Enumerative invariants and birational geometry
Spring 2024

Notes by Patrick Lei

Lectures by Various

Columbia University

Disclaimer

These notes were taken during the lectures using neovim. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. Also, notation may differe between lecturers. If you find any errors, please contact me at plei@math.columbia.edu.

Acknowledgements

I would like to thank Shaoyun Bai for co-organizing the seminar with me.

Seminar Website: https://math.columbia.edu/~plei/s24-birat.html

Contents

1

Preliminaries

1.1 Givental formalism (Patrick, Feb 01)

1.1.1 Introduction Let X be a smooth projective variety. Then for any $g, n \in \mathbb{Z}_{\geqslant 0}, \beta \in H_2(X, \mathbb{Z})$, there exists a moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ (Givental's notation is $X_{g,n,\beta}$) of *stable maps* $f \colon C \to X$ from genus-g, n-marked prestable curves to X with $f_*[C] = \beta$. It is well-known that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in A_\delta(\overline{\mathcal{M}}_{g,n}(X, \beta)), \qquad \delta = \int_\beta c_1(X) + (\dim X - 3)(1-g) + 3.$$

In addition, there is a universal curve and sections

$$\mathcal{C} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} \overline{\mathcal{M}}_{g,n}(X, \beta).$$

In this setup, there are tautological classes

$$\psi_i \coloneqq c_1(\sigma_i^* \omega_\pi) \in H^2(\overline{\mathcal{M}}_{g,n}(X, \beta)).$$

This allows us to define individual Gromov-Witten invariants by

$$\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \phi_i \cdot \psi_i^{a_i}.$$

These invariants satisfy various relations. The first is the *string equation:*

$$\langle \tau_0(1)\tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n+1,\beta} = \sum_{i=1}^n \left\langle \tau_{a_i - 1}(\phi_i) \prod_{j \neq i} \tau_{a_j}(\phi_j) \right\rangle^X_{g,n,\beta}.$$

The next is the *dilaton equation:*

$$\langle \tau_1(1)\tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n+1,\beta} = (2g - 2 + n)\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \rangle^X_{g,n,\beta}.$$

Finally, we have the *divisor equation* when one insertion is a divisor $D \in H^2(X)$:

$$\left\langle \tau_0(D)\tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n+1,\beta} = \left(\int_\beta D\right)\cdot\left\langle \tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n,\beta} + \sum_{i=1}^n \left\langle \tau_{a_i-1}(\phi_i\cdot D)\prod_{j\neq i}\tau_{a_j}(\phi_j)\right\rangle^X_{g,n,\beta}.$$

It is often useful to package Gromov-Witten invariants into various generating series.

Definition 1.1.1. The *quantum cohomology* $QH^*(X)$ of X is defined by the formula

$$(a\star_t b, c) := \sum_{\beta,n}\frac{Q^\beta}{n!}\langle a,b,c,t,\ldots,t\rangle^X_{0,3+n,\beta}$$

for any $t \in H^*(X)$. This is a commutative and associative product.

The *small quantum cohomology* is obtained by setting $t = 0$ and the ordinary cohomology is obtained by further setting $Q = 0$.

Remark 1.1.2. Convergence of the formula does not hold in general, so quantum cohomology needs to be treated as a formal object.

Definition 1.1.3. Let ϕ_i be a basis of $H^*(X)$ and ϕ^i be the dual basis. Then the J-*function* of X is the cohomology-valued function

$$J_X(t,z) := z + t + \sum_i\sum_{n,\beta}\frac{Q^\beta}{n!}\left\langle\frac{\phi_i}{z-\psi},t,\ldots,t\right\rangle^X_{0,n+1,\beta}\phi^i.$$

Definition 1.1.4. The *genus*-0 *GW potential* of X is the (formal) function

$$\mathcal{F}^X(t(z)) = \sum_{\beta,n}\frac{Q^\beta}{n!}\langle t(\psi),\ldots,t(\psi)\rangle^X_{0,n,\beta}.$$

The associativity of the quantum product is equivalent to the PDE

$$\sum_{e,f}\mathcal{F}^X_{abe}\eta^{ef}\mathcal{F}_{cdf} = \sum_{e,f}\mathcal{F}^X_{ade}\eta^{ef}\mathcal{F}^X_{bcf}$$

for any a, b, c, d, which are known as the *WDVV equations*. Here, we choose coordinates on $H^*(X)$ and set $z = 0$ (only consider primary insertions). In addition, set η_{ef} to be the components of the Poincaré pairing and let η^{ef} be the inverse matrix.

1.1.2 Frobenius manifolds

A Frobenius manifold can be thought of as a formalization of the WDVV equations.

Definition 1.1.5. A *Frobenius manifold* is a complex manifold M with a flat symmetric bilinear form $\langle -,-\rangle$ (meaning that the Levi-Civita connection has zero curvature) on TM and a holomorphic system of (commutative, associative) products $\star_t$ on T_tM satisfying:

1. The unit vector field $\mathbf{1}$ is flat: $\nabla\mathbf{1} = 0$;

2. For any t and $a, b, c \in T_tM$, $\langle a \star_t b, c\rangle = \langle a, b \star_t c\rangle$;
3. If $c(u,v,w) := \langle u \star_t v, w\rangle$, then the tensor $(\nabla_z c)(u,v,w)$ is symmetric in $u, v, w, z \in T_tM$.

If there exists a vector field E such that $\nabla\nabla E = 0$ and complex number d such that:

1. $\nabla\nabla E = 0$;
2. $\mathcal{L}_E(u \star v) - \mathcal{L}_E u \star v - u \star \mathcal{L}_E v = u \star v$ for all vector fields u, v;
3. $\mathcal{L}_E\langle u, v\rangle - \langle \mathcal{L}_E u, v\rangle - \langle u, \mathcal{L}_E v\rangle = (2-d)\langle u, v\rangle$ for all vector fields u, v,

then E is called an *Euler vector field* and the Frobenius manifold M is called *conformal*.

Example 1.1.6. Let X be a smooth projective variety. Then we can give $H^*(X)$ the structure of a Frobenius algebra with the Poincaré pairing and the quantum product. Note that the quantum product does not converge in general, so we must treat this as a formal object. The Euler vector field is given by

$$E_X = c_1(X) + \sum_i \left(1 - \frac{\deg \phi_i}{2}\right) t^i \phi_i,$$

where a general element of $H^*(X)$ is given by $t = \sum_i t^i \phi_i$. We will also impose that $\phi_1 = \mathbf{1}$. There is another very important structure, the *quantum connection*, which is given by the formula

$$\begin{aligned} \nabla_{t^i} &:= \partial_{t^i} + \frac{1}{z}\phi_i \star_t \\ \nabla_{z\frac{d}{dz}} &:= z\frac{d}{dz} - \frac{1}{z}E_X \star_t + \mu_X. \end{aligned}$$

Here, μ_X is the *grading operator*, defined for pure degree classes $\phi \in H^*(X)$ by

$$\mu_X(\phi) = \frac{\deg \phi - \dim X}{2}\phi.$$

Finally, in the direction of the Novikov variables, we have

$$\nabla_{\xi Q \partial_Q} = \xi Q\, \partial_Q + \frac{1}{z}\xi \star_t .$$

Remark 1.1.7. For a general conformal Frobenius manifold $(H, (-,-), \star, E)$, there is still a *deformed flat connection* or *Dubrovin connection* given by

$$\begin{aligned} \nabla_{t^i} &:= \frac{\partial}{\partial t^i} + \frac{1}{z}\phi_i \star \\ \nabla_{z\frac{d}{dz}} &:= z\frac{d}{dz} - \frac{1}{z}E\,\star. \end{aligned}$$

Definition 1.1.8. The *quantum* D-*module* of X is the module $H^*(X)[z][\![Q, t]\!]$ with the quantum connection defined above.

Remark 1.1.9. It is important to note that the quantum connection has a fundamental solution matrix $S^X(t, z)$ given by

$$S_X(t,z)\phi = \phi + \sum_i \sum_{n,\beta} \frac{Q^\beta}{n!}\phi^i \left\langle \frac{\phi_i}{z-\psi}, \phi, t, \ldots, t \right\rangle^X_{0,n+2,\beta}.$$

It satisfies the important equation

$$S_X^*(t,-z)S(t,z) = 1.$$

Using this formalism, the J-function is given by $S_X^*(t,z)\mathbf{1} = z^{-1}J_X(t,z)$.

1.1.3 Givental formalism The Givental formalism is a geometric way to package enumerative (CohFT) invariants cleanly. We begin by defining the symplectic space

$$\mathcal{H} := H^*(X, \Lambda)(\!(z^{-1})\!)$$

with the symplectic form

$$\Omega(f, g) := \mathrm{Res}_{z=0}(f(-z)g(z)).$$

This has a polarization by Lagrangian subspaces

$$\mathcal{H}_+ := H^*(X, \Lambda)[z], \qquad \mathcal{H}_- := z^{-1}H^*(X, \Lambda)[\![z^{-1}]\!]$$

giving $\mathcal{H} \cong T^*\mathcal{H}_+$ as symplectic vector spaces. Choose Darboux coordinates $\underline{p}, \underline{q}$ on $\mathcal{H}$. For example, there is a choice in Coates's thesis which gives a general element of $\mathcal{H}$ as

$$\sum_{k\geqslant 0}\sum_i q_k^i \phi_i z^k + \sum_{\ell\geqslant 0}\sum_j p_\ell^j \phi^j (-z)^{-\ell-1}.$$

Taking the *dilaton shift*

$$q(z) = t(z) - z = -z + t_0 + t_1 z + t_2 z^2 + \cdots,$$

we can now think of $\mathcal{F}^X$ has a formal function on $\mathcal{H}_+$ near $q = -z$. This convention is called the *dilaton shift*.

Before we continue, we need to recast the string and dilaton equations in terms of $\mathcal{F}^X$. Write $t_x = \sum t_k^i \phi_i$. Then the string equation becomes

$$\partial_0^1 \mathcal{F}(t) = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty}\sum_j t_{n+1}^j\, \partial_n^j \mathcal{F}(t)$$

and the dilaton equation becomes

$$\partial_1^1 \mathcal{F}(t) = \sum_{n=0}^{\infty} t_n^j\, \partial_n^j \mathcal{F}(t) - 2\mathcal{F}(t).$$

There are also an infinite series of topological recursion relations

$$\partial_{k+1}^i\, \partial_\ell^j\, \partial_m^k \mathcal{F}(t) = \sum_{a,b} \partial_k^i\, \partial_0^a \mathcal{F}(t) \eta^{ab}\, \partial_0^b\, \partial_\ell^j\, \partial_m^k \mathcal{F}(t).$$

We can make sense of these three relations for any (formal) function $\mathcal{F}$ on $\mathcal{H}_+$.

Now let

$$\mathcal{L} = \left\{(\underline{p}, \underline{q}) \in \mathcal{H} \mid \underline{p} = d_{\underline{q}}\mathcal{F}\right\}$$

be the graph of $d\mathcal{F}$. This is a formal germ at $q = -z$ of a Lagrangian section of the cotangent bundle $T^*\mathcal{H}_+$ and is therefore a formal germ of a Lagrangian submanifold in $\mathcal{H}$.

Theorem 1.1.10. *The function $\mathcal{F}$ satisfies the string equation, dilaton equation, and topological recursion relations if and only if $\mathcal{L}$ is a Lagrangian cone with vertex at the origin $q = 0$ such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along zL.*

Because of this theorem, $\mathcal{L}$ is known as the *Lagrangian cone*. It can be recovered from the J-function by the following procedure. First consider $\mathcal{L} \cap (-z + z\mathcal{H}_-)$. Via the projection to $-z + H$ along $\mathcal{H}_-$, this can be considered as the graph of the J-function. Next, we consider the derivatives $\frac{\partial J}{\partial t^i}$, which form a basis of $L \cap z\mathcal{H}_-$, which is a complement to zL in L. Then we know that

$$z\frac{\partial J}{\partial t^i} \in zL \subset \mathcal{L},$$

so

$$z\frac{\partial^2 J}{\partial t^i\,\partial t^j} \in L \cap z\mathcal{H}_-.$$

Writing these in terms of the first derivatives $\frac{\partial J}{\partial t^i}$ and using the fact that J is a solution of the quantum connection, so we recover the Frobenius structure of quantum cohomology.

We will now express some classical results in this formalism. Let X be a toric variety with toric divisors $D_1, \dots, D_N$ such that $D_1, \dots, D_k$ form a basis of $H^2(X)$ and Picard rank k. Then define the I-*function*

$$I_X = ze^{\sum_{j=1}^k t_i D_i} \sum_\beta Q^\beta \frac{\prod_{j=1}^N \prod_{m=-\infty}^0 (D_j + mz)}{\prod_{j=1}^N \prod_{m=-\infty}^{\langle D_j, \beta\rangle} (D_j + mz)}.$$

Theorem 1.1.11 (Mirror theorem)**.** *The formal functions* I_X *and* J_X *coincide up to some change of variables, which if* $c_1(X)$ *is semi-positive is given by components of the* I*-function.*

Theorem 1.1.12 (Mirror theorem in this formalism)**.** *For any* t*, we have*

$$I_X(t, z) \in \mathcal{L}.$$

Another direction in Gromov-Witten theory is the Virasoro constraints. In the original formulation, these involved very complicated explicit differential operators, but in the Givental formalism, there is a very compact formulation.

Define $\ell^{-1} = z^{-1}$ and

$$\ell_0 = z\frac{d}{dz} + \frac{1}{2} + \mu + \frac{c_1(X) \cup -}{z}.$$

Then define

$$\ell_n = \ell_0(z\ell_0)^n.$$

Theorem 1.1.13 (Genus-0 Virasoro constraints)**.** *Suppose the vector field on* $\mathcal{H}$ *defined by* ℓ_0 *is tangent to* $\mathcal{L}$*. Then the same is true for the vector fields defined by* ℓ_n *for any* $n \geqslant 1$.

Proof. Let L be a tangent space to $\mathcal{L}$. Then if $f \in zL \subset \mathcal{L}$, the assumption gives us $\ell_0 f \in L$. But then $z\ell_0 f \in zL$, so $\ell_0 z\ell_0 f = \ell_1 f \in L$. Continuing, we obtain $\ell_n f \in L$ for all n. □

Later, we will learn that the Quantum Riemann-Roch theorem can be stated in this formalism. Let $\mathcal{L}^{tw}$ be the twisted Lagrangian cone (where the twisted theory will be defined next week).

Theorem 1.1.14 (Quantum Riemann-Roch)**.** *For some explicit linear symplectic transformaiton* Δ*, we have* $\mathcal{L}^{tw} = \Delta\mathcal{L}$.

1.1.4 Quantization

In the last part of the talk, we will briefly discuss the quantization formalism, which encodes the higher-genus theory. In Darboux coordinates p_a, q_b, we will quantize symplectic transformations by the standard rules

$$\widehat{q_a q_b} = \frac{q_a q_b}{\hbar}, \qquad \widehat{q_a p_b} = q_a \frac{\partial}{\partial q_b}, \qquad \widehat{p_a p_b} = \hbar \frac{\partial^2}{\partial q_a \, \partial q_b}.$$

This determines a differential operator acting on functions on $\mathcal{H}_+$.

We also need the genus-g potential

$$\mathcal{F}_g^X \coloneqq \sum_{\beta, n} \frac{Q^\beta}{n!} \langle t(\psi), \dots, t(\psi) \rangle_{g,n,\beta}^X$$

and the *total descendent potential*

$$\mathcal{D} \coloneqq \exp\left(\sum_{g \geqslant 0} \hbar^{g-1} \mathcal{F}_g^X \right).$$

In this formalism, the Virasoro conjecture can be expressed as follows. Let $L_n = \widehat{\ell}_n + c_n$, where c_n is a carefully chosen constant.

Conjecture 1.1.15 (Virasoro conjecture)**.** *If* $L_{-1}\mathcal{D} = L_0\mathcal{D} = 0$, *then* $L_n\mathcal{D} = 0$ *for all* $n \geqslant 1$.

In this formalism, the higher-genus version of the Quantum Riemann-Roch theorem takes the very simple form

Theorem 1.1.16 (Quantum Riemann-Roch)**.** *Let* $\mathcal{D}^{\mathrm{tw}}$ *be the twisted descendent potential. Then*

$$\mathcal{D}^{\mathrm{tw}} = \widehat{\Delta}\mathcal{D}.$$

1.2 Quantum Riemann-Roch (Shaoyun, Feb 08)

We will state and prove the Quantum Riemann-Roch theorem in genus 0, following Coates-Givental.

1.2.1 Twisted Gromov-Witten invariants

Again, let X be a smooth projective variety. Let E be a vector bundle on X. We should note that

$$\overline{\mathcal{M}}_{0,n+1}(X, \beta) \xrightarrow{\pi} \overline{\mathcal{M}}_{0,n}(X, \beta)$$

is the universal curve, and the universal morphism is simply ev_{n+1}. We will consider the sheaf

$$E_{0,n,\beta} \coloneqq R\pi_* \,\mathrm{ev}_{n+1}^* E \in K^0(\overline{\mathcal{M}}_{0,n}(X, \beta)).$$

We need to check that this is a well-defined K-theory class. Choose an ample line bundle $L \to X$. By definition, for $N \gg 1$, the cohomology

$$H^i(X, E \otimes L^N) = 0$$

whenever $i \geqslant 1$. This gives us an exact sequence

$$0 \to \ker(\eqqcolon A) \to H^0(X, E \otimes L^N) \otimes L^{-N} (\eqqcolon B) \to E \to 0.$$

For any stable map $f\colon \Sigma \to X$ of positive degree, we obtain a long exact sequence

$$0 \to H^0(\Sigma, f^*E) \to H^1(\Sigma, f^*A) \to H^1(\Sigma, f^*B) \to H^1(\Sigma, f^*E) \to 0,$$

so we obtain

$$R^0\pi_* \operatorname{ev}_{n+1}^* E - R^1\pi_* \operatorname{ev}_{n+1}^* E = R^1\pi_* \operatorname{ev}_{n+1}^* B - R^1\pi_* \operatorname{ev}_{n+1}^* A.$$

This expresses $E_{0,n,\beta}$ as a difference of vector bundles.

We will now introduce a *universal characteristic class*

$$\mathbf{c}(-) = \exp\left(\sum_{k=0}^{\infty} s_k \operatorname{ch}_k(-)\right),$$

where $s_0, s_1, s_2, \dots$ are formal variables and ch_k is the k-th Chern character

$$\frac{x_1^k}{k!} + \cdots + \frac{x_r^k}{k!},$$

where x_i are the Chern roots.

Example 1.2.1. Let $E \to X$ be a vector bundle and equip it with the fiberwise $\mathbb{C}^*$-action by scaling. Let λ be the equivariant parameter and ρ_i be the Chern roots. Then

$$e(E) = \sum_i (\lambda + \rho_i).$$

We then rewrite

$$\begin{aligned}\prod(\lambda + \rho_i) &= \exp\left(\sum_i \left(\log\lambda - \sum_k \frac{(-\rho_i)^k}{k\lambda^k}\right)\right)\\ &= \exp\left(\operatorname{ch}_0(E)\log\lambda + \sum_{k>0} \frac{(-1)^{k-1}(k-1)!}{\lambda^k} \operatorname{ch}_k(E)\right),\end{aligned}$$

so for the (equivariant Euler class), we obtain

$$\begin{aligned} s_0 &= \log\lambda \\ s_k &= \frac{(-1)^{k-1}(k-1)!}{\lambda^k}, \qquad k > 0.\end{aligned}$$

We are now ready to define the $(E, \mathbf{c})$-twisted Gromov-Witten invariants.

Definition 1.2.2. Define the *twisted Gromov-Witten invariants* by

$$\left\langle \alpha_1\psi_1^{k_1}, \dots, \alpha_n\psi_n^{k_n} \right\rangle_{0,n,\beta}^{X,(E,\mathbf{c})} := \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \operatorname{ev}_i^*(\alpha_i)\psi_i^{k_i} \cup \mathbf{c}(E_{0,n,\beta})$$

for $\alpha_i \in H^*(X)$ and $k_i \in \mathbb{Z}_{\geqslant 0}$.

We will now construct the Lagrangian cone for the twisted theory. Let R be the coefficient ring containing $s_0, s_1, \dots$ and define

$$\mathcal{H}_X^{\mathrm{tw}} := H^*(X) \otimes R(\!(z^{-1})\!)[\![Q]\!].$$

We also introduce the *twisted Poincaré pairing*

$$(a, b)_{(E,\mathbf{c})} = \int_X a \cup b \cup \mathbf{c}(E).$$

The symplectic structure is defined by

$$\Omega_{tw}(f, g) = \mathrm{Res}_{z=0}(f(-z)g(z))_{(E,\mathbf{c})}.$$

There is a polarization

$$\mathcal{H}_X^{tw} = \mathcal{H}_+^{tw} \oplus \mathcal{H}_-^{tw}$$

with

$$\mathcal{H}_+^{tw} := H^*(X) \otimes R[z][\![Q]\!]$$
$$\mathcal{H}_-^{tw} := H^*(X) \otimes R[\![z]\!][\![Q]\!].$$

Finally, we have the *twisted genus-*0 *descendent potential*

$$\mathcal{F}^0_{X,tw}(t) := \sum_{\beta,n} \frac{Q^\beta}{n!} \langle t, \ldots, t \rangle^{X,(E,\mathbf{c})}_{0,n,\beta}.$$

Identifying $\mathcal{H}_X^{tw}$ with $T^*\mathcal{H}_+^{tw}$, we obtain the twisted Lagrangian cone $\mathcal{L}_X^{tw}$ as the graph of $d\mathcal{F}^0_{X,tw}$. Denote the untwisted Lagrangian cone as $\mathcal{L}_X$.

Theorem 1.2.3. *We have*

$$\mathcal{L}_X^{tw} = \Delta \mathcal{L}_X,$$

where

$$\Delta = \exp\left(\sum_{m \geqslant 0} \sum_{\ell \geqslant 0} s_{2m-1+\ell} \frac{B_{2m}}{(2m)!} \mathrm{ch}_\ell(E) z^{2m-1} \right).$$

Here, the Bernoulli numbers B_{2m} are defined by

$$\frac{t}{1 - e^{-t}} = \frac{t}{2} + \sum_{m \geqslant 0} \frac{B_{2m}}{(2m!)} t^{2m}.$$

1.2.2 Proof of Theorem 1.2.3

The idea is to use the Grothendieck-Riemann-Roch theorem.

Proposition 1.2.4. *We can write*

$$[\overline{\mathcal{M}}_{0,n}(X, \beta)]^{vir} \cap \mathrm{ch}_k(E_{0,n,\beta}) = \pi_*\left(\sum_{\substack{r+\ell=k+1 \\ r,\ell \geqslant 0}} \frac{B_r}{r!} \mathrm{ch}_\ell(\mathrm{ev}_{n+1}^* E) \Psi(r) \right),$$

where

$$\begin{aligned}\Psi(r) = \psi_{n+1}^r \cap [\overline{\mathcal{M}}_{0,n+1}(X, \beta)]^{vir} \\ - \sum_{i=1}^n (\sigma_i)_*(\psi_i^{n-1} \cap [\overline{\mathcal{M}}_{0,n}(X, \beta)]^{vir}) \\ + \frac{1}{2} j_*\left(\sum_{\substack{a+b=r-2 \\ a,b \geqslant 0}} (-1)^a \psi_+^a \psi_i^b \cap [\widetilde{Z}_{0,n+1,\beta}]^{vir} \right).\end{aligned}$$

Here, $Z_{0,n+1,\beta}$ is formed by the nodes of π, $\widetilde{Z}_{0,n+1,\beta}$ is a double cover of $Z_{0,n+1,\beta}$ formed by a choice of branch of the nodes, ψ_+ and ψ_- are the ψ-classes at the two branches of the nodes, and

$$j \colon \widetilde{Z}_{0,n+1,\beta} \to Z_{0,n+1,\beta} \to \overline{\mathcal{M}}_{0,n+1}(X,\beta)$$

is the "inclusion."

Proof. We will first assume that $\overline{\mathcal{M}}_{0,n+1}(X,\beta)$, $\overline{\mathcal{M}}_{0,n}(X,\beta)$, and $Z_{0,n+1,\beta}$ are all smooth and that $\pi(Z_{0,n+1,\beta})$ is a normal crossings divisor. In general, we need a Cartesian diagram

$$\begin{tikzcd}
 & \mathrm{ev}_{n+1}^* E \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
\overline{\mathcal{M}}_{0,n+1}(X,\beta) \arrow[rr] \arrow[dd] & & \mathcal{C} \arrow[dd] & \\
 & Z_{0,n+1,\beta} \arrow[ul, hook] & & Z \arrow[ul, hook] \\
\overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[rr] & & \mathcal{M}. &
\end{tikzcd}$$

Continuing in the ideal situation, we apply Grothendieck-Riemann-Roch[1] to obtain

$$\begin{aligned}
\mathrm{ch}(E_{0,n,\beta}) &= \mathrm{ch}(R\pi_* \,\mathrm{ev}_{n+1}^* E) \\
&= \pi_*(\mathrm{ch}(\mathrm{ev}_{n+1}^* E) \cdot \mathrm{td}^\vee \,\Omega_\pi),
\end{aligned}$$

where $\mathrm{td}^\vee$ is the dual Todd class, defined by $\frac{-x}{1-e^{tx}}$, and Ω_π is the sheaf of relative differentials.

We then have two short exact sequences

$$0 \to \Omega_\pi \to \omega_\pi \to \mathcal{O}_{Z_{0,n+1,\beta}} \to 0$$

and

$$0 \to \omega_\pi \to L_{n+1} \to \bigoplus_{i=1}^{n} \mathcal{O}_{D_i} \to 0,$$

where D_i is the divisor where the marked points $i, n+1$ collide and their component has exactly three special points. Now we obtain

$$\Omega_\pi = L_{n+1} - \sum_{i=1}^{n} \mathcal{O}_{D_i} - \mathcal{O}_{Z_{0,n+1,\beta}}$$

in K-theory. Using the facts that $c_1(L_{n+1}) = \psi_{n+1}$, $D_i \cap D_j = \emptyset$ for $i \neq j$, and $D_i \cap Z_{0,n+1,\beta} = \emptyset$, we see that L_{n+1} is trivial when restricted to D_i and $Z_{0,n+1,\beta}$. Now we apply the dual Todd class.

Lemma 1.2.5. *If $x_1 \cup x_2 = 0$, then*

$$(\mathrm{td}^\vee(x_1) - 1)(\mathrm{td}^\vee(x_2) - 1) = 0.$$

[1] We need to be careful about directly applying Grothendieck-Riemann-Roch in the stacky setting (and in general we are only quasi-smooth).

Using the lemma, we obtain

$$
\begin{aligned}
\mathrm{td}^\vee(\Omega_\pi) &= \mathrm{td}^\vee(L_{n+1}) \prod_{i=1}^n \mathrm{td}^\vee(-\mathcal{O}_{D_i})\, \mathrm{td}^\vee(\mathcal{O}_{Z_{0,n+1,\beta}})^{-1} \\
&= 1 + (\mathrm{td}^\vee(L_{n+1}) - 1) + \sum_{i=1}^n \left(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{D_i})} - 1 \right) + \left(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{Z_{n+1,\beta}})} - 1 \right).
\end{aligned}
$$

The first term in the statement comes from the dual Todd class of L_{n+1}, the second comes from

$$0 \to \mathcal{O}(-D_i) \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0$$

and the relation between $\mathcal{O}(-D_i)$ and L_i, and the last term can be found in Appendix A of Coates-Givental. $\square$

To obtain the Quantum Riemann-Roch theorem, we use the previous proposition and manipulate the generating function. If E is convex and $Y \subset X$ is a complete intersection defined by E, then $\mathcal{L}_X^{\mathrm{tw}}$ is closely related to $\mathcal{L}_Y$, so we are able to study the Gromov-Witten theory of Y using this.

1.3 Shift operators (Melissa, Feb 15)

Let X be a semiprojective smooth variety. This means that X is projective over its affinization. Also assume that X has an action by $T = (\mathbb{C}^\times)^m$ such that all T-weights in $H^0(X, \mathcal{O})$ are contained in a strictly convex cone in $\mathrm{Hom}(T, \mathbb{C}^\times)_\mathbb{R}$ and $H^0(X, \mathcal{O})^T = \mathbb{C}$. All such X imply that

(a) The fixed locus X^T is projective;

(b) The T-variety X is equivariantly formal. This means that $H_T^*(X)$ is a free module over $H_T^*(\mathrm{pt}) = \mathbb{Q}[\lambda] \coloneqq \mathbb{Q}[\lambda_1, \dots, \lambda_m]$ and there is a non-canonical isomorphism

$$H_T^*(X) \cong H^*(X) \otimes H_T^*(\mathrm{pt})$$

as $H_T^*(\mathrm{pt})$-modules.

(c) The evaluation maps $\mathrm{ev}_i \colon X_{0,n,d} \to X$ are proper.

Using (b), we may choose a basis $\{\phi_i\}_{i=0}^N$ of $H_T^*(X)$ over $H_T^*(\mathrm{pt})$. Let τ^i be the dual coordinates.

1.3.1 Equivariant big quantum cohomology

Let $(-,-)$ be the T-equivariant Poincaré pairing, which in general takes values in $\mathbb{Q}(\lambda)$. Then the T-equivariant big quantum product is defined by

$$
\begin{aligned}
(\phi_i \star_\tau \phi_j, \phi_k) &= \langle\!\langle \phi_i, \phi_j, \phi_k \rangle\!\rangle_{0,3}^{X,T} \\
&= \sum_{d,n} \frac{Q^d}{n!} \langle \phi_i, \phi_b, \phi_j, \tau, \dots, \tau \rangle_{0,n+3,d}^{X,T}.
\end{aligned}
$$

This can also be defined using the evaluation maps

$$(\mathrm{ev}_i)_* \colon H_T^*(X_{0,n+3,d}) \to H_T^{*-2(c_1(X)\cdot d + n)}(X)$$

as

$$\phi_i \star_\tau \phi_j = \sum_{d,n} \frac{Q^d}{n!} (\mathrm{ev}_3)_* \left(\mathrm{ev}_1^*(\phi_i)\, \mathrm{ev}_2^*(\phi_j) \prod_{i=4}^{n+3} \mathrm{ev}_i^*(\tau) \cap [X_{0,n+3,d}]^{\mathrm{vir}} \right) \in H_T^*(X)[\![Q]\!][\![\tau_0, \dots, \tau_n]\!].$$

1.3.2 Quantum connection We will define

$$\nabla_i \colon H_T^*(X)[z][\![Q]\!][\![\tau]\!] \to z^{-1}H_T^*(X)[z][\![Q]\!][\![\tau^0, \ldots, \tau^N]\!]$$

by setting

$$\nabla_i = \frac{\partial}{\partial \tau^i} + \frac{1}{z}(\phi_i \star).$$

We can view z as the loop variable by setting $\widehat{T} = T \times \mathbb{C}^\times$. If the extra copy of $\mathbb{C}^\times$ acts trivially on X, then

$$H_{\widehat{T}}^*(X) = H_T^*(X)[z].$$

This has a fundamental solution

$$M(\tau) \colon H_{\widehat{T}}^*(X)[\![Q, \tau]\!] \to H_{\widehat{T}}^*(X)_{\mathrm{loc}}[\![Q, \tau]\!]$$

where

$$H_{\widehat{T}}^*(X)_{\mathrm{loc}} \coloneqq H_{\widehat{T}}^*(X) \otimes_{\mathbb{Q}[\lambda, z]} \mathbb{Q}(\lambda(z)).$$

This satisfies the differential equation

$$z\frac{\partial}{\partial \tau^i} M(\tau) = M(\tau)(\phi_i \star),$$

which is equivalent to

$$\frac{\partial}{\partial \tau^i} \circ M(\tau) = M(\tau) \circ \nabla_i.$$

The solution has the form

$$(M(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j) + \left\langle\!\!\left\langle \phi_i, \frac{\phi_j}{z - \psi} \right\rangle\!\!\right\rangle_{0,2}^{X,T}.$$

1.3.3 Shift operators Let $k \colon \mathbb{C}^\times \to T$ be a cocharacter of T. Then define a $\widehat{T}$-action ρ_k on X by

$$\rho_k(t, x)x = tu^k \cdot x$$

for $t \in T, u \in \mathbb{C}^\times, x \in X$. Under the group automorphism

$$\phi_k \colon \widehat{T} \to \widehat{T} \qquad \phi_k(t, u) = (tu^{-k}, u),$$

the identity map $(X, \rho_0) \to (X, \rho_k)$ is $\widehat{T}$-equivariant, so we obtain isomorphisms

$$\Phi_k \colon H_{\widehat{T}, \rho_0}^*(X) \to H_{\widehat{T}, \rho_k}^*(X).$$

Now define the bundle

$$E_k = (X \times (\mathbb{C}^2 \setminus 0))/\mathbb{C}^\times,$$

where $\mathbb{C}^\times$ acts by

$$s \cdot (x, v_1, v_2) = (s^k x, s^{-1}v_1, s^{-1}v_2).$$

This is an X-bundle over $\mathbb{P}^1$ with an action on $\widehat{T}$ by

$$(t, u) \cdot [x, (v_1, v_2)] = [t \cdot x, (v_1, uv_2)].$$

Setting $0 = [1, 0]$ and $\infty = [0, 1]$, we see that $\widehat{T}$ acts on X_0 by ρ_0 and X_∞ by ρ_k.

Definition 1.3.1. A cocharacter $k\colon \mathbb{C}^\times \to T$ is *seminegative* if all weights of $H^0(X, \mathcal{O})$ are nonpositive with respect to k and is *negative* if all nonzero weights of $H^0(X, \mathcal{O})$ are negative.

Lemma 1.3.2. *If k is seminegative, then E_k is semiprojective.*

Now let $\pi\colon E_k \to \mathbb{P}^1$ be the projection. We now consider *section classes*, which are those effective classes in $H_2(E_k, \mathbb{Z})$ satisfying $\pi_* d = [\mathbb{P}^1]$. For the $\mathbb{C}^\times$-action on X given by k, there is a unique fixed component $F_{\min}$ whose normal weights are all positive (one way to see this is to consider the moment map of the corresponding circle action). Therefore, there is a minimal section class $\sigma_{\min}$ corresponding to $F_{\min}$.

Lemma 1.3.3. *Given $\tau \in H^*_T(X)$, there exists $\widehat{\tau} \in H^*_{\widehat{T}}(E_k)$ such that $\widehat{\tau}|_{X_0} = \tau$ and $\widehat{\tau}|_{X_\infty} = \Phi_k(\tau)$.*

Lemma 1.3.4. *If k is seminegative, then*

$$\mathrm{Eff}(E_k)^{\mathrm{sec}} = \sigma_{\min} + \mathrm{Eff}(X).$$

Definition 1.3.5. Let $k\colon \mathbb{C}^\times \to T$ be seminegative. Given $\tau \in H^*_T(X)$, we define the *shift operator*

$$\widetilde{S}_k\colon H^*_{\widehat{T},\rho_0}(X)[\![Q]\!] \to H^*_{\widehat{T},\rho_k}(X)[\![Q]\!]$$

by the formula

$$(\widetilde{S}_k(\tau)\alpha, \beta) = \sum_{\widehat{d} \in \mathrm{Eff}(E_k)^{\mathrm{sec}}} \frac{Q^{\widehat{d}-\sigma_{\min}}}{n!} \langle (\iota_0)_*\alpha, (\iota_\infty)_*\beta, \widehat{\tau}, \dots, \widehat{\tau} \rangle^{E_k, \widehat{T}}_{0, n+2, \widehat{d}},$$

where $\alpha \in H^*_{\widehat{t},\rho_0}(X)$ and $\beta \in H^*_{\widehat{T},\rho_k}(X)$. We also define

$$S_k(\tau) = \Phi_k^{-1} \circ \widehat{S}_k(\tau).$$

Theorem 1.3.6. *We have the formula*

$$M(\tau) \circ S_k(\tau) = \mathcal{S}_k \circ M(\tau),$$

where $\mathcal{S}_k$ is defined via the commutative diagram

$$\begin{tikzcd}
H^*_{\widehat{T}}(X)_{\mathrm{loc}} \arrow[r, "\mathcal{S}_k"] \arrow[d] & H^*_{\widehat{T}}(X)_{\mathrm{loc}} \arrow[d, "\iota^*"] \\
H^*_{\widehat{T}}(X^T)_{\mathrm{loc}} \arrow[r, "\bigoplus_i \Delta_i(k) e^{-2k\delta_\lambda}"] & H^*_{\widehat{T}}(X^T)_{\mathrm{loc}}.
\end{tikzcd}$$

Here, we define

$$\Delta_i(k) = Q^{\sigma_i - \sigma_{\min}} \prod_{\alpha} \prod_{j=1}^{\mathrm{rk}\, N_{i,\alpha}} \frac{\prod_{c=-\infty}^{0} (\rho_{i,\alpha,j} + \alpha + cz)}{\prod_{c=-\infty}^{-\alpha\cdot k} (\rho_{i,\alpha,j} + \alpha + cz)} \in H^*_{\widehat{T}}(F_i)_{\mathrm{loc}}[\![Q]\!],$$

where

$$N_i = N_{F_i/X} = \bigoplus_{\alpha} N_{i,\alpha}$$

is the normal bundle of F_i in X and $\rho_{i,\alpha,j}$ are its Chern roots.

The idea of the proof is to decompose

$$E^{\widehat{T}}_{k,0,n+2,\widehat{d}} = \bigsqcup_{i} \bigsqcup_{I_1 \cup I_2 = [n+2]} \bigsqcup_{d_0 + d_\infty + \widehat{\sigma} = \widehat{d}} (X_0)^T_{0,I_1 \sqcup p, d_0} \times_{F_i} (X_\infty)^T_{0, I_2 \sqcup q, d_\infty}.$$

Using the exact sequence

$$0 \to \mathrm{Aut}(C,x) \to \mathrm{Def}(f) \to T^1 \to \mathrm{Def}(C,x) \to \mathrm{Obs}(f) \to T^2 \to 0,$$

we obtain the explicit formulae

$$\begin{aligned} \mathrm{Aut}(C,x)^m &= \mathrm{Aut}(C_0,x_0)^m + \mathrm{Aut}(C_\infty,x_\infty)^m \\ \mathrm{Def}(C,x)^m &= \mathrm{Def}(C_0,x_0)^m \oplus \mathrm{Def}(C_0,x_0)^m \oplus T_p C_0 \otimes T_p \mathbb{P}^1 \oplus T_q C_\infty \otimes T_q \mathbb{P}^1. \end{aligned}$$

This gives the virtual normal bundle, and using virtual localization, we obtain

$$(\widetilde{\mathbb{S}}_k(\tau)\alpha, \beta) = (\widetilde{\mathbb{S}}_k M(\tau,z)\alpha, M'(\tau',-z)\beta),$$

where

$$M'(\tau',z) = \Phi_k \circ M(\tau,z) \circ \Phi_k^{-1}.$$

Using the unitarity property of M, we obtain the desired result.

2

Quantum cohomology of projective bundles

2.1 Mirror theorem (Che, Feb 22)

2.1.1 Setup Let X be a smooth projective variety, $\{\phi_i\}_{i=0}^s$ be a basis of $H^*(X)$, $\left\{\phi^i\right\}_{i=0}^s$ be the dual basis, and

$$\tau = \sum_{i=0}^{s} \tau^i \phi_i \in H^*(X).$$

We will let

$$J_X(\tau) = 1 + \frac{\tau}{z} + z^{-1} \sum_{d,n} \sum_{j=0}^{s} \left\langle \tau, \dots, \tau \frac{\phi_j}{z-\psi} \right\rangle_{0,n+1,d}^{X} \frac{Q^d}{n!},$$

which is the J-function in Definition 1.1.3 multiplied by z^{-1}.[1]. Also, recall the inverse of the fundamental solution of the quantum D-module

$$M_X(\tau) \in \mathrm{End}(H^*(X))[z^{-1}][\![Q,\tau]\!],$$

which is defined by

$$(\overline{M}_X(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j)_X + \sum_{d,n} \left\langle \phi_i, \tau, \dots, \tau, \frac{\phi_j}{z-\psi} \right\rangle_{0,n+2,d}^{X} \frac{Q^d}{n!}.$$

Remark 2.1.1. By the string equation, we have

$$J_X(\tau) = M_X(\tau) \cdot 1.$$

2.1.2 The vector bundle case Now let $V \to B$ be a vector bundle with $\mathrm{rk}\, V \geqslant 2$. This has an action of $\mathbb{C}^\times$ scaling the fibers. Then we have

$$H^*_{\mathbb{C}^\times}(V) = H^*(B) \otimes \mathbb{C}[\lambda].$$

Now we may take $\tau^0, \dots, \tau^s$ to be $\mathbb{C}[\lambda]$-valued coordinates.

Remark 2.1.2. Equivariant localization is required to define the Gromov-Witten invariants of V, which lie in $\mathbb{C}[\lambda, \lambda^{-1}]$.

[1] This is in fact the older definition of the J-function, but the one in Definition 1.1.3 lies on the Lagrangian cone

In order to avoid this issue, we will assume that $V^\vee$ is globally generated. This implies that V is semiprojective, meaning that the evaluation maps $\mathrm{ev}\colon V_{0,n,d} \to V$ are proper. As before, we may define the fundamental solution

$$M_V(\tau) \in \mathrm{End}(H^*(B))[\lambda, z^{-1}][\![Q,\tau]\!]$$

and the J-function

$$J_V^\lambda(\tau) = M_V(\tau)\cdot 1.$$

Because the evaluation maps are proper, they can be defined without localization.

2.1.3 Statement and discussion of the mirror theorem

Theorem 2.1.3. *Define the* $H^*(\mathbb{P}(V))$*-valued function*

$$I_{\mathbb{P}(V)}(\tau,t) = \sum_{k=0}^{\infty} \frac{e^{pt/z} q^k e^{kt}}{\prod_{c=1}^{k}\prod_\delta (p+\delta+cz)} J_V^{p+kz}(\tau),$$

where δ *are the Chern roots of* V*,* q *is the Novikov variable, and* $p = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$*. Then* $zI_{\mathbb{P}(V)}(\tau,t)$ *lies on the Lagrangian cone of* $\mathbb{P}(V)$*.*

Let $\mathcal{L}_X^{\mathrm{orig}}$ be the Lagrangian cone for X, which has the explicit form

$$-z + t(z) + \sum_{d,n}\sum_{k\geqslant 0}\sum_{i=0}^{s} \frac{\phi^i}{(-z)^{k+1}} \left\langle t(\psi),\ldots,t(\psi),\phi_i\psi^k\right\rangle_{0,n+1,d}^{X} \frac{Q^d}{n!}. \tag{2.1}$$

Definition 2.1.4. For a set of variables $x = (x_1, x_2, \ldots)$, we say that $f \in \mathcal{H}_X[\![x]\!]$ is a $\mathbb{C}[\![Q,x]\!]$*-valued point* on $\mathcal{L}_X^{\mathrm{orig}}$ if f is of the form 2.1 for some $t(z) \in \mathcal{H}_+[\![x]\!]$ with $t(z)|_{Q=x=0} = 0$.

Example 2.1.5. The point $zJ_X(\tau)|_{z\mapsto -z}$ is a $\mathbb{C}[\![Q,\tau]\!]$-valued point on $\mathcal{L}_X^{\mathrm{orig}}$.

Given this, define $\mathcal{L}_X \coloneqq \mathcal{L}_X^{\mathrm{orig}}|_{z\mapsto -z}$. By Theorem 1.1.10, we obtain

$$L_X = \bigcup_\tau zM_X(\tau)\mathcal{H}_+,$$

which means that any $\mathbb{C}[\![Q,x]\!]$-valued point on $\mathcal{L}_X$ can be written as $zM_X(\tau)f$ for some $\tau \in H^*(X)[\![Q,x]\!]$ and $f \in \mathcal{H}_+[\![x]\!]$ such that $\tau|_{Q=x=0} = 0$ and $f|_{Q=x=0} = 1$. This property will be used to construct the Fourier transform later.

2.1.4 Proof of Theorem 2.1.3 We will now sketch a proof of Theorem 2.1.3. First, we will need Quantum-Riemann-Roch for a vector bundle $W \to X$ in two cases:

(a) When the vector bundle W is convex, which means that $H^1(C, f^*W) = 0$ for all stable maps $f\colon C \to X$ of genus 0, and $\mathbf{c} = e(\lambda)$ is the equivariant Euler class, which corresponds to setting

$$s_k = \begin{cases} \log\lambda & k = 0 \\ (-1)^{k-1}(k-1)!\lambda^{-k} & k > 0. \end{cases}$$

(b) When W is globally generated and $\mathbf{c} = e_\lambda^{-1}$.

In the first case, we obtain the Gromov-Witten invariants of the zeroes of a regular section $Z \subset X$ of W via

$$\lim_{\lambda\to 0}\left\langle \alpha_1\psi^{k_1},\ldots,\alpha_n\psi^{k_n}\right\rangle_{0,n,d}^{X,(W,e_\lambda)} = \sum_{i_*d'=d}\left\langle i^*\alpha_1\psi_1^{k_1},\ldots,\alpha_n\psi_n^{k_n}\right\rangle_{0,n,d'}^{Z}.$$

In the second case, we obtain the Gromov-Witten invariants of W via

$$\left\langle \alpha_1\psi^{k_1},\ldots,\alpha_n\psi^{k_n}\right\rangle_{0,n,d}^{X,(W,e_\lambda^{-1})} = \left\langle i^*\alpha_1\psi_1^{k_1},\ldots,\alpha_n\psi_n^{k_n}\right\rangle_{0,n,d}^{W}.$$

We are now ready to begin the proof. Because $V^\vee$ is globally generated, there is a surjection

$$\mathcal{O}^{\oplus N} \to V^\vee.$$

This gives an exact sequence

$$0 \to V \to \mathcal{O}^{\oplus N} \to Q \to 0$$

embedding $\mathbb{P}(V) \hookrightarrow B \times \mathbb{P}^{N-1}$. By a result of Brown-Elezi, we have

$$J_{B\times\mathbb{P}^{N-1}}(\tau,t) = \sum_{k=0}^{\infty}\frac{e^{pt/z}q^k e^{kt}}{\prod_{c=1}^{k}(p+cz)^N}J_B(\tau).$$

Now define

$$Q(1) := \pi_1^*Q \otimes \pi_2^*\mathcal{O}(1)$$

on $B \times \mathbb{P}^1$. This has a section s given by

$$\pi_2^*\mathcal{O}(-1) \to \mathcal{O}_{B\times\mathbb{P}^{N-1}}^{\oplus N} \to \pi_1^*Q$$

which satisfies $s^{-1}(0) = \mathbb{P}(V)$. Because $Q(1)$ is convex, we use Quantum-Riemann-Roch in case (a) to relate the Gromov-Witten theory of $\mathbb{P}(V)$ to the $(Q(1), e_\lambda)$-twisted Gromov-Witten theory. We now require two more techinical ingredients.

Moving points on the Lagrangian cone via differential operators

Lemma 2.1.6. *Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be formal variables. Let*

$$F \in \mathbb{C}[z][\![x]\!]\langle z\,\partial_{x_1}, z\,\partial_{x_2},\ldots\rangle[\![Q,y]\!]$$

be a differential operator. Then $\exp(F/z)$ preserves $\mathbb{C}[\![Q,x,y]\!]$-valued points on $\mathcal{L}_X$.

Definition 2.1.7. A $\mathbb{C}[\![Q,\tau,y]\!]$-valued point f on $\mathcal{L}_X$ is called a *miniversal slice* if

$$f|_{Q=y=0} = z + \tau + \mathcal{O}(z^{-1}).$$

For example, the J-function is a miniversal slice.

Lemma 2.1.8. *Any miniversal slice on $\mathcal{L}_X$ can be obtained from $zJ_X(\tau)$ be applying $\exp(F/z)$ for some differential operator F as in the previous lemma satisfying $F|_{Q=y=0} = 0$.*

The rest of the proof (ignoring convergence issues) First, we introduce

$$\Delta_W^\lambda \coloneqq e^{\mathrm{rk}(W)(\lambda\log\lambda-\lambda)/z}\Delta_{(W,e_\lambda^{-1})}.$$

Because $\log\Delta_W^\lambda$ and $\log\Gamma(x)$ have similar asymptotic expansions, we have

$$\Delta_W^{\lambda+kz}/\Delta_W^\lambda = \prod_{c=1}^{k}\prod_{\delta}(\lambda+\delta+cz).$$

Using the exact sequence

$$0\to V\to \mathcal{O}^{\oplus N}\to Q\to 0,$$

we see that

$$\Delta_V^\lambda\Delta_Q^\lambda = \Delta_{\mathcal{O}^{\oplus N}}^\lambda,$$

which preserves the Lagrangian cone $\mathcal{L}_B$. We see that

$$\Delta_Q^\lambda : \mathcal{L}_{B,(V,e_\lambda^{-1})}\to\mathcal{L}_B.$$

Applying Quantum-Riemann-Roch in case (b), we see that

$$zJ_V^\lambda(z)\in L_{B,(V,e_\lambda^{-1})},$$

and thus

$$\Delta_Q^\lambda zJ_F^\lambda(z)\in\mathcal{L}_B.$$

By Lemma 2.1.8, there exists F such that

$$\Delta_Q^\lambda zJ_V^\lambda(z) = e^{F(\lambda)/z}zJ_B(\tau).$$

By Lemma 2.1.6, we obtain

$$e^{F(\lambda+z\partial_t)/z}J_{B\times\mathbb{P}^{N-1}}(\tau,t)\in\mathcal{L}_{B\times\mathbb{P}^{N-1}}.$$

Now we compute

$$\begin{aligned}
I^\lambda(\tau,t) &\coloneqq (\Delta_{Q(1)}^\lambda)^{-1}e^{F(\lambda+z\partial_t)/z}J_{B\times\mathbb{P}^{N-1}}(tau)\\
&= \sum_{k=0}\frac{e^{pt/z}q^ke^{kt}}{\prod_{c=1}^k(p+cz)^N}(\Delta_{Q(1)}^\lambda)^{-1}e^{F(\lambda+p+kz)}J_B(\tau)\\
&= \sum_{k=0}\frac{e^{pt/z}q^ke^{kt}}{\prod_{c=1}^k(p+cz)^N}(\Delta_{Q(1)}^\lambda)^{-1}(\Delta_Q^{\lambda+p})^{-1}\Delta_Q^{\lambda+p+kz}J_V^{\lambda+p+kz}\\
&= \sum_{k\geqslant 0}e^{pt/z}q^ke^{kt}\frac{\prod_{c=1}^k\prod_\varepsilon(\lambda+p+\varepsilon+cz)}{\prod_{c=1}^k(p+cz)^N}J_V^{\lambda+p+kz},
\end{aligned}$$

where ε runs over the Chern roots of Q. Taking the non-equivariant limit $\lambda\to 0$, we obtain the $I(\tau,t)$ in the statement of Theorem 2.1.3.

2.2 Fourier transform (Kostya, Feb 29)

Technically, there are two different Fourier transforms:

1. The discrete Fourier transform $\mathrm{QDM}_{S^1}(V) \xrightarrow{\mathrm{FT}} \mathrm{QDM}(\mathbb{P}(V))$;
2. The continuous Fourier transform $\mathrm{QDM}(\mathbb{P}(V))_{\mathrm{loc}} \to \bigoplus_{i=0}^{r-1} \mathrm{QDM}(B)$.

2.2.1 Quantum D-modules and symplectic spaces Let $V \to B$ be a rank r vector bundle. The quantum D-module of V will be

$$H^*_{S^1}(V) \otimes \mathbb{C}[z,\lambda][\![Q,\boldsymbol{\tau}]\!],$$

where λ is the equivariant variable, Q is the Novikov variable, and $\boldsymbol{\tau} = \{\tau^{i,k}\}$ for i counting a basis of $H^*(V)$ and k records the degree of λ. It is equipped with the Dubrovin connection

$$\nabla\colon\ \mathrm{QDM}_{S^1}(V) \to z^{-1}\,\mathrm{QDM}_{S^1}(V)$$

given by

$$\begin{aligned}
\nabla_{\tau^{i,k}} &= \frac{\partial}{\partial \tau^{i,k}} + z^{-1}\lambda^k(\phi_i \star -)\\
\nabla_{\xi Q\partial_Q} &= \xi Q\,\partial_Q + z^{-1}(\xi \star -)\\
\nabla_{z\partial_z} &= z\,\partial_z - z^{-1}(E_{S^1} \star -) + \mu_{S^1}.
\end{aligned}$$

Note the last line is only $\mathbb{C}$-linear. Recall the fundamental solution $M_V^{-1}(\boldsymbol{\tau}, z)$ is a fundamental solution in the cohomology directions (but not the conformal direction) in the sense that

$$\begin{aligned}
\partial_{\tau^{i,k}} M_V &= M_V \nabla_{\tau^{i,k}}\\
(\varepsilon Q\,\partial_Q + z^{-1\xi})M_V &= M_V \nabla_{\varepsilon Q\partial_Q}
\end{aligned}$$

and intertwines the shift operator by

$$\mathcal{S} M_V = M_V \mathcal{S}(\boldsymbol{\tau}),$$

where

$$\mathcal{S} = e_\lambda(V) e^{z\partial_\lambda}.$$

Now, the symplectic space for V is

$$\mathcal{H}_V^{S^1} \coloneqq H^*_{S^1}(V)(\!(z^{-1})\!)[\![Q,\tau]\!]$$

with its Lagrangian cone $\mathcal{L}_V$. It has the important property that $f(\boldsymbol{\tau})$ is in $\mathcal{L}_V$ means that there exists $\widehat{\tau}(\boldsymbol{\tau})$ and $\widetilde{f} \in \mathrm{QDM}_{S^1}(V)$ such that

$$f = zM_V(\widehat{\tau}(\boldsymbol{\tau}), z)\widetilde{f},$$

which can be seen as a Birkhoff factorization.

We now turn to $\mathbb{P}(V)$. There is a decomposition

$$H^*(\mathbb{P}(V)) = H^*(V)[p]/\prod_\delta(\delta + p),$$

where δ runs over the Chern roots of V. This receives the Kirwan map

$$\kappa\colon H^*_{S^1}(V) \twoheadrightarrow H^*(\mathbb{P}(V)), \qquad \kappa(\lambda) = p.$$

Thus the quantum D-module for $\mathbb{P}(V)$ is

$$\mathrm{QDM}(\mathbb{P}(V)) = H^*(\mathbb{P}(V)) \otimes \mathbb{C}[z, q][\![Q, \widehat{\boldsymbol{\tau}}]\!],$$

where $\widehat{\boldsymbol{\tau}} = \{\widehat{\tau}_i\}$ is a basis for $H^*(\mathbb{P}(V))$ and q is the Novikov variable of the fiber curve class. It is equipped with the connections $\nabla_{\widehat{\tau}_i}, \nabla_{\xi Q \partial_Q}, \nabla_{q\partial_q}, \nabla_{z\partial_z}$.

Remark 2.2.1. Note there are no shift operators, but there is an additional q-direction in the quantum D-module for $\mathbb{P}(V)$

We also have the symplectic space $\mathcal{H}_{\mathbb{P}(V)}$, the Lagrangian cone $\mathcal{L}_{\mathbb{P}(V)}$, and the fundamental solution $M_{\mathbb{P}(V)}(\widehat{\boldsymbol{\tau}}, z)$. Finally, we will recall the mirror theorem in the form that

$$I_{\mathbb{P}(V)} = \sum_{k \geqslant 0} \kappa(\mathcal{S}^{-k} J^{\lambda + kz}) q^k$$

lies on $\mathcal{L}_{\mathbb{P}(V)}$.

2.2.2 Discrete Fourier transform

Definition 2.2.2. The *discrete Fourier transform* $\mathcal{H}_V \to \mathcal{H}_{\mathbb{P}(V)}$ is the transform

$$J^\lambda \mapsto \widehat{J} = \sum_{k \geqslant 0} \kappa(\mathcal{S}^{-k J^{\lambda+kz}}) q^k.$$

In this framing, the mirror theorem states that the discrete Fourier transform of the J-function of V lies on the Lagrangian cone of $\mathbb{P}(V)$.

Theorem 2.2.3. *There exists a "mirror map"*

$$\widehat{\boldsymbol{\tau}} = \widehat{\boldsymbol{\tau}}(\boldsymbol{\tau}) \in H^*(\mathbb{P}(V))[q][\![Q, \tau]\!]$$

and an isomorphism

$$\mathrm{FT}\colon\ \mathrm{QDM}_{S^1}(V) \to \widehat{\boldsymbol{\tau}}^*\,\mathrm{QDM}(\mathbb{P}(V))$$

of $\mathbb{C}[z][\![Q, \boldsymbol{\tau}]\!]$*-modules intertwining the connections in the natural ways.*

Remark 2.2.4. One has to be careful with the Novikov variables and think about approximately eight other points of the theorem, but we will ignore these for now.

Because the Fourier transform intertwines the connections, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{QDM}_{S^1}(V) \arrow[r, "\mathrm{FT}"] \arrow[d, "M_V(\boldsymbol{\tau})"] & \boldsymbol{\tau}^*\,\mathrm{QDM}(\mathbb{P}(V)) \arrow[d, "M_{\mathbb{P}(V)}(\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau}))"] \\
\mathcal{H}_V \arrow[r, "J \to \widehat{J}"] & \mathcal{H}_{\mathbb{P}(V)}.
\end{tikzcd}$$

Idea of proof. The idea of the proof is to start from the mirror theorem (the bottom row) and apply Birkhoff factorization. The mirror theorem states that

$$(M_V(\boldsymbol{\tau})1)^{\wedge} = M(\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau}))\Upsilon \in \mathcal{L}_{\mathbb{P}(V)}$$

for some mirror map $\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau})$ and $\Upsilon \in \mathrm{QDM}(\mathbb{P}(V))$. Using the intertwining properties of M, we see that

$$(M_V(\boldsymbol{\tau})(\phi_i\lambda^k))^{\wedge} = M(\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau}))z\boldsymbol{\tau}^*\nabla_{\frac{\partial}{\partial t^{i,k}}}\Upsilon.$$

Defining

$$\mathrm{FT}(\phi_i\lambda^k) := z\boldsymbol{\tau}^*\nabla_{\partial_{\tau^{i,k}}}\Upsilon$$

and $\widehat{\boldsymbol{\tau}}$ to be the mirror map appearing in the Birkhoff factorization, we are done. □

Remark 2.2.5. The mirror map satisfies

$$\widehat{\boldsymbol{\tau}}(\boldsymbol{\tau})|_{q=Q=0} = \kappa(\boldsymbol{\tau})$$

and the Fourier transform satisfies

$$\mathrm{FT}(\phi_i\lambda^k)|_{Q=\tau=0} = \phi_i p^k.$$

Remark 2.2.6. The Fourier transform intertwines the natural pairings on the quantum D-modules.

2.2.3 Continuous Fourier transform

Definition 2.2.7. Define

$$\mathrm{QDM}(\mathbb{P}(V))_{\mathrm{loc}} := \mathrm{QDM}(\mathbb{P}(V)) \otimes \mathbb{C}[z](\!(q^{-\frac{1}{r'}}[\![Q, \widehat{\boldsymbol{\tau}}]\!],$$

where $r' = r$ or $2r$ depending on parity.

Theorem 2.2.8. *For* $j = 0, \ldots, r-1$, *there exist maps* $H^*(\mathbb{P}(V)) \to H^*(B)$ *given by*

$$\widehat{\boldsymbol{\tau}} \mapsto \zeta_j(\widehat{\boldsymbol{\tau}}) \in -c_1(V)\log\left(e^{\frac{2\pi\sqrt{-1}j}{r}} q^{\frac{1}{r}}\right) + H^*(B)(\!(q^{-\frac{1}{r}}[\![Q, \widehat{\boldsymbol{\tau}}]\!]$$

and an isomorphism

$$\Phi\colon \mathrm{QDM}(\mathbb{P}(V))_{\mathrm{loc}} \cong \bigoplus_{j=0}^{r-1} \zeta_j^* \,\mathrm{QDM}(B)_{\mathrm{loc}}$$

intertwining the pairings and quantum connections in a natural way, namely that

$$\Phi\Delta = \bigoplus \zeta_j^* \Delta\Phi.$$

Writing $\Phi = (\Phi_0, \ldots, \Phi_j)$, *we have*

$$\Phi_j(\phi_i p^k)|_{Q=\widehat{\boldsymbol{\tau}}=0} = \frac{1}{\sqrt{r}}\lambda_j^{k-\frac{r-1}{2}}(\phi_i + O(q^{-\frac{1}{r}})).$$

Idea of proof. We use another realization of the Fourier transform on $\mathrm{QDM}_{S^1}(V)$ and

$$\mathrm{FT}\colon\ \mathrm{QDM}_{S^1} \cong \widehat{\boldsymbol{\tau}}^* \,\mathrm{QDM}(\mathbb{P}(V)).$$

If we consider Δ_V^λ arising from Quantum Riemann-Roch, it is given as

$$\Delta_V^\lambda \asymp \prod_\rho \sqrt{\frac{z}{2\pi}} z^{\frac{\lambda+\rho}{z}} \Gamma\left(\frac{\rho+\lambda}{z}+1\right).$$

Shifting by $-z$, we see that

$$\Delta_V^{\lambda-z} = \Delta_V^\lambda \prod_\rho \frac{1}{\rho+\lambda} = \Delta_V^\lambda \frac{1}{e_{S^1}(V)}.$$

We now consider the transformation

$$s \mapsto \int q^{\frac{\lambda}{z}} (\Delta_V^\lambda)^{-1} M_V(\boldsymbol{\tau}) \cdot s \, d\lambda$$

for $s \in \mathrm{QDM}_{S^1}(V)$. Because this integral intertwines $\mathcal{S}$ with q and λ with $z\nabla_{q\partial_q}$, it formally gives a solution to $\mathrm{QDM}(\mathbb{P}(V))$.

To make sense of this terrible expression, we use the stationary phase expansion of the integral. Setting

$$I(s) = \int e^{-\frac{\varphi(\lambda)}{z}} \lambda^{-\frac{c_1(V)}{z}} \lambda^{-\frac{r}{2}} (\widetilde{\Delta}_V^\lambda)^{-1} J^\lambda \, d\lambda,$$

where $\frac{\varphi(\lambda)}{z}$ is the Stirling asymptotics of Δ_V^λ, given by

$$\varphi(\lambda) = r(\lambda \log \lambda - \lambda) - \lambda \log q.$$

The critical points of φ are given by

$$\frac{\partial}{\partial \lambda} \varphi(\lambda) = r(\log \lambda) - \log q = 0,$$

which tells us that $\lambda^r = q$. Thus, we obtain r solutions

$$\lambda_j = e^{\frac{2\pi\sqrt{-1}j}{r}} q^{\frac{1}{r}}.$$

We now consider the formal expansions around λ_j. These produce a "continuous Fourier transform"

$$J \mapsto \mathcal{F}_j(J)$$

such that

$$I(M^{-1}J) \asymp \sqrt{2\pi z} e^{r\frac{\lambda_j}{z}} \mathcal{F}_j(J).$$

These intertwine the quantum connection and multiplication by λ, as in

$$\mathcal{F}_j(\lambda J) = (\lambda_j + zq\,\partial_q)\mathcal{F}_j(J)$$
$$\mathcal{F}_j(\mathcal{S}J) = q\mathcal{F}_j(J),$$

so $z\mathcal{F}_j(J_V(\boldsymbol{\tau}))$ is on the Lagrangian cone of B. We then use the following result:

Proposition 2.2.9. *We have*

$$\mathcal{F}_j(J_V(\boldsymbol{\tau})) = M_B(\sigma_j(\boldsymbol{\tau}))\nu_j$$

for some σ_j, ν_j.

Unfortunately, $\mathcal{F}_j$ does not intertwine $\nabla_{z\partial_z}$ correctly. To fix this, define

$$\zeta_j(\widehat{\boldsymbol{\tau}}) = \sigma_j(\boldsymbol{\tau}(\widehat{\boldsymbol{\tau}})) + r\lambda_j$$

and Φ_j by a shift of ν_j. □

2.2.4 Discrete equals continuous

Warning 2.2.10. **Everything in this subsection may be false.**

Consider the Fourier transform

$$\int \prod_{\rho} \Gamma\left(-\frac{\rho+\lambda}{z}\right) J_V^{\lambda} q^{\frac{\lambda}{z}}\, d\lambda.$$

This can be computed either using residues or using stationary phase asymptotics. Using residues, we obtain

$$\sum_{k \geqslant 0} \mathrm{Res}_{p=0}\, \Gamma\left(-k - \frac{\rho+\lambda}{z}\right) J_V^{\lambda+kz} q^{\frac{p}{z}} q^{k},$$

which is precisely

$$\frac{1}{\prod_{c=0}^{k} e_{p+\lambda z}(V)} \Gamma\left(-\frac{\rho-\lambda}{z}\right).$$

Using stationary phase asymptotics, we obtain the $I(s)$ defined previously.