Moduli Spaces and Hyperkähler Manifolds
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Notes by Patrick Lei

Lectures by Giulia Saccà

Columbia University

Disclaimer

Unless otherwise noted, these notes were taken during lecture using the `vimtex` package of the editor `neovim`. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the instructor. If you find any errors, please contact me at `plei@math.columbia.edu`.

Acknowledements I would like to acknowledge Kevin Chang and Che Shen for pointing out mistakes in these notes.

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Hyperkähler Manifolds

Some useful references for K3 surfaces are the book by Huybrechts and the Barth-Peters-van de Ven book *Compact Complex Surfaces* and another book. For Hilbert schemes some references are Chapter 7 of *FGA Explained*, Huybrechts-Lehn, some lectures of notes of Lehn, and Nakajima's *Lectures on Hilbert Schemes*. For Hilbert schemes of K3 surfaces and abelian varieties, there is Beauville's *Varietés Kahlerienne dont la premiere classe de Chern est nulle*.

1.1 Motivation

Giulia believes that hyperkähler manifolds are some of the most interesting objects in algebraic geometry because one can actually prove results about high-dimensional hyperkähler varieties, unlike the usual situation in algebraic geometry. Because these objects are of a differential-geometric nature, through the course we will work over $\mathbb{C}$.

Recall that in order to classify curves, for a given curve C, we want to consider the positivity of the canonical bundle. In the first case, we know $\omega_{\mathbb{P}^1} = \mathcal{O}(-2) < 0$, in the second case of an elliptic curve, we have $\omega_C = \mathcal{O}_C$, and finally for a higher genus curve the canonical sheaf $\omega_C > 0$ is ample.

In higher dimension, let X be a smooth projective variety. Then there exists an integer $\kappa(X)$, the *Kodaira dimension* of X such that

$$h^0(\omega_X^{\otimes m}) \sim m^{\kappa(X)}$$

for $m \gg 0$ sufficiently divisible. Of course, this is a birational invariant.

There is a classification of surfaces. Each smooth surface is birational to a *minimal* surface. Here, a surface S is minimal if any birational morphism from S to a smooth surface is a birational curve. By Castelnuovo, we know that S is minimal if and only if it does not contain a (-1)-curve. Also, any surface dominates a minimal surface.

1.2 Hyperkähler manifolds

Example 1.2.1. If S is a surface, then $\kappa(S) = -\infty$ if and only if

$$P_m(S) := h^0(\omega_S^{\otimes m}) = 0$$

for all $m \geqslant 1$. Some examples of these are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 1.2.2. If S is a surface, then $\kappa(S) = 0$ if and only if $P_m(S) = 0$ generically and there exists m such that $P_m(S) = 1$. In this case, there are two cases. First, $h^0(\omega_S) = 1$, in which case $\omega_S = \mathcal{O}_S$, and thus either $h^1(\mathcal{O}_S) = 0$ (in which case we have a *K3 surface*) or $\mathfrak{h}^1(\mathcal{O}_S) = 2$ (in which case we have an *abelian surface*).

Alternatively, we may have $h^0(\omega_S) = 0$, in which case there exists $m \geqslant 2$ (where $m \in \{2, 3, 4, 6\}$) such that $\omega_S^{\otimes m} = \mathcal{O}_S$. Either $h^1(\mathcal{O}_S) = 0$, in which case we have an Enriques surface, or $h^1(\mathcal{O}_S) = 1$, in which case we have a bi-elliptic surface.

If T is an Enriques surface, then there exists a K3 surface S with a $2:1$ étale cover $S \to T$. On the other hand, any bi-elliptic surface has an $m:1$ étale cover from an abelian surface.

Theorem 1.2.3 (Beauville-Bogomolov)**.** *Let M be a compact Kähler manifold with $c_1(\omega_M) = 0$. The there exists a finite étale cover of M by a product*

$$T^n \times \prod Y_i \times \prod X_i \to M,$$

where T^n is a complex torus, the Y_i are strict Calabi-Yau, and the X_i are irreducible holomorphic symplectic (or hyperkähler).

Definition 1.2.4. Let Y be a compact Kähler manifold. Then Y is *strict Calabi-Yau* if $\pi_1(Y) = 1$ and $H^0(\Omega_Y^p) = \mathbb{C}$ when $p = 0, \dim Y$ and $H^0(\Omega_Y^p)$ vanishes elsewhere.

Definition 1.2.5. A compact Kähler manifold X is *irreducible holomorphic symplectic* if $\pi_1(X) = 1$ and $H^0(\Omega_X^2) = \mathbb{C}\sigma_X$, where σ_X is an irreducible symplectic form. In particular, σ^n is a nonzero top form and thus trivializes the canonical bundle. In addition, σ induces an isomorphism of holomorphic vector bundles $\Omega_X \simeq T_X$.

1.3 Some surfaces

Returning to the simplest case, we will define K3 surfaces.

Definition 1.3.1. A smooth projective surface S is a *K3 surface* if $\omega_S = 0$ and $h^1(\mathcal{O}_S) = 0$.

It follows from the definition that K3 surfaces are simply connected, so they are in fact both strict Calabi-Yau and irreducible holomorphic symplectic. Later in the course, we will see that irreducible holomorphic symplectic varieties are the true higher-dimensional analogues of K3 surfaces.

Lemma 1.3.2. *Let S be a K3 surface and $f\colon S \to C$ be a dominant morphism to a smooth projective curve C with connected fibers. Then $C = \mathbb{P}^1$ and the general fiber of f is an elliptic curve.*

Proof. The proof is left as an exercise to the reader. □

Any K3 surface S with a dominant map to a curve is called an *elliptic K3*. As a consequence, any surjective map $f\colon S \to B$ where B is not a point and f has connected fibers has either $B = \mathbb{P}^1$ or B is a singular K3. This is generalized by the following remarkable result:

Theorem 1.3.3 (Matsushita)**.** *Let X^{2n} be an irreducible holomorphic symplectic manifold and $f\colon X \to B$ be a proper surjective morphism with connected fibers with B a normal variety. If B is not a point, then either $\dim B = n$ and f is a Lagrangian fibration where the general fiber is an abelian n-fold or $\dim B = 2n$ and B is a singular symplectic variety if f is not an isomorphism. In the second case, f is called a symplectic resolution.*

Remark 1.3.4. There is another extremely difficult result of Hwang, which says that if B is smooth, then $B = \mathbb{P}^n$ (if $\dim B = 3$, apparently B is a $\mathbb{Q}$-factorial Fano threefold with klt singularities).

Now we will consider some examples. Beginning in the simplest case, consider a general section $f_4 \in |\mathcal{O}_{\mathbb{P}^3}(4)|$. By the Bertini theorem, the general $S = (f_4 = 0)$ is smooth, and by the adjunction formula, $\omega_S = \mathcal{O}_S$. Then we consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0,$$

and by the long exact sequence of cohomology and the known values of cohomology for projective space, we have $H^1(\mathcal{O}_S) = 0$.

Example 1.3.5. A concrete example of this is the Fermat quartic, which has the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

We will see that this is an elliptic K3. The first step is to see that S contains a line $\ell \subseteq S \subseteq \mathbb{P}^3$, so we choose a primitive ζ_8 and set $x_0 = \zeta_8 x_1$ and $x_2 = \zeta_8 x_3$. Now we project S from ℓ, and considering planes that contain ℓ, we obtain a rational map $S \dashrightarrow \mathbb{P}^1$. This extends over ℓ. Finally, we know that $S \cap \mathbb{P}^2$ is a quartic curve containing a line ℓ, so in fact the generic fiber of this map is an elliptic curve.

Similarly, we may consider other complete intersections, such as the $(2,3)$ complete intersection in $\mathbb{P}^4$ (intersection of a quadric and a cubic) and the $(2,2,2)$ complete intersection in $\mathbb{P}^5$. In higher dimensions, any degree $(n+1)$ hypersurface Y in $\mathbb{P}^n$ has $\omega_Y = \mathcal{O}_Y$. By the Lefschetz hyperplane theorem, this is a strict Calabi-Yau.

Example 1.3.6. Let $\Gamma \in |\mathcal{O}_{\mathbb{P}^2}(6)|$ be a general sextic and S be a $2:1$ cover of $\mathbb{P}^2$ branched along Γ. We will use the *covering trick*, which holds for any variety X, line bundle L, and $0 \neq s \in H^0(L^{\otimes m})$ for some $m \geqslant 1$. Then if we set $D = (s = 0)$, there exists a finite flat morphism $f\colon Y \to X$ that is a $\mathbb{Z}/m$-cover away from D and ramified along D. In this case, f^*L has a section t such that $(t = 0) \simeq D$. Finally if X and D are smooth, so is Y, and $\omega_Y = f^*\omega_X((m-1)(t=0))$.

In our example, we have $\omega_S = f^*\omega_{\mathbb{P}^2} \otimes \mathcal{O}_X(y^2 = \Gamma) = \mathcal{O}_S$, so S is a K3 surface.

Example 1.3.7 (Kummer K3 surfaces)**.** Let A be an abelian surface. It has an involution -1 with fixed locus $A[2]$. Thus $A/\pm 1$ has 16 singular points that look like $\mathbb{C}^2/\pm 1 = \operatorname{Spec}\mathbb{C}[x^2, xy, y^2] = \operatorname{Spec}\mathbb{C}[a,b,c]/(ab = c^2)$ (the A_1 singularity). Now the surface $S = \mathrm{Bl}_{A[2]}A/\pm 1$ is a K3 surface.

It is easy to see that the smooth locus of $A/\pm 1$ has a holomorphic symplectic form $\sigma_{A/\pm 1}$. Then we can bull back $f^*\sigma_{A/\pm 1}$ to a holomorphic symplectic form on $f^{-1}(U) \subseteq S$, and this form extends to S. The reason for this is that $\mathrm{Bl}_{A[2]} A$ still has the involution -1, and S is the quotient of $\mathrm{Bl}_{A[2]} A$ by this involution. If we denote this diagram by

$$\begin{tikzcd} \mathrm{Bl}_{A[2]} A \arrow[r, "q"] \arrow[d, "g"] & S \arrow[d, "f"] \\ A \arrow[r, "p"] & A/\pm 1 \end{tikzcd} \tag{1.1}$$

and denote $\widetilde{A} := \mathrm{Bl}_{A[2]} A$, then we obtain

$$\begin{aligned} \omega_{\widetilde{A}} &= f^*\omega_A \otimes \mathcal{O}\left(\sum E_i\right) = \mathcal{O}_{\widetilde{A}}\left(\sum E_i\right) \\ &= q^*\omega_S \otimes \mathcal{O}_{\widetilde{A}}\left(\sum E_i\right) \end{aligned}$$

and therefore $q^*\omega_S = \mathcal{O}_{\widetilde{A}}$, so $\omega_S = \mathcal{O}_S$. The morphism f is called a *symplectic resolution*.

Before we proceed, we will discuss crepant and symplectic resolutions. Let Y be a smooth variety, so Ω^1_Y is locally free. Then $\omega_Y \coloneqq \bigwedge^{\dim Y} \Omega^1_Y$ is called the *canonical bundle*. Then if $f\colon X \to Y$ is a birational morphism of smooth varieties, we have an exact sequence

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0.$$

Then we know $\Omega^1_{X/Y}$ is supported on the exceptional locus of f. Because Y is smooth, then the exceptional locus is divisorial, and thus

$$\omega_X = f^*\omega_Y \otimes \mathcal{O}_X\Big(\sum a_i E_i\Big).$$

Now suppose that Y is just normal with smooth locus U. Also suppose that $Y \setminus U$ has codimension at least 2, so Weil divisors on U and Y are the same. There are two ways to extend ω_U to Y. The first is to denote the inclusion $j\colon U \subseteq Y$ and consider the sheaf $j_*\omega_U$, which is generally not locally free. On the other hand, we can extend the Weil divisor K_U to Y, which determines a Weil divisor K_Y on Y, called the *canonical class*.

Remark 1.3.8. In general, the Weil divisor K_Y is not Cartier. In fact, K_Y is Cartier if and only if $j_*\omega_U$ is locally free.

Now let $f\colon X \to Y$ be a reolution of Y. This means f is proper and an isomorphism over U. We want a formula relating of the form $K_X = f^*K_Y + \sum a_i E_i$. Unfortunately, we can only pull back Cartier divisors, so we will assume that K_Y is $\mathbb{Q}$-Cartier, which means that there exists $m \geqslant 1$ such that mK_Y is Cartier. We know that $f^{-1}(U) \simeq U$, so $K_X\big|_{f^{-1}(U)} = f^*K_Y\big|_{f^{-1}(U)}$. Thus there exist integers a_i such that

$$mK_X = f^*mK_Y + \sum a_i E_i,$$

where the E_i are the divisorial components of $X \setminus f^{-1}(U)$. Formally dividing by m, we have

$$K_X = f^*K_Y + \sum a_i E_i.$$

Here, the a_i are known as the *discrepancies* and if $a_i = 0$, then the resolution is called *crepant*.

Example 1.3.9. One example of a crepant resolution is $S \to A/\pm 1$.

Example 1.3.10. Consider $Y = \mathbb{C}^{2N}/\pm 1$. This is the cone over the degree 2 Veronese embedding of $\mathbb{P}^{2N-1}$. Now we will write $f\colon X = \mathrm{Bl}_0 Y \to Y$, and the exceptional divisor is a $\mathbb{P}^{2N-1}$. We know that X is the total space of $\mathcal{O}(-2)$, so there is a projection $X \to \mathbb{P}^{2N-1}$. Now we need to compute a in the formula

$$K_X = f^*K_Y + aE.$$

First note that $K_Y = 0$. This is because the standard holomorphic symplectic form on $\mathbb{C}^{2N}$ descends to the smooth locus $U \subseteq Y$, so we have a symplectic form on $X \setminus E$. Now by the adjunction formula, we have

$$K_E = (K_X + E)\Big|_E,$$

and thus because $E = \mathbb{P}^{2N-1}$, we have

$$\mathcal{O}_E(-2N) = (a+1)E\Big|_E.$$

Finally, we see that $\mathcal{O}_X(E)\big|_E = \mathcal{O}_{\mathbb{P}^{2N-1}}(-2)$, and thus $a+1 = N$, so $a = N-1$. In particular, f is a crepant resolution if and only if $N = 1$. For $N \geqslant 2$, $f^*\omega_U$ extends to X with a zero of order $N-1$ along E. Therefore the form

$$f^*\sigma_U \wedge \cdots \wedge f^*\sigma_U$$

has a zero of order $N-1$ along E, so $f^*\sigma_U$ does not extend over E.[1]

1.4 Hilbert Schemes of points on surfaces

Let X be a smooth quasiprojective surface. Consider the functor

$$\mathrm{Hilb}^n_X \colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$$

associating a scheme T to isomorphism classes of flat proper morphisms $T \times X \supseteq Z \to T$ satisfying $p_{Z_t}(t) = n$.

Theorem 1.4.1 (Grothendieck)**.** *The functor* Hilb^n_X *is representable by a quasiprojective scheme* $X^{[n]}$. *If* X *is projective, so is* $X^{[n]}$.

Later in the course, we will sketch a construction of the Hilbert scheme, but for now we will simply assume that it exists. A fundamental resolt about Hilbert schemes is

Theorem 1.4.2 (Fogarty)**.** *Let* X *be a smooth quasiprojective surface. Then* $X^{[n]}$ *is a smooth connected quasiprojective variety of dimension* $2n$ *and there exists a morphism* $h\colon X^{[n]} \to X^{(n)}$, *called the Hilbert-Chow morphism,*[2] *which is a resolution of singularities. Here, if* Z *is a length* n *subscheme of* Z, *we have*

$$h(Z) = \sum_{p \in X} \ell(\mathcal{O}_{Z,p}) \cdot p.$$

Example 1.4.3. For $n = 2$, we are looking for ideal sheaves $I \subseteq \mathcal{O}_X$ with quotient $\mathcal{O}_Z$ of length 2. At a point p, we know $I/\mathfrak{m}^2 \subseteq \mathfrak{m}/\mathfrak{m}^2$, and thus subschemes of length 2 supported on p form a $\mathbb{P}\mathfrak{m}/\mathfrak{m}^2 = \mathbb{P}^1$.

Sketch of smoothness. We need to compute the Zariski tangent space at a given point, so we have

$$T_{[Z]}X^{[n]} = \mathrm{Hom}_{0 \mapsto Z}(\operatorname{Spec}\mathbb{C}[\varepsilon], X^{[n]}).$$

By definition, these are flat proper families of length n subschemes $\mathcal{Z} \to \operatorname{Spec}\mathbb{C}[\varepsilon]$ such that $\mathcal{Z}\big|_{\varepsilon=0} = Z$, and by a computation (for example in FGA Explained) we have

$$T_{[Z]} = \mathrm{Hom}_X(I_Z, \mathcal{O}_Z).$$

To compute the dimension, we begin by considering the exact sequence

$$0 \to I_Z \subseteq \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

and applying the functor $\mathrm{Hom}_X(-, \mathcal{O}_Z)$, we have an exact sequence

$$0 \to \mathrm{Hom}_X(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Hom}_X(I_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z),$$

[1] In fact, in this case, no crepant resolution exists. A necessary condition for f to be a symplectic resolution is that it is crepant. In dimension 2, the two notions are the same.

[2] This may be the most studied morphism in algebraic geometry besides $\mathbb{P}^n \to \operatorname{Spec} k$.

and thus because $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Hom}_X(I_Z, \mathcal{O}_Z)$ is the zero morphism, and $\mathrm{Ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z) = H^1(\mathcal{O}_Z) = 0$, we can simply compute the Ext group. Here, we have

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) \coloneqq \sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(\mathcal{O}_Z, \mathcal{O}_Z).$$

We simply need to show that the Euler characteristic vanishes because

$$\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = \mathrm{Hom}(\mathcal{O}_Z, \omega_X \otimes \mathcal{O}_Z)^\vee$$

has dimension n, as does $\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)$. To do this, we use Grothendieck-Riemann-Roch, which says that

$$\chi(\mathcal{F}, \mathcal{G}) = \mathrm{ch}(\mathcal{F}^\vee) \cdot \mathrm{ch}(\mathcal{G}) \sqrt{\mathrm{td}(X)},$$

and here we see that $\chi(\mathcal{O}_X, \mathcal{G}) = \chi(\mathcal{F})$ where $\mathcal{F} = \widetilde{\mathcal{O}}_X$. Now because $\mathrm{supp}(\mathcal{F})$ has dimension 0, then $\mathrm{ch}(\mathcal{F}) = [0, \dots, \pm\ell(\mathcal{F})]$. $\square$

Exercise 1.4.4. Prove that $\chi(\mathcal{O}_Z, \mathcal{O}_Z) = 0$ using a locally free resolution in the first variable.

Now we will review some basic theory of Hilbert schemes for quasiprojective varieties. Here, if X is quasiprojective and $p(t) \in \mathbb{Q}[t]$ is some Hilbert polynomial, consider $[Z] \in \mathrm{Hilb}_X^{p(t)}$.

Proposition 1.4.5. *If $I \subseteq \mathcal{O}_X$ is the ideal sheaf of Z, then $T_{[Z]}\mathrm{Hilb} = \mathrm{Hom}_X(I, \mathcal{O}_Z) = \mathrm{Hom}_Z(I/I^2, \mathcal{O}_Z) = H^0(Z, N_{Z/X})$.*

Sketch of proof. We know that $T_{[Z]}\mathrm{Hilb} = \mathrm{Hom}(\mathrm{Spec}\, k[\varepsilon], \mathrm{Hilb}, 0 \mapsto [Z])$. This set of morphisms is the same as the set of $\mathcal{Z} \subseteq X \times \mathrm{Spec}\, k[\varepsilon]$ flat over $k[\varepsilon]$. And a module M is flat over $k[\varepsilon]$ if and only if $M \otimes (\varepsilon) \simeq \varepsilon \cdot M$. Now we want an ideal sheaf $\widetilde{I} \subseteq \mathcal{O}_X[\varepsilon]$ such that

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & I \arrow[r] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u] & \mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \widetilde{I} \arrow[r, hook] \arrow[u] & \mathcal{O}_X[\varepsilon] \arrow[r] \arrow[u] & \mathcal{O}_{\widetilde{\mathcal{Z}}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u, hook, "\cdot\varepsilon"] & \mathcal{O}_X \arrow[r] \arrow[u, "\cdot\varepsilon"] & \mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

But now we can see that $I = \widetilde{I}/\varepsilon I \subseteq \mathcal{O}_X \oplus \varepsilon\mathcal{O}_Z$, and thus giving $\widetilde{I}$ is the same as giving an element of $\mathrm{Hom}_X(I, \mathcal{O}_Z)$. $\square$

Exercise 1.4.6. Let X be a smooth quasiprojective curve. Show that $X^{[n]} = X^{(n)}$ is smooth of dimension n.[3]

[3] Newton actually proved way back in the day that the symmetric powers of $\mathbb{A}^1$ are smooth (and equal to $\mathbb{A}^n$).

Theorem 1.4.7. *Let X be a quasiprojective variety. Then there exists a regular proper morphism*

$$h_n \colon X^{[n]} \to X^{(n)} \qquad Z \mapsto \sum_{p \in X} \ell(\mathcal{O}_Z, p)p$$

which is surjective and birational. By a result of Fogarty, the fibers of h_n are connected, so if X is connected, so is $X^{[n]}$.

From now on, we will assume that X is projective. Therefore, for all $Z \subseteq X$ with $\ell(Z) = n$, there exists an open affine neighborhood $U \subseteq X$ containing Z. Therefore we have $[Z] \in U^{(n)} \subseteq X^{(n)}$. Now if $Z = \sum \alpha_i p_i$ and $U_i \ni p_i$ are open neighborhoods, then $Z \in \prod U_i^{(\alpha_i)}$.

Remark 1.4.8. If X is a smooth surface, then the local structure of $X^{(n)}$ at np is the same as the local structure of $(\mathbb{A}^n)^{(n)}$ at $n \cdot \{0\}$. In particular, when $n = 2$, we have

$$(X^{(2)}, 2p) \simeq (Q, 0) \times \Delta,$$

where Δ is smooth of dimension 2 and Q is the quadric cone.

Now for any partition $n = \sum \alpha_i$ of n into positive integers with length k, write $\underline{\alpha} = (\alpha_i)$. Then define

$$X_\alpha^{(n)} = \left\{ \sum \alpha_i z_i \mid z_i \neq z_j \right\}.$$

These $X_\alpha^{(n)}$ give a stratification of $X^{(n)}$ into locally closed subsets, where the open stratum is $X_{(1,1,\dots,1)}^{(n)}$ and the closed stratum is $X_{(n)}^{(n)}$. It is easy to see that $\dim X_{\underline{\alpha}}^{(n)} = 2\ell(\alpha)$. Another important stratum is $X_{(2,1,\dots,1)}^{(n)}$, where exactly two points come together. Now note that

$$h_n^{-1}\left(\sum \alpha_i z_i\right) = \prod h_{\alpha_i}^{-1}(\alpha_i z_i),$$

where the $h_{\alpha_i}^{-1}$ are the *punctual Hilbert schemes* $\mathrm{Hilb}^{\alpha_i}(\mathcal{O}_X, z_i) \simeq \mathrm{Hilb}^{\alpha_i}(k[x_1, x_2], 0)$. For $\alpha = 2$, the punctual Hilbert scheme is simply $\mathbb{P}\mathfrak{m}/\mathfrak{m}^2$.[4]

Theorem 1.4.9 (Briançon)**.** *The fiber $h_n^{-1}(nz)$ is irreducible of dimension at most $n-1$.*

In particular, this tells us that $X_{(1,\dots,1)}^{[n]} \to X_{(1,\dots,1)}^{(n)}$ has fibers of dimension 0 and is thus an isomorphism.

Proposition 1.4.10. *The exceptional locus of h_n is an irreducible divisor E.*

Proof. Because $X^{(n)}$ is normal and $\mathbb{Q}$-factorial,[5] then any birational $Y \to X^{(n)}$ from a smooth variety Y has divisorial exceptional divisor.

Now the exceptional locus $E_{(2,1,\dots,1)}$ to $X_{(2,1,\dots,1)}^{(n)}$ has fibers $\mathbb{P}^1$, while for a general $\underline{\alpha}$, we have

$$\dim E_{\underline{\alpha}} = \dim X_{\underline{\alpha}}^{(n)} + \sum \dim h_{\alpha_i}^{-1}(\alpha_i z_i) \leqslant n + \ell(\underline{\alpha}).$$

Because the strata are irreducible and so are the fibers, we obtain irreducibility for the exceptional divisor. ☐

[4] Apparently these are useful in representation theory.
[5] Every finite quotient of something smooth is $\mathbb{Q}$-factorial.

Proposition 1.4.11. *Let X be a projective variety. Then there exists a birational surjective morphism* $h\colon X^{[n]} \to X^{(n)}$.

Proof. We will show that for all $Z \subseteq T \times X$ proper and flat over T with $\ell(Z_t) = n$ for all t, there exists a natural morphism $T \to X^{(n)}$ given by

$$t \mapsto \sum_{p \in X} \ell(\mathcal{O}_{Z_t}, p) \cdot p.$$

Fix $t_0 \in T$. Because X is projective, there exists $U \subseteq X$ affine with $Z_{t_0} \in U = \operatorname{Spec} A$. Then because $p\colon Z \to T$ is proper, there exists $t_0 \in V \subseteq T$, where $V = \operatorname{Spec} B$ is affine, and for all $t \in V$, $Z_t \in U$. In conclusion, we have a family $Z_V \subseteq V \times U$. At the level of rings, we have a diagram

$$\begin{tikzcd} C & B \otimes A \arrow[l, two heads, "\varphi"'] \\ B. \arrow[u] & \end{tikzcd}$$

Now we need a map $(A^{\otimes n})^{S_n} \to B$. Because $Z \to V$ is flat, C is a rank n projective B-module. Clearly we have a map $A \to \operatorname{End}_B(C)$ given by $A \mapsto \varphi(1 \otimes a)$, and thus $A^{\otimes n}$ acts on $C^{\otimes n}$. Then we obtain an action of $(A^{\otimes n})^{S_n}$ on $\bigwedge^n C$, which is just a map

$$(A^{\otimes n})^{S_n} \to \operatorname{End}_B\left(\bigwedge\nolimits^n C\right) \simeq B,$$

which is the map we want. □

As an example, consider $B = k$. Then $C = \prod C_i$ is Artinian, hence a product of Artinian local rings C_i of length α_i. Then the map we defined at the end of the proof factors through $\prod \operatorname{Sym}^{\alpha_i}(C_i)$.

Theorem 1.4.12 (Beauville-Fujiki)**.** *Let X be a smooth surface. Then the Hilbert-Chow morphism is a crepant resolution and if X has a holomorphic symplectic form, so does $X^{[n]}$.*

Exercise 1.4.13. If X is a smooth surface, prove that

$$X^{[n]} = \operatorname{Bl}_\Delta X^{(2)} = (\operatorname{Bl}_\Delta X \times X)/S_2.$$

Proof. Consider $X_*^{(n)} = X_{(1,\dots,1)}^{(n)} \cup X_{(2,1,\dots,1)}^{(n)}$ and define $X_*^n, X_*^{[n]}$ similarly. In X^n consider $\Delta = \bigcup \Delta_{ij}$, where the i-th and j-th points coincide. Now consider the diagram

$$\begin{tikzcd} \operatorname{Bl}_\Delta X_*^n \arrow[r, "\eta"] \arrow[d, "\rho"] & X_*^n \arrow[d] \\ X_*^{[n]} \arrow[r, "h"] & X_*^{(n)}, \end{tikzcd}$$

which quite clearly commutes by the same argument showing that $X^{[n]}$ is the blowup of $X^{(n)}$ along the diagonal. Then the exceptional divisor $\bigcup E_{ij}$ is fixed by S_2, so it maps to $E_* \subset X_*^{[n]}$.

Now it suffices to prove that $X_*^{[n]} \to X_*^{(n)}$ is crepant because the complement has codimension 2. To see this, the quotients by S_n have simple ramification, and thus we have

$$\begin{aligned} K_{\mathrm{Bl}_\Delta X_*^n} &= \rho^*(K_{X_*^{[n]}}) + \sum E_{ij} \\ &= \rho^* h^* K_{X_*^{(n)}} + (a+1)\sum E_{ij} \\ &= \eta^* K_{X_*^n} + \sum E_{ij}, \end{aligned}$$

so $a = 0$ because $\pi \colon X_*^n \to X_*^{(n)}$ is étale away from codimension 2 and thus $K_{X_*^n} = \pi^* K_{X_*^{(n)}}$. Therefore the Hilbert-Chow morphism is a crepant resolution.

Now suppose that X has a holomorphic symplectic form $\omega_X \in H^0(\Omega_X^2)$. By codimension reasons, it is enough to produce a holomorphic symplectic form on $X_*^{[n]}$. Clearly we have a symplectic form $\omega := \sum_i p_i^*(\omega_X)$ on X_*^n, which is clearly S_n-invariant. Therefore, we obtain a symplectic form $\sigma_{X^{(n)}_{(1,\ldots,1)}}$ on $X^{(n)}_{(1,\ldots,1)}$ and a symplectic form $\eta^*\omega$ on $\mathrm{Bl}_\Delta X_*^n$, which is degenerate along $\bigcup E_{ij}$ and S_n-invariant. This induces a holomorphic 2-form $\sigma_{X_*^{[n]}}$ on $X_*^{[n]}$ (as in there exists such a σ such that $\eta^*\omega = \rho^*\sigma$). We know that $\sigma_{X_*^{[n]}}$ is generically nondegenerate.

We now show that $\sigma := \sigma_{X_*^{[n]}}$ is symplectic. We know σ^n is a section of $\omega_{X_*^{[n]}}$, so the degeneracy locus of σ is the zero locus of σ^n. However, we know $K_{X_*^{(n)}} = 0$ by the existence of ω_X, and because h is crepant, we see that $K_{X_*[n]} = 0$, and thus σ^n must be nonzero everywhere. $\square$

We will now discuss some invariants of $X^{[n]}$.

Proposition 1.4.14. *There is an isomorphicm of Hodge structures*

$$H^2(X^{[n]}, \mathbb{Q}) = h^* H^2(X^{(n)}) \oplus \mathbb{Q}E.^6$$

Now we know that $H^2(X^{(n)}) = H^2(X^n)^{S_n}$. By the Künneth formula, we have

$$H^2(X^n) = \bigoplus_{i=1}^n H^2(X) \otimes H^0(X) \oplus \bigoplus_{i,j} H^1(X) \otimes H^1(X),$$

and therefore

$$H^2(X^{(n)}) = H^2(X) \oplus \bigwedge^2 H^1(X).$$

Now

$$\begin{aligned} H^2(X^{[n]}) &= H^2(X_*^{[n]}) \\ &= H^2(\mathrm{Bl}_\Delta X_*^n)^{S_n}] \\ &= (\mathrm{Im}\,\eta^*)^{S_n} \oplus \left(\bigoplus \mathbb{Q}E_{ij}\right)^{S_n} \\ &= \eta^*(H^2(X_*^n))^{S_n} \oplus \mathbb{Q}E \\ &= H^2(X) \oplus \bigwedge^2 H^1(X) \oplus \mathbb{Q}E. \end{aligned}$$

[6] Note that $X^{(n)}$ is a finite quotient of something smooth and thus has a pure Hodge structure.

Over $\mathbb{Z}$, we can check in local coordinates that there exists a class $\delta \in H^2(X^{[n]}, \mathbb{Z})$ such that $2\delta = E$.[7]

Corollary 1.4.15. *If X is a K3 surface, then there is an isomorphism*

$$H^2(X^{[n]}) = H^2(X) \oplus \mathbb{Q}E$$

as Hodge structures, and in particular,

$$H^0(\Omega^2_{X^{[n]}}) = H^{2,0}(X^{[n]}) \simeq H^2(X) = \mathbb{C}.$$

We will now sketch the computation of the fundamental group of $X^{[n]}$. One fact is that

$$h_*\colon \pi_1(X^{[n]}) \to \pi_1(X^{(n)}) = \pi_1(X)/[\pi_1(X), \pi_1(X)] = H_1(X, \mathbb{Z})$$

is an isomorphism. In particular, if X is a K3 surface, then $\pi_1(X^{[n]}) = 0$, so $X^{[n]}$ is irreducible holomorphic symplectic.

If A is an abelian surface, we know A has a holomorphic symplectic form σ_A, and thus $A^{[n]}$ has a holomorphic symplectic form $\sigma_{A^{[n]}}$, so $\omega_{A^{[n]}} = \mathcal{O}_{A^{[n]}}$. However, we know that $H^1(A^{[n]}) = H^1(A) = \mathbb{Z}^4$, so it is not simply connected. But then we know that

$$H^2(A^{[n]}) = H^2(A) \oplus \bigwedge^2 H^1(A) = H^2(A) \oplus H^2(A),$$

so $A^{[n]}$ has larger $H^{2,0}$. By Beauville-Bogomolov, we know that $A^{[n+1]}$ has an étale cover by a product of complex tori, irreducible holomorphic symplectics, and strict Calabi-Yaus. There exists a natural morphism

$$\mathrm{alb}\colon A^{[n+1]} \xrightarrow{h} A^{(n+1)} \to A$$

and an action of A on $A^{[n+1]}$ given by translation. Of course, this is not equivariant because $\sum(z_i + a) \mapsto \sum z_i + (n+1)a$, and by generic smoothness all fibers are isomorphic and smooth. Now if we consider the diagram

$$\begin{tikzcd} A \times_A A^{[n+1]} \arrow[r] \arrow[d] & A^{[n+1]} \arrow[d, "\mathrm{alb}"] \\ A \arrow[r, "n+1"] & A, \end{tikzcd}$$

we see that $A \times_A A^{[n+1]} = K_n(A) \times A$. Later, we will show that $K_n(A)$ is irreducible holomorphic symplectic.

Example 1.4.16. If $n = 1$, then $K_1(A)$ is the Kummer K3 surface associated to A.

Proposition 1.4.17.

1. $\omega_{K^n(A)} = \mathcal{O}_{K^n(A)}$;
2. *The restriction of teh holomorphic symplectic form* $\sigma_{A^{[n+1]}}\big|_{K^n(A)}$ *is a symplectic form.*

[7] We had a lengthy discussion checking the computations above, and the moral is that algebraic geometers are bad at basic algebra. Also, to avoid sign problems, work in characteristic 2.

3. $H^2(K^n(A)) = H^2(A) \oplus \mathbb{Q}F$ *and* $\pi_1(K^n(A)) = 1$*, where* F *is one of the fibers of* $E \to A$*, where* $E \subset A^[n+1]$ *is the exceptional divisor of* h*.*

We will prove a result that

Proposition 1.4.18. *There exists a (non-effective) line bundle* $\mathcal{L}$ *on* $X^{[n]}$ *such that* $\mathcal{L}^{\otimes 2} = \mathcal{O}_{X^{[n]}}(E)$*.*

Proof. Consider $\mathrm{Bl}_\Delta X^n_*/A_n$. This has simple ramification over $E_* \subseteq X^{[n]}_*$, and thus $f_*\mathcal{O}_Z = \mathcal{O}_{X^{[n]}_*} \oplus \mathcal{L}$. This is the desired line bundle. ☐

Corollary 1.4.19. *If* X *is a K3 surface, then* $\mathrm{Pic}(X^{[n]}) = \mathrm{Pic}(X) + \mathbb{Z}\delta$*, where* $\delta = c_1(\mathcal{L})$*.*

1.5 Generalized Kummers

Recall the construction of the varieties $K^n(A)$ for an abelian surface A. Recall the diagram

$$\begin{tikzcd}
K^n(A) \arrow[r, hook] \arrow[dd] & A^{[n+1]} \arrow[dd, "a"] \arrow[dr] & A^{n+1} \arrow[d] \\
 & & A^{(n+1)} \arrow[dl, "\varepsilon"] \\
0_A \arrow[r, hook] & A. &
\end{tikzcd}$$

Also recall that $\pi_1(A^{[n+1]}) = \pi_1(A)$. Using the long exact sequence of homotopy groups and the fact that A is a $K(\mathbb{Z}^4, 1)$, we see that $\pi_1(K^n(A)) = 0$.

Proposition 1.5.1. $K^n(A)$ *is an irreducible holomorphic symplectic manifold. In particular,*

1. *If* $\sigma_{A^{[n+1]}}$ *is the holomorphic symplectic form on* $A^[n+1]$*, then its restriction to* $K^n(A)$ *is a holomorphic symplectic form.*

2. $H^2(K^n(A)) = H^2(A) \oplus \mathbb{Q}F$*, where* $F = E \cap K^n(A)$*.*

Proof. Consider the Leray filtration on $H^2(A^{[n+1]})$ induced by the map a. Here, we have

$$H^2(A) = H^2(a_*\mathbb{Q}) \subseteq H^2(A^{[n+1]}) \to H^0(A, R^2 a_*\mathbb{Q}) = H^2(K^n(A))^{\mathrm{inv}},$$

where invariants are taken with respect to the monodromy group of a, which is $A[n+1]$ because base change by $A \xrightarrow{n+1} A$ trivializes a. We will show that the last inclusion is an equality. Also, note that

$$H^2(A^{[n+1]}) = \bigwedge^2 H^1(A) \oplus H^2(A) \oplus \mathbb{Q}E$$

and that if $\alpha, \beta \in H^1(A)$, then

$$a^*(\alpha \wedge \beta) = a^*\alpha \wedge a^*\beta = \sum p_i^*\alpha \wedge \sum p_i^*\beta.$$

Next, write $K^n_*(A)$ analogously to $A^{[n]}_*$ and let $N = \ker(A^{n+1} \to A)$. Then we have a diagram

$$\begin{tikzcd} & \mathrm{Bl}\, N_* \arrow[dl] \arrow[dr] & \\ K^n_*(A) \arrow[dr] & & N_* \arrow[dl] \\ & K^{(n)}_*(A). & \end{tikzcd}$$

Note that N has an action of S_{n+1} and an action of $A[n+1]$ given by adding ε to all elements that preserves N and Δ. Then we know that

$$H^2(N_*) = H^2(N) = H^2(A^n)$$

has an action of $A[n+1]$ has an action by translation, which is trivial in cohomology. Finally, we conclude that

$$H^2(K^n(A)) = H^2(K^n_*(A)) = H^2(\mathrm{Bl}\, N_*)^{S_{n+1}}$$

and obtain the desired result. □

Now we have two examples of irreducible holomorphic symplectic manifolds. The first is $K3^{[n]}$ with $b_2 = b_2(K3) + 1 = 23$ and the second is $K^n(A)$ with $b_2 = b_2(A) + 1 = 7$.

Proposition 1.5.2. *Let* $f\colon \mathcal{X} \to B$ *be a smooth proper morphism of complex manifolds such that for some* $0 \in B$, $\mathcal{X}_0(B)$ *is a Kähler irreducible holomorphic symplectic manifold. Then there exists an analytic neighborhood* $0 \in V \subseteq B$ *such that for all* $t \in U$, $\mathcal{X}_t$ *is Kähler and holomorphic symplectic.*

Proof. By a result of Kodaira, being Kähler is an open condition, so there exists an open $U \subseteq B$ such that for all $t \in U$, $\mathcal{X}_t$ is Kähler. Therefore, for all $t \in U$, the map $t \mapsto h^p(\mathcal{X}_t, \Omega^q_{\mathcal{X}_t})$ is constant by Ehresman's theorem that this family is topologically trivial and upper semicontinuity.

This implies that up to further restricting U, $f_*\Omega^2_{\mathcal{X}/B}|_U$ is free. This implies that $\sigma_0 \in H^0(\Omega^2_{\mathcal{X}_0})$ extends locally to a section $\widetilde{\sigma} \in H^0(\mathcal{X}_U, \Omega^2_{\mathcal{X}_U/U})$. To check that this is symplectic, we know that $\widetilde{\sigma}^n \in H^0(K_{\mathcal{X}/U})$ has closed zero locus which does not intersect the zero fiber, and so we obtain an open set where this form is nondegenerate. □

Proposition 1.5.3. *Let* $f\colon \mathcal{X} \to B$ *be a smooth proper family of Kähler manifolds. Then if* $\mathcal{X}_0$ *is irreducible holomorphic symplectic, so is* $\mathcal{X}_t$ *for all* $t \in B$.

Sketch of proof. First, note that the relative canonical bundle $K_{\mathcal{X}/B} \cong f^*\mathcal{L}$, where $\mathcal{L}$ is a line bundle on B. By the same proof as before, there exists $Z \subseteq B$ such that for all $t \in B \setminus Z$, $\mathcal{X}_t$ is irreducible holomorphic symplectic.

Now suppose $t_0 \in Z$. Then $K_{\mathcal{X}_{t_0}}$ is trivial and $\mathcal{X}_{t_0}$ is simply connected, so $\mathcal{X}_{t_0}$ is a product of irreducible holomorphic symplectic varieties and strict Calabi-Yau manifolds.

Now we will state without proof the fact that if X is a complex manifold with $K_X = \mathcal{O}_X$, then Def(X) is smooth (as a germ of complex manifold). This is a nontrivial result of Kodaira and Spencer. Note that if $X_{t_0} = \prod X_i \times \prod Y_i$, then

$$\mathrm{Def}(X_{t_0}) = \prod \mathrm{Def}(X_i) \times \prod \mathrm{Def}(Y_i)$$

because all X_i, Y_i satisfy $h^{1,0} = 0$. Thus the splitting situation is impossible. □

It is known that if S is a K3 surface, then $\mathrm{Def}(S)$ has dimension 20. Also, note that projective K3 surfaces are a 19-dimensional locus. Here, note that $\mathrm{Def}(X) = h^1(T_X) = h^1(\Omega^1_X) = h^{1,1}$.

In the next case, if $X = S^{[n]}$, then $\mathrm{Def}(X)$ has dimension 21, and there is a 20-dimensional locus of genuine Hilbert schemes of K3 surfaces. There are also higher-codimension loci parameterizing the spaces $M_v(S, h)$. Note that in both of these situations, the very general object is Kähler but not projective.

Now we will discuss some examples of Lagrangian fibrations.

Example 1.5.4. Let $f\colon S \to \mathbb{P}^1$ be an elliptic K3 surface. Then we have a morphism

$$S^{[n]} \xrightarrow{f} S^{(n)} \xrightarrow{f^{(n)}} (\mathbb{P}^1)^{(n)} = \mathbb{P}^n.$$

This is clearly a Lagrangian fibration.

Example 1.5.5. Let $A = E \times F$ be the product of two elliptic curves and let $\varphi\colon A \to F$ be the second projection. Then we have a diagram

$$\begin{tikzcd}
K^2(A) \arrow[r, hook] \arrow[d] & A^{[3]} \arrow[d] & & \\
K^{(2)}(A) \arrow[r, hook] & A^{(3)} \arrow[r, "\varphi^{(3)}"] \arrow[d] & F^{(3)} \arrow[d, "\varepsilon"] & \varepsilon^{-1}(0) \arrow[l, hook'] \arrow[d] \\
& A & F & 0_F. \arrow[l, hook']
\end{tikzcd}$$

Here, we see that $\varepsilon^{-1}(0) = \check{\mathbb{P}}^2$, and so in general there is a Lagrangian fibration $K^n(A) \to \mathbb{P}^n$.

1.6 Some operations

Now we will consider some birational transformations.

Example 1.6.1 (Atiyah flop)**.** Let $f\colon \mathcal{S} \to \Delta$ be a family of quartic surfaces in $\mathbb{P}^3$. Suppose that $\mathcal{S}_t$ is smooth and $\mathcal{S}_0$ has one simple node $p \in \mathcal{S}_0$. This simple node is given locally by $x^2 + y^2 + z^2 = t$.

Note that $\mathrm{Bl}_p \mathcal{S}_0 =: \widetilde{\mathcal{S}}_0$ is a smooth K3 surface. We would like to modify the family such that we get smooth fibers for all $t \in \Delta$. Now if we take a base change of Δ by $t \mapsto t^2$, locally at $p \in \overline{\mathcal{S}}$ we have the equation $x^2 + y^2 + z^2 = t^2$ is a singular point of $\overline{\mathcal{S}}$. But then $\mathcal{X} := \mathrm{Bl}_p \overline{\mathcal{S}}$ is smooth, and $\mathcal{X}_0 = \widetilde{\mathcal{S}}_0 \cup Q$, where $Q = \mathbb{P}^1 \times \mathbb{P}^1$.

Unfortunately, the discrepancy of $\nu\colon \mathcal{X} \to \overline{\mathcal{S}}$ is 1, so $K_{\mathcal{X}} = \nu^* K_{\overline{\mathcal{S}}} + Q$, and so by adjunction we see that

$$\omega_Q = (K_{\mathcal{X}} + Q)|_Q = \mathcal{O}(2Q)|_Q,$$

and thus $\mathcal{O}_X(Q) = \mathcal{O}(-1, -1)$. This tells us that we can contract Q along both of the factors and produce $\mathcal{S}^+, \mathcal{S}^-$ with maps to $\overline{\mathcal{S}}$. Then there is a birational map $\varphi\colon \mathcal{S}^+ \dashrightarrow \mathcal{S}^-$ which is an isomorphism away from the central fiber.

We conclude that $\mathcal{S}^{\pm}_0 = \widetilde{\mathcal{S}}_0$ and that φ is an isomorphism outside of the copies of $\mathbb{P}^1$ that we contracted Q onto but does not extend over those copies of $\mathbb{P}^1$. Also, note that $\mathcal{X} = \Gamma_\varphi$ and that $\mathcal{X}_t = \Gamma_{\varphi_t}$ for all $t \neq 0$, and $\mathcal{X}_0 = \widetilde{\mathcal{S}}_0 \cup \mathbb{P}^1 \times \mathbb{P}^1$.

The next observation is that $H^2(\mathcal{S}^{\pm}_{t_0}) \simeq H^2(\widetilde{\mathcal{S}}_0)$, but passing between the two identifications is actually reflection across the (-2)-curve produced from the Atiyah flop.

Here, we have used the following result of Nakano and Fujiki: Let $\widetilde{M}$ be a complex manifold and $E \subseteq \widetilde{M}$ be a smooth divisor that is a $\mathbb{P}^n$-bundle over some Z.. Then there exists a complex manifold $M \supset Z$ and $\pi\colon \widetilde{M} \to M$ such that $\widetilde{M} = \mathrm{Bl}_Z M$ if and only if $\mathcal{O}_{\widetilde{M}}(E)|_E = \mathcal{O}_X(-1)$.

Another fact that we used to show that $\mathcal{S}_0^+$ and $\mathcal{S}_0^-$ are isomorphic is that two birational K3 surfaces are isomorphic.

Now let $X \supseteq \mathbb{P}^n$ be a holomorphic symplectic manifold of dimension $2n$. For example, some K3 surfaces contain (-2)-classes, which are isomorphic to $\mathbb{P}^1$.

Lemma 1.6.2. *Any such $\mathbb{P}^n \subseteq X$ is a Lagrangian submanifold of X. Moreover, if $Z \subseteq X$ is any Lagrangian, $N_{Z/X} \cong \Omega^1_Z$.*

Proof. The first part is clear because $H^0(\Omega^2_{\mathbb{P}^n}) = 0$. Next, consider the exact sequencec

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega^1_{X|Z} \arrow[r] & \Omega^1_Z \arrow[r] & 0 \\ 0 \arrow[r] & T_Z \arrow[r] \arrow[u, hook] & T_{X|Z} \arrow[r] \arrow[u, "\sim"] & N_{X/Z} \arrow[r] \arrow[u, two heads] & 0. \end{tikzcd}$$

Note that the rightmost vertical morphism is generically injective with torsion kernel, but because $N_{X|\mathbb{P}^n}$ is torsion free, we have an isomorphism. $\square$

Now consider $\mathrm{Bl}_{\mathbb{P}^n} X$ and let E be the exceptional divisor. Denote $\mathbb{P}^n = \mathbb{P}V$ for some vector space V.

Lemma 1.6.3. *We have an isomorphism $E \simeq I \subseteq \mathbb{P}V \times \mathbb{P}V^\vee$, where I is the incidence subscheme. Moreover, we have $\mathcal{O}_{\widetilde{X}}(E)|_E \cong \mathcal{O}_E(-1,-1)$.*

Proof. We know that $E = \mathbb{P}N_{\mathbb{P}^n/X} \simeq \mathbb{P}\Omega^1_{\mathbb{P}^n}$. Now if we consider the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to V^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^n} \to 0,$$

we obtain an embedding

$$\mathbb{P}\Omega^1_{\mathbb{P}^n} \subseteq \mathbb{P}V^\vee \times \mathbb{P}V$$

as the locus $\{(s,x) \mid s(x) = 0\}$. Next, we use adjunction in $\widetilde{X}$ and in $\mathbb{P}V \times \mathbb{P}V^\vee$ to see that

$$\mathcal{O}_X(-n,-n) = \omega_E = \omega_{\widetilde{X}}(E)|_E = \omega_{\widetilde{X}(nE)}|_E.$$

$\square$

Now by the Nakano-Fujiki criterion, there exists $\widetilde{X}' \supseteq \mathbb{P}V$ and $q'\colon \widetilde{X} \to X'$ such that we have the following diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr, "q'"] & \\ X \arrow[rr, dashed, "\varphi"] & & X' \end{tikzcd}$$

such that q' takes E to $\mathbb{P}V^\vee$.

Definition 1.6.4. Such an X' is called the *Mukai flop* of X at $\mathbb{P}^n$.

Remark 1.6.5. We can perform the Mukai flop whenever we have $Z \subseteq X$ such that there exists some $\mathbb{P}^r$-bundle structure $Z \to B$ and Z has codimension r in X. We also require that $N_{Z/X} \simeq \Omega^1_{Z/B}$.

Remark 1.6.6. We have a diagram

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[dl] \arrow[dr] & \\
\mathbb{P}^n \arrow[r, hook] \arrow[dr] & X \arrow[rr, dashed] \arrow[dr, "\pi"] & & X' \arrow[dl] \\
 & p \arrow[r, hook] & X_0. &
\end{tikzcd}$$

Remark 1.6.7. If X' and X are isomorphic in codimension 2, they have isomorphic H^2 and X' is holomorphic symplectic.

The local structure of (X_0, p) is isomorphic to that of the cone $C^\bullet(I)$. In particular, X_0 is not $\mathbb{Q}$-factorial because the exceptional locus of π is $\mathbb{P}^n$, which is not a divisor. In addition, π is a crepant (symplectic resolution).

Proposition 1.6.8. *A birational map* $f\colon X \dashrightarrow X'$ *of compact complex manifolds (or projective varieties) with trivial canonical bundles is an isomorphism in codimension* 2*. In particular,* $\pi_1(X) = \pi_1(X')$ *and* $H^2(X, \mathbb{Z}) = H^2(X', \mathbb{Z})$*.*

Proof. Let Γ be the graph of f and consider the diagram

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "p"'] \arrow[dr, "p'"] & \\
X \arrow[rr, dashed, "f"] & & X.
\end{tikzcd}$$

Then if E, F are the exceptional divisors of p, p', we have $K_\Gamma = E = F$ up to linear equivalence. However, we know that $H^0(mF) = H^0(mE) = H^0(mK_\Gamma) = 1$, but these $h^0(\omega_X^{\otimes m})$ are birational invariants, so E, F do not move in their equivalence class. In particular, we have an isomorphism $X \setminus p(E) \simeq X' \setminus p'(F)$. ☐

Corollary 1.6.9. *Suppose that* $f\colon X \twoheadrightarrow S'$ *is a birational map of K3 surfaces. Then* f *is an isomorphism.*

Proof. Consider the graph Γ and diagram

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed, "f"] & & S'.
\end{tikzcd}$$

We know that f is an isomorphism away from finitely many points. We know that f is not defined at x if and only if $p^{-1}(x)$ is a curve. But then there exists a curve $C' \subseteq S'$ contracted by f^{-1}, which is impossible. ☐

Example 1.6.10 (Beauville)**.** This example comes from the paper *Some remarks on Kähler manifolds with* $c_1 = 0$ by Beauville.[8] Let $S \subseteq \mathbb{P}^3$ be a quartic K3 surface. Choose a length 2 point $z \in S^{[2]}$, which has linear span a line. But then $\ell \cap S = z + w$, and so we define a rational map $\varphi\colon S^{[2]} \dashrightarrow S^{[2]}$ given by $z + w$.

[8]This paper is written in English, but Giulia suggests that we read some math papers in French.

Proposition 1.6.11.

1. *φ is regular at $[Z] \in S^{[2]}$ if and only if $\ell = \langle Z \rangle \not\subseteq S$.*

2. *If $S \supseteq \ell_1, \dots, \ell_k$ where the ℓ_i are disjoint lines, then φ is the Mukai flop at $\ell_1^{[2]}, \dots, \ell_k^{[2]}$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} & \Gamma \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ S^{[2]} \arrow[rr, dashed] \arrow[dr, "p"'] & & S^{[2]} \arrow[dl] \\ & G = Gr(2,4), & \end{tikzcd}$$

and clearly p is finite over $[\ell] \in G$ if and only if $\ell \not\subseteq S$. In particular, if $\ell \subseteq S$, we have $p^{-1}([\ell]) = \ell^{[2]}$. Now consider the graph Γ and note that $\Gamma \subseteq S^{[2]} \times_G S^{[2]} \subseteq S^{[2]} \times S^{[2]}$. Because $S^{[2]}$ is smooth, we know φ is regular at $[Z]$ if and only if $q_1^{-1}(Z)$ is finite.

But now $q_1^{-1}(Z) \subseteq S^{[2]} \times S^{[2]}$ is contained in $[Z] \times p^{-1}(\ell)$. Thus, if $p^{-1}(\ell)$ is finite, so is $q_1^{-1}(Z)$. For dimension reasons, if $\ell \subseteq S$ is a line, then $\ell^{[2]} \times \ell^{[2]}$ is an irreducible component of $S^{[2]} \times_G S^{[2]}$. But then $q_1^{-1}(\ell^{[2]}) = \Gamma \cap \ell^{[2]} \times \ell^{[2]}$, and then $S^{[2]} \times_G S^{[2]} \subseteq S^{[2]} \times S^{[2]}$ is a local complete intersection. But then irreducible components intersect in the correct dimension, so we are done.

It remains to show that f is the Mukai flop. We may assume that there is a unique line $\ell \subseteq S$. The key technical lemma is that φ extends to $\mathrm{Bl}_{\ell^{[2]}} S^{[2]}$, which means we have a map

$$\begin{tikzcd} \mathrm{Bl}_{\ell^{[2]}} S^{[2]} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi"] & \mathrm{Bl}_{\ell^{[2]}} S^{[2]} \arrow[d, "\pi"] \\ S^{[2]} \arrow[r, dashed, "\varphi"] & S^{[2]}. \end{tikzcd}$$

But then $\widetilde{\varphi}$ takes E to itself, which means that it must swap the two rulings on E. But this means that the two copies of π contract E along the two rulings, as desired.

To prove the lemma, $S^{[2]} \to G$ factors as $S^{[2]} \to Z \to G$, where Z is normal and $Z \to G$ is finite. But then φ descends to an honest morphism $\overline{\varphi}\colon Z \to Z$, and thus if $\ell^{[2]}$ is contracted to z_0, $\overline{\varphi}$ lifts to $\mathrm{Bl}_{z_0} Z = \mathrm{Bl}_{\ell^{[2]}} S^{[2]}$. □

Proposition 1.6.12 (Huybrechts)**.** *This proposition comes from the paper Birational symplectic manifolds and their deformations. Let $\mathbb{P}^n \subseteq X^{2n}$, where X is Kähler and symplectic and $f\colon X \dashrightarrow X'$ be the Mukai flop. Then there exist two birational smooth proper families*

$$\begin{tikzcd} \mathcal{X} \arrow[rr, dashed, "\phi"] \arrow[dr] & & \mathcal{X}' \arrow[dl] \\ & \Delta & \end{tikzcd}$$

such that ϕ_t is an isomorphism for all $t \neq 0$, $\mathcal{X}_0 = X$, and $\mathcal{X}'_0 = X'$.

Corollary 1.6.13. *There exists an isomorphism of Hodge structure $H^*(X) \simeq H^*(X')$.*

Proof. Let $\Gamma \subseteq \mathcal{X} \times_\Delta \mathcal{X}'$ be the fiber product. Then we know $\Gamma_t = \Gamma_{\varphi_t} \subseteq \mathcal{X}_t \times \mathcal{X}'_t$, and this implies that

$$\gamma_t^* \colon H^*(\mathcal{X}'_t) \to H^*(X_t) \qquad \alpha \mapsto p_1^*[\Gamma] \smile p_2^*(\alpha)$$

is an isomorphism. But then we know $H^*(\mathcal{X}'_t) \simeq H^*(\mathcal{X}'_0)$ and similarly for $\mathcal{X}$. We also have a correspondence Γ_0^*, and this is an isomorphism. □

Example 1.6.14. Let $S \to \mathbb{P}^2$ be a degree 2 K3 surface. Then we obtain some $\mathbb{P}^2 \subseteq S^{[2]}$. Then the Mukai flop of $S^{[2]}$ is a hyperkähler manifold M with a Lagrangian fibration over $\check{\mathbb{P}}^2$. Here, if $\ell \subset \mathbb{P}^2$ is a line, we consider $C \in |f^*\mathcal{O}_{\mathbb{P}^2}(1)|$, and the fiber over $[C] \in |C| = \check{\mathbb{P}}$ is simply $\operatorname{Pic}^2(C)$. In addition, the Mukai flop takes $z \in S^{[2]}$ to the line bundle $\mathcal{O}_C(Z)$.