Defomation Theory Graduate Student Seminar Spring 2021

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Lectures by Various

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Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Seminar Website: `https://www.math.columbia.edu/~dejong/seminar.html`

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Johan (Sep 24): Schlessinger's Paper

The paper by Schlessinger is titled *Functors of Artin Rings*. Throughout this lecture, k is a field, $\mathcal{C}$ is the category of Artinian local k-algebras $A, B, C, \dots$ with residue field k, and $\widehat{\mathcal{C}}$ is the category of Noetherian complete local k-algebras $R, S, \dots$ with residue field k.

Remark 1.0.1. Every $R \in \widehat{\mathcal{C}}$ is of the form $k[[x_1, \dots, x_n]]/(f_1, \dots, f_m)$ by the Cohen structure theorem. Then $R \in \mathcal{C}$ if and only if and only if $(f_1, \dots, f_m)$ contains $(x_1, \dots, x_n)^N$ for some N.

Remark 1.0.2. In the paper, there is a more general setup, where Λ is a complete local Noetherian ring with residue field k. Then $\mathcal{C}_\Lambda, \widehat{\mathcal{C}}$ are defined analogously, which will allow things like $\Lambda = \mathbb{Z}_p$.

The idea of deformation theory is to look at functors $F\colon \mathcal{C} \to \mathsf{Set}$.

Example 1.0.3. Given $R \in \widehat{\mathcal{C}}$, we set $h_R\colon \mathcal{C} \to \mathsf{Set}$ sending $A \mapsto \mathrm{Hom}_{\widehat{\mathcal{C}}}(R, A)$. This is not necessarily representable because $R \notin \mathcal{C}$ in general, but it is pro-representable.

Definition 1.0.4. A functor F is *pro-representable* if $F \simeq h_R$ for some $R \in \widehat{\mathcal{C}}$.

Example 1.0.5. Let M be a variety over k and $m \in M(k)$. Then define

$$\mathrm{Def}_{M,m}(A) = \left\{ \mathrm{Spec}\, A \xrightarrow{m_A} M \mid m_A|_{\mathrm{Spec}\, k} = m \right\}.$$

It is easy to see that $\mathrm{Def}_{M,m}(A)$ is pro-representable by $\widehat{\mathcal{O}}_{M,m}$.

Observe that $h_R(k) = \{*\}$ is a singleton. Also note that $h_R(A \times_B C) = h_R(A) \times_{h_R(B)} h_R(C)$. Here, $A \times_B C$ is the fiber product of rings and not the tensor product.

Now consider the following conditions on F: let $A \to B \leftarrow C$ be a diagram in $\mathcal{C}$ and consider the morphism

$$F(A \times_B C) \xrightarrow{(*)} F(A) \times_{F(B)} F(C).$$

- (H_1) The morphism $(*)$ is surjective if $C \twoheadrightarrow B$;
- (H_2) The morphism $(*)$ is bijective if $C = k[\varepsilon] \twoheadrightarrow k = B$;
- (H_3) $\dim_k(t_F) < \infty$ (later, we will see that we need H_2 for formulate this). Here, t_F is the tangent space to F;
- (H_4) The morphism $(*)$ is bijective if $C \twoheadrightarrow B$.

Example 1.0.6. Fix a group G and a representation $\rho_0 \colon G \to GL_n(k)$. Now define

$$\mathrm{Def}^{\mathrm{naive}}_{\rho_0}(A) = \left\{\rho \colon G \curvearrowright A^{\oplus n} \mid \rho \pmod{m_A} \cong \rho_0\right\}/\cong.$$

Better, we will define

$$\mathrm{Def}_{\rho_0}(A) = \left\{\rho \colon G \curvearrowright A^{\oplus n} \mid \rho \pmod{m_A} = \rho_0\right\}/\ker(GL_n(A) \to GL_n(k)).$$

In general these functors fail (H_4) and $\mathrm{Def}^{\mathrm{naive}}_{\rho_0}$ even fails (H_2).

Namely, if $H = \mathbb{Z}$ and ρ_0 is the trivial representation, then for $\mathrm{Def}^{\mathrm{naive}}_{\rho_0}$, we are looking at subsets of

$$GL_n(A \times_B C)/\mathrm{conj} \to GL_n(A)/\mathrm{conj} \times_{GL_n(B)/\mathrm{conj}} GL_n(C)/\mathrm{conj}.$$

This morphism is always surjective, but in general it is not injective.

For example, if $A = k[\varepsilon_1], B = k, C = k[\varepsilon_2]$, we can look at elements of the form $1 + \varepsilon_1 T_1 + \varepsilon_2 T_2$ and see that on the left we can only conjugate together, while on the right we can conjugate both T_1, T_2 arbitrarily. Here $A \times_B C = k[\varepsilon_1, \varepsilon_2] = k[x_1, x_2]/(x_1^2, x_1 x_2, x_2^2)$.

Definition 1.0.7. A natural transformation $t \colon F \to G$ of functors on $\mathcal{C}$ is *smooth* if for all surjections $B \twoheadrightarrow A$ the map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

Note that this is equivalent to the existence of a lift in the diagram below:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d, hook] & M \arrow[d, "f"] \\
\operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & N.
\end{tikzcd}$$

This definition is motivated by the following example: let $f \colon M \to N$ be a morphism of varieties over k. Let $m \in M(k), n = f(m) \in N(k)$. Then the following are equivalent:

1. $\mathrm{Def}_{M,m} \to \mathrm{Def}_{N,n}$ is smooth.

2. f is smooth at m.

Definition 1.0.8. We say F has a *hull* if and only if $F(k) = \{*\}$ and there exists a smooth $t \colon h_R \to F$ for some $R \in \widehat{\mathcal{C}}$ which induces an isomorphism $t_R \cong t_F$.

Now we will say a bit about tangent spaces.

1. When $F(k) = \{*\}$, then $t_F = F(k[\varepsilon])$.

2. If F satisfies (H_2) and $F(k) = \{*\}$, then t_F has a natural k-vector space structure. Here, H_2 gives $F(k[\varepsilon_1, \varepsilon_2]) \to F(k[\varepsilon]) \times F(k[\varepsilon])$ is a bijection, and then we take $\varepsilon_1 \mapsto \varepsilon, \varepsilon_2 \mapsto \varepsilon$, which defines addition.

3. $t_R = \mathrm{Hom}_k(m_R/m_R^2, k) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(R, k[\varepsilon]) = h_R(k[\varepsilon]) = t_{(h_R)}$.

Theorem 1.0.9 (Schlessinger)**.** *Asssume that $F(k) = \{*\}$. Then the conditions $(H_1), (H_2), (H_3)$ hold for F if and only if F has a hull. In addition, (H_3) and (H_4) hold if and only if F is pro-representable.*

Very rough idea of proof of $\Rightarrow$ for the hull case. Let $n = \dim_k(t_F)$. Then (H_2) and $n < \infty$ imply the following: Let $S = k[[x_1, \dots, x_n]]$ and $\mathfrak{m} = \mathfrak{m}_S = (x_1, \dots, x_n)$. We can find $\xi_1 \in F(S/\mathfrak{m}^2)$ such that

$$t_S = \operatorname{Hom}_{\widehat{\mathcal{C}}}(S, k[\varepsilon]) \xrightarrow{\xi_1} t_F$$

is an isomorphism.

Next, we will choose $q \geqslant 2$ and consider pairs (J, ξ) where $\mathfrak{m}^{q+1} \subset J \subset \mathfrak{m}^2$ and $\xi \in F(S/J)$ such that $\xi \mapsto \xi_1 \in F(S/\mathfrak{m}^2)$. Say that $(J, \xi) \leqslant (J', \xi')$ if $J \subset J'$ and $\xi \mapsto \xi'$. Choose a minimal pair (J, ξ) for this ordering. We can choose J_q so that $\mathfrak{m}^{q+1} + J_{q+1} = J_q$ and ξ_{q+1} maps to ξ_q for bookkeeping purposes.

Choose $R = \lim S/J_q$, which is a quotient of S. Set $t\colon h_R \to F$ given by sending $\varphi\colon R \to A$ to the following: choose q such that φ factors as $R \to S/J_q \xrightarrow{\varphi_q} A$ and take $\xi_q \mapsto F(\varphi_q)(\xi_q) \in F(A)$.

Finally, we must show that t is smooth. Consider the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[dd] \arrow[drrr, dashed, bend left, "\varphi"] & S/\mathfrak{m}^{q+1} \arrow[dr, dashed, "\overline{\varphi}'"'] \arrow[drr, dashed, "\overline{\varphi}"] & & \\
& & S/J_q \times_A B \arrow[r] \arrow[dl, "pr_1"] & B \arrow[d, two heads] \\
R \arrow[r] & S/J_q \arrow[rr] \arrow[ur, dashed, bend left, "\psi"] & & A
\end{tikzcd}$$

with $B \ni \widetilde{\xi} \mapsto \xi \in A$ and $S/J_q \ni \xi_q \mapsto \xi$. First, choose $\varphi\colon S \to B$ making the diagram commute. We may increase q such that $\varphi(\mathfrak{m}^{q+1}) = 0$, so we now have $\overline{\varphi}\colon S/\mathfrak{m}^{q+1} \to B$. Now consider the fiber product $S/J_q \times_A B$ and $pr_1\colon S/J_q \times_A B \to S/J_q$, so we obtain $\overline{\varphi}'\colon S/\mathfrak{m}^{q+1} \to S/J_q \times_A B$. By (H_1), we obtain some $\widetilde{\widetilde{\xi}} \in F(S/J_q \times_A B)$ mapping to $\widetilde{\xi}$ and ξ_q. We may now assume that $B \to A$ is a small extension, which means that $\dim_k \ker(B \to A) = 1$, and thus pr_1 is a small extension. Therefore, either $\overline{\varphi}'$ is surjective or its image maps isomorphically via pr_1 to S/J_q., so we have ψ which gives $R \to B$ lifting our given $r \to A$.

The tricky part is to show that $F(\psi)(\psi_q) = \widetilde{\widetilde{\xi}}$, and this step is deliberately omitted. $\square$

A generalization of this is as follows. Consider a functor $\mathcal{F}\colon \mathcal{C} \to \mathsf{Grpd}$. We say that $\mathcal{F}$ satisfies the *Rim-Schlessinger condition* (RS) if

$$\mathcal{F}(A \times_B C) \to \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(C)$$

is an equivalence whenever $C \twoheadrightarrow B$. Let $x_0 \in \mathcal{F}(k)$ and set

$$\overline{\mathcal{F}}_{x_0}\colon \mathcal{C} \to \mathsf{Set} \qquad A \mapsto \{(x, \alpha) \mid x \in \mathcal{F}(A), \alpha\colon X_0 \to x|_k\}/\cong,$$

where $(x, \alpha) \cong (x', \alpha')$ means that $\varphi\colon x \to x'$ such that the diagram

$$\begin{tikzcd}
x|_k \arrow[r, "\varphi"] & x'|_k \\
x_0 \arrow[u, "\alpha"] \arrow[r, "\mathrm{id}"] & x_0 \arrow[u, "\alpha'"']
\end{tikzcd}$$

commutes.

Theorem 1.0.10. *If $\mathcal{F}$ has (RS) then $\overline{\mathcal{F}}_{x_0}$ has (H_1) and (H_2). Therefore, if $\dim t_{\overline{\mathcal{F}}_{x_0}} < \infty$ then $\overline{\mathcal{F}}_{x_0}$ has a hull.*

In this situation, $\overline{\mathcal{F}}_{x_0}$ has (H_4) if and only if $\mathrm{Aut}_A(x) \twoheadrightarrow \mathrm{Aut}_B(x|_B)$ whenever $A \twoheadrightarrow B$ and $x \in \mathcal{F}_{x_0}(A)$.

Example 1.0.11. Let $\mathcal{F}(A)$ be the category of representations $G \curvearrowright A^{\oplus n}$ with morphisms being isomorphisms of representations. This has (RS).

Example 1.0.12. Let $\mathcal{F}(A)$ be the category of smooth projective families of curves of genus g over A with morphisms being isomorphisms. This has (RS).

Returning to the example of representations, it turns out that $t_{\mathrm{Def}_{\rho_0}} = H^1(G, M_{n\times n}(k))$, where G acts on $M_{n\times n}(k)$ via ρ_0 by conjugation.

Example 1.0.13. Consider $G = \mathbb{Z} \oplus \mathbb{Z}$ and ρ_0 to be the trivial representation on $k^{\oplus 2}$. Then $t_{\mathrm{Def}_{\rho_0}} = H^1(\mathbb{Z}^2, M_2(k)) = M_2(k) \oplus M_2(k)$. Given two matrices A, B, we have the representation

$$
\begin{aligned}
\mathbb{Z}^2 \to \mathrm{GL}_2(k[\varepsilon])(1,0) &\mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \varepsilon A \\
(0,1) &\mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \varepsilon B.
\end{aligned}
$$

We get a hull R with $h_R \to \mathrm{Def}_{\rho_0}$. We know that R is a some quotient of $k[[a_{11}, \ldots, a_{22}, b_{11}, \ldots, b_{22}]]$ with ρ looking like

$$
(1,0) \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + A \qquad (0,1) \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + B,
$$

and of course R is the quotient of the power series ring by the ideal generated by the coefficients of $AB - BA$.

2

Ivan and Cailan (Oct 1): Deformations of Schemes

2.1 Deformations of affine schemes

We are looking for a Cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[r] & S \end{tikzcd}$$

where π is flat and surjective and S is surjective. This is called an *deformation* of X over S. For the beginning of this lecture (the part given by Ivan), we are interested in $S = \operatorname{Spec} A$, where $A \in \mathcal{C}^*$ (this category was defined in the previous lecture). This case is called a *local deformation*, and in the face where A is Artinian, it is called an *infinitesimal deformation*.

For the ring theorists, we will make the following digression. Let A be a ring and $I \subset A$ be an ideal with $I^2 = 0$. Suppose that $\overline{B}$ is an A/I-algebra, J is an $\overline{B}$-module, and $h \colon I \to J$ is an A-module map. Then we are interested in a diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & ? \arrow[r] & \overline{B} & \\ 0 \arrow[r] & I \arrow[r] \arrow[u, "h"] & A \arrow[r] \arrow[u] & A/I \arrow[r] \arrow[u] & 0, \end{tikzcd}$$

which we will call a deformation of A. Here are some interesting questions:

1. Is such a deformation unique?
2. If $\overline{B}$ is flat over A/I, does that mean that B is flat over A?

Returning to the case of schemes, we will say that two deformations $\mathcal{X}, \mathcal{X}'$ of X over S are isomorphic if there exists an S-isomorphism $\phi \colon \mathcal{X}' \to \mathcal{X}$ commuting with the inclusions of the central fibers $X \to \mathcal{X}, \mathcal{X}'$.

Example 2.1.1. The most basic example of a family is the trivial deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times_k S \arrow[d] \\ \operatorname{Spec} k \arrow[r] & S. \end{tikzcd}$$

Definition 2.1.2. A scheme X is *rigid* if all deformations of X are isomorphic to the trivial deformation.

Theorem 2.1.3. *If X is a smooth affine k-scheme and $S = \operatorname{Spec} A$ for some local Artinian ring, then X is rigid.*

Definition 2.1.4. A closed immersion $i\colon S_0 \hookrightarrow S$ of schemes is called a *first (resp. nth) order thickening* if the ideal sheaf $\mathcal{I} = \ker(i^\flat\colon \mathcal{O}_S \to \mathcal{O}_{S_0})$ satisfies $I^2 = 0$ (resp. $I^{n+1} = 0$).

Definition 2.1.5. A morphism $f\colon X \to S$ is called *formally smooth* (resp. unfamified, resp. étale) if for all first order thickenings $i\colon T_0 \to T$ of affine schemes and diagrams

$$\begin{tikzcd} T_0 \arrow[r, "u_0"] \arrow[d, "i"] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "\widetilde{u_0}"] & S \end{tikzcd}$$

there exists a lift $\widetilde{u_0}$ (resp. there is at most one such $\widetilde{u_0}$, resp. there exists a unique $\widetilde{u}_0$).

Example 2.1.6.

1. Open immersions are formally étale. This is cleaer because T_0, T have the same underlying topological space.

2. Closed immersions are formally unramified. This is clear because $X \to S$ induces an injection on T-points.

3. $\mathbb{A}^n_S \to S$ is formally smooth. To see this, assume $S = \operatorname{Spec} R$ is affine and then consider the corresponding lifting problem in commutative algebra.

Proposition 2.1.7. *The classes of formally smooth (resp. étale, resp. unramified) morphisms are closed under base change, composition, and products and local on both source and target.*

Definition 2.1.8. A $f\colon X \to S$ is *smooth* if it is formally smooth and locally of finite presentation.

We will now consider differentials. Let $X = \operatorname{Spec} A$ be an affine scheme over k and choose a k-point and consider the diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k[\varepsilon] \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

If X is smooth, then there exists a lift $\operatorname{Spec} k[\varepsilon] \to X$. But this is given by a morphism

$$\widetilde{\phi}\colon A \to k[\varepsilon]/\varepsilon^2 \qquad a \mapsto \phi(a) + d(a)\varepsilon.$$

This motivates the following definition:

Definition 2.1.9. Let $R \to A$ be a morphism of rings and M be an A-module. A *derivation* $d\colon A \to M$ is an A-linear map satisfying the Leibniz rule.

Proposition 2.1.10. *There exists an* A*-module* $\Omega^1_{A/k}$ *equipped with a derivation* $d\colon \Omega^1_{A/k}$ *that is universal among derivations from* A*. This means that all derivations* $\widetilde{d}\colon A \to M$ *factor through* d*, and formally, we have an identity*

$$\mathrm{Der}_R(A, M) \simeq \mathrm{Hom}_A(\Omega^1_{A/k}, M).$$

Definition 2.1.11. For an A-module M with derivation $d\colon A \to M$, define the ring $A[M]$ as the module $A \oplus M$ with the multiplication

$$(a, m) \cdot (a', m') = (aa', am' + a'm).$$

There is a sequence $\phi\colon A \to A[M] \to A$.

Proposition 2.1.12. *Let* $S \leftarrow R \to A \to B$ *be a diagram of rings. Then*

1. $\Omega^1_{A \otimes S/S} \simeq \Omega^1_{A/R} \otimes_R S$*;*
2. *The sequence* $\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$ *is exact.*
3. *If* $B = A/I$ *for some ideal* I*, we have an exact sequence*
$$I/I^2 \to \Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to 0.$$
4. *For all* $f \in A$*, we havae* $\Omega^1_{A[f^{-1}]/R} \simeq \Omega^1_{A/R} \otimes_A A[f^{-1}]$*.*

Remark 2.1.13. If $J = \ker(A \otimes_R A \to A)$, then $\Omega^1_{A/R} = J/J^2$.

Theorem 2.1.14. *Let* $f\colon X \to S$ *be locally of finite presentation. The following are equivalent:*

1. f *is smooth;*
2. f *is flat with smooth fibers;*
3. f *is flat and has smooth geometric fibers.*

We will finally return to deformation theory.

Lemma 2.1.15. *Let* Z_0 *be a closed subscheme of* Z *determined by a nilpotent ideal sheaf* N*. If* Z_0 *is affine, then so is* Z*.*

Proof of this result can be found in EGA, Chapter I.5.9.

Proof of Theorem 2.1.3. Recall that we have a diagram of the form

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \uparrow & & \uparrow \\ A & \longrightarrow & k, \end{array}$$

where $A \to B$ is flat and $B_0 \simeq B \otimes_A k$ is a smooth k-algebra. We need to prove that $B_0 \simeq B \otimes_A k$. The first step is to prove this result for first-order deformations. Suppose that $A = k[\varepsilon]$ is a square-zero extension.

Lemma 2.1.16. *For a ring* R *with* M, N *flat over* R*, nilpotent ideal* $I \subset R$*, and* $f\colon M \to N$*, then if* $f \otimes_R R/I$ *is an isomorphism, then so is* f*.*

To prove the lemma, note that the cokernel of f is preserved by I, so it must vanish. Returning to our case, we know that B is a smooth $k[\varepsilon]$-algebra. Now we obtain a square-zero extension $B_0[\varepsilon]$ of B_0 and a diagram

$$\begin{tikzcd} B \arrow[r] & B_0 \\ k[\varepsilon] \arrow[r, "f"] & B_0[\varepsilon] \arrow[u] \end{tikzcd}$$

with a lift $B \to B_0[\varepsilon]$. But now by the lemma, we have $B \otimes_{k[\varepsilon]} k = B_0[\varepsilon] \otimes_{k[\varepsilon]} k$. The rest of the proof follows using an inductive argument that was verbalized but now written down. □

2.2 Deformations of schemes

The main theorem of this section is

Theorem 2.2.1. *Assume X is a smooth R-scheme. Then there is a bijection*

$$\mathrm{Def}_X^{\mathrm{sm}}(k[I]) \simeq H^1(X, T_{X/k} \otimes I).$$

Proof. Let $\mathcal{X}'$ be a smooth deformation over $k[I]$. Then the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[I] \end{tikzcd}$$

is cartesian. Then if $U_k = \operatorname{Spec} B_k$ is an affine cover of X and $U'_k = \operatorname{Spec} D_k$ is an affine cover of X', we have a $k[I]$-linear ring isomorphism

$$\varphi_k \colon k[I] \otimes_k B_k \to D_k \qquad (k, i) \otimes b \mapsto s(b) + i.$$

Modulo I, φ_k is the identity on B_k. Without loss of generality, we may assume that $U_{kj} = U_k \cap U_j$ is a distinguished open for both U_k and U_j, so let $U_{kj} = \operatorname{Spec} B_{kj}$ and $U'_{kj} = \operatorname{Spec} D_{kj}$. Now note that both

$$\varphi_k, \varphi_j \colon k[I] \otimes_k B_{kj} \to D_{kj}$$

induce the identity on B_{kj} modulo I. Now we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{id}"] & D_{kj} \arrow[r] \arrow[d, "\varphi_j^{-1}\varphi_k"] & B_{kj} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & I \arrow[r] & D_{kj} \arrow[r] & B_{kj} \arrow[r] & 0. \end{tikzcd}$$

Lemma 2.2.2. *The morphism $g = \varphi_j^{-1}\varphi_k$ must be of the form*

$$g(i + b) = i + b + \delta(b),$$

where $\delta \colon B_{kj} \to I$ is a derivation.

In particular, this means that $\varphi_j^{-1} \circ \varphi_k(b, b') = (b, \alpha_{kj}(b) + b')$, where $\alpha_{kj} \colon B_{kj} \to I \otimes_k B_{kj}$ is a derivation.

By definition, we have

$$\begin{aligned}(T_{X/k} \otimes_k I)(B_{kj}) &= \mathrm{Hom}_{B_{kj}}(\Omega^1_{B_{kj}/k}, B_{kj}) \otimes_k I \\ &= \mathrm{Hom}_{B_{kj}}(\Omega^1_{B_{kj}/k}, B_{kj} \otimes_k I) \\ &= \mathrm{Der}_k(B_{kj}, B_{kj} \otimes_k I).\end{aligned}$$

Therefore, $\alpha_k \in H^0(B_{kj}, T_X \otimes_k I)$. Note that

$$\varphi_\ell^{-1} \circ \varphi_j \circ \varphi_j^{-1} \circ \varphi_k^{-1} = \varphi_\ell^{-1} \circ \varphi_k,$$

which implies that

$$(b, \alpha_{j\ell}(b) + \alpha_{kj}(b) + b') = (b, \alpha_{k\ell}(b) + b')$$

and thus $\{\alpha_{kj}\} \in Z^1(\{U_k\}, T_X \otimes_k I)$.

If two deformations are the same, note that φ_k is defined using a ring section $s_k \colon B_k \to D_k$ of the canonical map $\pi_k \colon D_k \to B_k$. If φ_k' is defined using another section s_k', then define $\theta_k = s_k' - s_k \in \mathrm{Der}(B_k, I \otimes_k B_k)$. We now compute

$$((\varphi_j')^{-1} \circ \varphi_k' - \varphi_j^{-1} \circ \varphi_k)(b, b') = (0, \theta_k(b) - \theta_j(b)),$$

and thus the two differ by the desired coboundaries. $\square$

We will now consider some obstructions. We are looking for a diagram of the form

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & X'' \arrow[d] \\ \mathrm{Spec}\, A' \arrow[r] & \mathrm{Spec}\, A''. \end{tikzcd}$$

for each pair (j, k), we have a isomorphism $\psi_{jk} \colon V_j' \to V_k'$ and a cocycle

$$c_{jk\ell} = \psi_{k\ell} \circ \psi_{jk} \circ \psi_{j\ell}^{-1}.$$

This induces $B_{jk\ell} \in \mathrm{Der}_A(D_{jk\ell}, J \otimes_A D_{k\ell}) = Z^2(U, T_{X'/A} \otimes_A J)$.

Now we will discuss some examples.

Theorem 2.2.3. *Let C be a smooth projective curve, $T = T_C$, and $K = \Omega^1_C$. We have the following table:*

Table 2.1: Cohomology

	degree	h^0	h^1	h^2
K	$2g-2$	g	1	0
T	$2-2g$	ε	$\varepsilon+3g-3$	0

where $\varepsilon = 0$ where $g \geqslant 2$, $\varepsilon = 1$ if $g = 1$, and $\varepsilon = 3$ if $g = 0$.

For $g \geqslant 2$, $\deg T < 0$, and by Riemann-Roch and Serre duality, we have $h^1(C, T_C) = 3g - 3$.

Theorem 2.2.4. $\mathbb{P}^n$ *has no infinitesimal deformations.*

Proof. Consider the Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

and use the long exact sequence in cohomology. Because positive degree line bundles have no higher cohomology, we have $H^1(T_{\mathbb{P}^n}) = 0$. $\square$