GENERALITIES ON ORBIFOLD COHOMOLOGY AND TORIC DM STACKS

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Abstract. I will explain various technicalities in Gromov-Witten theory for Deligne-Mumford stacks and how to construct toric Deligne-Mumford stacks from (extended) stacky fans.

Contents

1. Orbifold Gromov-Witten theory

Let X be a smooth and separated Deligne-Mumford stack of finite type over $\mathbb{C}$.

Definition 1.1. The *inertia stack* of X is the fiber product in the diagram

$$\begin{tikzcd} IX \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

More concretely, we may think about $|X|$ as parameterizing pairs (x, g), where $x \in X$ and $g \in \operatorname{Aut}(x)$. There is another description of IX if X lives over $\mathbb{C}$. In general, IX is disconnected. We will write

$$IX = \bigsqcup_{i \in I} X_i.$$

It also has an important morphism $\operatorname{inv}\colon IX \to IX$ given by $(x, g) \mapsto (x, g^{-1})$.

Definition 1.2. A morphism $X \to Y$ of algebraic stacks is *representable* if for all schemes S and morphisms $S \to Y$, the fiber product $X \times_S Y$ is an algebraic space.

Theorem 1.3. *Let*

$$I_\mu X \coloneqq \bigsqcup_{r \geqslant 0} \operatorname{Hom}_{\mathrm{rep}}(B\mu_r, X)$$

Date: April 4, 2024.

denote the stack of representable morphisms from classifying stacks of roots of unity to X *(the cyclotomic inertia stack). Then* $I_\mu X \simeq IX$.

We need to make one more definition, which will appear as a degree shift on cohomology. Let $(x, g) \in X_i$. Because $\langle g\rangle \subset \mathrm{Aut}(x)$ is cyclic, there is a decomposition

$$T_x X = \bigoplus_{0 \leqslant \ell < r_i} V_\ell,$$

where V_ℓ is the eigenspace with eigenvalue $e^{2\pi\sqrt{-1}\frac{\ell}{r_i}}$ and r_i is the order of g. Then the function

$$\mathrm{age} := \frac{1}{r_i} \sum_{0 \leqslant \ell < r_i} \ell \cdot \dim V_\ell$$

is constant on X_i, so we denote its value by $\mathrm{age}(X_i)$.

Recall that by the Keel-Mori theorem, X (which has finite inertia) has a coarse moduli space $|X|$, which is an algebraic space satisfying two properties:

- The morphism $\pi\colon X \to |X|$ is bijective on k-points whenever k is an algebraically closed field;
- $|X|$ is initial for morphisms from X to any algebraic space.

From now on, we will assume that $|X|$ is quasiprojective, and in particular that it is a scheme.

1.1. **Moduli of stable maps.**

Definition 1.4. The moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parameterizes objects

$$\begin{array}{ccc} (C, \{\Sigma_i\}) & \xrightarrow{f} & X \\ \downarrow & & \\ T, & & \end{array}$$

where

(1) C is a prestable balanced twisted curve of genus g. This means that C has stacky structure only at nodes and marked points, and the nodes are formally locally $[(\mathbb{C}[x, y]/xy)/\mu_r]$, where μ_r acts by $\zeta(x, y) = (\zeta x, \zeta^{-1} y)$;
(2) $\Sigma_i \subset C$ is an étale cyclotomic gerbe over T with a trivialization for all i;
(3) $f\colon C \to X$ is representable and the induced morphism between coarse moduli spaces is a stable map of degree β with n marked points.

We see that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has evaluation maps $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to IX$. It is also disconnected, with the connected components being indexed by components of IX. Let

$$\overline{\mathcal{M}}_{g,n}(X, \beta, i_1, \ldots, i_n) := \bigcap_{j=1}^{n} \mathrm{ev}_j^{-1}(X_{i_j}).$$

Then

$$\overline{\mathcal{M}}_{g,n}(X, \beta) = \bigsqcup_{i_1,\ldots,i_n} \overline{\mathcal{M}}_{g,n}(X, \beta, i_1, \ldots, i_n).$$

Each component has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X, \beta, i_1, \ldots i_n)]^{\mathrm{vir}} \in H_*(\overline{\mathcal{M}}_{g,n}(X, \beta, i_1, \ldots, i_n), \mathbb{Q})$$

of virtual dimension

$$\int_\beta c_1(X) + (1-g)(\dim X - 3) + n - \sum_{j=1}^{n} \mathrm{age}(X_{i_j}).$$

given by the relative perfect obstruction theory $(R\pi_* f^* TX)^\vee$, where $\pi\colon C \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ is the universal curve, over the moduli stack $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ of prestable twisted curves. Because we chose to work with trivialized gerbe markings, we need to multiply the virtual fundamental class as follows. Note that the j-th marked point is

$$\Sigma_j \cong \overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots,i_n) \times B\mu_{r_{i_j}}.$$

Here, if $x = [B\mu_r \to X] \in X_{i_j} \subset IX$, then $r_{i_j} = r$. Then set

$$[\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots i_n)]^w := \left(\prod_{j=1}^{n} r_{i_j}\right)[\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots i_n)]^{\mathrm{vir}}.$$

Now consider the morphism $p\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(|X|,\beta)$ given by taking the coarse moduli space. Let $C_{|X|} \to \overline{\mathcal{M}}_{g,n}(|X|,\beta)$ be the universal curve and $\sigma_{i,|X|}$ be the marked points. Then the descendant classes[1] are defined to be

$$\psi_j := p^* c_1(\sigma_j^* \omega_{C_{|X|}/\overline{\mathcal{M}}_{g,n}(|X|,\beta)}).$$

1.2. **Quantum cohomology.** We are now able to define Gromov-Witten invariants. Let $\alpha_j \in H^{p_j}(X_{i_j},\mathbb{C})$. Then define

$$\left\langle \alpha_1\psi^{k_1},\dots,\alpha_n\psi^{k_n}\right\rangle^X_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta,i_1,\dots,i_n)]^w} \prod_{j=1}^{n} \mathrm{ev}_j^*\,\alpha_j\psi_j^{k_j}.$$

We are still able to form generating series $\mathcal{F}_g, J_X, \dots$ as before, and the invariants satisfy the string, dilaton, and divisor equations (although we have to be careful that the marked point we delete is a scheme point), so the orbifold Gromov-Witten theory has a Lagrangian cone $\mathcal{L}_X \subset \mathcal{H}$.

The *orbifold Poincaré pairing* is defined by the formula

$$(\alpha,\beta) := \int_{IX} \alpha \cup \mathrm{inv}^*\,\beta,$$

where $\cup$ denotes the usual cup product. This is well-defined because of the formula

$$\mathrm{age}(X_i) + \mathrm{age}(X_{\mathrm{inv}(i)}) = \dim X - \dim X_i$$

when X is proper. When X is not proper, we will assume we are working equivariantly. Now we may define the *quantum product* by the formula

$$(a \star_\tau b, c) := \sum_{n,\beta} \frac{Q^\beta}{n!}\langle a,b,c,\tau,\dots,\tau\rangle^X_{0,n+3,\beta}$$

for $a,b,c,\tau \in H^*(IX,\mathbb{C})$. Restricting to the degree 0 part and setting $\tau = 0$, we obtain the *orbifold cup product*, which is given by

$$(a \star b, c) = \langle a,b,c\rangle^X_{0,3,0}.$$

[1] Most people call these $\overline{\psi}$, but I am extremely lazy.

Denote $H^*_{CR}(X) \coloneqq (H^*(IX, \mathbb{C}), \cup)$. Note that this product is graded for the grading $\deg(a) = p + 2\,\mathrm{age}(X_i)$ for $a \in H^p(X_i)$. Using the quantum product, we may define the quantum connection and its fundamental solution.

2. Toric Deligne-Mumford stacks

We will assume the reader is familiar with the fan presentation of a toric variety. If you are not, there are many references.

Definition 2.1. An *extended stacky fan* is a quadruple $\boldsymbol{\Sigma} = (N, \Sigma, \beta, S)$ of

(1) A finitely generated abelian group N of rank n;
(2) A rational simplicial fan Σ in $N_{\mathbb{R}} = N \otimes \mathbb{R}$;
(3) A homomorphism $\beta \colon \mathbb{Z}^m \to N$. We will write $b_i = \beta(e_i) \in N$ for the image of the standard basis vector $e_i \in \mathbb{Z}^m$ and $\overline{b}_i$ for its image in $N_{\mathbb{R}}$;
(4) A subset $S \subset \{1, \dots, m\}$

satisfying the following conditions:

(1) The set $\Sigma(1)$ of 1-dimensional cones is exactly the set $\left\{\mathbb{R}_{\geqslant 0} \cdot \overline{b}_i \mid i \notin S\right\}$;
(2) For all $i \in S$, $\overline{b}_i \in |\Sigma|$.

We will now assume that $|\Sigma|$ is convex and full-dimensional and, that there is a strictly convex piecewise linear function $f \colon |\Sigma| \to \mathbb{R}$ which is linear on each cone, and that β is surjective. From this data, we will now obtain a GIT presentation. Define $\mathbb{L}$ by the exact sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\beta} N \to 0.$$

Then define $K \coloneqq \mathbb{L} \otimes \mathbb{C}^\times$. Then define $D_i \in \mathbb{L}^\vee$ to be the image of the i-th standard basis vector in $(\mathbb{Z}^m)^\vee$ under the last arrow in the exact sequence

$$0 \to N^\vee \to (\mathbb{Z}^m)^\vee \to \mathbb{L}^\vee$$

Finally, set

$$\mathcal{A}_\omega = \left\{I \subset \{1, \dots, m\} \mid S \subset I, \sigma_{\overline{I}} \text{ is a cone of } \Sigma\right\}.$$

Choose a stability condition

$$\omega \in C_\omega \coloneqq \bigcup_{I \in \mathcal{A}_\omega} \left\{\sum_{i \in I} a_i D_i \mid a_i \in \mathbb{R}_{>0}\right\}.$$

Then we define

$$X_{\boldsymbol{\Sigma}} \coloneqq [(\mathbb{C}^m)^s / K].$$

The ample cone is $C'_\omega \subset \mathbb{L}^\vee_{\mathbb{R}} / \sum_{i \in S} \mathbb{R} D_i \cong H^2(X_{\boldsymbol{\Sigma}}, \mathbb{R})$, which is defined in the same way as C_ω after deleting S from the extended stacky fan, and the cone of effective curve classes is its dual.

2.1. **Orbifold cohomology.**

First, we will describe the equivariant cohomology of $X_{\boldsymbol{\Sigma}}$. Let $\mathcal{Q} = (\mathbb{C}^\times)^m / K$. Then if u_i is Poincaré dual to $(x_i = 0 \subset (\mathbb{C}^m)^s)/K$, we have

$$H^*_{\mathcal{Q}}(X_{\boldsymbol{\Sigma}}, \mathbb{C}) = H^*_{\mathcal{Q}}(\mathrm{pt}, \mathbb{C})[u_1, \dots, u_m] / (\mathfrak{I} + \mathfrak{J}),$$

where

$$\mathfrak{I} := \left\langle \chi - \sum_{i=1}^{m} \langle \chi, b_i \rangle u_i \mid \chi \in N_{\mathbb{C}}^{\vee} \right\rangle$$

$$\mathfrak{J} := \left\langle \prod_{i \notin I} u_i \mid I \notin \mathcal{A}_\omega \right\rangle.$$

There is a combinatorial description of the components of the inertia stack $IX_{\boldsymbol{\Sigma}}$. Because $X_{\boldsymbol{\Sigma}}$ is a global quotient, the components of the inertia stack correspond to elements $g \in K$ such that $((\mathbb{C}^m)^s)^g$ is nonempty. Equivalently, if we define

$$\mathbb{K} := \{ f \in \mathbb{L} \otimes \mathbb{Q} \mid \{ i \in \{1, \dots, m\} \mid D_i \cdot f \in \mathbb{Z} \} \in \mathcal{A}_\omega \},$$

then the components of $IX_{\boldsymbol{\Sigma}}$ are in bijection with $\mathbb{K}/\mathbb{L}$. To give a description in terms of the fan, for any $\sigma \in \Sigma(n)$, define

$$\mathrm{Box}(\sigma) := \left\{ v \in N \mid \bar{v} = \sum_{\rho_i \subseteq \sigma} a_i \bar{b}_i \mid 0 \leqslant a_i < 1 \right\}$$

and then

$$\mathrm{Box}(\boldsymbol{\Sigma}) := \bigcup_{\sigma \in \Sigma(n)} \mathrm{Box}(\sigma).$$

Then there is a natural bijection $\mathbb{K}/\mathbb{L} \cong \mathrm{Box}(\boldsymbol{\Sigma})$. For any $f \in \mathbb{K}/\mathbb{L}$, X_f is a toric DM stack with $K, \mathbb{L}, \omega$ the same as for X_ω and characters D_i for i such that $D_i \cdot f \in \mathbb{Z}$. At the level of fans, this corresponds to killing the minimal cone of Σ containing the corresponding $\bar{v}$.

We will now give the orbifold cohomology of $X_{\boldsymbol{\Sigma}}$. Define the *deformed group ring* $\mathbb{C}[N]^{\boldsymbol{\Sigma}}$ as the vector space $\mathbb{C}[N]$ with product given by

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1 + c_2} & \text{there exists } \sigma \in \Sigma \text{ such that } \bar{c}_1, \bar{c}_2 \in \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Then there is an isomorphism of rings

$$H^*_{\mathrm{CR}}(X_{\boldsymbol{\Sigma}}) \cong \frac{\mathbb{C}[N]^{\boldsymbol{\Sigma}}}{\left\langle \sum_{i \notin S} \chi(b_i) y^{b_i} \mid \chi \in N^\vee \right\rangle}.$$

Remark 2.2. This result also works in families over a base B, where $\mathbb{C}^m$ is replaced by a direct sum of m line bundles on B. Then we need to add a $c_1(L_\chi)$ to the relations and obtain

$$H^*_{\mathrm{CR}}(X^B_{\boldsymbol{\Sigma}}) := \frac{H^*(B)[N]^{\boldsymbol{\Sigma}}}{\left\langle c_1(L_\chi) + \sum_{i \notin S} \chi(b_i) y^{b_i} \mid \chi \in N^\vee \right\rangle}.$$

3. Gamma-integral structure

Let $IX = \coprod_{v \in B} X_v$ and $q_v \colon X_v \to X$ be the restriction of $IX \to X$. Let E be a T-equivariant vector bundle on X. Recall that v corresponds to some $g_v \in K$, so we obtain an eigenbundle decomposition

$$q_v^* E = \bigoplus_{0 \leqslant f < 1} E_{v,f},$$

where $E_{\nu,f}$ is the subbundle where g_ν acts by $e^{2\pi i f}$. We now define the orbifold Chern character to be

$$\widetilde{\mathrm{ch}}(E) = \bigoplus_{\nu\in B} \sum_{0\leqslant f<1} e^{2\pi i f} \mathrm{ch}(E_{\nu,f}).$$

Now let $\delta_{\nu,f,i}$ be the Chern roots of $E_{\nu,f}$. We define the orbifold Todd class to be

$$\widetilde{\mathrm{Td}}(E) := \bigoplus_{\nu\in B} \left(\prod_{0<f<1} \prod_i \frac{1}{1-e^{-2\pi i f - \delta_{\nu,f,i}}} \right) \prod_i \frac{\delta_{\nu,0,i}}{1-e^{-\delta_{\nu,0,i}}}.$$

The $\widehat{\Gamma}$-class should be a square root of this and is defined by

$$\widehat{\Gamma}(E) = \bigoplus_{\nu\in B} \prod_{0\leqslant f<1} \prod_i \Gamma(1-f+\delta_{\nu,f,i}),$$

where we expand Γ around $1-f$. The reflection formula for the Γ-function implies that the X_ν-component of $\widehat{\Gamma}(E^\vee) \cup \widehat{\Gamma}(E)$ is given by

$$[\widehat{\Gamma}(E^\vee) \cup \widehat{\Gamma}(E)]_\nu = (2\pi i)^{\mathrm{rk}(q_\nu^* E)^{\mathrm{mov}}} \left[e^{-\pi i (\mathrm{age}(q^*E) + c_1(q^*E))} (2\pi i)^{\frac{\deg_0}{2}} \widetilde{\mathrm{Td}}(E) \right]_{\mathrm{inv}(\nu)}.$$

Here, $\deg_0$ is the grading operator given by the degree without age shifting.

Definition 3.1. Define the K-*group framing* $\mathfrak{s} \colon K_T(X) \to H^*_{CR,T}(X) \otimes_{R_T} R_T[\log z](\!(z^{-\frac{1}{k}})\!)[\![Q,\tau]\!]$ by the formula

$$\mathfrak{s}(E)(\tau,z) := \frac{1}{(2\pi)^{\frac{\dim X}{2}}} L(\tau,z) z^{-\mu} z^{\rho} \widehat{\Gamma}_X \cup (2\pi i)^{\frac{\deg_0}{2}} \mathrm{inv}^* \widetilde{\mathrm{ch}}(E),$$

where $L(\tau,z)$ is the fundamental solution to the quantum connection, μ is the usual grading operator given by $\frac{1}{2}(\deg - \dim X)$ on homogeneous elements, and $\rho = c_1(TX) \in H^2(X)$.

Proposition 3.2. *Define the equivariant Euler pairing by*

$$\chi(E,F) := \sum_i (-1)^i \mathrm{ch}^T(\mathrm{Ext}^i(E,F))$$

and the modified version $\chi_z(E,F)$ *by replacing the equivariant parameters* λ_j *by* $\frac{2\pi i \lambda_j}{z}$. *Then*

$$(\mathfrak{s}(E)(\tau, e^{-i\pi} z), \mathfrak{s}(F)(\tau,z)) = \chi_z(E,F).$$

Remark 3.3. Everything we have discussed so far makes sense for toric DM stacks after specializing $Q = 1$.