Math 797RT
Representation Theory

Notes by Patrick Lei, May 2020

Lectures by Ivan Mirkovic, Fall 2018

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Disclaimer

These notes are a May 2020 transcription of handwritten notes taken during lecture in Fall 2018. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style (omit lengthy computations, use category theory) and that of the instructor. If you find any errors, please contact me at `plei@umass.edu`.

Additional Disclaimers

1. Ivan was in China for a week, and I missed one of the two makeup lectures. If there are any gaps in the material, this is probably a big reason why.
2. I did not take notes for the final lecture. I believe only the classification of semisimple Lie algebras was given.
3. The course has been redesigned since I took it, so these notes may not be useful. In fact, I'm told that there is no set syllabus for this course, so it may continue to change in the future.
4. Ivan has notes at `https://people.math.umass.edu/~mirkovic/C.Graduate/TopicsCourses/7.RepresentationTheory/8.pdf`, but his notes are much more extensive and in depth than the lectures. My notes are a more faithful representation of what actually occurred during lecture.
5. Ivan's lectures assumed knowledge far beyond the stated prerequisites of the course, which were just graduate algebra. It may be clear that I did not understand what was going on half the time.

Contents

1

Groups

1.1 Overview

We begin with an overview of various aspects of representation theory.

1. First, the principle that any subject should be organized by its symmetries (harmonic analysis). We want to spot symmetries in a problem, and this is related to physics and QFT.

2. We want to consider symmetries related to subtle actions of space. For example, for a scheme X and sufficiently nice group scheme G, we can consider the quotient stack X/G. The projection from X is a principle G-bundle, and in fact, we can do this for $X = \operatorname{Spec} k$, which is a point. Also, we will need the idea of the fiber product $X \times_Z Y$ later.

3. Groups are used to describe objects that are locally trivial but not globally (for example cohomology classes).

4. Groups can be recovered from their categories of representations via Tannaka-Krein duality by considering automorphisms of the forgetful functor $\operatorname{Rep} G \to \mathrm{Vec}$.

5. The most important class of groups is the compact Lie groups.

Recall the notion of a group action on a set X. If we view $X \in \mathscr{C}$ for some category $\mathscr{C}$, then we want our group G to act by automorphisms in $\mathscr{C}$. Therefore we can define topological group actions on topological spaces and actions of groups on vector spaces.

Lemma 1.1. *For a group G and vector space V, then an action of G on V is equivalent to a morphism $\pi : G \to GL(V)$.*

Example 1.2. If G acts on a set X, then the space $\mathscr{O}(X) = \operatorname{Hom}(X, k)$ of functions on X is a representation of G.

We can generalize the above example to where X has additional structure, where we can consider functions in various categories. For example, if we consider $G = SL_2(\mathbb{R})$ acting on the upper half plane $\mathscr{H}$, then we can consider the space $\mathscr{O}_{\mathscr{H}}(\mathscr{H})$ of holomorphic functions on $\mathscr{H}$.

1.2 Category of Representations

Lemma 1.3. *1. Representations of G on vector spaces over a field k form a category* $\mathrm{Rep}_G(k) = \mathrm{Rep}(G)$. *Note that morphisms are*

$$\mathrm{Hom}_{\mathrm{Rep}\,G}(U, V) = \{\varphi : U \to V \mid g_V \circ \varphi = \varphi \circ g_U \text{ for all } g \in G\}.$$

2. $\mathrm{Rep}_G(k)$ *is an abelian category.*

   Recall that an abelian category is a category with the following:

   - *A zero object* 0. *Here, just take the zero representation.*
   - *A direct sum (coproduct) operation $\oplus$. Here, take $g(u \oplus v) = gu \oplus gv$.*
   - *Subobjects, which in this case are G-invariant subspaces.*
   - *Quotients by subobjects. This is obvious.*
   - $\mathrm{Rep}\,G$ *has all (co)kernels and (co)images, and images and coimages are isomorphic.*

   Some examples of abelian categories are categories of modules over a ring, but in fact $\mathrm{Rep}\,G$ *is equivalent to* $k[G]$-Mod.

3. $\mathrm{Rep}\,G$ *is a monoidal category. For module categories, this is only true in general for commutative rings. For two representations U, V of G, we can define an action on $U \otimes V$ by $g(u \otimes v) = gu \otimes gv$.*

   Recall that a monoidal structure on a category $\mathscr{C}$ is an associative and unital functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$. The unit in $\mathrm{Rep}\,G$ *is k with the trivial action.*

4. $\mathrm{Rep}(G)$ *is a closed monoidal category. This means that there is an internal Hom functor*

   $$- \otimes - : \underline{\mathrm{Hom}} : C^{\mathrm{op}} \times C \to C$$

   that is the right adjoint to the tensor product.

Examples 1.4. We give some examples of closed monoidal categories.

1. In the category of sets, we see that

   $$\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

   by the maps

   $$f := A \times B \to C \leftrightarrow (a \mapsto f(a, -)).$$

2. In the category of vector spaces, we again note that

   $$\mathrm{Hom}(A \otimes B, C) \simeq \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

   by the same currying morphism.

We can check that the internal Hom in the category of representations is simply the vector space Hom. To see that this is a representation, define the action $(g\varphi)(u) = g\varphi(g^{-1}u)$.

Recall that a right adjoint $G : B \to A$ to a functor $F : A \to B$ satisfies

$$\mathrm{Hom}_B(Fa, b) \simeq \mathrm{Hom}_A(a, Gb)$$

as sets in a manner that is natural in a, b. Being a closed monoidal category means that $- \otimes b$ is a left adjoint to $\underline{\mathrm{Hom}}(b, -)$.

1.3 Representations of Finite Groups

First we consider invariants of $V \in \operatorname{Rep} G$, which correspond to morphisms from the trivial representation.

Lemma 1.5. *We have the two identities* $\operatorname{Hom}_G(k, V) = V^g$ *and* $\operatorname{Hom}_k(U, V)^G = \operatorname{Hom}_G(U, V)$.

Definition 1.6. A representation $V \in \operatorname{Rep} G$ is *irreducible* if $V \neq 0$ and has no nontrivial subrepresentations.

Then define Irr_G to be the set of isomorphism classes of irreducible representations of G. We will use the following strategy to understand $\operatorname{Rep} G$:

1. Find all irreducible representations;
2. Study each irreducible representation;
3. For any interesting representation, decompose it into irreducibles.

For a finite group G, consider the set of functions $\mathcal{O}(G) = \operatorname{Hom}(G, k)$. Then we have two subrepresentations $\mathcal{O}_+(G) := \mathcal{O}(G)^G$ and $\mathcal{O}_-(G) = \left\{f : G \to k \mid \sum_{g \in G} f(g) = 0\right\}$.

Lemma 1.7. *If the characteristic of* k *does not divide* $|G|$, *then* $\mathcal{O}(G) \simeq \mathcal{O}_+(G) \oplus \mathcal{O}_-(G)$.

Proof. The dimensions match up, so the only constant satisfying $c|G| = 0$ is $c = 0$ by assumption. □

Example 1.8. If $G = S_2 = \{e, a\}$, then $\mathcal{O}_\pm(S_2)$ are irreducible, but they are actually the same subrepresentation of $\mathcal{O}(S_2)$ over $\mathbb{F}_2$.

Definition 1.9. A representation V is *semisimple* if it is a direct sum of irreducibles.

Theorem 1.10 (Schur Lemma)**.** *Let* U, V *be two irreducible finite-dimensional representations of* G.

1. *If* $0 \neq \alpha \in \operatorname{Hom}_G(U, V)$, *then* α *is an isomorphism;*
2. $\operatorname{End}_G(U)$ *is a finite-dimensional division algebra;*
3. *If* $k = \bar{k}$, *then* $\operatorname{End}_G(U) = k$.

Proof.

1. Note that $\ker \alpha$ is a subrepresentation and U is irreducible, so $\ker \alpha = 0$. Similarly, the image is nonzero and is a subrepresentation of V, so it must be all of V.
2. Any nonzero endomorphism must be an isomorphism.
3. If $\alpha \in \operatorname{End}_G(U)$ is nonzero, it has an eigenvalue λ with eigenvector v. Then $\ker(\alpha - \lambda)$ is a nonzero subrepresentation, so it must be everything.

□

Corollary 1.11.

1. *V is semisimple if and only if it can be written as a direct sum*

$$V = \bigoplus_{U \in \operatorname{Irr}(G)} M_U \otimes U.$$

2. *Let* U *be irreducible. Then* $\operatorname{Hom}_G(U, V) = \operatorname{Hom}_G(U, U) \otimes M_U$.

Proof.

1. Note that if V is a direct sum of irreducibles, then we have

$$V = \bigoplus_{U \in \mathrm{Irr}(G)} \left(\bigoplus_i U_i \right) = \bigoplus U \otimes M_u.$$

2. Hom is an additive functor, so just use Schur's Lemma.

$\square$

Corollary 1.12. *If k is algebraically closed, then $\mathrm{Hom}_G(U, V) = M_U$ for V semisimple and U irreducible.*

Theorem 1.13. *Let G be abelian and k be algebraically closed. Then all finite-dimensional irreducible representations of G are one-dimensional.*

Proof. Let $V \in \mathrm{Irr}(G)$ and $g \in G$. Then g has an eigenvalue λ with eigenvector v. Therefore, $\ker(g - \lambda) = V$ is a subrepresentation because G is abelian, so all subspaces are invariant and thus $\dim V = 1$. $\square$

This tells us that $\mathrm{Irr}(G) = \mathrm{Hom}(G, k^*) =: \widehat{G}$. This is known as the group of *characters*[1] of G.

Lemma 1.14. *Subrepresentations of semisimple representations are semisimple.*

Proof. Let $V' \subseteq V$ be a subrepresentation of an irreducible representation. Then V' has an irreducible subrepresentation V_1' and thus $0 \neq \mathrm{Hom}(V_1', V) = \bigoplus_{i \in I} \mathrm{Hom}(V_1', V_i)$, so there exists i such that $\mathrm{Hom}(V_1', V_i) \neq 0$, so $V_1' \subseteq V$. Then $V'/V_1' \subseteq V/V_1' = \bigoplus_{j \neq i} V_j$, so we have an exact sequence

$$0 \to V_1' \to V' \to V'/V_1' \to 0.$$

Therefore $V = V_i \oplus C$, so $V' = (V_i \oplus C) \cap V'$. $\square$

Theorem 1.15.

1. *Sums of semisimple representations are semisimple.*
2. *V is semisimple iff and only if for every subrepresentation $V' \subseteq V$, there is another subrepresentation V'' such that $V = V' \oplus V''$.*

Proof. We will only prove the second part (the first part is obvious). Note that if the condition holds, then choose an irreducible subrepresentation V', apply the condition, and use induction on the dimension.

Now suppose V is semisimple. Then apply Lemma 1.14 to $V' = \bigoplus \mathrm{Hom}_G(U, V') \otimes U \subseteq \bigoplus \mathrm{Hom}_G(U, V) \otimes U$ and we see that there is a complement. $\square$

Theorem 1.16. *Let G be a final group and $k = \mathbb{C}$. Then every $V \in \mathrm{Irr}^{\mathrm{fd}}(G)$ has a G-invariant Hermitian inner product and is semisimple.*

Proof. To construct an invariant inner product $(-,-)$, just take any Hermitian inner product $\langle -,- \rangle$ and define

$$(u, v) = \sum_{g \in G} \langle gu, gv \rangle.$$

If we have an inner product, let $W \subseteq V$ be a subrepresentation. Then $W^\perp$ is also a subrepresentation, and clearly $W \oplus W^\perp = V$, so V is semisimple by Theorem 1.15. $\square$

[1]Note there is another notion of character that will appear in this course.

Theorem 1.17 (Maschke)**.** *If* $\operatorname{char} k \nmid |G|$*, then all finite-dimensional representatons of* G *are semisimple.*

We will now study in detail the characters $\operatorname{Hom}(G, k^*)$ of G.

Lemma 1.18.

1. $\operatorname{Hom}(G, k^*)$ *is a group with operation the multiplication of functions.*
2. $\operatorname{Hom}(G, k^*) = \operatorname{Irr}^{1d}(G)$ *because* $GL_1(k) = k^*$.
3. $\operatorname{Hom}(G, k^*) = \operatorname{Hom}(G^{ab}, k^*)$.

If A is abelian, then we call the group of characters $\widehat{A} \coloneqq \operatorname{Hom}(A, k^*)$ the *Pontryagin dual of* A.

Examples 1.19.

1. Note that $\widehat{\mathbb{Z}} = k^*$ because $\mathsf{Ab} = \mathbb{Z}\text{-}\mathsf{Mod}$.
2. We can see that $\widehat{\mathbb{Z}/n\mathbb{Z}} = \mu_n(k)$.
3. If k is algebraically closed, and $\operatorname{char} k \nmid n$, then $\widehat{\mathbb{Z}/n\mathbb{Z}} \simeq \mathbb{Z}/n\mathbb{Z}$.

Note that $\operatorname{Hom}(G, k^*) \times \operatorname{Hom}(H, k^*) \cong \operatorname{Hom}(G \times H, k^*)$ by the categorical properties of the product. Consider the functor $\mathsf{Ab}^{op} \to \mathsf{Ab}$ given by $A \mapsto \widehat{A} = \operatorname{Hom}(A, k^*)$. Note that there is a natural[2] morphism $\iota : A \to \widehat{\widehat{A}}$ given by $\iota(g)(\chi) = \chi(g)$ for $g \in A$.

Lemma 1.20 (Pontryagin Duality)**.** *If* k *is an algebraically closed field of characteristic* 0*, then* $\widehat{\ } : (\mathsf{Ab}^f)^{op} \to \mathsf{Ab}^f$ *is an equivalence of categories. In fact, the natural transformation* ι *defined above is an isomorphism.*

Proof. This is obviously true for finite cyclic groups, and then use the structure theorem for finite abelian groups and the properties of the product. ☐

Remark 1.21. If $\operatorname{char} k > 0$, then the result extends to finite abelian group schemes. If we take locally compact abelian groups and $\operatorname{Hom}(A, S^1)$, then this result still holds.

Now we will study representations of products of groups. Consider the box product

$$- \boxtimes - : \operatorname{Rep} G \times \operatorname{Rep} H \to \operatorname{Rep}(G \times H)$$

with the operation the tensor product with G acting from the left and H acting from the right.

Lemma 1.22. *Let* $W \in \operatorname{Irr}(G \times H)$*. Then there exist* $V \in \operatorname{Irr}(G)$ *and* $U \in \operatorname{Irr}(H)$ *such that* $W = V \boxtimes U$*.*

Proof. Let $V \in \operatorname{Rep} G$. Then consider the morphism $ev : V \otimes \operatorname{Hom}_G(V, W) \to W$. We claim this is a representation of H. To see this, define $(h\varphi)(v) = (1, h)(\varphi v)$, which commutes with the action of G on W. Clearly, evaluation is surjective, so we can consider $\ker(ev) = V \otimes K'$ for some representation k' of H. Therefore we can write

$$W = \frac{V \otimes \operatorname{Hom}(V, W)}{\ker(ev)} = V \otimes \frac{\operatorname{Hom}(V, W)}{K'} = V \otimes H'$$

for some $H' \in \operatorname{Irr}(H)$ (if H is reducible, then so is the entire representation). ☐

Lemma 1.23. *If* $V \in \operatorname{Irr} G$ *and* $U \in \operatorname{Irr} H$*, then* $V \boxtimes U \in \operatorname{Irr} G \times H$*.*

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[2] Yes, a natural transformation.

Proof. Consider an irreducible subrepresentation $V' \boxtimes U'$. Then

$$0 \neq \mathrm{Hom}_{G\times H}(V' \boxtimes U', V \boxtimes U) = \mathrm{Hom}_G(V', V) \otimes \mathrm{Hom}_H(U', U),$$

so $V' = V$ and $U' = U$. □

Theorem 1.24. *The box product gives an isomorphism* $\mathrm{Irr}\, G \times \mathrm{Irr}\, H \xrightarrow{\boxtimes} \mathrm{Irr}\, G \times H$.

Now let V be a representation of G and consider the *matrix coefficient* map

$$c^V : V \otimes V^k \to \mathscr{O}(G), \qquad c^V_{u\otimes\alpha}(g) = \langle gu, \alpha\rangle.$$

This is equivariant with respect to G^2, where $(g,h)f(x) = f(h^{-1}xg)$. Also, the matrix coefficient map $c^V : V \otimes V^* \to \mathscr{O}(G)$ is an embedding because V, V^* are both irreducible.

Corollary 1.25. *Any finite group G has finitely many irreducible representations.*

Theorem 1.26. *Let k be an algebraically closed field of characteristic* 0*. Then*

$$\mathscr{O}(G) \cong \bigoplus_{V \in \mathrm{Irr}\, G} V^* \boxtimes V$$

as a representation of G^2

Proof. We compute that as representations of $1 \times G$,

$$\mathrm{Hom}_{1\times G}(V, \mathscr{O}(G)) \otimes V = \mathrm{Hom}_1(V, k) = V^*$$

by Frobenius reciprocity. □

It is not hard to see that an explicit isomorphism can be given by c^V, which is the same thing as evaluation.

1.4 Characters

From now on, we will always work over the complex numbers. It turns out that each irreducible representation of G gives us one conjugation-invariant function $G \to \mathbb{C}$, the trace. Denote the *character* of a representation V by $\chi_V(g) := \mathrm{Tr}_V\, g$.

Lemma 1.27.

1. $\chi_{U\oplus V} = \chi_U + \chi_V$;
2. $\chi_{U\otimes V} = \chi_U \cdot \chi_V$.

These results follow from linear algebra of direct sums and tensor products.

2

Representation Theory of the Symmetric Group

Now we will construct irreducible representations of the symmetric group. Because irreducible representations correspond to conjugacy classes, there are bijections between the following:

1. Irreducible representations of S_n;
2. Conjugacy classes of S_n;
3. Partitions λ of n;
4. Young diagrams with n squares;
5. Conjugacy classes of nilpotent matrices of size n;
6. Finite (length n) subschemes of $\mathbb{A}^2$.

We want to construct an irreducible representation for each partition λ. For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$, write $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_m}$. We will write $M^\lambda = \mathrm{Coind}_{S_\lambda}^{S_n} \tau_\lambda$, where τ is the trivial representation. Similarly, define $N_\lambda = \mathrm{Coind}_{S_\lambda}^{S_n} \sigma_\lambda$, where σ is the sign representation.

Theorem 2.1. *If C, R are conjugate partitions, then there exists a unique irreducible representation π_C that is common to M^R and N^C.*

Now, we will consider the *dominance order* on the set Π_n of partitions of n. We say that $\lambda \geq \mu$ if $\lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$, and so on.

We may also consider a more geometric notion. Let $\mathcal{N}_n$ be the cone of nilpotent $n \times n$ matrices. Then every nilpotent matrix has a Jordan canonical form that corresponds to a shift operator e_λ (shift left on a Young diagram λ). Then $\mathcal{N}_n$ is a stratified space with strata given by orbits $\mathcal{O}_\lambda$ corresponding to partitions λ. Then we say that $\lambda \geq \mu$ if $\overline{\mathcal{O}}_\lambda \supseteq \mathcal{O}_\mu$.

Lemma 2.2. *The dominance and closure orders coincide.*

2.1 Geometry of GL_n

The correspondence between irreducible representations of S_n and orbits of nilpotent matrices under conjugation by GL_n, due to Springer, is the origin of *geometric representation theory*. This also extends to other reductive algebraic groups. For this, we will need to do some geometry.

First recall the *Grassmannian* $Gr_p(V)$ which parameterizes dimension p subspaces of V. Then we can consider generalized flag varieties $Gr_{1\leq p_1 \leq \cdots \leq p_k \leq n}$, which parameterizes flags of subspaces with dimension $p_1, \ldots, p_n$. Finally, we have the *flag variety*

$$\mathscr{F} = Gr_{1<2<\ldots<n} = \{U_1 \subset U_2 \subset \cdots \subset U_n = V \mid \dim U_i = i\}.$$

Lemma 2.3. *These spaces are all homogeneous spaces of GL_n which are smooth projective varieties.*

Now recall that the nilpotent cone $\mathscr{N}_n = \bigsqcup_{\lambda \in \Pi_n} \mathscr{O}_\lambda$ is a disjoint union of smooth pieces. By definition of the dominance order, we know that

$$\overline{\mathscr{O}}_\lambda = \bigcup_{\mu \leq \lambda} \mathscr{O}_\mu.$$

However, note that $\mathscr{N}_n$ itself is a singular space, so we should be able to extract information from the singularities.

Definition 2.4. Let X be a singular variety. Then a *resolution* of X is a proper birational map $\pi : \widetilde{X} \to X$ from a smooth variety $\widetilde{X}$.

Example 2.5. We can resolve the variety $(xy = 0) \subseteq \mathbb{C}^2$ by taking the disjoint union of two lines.

If we consider $\mathscr{N}_2$, then this is just a quadric cone in $\mathbb{A}^3$, so we can resolve by blowing up the origin to get $\widetilde{\mathscr{N}_2} = \left\{(e, L) \in \mathscr{N}_2 \times \mathbb{P}^1 \mid e \in L\right\}$.

Lemma 2.6. *$\mathscr{F} = GL_n/B_0$, where $B_0 \subseteq GL_n$ is the subgroup of upper-triangular matrices, also called the standard Borel. In addition, $\mathscr{F}$ has a natural identification with the space of Borel subgroups of GL_n.*

Remark 2.7. Note that GL_n is a nonabelian reductive group. G has a maximal Borel B_0, and we call the subgroup of B_0 with 1s on the diagonal N_0. Finally, we write $T_0 = B_0/N_0$, and this is called the *standard Cartan*, or the *maximal torus*, because $T_0 \simeq \mathbb{G}_m^n$.

2.2 Springer Theory for GL_n

We want to resolve $\mathscr{N}_n$ in general. To do this, we will consider $\mathfrak{g} = M_n(\mathbb{C}) = \mathfrak{gl}_n$. Then define

$$\tilde{\mathfrak{g}} = \{(x, F) \in \mathfrak{g} \times \mathscr{F} \mid xF_i \subseteq F_i\},$$

so we have a natural morphism $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ that forgets the flag. Similarly, define

$$\widetilde{\mathscr{N}} = \{(x, F) \in \mathscr{N} \times \mathscr{F} \mid xF_i \subseteq F_{i-1}\}.$$

Again, there is a map $\pi : \widetilde{\mathscr{N}} \to \mathscr{N}$. Write $\mathscr{F}_x = \pi^{-1}x$.

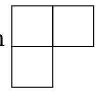
Lemma 2.8. *Let $s \in \mathfrak{g}_{rs}$ be a regular semisimple matrix. In other words, s has n distinct eigenvalues. Then $|\mathscr{F}_s| = n!$ and $\mathscr{F}_s$ is a torsor for S_n.*

Here are some examples of torsors:

1. The set of bases of $\mathbb{C}^n$ is a torsor for GL_n.
2. The Möbius strip with the zero section removed is a torsor for $\mathbb{R}^*$.
3. Rank n vector bundles over a base space B are the same thing as GL_n-torsors over B.

If G acts on X and Y, we can create a new G-set $X \times_G Y := (X \times Y)/\Delta G$, where the diagonal action of G is $g(x, y) = (xg^{-1}, gy)$. We can see that $X \times_G Y$ is a Y-bundle over X/G, so if X is a G-torsor, then $X \times_G Y \simeq Y$.

Note that the set of regular semisimple matrices is a Zariski-open subset of $\mathfrak{g}$. We now consider the special fibers:

1. Above $x = 0$, the fiber is simply all of $\mathscr{F}$.
2. If x corresponds to the Young diagram , then $\mathscr{F}_x$ is a point.
3. For $n = 3$, the fiber above the Young diagram  is two copies of $\mathbb{P}^1$ intersecting at a point.

Lemma 2.9. *As spaces, $\tilde{\mathfrak{g}} = G \times_B \mathfrak{b}$. This means that $\tilde{\mathfrak{g}}$ is a vector bundle over $G/B = \mathscr{F}$. Because $\mathfrak{b}$ is a representation of B, $H^0(G/B, \tilde{\mathfrak{g}})$ is a representation of G.*

Remark 2.10. These *Associated Bundle* constructions allow us to construct G-equivariant vector bundles from representations of B. If we consider the character (m, n) that sends $(\mathbb{C}^*)^2 \ni (a, b) \mapsto a^m b^n$ as a representation $\mathbb{C}_{m,n}$ of B, then we can define the vector bundle $V_{m,n} = G \times_B \mathbb{C}_{m,n}$. Then we can consider the representation $L_{m,n} = H^0(G/B, V_{m,n})$. Then each $L_{m,n}$ is either zero or irreducible or zero and all irreducible representations of G are of this type. This is the *Borel-Weil theorem.*

Example 2.11. For $G = GL_2$, then it is easy to see that $G/B = \mathbb{P}^1$.

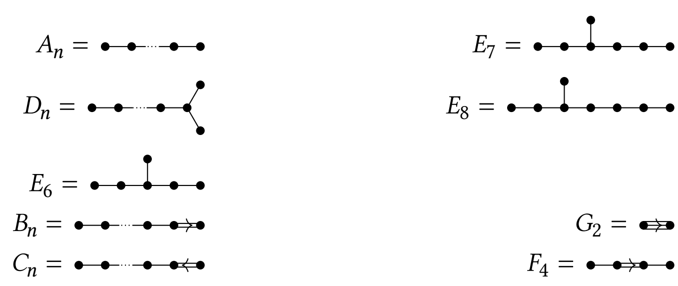
Lemma 2.12. *As spaces, $\widetilde{\mathcal{N}} = G \times_B \mathfrak{n} = T^*\mathscr{F}$.*

Lemma 2.13. *If G acts on X, then any G-equivariant map $X \to G/A$ identifies X with an associated bundle $G \times_A X_{eA}$.*

Note that there is a natural symplectic structure on $T^*\mathscr{F}$ and that $\widetilde{\mathcal{N}} \to \mathcal{N}$ is a resolution. Here, over a regular nilpotent matrix, the map is an isomorphism and the map is proper because fibers $\mathscr{F}$ is projective.

We will call the partition $(n-1, 1)$ *subregular* and the partition $2, 1^{n-2}$ minimal. In general, these classes exist for all semisimple Lie algebras and we can perform the Springer construction for all of them. We now guess that $\dim \pi_\lambda$ is the number of irreducible components of $\mathscr{F}_\lambda$.

Remark 2.14. Semisimple Lie algebras are the same thing as Dynkin diagrams, which are displayed below:



Theorem 2.15. *All irreducible components of Springer fibers are equidimensional.*

We now guess that $\pi_\lambda = \mathbb{C}[\mathrm{Irr}(\mathscr{F}_\lambda)]$. However, it is not clear what the action S_n is. For this, we will need a continuity argument. For this, we will need sheaf cohomology.

Theorem 2.16. *$\mathscr{F}_\lambda$ is paved. This means that it is a disjoint union of affine spaces.*

Being paved means that it is easy to compute the cohomology with coefficients in $\mathbb{Z}$.

2.3 Sheaves

A *presheaf* on a space X valued in a category $\mathscr{C}$ is a presheaf $X^{\mathrm{op}} \to \mathscr{C}$. A presheaf $\mathscr{F}$ is a *sheaf* if for an open set U and a cover U_i with sections $f_i \in \mathscr{F}|_{U_i}$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then there exists a unique section $f \in \mathscr{F}|_U$ such that $f|_{U_i} = f_i$.

Let $f : X \to Y$ be a morphism of spaces. Then there is a *pushforward*

$$(f_* \mathscr{F})(V) = \mathscr{F}(f^{-1}V)$$

and a *pullback*

$$(f^* \mathscr{G}(U)) = \varprojlim_{f(U) \subseteq V} \mathscr{G}(V).$$

There operations define functoriality for categories of sheaves.

If $\mathscr{C}$ is an abelian category, then so is $Sh(X, \mathscr{C})$, so we may consider complexes of sheaves. Then we may consider the derived category of complexes. In this setting, we have derived versions of f_*, f^* as well as the compactly supported $Rf_!$ and exceptional inverse image functor $Rf^!$.

Given a sheaf $\mathscr{F}$ and a point x, the *stalk* $\mathscr{F}_x$ is defined as the colimit

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}|_U.$$

Alternatively, given the inclusion $\iota : x \to X$, the stalk is simply $\iota^* \mathscr{F}$. Then we can define the *support* of a sheaf as the closure of the set of points with nonzero stalks.

Remark 2.17. The derived category is not an abelian category, but it is an *triangulated category*. This means we have exact triangles

$$A \to B \to C \to A[1].$$

There is a *Verdier duality*[1] functor $\mathbb{D}_X : D(X) \to D(X)^{\mathrm{op}}$ such that $\mathbb{D} f_* \mathbb{D} = f_!$ and $\mathbb{D} f^* \mathbb{D} = f^!$.

Example 2.18. If $a : X \to \mathrm{pt}$, then $a^* \mathscr{F} = \mathscr{F}_x$ and $a^! k = \omega_X$, the dualizing sheaf.

Under base change, we have the standard push-pull trick, and if f is proper, then $f_! = f_*$. If f is a smooth morphism, then $f^! = f^*[d_y - d_x] \otimes \mathrm{or}_f$, where *or* is the orientation sheaf.

Note that cohomology is simply the derived functor of global sections, so we will treat everything as a derived functor.

Definition 2.19. A *local system* on X is a locally constant sheaf, and a sheaf $\mathscr{F}$ is *constructible* if there exists a stratification $X = \bigsqcup S_i$ such that $\mathscr{F}|_{S_i}$ is a local system.

Remark 2.20. All of the derived category and functoriality constructions apply in the case of constructible sheaves.

2.4 Springer Fibers Again

Returning to our representation theory, we want to produce $a_* a^! \mathbb{C}$ from our setup. Therefore, we will push the constant sheaf $\mathbb{C}$ forward from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$. The resulting complex of sheaves is called the *Grothendieck sheaf*.

Remark 2.21. Note that S_n is embedded in GL_n and is embedded in the normalizer of the standard Cartan. In addition, $S_n \cong N_G(T)/T$.

[1] Ivan says this was probably invented by Grothendieck because it required a great mathematician

If $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is the projection that forgets the flag, denote the Grothendieck sheaf by $\mathscr{G} := \pi_* k$. Then the stalk $\mathscr{G}_x$ is the same as the cohomology $H^*(\mathscr{F}_x)$. We know that S_n acts on $\mathscr{G}_{\mathrm{rs}} = \mathscr{G}|_{\mathfrak{g}_{\mathrm{rs}}}$, so we need to extend this action.

Proposition 2.22. *$\mathscr{G}$ is obtained from its restriction to the regular semisimple matrices by a procedure called "intersection cohomology extension," and this works because π is "small," which means that the fibers increase slowly in some sense.*

Definition 2.23. A morphism $\pi : Y \to X$ is *semi-small* if $d_x = \dim \pi^{-1} x$ is generically zero and the closed set where $d_x \geq k$ has codimension at least $2k$. The morphism π is *small* if it is semi-small and the set of points with $d_x > 0$ has codimension at least 3.

Theorem 2.24. *The morphism $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is small.*

This means that S_n acts on the entire Grothendieck sheaf $\mathscr{G}$, so in particular, it acts on the stalks. This tells us that each stalk is a representation of S_n.

Lemma 2.25. *There is an isomprhism $\mathbb{C}[S_n] \simeq \mathrm{End}(\mathscr{G}) = H_{\mathrm{top}}(\widetilde{mfg} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})$.*

We conclude from our geometric construction that

1. Interesting objects have geometric realizations.
2. Geometric methods then give construction of irreducible representations.
3. This allows us to study Irr more efficiently.

3

Lie Algebras

3.1 Basic Notions

First, we will define manifolds over $k = \mathbb{R}, \mathbb{C}$. These are topological spaces M locally isomorphic to k^n with transition functions that are (C^k, analytic, holomphic). An example of a manifold is a Riemann surface. Vector fields are simply derivations of $\mathcal{O}_M$ and the tangent space at a is simply the derivations at that point.

Example 3.1. If $M = k^n$, then the space of vector fields is simply $V(M) = \sum \mathcal{O}_M(M)\frac{\partial}{\partial x_i}$ and the tangent spaces are $T_a M = \bigoplus k \left.\frac{\partial}{\partial x_i}\right|_a$.

The tangent sheaf is locally free, so it is the sheaf of sections of the *tangent bundle.* If $\mathcal{O}_{M,x}$ is the local ring at the point x and M_x is the maximal ideal, the cotangent space is isomorphic to M_x/M_x^2.

Definition 3.2. A *Lie Group* is a group object in the category of manifolds.

Example 3.3. The group GL_n is a Lie group, given by the equation $\det g \cdot z - 1 = 0$ in ambient space $M_n(k) \times \mathbb{A}^1_z$. Similarly, the group SL_n is given by $\det g = 1$, and the group $Sp(V)$ is the set of invertible matrices that preserve the symplectic form. Finally, the group $O(V)$ is the group of matrices that preserve a nondegenerate symmetric bilinear form.

Remark 3.4. A *symplectic form* is a nondegenerate alternating bilinear form.

Definition 3.5. A *Lie algebra* over k is a vector space $\mathfrak{g}$ together with an anticommutative product $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity.

Example 3.6. Any associative algebra can be made into a Lie algebra with the commutator as Lie bracket. Similarly, if A is an associative algebra, then $\operatorname{Der} A$ is a Lie algebra with the commutator as Lie bracket.

Example 3.7. Let G be a Lie group. Then the space T_eG is a Lie algebra and is the same as the left-invariant vector fields. We will denote this Lie algebra by $\operatorname{Lie} G$.

1. It is easy to see that $\operatorname{Lie} GL_n = \mathfrak{gl}_n = M_n$.
2. The Lie algebra $\mathfrak{sl}_n$ of SL_n is simply the set of trace zero matrices.
3. The Lie algebra $\mathfrak{sp}_{2n}$ is the set of matrices where $\omega(xu, v) + \omega(u, xv) = 0$.
4. The *orthogonal Lie algebra* $\mathfrak{o}_n$ is the set of matrices x where $x + x^T = 0$.

Theorem 3.8. *There is an exponential map* $\exp : \mathfrak{g} \to G$ *such that*

1. $\exp(0) = e$*;*
2. $\mathrm{d}_0 \exp : \mathfrak{g} \to \mathfrak{g}$ *is the identity. In particular,* $\exp$ *is an isomorphism near* 0 *and* e*;*
3. $\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \exp(sx)\exp(sy)\exp(sx)^{-1}\exp(sy)^{-1} = [x, y]$. *This tells us that the Lie algebra commutator is an infinitesimal version of the group commutator.*

Theorem 3.9. *We will see that* $\mathfrak{g}$ *controls* $G_e/Z(G_e)$*.*

1. *Any morphism* $\pi : G \to G'$ *of Lie groups is determined on* G_e *by* $\mathrm{d}_e\pi : \mathfrak{g} \to \mathfrak{g}'$*.*
2. *If* $\tau : \mathfrak{g} \to \mathfrak{g}'$ *is a map of Lie algebras and* $\pi_1(G) = 0$*, then* $\tau = \mathrm{d}_e\pi$ *for a unique map of groups* $\pi : G \to G'$*.*

The universal cover of any connected Lie group is still a Lie group, so we can preserve most of the information by studying simply connected Lie groups.

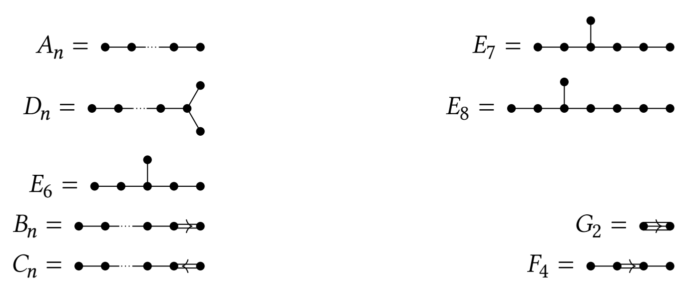
3.2 Representation Theory and Classification

A representation of a Lie group is a morphism $\pi : G \to GL_n$ and a representation of a Lie algebra is a morphism $\mathfrak{g} \to \mathfrak{gl}_n$.

Lemma 3.10. *Representations of G differentiate to representations of* $\mathfrak{g}$*. In particular, if G is simply connected, differentiation is an equivalence of categories.*

A Lie algebra is *simple* if it has no nontrivial ideals $\mathfrak{a}$ such that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$. A solvable Lie algebra is a Lie algebra where the series $[\mathfrak{g}, \mathfrak{g}], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], \ldots$ eventially arrives at 0.

We will now state the classification of finite-dimensional semisimple Lie algebras over $\mathbb{C}$. These are simply sums of simple Lie algebras, so we simply need to classify the simple Lie algebras, which are simply given by Dynkin diagrams:



The correspondence is given by the vertices corresponding to *simple roots* in $\mathfrak{g}$.

Definition 3.11. A connected Lie group is *semisimple* if its Lie algebra is semisimple.

Our slogan will be that $\mathfrak{sl}_2 \in \mathfrak{sl}_n \subset$ semisimple $\subset$ Kac-Moody. A Kac-Moody Lie algebra is a generalization of the semisimple or affine Lie algebras. An important example of such an object is the *loop group*. For a real manifold M, the *loop space* of M is $LM = \mathrm{Map}(S^1, M)$. For a complex manifold $\mathscr{M}$, then $LM = \mathrm{Hom}(\mathbb{C}^*, \mathfrak{M})$, and finally if X is an algebraic variety, $LX = \mathrm{Hom}(\mathbb{A}^1 \setminus 0, X)$.

If G is a Lie group, then LG is an infinite-dimensional version of a Lie group and $\mathrm{Lie}(LG) = L\,\mathrm{Lie}(G)$. If G is an algebraic group, then $L(G)(R) = G(R((t)))$ for any ring R, where we think of G and LG as functors.

Now we will consider the representation theory of SL_n. Note that $GL_1(\mathbb{C}) = \mathbb{G}_m = S^1 \times \mathbb{R}$. The torus is a complex Lie group $T \simeq \mathbb{G}_m^n$. We will consider a maximal torus $T \subseteq SL_n$. There are also called *Cartans*.

Theorem 3.12. *All maximal tori are conjugate to the set of diagonal matrices.*

We will denote the set of roots of T in $G = SL_n$ by Δ. We will aim to describe $\mathfrak{g}$ in terms of $\mathfrak{t} = \operatorname{Lie} T$.

Lemma 3.13. *Let U be a compact Lie group. Then it has a left-invariant Haar measure that is unique up to constants and* $\operatorname{Rep}^{\mathrm{fd}}(U, \mathbb{C})$ *is semisimple.*

Proof. We need to find a left-invariant differential form. Fortunately, the tangent and cotangent bundles are trivial, so in particular, the canonical bundle is trivial.

Once we have a Haar measure, choose an inner product h_0. Then we can write an invariant inner product by

$$h(x, y) = \int_U h_0(u^{-1}x, u^{-1}y)\, \mathrm{d}\mu(u)\,.$$

Then we finish the proof as in the case of finite groups. □

Corollary 3.14. *Let G be a connected complex Lie group with a compact real form U. Then let $V \in \operatorname{Rep} G$ and h be the U-invariant hermitian inner product on V. Then h is invariant under G.*

Proof. Because G is connected, we can transfer the problem to the Lie algebra. The condition that $h(gx, gy) = h(x, y)$ becomes $h(\alpha x, y) + h(x, \alpha y) = 0$. The other direction is true when $g \in \exp(\mathfrak{g})$ because $\exp(\mathfrak{g})$ generates G. Therefore, we can reduce the problem to $\mathfrak{u}$ because $\mathfrak{g}$ is its complexification. However, we can finally reduce from $\mathfrak{u}$ to U, so we are done. □

Example 3.15. Let $T = (\mathbb{C}^*)^n$, Then $T_{\mathbb{R}} = (S^1)^n$ is compact.

Lemma 3.16. *For any finite dimensional representation V of a torus,*

$$V = \bigoplus_{\chi \in X^*(T)} \chi \otimes [V : \chi].$$

Here $X^*(T) := \operatorname{Hom}(T, \mathbb{C}^*)$.

Note that any character $\chi : \mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto z^n$ for some $n \in \mathbb{Z}$, so we see that $\chi^*(T) \simeq \mathbb{Z}^n$.

Because $SL_n(\mathbb{C})$ has a compact real form $SU(n)$, this means that $\operatorname{Rep} SL_n$ is a semisimple category. Then $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$, and there is an isomorphism $G = U \times \mathfrak{u}$ given by $(u, x) \mapsto u \exp(x)$. Therefore $\pi_1(G) = \pi_1(U)$. In particular, for $n = 2$, $SU(2) = S^3$, so it is simply connected. We also have a fiber sequence as below:

$$\begin{tikzcd} SU(n-1) \arrow[r, hook] & SU(n) \arrow[d, two heads] \\ & S^{2n-1} \end{tikzcd}$$

Given a maximal torus T, its Lie algebra $\mathfrak{t}$ acts on $V \in \operatorname{Rep} T$. If T acts by χ, then $\mathfrak{t}$ acts by $\mathrm{d}_e\chi : \mathfrak{t} \to \mathbb{C}$, so $\mathrm{d}_e\chi \in \mathfrak{t}^*$. We will denote these functionals by λ, so define

$$V_\lambda = \{v \in V \mid s \cdot v = \langle \lambda, s\rangle\, v \text{ for all } s \in \mathfrak{t}\}.$$

Lemma 3.17. *The eigenspaces of χ and $\mathrm{d}_e\chi$ are the same. In particular, differentiation is injective.*

Lemma 3.18. *For all $x \in \mathfrak{g}$ there exists a unique $\Theta_x : (\mathbb{R}, +) \to G$ such that $d_0(\Theta_x)(1) = x$. We denote* $\exp x := \Theta_x(1)$.

For $G = \mathbb{C}^*$, we see that $\exp\tau = e^\tau$, so $e^{sz} = \chi_s(e^z)$. If we set $z = 2\pi i$, then we see that $s \in \mathbb{Z}$. Alternatively, we can differentiate $z \mapsto z^n$ at $z = 1$.

We will define the set of *coroots* by $X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$. Then it is easy to see that $X^*(T)$ and $X_*(T)$ are dual lattices under the pairing

$$X^*(T) \otimes X_*(T) \to \mathbb{Z} \qquad \text{given by} \qquad \chi \otimes \eta \mapsto \chi \circ \eta \in \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}.$$

Corollary 3.19.

1. *Differentiation at* 1 *gives an injection* $X_*(T) \hookrightarrow \mathfrak{t}$*;*
2. *Differentiation at the identity gives an injection* $X^*(T) \hookrightarrow \mathfrak{t}^*$*;*
3. *The map* $X_*(T) \otimes \mathbb{C} \to \mathfrak{t}$ *given by* $\eta \otimes z \mapsto z d_1 \eta$ *is an isomorphism.*

We will now study the system of roots. Let $\mathbb{V} = X^*(T) \otimes \mathbb{R} =: \mathfrak{t}^*_{\mathbb{R}}$. Then for $V \in \mathrm{Rep}\, T$, write

$$\mathscr{W}^{\mathfrak{t}}(V) = \{\lambda \in \mathfrak{t}^* \mid V_\lambda \neq 0\} \subseteq X^*(T) \subseteq \mathfrak{t}^*.$$

A *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$ is a maximal toral subalgebra. Then for all $s \in \mathfrak{t}$, the Lie bracket $[s, -] : \mathfrak{g} \to \mathfrak{g}$ is a semisimple operator.

Lemma 3.20. *If* $T \subseteq G$ *is a Cartan, then* $\mathfrak{g} = \bigoplus_{\alpha \in \mathscr{W}(g)} \mathfrak{g}_\alpha$

Example 3.21. In both GL_n and SL_n, the maximal torus is the set of diagonal matrices, and for $G = GL_n$, $\mathfrak{t}$ is the set of diagonal matrices. In SL_n, the Cartan subalgebra is the set of trace zero diagonal matrices.

Define the set of roots $\Delta_{\mathfrak{t}}(\mathfrak{g}) = \mathscr{W}(\mathfrak{g}) \setminus \{0\}$. This tells us that $\mathfrak{g} = \bigoplus_{x \in \Delta} \mathfrak{g}_\alpha \oplus Z_{\mathfrak{g}}(\mathfrak{t})$ and that for $V \in \mathrm{Rep}\, G$, if $\lambda \in \mathscr{W}(V)$ and $\alpha \in \mathscr{W}(\mathfrak{g})$, then $g_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$.

3.3 Representation Theory of $\mathfrak{sl}_n$

Lemma 3.22.

1. *For* $\mathfrak{g} = \mathfrak{sl}_n$, $\Delta = \left\{\alpha_{ij} = \varepsilon^i - \varepsilon^j \mid i \neq j\right\}$*;*
2. *The weight spaces are simply* $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij}$*;*
3. $\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \oplus \mathfrak{t}$*;*
4. *For all* $\alpha \in \Delta$, $\dim \mathfrak{g}_\alpha = 1$*;*
5. *If* $\alpha, \beta, \alpha + \beta \in \Delta$*, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Lemma 3.23.

1. *If* $\alpha, \beta \in \Delta$ *but* $0 \neq \alpha + \beta \notin \Delta$*, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$*;*
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}(\alpha_{ij})^\vee \subseteq \mathfrak{t}$.

In the case $n = 2$, we can write a basis for $\mathfrak{g}$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the identities

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h.$$

Lemma 3.24. *Any root $\alpha \in \Delta$ defines an $\mathfrak{sl}_2$-subalgebra $S_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.*

1. *There exists a Lie algebra map $\varphi : \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ with $\varphi e \in \mathfrak{g}_\alpha$, $\varphi f \in \mathfrak{g}_{-\alpha}$;*
2. *The image is a Lie subalgebra;*
3. *The image is isomorphic to $\mathfrak{sl}_2$;*
4. *φh is independent of φ.*

Proof. The first three parts are easy. Suppose $\psi e = a\varphi e$ and $\psi f = b\varphi f$. Then

$$\begin{aligned} 2\psi(e) &= \psi(2e) \\ &= \psi([\psi, e]) \\ &= [\psi h, \psi e] \\ &= a^2 b[\varphi h, \varphi e] \\ &= a^2 \cdot 2\varphi e, \end{aligned}$$

so $ab = 1$. □

Corollary 3.25. *We see that $\langle \alpha, \check{\alpha} \rangle = 2$.*

Now we will consider reflections. If $v \in V$ and $v^* \in V^*$, then if $\langle v, v^* \rangle = 2$, there is an involution $s_{v,v^*}(x) = x - \langle v^*, x \rangle v$. Then we see that

$$s_{\alpha_{ij}}(\varepsilon^k) = \begin{cases} \varepsilon^k & k \neq i, j \\ \varepsilon^j & k = i \\ \varepsilon^i & k = j \end{cases}.$$

The group W generated by all reflections is called the *Weyl group* of Δ. For $G = SL_n$, $W = S_n$.

Lemma 3.26. *For $w \in W$ and $\alpha \in \Delta$, $w(S_\alpha) = S_{w\alpha}$.*

Lemma 3.27. *V has a W-invariant inner product.*

The inner product is simply the standard inner product on $V_0^* = \bigoplus \mathbb{R}\varepsilon^i = \mathbb{R}^n$. This restricts to V^*, so we can compute the inner product of the indices. An inner product of 0 corresponds to an angle of $\pi/2$, 1 corresponds to an angle of $\pi/6$, and 2 corresponds to an angle of 0.

Proposition 3.28. *Define the coroot $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)} \in V \simeq V^*$.*

1. *For all $\alpha \in \Delta$, $s_\alpha \Delta = \Delta$;*
2. *For all $\alpha, \beta \in \Delta$, $(\alpha, \check{\beta}) \in \mathbb{Z}$.*
3. *Δ spans V.*

We will now describe the irreducible representations of $\mathfrak{sl}_2$. Note that $\Delta = \{\alpha, -\alpha\}$ and $X^*(T) = \mathbb{Z}\alpha$. Thus we will see that the irreducible representations of $\mathfrak{sl}_2$ are simply natural numbers, where

$$h = \begin{pmatrix} n & & & & \\ & n-2 & & & \\ & & \ddots & & \\ & & & 2-n & \\ & & & & -n \end{pmatrix}.$$

We see that e increses the weight by 2 and f decreases the weight by 2. In addition, if $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in T$, then

$$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} v = \exp \begin{pmatrix} A & \\ & -A \end{pmatrix} v = e^{Ak} v = a^k v = (k\rho) \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} v,$$

where $\rho \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} = a$. The representation given above is the only irreducible representation of $\mathfrak{sl}_2$ with highest weight n. In general, we have irreducible representations with highest weight λ.

Note that $\mathbb{N}\rho \subset \mathbb{Z}\rho$ is a direction in $X^*(T)$, which comes from a choice of Borel, so the roots come from $\mathfrak{n}, \mathfrak{b}$.

For $\mathfrak{sl}_n$, define the *dominant cone* $X^*(T)^+ \subset X^*(T)$ as the set of weights $\lambda \in X^*(T)$ which are positive with respect to all roots in $\mathfrak{n}, \mathfrak{b}$.

Definition 3.29. In $X^*(T)$, we say that $\lambda \geq \mu$ if $\lambda - \mu$ is in the $\mathbb{N}$-span of Δ^+.

Definition 3.30. In a representation $V \in \operatorname{Rep} \mathfrak{sl}_n$, v is *primitive of weight* λ if $v \in V_\lambda$ for some maximal weight λ.

Theorem 3.31. *For all $\lambda \in X^*(T)^+$, there exists a unique representation L of $\mathfrak{g}$ such that*

1. *L has highest weight λ;*
2. *There exists a primitive vector v of weight λ, where $\mathfrak{n} \cdot v = 0$.*

Proof. We know that

$$\begin{aligned} \mathfrak{n} \cdot v &= \sum \mathfrak{g}_\alpha v \\ &\subseteq \sum \mathfrak{g}_\alpha V_\lambda \\ &\subseteq \sum V_{\lambda+\alpha}. \end{aligned}$$

Because λ is maximal, then $V_{\lambda+\alpha} = 0$. □