Complex Analysis
Math 621

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Disclaimer

These notes are a transcription of handwritten notes that were taken during lecture. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style (omit lengthy computations, use category theory) and that of the instructor. If you find any errors, please contact me at `plei@umass.edu`.

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Basics of Complex Analysis

A *complex number* is a sum $z = x + iy$, where $x, y \in \mathbb{R}$ and i is a symbol satisfying the identity $i^2 = -1$. Addition and multiplication work as one would expect. The set $\mathbb{C}$ of complex numbers is a *field*. This means that $(\mathbb{C}, +)$ is an abelian group, $(\mathbb{C} \setminus \{0\}, \cdot)$ is an abelian group, and that multiplication distributes over addition.

Remark 1.0.1. If F is a field and $f \in F[x]$ is irreducible, then we can construct a field extension K/F such that K has a root of f by setting $K = F[x]/(f)$. In this way, we have $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$. The Galois group $\operatorname{Gal} \mathbb{C}/\mathbb{R}$ is generated by complex conjugation.

1.1 Holomorphic Functions

Let $\Omega \subset \mathbb{C}$ be an open set. Here, the topology is the Euclidean topology on $\mathbb{C} = \mathbb{R}^2$. Then $f : \Omega \to \mathbb{C}$ is *holomorphic* at a point $z_0 \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we write $f'(z_0)$ for the derivative at z_0.

Example 1.1.1. The function $f(z) = \bar{z}$ is **not** holomorphic. To see this, the difference quotient has different limits on the real and imaginary axes.

Example 1.1.2. The function $f(z) = z^n$ is holomorphic for $n \in \mathbb{N}$, and $f'(z) = nz^{n-1}$.

Remark 1.1.3. The usual formulas for differentiation (chain rule, product rule, linearity) hold in this case.

We will now compare holomorphic and real differentiability. Rewrite $F = u + iv$. Recall that F is *real differentiable* at $\mathbf{a} = (x_0, y_0)$ if

$$\lim_{\mathbf{h} \to 0} \frac{\|F(\mathbf{a} + \mathbf{h}) - F(\mathbf{a}) - A\mathbf{h}\|}{\|\mathbf{h}\|} = 0$$

for some linear map A. We say that A is the derivative of F at $\mathbf{a}$. Moreover, A is given by the *Jacobian matrix*

$$J_F = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$