natural home for derived phenomena. We will not aim to cover the entire theory but instead to give an idea of how to work with and think about ∞-categories.

Convention 2.1. We will resolve size issues by assuming the existence of a Grothendieck universe $\mathcal{U}$. Elements of $\mathcal{U}$ will be called *small* sets. We will take categories to have large object sets and large morphism sets. Therefore, for us we will take Set to mean the category of **small** sets.

2.1. ∞-categories.

Definition 2.2. An ∞-*category* is is a (large) simplicial set $\mathcal{C}$ with the property that any diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

admits a filling for all $0 < i < n$. Here, Λ_i^n is the i-th horn given by taking the boundary of Δ^n and removing the facet opposite to the i-th vertex.

A *functor* $\mathcal{C} \to \mathcal{D}$ is a map of simplicial sets.

It is not clear why this definition should be related to categories at all, but we will construct an example of an ∞-category from an ordinary category.

Example 2.3. Let $\mathcal{C}$ be an ordinary category. Define the *nerve* of C to be the simplicial set given by

$$(N\mathcal{C})_n = \mathsf{Fun}([n], \mathcal{C}),$$

where $[n]$ is the poset given by $0 \le 1 \le \cdots \le n$.

Exercise 2.4.

(1) Show that $N\mathcal{C}$ is an ∞-category;
(2) Show that functors $N\mathcal{C} \to N\mathcal{D}$ are in bijection with functors $\mathcal{C} \to \mathcal{D}$.

Now let $\mathcal{C}$ be an ∞-category. We will call $\mathcal{C}_0$ the set of *objects* and $\mathcal{C}_1$ the set of *morphisms*. Note that it is not clear what the composition of two morphisms should be. For any $f \in \mathcal{C}_1$, we will view f as a morphism

$$f\colon a \to b, \qquad a = \partial_1 f, b = \partial_0 f.$$

Definition 2.5. Two morphisms $f, g\colon a \to b$ are *equivalent* if there exists a 2-simplex $\sigma \in \mathcal{C}_2$ of the form

$$\begin{tikzcd} & b \arrow[dr, "\mathrm{id}"] & \\ a \arrow[ur, "f"] \arrow[rr, "g"'] & & b. \end{tikzcd}$$

In this case, we write $f \simeq g$.

This notion is extremely boring for the nerve of a 1-category. It is more interesting in the next example.

Example 2.6. Let $\mathcal{C}$ be a Kan complex. This means that all horns can be filled in, so in particular $\mathcal{C}$ is an ∞-*category*. For example, if X is a topological space, then the singular simplicial set $\mathsf{Sing}(X)$ is a Kan complex. Here, objects are points of X, morphisms are paths in X, and equivalence is simply homotopy of paths relative to the endpoints. If we pass to equivalence classes, we obtain the fundamental groupoid of X, so we can think of $\mathsf{Sing}(X)$ as being an ∞-categorical version of the fundamental groupoid. Note here that composition of paths is only associative up to homotopy.

Definition 2.7. Let $f: a \to b$ and $g: b \to c$ be morphisms. A *composition* of f and g is a 2-simplex σ of the form

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g"'] & & c. \end{tikzcd}$$

In this case, we will write $h \simeq g \circ f$.

Exercise 2.8. Check that two different choices of compositions, $h \simeq g \circ f$ and $h' \simeq g \circ f$, are equivalent.

The non-existence of uniqueness may seem like a defect, but if we consider coproducts in ordinary categories, being unique up to unique isomorphism is enough because it allows us to not have to think about a coproduct functor being associative (or maybe only in a weak sense).

Definition 2.9. Let $f: a \to b$ be a morphism. It is an *equivalence* if there exists a morphism $g: b \to a$ such that $\mathrm{id}_a \simeq g \circ f$ and $\mathrm{id}_b \simeq f \circ g$.

Recall that the functors between ordinary categories form a category. We will upgrade this to ∞-categories.

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets. Then we define the simplicial set $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ by the formula

$$\mathsf{Fun}(\mathcal{C}, \mathcal{D})_n = \{\mathcal{C} \times \Delta^n \to \mathcal{D}\}.$$

Proposition 2.11. *Let $\mathcal{C}$ be an arbitrary simplicial set and $\mathcal{D}$ be an ∞-category. Then $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is an ∞-category.*

Note that a morphism $\eta: f \to g$ is a functor $\mathcal{C} \times \Delta^1 \to \mathcal{D}$ which restricts to f and g on the boundaries. We will call these *natural transformations*. A natural transformation η is an equivalence in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ if and only if it is objectwise an equivalence in $\mathcal{D}$.

Definition 2.12. A functor $f: \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists a functor $g: \mathcal{D} \to \mathcal{C}$ and natural equivalences $f \circ g \simeq \mathrm{id}_{\mathcal{D}}$ and $g \circ f \simeq \mathrm{id}_{\mathcal{C}}$. In this case, we will write $\mathcal{C} \simeq \mathcal{D}$.

We would now like to discuss something that is like the fact that equivalences of ordinary categories are fully faithful and essentially surjective. Unfortunately, we don't really have the machinery to do this yet.

Definition 2.13. Let $a, b \in \mathcal{C}$ be two objects in an ∞-category. We define the *mapping space* to be the simplicial set (in fact a Kan complex) which is the pullback in the diagram

$$\begin{tikzcd} \mathsf{Map}_{\mathcal{C}}(a,b) \arrow[r] \arrow[d] & \mathsf{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(a,b)}"] & \mathcal{C} \times \mathcal{C}. \end{tikzcd}$$

Note that the 0-simplices in this mapping space are simply morphisms $f \colon a \to b$ and that 1-simplices are equivalences. Thus $\pi_0 \mathsf{Map}_{\mathcal{C}}(a,b)$ is the set of equivalence classes of morphisms $a \to b$.

Exercise 2.14.

(1) Check that in $N\mathcal{C}$ for a 1-category $\mathcal{C}$, mapping spaces are discrete and agree with morphism sets, and that the composition map recovers ordinary composition.
(2) Check that in $\mathsf{Sing}(X)$, the mapping space
$$\mathsf{Map}_{\mathsf{Sing}(X)}(a,b)$$
is homotopy equivalent to the space of paths from a to b in X and that the composition map comes from path composition.

Definition 2.15. A functor $f \colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is *fully faithful* if for any pair of objects $a, b \in \mathcal{C}$, the map

$$\mathsf{Map}_{\mathcal{C}}(a,b) \to \mathsf{Map}_{\mathcal{D}}(f(a), f(b))$$

is a homotopy equivalence.

Definition 2.16. A functor $f \colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is *essentially surjective* if for all $d \in \mathcal{D}$, there exists some $c \in \mathcal{C}$ and an equivalence $d \simeq f(c)$.

Proposition 2.17. *A functor $f \colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is an equivalence if and only if f is fully faithful and essentially surjective.*

Recall the notion of a full subcategory. We will define an ∞-categorical version of this.

Definition 2.18. Let $S \subset \mathcal{C}_0$. We define the *full subcategory* $\mathcal{C}_S \subseteq \mathcal{C}$ to consist of all simplices with vertices in S.

Note that if we first saturate S under equivalence of objects to form $S \subseteq \bar{S}$, we obtain an equivalence

$$\mathcal{C}_S \to \mathsf{C}_{\bar{S}}.$$

We will now discuss a way to upgrade compositions to mapping spaces. Consider the pullback

$$\begin{tikzcd} \mathsf{Map}_{\mathcal{C}}(a,b,c) \arrow[r] \arrow[d] & \mathsf{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(a,b,c)}"] & \mathcal{C} \times \mathcal{C} \times \mathcal{C}, \end{tikzcd}$$

which is a triple mapping space.

Lemma 2.19. *The triple mapping space* $\mathsf{Map}_{\mathcal{C}}(a,b,c)$ *is a Kan complex and fits in a diagram*

$$\mathsf{Map}_{\mathcal{C}}(b,c) \times \mathsf{Map}_{\mathcal{C}}(a,b) \xleftarrow[\simeq]{(\partial_0,\partial_2)} \mathsf{Map}_{\mathcal{C}}(a,b,c) \xrightarrow{\partial_1} \mathsf{Map}_{\mathcal{C}}(a,c)$$

induced by the horn-filling

$$\Lambda^2_1 \to \Delta^2 \leftarrow \Delta^1.$$

A choice of a homotopy inverse S gives us a composition map

$$\mathsf{Map}_{\mathcal{C}}(b,c) \times \mathsf{Map}_{\mathcal{C}}(a,b) \xrightarrow{S} \mathsf{Map}_{\mathcal{C}}(a,b,c) \xrightarrow{\partial_1} \mathsf{Map}_{\mathcal{C}}(a,b).$$

This seems to depend on the choice of S, but it is unique up to unique homotopy. In fact, the space of all possible homotopy inverses is contractible.

Similarly, we can define $\mathsf{Map}_{\mathcal{C}}(a,b,c,d)$, which parameterizes the associativity of composition. We can compare this to Segal categories, where data like this is an axiom and gives us categories weakly enriched in spaces. In our approach, we get this for free.

Until now, we have seen two kinds of ∞-categories: (nerves of) ordinary categories and Kan complexes (where all morphisms are invertible, sometimes called ∞-groupoids). We will now discuss another example, the ∞-category of spaces, which is the prototypical example of an ∞-category.

Recall that the category of Kan complexes is enriched in simplicial sets. This enrichment actually lands in Kan complexes, so we would like to form an ∞-category.

Construction 2.20. For a finite, nonempty, totally ordered set J, define a simplicially enriched category

$$\mathfrak{C}[\Delta^J]$$

where objects are elements of J and for any $i, j \in J$, the mapping simplicial set is given by

$$\mathrm{Hom}(i,j) = N\{K \subseteq J \mid \min K = i, \max k = j\}.$$

Here, composition is induced by union of subsets of J.

Definition 2.21. Define a simplicial set $\mathcal{S}$ by the formula

$$\mathcal{S}_n = \mathsf{Fun}(\mathfrak{C}[\Delta^n], \mathsf{Kan}).$$

The simplicies look like the follows:

- 0-simplices are Kan complexes;
- 1-simplices are simply morphisms of Kan complexes;
- 2-simplices correspond to diagrams

$$\begin{tikzcd} & X_1 \arrow[dr, "f_2"] & \\ X_0 \arrow[ur, "f_0"] \arrow[rr, "f_1"'] \arrow[rr, phantom, bend left=40, "\Uparrow"] & & X_2 \end{tikzcd}$$

2.2. **Limits.**

2.3. **Colimits.**

2.4. **Derived catgories.**

2.5. **Slices and filtered colimits.**

2.6. **Derived functors.**

2.7. **Nonabelian derived functors.**

3. Higher algebra

3.1. **Spectra.**

3.2. **Symmetric monoidal ∞-categories.**

3.3. $\mathbb{E}_n$**-algebras.**

3.4. p**-adic completion.**

3.5. **Tate construction.**

3.6. **Tate diagonal.**

4. Classical Hochschild and cyclic homology

In this section, we discuss Hochschild homology, (negative, periodic) cyclic homology, the HKR theorem, and related topics.

4.1. **Hochschild homology.** Let k be a field and R be a k- algebra.

Definition 4.1. The *Hochschild homology* of R is the homology of the complex

$$\mathsf{HH}(R/k) \coloneqq (\cdots \xrightarrow{\partial} R \otimes_k R \xrightarrow{\partial} R),$$

where the differential is given by

$$\begin{aligned}\partial(a_0 \otimes \cdots \otimes a_n) \coloneqq\ & a_0 a_1 \otimes \cdots \otimes a_n \\ & - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n \\ & \cdots \\ & + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.\end{aligned}$$

Remark 4.2. This seems related to what happens when you multiply elements placed on an S^1 as in Figure 1. In fact, we will see later that there is a connection to the actual S^1.

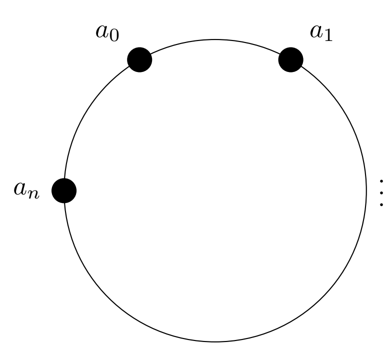


Figure 1. Components of a basic tensor placed along a circle.

Example 4.3. For now, set $R = k$. In this case, we have

$$\mathsf{HH}(k/k) = (\cdots \to k \to k),$$

where the differentials are alternating sums of 1 and -1. In fact, we see that the complex is

$$\mathsf{HH}(k/k) = (\cdots \xrightarrow{0} k \xrightarrow{\mathrm{id}} k \xrightarrow{0} k),$$

and therefore the Hochschild homology is

$$\mathsf{HH}_*(k/k) = \begin{cases} k & * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.4. Because the last differential is given by $a \otimes b \mapsto ab - ba$, we see that

$$\mathsf{HH}_0(R/k) = R/[R,R].$$

In particular, if R is commutative, we have

$$\mathsf{HH}_0(R/k) = R.$$

In this case, we have

$$\mathsf{HH}_1(R/k) = \frac{R \otimes_k R}{\operatorname{Im} \partial} \cong \Omega^1_{R/k}.$$

We will now recall the definition of the module of Kähler differentials.

Definition 4.5. The *module of Kähler differentials* of a commutative ring R is the module $\Omega^1_{R/k}$ generated by $\mathrm{d}x$ for all $x \in R$ subject to the following relations:

(1) For all $x, y \in R$, we have $\mathrm{d}(x+y) = \mathrm{d}x + \mathrm{d}y$;
(2) For all $x, y \in R$, we have $\mathrm{d}(xy) = x\,\mathrm{d}y + y\,\mathrm{d}x$;
(3) For all $x \in k$, we have $\mathrm{d}x = 0$.

Therefore, we see that $\Omega^1_{R/k}$ receives a universal k-linear derivation from R.

There is at most one map

$$\mathsf{HH}_1(R/k) \to \Omega^1_{R/k} \qquad x \otimes y \mapsto x \cdot \mathrm{d}y.$$

Exercise 4.6. Check that this is an isomorphism.

We will now prove some more properties of Hochschild homology.

Lemma 4.7. *If R is commutative, then $\mathsf{HH}_*(R)$ has the structure of a strictly graded-commutative ring. Here, this means that $x^2 = 0$ for any x of odd degree, which is only relevant in characterisic 2.*

Proof. The Hochschild complex $\mathsf{HH}(R/k)$ in fact comes from a simplicial commutative ring

$$\cdots \; R \otimes_k \otimes_k R \; R \otimes_k R \; R,$$

where the face maps are the components of the differential ∂. The desired result then follows by Theorem A.2. $\square$

On the de Rham side, there is also a multiplicative structure.

Definition 4.8. The *de Rham complex* is the exterior algebra

$$\begin{aligned}\Omega^*_{R/k} &\coloneqq \Lambda^*_R \Omega^1_{R/k} \\ &= \bigoplus_{k \geq 0} \Lambda^k_R \Omega^1_{R/k} \\ &= \frac{\operatorname{Sym}_R \Omega^1_{R/k}[1]}{x^2 = 0, x \in \Omega^1_{R/k}}.\end{aligned}$$

By extending from degree 1, we obtain a map $\Omega^*_{R/k} \to \mathsf{HH}_*(R/k)$.

Theorem 4.9 (Hochschild-Kostant-Rosenberg)**.** *If R/k has cotangent complex concentrated in degree 0, then the map*

$$\Omega^*_{R/k} \to \mathsf{HH}_*(R/k)$$

is an isomorphism.

We will discuss the cotangnet complex later, but for now we may think that R is a smooth or ind-smooth k-algebra.

Remark 4.10. The differential on $\Omega^*_{R/k}$ does not play a role here. Studying what it corresponds to on the Hochschild side will leade us to cyclic homology.

Exercise 4.11. Find an explicit description of $\Omega^*_{k[x_1,\dots,x_n]/k}$ and in particular show that it is concentrated in degrees at most n.

We will now extend to the case when k is an arbitrary commutative ring. This will lead to flatness issues, so we will fix this by taking derived tensor products. For now, we will let R be a dg-algebra over k which is K-flat. This is a flatness condition, but when R is bounded below, it agrees with levelwise flatness. Over $\mathbb{Z}$, it always agrees with levelwise flatness. For such an R, we define the Hochschild complex by

$$\mathsf{HH}(R/k) \coloneqq \operatorname{Tot}(\cdots \to R \otimes_k R \to R).$$

If R is not K-flat, we will simply replace it by a K-flat resolution $R^\flat$ and define

$$\mathsf{HH}(R/k) \coloneqq \mathsf{HH}(R^\flat.k).$$

This is well-defined up to quasi-isomorphism.

Proposition 4.12. *For a dg-algebra R, there is a quasi-isomorphism*

$$\mathsf{HH}(R/k) \simeq R \otimes^{\mathbb{L}}_{R \otimes^{\mathbb{L}}_k R^{\mathrm{op}}} R.$$

Remark 4.13. This is usually called Shukla homology, but we will not make this distinction here.

Proof. Replacing R by a flat resolution, we consider the complex

$$\operatorname{Tot}(\cdots \to R \otimes_k R \otimes_k R \to R \otimes_k R) \simeq R.$$

The differentials are given by multiplying adjacent terms, but without the $a_n a_0$ term (think configurations on an interval). This complex is usually called the *bar complex.* Applying $- \otimes_{R \otimes_k R^{\mathrm{op}}} R$, the resulting total complex computes the desired complex $R \otimes^{\mathbb{L}}_{R \otimes_k R^{\mathrm{op}}} R$, but it also computes $\mathsf{HH}_*(R/k)$. $\square$

Lemma 4.14. *If R is a dg-algebra which arises from a simplicial commutative k-algebra, then $\mathsf{HH}(R/k)$ also comes from a simplicial commutative k-algebra. In particular, the Hochschild homology groups $\mathsf{HH}_*(R/k)$ form a strictly graded-commutative k-algebra.*

Proof. We replace R by a flat simplicial commutative k-algebra. Then $\mathsf{HH}(R_n/k)$ gives a **bisimplicial** commutative k-algebra. The total complex is then quasi-isomorphic to the diagonal, which is clearly simplicial. □

Lemma 4.15. *If A, B are dg-algebras, then*

$$\mathsf{HH}(A \otimes^{\mathbb{L}} B/k) \simeq \mathsf{HH}(A/k) \otimes_k^{\mathbb{L}} \mathsf{HH}(B/k).$$

Remark 4.16. If R is commutative, then setting $A = B = R$ gives a multiplication map on HH via the multiplication of R.

Proof. Because $\mathsf{HH}(A/k)$ and $\mathsf{HH}(B/k)$ are simplicial dg-algebras, the tensor product

$$\mathsf{HH}(A/k) \otimes_k^{\mathbb{L}} \mathsf{HH}(B/k)$$

is a bisimplicial dg-algebra. Taking the total complex in all three directions, the result is quasi-isomorphic to the diagonal, which is precisely

$$\mathsf{HH}(A \otimes_k^{\mathbb{L}} B).$$ □

We are now ready to prove the HKR theorem for polynomial rings. As we will see later, we will reduce the general case to this one.

Proof of Theorem 4.9 for polynomial rings. In the case of $R = k[x]$, we have

$$\Omega^*_{k[x]/k} = k[x] \otimes \Lambda(\mathrm{d}x).$$

Then we only need to check that $\mathsf{HH}(k[x]/k)$ vanishes in degree greater than 1. This is given by

$$\begin{aligned} \mathsf{HH}(k[x]/k) &= k[x] \otimes^{\mathbb{L}}_{k[x]\otimes_k k[x]} k[x] \\ &= k[x] \otimes^{\mathbb{L}}_{k[a,b]} k[x]. \end{aligned}$$

We have the resolution

$$k[a,b] \xrightarrow{\cdot(a-b)} k[a,b]$$

of $k[x]$, and this gives the desired result.

For $R = k[x_1, \dots, x_n]$ This is simply a tensor product

$$R = k[x_1] \otimes_k \cdots \otimes_k k[x_n],$$

so because both sides of the HKR theorem commute with tensor products, we obtain the desired result.

Finally, if $R = k[x_i \mid i \in I]$, we write R as a filtered colimit of finitely-generated polynomial algebras by writing the set $\{x_i \mid i \in I\}$ of generators as a filtered colimit of its finite subsets. Because both sides commute with filtered colimits, we are done. □

We will now turn to an extremely non-smooth example.

Proposition 4.17. *We have*

$$\mathsf{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle x\rangle,$$

where the $\langle x\rangle$ means the free divided power algebra generated by x.

Proof. We will compute the derived tensor product

$$\mathbb{F}_p \otimes_{\mathbb{F}_p \otimes^{\mathbb{L}} \mathbb{F}_p} \mathbb{F}_p.$$

We will resolve $\mathbb{F}_p$ by the dg-algebra

$$\mathbb{Z}[\varepsilon]/\varepsilon^2, \qquad \partial\varepsilon = p, \qquad |\varepsilon| = 1.$$

The tensor product is then given by

$$A \simeq \mathbb{F}_p \otimes^{\mathbb{L}} \mathbb{F}_p \simeq \mathbb{F}_p[\varepsilon]/\varepsilon^2.$$

We will resolve $\mathbb{F}_p$ as an A-algebra by

$$A\langle x\rangle = \frac{A[x_1, x_2, \ldots]}{x_i x_j = \binom{i+j}{i} x_{i+j}}, \qquad \partial x_i = \varepsilon x_{i-1}, \qquad |x_i| = 2i,$$

and the desired result follows by taking the tensor product. □

4.2. **The Connes operator.** We will describe an operator on HH which corresponds to the de Rham differential. Recall that the de Rham differential

$$\mathrm{d}\colon \Omega^*_{R/k} \to \Omega^{*+1}_{R/k}$$

is induced from the universal derivation $\mathrm{d}\colon R \to \Omega^1_{R/k}$ by the Leibniz rule.

Definition 4.18. Let R be an associative k-algebra. We define a k-linear map

$$B\colon \mathsf{HH}(R/k)_n \to \mathsf{HH}(R/k)_{n+1}$$

by the formula

$$r_0 \otimes \cdots \otimes r_n \mapsto \sum_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)}(1 \otimes r_\sigma + (-1)^n r_\sigma \otimes 1),$$

where we define

$$r_\sigma \coloneqq r_{\sigma(0)} \otimes \cdots \otimes r_{\sigma(n)}.$$

This formula is related to the *cyclic objects* introduced by Connes, and there is a way to think about it systematically in the topological setting, which we will discuss later.

Exercise 4.19. Check that
(1) $B^2 = 0$;
(2) $\mathrm{d}B + B\,\mathrm{d} = 0$.

The operator B equips $\mathsf{HH}(R/k)$ with the structure of a dg-module over the dg-algebra

$$A \coloneqq k[b]/b^2, \qquad |b| = 1, \qquad \partial = 0.$$

Note that if $\mathbb{T} = U(1)$ is the circle group, then we have

$$A = H_*(\mathbb{T}, k).$$

We then see that $\mathsf{HH}_*(R/k)$ is a graded module over $k[b]/b^2$, and in particular that there is an operator

$$B\colon \mathsf{HH}_*(R/k) \to \mathsf{HH}_{*+1}(R/k)$$

satisfying $B^2 = 0$.

Warning 4.20. One might object that we only defined the Connes operator only for algebras concentrated in degree 0, but with some careful thought we can define B for dg-algebras R/k.

We need the following fact.

Lemma 4.21. *If R is commutative, then B is a derivation. In particular, it satisfies the graded Leibniz rule*

$$B(xy) = B(x)y + (-1)^{|x|}xB(y).$$

Warning 4.22. The Connes operator is not true on the complex $\mathsf{HH}(R/k)$. It is true up to coherent homotopy, but proving this is very hard. Instead, we will later use the topological enhancement to understand this.

Proposition 4.23. *The map*

$$\Omega^*_{R/k} \to \mathsf{HH}_*(R/k)$$

sends the de Rham differential d *to the Connes operator B. In other words, the diagram*

$$\begin{tikzcd} \Omega^*_{R/k} \arrow[r] \arrow[d, "\mathrm{d}"] & \mathsf{HH}_*(R/k) \arrow[d, "B"] \\ \Omega^{*+1}_{R/k} \arrow[r] & \mathsf{HH}_{*+1}(R/k) \end{tikzcd}$$

commutes.

Proof. The map $\Omega^*_{R/k} \to \mathsf{HH}_*(R/k)$ is determined by its effect in degrees 0 and 1. In degree 0, it is given by the identity

$$R \to \mathsf{HH}_0(R/k) = R,$$

and in degree 1, it is given by

$$\Omega^1_{R/k} \to \mathsf{HH}_1(R/k) = \frac{R \otimes_k R}{\sim} \qquad x\,\mathrm{d}y \mapsto [x \otimes y].$$

Therefore, it is enough to check that the diagram

$$\begin{tikzcd} R = \Omega^0_{R/k} \arrow[r] \arrow[d, "\mathrm{d}"] & \mathsf{HH}_0(R/k) \arrow[d, "B"] \\ \Omega^1_{R/k} \arrow[r] & \mathsf{HH}_*(R/k) \end{tikzcd}$$

commutes. Note that for any $r \in R$, we see that all signs in the definition of B are $+1$, so we have

$$B(r) = [1 \otimes r] + [r \otimes 1].$$

This corresponds to the element

$$1 \cdot \mathrm{d}r + r \cdot d1 = \mathrm{d}r \in \Omega^1_{R/k}.$$

We conclude by using the fact that the de Rham differential is uniquely determined by being a derivation and its effect on R. $\square$

Remark 4.24. Sometimes in the rest of these notes, we will denote the Connes operator by d. If this causes confusion, we will call it B.

Example 4.25. For $\mathsf{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle x\rangle$ with x in degree 2, the Connes operator acts trivially for degree reasons.

Question 4.26. *Does this mean that it acts trivially (maybe up to homotopy or quasi-isomorphism) on $\mathsf{HH}(\mathbb{F}_p, \mathbb{Z})$?*

The answer to this question is in fact **no**, and we will see this by taking a (derived) reduction mod p. This means that we will consider

$$\mathsf{HH}(\mathbb{F}_p) /\!\!/ p \coloneqq \mathsf{HH}(F_p) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_p.$$

Letting A act trivially on $\mathbb{F}_p$, we obtain a structure of an A-module on $\mathsf{HH}(\mathbb{F}_p) /\!\!/ p$. We compute

$$\begin{aligned} \mathsf{HH}(\mathbb{F}_p) /\!\!/ p &\simeq \mathsf{HH}(F_p) \otimes_{\mathbb{Z}} (\Lambda_{\mathbb{Z}}(\varepsilon), |\varepsilon| = 1, \partial\varepsilon = p) \\ &\simeq \mathsf{HH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon). \end{aligned}$$

This implies that

$$\begin{aligned} H_*(\mathsf{HH}(\mathbb{F}_p) /\!\!/ p) &\simeq \mathsf{HH}_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon) \\ &\simeq \mathbb{F}_p\langle x \rangle \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon). \end{aligned}$$

Proposition 4.27. *Writing the divided power $\gamma_n(x)$ as $x^{[n]}$, we have $B(x^{[n]}) = 0$ and $B(\varepsilon) = x$.*

Note that this implies that

$$B(\varepsilon x^{[n]}) = x x^{[n]} = (n+1) x^{[n+1]}.$$

Proof. The map $\mathsf{HH}(\mathbb{F}_p) \to \mathsf{HH}(\mathbb{F}_p) /\!\!/ p$ is compatible with B, and therefore we obtain $B(x^{[n]}) = 0$ in homology.

Now recall that we computed

$$\mathsf{HH}_*(\mathbb{F}_p) = \mathbb{F}_p\langle x \rangle$$

by replacing $\mathbb{F}_p$ by $(\Lambda_{\mathbb{Z}}(\varepsilon), \partial\varepsilon = p)$, so we write the complex

$$\cdots \xrightarrow{\partial} \Lambda_{\mathbb{Z}}(\varepsilon) \otimes \Lambda_{\mathbb{Z}}(\varepsilon) \xrightarrow{\partial} \Lambda_{\mathbb{Z}}(\varepsilon).$$

Writing out the full double complex, we obtain

$$\begin{tikzcd}
& \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& \cdots \arrow[r] \arrow[d] & \mathbb{Z}(\varepsilon \otimes \varepsilon) \arrow[r] \arrow[d, "{(-p,p)}"] & 0 \arrow[d] \\
& \cdots \arrow[r] \arrow[d] & \mathbb{Z}(\varepsilon \otimes 1) \oplus \mathbb{Z}(1 \otimes \varepsilon) \arrow[r, "0"] \arrow[d, "{(p,p)}"] & \mathbb{Z}\varepsilon \arrow[d, "p"] \\
{} \arrow[r] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}.
\end{tikzcd}$$

The class x is given by $1 \otimes \varepsilon - \varepsilon \otimes 1$. Applying B, we obtain

$$B(\varepsilon) = 1 \otimes \varepsilon - \varepsilon \otimes 1.$$

After reducing mod p, we can replace all copies of $\mathbb{Z}$ by $\mathbb{F}_p$ and ε becomes a cycle, and we obtain $B(\varepsilon) = x$. $\square$

4.3. **Periodic and cyclic homology.** In the anology between Hochschild homology and the de Rham complex, we will now discuss cyclic homology, which will be analogous to algebraic de Rham cohomology.

We will work with derived functors in the category DGMod_A. The homotopy theory we will work with is inverting quasi-isomorphisms, which means we will consider the ∞-category

$$\mathsf{Mod}_A(D(k)).$$

Definition 4.28. Let R be a k-algebra.

(1) The *cyclic homology* of R is

$$\mathsf{HC}_*(R/k) \coloneqq H_*(k \otimes_A^{\mathbb{L}} \mathsf{HH}(R/k)) = \mathrm{Tor}_*^A(k, \mathsf{HH}(R/k)).$$

This will not play any role in these notes.

(2) The *negative cyclic homology* of R is

$$\mathsf{HC}_*^-(R/k) \coloneqq \mathrm{RHom}_A(k, \mathsf{HH}(R/k)) = \mathrm{Ext}_A^{-*}(k, \mathsf{HH}(R/k)).$$

This is more important than cyclic homology, and topological cyclic homology will be an analog of this rather than of cyclic homology. Also, note that $\mathsf{HC}_*^-(R/k)$ is a module over $\mathrm{Ext}_A^*(k,k) = k[t]$, where $|t| = -2$.

(3) The *periodic cyclic homology* (or *periodic homology*) of R is

$$\mathsf{HP}_*(R/k) = \mathsf{HC}_*^-(R/k)[t^{-1}].$$

Exercise 4.29. Show that $\mathrm{Ext}_A^*(k,k) \cong k[t]$, where $|t| = -2$.

Note that the definition we gave is not the standard one in the literature, which involves double complexes. We may wonder if these definitions are equivalent.

Proposition 4.30. *For any k-algebra R, we have*

$$\mathsf{HC}^-(R/k) \simeq (\mathsf{HH}(R/k)[\![t]\!], \partial + tB),$$

where $\partial + tB$ is defined as

$$xt^i \mapsto (\partial x)t^i + (Bx)t^{i+1}$$

for $x \in \mathsf{HH}(R/k)$ (this is basically saying that $\partial(t) = 0$). Moreover, we have

$$\mathsf{HP}(R/k) \simeq (\mathsf{HH}(R)(\!(t)\!), \partial + tB).$$

Proof. We will resolve k as an A-algebra by

$$C = A\{x_0, x_1, \ldots\}, \qquad |x_k| = 2k,$$

where we declare that

$$\partial(x_k) = bx_{k-1}.$$

Note that C also has the structure of a coalgebra with coproduct given by

$$\Delta(x_k) = \sum_{i+j=k} \sum_{i+j=k} x_i \otimes x_j.$$

This implies that

$$\begin{aligned} \mathrm{RHom}_A(k, \mathsf{HH}(R/k)) &= \underline{\mathrm{Hom}}_A(C, \mathsf{HH}(R/k)) \\ &= (\mathsf{HH}(R/k)[\![t]\!], \partial + tB). \end{aligned}$$

Here, we use the fact that t^i is the dual of the element x_i, and therefore because the differential on C is just b and lowers the x-degree by 1, dually it will be B and raise the t-degree by 1. The formula for $\mathsf{HP}(R/k)$ follows by inverting t. $\square$

Remark 4.31. This argument works more generally to obtain $\mathrm{RHom}(k, H)$ for an arbitrary $H \in \mathsf{DGMod}_A$.

Remark 4.32. A is a Hopf algebra with couint

$$\varepsilon\colon A \to k \qquad \varepsilon(b) = 0$$

and coproduct $\Delta\colon A \to A \otimes A$ given by

$$\Delta(b) = 1 \otimes b + b \otimes 1.$$

This implies that $\underline{\mathrm{RHom}}_A(k, -)$ is a lax symmetric monoidal functor (here, note that k is the unit in the category of dg A-modules) and as such is given by

$$H \mapsto (H[\![t]\!], \partial + tB)$$

as a lax symmetric monoidal functor. Moreover, if C is an algebra object in DGMod_A, i.e. C is a dg algebra and B satisfies the Leibniz rule, then $\underline{\mathrm{RHom}}_A(k, C)$ is a dg algebra.

Warning 4.33. Even if R is commutative, B is **not** a derivation on $\mathsf{HH}(R)$, so we cannot simply argue that negative cyclic and periodic homology have products. This is true up to chain homotopy, however, but this is still not enough to obtain a product. We will see that B is a derivation up to coherent homotopy, but we run into issues with the formality of A, which will require working with S^1 to resolve.

Instead of doing all of that, we will use the hack that $(\mathsf{HH}(R/k)[\![t]\!], \partial + tB)$ has a product up to chain homotopy. This implies that $\mathsf{HC}^-_*(R/k)$ and $\mathsf{HP}_*(R/k)$ are graded commutative algebras.

Our goal now is to enhance the HKR theorem (Theorem 4.9). For this, we will need de Rham cohomology.

Definition 4.34. Let R be a commutative k-algebra. The *de Rham cohomology* of R relative to k is defined as

$$H^*_{\mathsf{dR}}(R/k) \coloneqq H_*(\Omega^*(R/k), d).$$

Remark 4.35. Since we are working with commutative rings (equivalently, affine schemes), every term in the de Rham complex is acyclic by vanishing of higher cohomology on affines, so we are just taking global sections. However, this invariant is more interesting for schemes X/k by upgrading the de Rham complex to a complex of sheaves.

Theorem 4.36. *Assume that $\mathbb{Q} \subseteq k$ and that $L_{R/k}$ is flat and concentrated in degree 0. Then there are natural isomorphisms*

$$\mathsf{HP}_*(R/k) \cong H^*_{\mathsf{dR}}(R/k)(\!(t)\!),$$

where $|t| = -2$ and

$$\mathsf{HC}^-_n(R/k) \cong Z^n_{\mathsf{dR}}(R/k) \oplus \prod_{i \geq 1} H^{n+2i}_{\mathsf{dR}}(R/k).$$

Exercise 4.37. Describe the map $\mathsf{HC}^-_* \to \mathsf{HP}_*$ in this case and check that it exhibits

$$\mathsf{HP}_* \cong \mathsf{HC}^-_*[t^{-1}].$$

Remark 4.38. Because this is again relatively boring for rings, the interesting statement is that for a scheme X/k with $L_{X.k}$ being flat and concentrated in degree 0, we have

$$\mathsf{HP}_*(X/k) \cong H^*_{\mathsf{dR}}(X,k)(\!(t)\!).$$

Remark 4.39. We can drop the flatness assumption on R if we replace de Rham cohomology with a Hodge-completed derived de Rham cohomology following Bhatt-Morrow-Scholze, but this is of course extremely complicated.

Remark 4.40. The statement of Theorem 4.36 is false if k is not characteric 0.

Proof. We will give a classical proof. It is possible to give a fancier proof using nonabelian derived functors.

Consider a chain-level map

$$\mu\colon \mathsf{HH}(R/k) \to \Omega^*_{R/k}$$

given by

$$r_0 \otimes r_1 \otimes \cdots \otimes r_n \mapsto \frac{1}{n!} r_0\,\mathrm{d}r_1 \cdots \mathrm{d}r_n.$$

Lemma 4.41. *This μ is an A-linear map of commutative dg algebras*

$$(\mathsf{HH}(R/k), \partial, B) \to (\Omega^*_{R/k}, 0, \mathrm{d}).$$

Proof of Lemma. We compute

$$\begin{aligned}
&\mu(\partial(r_0 \otimes \cdots \otimes r_n))\\
&= \mu(r_0r_1 \otimes r_2 \otimes \otimes r_n - r_0 \otimes r_1r_2 \otimes \cdots \otimes r_n + \cdots \pm r_nr_0 \otimes r_1 \otimes \cdots \otimes r_{n-1})\\
&= r_0r_1\,\mathrm{d}r_2 \cdots \mathrm{d}r_n\\
&\quad - r_0\,\mathrm{d}(r_1r_2)\,\mathrm{d}r_3 \cdots \mathrm{d}r_n\\
&\quad + \cdots\\
&= r_0r_1\,\mathrm{d}r_2 \cdots \mathrm{d}r_n\\
&\quad - r_0r_1\,\mathrm{d}r_2 \cdots \mathrm{d}r_n\\
&\quad + \cdots\\
&\quad = 0.
\end{aligned}$$

Compatibility with B and d is given by the equation

$$\begin{aligned}
\mu(B(r_0 \otimes \cdots \otimes r_n)) &= \sum_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)} \mu(1 \otimes r_\sigma \pm r_\sigma \otimes 1)\\
&= \frac{1}{(n+1)!} \sum_{\sigma \in C_{n+1}} \operatorname{sign}(\sigma)\,\mathrm{d}r_{\sigma(0)} \cdots \mathrm{d}r_{\sigma(n)}\\
&= \frac{1}{n!}\,\mathrm{d}r_0 \otimes \cdots \otimes \mathrm{d}r_n\\
&= \mathrm{d}\left(\frac{1}{n!} r_0\,\mathrm{d}r_1 \cdots \mathrm{d}r_n\right)\\
&= \mathrm{d}(\mu(r_0 \otimes \cdots \otimes r_n)).
\end{aligned}$$

We conclude with the following exercise.

Exercise 4.42. Check that μ is multiplicative with respect to the shuffle product on the left and the wedge product of forms on the right. $\square$

Using the lemma, we see that

$$\Omega^*_{R/k} \to \mathsf{HH}_*(R/k) \xrightarrow{H_*(\mu)} \Omega^*_{R/k}$$

is the identity, so the statement follows from Theorem 4.9. □

We are now ready to prove Theorem 4.36.

Proof of Theorem 4.36. We simply compute

$$\begin{aligned}\mathsf{HP}_*(R/k) &= H_*(\mathsf{HH}(R/k)(\!(t)\!), \partial + tB) \\ &\simeq H_*(\Omega^*_{R/k}(\!(t)\!), t\,\mathrm{d}) \\ &\cong H^*_{\mathsf{dR}}(R/k)(\!(t)\!).\end{aligned}$$

Applying the same strategy to negative cyclic homology, we obtain

$$\mathsf{HC}^-_*(R/k) \cong H_*(\Omega^*_{R/k}[\![t]\!], t\,\mathrm{d}).$$ □

We still want to deal with arbitary k (so not in characteristic 0).

Construction 4.43. We have the *Postnikov filtration*

$$\tau_{\geq \bullet}\mathsf{HH}(R/k)$$

of $\mathsf{HH}(R/k)$ as an A-module, which is compatible with B and leads to a filtration of

$$(\tau_{\geq \bullet}\mathsf{HH}(R/k)(\!(t)\!), \partial + tB).$$

$\mathsf{HP}(R/k)$.

This leads to a multiplicative, conditionally convergent spectral sequence

$$E_2 = \mathsf{HH}_*(R/k)(\!(t)\!) \Longrightarrow \mathsf{HP}_*(R/k).$$

The E_3-page is given by

$$H_*(\mathsf{HH}_*(R/k), B)(\!(t)\!) \Longrightarrow \mathsf{HP}_*(R/k).$$

If R has flat cotangent complex, then this is

$$E^3 = H^*_{\mathsf{dR}}(R/k)(\!(t)\!) \Longrightarrow \mathsf{HP}_*(R/k).$$

We will now return to the example of $\mathbb{F}_p$ over $\mathbb{Z}$. First, we need to upgrade divided powers to divided power series.

Definition 4.44. Let R be a commutative ring. We define the *divided power series algebra* $R\langle\!\langle x\rangle\!\rangle$ as the completion of $R\langle x\rangle$ at the filtration generated by the divided powers of x (note this is not an adic completion).

Proposition 4.45. *For $R = \mathbb{F}_p$, we have*

$$\mathsf{HC}^-_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{Z}_p[t]\langle\!\langle x\rangle\!\rangle/(xt - p), \qquad |x| = 2, |t| = -2.$$

In addition, we have

$$\begin{aligned}\mathsf{HP}_*(\mathbb{F}_p/\mathbb{Z}) &\cong \mathbb{Z}_p[t^{\pm}]\langle\!\langle x\rangle\!\rangle/(xt - p) \\ &\cong (\mathbb{Z}_p\langle\!\langle y\rangle\!\rangle/(y-p))[t^{\pm}],\end{aligned}$$

where y has degree 0.

Remark 4.46. Note that we have

$$\mathsf{HP}_0(\mathbb{F}_p/\mathbb{Z}) \cong \mathsf{HC}_0^-(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{Z}_p\langle\langle y\rangle\rangle/(y-p),$$

which is obtained by adjoining divided powers of p to $\mathbb{Z}_p$. Because $\mathbb{Z}_p$ already has divided powers of p, the element $y^{[p]} - \frac{p^p}{p!}$ is p-torsion. In fact, we have

$$\mathbb{Z}_p\langle y\rangle/(y-p) \cong \mathbb{Z}_p\langle z\rangle/z,$$

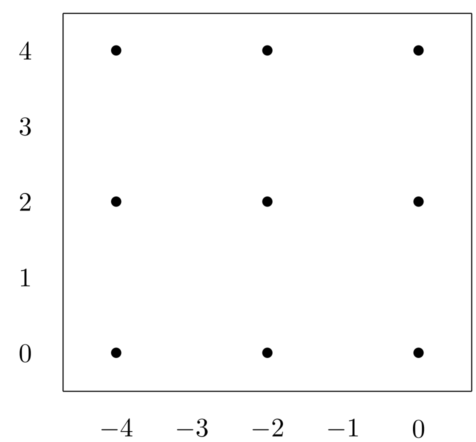
where we should think of z as $y-p$.

What we have seen is that the negative cyclic and periodic cyclic homology of $\mathbb{F}_p$ is very unpleasant. The situation will improve when we consider the topological version of everything. Also, in fact, $\mathsf{HP}_*(\mathbb{F}_p/\mathbb{Z})$ is the 2-periodic derived de Rham cohomology of $\mathbb{F}_p$.

Proof. Recall that $\mathsf{HH}_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{F}_p\langle x\rangle$. We also recall the spectral sequence coming from the Postnikov filtration, which reads

$$\mathbb{F}_p\langle x\rangle[t] \Longrightarrow \mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z}).$$

The E_2-page of the spectral sequence is given below, where the horizontal direction is powers of t and the vertical direction is divided powers of x. We also use dots to denote copies of $\mathbb{F}_p$.



Because everything is in even degree, the spectral sequence collapses at the E_2-page. Therefore, $\mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z})$ has a filtration with associated graded given by $\mathbb{F}_p\langle x\rangle[t]$.

There is an additive extension problem and a multiplicative extension problem. We will obtain a copy of $\mathbb{Z}_p$ by using the diagonal (with slope 1) entries, and we also need to check that divided powers lift from the associated graded.

The connective cover of $\mathsf{HC}^-(\mathbb{F}_p/\mathbb{Z})$ can be represented by a simplicial commutative ring. Later, we will study everything using homotopy theory and see that the Connes operator comes from an S^1-action on animated rings, so we can just take homotopy fixed points to obtain HC^-. In any case, the connective cover admits divided powers on positive degree homotopy groups. In particular, every choice of $x, t \in \mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z})$ yields a map

$$\mathbb{Z}\langle x\rangle[t] \to \mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z}).$$

If we can find x, t such that $\mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z})$, then this map induces an isomorphism on associated gradeds of the map

$$\mathbb{Z}\langle x\rangle/(xt - p) \to \mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z}).$$

Because the filtration on $\mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z})$ is complete, completing $\mathbb{Z}\langle x\rangle/(xt - p)$ yields the desired result.

Now we will recall the computation of $\mathsf{HH}(\mathbb{F}_p/\mathbb{Z})$. We first resolved $\mathbb{F}_p$ by

$$\mathbb{F}_p \simeq (\Lambda_{\mathbb{Z}}(\varepsilon), \partial\varepsilon = p).$$

In $\mathsf{HH}(\mathbb{F}_p/\mathbb{Z})$, we have $x = B\varepsilon$, $\partial x = 0$, and $Bx = 0$. It follows that in $\mathsf{HH}(\mathbb{F}_p/\mathbb{Z})$, we have a cycle x representing $x \in HH_*(\mathbb{F}_p/\mathbb{Z})$. Using the fact that

$$\mathsf{HC}^-(\mathbb{F}_p/\mathbb{Z}) = (\mathsf{HH}(\mathbb{F}_p/\mathbb{Z})[\![t]\!], \partial + tB),$$

we calculate

$$(\partial + tB)\varepsilon p + tx,$$

and therefore $p = tx$ in $\mathsf{HC}_*^-(\mathbb{F}_p/\mathbb{Z})$. $\square$

Remark 4.47. One can also deduce this entire computation using the fact that

$$(\mathsf{HH}(\mathbb{F}_p/\mathbb{Z}), \partial, B) \simeq \left(\frac{\mathbb{Z}[\varepsilon]}{\varepsilon^2}\langle x\rangle, \begin{matrix}\partial\varepsilon = p & B\varepsilon = x \\ \partial x = 0\text{'} & Bx = 0\end{matrix}\right).$$

Remark 4.48. We have not talked at all about cyclic homology, but there is a long exact sequence

$$\begin{tikzcd} & & \mathsf{HP}_{*-1}(R/k) \arrow[dll] \\ \mathsf{HC}_{*-1}(R/k) \arrow[r] & \mathsf{HC}_*^-(R/k) \arrow[r] & \mathsf{HP}_*(R/k) \arrow[dll] \\ \mathsf{HC}_{*-2}(R/k) \arrow[r] & \cdots & \end{tikzcd}$$

which computes $\mathsf{HC}(R/k)$. This comes from a cofiber sequence, or distinguished triangle.

4.4. **The HKR theorem.**

We will now discuss the cotangent complex using nonabelian derived functors and prove the HKR theorem.

Let $\mathcal{C}$ be a 1-category, which is generated by compact projective objects. Here, an object K in *compact projective* if

$$\mathrm{Hom}(K, -)$$

preserves filtered colimits and reflexive coequalizers, which are diagrams of the form

$$\bullet \Longleftarrow\!\!\!\Longrightarrow \bullet.$$

Note that this notion doesn't really make sense for ∞-categories. Being generated by compact projective objects means that all

$$\mathrm{Hom}(K, -)$$

detect isomorphisms.

We can now pass from $\mathcal{C}$ to the full subcategory on compact projective objects. If $\mathcal{C} = \mathsf{Set}$, then we get finite sets, and if $\mathsf{C} = \mathsf{Mod}_R$, then we get finitely generated projective modules. We can then construct the *animation* of $\mathcal{C}$, which is given by

$$\mathsf{Anim}(\mathcal{C}) = \mathsf{Fun}^{\Pi}((\mathcal{C}^{\mathsf{cp}})^{\mathrm{op}}, \mathcal{S}),$$

where $\mathcal{S}$ denotes the ∞-category of anima (spaces or simplicial sets). Some properties of this include the following:

- The category $\mathcal{C}^{\mathsf{cp}}$ of compact projective objects is a full subcategory of $\mathsf{Anim}(\mathcal{C})$, and so is the ind-category $\mathsf{Ind}(\mathcal{C}^{\mathsf{cp}})$.
- A general object of $\mathcal{C}$ can be represented as the geometric realization of a simplicial diagram with entries in $\mathsf{Ind}(\mathcal{C}^{\mathsf{cp}})$.
- If X is compact projective object of $\mathcal{C}$ and Y_j are any ind-compact projective objects, then the mapping space
$$\mathsf{Map}_{\mathsf{Anim}(\mathcal{C})}(X, \operatorname*{colim}_{j\in\Delta^{\mathrm{op}}} Y_j) = \operatorname*{colim}_{j\in\Delta^{\mathrm{op}}} \mathsf{Map}_{\mathsf{Anim}(\mathcal{C})}(X, Y_j)$$
is a simplicial set, so taking the geometric realization gives us a space.

Example 4.49. If $\mathcal{C} = \mathsf{Set}$, functors are determined by their effect on the singleton set, so we just get $\mathsf{Anim}(\mathsf{Set}) = \mathsf{Spaces}$.

Example 4.50. The *non-negative derived category* $D(\mathcal{A})_{\geq 0}$ of an abelian category with enough compact projectives is equivalent to the animation $\mathsf{Anim}(\mathcal{A})$. One advantage of this description is that any functor out of $\mathsf{Anim}(\mathcal{A})$ preserving filtered colimits and reflexive coequalizers is simply a functor out of the 1-category $\mathcal{C}^{\mathsf{cp}}$. For example, if we have a nonadditive functor

$$F\colon \mathcal{A} \to \mathcal{B}(\hookrightarrow \mathsf{Anim}(\mathcal{B})),$$

this will extend to a functor preserving filtered colimits and geometric realizations denoted by

$$LF\colon \mathsf{Anim}(\mathcal{A}) \to \mathsf{Anim}(\mathcal{B}).$$

Example 4.51. For any commutative ring R, there is a functor

$$\Lambda^n_R\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$$

taking a module to its n-th exterior power. This is clearly nonadditive, but there is a derived functor

$$L\Lambda^n_R\colon D(R)_{\geq 0} \to D(R)_{\geq 0}.$$

We cannot compute by resolving chain complexes, so instead we will represent an object by a simplicial diagram of projective modules and then applying Λ^n_R levelwise.

For example, consider $R = \mathbb{Z}$. We will attempt to resolve $\mathbb{Z}/n$, and this is resolved by the chain complex

$$\mathbb{Z} \xrightarrow{n} \mathbb{Z}.$$

Applying the Dold-Kan correspondence, we obtain a simplicial object

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathbb{Z}\oplus\mathbb{Z} \;\substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \mathbb{Z}.$$

Exercise 4.52.

(1) Find a levelwise projective simplicial abelian group whose complex is quasi-isomorphic to $\mathbb{Z}/n[0]$.
(2) Compute $L\Lambda^2_{\mathbb{Z}}(\mathbb{Z}/n)$. *Hint: compute H_i for $i \leq 2$ by hand and then find a different reason for the vanishing for higher i.*

If we apply $\Lambda^2_{\mathbb{Z}}$ to this, we obtain a simplicial diagram

$$\cdots \,\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow}\, \Lambda^2_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}) \,\substack{\longleftarrow\\\longrightarrow}\, \mathbb{Z},$$

which is simply given by

$$\cdots \,\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow}\, \mathbb{Z} \,\substack{\longleftarrow\\\longrightarrow}\, 0,$$

which is equivalent to $\mathbb{Z}/n[1]$. Note that this is not equivalent to applying $\Lambda^2_{\mathbb{Z}}$ levelwise to the chain complex we started with, and that a functor commutes with Dold-Kan only if it is additive.

One advantage of this animation language is that we can discuss animation for categories which have no linear structure. For example, we have

$$\begin{aligned} \mathsf{Anim}(\mathsf{cRing}) &= \mathsf{Fun}^{\Pi}((\mathsf{cRing}^{\mathsf{cp}})^{\mathrm{op}}, \mathcal{S}) \\ &\cong \mathsf{Fun}^{\Pi}((\mathsf{Poly}^{\mathsf{fg}})^{\mathrm{op}}, \mathcal{S}). \end{aligned}$$

Clearly we see that the ind-compact projective objects

$$\mathsf{Ind}(\mathsf{cRing}^{\mathsf{cp}}) = \mathsf{Poly}$$

are simply all polynomial rings. Therefore, we will represent objects by simplicial diagrams of polynomial rings. For example, we have

$$\mathsf{Map}_{\mathsf{Anim}(\mathsf{cRing})}(\mathbb{Z}[x], \operatorname*{colim}_{\Delta^{\mathrm{op}}} Y_i) = \operatorname*{colim}_{\Delta^{\mathrm{op}}} Y_i$$

extracts the underlying simplicial set (in fact a simplicial abelian group). This give some version of non-negative complexes with ring structure.

Warning 4.53. This **does not** agree with commutative ring objects in $D(\mathbb{Z})$.

The universal property is that

$$\begin{aligned} \mathsf{Fun}^{\mathrm{geom,filtered}}(\mathsf{Anim}(\mathsf{cRing}), \mathcal{D}) &= \mathsf{Fun}(\mathsf{Poly}^{\mathsf{fg}}, \mathcal{D}) \\ &= \mathsf{Fun}^{\mathrm{filtered}}(\mathsf{Poly}, \mathcal{D}). \end{aligned}$$

This allows us to define nonabelian derived functors by defining functors on polynomial rings which preserve filtered colimits.

Example 4.54. The functor $\mathsf{HH}(-/\mathbb{Z}) \colon \mathsf{Poly} \to D(\mathbb{Z})_{\geq 0}$ commutes with filtered colimits. Therefore, it extends to a functor

$$L\mathsf{HH} \colon \mathsf{Anim}(\mathsf{cRing}) \to D(\mathbb{Z})_{\geq 0}.$$

If we apply it to an ordinary commutative ring, this agrees with our previous definiion of Hochschild homology. In fact, it agrees for all animated rings by considering the associated dg-algebra.

Remark 4.55. There is a way to define Hochschild homology in ∞-categories in much greater generality which does not require animated rings.

Question 4.56. *Is there a way to extend the HKR theorem to general rings using non-abelian derived functors?*

Construction 4.57. For any $C \in D(\mathbb{Z})$, we can cut off homology groups using the functor $\tau_{\geq n} \colon D(\mathbb{Z}) \to D(\mathbb{Z})$. This produces a map

$$\tau_{\geq n} C \to C$$

which has the following properties:

- The map is an isomorphism on H_i for $i \geq n$;
- For all $i < n$, we have $H_i(\tau_{\geq n} C) = 0$.

One explicit construction of this is to replace everything in degree $< n$ by 0 and to replace the degree n part with the cycles.

Exercise 4.58. Construct the functor $\tau_{\geq n} \colon D(\mathbb{Z}) \to D(\mathbb{Z})$ using an adjunction.

We in fact have a map $\tau_{\geq n+1} C \to \tau_{\geq n} C$, and the cofiber is given by

$$\mathrm{Cofiber}(\tau_{\geq n+1} C \to \tau_{\geq n} C) \simeq H_n(C)[n].$$

For polynomial rings R, we have a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \tau_{\geq n+1}\mathsf{HH}(R/\mathbb{Z}) \arrow[d] & \\
 & \tau_{\geq n}\mathsf{HH}(R/\mathbb{Z}) \arrow[r] \arrow[d] & \Omega^n_{R/\mathbb{Z}}[n] \\
 & \vdots \arrow[d] & \\
\mathsf{HH}(R/\mathbb{Z}) \arrow[r, "\sim"] & \tau_{\geq 0}\mathsf{HH}(R/\mathbb{Z}). &
\end{tikzcd}$$

Therefore, we can try to compute Hochschild homology using the machinery of nonabelian derived functors by levelwise using the HKR theorem for this "filtration."

Definition 4.59. We define

$$F^n_{\mathsf{HKR}}\mathsf{HH}(-/\mathbb{Z}) \colon \mathsf{Anim}(\mathsf{cRing}) \to D(\mathbb{Z})_{\geq 0}$$

as the geometric realization preserving extension of $\tau_{\geq n}\mathsf{HH}(-/\mathbb{Z})$ from Poly.

We see that there is a cofiber sequence

$$F^n_{\mathsf{HKR}}\mathsf{HH}(R/\mathbb{Z}) \to F^n_{\mathsf{HKR}}(R/\mathbb{Z}) \to L\Omega^n_{R/\mathbb{Z}}[n]$$

for any animated ring R.

Lemma 4.60. *The nonabelian derived functor $F^n_{\mathsf{HKR}}(-/\mathbb{Z})$ lands in $D(\mathbb{Z})_{\geq n}$. More precisely, for any animated ring R, we have*

$$F^n_{\mathsf{HKR}}\mathsf{HH}(R/\mathbb{Z}) \in D(\mathbb{Z})_{\geq n}.$$

Proof. The lemma clearly holds for polynomial rings, and the extension to everything else follows from the fact that $D(\mathbb{Z})_{\geq n}$ is closed under filtered colimits. □

Lemma 4.61. *If a functor $F \colon \mathsf{cRing} \to \mathsf{Ab}$ commutes with reflexive coequalizers, then*

$$H_0(LF(R)) = F(R).$$

Proof. We resolve R by a simplicial diagram of polynomial rings as

$$R_0 \Lleftarrow R_1 \Lleftarrow \cdots$$

and note that R is the reflexive coequalizer of

$$R_0 \Lleftarrow R_1.$$

Therefore, $LF(R)$ is the total complex of

$$F(R_0) \Lleftarrow F(R_1) \Lleftarrow \cdots$$

so taking H_0, we obtain the reflexive coequalizer of

$$F(R_0) \Lleftarrow F(R_1).$$

$\square$

Exercise 4.62. Show that the following functors commute with reflexive coequalizers:

(1) The functor $\mathsf{cRing} \to \mathsf{Set}$ given by $R \mapsto R^{\times n}$;
(2) The functor $\mathsf{cRing} \to \mathsf{Ab}$ given by $R \mapsto \mathbb{Z}[R^{\times n}]$;
(3) The functor $\Omega^i_{-/\mathbb{Z}} \colon \mathsf{cRing} \to \mathsf{Ab}$. *Hint: think about the generators and relations presentation for* $\Omega^1_{R/\mathbb{Z}}$.

We are now ready to give an enhanced version of Theorem 4.9.

Theorem 4.63. *If $L\Omega^n_{R/\mathbb{Z}}$ is concentrated in degree 0 for each n, then Theorem 4.9 holds for R.*

Proof. Using the long exact sequence in homology associated to

$$F^{n+1}_{\mathsf{HKR}} \to F^n_{\mathsf{HKR}} \to L\Omega^n_{R/\mathbb{Z}}[n],$$

This implies that

$$H_n(F^n_{\mathsf{HKR}}) \to H_0(L\Omega^n_{R/\mathbb{Z}}) = \Omega^n_{R/\mathbb{Z}}$$

is an isomorphism and that for all $i > n$, the induced map

$$H_i(F^{n+1}_{\mathsf{HKR}}) \to H_i(F^n_{\mathsf{HKR}})$$

is an isomorphism. Descending to R^0, we obtain

$$\Omega^n_{R/\mathbb{Z}} \simeq H_n F^n_{\mathsf{HKR}} \simeq H_n F^0_{\mathsf{HKR}} = \mathsf{HH}_n(R/\mathbb{Z}).$$

We will omit checking that this is compatible with multiplication, and that in fact it is the map we had before. $\square$

To better understand this, we need to better understand $L\Omega^n_{R/\mathbb{Z}}$. This will in fact agree with the value on R of the nonabelian derived functor

$$\mathsf{Anim}(\mathsf{cRing}/R) \to D(R)_{\geq 0} \to D(\mathbb{Z})_{\geq 0}$$

which is given by

$$A \mapsto R \otimes_A \Omega^n_{A/\mathbb{Z}}$$

for $A \in \mathsf{Poly}/R$. By definition, we have

$$R \otimes_A \Omega^n_{A/\mathbb{Z}} \simeq \Lambda^n_R(R \otimes_A \Omega^1_{A/\mathbb{Z}}).$$

Exercise 4.64. Check that $A \mapsto R \otimes_A \Omega^1_{A/k}$ takes compact projective objects in $k\text{−}\mathsf{Alg}/R$ to compact projective objects in Mod_R.

Applying nonabelian derived functors, we obtain

$$L\Omega^n_{R/\mathbb{Z}} = L\Lambda^n_R(L\Omega^1_{R/\mathbb{Z}}).$$

Proposition 4.65. *If $L\Omega^1_{R/\mathbb{Z}}$ is concentraded in degree 0 and $\Omega^1_{R/\mathbb{Z}}$ is a flat R-module, then $L\Omega^n_{R/\mathbb{Z}}$ is also concentrated in degree 0.*

Proof. If $\Omega^1_{R/\mathbb{Z}}$ is projective, then

$$L\Lambda^n_R(\Omega^1_{R/\mathbb{Z}}) = \Lambda^n_R(\Omega^1_{R/\mathbb{Z}}).$$

In the general case, we use Lazard's theorem, which states that every flat R-module is a filtered colimit of finitely generated projective modules. □

We are now able to state the correct form of Theorem 4.9.

Theorem 4.66 (Hochschild-Kostant-Rosenberg, ultimate version)**.** *If $L\Omega^1_{R/k}$ is concentrated in degree 0 and $\Omega^1_{R/\mathbb{Z}}$ is a flat R-module, then*

$$\mathsf{HH}_n(R/\mathbb{Z}) = \Omega^n_{R/\mathbb{Z}}.$$

Remark 4.67. For simplicity, we worked over $\mathbb{Z}$, but everything still holds if we replace $\mathbb{Z}$ be an arbitrary commutative ring k.

4.5. **Hochschild homology of schemes.** Let k be a commutative ring and X be a scheme over k. We could in principle allow the base to be non-affine, but for simplicity we will ignore this. There are two approaches:

(1) Extending from the affine case (by a right Kan extension);
(2) Generalize the functor $\mathsf{HH}(-/k)$ to k-linear dg categories (or stable ∞-categories) and then define define

$$\mathsf{HH}(X/k) = \mathsf{HH}(\mathsf{Perf}(X)/k).$$

This is essentially an approach in the spirit of non-commutative geometry and is actually better in many cases (for example, we see that Hochschild homology is really an invariant of a stable ∞-category), but we will take the first approach in these notes since it is closer to the case of rings.

Definition 4.68. For a scheme X, we define its *Hochschild homology* by

$$\mathsf{HH}(X/k) = \lim_{\substack{U\subseteq X \\ U \text{ affine open}}} \mathsf{HH}(\mathcal{O}(U)/k) \in D(k).$$

Note that if $X = \operatorname{Spec} R$ is affine, then

$$\begin{aligned}\mathsf{HH}(\operatorname{Spec}(R)/k) &= \lim_{U\subseteq \operatorname{Spec} R} \mathsf{HH}(\mathcal{O}(U), k) \\ &= \mathsf{HH}(\mathcal{O}(\operatorname{Spec} R)/k) \\ &= \mathsf{HH}(R/k)\end{aligned}$$

because $U = \operatorname{Spec} R$ is terminal in the poset of affine open subsets of $\operatorname{Spec} R$.

Remark 4.69. We can also define Hochschild homology for stacks. For simplicity, we will not do this.

We will now discuss an example of a non-affine scheme. Let $X = \mathbb{P}^1_k$. It has an open cover by

$$(\mathbb{A}^1_k)^+ = \operatorname{Spec} k[x] \qquad \text{and} \qquad (\mathbb{A}^1_k)^- = \operatorname{Spec} k[y],$$

whose intersection is given by

$$(\mathbb{A}^1_k)^+ \cap (\mathbb{A}^1_k)^- = \mathbb{G}_m = \operatorname{Spec} k[x^{\pm}].$$

We will hope for some kind of descent of Mayer-Vietoric sequence.

Theorem 4.70. *For any pair of open subsets $U, V \subseteq X$ such that $X = U \cup V$, the square*

$$\begin{tikzcd} \mathsf{HH}(X/k) \arrow[r] \arrow[d] & \mathsf{HH}(U/k) \arrow[d] \\ \mathsf{HH}(V/k) \arrow[r] & \mathsf{HH}((U \cap V)/k) \end{tikzcd}$$

is a pullback in $D(k)$. In other words, Hochschild homology satisfies Zariski descent.

Remark 4.71. Later, we will see that $\mathsf{HH}(-/k)$ satisfies fpqc descent.

Corollary 4.72. *There is a long exact sequence*

$$\begin{tikzcd} & \cdots \arrow[r] & \mathsf{HH}_{n-1}(U \cap V) \arrow[dll, out=-30, in=150] \\ \mathsf{HH}_n(X) \arrow[r] & \mathsf{HH}_n(U) \oplus \mathsf{HH}_n(V) \arrow[r] & \mathsf{HH}_n(U \cap V) \arrow[dll, out=-30, in=150] \\ \mathsf{HH}_{n-1}(X) \arrow[r] & \cdots & \end{tikzcd}$$

Lemma 4.73. *For any commutative ring R and $x \in R$, we have*

$$\mathsf{HH}(R[x^{-1}]/k) \cong \mathsf{HH}(R/k) \otimes_R R[x^{-1}].$$

Exercise 4.74. Prove this.

In the example of $\mathbb{P}^1$, we have a pullback

$$\begin{tikzcd} \mathsf{HH}(\mathbb{P}^1/k) \arrow[r] \arrow[d] & \mathsf{HH}(k[x]/k) \arrow[d, "x \mapsto x"] \\ \mathsf{HH}(k[y]/k) \arrow[r, "y \mapsto x^{-1}"] & \mathsf{HH}(k[x^{\pm}]/k). \end{tikzcd}$$

This becomes the long exact sequence

$$\begin{tikzcd}
& & 0 \arrow[dll, rounded corners, to path={...}] \\
\mathsf{HH}_*(\mathbb{P}^1/k) \arrow[r] & k[x]\,\mathrm{d}x \oplus k[y]\,\mathrm{d}y \arrow[r, "{(f\mathrm{d}x, g\mathrm{d}y) \mapsto f(x)\mathrm{d}x + g(x^{-1})\left(-\frac{1}{x^2}\right)\mathrm{d}x}"'] & k[x^{\pm}]\,\mathrm{d}x \arrow[dll] \\
\mathsf{HH}_0(\mathbb{P}^1/k) \arrow[r] & k[x] \oplus k[y] \arrow[r, "{(f,g) \mapsto f(x) + g(x^{-1})}"'] & k[x^{\pm}] \arrow[dll] \\
\mathsf{HH}_{-1}(\mathbb{P}^1/k) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, it is easy to compute

$$\mathsf{HH}_1(\mathbb{P}^1/k) = 0, \qquad \mathsf{HH}_0(\mathbb{P}^1/k) = k \oplus k, \qquad \text{and} \qquad \mathsf{HH}_{-1}(\mathbb{P}^1/k) = 0.$$

Proof of Theorem 4.70. We always have the square

$$\begin{tikzcd}
\mathsf{HH}(X) \arrow[r] \arrow[d] & \mathsf{HH}(U) \arrow[d] \\
\mathsf{HH}(V) \arrow[r] & \mathsf{HH}(U \cap V).
\end{tikzcd}$$

For any open set $U \subset X$, a cofinality argument gives

$$\mathsf{HH}(U) = \lim_{A \subseteq X} \mathsf{HH}(A \cap U).$$

Therefore, without loss of generality that X is affine. Unfortunately, affine schemes can have non-affine open subsets, but another cofinality argument allows us to assume that U and V are affine and are standard opens. Therefore, we have reduced to the case of

$$\begin{tikzcd}
\mathsf{HH}(R/k) \arrow[r] \arrow[d] & \mathsf{HH}(R[x^{-1}]/k) \arrow[d] \\
\mathsf{HH}(R[y^{-1}]/k) \arrow[r] & \mathsf{HH}(R[x^{-1}, y^{-1}]/k),
\end{tikzcd}$$

where $1 \in (x, y)$. Using the fact that Hochschild homology commutes with localization, this becomes

$$\begin{tikzcd}
\mathsf{HH}(R/k) \arrow[r] \arrow[d] & \mathsf{HH}(R]/k) \otimes_R R[x^{-1}] \arrow[d] \\
\mathsf{HH}(R/k) \otimes_R R[y^{-1}] \arrow[r] & \mathsf{HH}(R/k) \otimes_R R[x^{-1}, y^{-1}].
\end{tikzcd}$$

Therefore, it suffices to prove that

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R[x^{-1}] \arrow[d] \\
R[y^{-1}] \arrow[r] & R[x^{-1}, y^{-1}],
\end{tikzcd}$$

which is a standard fact. This is true because

$$R[x^{-1}]/R \cong R[x^{-1}, y^{-1}]/R[y^{-1}],$$

which is true because y acts invertibly on $R[x^{-1}]/R$. This is true because x is invertible on R/y. □

Our goal now is to prove an analog of Theorem 4.63 and Theorem 4.66 for schemes. We begin by defining a filtration on $\mathsf{HH}(X/k)$ by the formula

$$F^n_{\mathsf{HKR}}\mathsf{HH}(X/k) = \lim_{\substack{U \subseteq X \\ \text{affine open}}} F^n_{\mathsf{HKR}}(\mathsf{HH}(\mathcal{O}(U)/k)).$$

Remark 4.75. Note that this cannot be done using the non-commutative approach.

Proposition 4.76. *This defines a complete filtration on* $\mathsf{HH}(X/k)$*, namely that*

$$\varinjlim_n F^n_{\mathsf{HKR}}\mathsf{HH}(X/k) = 0.$$

Moreover, the associated graded is given by

$$\lim_{U \subseteq X} L\Omega^n_{U/k}[n] = R\Gamma(X, L\Omega^n_{X/k})[n],$$

which is usually called the derived Hodge cohomology of X*.*

Note that if X is smooth, we just get the usual Hodge cohomology.

Proof. Completeness follows from the fact that it is true for affines and that inverse limits commute with inverse limits. The assertion about (derived) Hodge cohomology follows again from the fact that cofibers commute with limits. □

Using the proposition, we obtain a spectral sequence

$$R\Gamma(X, L\Omega^n_{X/k}[n]) \Longrightarrow \mathsf{HH}_*(X/k).$$

Corollary 4.77. *If* $\mathbb{Q} \subseteq k$*, then there is a canonical isomorphism*

$$\mathsf{HH}(X/k) \cong \prod R\Gamma(X, L\Omega^n_{X/k}[n]).$$

Proof. This is clearly true if $X = \operatorname{Spec} k[x_1, \ldots, x_n]$. Using non-abelian derived functors, this is also true for any affine X. The general case follows. □

Remark 4.78. This isomorphism is in fact compatible with the B operator, so we get the desired results for periodic and negative cyclic homology.

5. Topological Hochschild homology and cyclic homology

Appendix A. Simplicial rings and divided power structures

Definition A.1. The category ScRing of simplicial commutative rings is defined to be the functor category

$$\mathsf{scRing} := \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{cRing}).$$

More precisely, this is a diagram

$$\cdots \; R_2 \; R_1 \; R_0$$

satisfying the usual conditions for face and degeneracy maps.

Theorem A.2 (Eilenberg-Zilber)**.** *The functor*

$$\mathsf{sAb} \to \mathsf{Ch}$$

sending a simplicial abelian group to the associated chain complex is lax symmetric monoidal with respect to the pointwise tensor product on simplicial abelian groups and the usual tensor product on chain complexes.

This implies that every simplicial commutative ring produces a commutative dg algebra. We will now make this product structure explicit.

Definition A.3. An (m, n)-shuffle is a function

$$(\mu, \nu) \colon [m+n] \to [m] \times [n]$$

satisfying the following conditions:

- Both μ and ν are surjective and order preserving;
- The maps μ and ν jump at disjoint positions. For example, consider the case when $m = n - 2$, as in Figure 2. We can see that an (m, n)-shuffle is

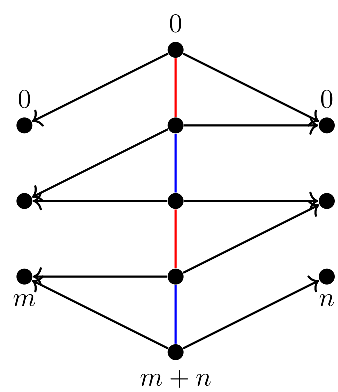


FIGURE 2. A $(2, 2)$ shuffle. Red lines denote where μ jumps and blue lines denote where ν jumps.

simply a way to shuffle m red and n blue cards together. There is a unique permutation which takes the configuration having all red cards on top to our given shuffle.

Proposition A.4. *Let $R \in \mathsf{scRing}$. Then there is a map $R_m \otimes R_n \xrightarrow{\cdot} R_{m+n}$ given by the formula*

$$x \cdot y = \sum_{(\mu,\nu)} \operatorname{sign}(\mu, \nu) s_\mu(x) s_\nu(y),$$

where $s_\mu(x)$ and $s_\nu(y)$ correspond to applying the degeneracies making up μ and ν to obtain elements of R_{m+n}. This gives R the structure of a strict cdga.

Proof. We will not go through all of the combinatorics explicitly.

- To prove associativity, we can express both $(xy)z$ and $x(yz)$ as a sum

$$\sum_{(\lambda,\mu,\nu)} \operatorname{sign}(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y) s_\nu(z).$$

- To check commutativity, the fact that R was a simplicial commutative ring means we can exchange the roles of μ and ν up to
$$\frac{\operatorname{sign}(\mu,\nu)}{\operatorname{sign}(\nu,\mu)} = (-1)^{mn},$$
which is exactly what we desire.
- A variant of this argument also gives strictness. When we compute x^2, the summands corresponding to (μ,ν) and (ν,μ) appear with opposite signs, so we see that $x^2 = 0$.
- The differential is given by the alternating sum of the face maps. Applying this to $x \cdot y$ will decompose the result into $(m-1,n)$-shuffles and $(m,n-1)$-shuffles, which yields the desired result. □

We will now discuss divided power structures. In fact, every simplicial commutative ring will give us one of these.

Proposition A.5. *Let $R \in \mathsf{scRing}$ be a simplicial commutative ring. Then there exist maps $\gamma_k \colon R_n \to R_{nk}$ for all $n, k \geq 1$ such that*
- *We have $\gamma_1(x) = x$ for all $x \in R$;*
- *We also have*
$$\gamma_k(x)\gamma_\ell(x) = \binom{k+\ell}{\ell}\gamma_{k+\ell}(x)$$
for all k, ℓ. This means that we should think that $x^k = k!\gamma_k(x)$;
- *For all $x, y \in R_n$, we have*
$$\gamma_k(x+y) = \sum_{i+k=k} \gamma_i(x)\gamma_j(y);$$
- *Similarly, we have*
$$\gamma_k(xy) = x^k\gamma_k(y);$$
- *We have the identity*
$$\gamma_k(\gamma_\ell(x)) = \frac{(k\ell)!}{k!(\ell!)^k}\gamma_{k\ell}(x).$$

Proof. If x is odd, we define $\gamma_1(x) = x$ and $\gamma_k(x) = 0$ for all $k \geq 2$. This is fine because $x^2 = 0$ already. If x is even, we have
$$x^k = \sum_{(\mu_1,\dots,\mu_k)} \operatorname{sign}(\mu_1,\dots,\mu_k) s_{\mu_1}(x)\cdots s_{\mu_k}(x).$$
We can permute the factors $\mu_1,\dots,\mu_k$ (all of the signs will become $+1$), and therefore for all $\sigma \in \Sigma_k$, $(\mu_1,\dots,\mu_k)$ and $(\mu_{\sigma(1)},\dots,\mu_{\sigma(k)})$ give the same summand. Choosing one element from each equivalent class, we obtain an element $\gamma_k(x)$ such that $x^k = k!\gamma_k(x)$.

We will omit checking that the properties are satisfied. □

We now need to make sure that this divided power structure behaves well with respect to the chain differential.

Proposition A.6. *Let $R \in \mathsf{scRing}$ be a simplicial commutative ring. Then*
- *If x is even, then $\mathrm{d}\gamma_k(x) = \gamma_{k-1}(x) \cdot \mathrm{d}x$;*
- *The maps γ_k preserve boundaries;*
- *The γ_k give a well-defined divided power structure on $H_*(R)$.*

Proof. Suppose that x is a boundary. Then $x = \mathrm{d}y$ for x with $\mathrm{d}_i y = 0$ for all $i > 0$. Define

$$\gamma'_k(y) := \sum_{\substack{(\mu_1,\ldots,\mu_k)\\ \Sigma_k\text{-representatives}}} \operatorname{sign}(\mu_1,\ldots,\mu_k) s_{\mu'_1}(y)\cdots s_{\mu'_k}(y).$$

Here, for maps $\mu_i\colon [kn] \to [n]$, we set $\mu'_i\colon [1+kn] \to [1+n]$ to be the maps given by sticking a 1 in front. Then $\mathrm{d}_0\gamma'_k(y) = \gamma_k(x)$ and $\mathrm{d}_i\gamma'_k(y) = 0$ for all $i > 0$, and therefore $\mathrm{d}\gamma'_k(y) = \gamma_k(x)$. ☐

Appendix B. Cotangent complexes and obstruction theory

B.1. **Definition of the cotangent complex.**

Recall that we defined

$$L\Omega^1_{-/k}\colon \mathsf{Anim}(k\text{−}\mathsf{alg}) \to D(\mathbb{Z})_{\geq 0}$$

as the nonabelian derived functor of

$$\Omega^1\colon k\text{−}\mathsf{alg} \to \mathsf{Ab}.$$

Lemma B.1. *The following constructions agree:*

(1) The functor

$$L\Omega^1_{-/k}\colon \mathsf{Anim}(k\text{−}\mathsf{alg}) \to D(\mathbb{Z})_{\geq 0}$$

evaluated on R;

(2) The functor

$$L(A \mapsto R \otimes_A \Omega^1_{A/k})\colon \mathsf{Anim}(k\text{−}\mathsf{alg}/R) \to D(R)_{\geq 0}$$

evaluated on R.

Proof. If we begin with the second construction, we obtain

$$\begin{aligned}
\operatorname*{colim}_{i\in\Delta^{\mathrm{op}}} R \otimes_{A_i} \Omega^1_{A_i/k} &\simeq \operatorname*{colim}_{i\in\Delta^{\mathrm{op}}}\left(\operatorname*{colim}_{j\in\Delta^{\mathrm{op}}} A_j\right) \otimes^{\mathbb{L}}_{A_i} \Omega^1_{A_i/k}\\
&\simeq \operatorname*{colim}_{i\in\Delta^{\mathrm{op}}}\left(\operatorname*{colim}_{j\in\Delta^{\mathrm{op}}_{i/}} A_j\right) \otimes^{\mathbb{L}}_{A_i} \Omega^1_{A_i/k}\\
&\simeq \operatorname*{colim}_{i\to j\in(\Delta^{\mathrm{op}})^{\Delta^1}} A_j \otimes^{\mathbb{L}}_{A_i} \Omega^1_{A_i/k}\\
&\simeq \operatorname*{colim}_{i\in\Delta^{\mathrm{op}}} \Omega^1_{A_i/k}\\
&\simeq L\Omega^1_{R/k}.
\end{aligned}$$

with A_i polynomial rings. Here, we have used the fact that identity morphisms are cofinal in the arrow category. ☐

The second functor has better categorical properties than the first. Namely, it commutes with coproducts. By definition, this implies that it preserves all colimits.

Example B.2. If $R = k[x_1,\ldots,x_n]/(f_1,\ldots,f_m)$ is defined by a regular sequence, then the diagram

$$\begin{tikzcd}
k[f_1,\ldots,f_m] \arrow[r] \arrow[d] & k[x_1,\ldots,x_m] \arrow[d]\\
k \arrow[r] & R
\end{tikzcd}$$

is a pushout in $\mathsf{Anim}(k\text{−}\mathsf{alg}/R)$. The functor $L(R \otimes_\bullet \Omega^1_{\bullet/k})$ takes this to

$$\begin{tikzcd} R\{\mathrm{d}f_1, \dots, \mathrm{d}f_m\} \arrow[r] \arrow[d] & R\{\mathrm{d}x_1, \dots, \mathrm{d}x_n\} \arrow[d] \\ 0 \arrow[r] & L\Omega^1_{R/k}, \end{tikzcd}$$

where the top arrow is the Jacobian matrix of $f_1, \dots, f_m$. This is also a pushout in $D(R)_{\geq 0}$, and therefore $L\Omega^1_{R/k}$ has as H_0 and H_1 the cokernel and kernel of the Jacobian matrix.

Definition B.3. We will call $\Omega^1_{R/k}$ the *cotangent complex* of R relative to k and denote it by $L_{R/k}$.

B.2. **Applications to obstruction theory of rings.** Let S be a ring and M be an S-module. We can form a *split square-zero extension* on the S-module $S \oplus M$ where

- If $m_1, m_2 \in M$, we have $m_1 m_2 = 0$;
- Elements of s multiply as usual;
- If $s \in S$ and $m \in M$, then sm is given by the usual action of s on M.

We would like to understand when there is a lift in the diagram

$$\begin{tikzcd} & S \oplus M \arrow[d] \\ R \arrow[r, "\varphi"] \arrow[ur, dashed] & S. \end{tikzcd}$$

These lifts correspond to φ-linear derivations $R \to M$, which are simply R-module maps

$$\Omega^1_{R/k} \to \varphi_* M.$$

Proposition B.4. *There exists a functor*

$$D(S)_{\geq 0} \to \mathsf{Anim}(k\text{−}\mathsf{alg})/S$$

which sends a projective module P to the square-zero extension $S \oplus P$. We have equivalences of mapping spaces

$$\mathsf{Map}_{D(R)_{\geq 0}}(L_{R/k}, \varphi_* M) \simeq \mathrm{Fiber}\begin{pmatrix} \mathsf{Map}_{\mathsf{Anim}(k-\mathsf{alg})}(R, S \oplus M) \\ \downarrow \\ \mathsf{Map}_{\mathsf{Anim}(k-\mathsf{alg})}(R, S) \end{pmatrix}.$$

We would now like to apply this to study ordinary rings. Given a surjective $\tilde{R} \to R$ whose kernel I satisfies $I^2 = 0$, we will call this a *(not necessarily split) square-zero extension.* One example of a non-split square-zero extension is

$$\mathbb{Z}/p^2 \to \mathbb{Z}/p,$$

which does not even split as a morphism of abelian groups. Note that the condition that $I^2 = 0$ implies that the $\tilde{R}$-action on I factors through R.

It turns out that $L_{R/\tilde{R}}$ has $H_0 = 0$ and

$$H_1 = R \otimes_{\tilde{R}} I = I.$$

There is a tautological morphism

$$L_{R/\tilde{R}} \to I[1]$$

of R-modules which induces an isomorphism on H_1. This corresponds to a map

$$\delta\colon R \to R \oplus I[1]$$

of animated $\tilde{R}$-algebras.

Proposition B.5. *$\tilde{R}$ is the pullback of the diagram*

$$\begin{tikzcd} \tilde{R} \arrow[r] \arrow[d] & R \arrow[d, "s"] \\ R \arrow[r, "\delta"] & R \oplus I[1] \end{tikzcd}$$

in the category of animated $\tilde{R}$-algebras.

We see that (not necessarily split) square-zero extensions $\tilde{R} \to R$ along I correspond to maps

$$L_{R/k} \to I[1].$$

Maps $S \to \tilde{R}$ lifting a given $S \to R$ correspond to diagrams of the form

$$\begin{tikzcd} & \tilde{R} \arrow[r] \arrow[d] & R \arrow[d, "s"] \\ S \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & R \arrow[r, "\delta"] & R \oplus I[1], \end{tikzcd}$$

which are simply lifts

$$\begin{tikzcd} & R \arrow[d, "s"] \arrow[dd, bend left=60, "\mathrm{Id}"] \\ S \arrow[r] \arrow[ur, dashed] \arrow[ur, phantom, shift right=2, "\Rightarrow"] & R \oplus I[1] \arrow[d] \\ & R. \end{tikzcd}$$

If we unwrap everything, we see that a lift exists if and only if the map

$$L_{S/k} \to I[1]$$

is nullhomotopic. Furthermore, lifts are in correspondence to choices of nullhomotopies. These form a torsor over $\pi_1(\mathsf{Map}(L_{S/k}, I[1]))$. We therefore see that

- The possible square-zero extensions $\tilde{R}$ are classified by
$$\pi_0 \mathsf{Map}_{D(R)}(L_{R/k}, I[1]);$$
- For a given $\tilde{R}$ and $S \to R$, there is an obstruction living in
$$\pi_0 \mathsf{Map}_{D(S)}(L_{S/k}, I[1]).$$
- If lifts do exist, they are parameterized by
$$\pi_1 \mathsf{Map}_{D(S)}(L_{S/k}, I[1]) \simeq \pi_0 \mathsf{Map}_{D(S)}(L_{S/k}, I).$$

Exercise B.6. Let S be a k-algebra. Show that the following are equivalent:

(1) For every square-zero extension $\tilde{R} \to R$ and every $S \to R$, there is a lift (this is usually called being *formally smooth*).
(2) We have $H_1(L_{S/k}) = 0$ and $H_0(L_{S/k})$ is a projective S-module.

We are now ready to apply this machinery.

Theorem B.7. *If R_1 is a perfect $\mathbb{F}_p$-algebra (this means that Frobenius is an automorphism), then there exist flat $\mathbb{Z}/p^n$-algebras R_n, unique up to isomorphism, with*

$$R_1 \simeq R_n \otimes_{\mathbb{Z}/p^n} \mathbb{F}_p.$$

In particular, we have

$$R_{n-1} \cong R_n \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^{n-1},$$

so we have a tower

$$\cdots \to R_n \to R_{n-1} \to \cdots$$

whose lift

$$R \coloneqq \lim R_n$$

is the unique flat, p-complete $\mathbb{Z}_p$-algebra satisfying

$$R_1 \cong R \otimes_{\mathbb{Z}_p} \mathbb{F}_p.$$

This theorem tells us that there is a way to escape from the positive characteristic to the mixed characteristic setting under the assumption that R_1 is perfect.

We will characterize R_n as a non-split square-zero extension of R_{n-1} by R_1. This is because if it exists, tensoring

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to 0$$

with R_n gives us

$$0 \to R_1 \to R_n \to R_{n-1} \to 0.$$

These extensions are classified by

$$\pi_0 \mathsf{Map}_{D(R_{n-1})}(L_{R_{n-1}/\mathbb{Z}}, R_1[1]).$$

They also need to be compatible with the element of

$$\pi_0 \mathsf{Map}_{D(\mathbb{Z}/p^{n-1})}(L_{(\mathbb{Z}/p^{n-1})/\mathbb{Z}}, \mathbb{Z}/p[1])$$

classifying $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1}$.

Lemma B.8. *Isomorphism classes of such R_n form a torsor over*

$$\pi_0 \mathsf{Map}_{D(R_{n-1})}(L_{R_{n-1}/\mathbb{Z}/p^{n-1}}, R_n[1]) \simeq \pi_0 \mathsf{Map}_{D(R_1)}(L_{R_1/\mathbb{F}_p}, R_1[1]).$$

Lemma B.9. *If R_1 is a perfect $\mathbb{F}_p$-algebra, then $L_{R_1/\mathbb{F}_p} \simeq 0$.*

Proof. For any perfect $\mathbb{F}_p$-algebra A, we will consider the effect of the Frobenius $\varphi\colon A \to A$ on the contangent complex. This induces the zero morphism on $\Omega^1_{A/\mathbb{F}_p}$ (in particular for polynomial rings). Therefore, if we resolve R_1 by polynomial rings, φ acts by 0 on $L_{R_1/\mathbb{F}_p}$. However, because we assumed that R_1 is perfect, it must also act by an isomorphism. The desired result follows immediately. □

Putting together the two lemmas, we see that there is a unique isomorphism class of possible R_n, which proves Theorem B.7.

Definition B.10. The R in Theorem B.7 is called the *Witt vectors* of R and denoted by $W(R_0)$.

Remark B.11. Usually the Witt vectors are given by an explicit construction and exist for any $\mathbb{F}_p$-algebra, not just perfect ones, but we believe this explanation is much more satisfying.

Example B.12. Consider the finite field $\mathbb{F}_{p^n}$. The Witt vectors $W(\mathbb{F}_{p^n})$ is a $\mathbb{Z}_p$-algebra with $W(\mathbb{F}_{p^n})/p \simeq \mathbb{F}_{p^n}$. One way to construct this is to consider $\mathbb{F}_{p^n} = \mathbb{F}_p[x]/f(x)$ and take the ring

$$\mathbb{Z}_p[x]/\tilde{f}(x)$$

for some lift $\tilde{f}$ of f.