Minimal Model Program Learning Seminar Spring 2021

Notes by Patrick Lei

Lectures by Joaquín Moraga

Zoom University

Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Seminar Website: `https://web.math.princeton.edu/~jmoraga/Learning-Seminar-MMP`

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Overview

The goal of the minimal model program is to classify smooth projective complex varieties $X \subseteq \mathbb{P}^n$. Let T_X be the tangent byndle and Ω_X be the cotangent bundle. Then $\omega_X = \Omega_X^n$ is called the *canonical bundle*, and can be written as $\mathscr{O}_X(K_X)$ for some Cartier divisor K_X.

Question 1.0.1. *Can we understand the geometry of X using numerical properties of K_X?*

For a curve $C \subseteq X$ and a line bundle $\mathscr{L}$ on X, then $\mathscr{L}.C = \deg_C(i^*\mathscr{L})$. Then K_X is *ample (resp. antiample)* if $K_X.C > 0$ (resp < 0) for all curves $C \subseteq X$. Similarly, K_X is *numerically trivial* if $K_X.C = 0$ for all curves $C \subseteq X$.

Definition 1.0.2. We say that X is *Fano* if K_X is antiample, *Calabi-Yau* if K_X is numerically trivial, and *canonically polarized* if K_X is ample.

Example 1.0.3. If C is a Fano curve, then $C \simeq \mathbb{P}^1$. If C is a CY curve, then C is an elliptic curve. If C is a canonically polarized curve, then $g(C) \geq 2$.

Example 1.0.4. Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d. Then by the adjunction formula, we have $K_X \simeq (K_{\mathbb{P}^n} + X)\big|_X \simeq (d-n-1)H\big|$. Therefore, X is Fano if $d \leq N$, CY if $d = n+1$, and canonically polarized if $d \geq n+2$.

Remark 1.0.5. If E is an elliptic curve, then $E \times \mathbb{P}^1$ has $K_X.C = 0$ for some curves and $K_X.C < 0$ for others.

Now we consider various properties of different varieties:

Table 1.1: Properties of varieties of different classes

	Fano	CY	Canonically polarized
π_1	trivial	?	generally infinite
Automorphisms	linear algebraic groups	?	finite groups
Birational automorphisms	monstrous	?	finite groups
Geometry	simple geometry	?	complicated, rich
Arithmetic	a lot of $\mathbb{Q}$-points	?	$\mathbb{Q}$-points in a proper closed

We have said nothing about Calabi-Yaus, but of course by the Beauville-Bogomolov decomposition, we can reduce to pure Calabi-Yaus ($h^{i,0} = 0$ for $0 < i < \dim X$), hyperkählers, and abelian varieties up to taking a finite cover.

Now let x be a closed point on X. Then there is a variety $\mathrm{Bl}_x X \to X$ that is an isomorphism away from x where the fiber above x is an exceptional divisor E parameterizing tangent directions at x.

Example 1.0.6. Consider points $p_1, \ldots, p_n, \ldots \in \mathbb{P}^2$. Then if we blow up these points in a sequence, we obtain a sequence of varieties $X_1, \ldots, X_2, \ldots, X_n, \ldots$ Over $\mathbb{P}^2 \setminus \{p_1, \ldots, p_{i-1}\}$, the morphism $X_i \to \mathbb{P}^2$ is an isomorphism. For $i \neq j$, clearly X_i is not isomorphic to X_j (they have different Picard ranks), but they are *birational*. We say that $X_1 \sim_{\mathrm{bir}} X_2$ if they have isomorphic dense open subsets.

Now we can state the goal of the Minimal Model Program. If X is projective and has "mild singularities" the goal is to prove that there exists a birational map $\pi \colon X \dashrightarrow X'$ and a fibration ($\varphi_* \mathscr{O}_{X'} = \mathscr{O}_Z$ and positive dimensional general fiber) $X' \xrightarrow{\varphi} Z$ such that one of the following holds:

1. F is Fano;
2. F is Calabi-Yau;
3. $Z = \operatorname{Spec} \mathbb{C}$ and X' is canonically polarized.

The way we will construct this birational morphism is by studying the geometry of curves on X which intersect K_X negatively. If $K_X.C < 0$ under some hypotheses (extremity on $NE(X)$), we can find $\varphi_C \colon X \to X_1$ contracting precisely the curves which are numerically equivalent to a positive multiple of C.

1. If the curves numerically equivalent to a positive multiple of C cover X, then φ_C has positive-dimensional fibers, is a contraction, and the general fiber F is Fano. This is called a *Mori fiber space*.
2. If the curves numerically equivalent to a positive multiple of C cover a divisor on X, then we say that $\varphi_C \colon X \to X_1$ is a divisorial contraction and thus $\rho(X_1) = \rho(X) - 1$. X_1 still has nice singularities, so we can iterate this process.
3. The last case is called a small contraction. The curves which are numerically equivalent to a positive multiple of C cover a set of codimension at least 2. In this case, X_1 may have very bad singularities (by this, we mean that K_{X_1} is not $\mathbb{Q}$-Cartier). We construct a new birational morphism $\varphi_C^+ \colon X^+ \to X_1$ which contracts K_{X^+}-positive curves.

Another type of surgery is a flip, which changes a locus of codimension at least 2. For example, consider $D = p_1^* \mathscr{O}(1) \otimes p_2^* \mathscr{O}(r)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then write

$$X = \operatorname{Spec} \left(\bigoplus_{m \geq 0} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(mD)) \right).$$

Then X is (locally) a cone over $\mathbb{P}^1 \times \mathbb{P}^1$, so K_X is not $\mathbb{Q}$-Cartier. Therefore, we can blow up the vertex, and the exceptional divisor is $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Now the $\mathbb{P}^1 \times \mathbb{P}^1$ can be collapsed onto each of the two factors, so we obtain a birational map $\pi \colon X_1 \dashrightarrow X_1^+$. Here, if C, C^+ are the resulting curves, we have $K_{X_1}.C < 0$ and $K_{X_1^+}.C^+ > 0$.

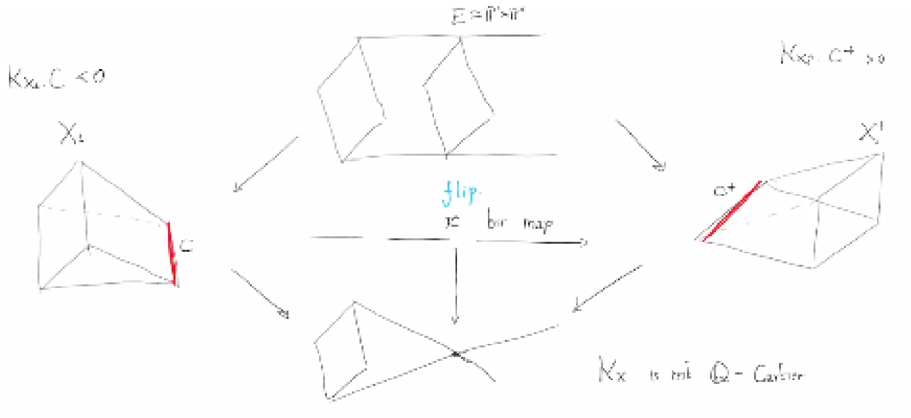

Figure 1.1: A Flip

This gives us the question:

Question 1.0.7. *Do flips always exist?*

Now either the algorithm given by iterating this process always terminates or it continues infinitely. We are fine once we reach the Mori fiber space case, and there are only $\rho(X) - 1$ divisorial contractions, so the only possible problem is that there is an infinite sequence of flips.

Conjecture 1.0.8 (Termination of flips)**.** *This algorithm always terminates after finitely many flips with either a Mori fiber space $X_n \to Z$ or a variety X_n such that $K_{X_n}.C \geq 0$ for every curve C (in other words, K_{X_n} is nef).*

Conjecture 1.0.9 (Abundance)**.** *X has mild singularities and K_X is nef. Then $|mK_X|$ is basepoint-free for some $m \gg 0$.*

If this is true, and $X \xrightarrow{\varphi} X_1$ contracts all K_X-trivial curves, then either

1. The general fiber has positive dimension. In this case, $K_F \equiv 0$.
2. $\dim X = \dim X_1$. Then $X \to X_1$ is birational and X_1 is canonically polarized.

Therefore, the goal of the MMP is achieved if we can solve the conjectures of existence of flips, termination of flips, and abundance. Existence of flips was proved by Birkar, Cascini, Hacon, and McKernan in 2006 and termination is known in dimension at most 3 and in some cases in dimension 4. Finally, abundance is known in dimension at most 3.

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MMP in Dimension 3

2.1 Rational Curves

Proposition 2.1.1 (Bend and Break). *Let X be proper and C be a smooth proper curve. Let $p \in C$ and $g_0 \colon C \to X$ be nonconstant. Next, let $0 \in D$ be a pointed curve and $G \colon C \times D \to X$ such that*

1. $G\big|_{C \times \{0\}} = g_0$.
2. $G(\{p\} \times D) = g_0(p)$.
3. $G\big|_{C \times \{t\}}$ *is different from g_0 for general t.*

All of these imply that this is a nontrivial deformation of g_0 fixing p. Then there exists $g_1 \colon C \to X$ and $Z = \sum a_i Z_i$ a union of rational curves such that $(g_0)_* C$ is algebraically equivalent to $(g_1)_*(C) + Z$ and $g_0(p) \in \bigcup_i Z_i$. In particular, there exists a rational curve through $g_0(p)$.

Proof. First, compactify D and let $\overline{G} \colon C \times \overline{D} \dashrightarrow X$ be the rational map. This map is undefined at $\{p\} \times \overline{D}$ by the rigidity lemma, so let S be the normalization of the graph of $\overline{G}$. So we have a map $\pi \colon S \to C \times \overline{D}$ and write $G_S \colon S \to$.

Then we define $h \colon S \to C \times \overline{D} \to \overline{D}$. Then there exist $d \in C \times \overline{D}$ such that π is not an isomorphism over d. Then we know that $h^{-1}(d) = C' + E$ where C' is a birational transform of C and E is π-exceptional. Then we set $g_1 \colon C \to X$ to be the restriction of G_S to C' and $Z = G_S(E)$.

By a lemma of Abhyankar, we know that E is a union of rational curves, and then by the Lüroth theorem, we know that Z is a union of rational curves and

$$(g_0)_* C \sim_{\text{alg}} (g_1)_* C + Z.$$

$\square$

Lemma 2.1.2 (Abhyankar). *Let X have mild singularities and $Y \xrightarrow{\pi} X$ be a proper birational morphism. For any $x \in X$, either $\pi^{-1}(x)$ is a point or is covered by rational curves.*

Here is an intuitive image of the bend-and-break process:

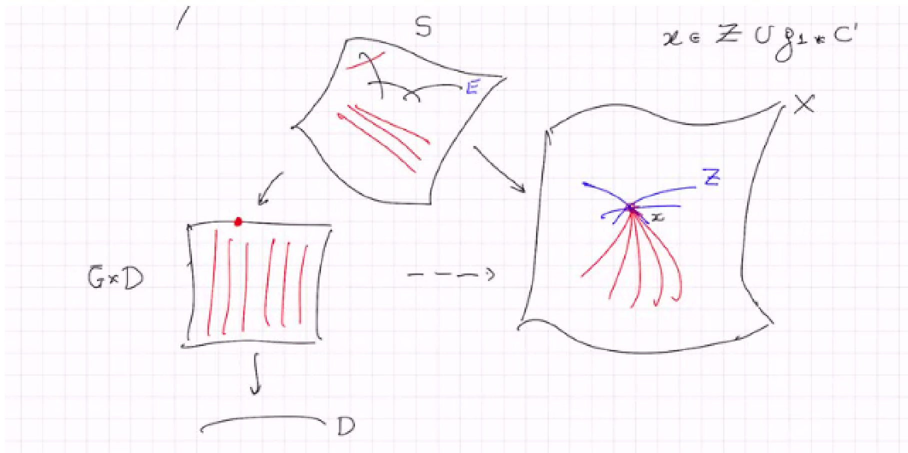

Figure 2.1: Bend and Break

Proposition 2.1.3 (Bend and Break II)**.** *Let X be a projective variety and $g_0 \colon \mathbb{P}^1 \to X$ be a nonconstant morphism. Let D be a smooth pointed curve and $G \colon \mathbb{P}^1 \times D \to X$ such that*

1. $G|_{\mathbb{P}^1 \times \{0_D\}} = g_0$;
2. $G(\{0\} \times D) = g_0(0), G(\{\infty\} \times D) = g_0(\infty)$;
3. $G(\mathbb{P}^1 \times D)$ *is a surface.*

Then $(g_0)_* \mathbb{P}^1$ is algebraically equivalent either to a reducible curve or a multiple curve.

Proof. Let S be a $\mathbb{P}^1$-bundle containing $\mathbb{P}^1 \times D$ and consider the rational map $\widetilde{G} \colon S \dashrightarrow X$. Then we can resolve the basepoints to obtain $\widetilde{G} \colon \widetilde{S} \to S$ and induct on $\rho(\widetilde{S}/S) =: \rho$.

Case 1: $\rho = 0$**:** Consider the sections C_0, C_∞ at 0 and ∞. Then let H is ample on X and then we see that $(\widetilde{G}^* H)^2 > 0$ and $(C_0 \cdot \widetilde{G}^* H) = (C_\infty \cdot \widetilde{G}^* H) = 0$ by the projection formula. By the Hodge index theorem, we see that $C_0^2 < 0, C_\infty^2 < 0$ (because $\widetilde{G}^* H, C_0, C_\infty$ are linearly independent). But then we know that $\rho(S) = 2$, which is a contradiction.

Inductive step: Consider the diagram

$$\begin{tikzcd}
\widetilde{S} \arrow[r, "\widetilde{G}"] \arrow[d, "r'"] \arrow[dd, bend right, "\widetilde{r}"'] & X \\
S' \arrow[d, "r"] \arrow[ur, dashed, "\overline{G}'"] & \\
S \arrow[d, "q"] \arrow[uur, dashed, "\overline{G}"'] & \\
D. &
\end{tikzcd}$$

Then r is the first blowup in $\widetilde{S} \to S \ni P$ and $y \in D$ will be a point such that $P \in q^{-1}(y)$. Let F_1 be the exceptional divisor of r and F_2 be the strict transform of $q^{-1}(y)$ in S'. Then

F_1, F_2 intersect at a point Q. But then $\overline{G}'$ is a morphism around F_2. But then $(g_0)_*\mathbb{P}^1 \sim \widetilde{G}_*((q \circ r)^*(y))$, which is reduced and irreducible. If $\overline{G}$ is not defined at $Q \neq P$, then

$$\widetilde{G}_*((q \circ r)^*(y)) = \widetilde{G}_*\mathrm{red}(\widetilde{r}^{-1}(p)) + \widetilde{G}_*\mathrm{red}(\widetilde{r}^{-1}(Q)) + (\text{effective}),$$

which is a contradiction and thus $\overline{G}$ is defined at $Q \neq P$. Then if $\overline{G}'$ is not defined at Q_0, then after blowing up Q_0, we see that $(q \circ r)^*(y)$ must contain a component of multiplicity at least 2. Now contracting F_2, we have the desired result by induction on ρ. □

This tells us that to produce rational curves, we simply need to deform them with enough fixed points and use bend and break. But now we need to actually find rational curves.

Theorem 2.1.4. *Let X be smooth and projective and $-K_X$ be ample. For every $x \in X$, there exists a rational curve C through x such that*

$$0 < -K_X.C \leq \dim X + 1.$$

Proof. Choose some curve $C \subseteq X$ through x. Then the space of deformations of C on X fixing x has dimension at least

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) - \dim X = -f_*C.K_X - g(C)\dim X.$$

We have several cases:

1. If $g(C) = 0$, then we are done.
2. If $g(C) = 1$, then we can replace f with the composition by an endomorphism of large degree n, then we see that

   $$-((f \circ h)_* C \cdot K_X) - \dim X = -n^2 f_*C.K_X - \dim X > 0$$

   whenever n is sufficiently large
3. Assume $g(C) \geq 2$. Then there are no endomorphisms of high degree, so assume X, C are defined over $\mathbb{Z}$. Then let X_p, C_p be the reduction to $\overline{F}_p$. Now we apply the Frobenius map F_p, which has degree p. By generic flatness, we know that $(f_p)_* C_p.K_{X_p}, g(C_p), \chi\Big(T_X\big|_{C_p}\Big)$ are the same for almost all p, so by the same argument as in the genus 1 case, we see have a rational curve A_p on X_p for almost all p. By bend and break II, we can find a rational curve of the desired degree.

   Then we use the fact that if a statement holds for all p large enough, then it holds for the complex numbers, and we obtain a curve. This is analogous to the idea that if $Z \subseteq \mathbb{P}^n_{\mathbb{Z}}$, then if the image of $\pi\colon Z \to \operatorname{Spec}\mathbb{Z}$ contains a Zariski-dense subset, then it contains the generic point. □

Theorem 2.1.5. *Let X be a smooth projective variety and let H be ample on X. Assume there exists $C' \subseteq X$ such that $-(C'.K_X) > 0$. Then there exists a rational curve E such that* $\dim X + 1 \geq -(E.K_X) > 0$ *and*

$$\frac{-(E.K_X)}{E.H} \geq \frac{-C'.K_X}{C'.H}.$$

Theorem 2.1.6 (Cone Theorem)**.** *Let X be smooth and projective. Then there exist countably many curves $C_r \subseteq X$ such that* $0 < -K_X.C_i \leq \dim X + 1$ *and*

$$\overline{\mathrm{NE}}(X) - \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i].$$

Proof. Choose C_i with $0 < -(C.K_X) \leq \dim X + 1$ and let W be the closure of $\overline{\mathrm{NE}}_{K\geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i]$. Now choose D positive on $W \setminus \{0\}$ and negative somewhere on $\overline{\mathrm{NE}}(X)$. Let H be ample and

$$\mu = \max \left\{ \mu' \mid H + \mu' D \text{ is nef} \right\}.$$

This means that $H + \mu D$ is nef. Then let $Z \in \overline{\mathrm{NE}}(X)$ with $(H + \mu D).Z = 0$ and $K_X.D < 0$. Let Z_k be a sequence of curves approximating Z. Then we see that

$$\max_j \frac{-(Z_{k_j}.K_X)}{(Z_{k_j}.(H + \mu' D))} \geq \frac{-Z_k.K_X}{Z_k.(H + \mu D)}$$

is obtained by Z_{k_0}. Now we will replace Z_k with rational curves $E_{i(k)}$ such that $\dim X + 1 \geq -E_{i(k)}.K_X > 0$ and

$$\frac{-E_{i(k)}.K_X}{E_{i(k)}.(H + \mu' D)} \geq \frac{-Z_{k_0}.K_X}{Z_{k_0}.(H + \mu' D)} \geq \frac{-Z_k.K_X}{Z_k.(H + \mu' D)}.$$

Because $E_{i(k)}.D \geq 0$, we have

$$\frac{-E_{i(k)}.K_X}{E_{i(k)}.H} \geq \frac{-Z_k.K_X}{Z_k.(H + \mu' D)}.$$

Fixing $M \gg 0$ such that $MH + K_X$ is ample, then we see that $(MH + K_X).E_{i(k)} > 0$, so

$$M > \frac{-E_{i(k)}.K_X}{E_{i(k)}.H} \geq \frac{-Z_k.K_X}{Z_k.(H + \mu' D)}.$$

Taking $k \to \infty, \mu' \to \mu$, we see that

$$M > \frac{Z.K_X}{Z.(H + \mu D)} \longrightarrow \infty,$$

a contradiction. $\square$

Example 2.1.7. Suppose K_X is not nef. Then there are no rational curves on X. For example, there are no rational curves of an abelian variety.

2.2 Singularities of the MMP

Consider pairs (X, Δ) such that X is normal quasiprojective and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. These are called *log pairs*.

Definition 2.2.1. Let $\pi \colon Y \to X$ is a resolution of singularities, $E \subseteq Y$ is exceptional, and $\pi^*(K_X + D) + \sum E_i$ has simple normal crossings. Define the *log discrepancy*

$$a_E(X, D) := 1 + \mathrm{coeff}_E(K_Y - \pi^*(K_X + D)).$$

Definition 2.2.2. We say that (X, Δ) is

1. *terminal* if $a_E(X, \Delta) > 1$ for every exceptional E over Y.
2. *canonical* if $a_E(X, \Delta) \geq 1$ for every exceptional E over Y.
3. *Kawamata log terminal* if $a_E(X, \Delta) > 0$ for every E.

4. *log canonical* if $a_E(X,\Delta) \geq 0$ for every E.

Now if X is smooth projective and K_X is pseudoeffective, then there exists $X \dashrightarrow X_{\text{ter}}$ such that $K_{X_{\text{ter}}}$ is nef. By abundance, $K_{X_{\text{ter}}}$ is semiample. Then there is a morphism $X_{\text{ter}} \to X_{\mathfrak{can}}$ such that $K_{X_{\text{can}}}$ is ample. Then terminal singularities are those that may appear in the terminal model, and canonical singularities are those that may appear on the canonical model.

Recall the *adjunction formula*: If (X,D) is log smooth, then $K_X + D\big|_D \sim K_X$. Usually, (X,D) is log canonical but not klt. But then $a_D(X,D) = 0$ but $a_E(X,D) > 0$ for every $E \neq D$. Terminal singularities are the smallest category of singularities that we need to understand to run the minimal model program. On the other hand, log canonical is the largest class of singularities in which we can expect the MMP to work.

Example 2.2.3 (Examples of klt singularities)**.** Both cone singularities and quotient singularities are klt.

Proposition 2.2.4 (Cones)**.** *Let (X,Δ) be a log pair and A an ample Cartier divisor on X. Then define*

$$C(X,\Delta) = \operatorname{Spec}(\bigoplus_{m \geq 0} H^0(X, \mathscr{O}_X(mA))).$$

Then $C(X,A)$ is

1. *terminal if and only if $rA \sim_{\mathbb{Q}} K_X + \Delta$ with $r < -1$ and (X,Δ) terminal;*

2. *canonical if and only if $ra \sim_{\mathbb{Q}} k_x + \delta$ with $r \leq -1$ and (X,Δ) canonical;*

3. *klt if and only if $ra \sim_{\mathbb{Q}} k_x + \delta$ with $r < 0$ and (X,Δ) is klt;*

4. *log canonical if and only if $ra \sim_{\mathbb{Q}} k_x + \delta$ with $r \leq 0$ and (X,Δ) is log canonical.*

In particular, the cone over a Fano is klt, the cone over a Calaby-Yau is log canonical, and cones over canonically polarized varieties are terrible.

Example 2.2.5. Consider the cone C_n over a rational normal curve of degree n. Then resolving $Y_n \to C_n$, we see that the exceptional $E_n \simeq \mathbb{P}^1$ and

$$\pi^*(K_{C_n}) = K_{Y_n} + \left(1 - \frac{2}{n}\right) E_n,$$

and therefore $a_{E_n}(C_n) = \frac{2}{n}$.

Consider $E \subseteq \mathbb{P}^3$ and elliptic curve. Then $\pi^*(K_{C_E}) = K_{Y_E} + E$, so $a_E(C_E) = 0$.

Now consider $G \subseteq GL_n$ a finite group. Then $\mathbb{C}^n/G = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]^G$ has klt singularities.

Example 2.2.6. In dimension 2, we have

1. terminal is equivalent to smooth;

2. canonical is equivalent to ADE;

3. klt is equivalent to quotient singularity;

4. log canonical is "equivalent" to quotient and/or elliptic cone.

Example 2.2.7. In dimension 3, terminal singularities are classified as quotients of hypersurface singularities, also known as *hyperquotient singularities*. Then are given by actions of finite groups G acting on hypersurface singularities of the form $\{x^2 + y^2 + f(z, w) = 0\}$. Even for canonical singularities, we have no idea what they look like.

Example 2.2.8. In dimension 4, there are examples of 4-fold terminal singularities with analytic embedding dimension n for every n (Kollar, 2010). By contrast, terminal singularities in dimension 3 all have analytic embedding dimension 4.

Theorem 2.2.9 (Prokhorov, Xu, 2019)**.** *Any klt singularity deforms to a klt cone singularity. For $x \in X$, there exists a flat morphism $\mathscr{X} \xrightarrow{\varphi} \mathbb{A}^1$ such that $\varphi(\mathbb{A}^1 \setminus 0) \sim (\mathbb{A}^1 \setminus 0) \times X$ and $\varphi^{-1}(0) = X_0$ is a klt cone singularity. This is just a deformation to the normal cone.*

We will apply the following philosophy:

Any theorem for smooth projective varieties should work with klt singularities.

Example 2.2.10. Let $K_X = 0$. By Beauville-Bogomolov (1970s), there exists a cover $X \leftarrow Y$, where Y is a product of abelian varieties, irreducible Calabi-Yau varieties, and hyperkählers. There is an analogue for klt singularities that was proved by Druel, Campana,. . . in 2020.

Example 2.2.11. Let X be smooth projective and $-K_X$ be nef. Then

$$\widetilde{X} \sim \mathbb{C}^q \times \prod Y_i \times \prod S_k \times Z,$$

where Y_i is strict Calabi-Yau, S_k is hyperkahler, and Z is rationally connected. There is a version for klt singularities in progress.

We will now discuss localization of singularities. Suppose X is a variety with terminal singularities and $Z \subseteq X$ a subvariety of codimension 1. Then Spec $\mathscr{O}_{X,Z}$ has terminal singularities if and only if Spec $\mathscr{O}_{X,Z}$ is a smooth local ring, and this is equivalent to smoothness at the generic point of Z. In particular, if X is terminal, the singularities must appear in codimension at least 3.

2.3 Vanishing

Let $C \subseteq X$ be a K_X-negative curve. Then if $K_X \sim_{\mathbb{Q}} E \geq 0$ for some E effective, we see that $K_X.C = E.C < -$, so $C \subseteq E$. Now if (X, D) is a log smooth pair, we have an exact sequence

$$0 \to \mathscr{O}_X(K_X) \xrightarrow{\otimes D} \mathscr{O}_X(K_X \times D) \to \mathscr{O}_D(K_D) \to 0.$$

If $H^1(X, \mathscr{O}_X(K_X)) = 0$, then $H^0(K_X + D) \twoheadrightarrow H^0(K_D)$. This tells us that vanishing theorems help us find sections of line bundles.

Theorem 2.3.1 (Kodaira Vanishing)**.** *Let X be smooth projective and $\mathscr{L}$ be ample. Then $H^i(X, \mathscr{L}^{-1}) = 0$ for all $i < \dim X$.*

Sketch of Proof. Let $s \in H^0(X, \mathscr{L}^m)$ and $D = (s = 0)$ be smooth. Then we have $\mathscr{O}_X \xrightarrow{s} \mathscr{L}^m$. Now we have

$$\mathscr{L}^{-i} \otimes \mathscr{L}^{-j} \simeq \mathscr{L}^{-i-k}\mathscr{O}_X \xrightarrow{\mathrm{id} \otimes s} \mathscr{L}^{-i-j} \otimes \mathscr{L}^m = \mathscr{L}^{-i-j+m}.$$

Setting $Z = \operatorname{Spec} \bigoplus_{i=0}^{m-1} \mathscr{L}^{-i}$ with projection $p\colon Z \to X$, we note that if X, D are smooth, then Z is smooth. Consider a morphism $\tau\colon H^i(Z, \mathbb{C}_Z) \twoheadrightarrow H^i(Z, \mathscr{O}_Z)$ and its pushforward $p_*\tau\colon H^i(X, p^*\mathscr{C}_Z) \twoheadrightarrow H^i(X, p_*\mathscr{O}_Z)$. Now we consider the surjection

$$\bigoplus_{r=0}^{m-1} H^i(X, \mathbb{C}[\zeta^r]) \twoheadrightarrow \sum_{r=0}^{m-1} H^i\left(X, \bigoplus_{r=0}^{m-1} \mathscr{L}^{-r}\right).$$

Now we use the result that $\mathbb{C}[\zeta^r] \hookrightarrow \mathscr{L}^{-r}$ factors through $\mathbb{C}[\zeta^r] \hookrightarrow \mathscr{L}^{-r}(-kD) \hookrightarrow \mathscr{L}^{-r}$. By Serre, we see that $H^i(X, \mathscr{L}^{-(r+mk)}) = 0$ for arbitrary k. $\square$

Theorem 2.3.2 (KV vanishing)**.** *Let X be a smooth projective complex and $\mathscr{L}$ be a line bundle on X such that $\mathscr{L} \equiv M + \sum a_i D_i$, where M is a big and nef $\mathbb{Q}$-divisor, $\sum D_i$ is a snc divisor, and $0 \le a_i < 1, a_i \in \mathbb{Z}$. Then $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < \dim X$.*

Proposition 2.3.3. *Let X be quasiprojective and normal, D Cartier, and $m > 0$ a positive integer. Suppose $Y \xrightarrow{p} X$ is finite and D' Cartier such that $p^*D \sim MD'$. If X is smooth and $\sum F_j$ is simple normal crossing, then Y is smooth and $\sum p^*F_j$ has simple normal crossings.*

Lemma 2.3.4. *Let $Y \to X$ be finite. Then $\mathscr{O}_X \to f_*\mathscr{O}_Y$ splits. If $\mathscr{F}$ is a coherent sheaf on X, then $\mathscr{F}$ is a direct summand of $f_*f^*\mathscr{F}$. Finally, $H^i(X, \mathscr{F})$ is a direct summand of $H^i(Y, f^*\mathscr{F})$.*

Sketch of KV Vanishing. Consider $\sum a_i D_i$ and write $a_1 = b/m$ for some $m \ge 0$. Then consider $p_1\colon X_1 \to X$ such that $p^*D_1 \sim mD$. Then $H^i(X, \mathscr{L}^{-1})$ is a direct summand of $H^i(X_1, p_1^*\mathscr{L}^{-1})$. Then $p_1^*D_1$ is a section of $\mathscr{O}_X(mD)$, so we can apply the index cover to obtain $X_2 \xrightarrow{p_2} X$ with X_2 smooth, $p_2^*(D_i)$ is smooth, and $\sum p_2^* p_1^* D_i$ has simple normal crossings. Then

$$(p_2)_*\mathscr{O}_{X_2} = \sum_{k=0}^{m-1} \mathscr{O}_{X_1}(-jD),$$

so

$$H^i(X_2, p_2^*p_1^*\mathscr{L}^{-1}(bD)) = \bigoplus_{j=0}^{m-1} H^i(X, p_1^*((b-j)D)).$$

Choosing $j = b$, we see that $H^i(X_1, p_1^*\mathscr{L}^{-1})$ is a direct summand of $H^i(X_2, p_2^*p_1^*\mathscr{L}^{-1}(bD))$. Therefore,

$$p_2^*p_1^*\mathscr{L}^{-1}(bD) = p_2^*p_1 * M + \sum_{i>1} a_1 p_2 * p_1 * (D_i).$$

But now we have reduced the number of components of D_i, so by induction, the cohomology of the pullback vanishes. Now we need to consider M. We know M is big and nef, so $M \sim_{\mathbb{Q}} A + E$, where A is ample and E is effective. Then there exists $f\colon Y \to X$ projective and birational such that $f^*\mathscr{L} = A + E$, where A is ample and E has simple normal crossings with $E = \sum a_i E_i, 0 \le a_i < 1$. Now let $H \subseteq X$ be an ample divisor. Then

$$H^i(X, \mathscr{L}(rH) \otimes R^j f_*\omega_Y) \Rightarrow H^{i+j}(Y, \omega_Y \otimes f^*\mathscr{L}(rH)).$$

But then $f^*\mathscr{L}(rH) = (A + rf^*H) + E$, where $A + rf^*H$ is ample. But now we know that

$$H^k(Y, f^*\mathscr{L}(rH) \otimes \omega_Y) = 0$$

for $k > 0$, so

$$H^0(X, \mathscr{L}(rH) \otimes R^j f_*\omega_Y) = H^j(Y, \omega_Y \otimes f^*\mathscr{L}(rH)) = 0$$

and therefore $R^j f_* \omega_Y = 0$ for $j > 0$. But now setting $r = 0$, we see that

$$H^i(X, \mathscr{L} \otimes f_* \omega_Y) \simeq H^i(Y, f^* \mathscr{L} \otimes \omega_Y) = 0. \qquad \square$$

Theorem 2.3.5. *Let (X, Δ) be a proper klt pair with N a $\mathbb{Q}$-Cartier divisor. Suppose that $N = M + \Delta$, where M is big and nef. Then $H^i(X, \mathscr{O}_X(-N)) = 0$ for $i < \dim X$. Equivalently, $H^{n-i}(X, K_X + N) = 0$ for $n - i > 0$.*

Remark 2.3.6. This philosophy fails for log canonical singularities.

2.4 Cone Theorems

Note that Kollar-Mori use outdated notation. What they call klt is actually sub-klt and what they call klt with $\Delta \geq 0$ we call klt. This still confuses people today, so let's blame the person with the Fields medal.

Theorem 2.4.1 (Non-vanishing)**.** *Let X be proper with (X, Δ) sub-klt*[1] *and D be a nef Cartier divisor. Assume $aD - (K_X + \Delta)$ is big and nef for some $a \geq 0$. Then for $m \gg 0$, we have*

$$H^0(X, mD - \lfloor \Delta \rfloor) \neq 0.$$

Theorem 2.4.2 (Basepoint-freeness)**.** *Let X be proper and (X, Δ) be klt. Suppose D is a nef Cartier divisor. Assume $aD - (K_X + \Delta)$ is big and nef for some $a \geq 0$. Then for $m \gg 0$, the linear sustem $|mD|$ is basepoint-free.*

Theorem 2.4.3 (Rationality)**.** *Let X be proper and (X, Δ) be klt. Suppose that $K_X + \Delta$ is not nef, $a(K_X + \Delta)$ is Cartier, and H is a nef and big Cartier. Define*

$$r := r(H) = \max \{ t \in \mathbb{R} \mid H + t(K_X + \Delta) \text{ is nef} \}.$$

Then r is rational and its denominator is at most $a(\dim X + 1)$.

Theorem 2.4.4 (Cone Theorem)**.** *Let (X, Δ) be a projective klt pair.*

1. *There are countably many $C_i \subseteq X$ such that $0 < -(K_X + \Delta).C_i \leq 2 \dim X$ and*

$$\overline{NE}(X) = \overline{NE}_{(K_X + \Delta) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

2. *For any H ample and $\varepsilon > 0$, we have*

$$\overline{NE}(X) = \overline{NE}_{(K_X + \Delta + \varepsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

3. *If $F \subseteq \overline{NE}(X)$ is extremal and $(K_X + \Delta)$-negative, then there exists a contraction morphism $\mathrm{cont}_F \colon X \to Z$ such that $C \subseteq X$ is mapped to a point if and only if $[C] \in F$.*

4. *Let $\mathrm{cont}_F \colon X \to Z$ be as above and $\mathscr{L}$ be a line bundle on X such that $\mathscr{L}.F = 0$. Then there exists $\mathscr{L}_Z$ on Z such that $\mathscr{L} \simeq \mathrm{cont}_F^* \mathscr{L}_Z$.*

The logical structure is this: We use non-vanishing to find sections, then use various techniques (Kodaira vanishing) to lift enough sections to get basepoint-freeness, then we prove rationality by studying linear systems of the form $|pH + qK_X|$, and finally we get the cone theorem as a formal consequence of convex geometry. However, we will follow the order of Kollar-Mori.

[1] The incorrect term was used initially

Remark 2.4.5. The entire discussion will be carried out using klt singularities. However, these results hold when (X, Δ) is log canonical at the cost of replacing nef and big with ample and at the cost of using significantly more machinery that was developed in the last 15 years.

Proof of basepoint-freeness. By non-vanishing, we know that $H^0(X, mD) \neq 0$ for $m \gg 0$. If $B(s)$ is the base locus of $|sD|$, it suffices to prove that for $B_s = B(m) \neq 0$. Next, we may consider a log resolution

$$f\colon Y \to X \qquad K_Y = f^*(K_X + \Delta) + \sum a_j F_j \qquad a_j > -1.$$

Now we may perturb K_Y such that

$$f^*(aD - (K_X + \Delta)) - \sum p_j F_j \qquad 0 < p_j \ll 1,$$

is ample. Therefore $f^*|mD| = |\mu| + \sum r_j F_j$, so $\sum r_j F_j$ is the fixed part. Therefore $B_s = \bigcup \{ f(F_j) \mid r_j > 0\}$ and $f^{-1} B_s |mD| = B_s |m f^* D|$.

We want to prove that there exists F_j with $r_j \geq 0$ such that for all $b \gg 0$, F_j is not contained in $B_s|bf^*D|$. Let $b > 0$ be an integer, $c > 0$ be rational, and $b > cm + a$. Then we define

$$\begin{aligned} N(b,c) &= bf^*D - K_Y + \sum_j (-cr_j + a_j - p_j) F_j \\ &= (b - cm - a) f^*D + c(mf^*D - \sum r_j F_j) + f^*(aD - (K_X + \Delta)) - \sum p_j F_j. \end{aligned}$$

But now we see that the first term is is nef, the second term is basepoint-free, and the final two terms form an ample divisor. Therefore $N(b,c)$is ample. By Kodaira vanishing, we see that $H^1(Y, \lceil N(b,c) \rceil + K_Y) = 0$ and $\lceil N(b,c) \rceil = bf^*D + \sum \lceil -cr_j + a_j - p_j \rceil F_j - K_Y$. Now

$$\sum \lceil -cr_j + a_j - p_j \rceil F_j = \lceil A \rceil - F,$$

where $A \geq 0$ is effective and $F = F'_k$ is prime. Therefore

$$K_Y + \lceil N(b,c) \rceil = bf^*D + \lceil A \rceil - F$$

and we have the exact sequence

$$0 \to \mathscr{O}_Y(bf^*D + \lceil A \rceil - F) \xrightarrow{\times F} \mathscr{O}_Y(bf^*D + \lceil A \rceil) \to \mathscr{O}_F(bf^*D + \lceil A \rceil) \to 0.$$

Therefore we have a surjection $H^0(Y, bf^*D + \lceil A \rceil) \twoheadrightarrow H^0\big(F, (bf^*D + \lceil A \rceil)\big|_F\big)$ for $b \geq cm + a$. Now $\lceil A \rceil$ is f-exceptional, so

$$N(b,c)\Big|_F = (bf^*D + A - F - K_Y)\Big|_F = (bf^*D + A)\Big|_F - K_F.$$

Now by non-vanishing, $H^0\big(F, (bf^*D + \lceil A \rceil)\big|_F\big) \neq 0$, so $H^0(Y, bf^*D + \lceil A \rceil)$ has a section not vanishing on F. Because $\lceil A \rceil$ is f-exceptional,

$$H^0(Y, bf^*D + \lceil A \rceil) = H^0(Y, bf^*D) = H^0(X, bD)$$

by the negativity lemma. Here, if $0 \leq E \sim bf^*D + \lceil A \rceil$, then $E - \lceil A \rceil \sim bf^*D \sim_{\mathbb{Q},X} 0$, so if $E - \lceil A \rceil \geq 0$, then $f_*(E - \lceil A \rceil) = f_*E = 0$. However, F must be disjoint from E, so we have found a section E of bf^*D disjoint from F. Therefore f_*E is a section of bD disjoint from W. ☐

Lemma 2.4.6 (Negativity lemma). *Let $h\colon Z \to Y$ be birational and proper between normal varieties and $-B$ be h-nef. Then B is effective if and only if $h_*B \geq 0$. If B is effective, either $h^{-1}(Y) \subseteq \operatorname{supp} B$ or $h^{-1}(Y) \cap \operatorname{supp} B = \varnothing$.*

Theorem 2.4.7. *Let (X, Δ) be a proper klt pair and $K_X + \Delta$ be big and nef. Then the graded ring*

$$\bigoplus_{m \geq 0}^{\infty} H^0(\mathscr{O}_X(mK_X + \lfloor m\Delta \rfloor))$$

is finitely-generated over $\mathbb{C}$.

This result holds even after dropping the big and nef assumption and was proved by Birkar, Cascini, Hacon, and McKernan in 2006.

Conjecture 2.4.8 (Abundance)**.** *Let (X, Δ) be projective and klt. If $K_X + \Delta$ is nef, then it is semiample.*

Conjecture 2.4.9 (Effectivity, folklore)**.** *Let (X, Δ) be projective klt. If $K_X + \Delta$ is pseudoeffective (in the closure of the effective cone), then $K_X + \Delta$ is effective.*

2.4.1 Proof of the Cone Theorem

We will now prove the cone theorem. This relies on the following result from convex geometry.

Theorem 2.4.10. *Let $N_{\mathbb{Z}} \subseteq N_{\mathbb{Q}} \subseteq N_{\mathbb{R}}$ and $\overline{NE} \subseteq N_{\mathbb{R}}$ be a closed strictly convex clone. Let $K \in N_{\mathbb{Q}}^*$ such that $(K.C) < 0$ for some $C \in \overline{NE}$. Assume there exists $\alpha(K) \in \mathbb{Z}_{\geq 0}$ such that for all $H \in N_{\mathbb{Z}}^*$ with $H > 0$ on $\overline{NE} \setminus \{0\}$ and that*

$$r := \max \left\{ b \in \mathbb{R} \mid H + tK \geq 0 \text{ on } \overline{NE} \right\}$$

is rational of the form $U/\alpha(K)$. Then

$$\overline{NE} = \overline{NE}_{K \geq 0} + \sum_{\text{countable}} \mathbb{R}_{\geq 0}[\xi_i]$$

with $\xi_i \in N_{\mathbb{Z}}$ with $(\xi_i.K) < 0$ and such that $\mathbb{R}_{\geq 0}[\xi_i]$ do not accumulate in $K_X < 0$.

Let H be ample and Cartier. Suppose L is nef and define $F_L = L^{\perp} \cap \overline{NE}$. Then for $n \in \mathbb{Z}_{\geq 0}$, we can set

$$r_L(n, H) = \max \left\{ t \in \mathbb{R} \mid nL + H + \frac{t}{\alpha(K)} K \text{ is nef} \right\}.$$

Then $r_K(n, H) \in \mathbb{Z}_{>0}$ is non-decreasing with respect to n. Now if $\xi \in F_L \setminus \overline{NE}_{K \geq 0}$, then

$$H \cdot \xi + \frac{r_L(n, H)}{\alpha(K)} \cdot K \cdot \xi \geq 0 \qquad r_L(n, H) \leq \alpha(K) \cdot \frac{H \cdot \xi}{-K \cdot \xi}.$$

Therefore $r_L(n, H)$ is bounded above, integral, and non-decreasing, so this sequences stabilizes for n large enough to $r_L(H)$. Now define the divisor

$$D(nL, H) = (n\alpha(K)L + \alpha(K)H + r_L(H)L) \cdot \xi = 0,$$

so $F_{D(nL.H)} \subseteq \overline{NE}_{K<0} \cup \{0\}$. To prove this, let $\xi \in F_{D(nL,H)}$ with $\xi \notin F_L$. Then we know

$$\xi.L > 0 \qquad \xi.(n\alpha(K)L + \alpha(K)H + r_L(H)L) = 0.$$

For $n' \gg n$, we have

$$\xi.(n'\alpha(K)L + \alpha(K)H + r_L(H)K) > 0,$$

so $\xi \notin F_{D(n'L,H)}$. Because L is nef, $F_{D(n'L,H)} \subsetneq F_{D(nL,H)}$. If $F_{D(n'L,H)} \subseteq F_L$, then we stop. If not, we can iterate the above process to decrease $\dim F_D(n'L,H)$ again, so the desired result eventually holds. Now $0 \neq F_D(nL.H) \subseteq F_L$ holds up to replacing n with a large multiple.

Now we will show that for some H, $\dim F_{D(nL.H)} < \dim F_L$. If H_i is a basis for F_L^*, the linear functions

$$\left(nL + H_i + \frac{r_L(H_i)}{\alpha(K)}K\right)\Big|_{F_L}$$

cannot all vanish, so $\dim F_{D(nL,H_i)} < \dim F_L$ for some i. Now we can reduce to $F_{L'} \subseteq F_L$ of dimension 1. This implies that $\overline{NE}$ and $\overline{NE}_{K\geq 0} + \sum_{\dim F_L=1} F_L$ have the same closure.

Now we need to show that the F_L do not accumulate in $K_{<0}$. This is a formal argument in linear algebra, so we skip it. Next, we need to prove that

$$\overline{NE}(X) = \overline{NE}(X)_{K+\varepsilon H} > 0 + \sum_{\text{finite}} F_L.$$

In the limit as $\varepsilon \to 0$, we produce countably many F_L (with a formal argument that is omitted here).

The next step is to prove that if $F \subseteq \overline{NE}(X)$ is a $(K_X + \Delta)$-negative face, then there exists a nef Cartier divisor D such that $F_D = F$. Let $\langle F \rangle$ be the linear span of F and $V \subseteq N(X)^*$ be the set of linear functions vanishing on $\langle F \rangle$. Because the generators of F are defined over $\mathbb{Q}$, then V is also defined over $\mathbb{Q}$. Take $\varepsilon > 0$ small enough such that $K_X + \Delta + \varepsilon H$ is negative on F. Because F is extremal, we know $\langle F \rangle \cap \overline{NE}(X) = F$. Therefore,

$$W_F := \overline{NE}(X)_{K_X+\Delta+\varepsilon H \geq 0} + \sum_{\substack{\dim F_L = 1 \\ F_L \not\subseteq F}} F_L$$

is a closed strictly convex cone intersecting $\langle F \rangle$ at the origin. We also note that $\overline{NE} = W_F + F$, so we can find a lattice point $p \in V$ such that $(p = 0) \supseteq \langle F \rangle$ and $(p = 1) \cap W_F = 0$. Therefore we can find a Cartier divisor D which gives a supporting function of $F \subseteq NE(X)$.

Now by assumption, $-(K_X + \Delta)$ is positive on F. This means that $mD - (K_X + \Delta)$ is strictly positive on $\overline{NE}(X) \setminus \{0\}$ for $m \gg 0$, so $|mD|$ is basepoint-free. Now let $g_F \colon x \to Z$ be the contraction associated by the Stein factorization to the linear system $|mD|$. Because g_F is not an isomorphism, it constracts some curve C. Similarly to teh smooth case, we may assume that

$$0 < -(K_X + \Delta).C \leq 2 \dim X.$$

Finally, we prove that any line bundle $\mathscr{L}$ on X such that $\mathscr{L}.F = 0$ descends to Z, which means there exists a line bundle $\mathscr{L}_Z$ on Z such that $\mathscr{L} = g_F^* \mathscr{L}_Z$. Now let D be a Cartier divisor supporting F. We know $W_F \subseteq \overline{NE}(X)$ and that g_F is defined by $\overline{mD}$. Therefore, both mD and $(m+1)D$ are pullbacks of Cartier divisors on Z. Write $mD = g_F^* D_1, (m+1)D = g_F^*(D_2)$, and therefore we see that $D = (m+1)D - mD = g_F^*(D_2 - D_1)$. This implies that D is the pullback of a Cartier divisor on Z. If $\mathscr{L}.F = 0$, then $\mathscr{L} + mD$ also supports F, so $\mathscr{L} + mD = g_F^* M_Z$ for some Cartier divisor M_Z of Z. We simply set $\mathscr{L}_Z = \mathscr{O}_Z(M_Z - D_1)$. □

2.4.2 Nonvanishing and Rationality

First we will prove rationality assuming non-vanishing and basepoint-freeness and then we will prove rationality.

Lemma 2.4.11. *Let Y be a smooth projective variety and $D_1, \dots, D_n$ be Cartier divisors. Then let A be a normal crossing divisor with $\lceil A \rceil \geq 0$. Define*

$$P(u_1, \dots, u_n) := \chi\left(\sum u_i D_i + \lceil A \rceil\right).$$

Assume that for certain u_i, $\sum u_i D_i$ is nef and $\sum u_i D_i + A - K_Y$ is ample. Then $P(u_1, \dots, u_n)$ is a nonzero polynomial of degree at most $\dim Y$.

Proof. For $m \gg 0$, the sub $\sum m u_i D_i + A - K_Y$ is still ample. Then we know $H^i(\sum m u_i D_i + \lceil A \rceil) = 0$ for $i > 0$ by KV vanishing. By non-vanishing, we know $h^0(\sum m u_i D_i + \lceil A \rceil) \neq 0$ and thus $\chi(\sum m u_i D_i + \lceil A \rceil) \neq 0$ and thus $P(mu_1, \dots, mu_n) \neq 0$. □

Lemma 2.4.12. *Let $P(x, y) \neq 0$ be a polynomial of degree at most n. Assume P vanishes for all sufficiently large integral solutions of $0 < ay - rx < \varepsilon$ for $a \in \mathbb{Z}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$. Then r is rational and in reduced form it has denominator at most $a(n+1)/\varepsilon$.*

This is a purely arithmetic fact, so proof is omitted.

Proof of Rationality. First, we reduce to the case in which H is basepoint-free. Define

$$H' = m(vH + da(K_X + \Delta)).$$

By basepoint-freeness, we know that $|H'|$ is basepoint-free. For $m \gg c \gg d > 0$, we know $r(H) = \frac{r(H') + mda}{mc}$ and thus rationality of $r(H)$ is equivalent to $r(H')$. If the denominator of $r(H')$ divides v, then the denominator of $r(H)$ divides mcv. Replacing H with H', now H is basepoint-free.

Now we need to study the base locus $L(p, q)$, which is the base locus of $|pH + qa(K_X + \Delta)|$. Then we know $L(p, q) = X$ if and only if $|pH + qa(K_X + \Delta)| = \emptyset$. If p, q are large enough to be in the strip between $ay - rx = 0, ay - rx = \varepsilon$, then $L(p, q)$ stabilizes. To see this, the xH direction is semiample and then the base locus stabilizes by Noetherian induction to some L_0. Now define $I \subseteq \mathbb{Z} \times \mathbb{Z}$ to be the set of (p, q) such that $0 < aq - rp < \varepsilon$ and $L(p, q) = L_0$. Now I contains arbitrary large lattice points.

Next, we will define the polynomial $P(x, y)$ and prove that it does not vanish. Define $p \colon Y \to X$ to be a log resolution of (X, Δ). Then if $D_1 = p^* H, D_2 = p^*(a(K_X + \Delta)), K_Y = p^*(K_X + \Delta) + A$ where $\lceil A \rceil \geq 0$ is p-exceptional, then $P(x, y) := \chi(xD_1 + yD_2 + \lceil A \rceil)$ is a polynomial of degree at most $\dim Y = \dim X = n$. Then note that if $y = 0, x \gg 0$, D_1 is big and nef, so $P \neq 0$. Furthermore, we know that

$$H^0(Y, pD_1 + qD_2 + \lceil A \rceil) = H^0(X, pH + qa(K_X + \Delta)).$$

From now on we will assume that r is not rational.

Now we show that $L_0 \neq X$. If $0 < ay - rx < 1$, then

$$xD_1 + yD_2 + A - K_X \equiv p^*(xH + (ay - 1)(K_X + \Delta))$$

is big and nef. Thus $H^i(Y, xD_1 + yD_2 + \lceil A \rceil) = 0$ for $i > 9$. For (p, q) large enough we know $P(p, q) \neq 0$ by the first lemma, so $h^0(Y, pD_1 + qD_2 + \lceil A \rceil) \neq 0$. But this implies that $|pH + qa(K_X + \Delta)| \neq \emptyset$, so $L_0 \neq \emptyset$.

We show that $L(p', q') \subsetneq L_0$ for (p', q') large in the strip. This will lead to a contradiction. Fixing $(p, q) \in I$, let $f \colon Y \to (X, \Delta)$ be a log resolution satisfying:

1. The divisor $f^*(pH + (qa - 1)(K_X + \Delta)) - \sum p_j F_j$ is ample;
2. $K_Y \equiv f^*(K_X + \Delta) + \sum a_j F_j$ for $a_j > -1$.
3. $f^*|pH + qa(K_X + \Delta)| = |L| + \sum r_j F_j$, where $|L|$ is the movable part and $\sum r_j F_j$ is the fixed part.

Then we can choose $c > 0$ and $p_j > 0$ such that

$$\sum(-cr_j + a_j - p_j)F_j = A' - F,$$

where F is prime and $\lceil A' \rceil \geq 0$ and A' does not contain F in its support. Now F maps to a component B of $L(p,q) = f\left(\bigcup_{r_j>0} F_j\right)$. Now define

$$\begin{aligned} N(p',q') &= f^*(p'H + q'a(K_X + \Delta)) + A' - F - K_Y \\ &\equiv cL + f^*(pH + (qa-1)(K_X+\Delta)) \\ &\quad - \sum p_jF_j + f^*((p' - (1+c)p)H + (q' - (1+c)q)a(K_X + \Delta)). \end{aligned}$$

We can choose (p',q') with $aq' - rp' < aq - rp$. Then $(q' - (1+c)q)a < r(p' - (1+c)p)$, so

$$(p' - (1+c)p)H + (q' - (1+c)q)a(K_X + \Delta)$$

is nef. We conclude that $N(p',q')$ is ample because cL is nef and the second term in the sum is ample. Therefore,

$$H^0(Y, f^*(p'H + q'a(K_X + \Delta)) + \lceil A \rceil) \twoheadrightarrow H^0\left(F, \left(f^*(p'H + q'a(K_X+\Delta))\right)\Big|_F\right).$$

By the adjunction formula, we know

$$\left(f^*(p'H + q'a(K_X + \Delta))\right)\Big|_F = \left(f^*(p'H + q'a(K_X+\Delta)) + A'\right)\Big|_F - K_F.$$

Applying the lemmas, we conclude that

$$H^0\left(F, \left(f^*(p'H + q'a(K_X+\Delta)) + \lceil A \rceil\right)\Big|_F\right) \neq 0$$

and thus $H^0(Y, f^*(p'H + q'a(K_X + \Delta)))$ contains a section $\Gamma \geq 0$ not vanishing at F. Running the same argument using the negativity lemma implies that Γ actually is disjoint from F and thus $0 \leq f_*\Gamma \sim |p'H + q'a(K_X+\Delta)|$ is a section disjoint from $B = f(F) \subseteq L_0$. Therefore $L(p',q') \subsetneq L_0$ and thus r is rational.

Now we need to control the denominator of r. Assume the denominator is larger than the constant given by the second lemma. Setting $\varepsilon = 1$, we choose (p,q) large with $0 < aq - rp < 1$. Then we have $P(p,q) = h^0(Y, pD_1 + qD_2 + \lceil A \rceil) \neq 0$, so $|pH + qa(K_X + \Delta)| \neq \emptyset$ for all $(p,q) \in I$. Now choose (p,q) such that $aq - rp$ is the maximum, equal to $\frac{d}{v}$. Then we can show that $\chi = h^0 \neq 0$ for $(f_*(p'H + q'a(K_X+\Delta)) + \lceil A' \rceil)|_F$. Then there exist (p',q') large enough in $0 < aq' - rp' < 1$ with $\varepsilon = 1$ and $aq' - rp' < \frac{d}{v} = aq - rp$. Running the same argument from the previous paragraph gives us the desired conclusion. $\square$

Proof of non-vanishing. First we will reduce to the case where X is smooth and $aD - (K_X + \Delta)$ is ample. Choose $f\colon X' \to X$ be a projective resolution and suppose $f^*(K_X + \Delta) - K_{X'} + \Delta'$ and (X', Δ') is a sub-klt pair. Then we know $af^*D - (K_{X'} + \Delta') = f^*(aD - (K_X + \Delta))$ is nef and big, so $af^*D - (K_{X'} + \Delta') - F$ is ample, so $(X', \Delta' + F)$ is sub-klt. Writing $\Delta'' = \Delta' + F$, we see that $f_*(\Delta'') \leq \Delta$ and

$$h^0(X', mf^*D - \lfloor \Delta'' \rfloor) \leq h^0(X, mD - \lfloor \Delta \rfloor).$$

Replacing X, Δ with (X', Δ''), we have the desired reduction.

Next, we need to rule out the case that D is numerically trivial. By KV vanishing, we know

$$h^0(X, mD - \lfloor \Delta \rfloor) = \chi(X, mD - \lfloor \Delta \rfloor) = \chi(X, -\lfloor \Delta \rfloor) = h^0(X, -\lfloor \Delta \rfloor) \geq 1.$$

Thus we may assume that D is not numerically trivial.

Now we show that there exists q_0 such that if $x \in X$ is not in the support of Δ, then for $q \geq q_0$ we can find $M(q) \equiv (qD - (K_X + \Delta))$ with multiplicity greater than $2 \dim X$. For A ample and $e > 0$, we have $D^e A^{d-e} \geq 0$, so we conclude that

$$(qD - (K_X + \Delta))^d = ((q-a)D + aD - (K_X + \Delta))^d \geq d(q-a)(D \cdot (aD - (K_X + \Delta))^{d-1}).$$

Because $aD - (K_X + \Delta)$ is ample, then $(aD - (K_X + \Delta))^{d-1} = C + \text{eff}$, where C is a curve satisfying $C \cdot D > 0$. Therefore, $(qD - (K_X + \Delta))^d \to \infty$ as $q \to \infty$. Now

$$h^0(e(qD - (K_X + \Delta))) \geq \frac{e^d}{d!}(qD - (K_X + \Delta))^d + O(e^{d-1}).$$

Thus if $M(q,e) \in |e(qD - (K_X + \Delta))|$, imposing that $M(q,e)$ has multiplicity greater than $2de$ at x imposes at most $\frac{e^d}{d!}(2d)^d + O(e^{d-1})$ conditions. As $q \to \infty$, then we know $(qD + (K_X + \Delta))^d > (2d)^d$, so for q large enough, some section satisfies the condition. Now we define $M(q) := M(q,e)/e$, so $M(q) \in |qD - (K_X + \Delta)|$ has multiplicity at least $2d$ at x.

Next, consider a log resolution of $(X, \Delta + M(q))$ that dominates $\mathrm{Bl}_x X$. Then

1. $K_Y \equiv f^*(K_X + \Delta) + \sum b_j F_j$, $b_j > -1$;
2. $f^*(aD - (K_X + \Delta)) - \sum p_j F_j$ is ample for $0 < p_j \ll 1$;
3. $f^*(M(q)) = \sum r_j F_j$ where F_0 corresponds to the exceptional divisor of the blowup at x.

Now we will perturb the coefficients and lift from lower dimension. Set

$$N(b,c) := bf^*D + \sum(-cr_j + b_j - p_j)F_j - K_Y.$$

This is ample as long as $c \leq \frac{1}{2}$ and $b \geq a + c(q-a)$. Because we can choose b arbitrarily large, the second condition is always achievable. But now because $x \notin \mathrm{Supp}(\Delta)$, we know $b_0 = d - 1$ and $r_0 > 2d$, so $c < \frac{1 + (d-1) - p_1}{2d} < \frac{1}{2}$. Therefore we can write $N(b,c) = bf^*D + A - F - K_Y$, so

$$H^0(Y, bf^*D + \lceil A \rceil - F) = H^0(Y, bf^*D - f^*\lfloor \Delta \rfloor) = H^0(X, bD - \lfloor \Delta \rfloor).$$

Because $N(b,c)$ is ample, then

$$H^1(Y, bf^*D + \lceil A \rceil - F) = H^1(Y, bf^*D + \lceil A - F \rceil) = 0,$$

so $H^0(X, bD - \lfloor \Delta \rfloor) \neq 0$ as long as $H^0\big(F, (bf^*D + \lceil A \rceil)\big|_F\big) \neq 0$. But by the non-vanishing theorem in lower dimension, this is true. $\square$