NOTES ON THE HIGHER-GENUS GROMOV-WITTEN THEORY OF COMPACT CALABI-YAU THREEFOLDS

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ABSTRACT. These notes, taken during the first two weeks of the program Recent developments in higher genus curve counting at the Simons Center for Geometry and Physics, explain how to prove important structural results in the higher-genus Gromov-Witten theory of compact Calabi-Yau threefolds.

CONTENTS

1. Introduction

One of the oldest problems in Gromov-Witten theory is to compute the Gromov-Witten invariants of compact Calabi-Yau threefolds. While there are a variety of physical methods which have allowed physicists to make spectacular predictions in a wide range of examples, mathematical progress has often been frustratingly slow due

to a lack of satisfactory tools to attack the problem with. To illustrate this, we will briefly outline some developments in mathematics regarding Calabi-Yau threefolds with $h^2 = 1$ which arise as complete intersections in weighted projective space.

- For complete intersections in projective space, a genus-zero mirror theorem was proved by Givental in 1996 and Lian-Liu-Yau in 1997 [Giv96; LLY97].
- For complete intersections in weighted projective space, a genus-zero mirror theorem was proved by Coates-Corti-Lee-Tseng in 2006 in the convex case and by Jun Wang in 2019 in the non-convex case [CCLT09; Wan20]. The main difficulty in the non-convex case is the failure of the quantum Lefschetz theorem [CG07], which is the main tool used in [Giv96; LLY97; CCLT09].
- Meanwhile, a genus-one mirror theorem was proved for the quintic by Zinger in 2007 and for complete intersections in projective space by Popa in 2010 [Zin09; Pop13]. This used the theory of reduced invariants developed by Vakil-Zinger and Li-Zinger [VZ08; LZ09], which is a technique of performing birational modifications to the main component of the moduli space of stable maps to force the quantum Lefschetz theorem to hold so that computations can be performed using virtual localization [GP99]. Unfortunately, to this date no computations have been successfully performed using reduced invariants since.
- Another way to force the quantum Lefschetz theorem to hold is to consider the theory of GLSMs [Wit92], which were constructed mathematically by various authors during the 2010s [FJR18; FK21]. Unfortunately, the virtual cycle is supported on the moduli space of stable maps to the threefold, which does not carry a torus action.
- For the quintic, the ambient space of the relevant GLSM is the total space of $\mathcal{O}_{\mathbb{P}^4}(-5)$. A standard way to gain a torus action is to compactify the moduli space at infinity and consider relative (or logarithmic) invariants. This leads to the theory of logarithmic GLSMs, which was introduced by Chen-Janda-Ruan in 2019 [CJR21]. Before this, Guo-Janda-Ruan used this theory [CJR23], including still conjectural foundational results [CJRa; CJRb], to prove a genus-two mirror theorem for the quintic in 2017 (up to the fact that the moduli space used in their 2017 paper is still not defined) and to prove the Yamaguchi-Yau finite generation conjecture, holomorphic anomaly equations, and orbifold regularity (as well as the conifold gap condition in low genus) for the quintic [GJR17; GJR18].
- Because a GLSM is an enumerative theory of a critical locus in a GIT quotient, it depends on a stability parameter. We can vary the stability parameter and construct a master theory following the master space construction of Thaddeus [Tha96]. This leads to the theory of Mixed-Spin-P fields, which was first constructed by Chang-Li-Li-Liu in 2015 (for the quintic only) [CLLL19; CLLL22]. Unfortunately, the original theory is impossible to compute with when $g \geq 2$, so a parameter N was introduced by Chang-Guo-Li-Li in 2018 [CGLL21]. In the same year, Chang-Guo-Li proved the Yamaguchi-Yau finite generation conjecture, BCOV Feynman rule, and holomorphic anomaly equations for the quintic threefold [CGL21; CGL19]. These results were generalized to hypersurfaces in weighted projective space by the author in 2024 [Lei24b; Lei24a].

- A genus-one mirror theorem for hypersurfaces in weighted projective space was finally proved in 2024 by the author using the theory of Mixed-Spin-P fields [Lei24b].
- The Castelnuovo bound was proved in 2022 by Liu-Ruan [LR22] for any Calabi-Yau threefold satisfying a conjectural Bogomolov-Gieseker-type inequality (including the quintic) due to Bayer-Macri-Toda [BMT14] and a weaker version of the result was proved by Zhiyu Liu in 2024 [Liu24] without assuming the conjectural inequality. These results were obtained using the GW/DT correspondence [MNOP06a; MNOP06b], which was proved for complete intersections in products of projective spaces by Pandharipande-Pixton in 2012 [PP17] and for all Calabi-Yau threefolds by Pardon in 2023 [Par24].

As we can see, two major breakthroughs were made around the year 2018, when mathematicians discovered ways which make proving results about Gromov-Witten invariants in arbitrary genus tractable.[1] These notes explain how to use these new ideas to prove the most important conjectures which are provided to us by physicists, namely the Yamaguchi-Yau finite generation conjecture and the holomorphic anomaly equations. Part 1 covers the B-model topological string and the Givental formalism, Part 2 covers log GLSMs, and Part 3 covers Mixed-Spin-P fields. We will cover both the foundational theory and the calculations in both approaches – in particular, Section 4 and Section 6 discuss the geometry of the log GLSM and MSP moduli spaces while Section 5 and Section 7 discuss calculations.

Author's note. In contrast to the genus zero situation, not much is known about higher-genus Gromov-Witten theory. This is in part because there are significantly fewer tools to study higher-genus invariants, and the ones which do exist seem to be considered extremely inaccessible. It is my sincere hope that these notes can help make the subject more accessible.

All errors are the sole responsibility of the author. Please email me if you find any mistakes or typos in these notes.

The names next to section titles indicate that the material in that part of the notes comes from talks given by that speaker. The notes from the lectures by Albrecht Klemm were taken partly during the lecture and partly by watching the videos on the SCGP website, the notes from my lectures are slightly expanded versions of my lecture notes, and the notes from all other lectures were taken during the lecture, with some material being provided by the speakers after their lectures. All references were added after the lectures.

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[1]Considering that the progress had been approximately one genus every decade, the existence of such breakthroughs is implied by the existence of these notes prior to the year 2027, or really at all.

Part 1. Mathematical and physical preliminaries

We begin these notes with discussion of the topological B-model. Using mirror symmetry, physicists are able to transport questions in Gromov-Witten theory (the A-model) to the B-model, which lacks a satisfactory mathematical definition. The benefit of doing this is that it is easier to calculate in the B-model, where genus-zero invariants are simply period integrals. Using other B-model techniques,[2] physicists are able to make far-reaching predictions about the Gromov-Witten invariants of Calabi-Yau threefolds. Note that there are proposed mathematical definitions of the B-model – the analytic approach of Costello-Li [CL12b; Li11] and the categorical enumerative invariants of Caldararu-Tu [CT24] – but they are extremely difficult to compute with.

We then proceed to a discussion of the mathematical prerequisites for the calculations which appear later in these notes. The most important tool for performing calculations in Gromov-Witten theory is the formalism introduced by Givental in a highly influential series of papers [Giv01b; Giv01a; Giv04] in the early 2000s. This formalism allows us to package large formulae, for example coming from virtual localization, in a compact way, which allows us to reason about higher-genus invariants in a relatively streamlined way. For example, polynomiality of GW generating series will be a direct corollary of polynomiality of the entries of an *R-matrix* which plays a central role in Givental's formalism.

2. Introduction to the topological B-model (Albrecht Klemm)

2.1. **Mirror symmetry and the role of Calabi-Yau threefolds.** Let $X(\Omega, \omega)$ be a Calabi-Yau n-fold, where here Ω is a holomorphic n-form and ω is a Kähler form. Recall that this is equivalent to having $SU(n)$ holonomy or to $K_X = 0$. By a result of Yau [Yau78], there exists a Kähler-Einstein metric g in the class of ω with vanishing Ricci curvature. For our purposes, we will mostly consider the case when $n = 3$.

We will really consider families of Calabi-Yau varieties

$$\mathcal{X} \to X \to \mathcal{M}(\underline{z}, \underline{t}),$$

where $\mathcal{M}(\underline{z}, \underline{t})$ is parametrized by a combination of complex structure moduli $\underline{z}$ and Kähler structure moduli $\underline{t}_{\mathbb{R}}$ complexefied by a Neveu-Schwarz harmonic 2-form B to $\underline{t}$. We will denote the fiber by

$$X(\Omega_{\underline{z}}, \omega_{\underline{t}}).$$

On the moduli of complex structures $\mathcal{M}(\underline{z})$, the tangent space $H^{(0,1)}(X, T_X)$ has a basis given by harmonic forms

$$A^{(k)} = A^{(k)j}_{\bar{\imath}} \frac{\partial}{\partial x^j} \, \mathrm{d}x^{\bar{\imath}}$$

for $k = 1, \ldots, h^{n-1,1}(X)$. Because we are in the Calabi-Yau setting, contracting with Ω gives us a basis

$$\chi^{(k)} = A^{(k)} \lrcorner \Omega$$

of $H^{n-1,1}(X)$. By a theorem of Tian-Todorov [Tia87; Tod89], the moduli space is unobstructed. The B-model is built from mathematical structures on $\mathcal{M}(\underline{z})$.

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[2]They also have the advantage of not needing to prove any theorems rigorously.

There is a moduli space $\mathcal{M}_g{}^3$ parameterized by δg subject to the condition

$$R_{i\bar{j}}(g + \delta g) = 0.$$

To first order, we have

$$\nabla^\rho \nabla_\rho \delta g_{\mu\nu} - 2R_{\mu\nu}^{\kappa\sigma} \delta g_{k\sigma} = 0.$$

The indices with pure Hodge type correspond to harmonic forms in $H^{(0,1)}(X, TX)$ with components

$$\delta g_{\bar{j}}^i = g^{i\bar{k}} \delta g_{\bar{k}\bar{j}}$$

yielding the Kuranishi family over $\mathcal{M}(\underline{z})$. The mixed indices correspond to real harmonic $(1,1)$-forms, and expanding the Kähler form linearly we obtain

$$\omega = \sum t_{\mathbb{R}}^k \omega^{(k)}$$

in terms of real Kähler parameters

$$\operatorname{Re}(t^k) = t_{\mathbb{R}}^k = \int_{\mathcal{C}_{(k)}} \omega > 0,$$

which are volumes of holomorphic curves.

The Kähler moduli space $\mathcal{M}(\underline{t}_{\mathbb{R}})$ is the real Kähler cone subject to positivity conditions from integration over k-dimensional holomorphic submanifolds, namely

$$\int_{\mathcal{D}^{(k)}} \omega^{\wedge k} > 0.$$

The bosonic part of the string action contains the harmonic antisymmetric Neveu-Schwarz background field $b_{i\bar{j}}$

$$S_{\mathsf{bos}} = \frac{1}{2\pi\alpha'} \int_\Sigma \sqrt{h}(h^{ab} g_{i\bar{j}} + \sqrt{-1} b_{i\bar{j}} \varepsilon^{ab}) \, \partial_{\sigma_a} x^i \, \partial_{\sigma_b} x^{\bar{j}},$$

where α' is the string coupling constant. Its critical values measure complexified volumes of holomorphic curves by

$$t^k = \int_{\mathcal{C}^k} (\omega + ib) = \operatorname{Re}(t^{(}k)) + \sqrt{-1} \operatorname{Im}(t^{(k)}).$$

Conjecture 2.1. *For non-rigid Calabi-Yau threefolds X with $h^{2,1} > 0$, there exists a mirror Calabi-Yau $\hat{X}$ with $h^{1,1}(\hat{X}) = h^{2,1}(x)$ and $h^{2,1}(\hat{X}) = h^{1,1}(X)$ such that the moduli spaces satisfy*

$$\mathcal{M}[\hat{X}](\underline{\hat{z}}) = \mathcal{M}[X](\underline{t}) \qquad and \qquad \mathcal{M}[\hat{X}](\underline{\hat{t}}) = \mathcal{M}[X](\underline{z})$$

and all relevant physical and mathematical structures can be identified using locally invertible mirror maps

$$\underline{\hat{z}}(\underline{t}) \qquad and \qquad \underline{\hat{t}}(\underline{z}).$$

Mirror symmetry identifies Type IIA compactifications on X with Type IIB compactifications on $\hat{X}$, and vice versa. Additional Ramond-Ramond background fields and axio-dilaton fields with modulus $\underline{j}$ and a extend the moduli spaces as

$$\mathcal{M}^{\mathsf{IIB}}[X] = \mathcal{M}[X](\underline{z}) \times \mathcal{Q}[X](\underline{t}, \underline{j}, a)$$

and

$$\mathcal{M}^{\mathsf{IIA}}[\hat{X}] = \mathcal{M}[X](\underline{\hat{t}}) \times \mathcal{Q}[\hat{X}](\underline{\hat{z}}, \underline{\hat{j}}, \hat{a}),$$

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[3]This is a moduli space of metrics, not the moduli space of smooth genus g curves.

where $\mathcal{Q}$ denotes a quaternionic extension of $\mathcal{M}$. The RR $(k+1)$-form fields are sourced from D_k branes. k is even for Type IIA and odd for Type IIB. The D_{2m} correspond to coherent sheaves and D_{2m+1} correspond to special Lagrangian branes. Here, $\mathcal{Q}$ is the 4d $N=2$ hyper-multiplet moduli space, and $\mathcal{M}$ is the 4d $N=2$ vector multiplet moduli space.

One kind of mathematical structure is Hodge numbers. In the traditional picture of the Hodge diamond, Poincaré duality corresponds to a reflection along the horizontal axis, Dolbeaut symmetry is a reflection along the vertical axis, and mirror symmetry is a reflection along a line with slope 1.

Superstring theory is defined by maps

$$X \colon \Sigma_g \to \mathcal{C}_\beta \subset \text{spacetime}$$

weighted by an action S which is a supersymmetric extension of the area of $\mathcal{C}_\beta$. It is easy to quantize the Neveu-Schwarz-Ramond action, and the Green-Schwarz action incorporates the RR fields. The *first quantized theory* is defined by a variatonal integral with partition function

$$Z(g, b, \phi) = \int \mathcal{D}X \mathcal{D}h \mathcal{D}\psi_{\mathsf{ferm}} e^{\frac{i}{\hbar} S_{\mathsf{NSR}}(X, h, \psi_{\mathsf{ferm}}, g, b, \phi)}.$$

Superstring theory is Weyl invariant in ten dimensions, or in other words

$$\int \mathcal{D}h \to \sum_{g=0}^{\infty} \int_{\mathcal{M}_{\Sigma_g}} \mu_{3g-3},$$

so we obtain a discrete sum of finite-dimensional integrals. This implies that the compact part X of the spacetime M must be a complex threefold. If X is Calabi-Yau, we obtain an extended $(2, 2)$ world-sheet SCFT, which has four nilpotent operators

$$Q_\pm^2 = \bar{Q}_\pm^2 = 0.$$

The A-twist corresponds to taking

$$Q_A = Q_- + \bar{Q}_+$$

and the B-twist corresponds to

$$Q_B = \bar{Q}_- + \bar{Q}_+.$$

The topological A-model yields a cohomological topological theory depending only on the Kähler structure, while the topological B-model is a homological topological theory depending only on the complex structure. Mirror symmetry then exchanges the A-model and B-model.

2.2. **The topological A-model and B-model.** In the A-model, terms depending on the complex structure are Q_A-exact, so the variatinoal integral simplifies to

$$Z = \sum_{g=0}^{\infty} \sum_{\beta \in H_2(X, \mathbb{Z})} g_s^{2g-2} Q^\beta \int_{\overline{\mathcal{M}}_g(X, \beta)} \mathbf{1}.$$

Here, we have

$$Q = e^{2\pi i \int_{\mathbf{e}_\beta} i\omega + b} = e^{t\beta}$$

and these holomorphic maps are stationary points of the action. Moreover, taking the logarithm, we obtain

$$\mathcal{F}(g_s, Q) = \log Z = \sum_{g,\beta} g_s^{2g-2} Q^\beta r_g^\beta = \sum_{g=0}^\infty g_s^{2g-2} \mathcal{F}_g(Q),$$

where r_g^β are the GW invariants. Rewriting these in terms of GV invariants, we obtain

$$\mathcal{F}(g_s, Q) = \frac{c(t)}{\lambda^2} + \ell(t) + \sum_{g,\beta} \sum_{m=1}^\infty \frac{n_g^\beta}{m} \left(2 \sin \frac{mg_s}{2}\right)^{2g-2} Q^{m\beta}.$$

In the B-model, the terms depending on the Kähler structure are Q_B-exact and the variational integral localizes to constant maps albeit with a nontrivial measure depending on the complex structure. Mirror symmetry is supposed to be an exact duality, so we should have

$$\left\langle \mathcal{O}_i^{(0)} \mathcal{O}_j^{(0)} \mathcal{O}_k^{(0)} \right\rangle_{g=0} = \int_{\hat{X}} \Omega(z)\, \partial_{z_i}\, \partial_{z_j}\, \partial_{z_k} \Omega(z)$$
$$= \partial_{t_i}\, \partial_{t_j}\, \partial_{t_k} \mathcal{F}_0(t).$$

Period integrals

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z})$$

define a nondegenerate pairing between middle homology and cohomology by Stokes' theorem. This pairing is antisymmetric if n is odd and symmetric if n is even. For example, if X is a K3 surface, then the lattice $H^2(X, \mathbb{Z})$ is

$$E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}.$$

If n is odd, we can fix an integral symplectic basis $\underline{\Gamma} = \{A_\ell, B^\ell\}$, which is defined only up to the action of $\mathrm{Sp}(b_n(X), \mathbb{Z})$.

Example 2.2. If we consider an elliptic curve $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$, then we obtain

$$\Omega(z) = \oint \frac{2\, \mathrm{d}x \wedge \mathrm{d}y}{p_3} = \frac{\mathrm{d}x}{y}$$

and

$$\partial_z \Omega(z) \sim \frac{x\, \mathrm{d}x}{y}.$$

Then the integrals

$$E_1(z) = \oint_A \Omega \qquad \text{and} \qquad E_2(z) = \oint_B \Omega$$

are elliptic integrals. The periods are annihilated by the Picard-Fuchs operator, and by definition thus satisfy the equation

$$\mathcal{L} \int_\Gamma \Omega = \left[(1-z)\, \partial_z^2 + (1-2z)\, \partial_z - \frac{1}{4}\right] \int_\Gamma \Omega = 0.$$

The main constraints which govern the periods of a Calabi-Yau n-fold are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_X \Omega \wedge \bar\Omega > 0.$$

This defines the exponential of the *Kähler potential* $K(z)$ for the Weil-Petersson metric

$$G_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{j}}} K(z)$$

on $\mathcal{M}[X](\underline{z})$. There are also holomorphic bilinear relations

$$\int_X \Omega \wedge \underline{\partial}_{\ell_k}^k \Omega = \begin{cases} 0 & k < n \\ C_{\ell_n}(z) & \ell = n \end{cases}$$

which follow from Griffiths transversality [Gri68a; Gri68b]. Here, the integrand in the left hand side are arbitrary combinations of derivatives of Ω with respect to the z_i. We will see later that these give rise to propagators, the holomorphic anomaly equations, and other structures. The $C_{\ell_n}(z)$ are rational functions labelled by ℓ_n up to permutations.They are also determined by differential ideals $\mathcal{L}\vec{\Pi}$ also determine the $C_{\ell_n}(z)$ up to normalization.

Remark 2.3. In terms of the periods $\vec{\Pi}$, if we write them in a basis compatible with the intersection form Σ, the quantities in the relations may be written as

$$\int_X \Omega \wedge \bar{\Omega} = \vec{\Pi}^\dagger \Sigma \vec{\Pi} \qquad \text{and} \qquad \int_X \Omega \wedge \partial_{\ell_k}^k \Omega = -\vec{\Pi}^\dagger \Sigma \underline{\partial}_{\ell_k}^k \vec{\Pi}.$$

2.3. The quintic. Consider the mirror quintic W, which is given by the equation

$$\hat{p}_5 = \sum_{i=0}^4 x_i^5 - 5z^{-\frac{1}{5}} \prod_{k=0}^4 z_i = 0 \subset \hat{\mathbb{P}}^4.$$

The period vector $\Pi(z)$ (with an appropriate choice of integral cycles) fulfills the Picard-Fuchs equation

$$\left[\theta^4 - 5^5 z \left(\theta + \frac{1}{5} \right) \left(\theta + \frac{2}{5} \right) \left(\theta + \frac{3}{5} \right) \left(\theta + \frac{4}{5} \right) \right] \Pi(z) = 0,$$

where $\theta = z \frac{\mathrm{d}}{\mathrm{d}z}$.

We want to find a basis where the monodromies around the singular points are integral symplectic matrices, so we look at the Riemann symbol and see that it is given by

$$\mathcal{P} \left\{ \begin{matrix} 0 & 5^{-5} & \infty & * \\ 0 & 0 & \frac{1}{5} & \\ 0 & 1 & \frac{2}{5} & z \\ 0 & 2 & \frac{3}{5} & \\ 0 & 1 & \frac{4}{5} & \end{matrix} \right\}.$$

Here, $z = 0$ is a point of maximal unipotent monodromy, $z = \infty$ is the orbifold (or Landau/Ginzburg, or Gepner) point, and $z = 5^{-5}$ is the conifold point. At a point of maximal unipotent monodromy, we can expand the mirror map and go to the large volume limit point in the A-model.

Using special geometry, Bryant and Griffiths [BG83] showed that the periods may actually be expressed using a prepotential $\mathcal{F}$ as

$$\begin{pmatrix} \int_{B_0} \Omega \\ \int_{B_1} \Omega \\ \int_{A_0} \Omega \\ \int_{A_1} \Omega \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ X^0 \\ X^1 \end{pmatrix} = \begin{pmatrix} 2\mathcal{F}_0 - t\,\partial_t\mathcal{F}_0 \\ \partial_t\mathcal{F}_0 \\ 1 \\ t \end{pmatrix}.$$

These correspond to triple logarithmic, double logarithmic, analytic, and logarithmic solutions, respectively. Using this, we can make the identification

$$\mathcal{F}(z) = \mathcal{F}_0(t(z)),$$

where the mirror map is given by

$$t = \frac{X^1}{X^0} = \log(z) + \mathcal{O}(z).$$

This was generalized to multi-parameter models by Hosono et. al [HKTY95], who related the classical terms to CTC Wall data $\kappa = D^3$ and $\sigma = \frac{\kappa \mod 2}{2}$ in the formula

$$\mathcal{F} = -\frac{\kappa}{6} + \frac{\sigma}{2}t^2 + \frac{c_2 \cdot D}{24}t + \frac{\chi(M)}{2}\frac{\zeta(3)}{(2\pi i)^3} - \frac{1}{(2\pi i)^3}\sum_{\beta \neq 0} n_0^\beta \operatorname{Li}_3(Q^\beta).$$

Later, we will use this to find the integral symplectic basis without calculating any monodromy.

We now turn our attention to calculating the numbers n_g^β. We already saw the genus-zero invariants, but we are also interested in the higher-genus ones. Some invariants in low genus and degree are given in Table 1.

TABLE 1. Low genus GV invariants of the quintic.

g	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$
0	2875	609250	317206375	242467530000	22930588887625
1	0	0	609250	3721431625	12129909700200
2	0	0	0	534750	75478987900
3	0	0	0	8625	-15663750
4	0	0	0	0	49250
5	0	0	0	0	1100
6	0	0	0	0	10

Physical predictions were made by Candelas et al. [COGP92] for $g = 0$, Bershadsky et al. [BCOV94] for $g \leq 3$, and Huang et al. [HKQ09] for $g \leq 53$. The numbers n_1^1, n_1^2, and n_1^3 were calculated mathematically by Schubert, Katz [Kat86], and Ellingsrud-Strømme [ES96], respectively. Kontsevich [Kon95] gave a mathematical proof of some numbers in $g = 0$ before the genus-zero invariants were completely determined by Givental and Lian-Liu-Yau [Giv96; LLY97]. The formula for the genus-one invariants was proved by Zinger [Zin09].

2.4. **Fourteen one-parameter hypergeometric families.** There are fourteen hypergeometric Picard-Fuchs operators for one-parameter families of Calabi-Yau threefolds $\hat{X}$ given by

$$\mathcal{L} = \theta^4 - \mu^{-1} z \prod_{k=1}^{4}(\theta + a_k),$$

where the values of μ and a_k are given in Table 2. Here, the notation for the mirror X means the complete intersection in the weighted projective space with weights given in parentheses with degrees given by the subscripts. The local exponents are all 0 at the MUM point, $0, 1, 1, 2$ at the conifold point μ, and are given by

TABLE 2. Data of one-parameter hypergeometric families

N	a_k	μ^{-1}	Mirror X	κ	$c_2 \cdot D$	$\chi(X)$
8	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	2^8	$X_{2,2,2,2}(1^8)$	16	64	-128
9	$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	$2^6 3^3$	$X_{4,3}(1^5 2^1)$	6	48	-156
16	$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	2^{10}	$X_{4,2}(1^6)$	8	56	-176
25	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	5^5	$X_5(1^4)$	5	50	-200
27	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	3^6	$X_{3,3}(1^6)$	9	54	-144
32	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	2^{12}	$X_{4,4}(1^4 2^2)$	4	40	-144
36	$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	$2^4 3^3$	$X_{3,2,2}(1^7)$	12	60	-144
72	$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	$2^8 3^3$	$X_{6,2}(1^5 3^1)$	4	54	-256
108	$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	$2^4 3^6$	$X_6(1^4 2^1)$	3	42	-204
128	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	2^{16}	$X_8(1^4 4^1)$	2	44	-296
144	$\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}$	$2^{10} 3^3$	$X_{6,4}(1^3 2^2 3^1)$	2	32	-158
200	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	$2^8 5^5$	$X_{10}(1^3 2^1 5^1)$	1	34	-288
216	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	$2^8 3^6$	$X_{6,6}(1^2 2^2 3^2)$	1	22	-120
864	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$2^{12} 3^6$	$X_{12,2}(1^4 4^1 6^1)$	1	46	-484

a_1, a_2, a_3, a_4 at the orbifold point ∞. The monodromy around a singular point $*$ is captured by the minimal exponent[4] $1 \leq k < \infty$ such that

$$(M_*^k - 1)^{p+1} = 0$$

for some $0 \leq p \leq \dim_{\mathbb{C}} X$. When $k > 1$ and $p = 0$, then we have an orbifold point. When $p > 0$ we have an infinite shift monodromy. If $p = 1$ and the local exponents take the form a, b, b, c, then we have a single vanishing period dual to a logarithmic degenerating period and hence a conifold point, and if the local exponents are take the form a, a, b, b, then we have two vanishing periods and two logarithmic degenerating periods, and this case is called a K-point. The case $p = 2$ cannot occur by Schmid's $\mathrm{SL}(2, \mathbb{C})$ orbit theorem, and when $p = 3$, we have a point of maximal unipotent monodromy with local exponents a, a, a, a.

At the point $z = 0$, there is a Frobenius basis of solutions

$$\Pi_0(z) = \begin{pmatrix} f_0(z) \\ f_0(z) \log z + f_1(z) \\ \frac{1}{2} f_0(z)(\log z)^2 + f_1(z) \log z + f_2(z) \\ \frac{1}{6} f_0(z)(\log z)^3 + \frac{1}{2} f_1(z)(\log z)^2 + f_2(z) \log z + f_3(z), \end{pmatrix}$$

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[4]This corresponds to the degree of the base change needed to make the monodromy unipotent.

where we normalize the power series to have $f_0(0) = 1$ and $f_{k>0}(0) = 0$. Therefore, we can find an integral basis

$$\Pi = (2\pi i)^3 \begin{pmatrix} \frac{\zeta_3\chi(M)}{(2\pi i)^3} & \frac{c_2 \cdot D}{24 \cdot 2\pi i} & 0 & \frac{\kappa}{(2\pi i)^3} \\ \frac{c_2 \cdot D}{24} & \frac{\sigma}{2\pi i} & \frac{\kappa}{(2\pi i)^2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 \end{pmatrix} \Pi_0.$$

In the integral basis, all monodromies are integral symplectic matrices. At the point $z = 0$, we have

$$M_0 = \begin{pmatrix} 1 & -1 & \frac{\kappa}{6} + \frac{c_2 \cdot D}{12} & \frac{\kappa}{6} + \sigma \\ 0 & 1 & \sigma - \frac{\kappa}{2} & -\kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and at the conifold point, we have

$$M_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

At $z = \infty$, we have $M_\infty = (M_0 M_\mu)^{-1}$.

The work of Alexandrov-Feyzbakhsh-Klemm-Pioline-Schimannek [Ale+24] solves the topological string on these models up to g_{mod} given in Table 3 using D_4 brane and wall-crossing.[5] Note that g_{avail} is the maximum genus for which data appears at http://www.th.physik.uni-bonn.de/groups/klemm/data.php and g_{integ} is the maximum genus solved by the work of Huang-Klemm-Quackenbush [HKQ09].

TABLE 3. Current state of the art in physics literature as of January 2023.

X	χ_D	n_1^p	n_1^c	Type	g_{integ}	g_{mod}	g_{avail}
$X_5(1^5)$	5	7	0	O	53	69	64
$X_6(1^4 2^1)$	4	4	0	O	48	66	48
$X_8(1^4 4^1)$	4	4	0	O	60	84	64
$X_{10}(1^3 2^1 5^1)$	2	7	0	O	50	70	68
$X_{4,3}(1^5 2^1)$	5	9	0	O	20	24	24
$X_{6,4}(1^3 2^2 3^1)$	3	3	0	O	14	17	17
$X_{3,3}(1^6)$	6	14	1	K	29	33	33
$X_{4,4}(1^4 2^1)$	4	6	1	K	26	34	34
$X_{6,6}(1^2 2^2 3^2)$	2	1	0	K	18	21	21
$X_{6,2}(1^5 3^1)$	5	7	0	C	63	84	49
$X_{4,2}(1^6)$	6	15	1	C	50	64	40
$X_{3,2,2}(1^7)$	7	21	1	C	14	?	14
$X_{2,2,2,2}(1^8)$	8	33	3	M	17	?	32

Remark 2.4. Even inserting at least one D_4 brane requires going beyond the topological string, which only gives the usual DT or PT invariants (one D_6 brane, no D_4 branes, and arbitrary D_2 and D_0 brane charges). Hopefully, we will be able to explore the entire space of Bridgeland stability conditions in the future.

----

[5]There have since been more improvements found by adding more D_4 brane charges [AFKP23].

2.5. **More on periods.** Consider the Picard-Fuchs operator

$$\mathcal{L}^{(n+1)}(z) = \sum_{i=1}^{n+1} a_i(z)\, \partial_z^i$$

of a one-parameter Calabi-Yau n-fold with middle Hodge structure of type $1, 1, \ldots, 1$. Defining the adjoint operator

$$\mathcal{L}^{(n+1)\vee}(z) = \sum_{i=1}^{n+1} (-\partial_z)^i a_i(z),$$

then Griffiths transversality implies that the operator is essentially self-adjoint, or in other words

$$\mathcal{L}^{(n+1)} C_{\underline{z}} = (-1)^{n+1} C_{\underline{z}} \mathcal{L}^{(n+1)\vee}.$$

Here, the Yukawa coupling $C_{\underline{z}}$ satisfies the differential equation

$$C_{\underline{z}}' = -\frac{2}{n+1} a_n C_{\underline{z}}.$$

For the quintic, we in fact obtain

$$C_{zzz} = \frac{5}{z^3(1 - 5^5 z)}.$$

Equivalently, self-adjointness implies

$$\sum_{j=k}^{n+1} \binom{j}{k} \left\{ \frac{C_{\underline{z}}^{(j-k)}}{C_{\underline{z}}} a_j + (-1)^{n+j} a_j^{j-k} \right\} = 0.$$

When $n = 3$, this yields

$$a_3^3 + 4a_3'' + 6a_3 a_3' + 8a_1 - 4a_2 a_3 - 8a_2' = 0.$$

In the multi-parameter case, the periods $\vec{\Pi}$ span the kernel of the Picard-Fuchs differential ideal

$$\{\mathcal{L}\} = \{\mathcal{L}_k^{(d_k)} \mid k = 1, \ldots, \ell\}$$

generated by the differential operators

$$\mathcal{L}_\ell^{(d_k)}(\underline{\theta}, \underline{z}) \underline{\Pi}(\underline{z}) = 0.$$

This system is complete if all three-point functions

$$C_{ijk}(\underline{z}) = C_{z_i z_j z_k}(\underline{z})$$

can be integrated from the Griffiths transversality relations.

Example 2.5. For the example $X_{18}(1^3 6^1 9^1)$ with middle Hodge structure $1, 2, 2, 1$, the Picard-Fuchs equation is generated by

$$\mathcal{L}_1^{(2)} = \theta_1(\theta_1 - 3\theta_2) - 12z_1(6\theta_1 + 1)(6\theta_1 + 5),$$

$$\mathcal{L}_2^{(3)} = \theta_2^3 + \prod_{i=0}^{2}(3\theta_2 - \theta_1 + i),$$

and the C_{ijk} are given by

$$C_{111} = \frac{9}{z_1^3 \Delta_1};$$

$$C_{112} = \frac{3\delta}{z_1^2 z_2 \Delta_1};$$

$$C_{122} = \frac{\Delta_2^2}{z_1 z_2^2 \Delta_1};$$

$$C_{222} = \frac{9(\delta^3 + (432z_1)^3)}{z_2^2 \Delta_1 \Delta_2},$$

where the components of the discriminant are given by

$$\Delta_1 = (1 - 432z_1)^3 - 27z_2(432z_1)^3 \qquad \text{and} \qquad \Delta_2 + 1 + 27z_2,$$

and $\delta = 1 - 432z_1$.

If we shift the holomorphic form by

$$\Omega \to e^{f(z)}\Omega,$$

this induces a Kähler gauge transformation

$$K \to K - f(z) - \overline{f(z)}.$$

Also note that the Yukawa couplings $C_{ijk}(\underline{z})$ are sections of

$$\mathcal{L}^2 \otimes \operatorname{Sym}^3 T^* \mathcal{M}(\underline{z})^{1,0},$$

where $\mathcal{L}$ is the dual of the Hodge bundle.

By the local Torelli theorem [Gri68a; Gri68b], the X^I components of the period vector

$$\begin{pmatrix} \int_{B_0} \Omega \\ \vdots \\ \int_{B_r} \Omega \\ \int_{A_0} \Omega \\ \vdots \\ \int_{A_r} \Omega \end{pmatrix} = \begin{pmatrix} F_0 \\ \vdots \\ F_r \\ X^0 \\ \vdots \\ X^r \end{pmatrix}$$

are homogeneous coordinates on $\mathcal{M}(\underline{z})$. Griffiths transversality implies that

$$F = \frac{1}{2} X^I F_I \in \mathcal{L}^2$$

is a homogeneous prepotential such that

$$e^{-K} = i(F_i \bar{X}^I - X^I \bar{F}_I)$$

and the Yukawa couplings are given by

$$C_{IJK} = \frac{\partial}{\partial X^I} \frac{\partial}{\partial X^J} \frac{\partial}{\partial X^K} F.$$

Dividing by X^0, we will get inhomogeneous coordinates $t^i = \frac{X^i}{X^0}$, and we obtain the inhomogeneous prepotential

$$\mathcal{F}(\underline{t}) = \frac{F(X)}{(X^0)^2}.$$

The Kähler potential and Yukawa coupling become

$$e^{-K} = i[(t^{\bar{\imath}} - t^i)(\mathcal{F}_i + \mathcal{F}_{\bar{\imath}}) + 2(\mathcal{F} - \bar{\mathcal{F}})]$$

$$C_{ijk}(\underline{t}) = \partial_i \partial_j \partial_k \mathcal{F}(\underline{t}).$$

In addition, we have

$$C_{ijk}(\underline{t}) = \frac{1}{(X^0)^2} \frac{\partial z_\ell}{\partial t^i} \frac{\partial z_m}{\partial t^j} \frac{\partial z_n}{\partial t^k} C_{\ell mn}(\underline{z}).$$

In inhomogeneous coordinates, we can obtain the period vector from $\mathcal{F}(\underline{t})$ by

$$\vec{\Pi}^T = X^0(2\mathcal{F}(\underline{t}) - t^i \, \partial_i \mathcal{F}(\underline{t}), \partial_i \mathcal{F}(\underline{t}), 1, t^i).$$

The Picard-Fuchs equation is equivalent to the Gauss-Manin connection

$$(\partial_i - A_i(\underline{z}))\vec{\Pi}(\underline{z}) = 0,$$

where $i = 1, \ldots, r$ and $A_i(\underline{z}) \in \mathbb{Q}[\underline{z}]$. If we set

$$\hat{\Omega}_0 = \alpha_0 + t^i \alpha_i - \partial_i \mathcal{F} \beta^i - (2\mathcal{F} - t^i \, \partial_i \mathcal{F})\beta^0$$
$$\hat{\chi}_i = \alpha_i - \partial_i \, \partial_j \mathcal{F} \beta^j - (\partial_a \mathcal{F} - t^i \, \partial_i \, \partial_j \mathcal{F})\beta^0$$
$$\hat{\chi}^i = -\beta^i + t^a \beta^0$$
$$\hat{\Omega}^0 = \beta^0,$$

this becomes

$$\partial_i \begin{pmatrix} \Omega_0 \\ \chi_j \\ \chi^j \\ \Omega^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_i^k & 0 & 0 \\ 0 & 0 & C_{ijk}(\underline{t}) & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Omega_0 \\ \chi_j \\ \chi^j \\ \Omega^0 \end{pmatrix}.$$

2.6. Special geometry.

On the moduli space of complex structures $\mathcal{M}_{\mathsf{cs}} \coloneqq \mathcal{M}(\underline{z})$, we have metric connections from the Weil-Petersson metric and the Kähler line bundle connection. On sections

$$V_{j\bar{\jmath}} \in T_{1,0}^* \mathcal{M}_{\mathsf{cs}} \otimes T_{0,1}^* \mathcal{M}_{\mathsf{cs}} \otimes \mathcal{L}^{\otimes n} \otimes \bar{\mathcal{L}}^{\otimes m},$$

the covariant derivatives are given by

$$D_i V_{j\bar{\jmath}} = \partial_i V_{j\bar{\jmath}} - \Gamma_{ij}^\ell V_{\ell\bar{\jmath}} + nK_i V_{j\bar{\jmath}}$$
$$D^i V_{j\bar{\jmath}} = \partial_{\bar{\imath}} V_{j\bar{\jmath}} - \Gamma_{\bar{\imath}\bar{\jmath}}^{\bar{\ell}} V_{i\bar{\ell}} + mK_{\bar{\imath}} V_{j\bar{\jmath}}.$$

Using the covariant derivatives, we see that $\chi_i \in H^{n-1,1}(X)$ and $\bar{\chi}_{\bar{\imath}} \in H^{1,n-1}(X)$.

Proceeding more systematically, repeated applications of D_i yield

$$D_i\Omega = (\partial_i + K_i)\Omega = \chi_i$$
$$D_i\chi_j = -ie^K C_{ijk} G^{k\bar{k}} \chi_{\bar{k}}$$
$$D_i\chi_{\bar{k}} = G_{i\bar{k}}\bar{\Omega}$$
$$D_i\bar{\Omega} = 0.$$

Using the relation

$$[D_i, D_j]\chi_k = -G_{i\bar{\jmath}}\chi_k + R_{i\bar{\jmath}k}^p \chi_p,$$

we further deduce

$$[D_i, D_{\bar{\jmath}}]\chi_k = G_{k\bar{\jmath}}\chi_i - e^{2K} C_{\bar{\jmath}\bar{m}\bar{n}} G^{m\bar{m}} G^{n\bar{n}} C_{ikm}\chi_n.$$

It follows that

$$[D_i, D_{\bar{\jmath}}]_\ell^k = -R_{i\bar{\jmath}\ell}^k = \partial_{\bar{\jmath}}\Gamma_{i\ell}^k = \delta_\ell^k G_{\bar{\jmath}i} + \delta_i^k G_{\bar{\jmath}\ell} - C_{\bar{\jmath}}^{km} C_{i\ell m},$$

where $C_{\bar{\jmath}}^{k\ell} = e^{2K} C_{\bar{\jmath}\bar{k}\bar{\ell}} G^{k\bar{k}} G^{\ell\bar{\ell}}$. This equation is known as the *special geometry equation* and is the integrability condition for the existence of the prepotential $\mathcal{F}$.

2.7. **Genus one predictions.** In a $(2,2)$ theory, the topological torus partition function is defined by

$$F(t,\bar{t}) = \frac{1}{2}\int_{\mathcal{F}_{\mathsf{fund}}} \frac{\mathrm{d}^2\tau}{\operatorname{Im}\tau} \operatorname{Tr}(-1)^F F_L F_R q^H \bar{q}^{\bar{H}}$$

as an integral of the fermion number projected partition function over the fundamental region of the torus. Using its relationship to the family index of Ray-Singer analytic torsion [FLY08], it satisfies the holomorphic anomaly equation

$$\partial_i \bar{\partial}_{\bar{j}} F_1 = \frac{1}{2} C_{ijk} C_{\bar{j}}^{k\ell} - \Big(\frac{\chi}{24} - 1\Big) G_{i\bar{j}}.$$

Using $G_{i\bar{j}} = \partial_{\bar{j}}\partial_i K = \partial_{\bar{j}} K_i$ and the special geometry equation, we can integrate the holomorphic anomaly equation to obtain

$$(1) \quad F_1 = -\frac{1}{2}\log\det G_{i\bar{j}} + \left(\frac{1}{2}(h_{11}+1) - \frac{\chi}{24} + 1\right)K + \log f_1(z) + \log f_1(\bar{z}).$$

2.8. **Propagators.** The *propagators* are **an-holomorphic** sections S^{ij}, S^i, and S of $\mathcal{L}^{-2}\otimes\operatorname{Sym}^2(T\mathcal{M}_{\mathsf{cs}}^{1,0})$, $\mathcal{L}^{-2}\otimes T\mathcal{M}_{\mathsf{cs}}^{1,0}$, and $\mathcal{L}^{-2}$, respectively, defined by

$$\partial_{\bar{i}} S^{jk} = C_{\bar{i}}^{jk}, \qquad \partial_{\bar{j}} S^k = G_{i\bar{j}} S^{ik}, \qquad \text{and} \qquad \partial_{\bar{j}} S = G_{i\bar{j}} S^i.$$

We can integrate the first equation using the special geometry equation and the observation that the only contributions to $C_{\bar{j}}^{km} C_{i\ell m}$ are derivatives in $\bar{\partial}_{\bar{j}}$. Therefore, if there exists an index i such that $[C_{(i)}]_m$ is invertible, we obtain

$$S^{km} = C^{(i)k\ell}(\delta_\ell^m K_{(i)} + \delta_{(i)}^m K_\ell + \Gamma_{(i)\ell}^m + q_{(i)\ell}^m),$$

where $q_{(i)\ell}^m$ is the holomorphic propagator ambiguity.

Following Alim-Länge [AL07], it is convenient to shift the remaining propagators as

$$\tilde{S}^i = S^i - S^{ij} K_j;$$

$$\tilde{S} = S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j.$$

Applying the special geometry equation, we obtain

$$\partial_i S^{jk} = C_{imn} S^{mj} S^{nk} + \delta_i^j \tilde{S}^k + \delta_i^k \tilde{S}^j - q_{im}^j S^{mk} - q_{im}^k S^{mj} + q_i^{jk};$$

$$\partial_i \tilde{S}^j = C_{imn} S^{mj}\tilde{S}^n + 2\delta_i^j\tilde{S} - q_{im}^j\tilde{S}^m - q_{ik}S^{jk} + q_i^j;$$

$$\partial_i\tilde{S} = \frac{1}{2}C_{imn}\tilde{S}^m\tilde{S}^n - q_{ij}\tilde{S}^j + q_i;$$

$$\partial_i K_j = K_i K_j - C_{ijn} S^{mn} K_m + q_{ij}^m K_m - C_{ijk}\tilde{S}^k + q_{ij}.$$

Here, all of the ambiguities are in fact rational functions in $\underline{z}$ with rational coefficients. This allows us to obtain explicit formulae

$$\tilde{S}^k = \frac{1}{2}(\partial_k S^{kk} - C_{k\ell m} S^{k\ell} S^{km} + 2q_{k\ell}^k S^{\ell k} - q_k^{kk});$$

$$\tilde{S} = \frac{1}{2}(\partial_\ell\tilde{S}^\ell - C_{k\ell m}\tilde{S}^k S^{\ell m} + q_{\ell m}^\ell\tilde{S}^m + q_{\ell m} S^{\ell m} - q_\ell^\ell).$$

Taking the holomorphic limit of the propagators, we obtain

$$\mathcal{S}^{km} = C^{(i)k\ell}(\delta_\ell^m\mathcal{K}_{(i)} + \delta_{(i)}^m\mathcal{L}_\ell - \Upsilon_{(i)\ell}^m + q_{(i)\ell}^m),$$

where we take the holomorphic limits $e^{-K} \to e^{-\mathcal{K}} = X_*^0$, $K_i \to \mathcal{K}_i = -\partial_i \log(X_*^0)$, and $\Gamma_{jk}^i \to \Upsilon_{jk}^i = \frac{\partial z^i}{\partial t_*^a} \frac{\partial^2 t_*^a}{\partial z^j \partial z^k}$.

Returning to the genus one situation, the holomorphic anomaly equation becomes

$$\partial_i F_1 = C_i - \left(\frac{\chi(X)}{24} - 1 \right) K_i,$$

where we define $C_i = \frac{1}{2} S^{jk} C_{ijk} + f_i^{(1)}$. This can be integrated to obtain (1). The holomorphic limit is given by

$$\mathcal{F}_1 = -\frac{1}{2} \log \det \left(\frac{\partial t_*^a}{\partial z_i} \right) + \left(\frac{\chi(X)}{24} - \frac{1}{2} h_{11} + 3 \right) \log \frac{X_*^0}{(2\pi i)^3} + f_1(z),$$

where the holomorphic ambiguity is given by

$$f_1(z) = \log \left(z^{-\frac{c_2 \cdot D + 12}{24}} \Delta_{\mathrm{con}}^{-\frac{1}{12}} \right).$$

2.9. **Higher-genus predictions.** In higher genus, the holomorphic anomaly equation becomes

$$\bar{\partial}_{\bar{k}} = \frac{1}{2} \bar{C}_{\bar{k}}^{ij} \left(D_i D_j F_{g-1} + \sum_{r=1}^{g-1} D_i F_r D_j F_{g-r} \right).$$

This can be rewritten as the system of equations

$$\frac{\partial F_g}{\partial S^{ij}} = \frac{1}{2} D_i D_j F_{g-1} + \frac{1}{2} \sum_{h=1}^{g-1} D_i F_h D_j F_{g-h}, \qquad \frac{\partial F_g}{\partial K_i} + S^i \frac{\partial F_g}{\partial S} + S^{ij} \frac{\partial F_g}{\partial S^j} = 0$$

assuming that S^{ij}, S^i, S, and K are algebraically independent. Using the shifted propagators, we obtain $\frac{\partial F_g}{\partial K_j} = 0$ and

$$\frac{\partial F_g}{\partial S^{jk}} - \frac{1}{2} \frac{\partial F_g}{\partial \tilde{S}^k} K_j - \frac{1}{2} \frac{\partial F_g}{\partial \tilde{S}^j} + \frac{1}{2} \frac{\partial F_g}{\partial \tilde{S}} K_j K_k + \frac{1}{2} D_j D_k F_{g-1} + \frac{1}{2} \sum_{h=1}^{g-1} D_j F_h D_k F_{g-h}.$$

Using the method of direct integration, due to Yamaguchi-Yau, Grimm-Klemm-Mariño-Weiss, and Alim-Länge [YY04; GKMW07; AL07], we obtain

$$\frac{\partial F_g}{\partial S^{ij}} = \frac{1}{2} \partial_i (\partial_j' F_{g-1}) + \frac{1}{2} (C_{ij\ell} \tilde{S}^{\ell k} - q_{ij}^k) \partial_k' F_{g-1} + \frac{1}{2} (C_{ijk} \tilde{S}^k - q_{ij}) c_{g-1}$$

$$+ \sum_{h=1}^{g-1} \partial_i' F_h \partial_j' F_{g-h};$$

$$\frac{\partial F_g}{\partial \tilde{S}^i} = (2g - 3) \partial_i' F_{g-1} + \sum_{h=1}^{g-1} c_h \partial_i' F_{g-h};$$

$$\frac{\partial F_g}{\partial \tilde{S}} = (2g - 3) c_{g-1} + \sum_{h=1}^{g-1} c_h c_{g-h}.$$

Here, we set

$$c_g = \begin{cases} \frac{\chi}{24} - 1 & g = 1 \\ (2g - 2) F_g & g > 1 \end{cases} \qquad \text{and} \qquad \partial_i' F_g = \begin{cases} \partial_i F_g + \left(\frac{\chi}{24} - 1 \right) K_i & g = 1 \\ \partial_i F_g & g > 1. \end{cases}$$

Therefore, we can solve for a polynomial $F_g(S^{ij}, \tilde{S}^i, \tilde{S}, z)$, which is a weighted polynomial of degree $3g - 3$ with the weights $1, 2, 3, 0$. For example, in the one-parameter case, we obtain

$$
\begin{aligned}
F_2 &= \frac{5}{24} C_{111}^2 (S^{11})^3 + \frac{1}{8}(\partial_1 C_{111} - 3 C_{111} q_{11}^1 + 4 C_{111} f_1^{(1)})(S^{11})^2 \\
&\quad + \left(\frac{1}{4} q_1^{11} C_{111} + \frac{1}{2} \partial_1 f_1^{(1)} + \frac{1}{2} f_1^{(1)}(f_1^{(1)} - q_{11}^1) + \frac{1}{2}\left(1 - \frac{\chi}{24}\right) q_{11} \right) S^{11} \\
&\quad + \frac{\chi}{48}(C_{111} S^{11} + 2 f_1^{(1)}) \tilde{S}^1 + \frac{\chi}{24}\left(\frac{\chi}{24} - 1\right) \tilde{S} + f_2(z).
\end{aligned}
$$

2.10. **Boundary conditions.** The most difficult part of computing F_g is the degree-zero part $f_g(\underline{z})$, which is known as the *holomorphic or modular ambiguity*. In the hypergeometric cases, we have

$$
f_{g>1}(z) = \frac{\sum_{k=0}^{2g-2} a_k z^k}{(1 - \mu^{-1} z)^{2g-2}} + \sum_{k=1}^{c_\infty} b_k z^k,
$$

where the number c_∞ depends on the type of singularity at $z = \infty$. At $z = 0$, the boundary conditions come from the asymptotics

$$
\begin{aligned}
\left(\frac{\lambda}{g_2}\right)^{2g-2} \mathcal{F}_g &= \sum_\beta \left((-1)^{g-1} \frac{(2g-2) B_{2g}}{(2g)!} n_0^\beta + \frac{2(-1)^g n_2^\beta}{(2g-2)!} + \cdots \right) \mathrm{Li}_{3-2g}(Q_\beta) \\
&= \frac{(-1)^g (2g)!}{(2\pi)^{4g-2} g(2g-2)} \chi(X) + \mathcal{O}(Q_\beta).
\end{aligned}
$$

The second equality requires the computation of degree-zero GW invariants due to Faber-Pandharipande.

In the one-parameter case, the Castelnuovo bound implies that the GV invariants n_g^d are nonzero only when the genus is strictly less than $g_{\max}(d) \le \left\lfloor \frac{d^2}{2\kappa} + \frac{d}{2} \right\rfloor + 1$ in general and $g_{\max}(d) \le \left\lfloor \frac{2d^2}{3\kappa} + \frac{d}{3} \right\rfloor + 1$ when $0 < d < \kappa$. This was first observed by Huang-Klemm-Quackenbush [HKQ09] and proved in 2023 by Feyzbakhsh (in the paper [Ale+24] of Alexandrov et. al). Because curves of degree β of genus $g_{\max}(\beta)$ are smooth, the associated invariants may be obtained by the formula

$$
n_{g,\beta} = (-1)^{\dim_\mathbb{C} \mathcal{M}_\beta} \chi(\mathcal{M}_\beta),
$$

where $\mathcal{M}_\beta$ denotes the deformation space of the curve. For the one-parameter hypergeometric cases, we obtain

$$
n_{g_{\max}(\kappa d), \kappa d} = \begin{cases} \frac{\omega(\omega-1)}{2} & d = 1 \\ (-1)^{\kappa \frac{d(d-1)}{2}} \omega\left(\omega + \frac{\kappa d(d-1)}{2}\right) & \text{otherwise.} \end{cases}
$$

The most important boundary condition is the gap condition at the conifold point $z = \mu$, where a 3-cycle with the topology of a lens space vanishes. Here, this says that we have

$$
(X_{\mathsf{con}}^0)^{2g-2} \mathcal{F}_g = \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{\frac{1}{2}}}{t_{\mathsf{con}}} \right)^{2g-2} + \mathcal{O}(t_{\mathsf{con}}^0)
$$

for all $g > 1$, where t_{con} is the local coordinate at the conifold point given by the ratio of a vanishing and non-vanishing period.

3. An axiomatic approach to enumerative geometry (Patrick Lei)

3.1. **Moduli of curves and CohFTs.** We will denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable curves of genus g with n marked points. This is nonempty if and only if $2g - 2 + n > 0$, in which case it is a Deligne-Mumford stack of dimension $3g - 3 + n$. There is a combinatorial structure to the collection of $\overline{\mathcal{M}}_{g,n}$, which is given by a collection of morphisms.

- There is the gluing morphism
$$q \colon \overline{\mathcal{M}}_{g,n+1} \times \overline{\mathcal{M}}_{h,m+1} \to \overline{\mathcal{M}}_{g+h,n+m},$$
  which takes two curves and glues them along the last marked point to form a node;
- There is the self-gluing morphism
$$s \colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$
  which glues the last two marked points.

There is of course another interesting map, which is the forgetful map
$$\mathrm{pr} \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$
given by deleting the last marked point and then stabilizing. We are now able to define our enumerative theories of interest, which include Gromov-Witten theory.

Definition 3.1. Given a vector space V with a nondegenerate symmetric bilinear form η and a unit element $\mathbf{1} \in V$, a *Cohomological Field Theory (CohFT)* [KM94] Ω on V is a collection $\Omega_{g,n}$ of S_n-equivariant linear maps
$$\Omega_{g,n} \colon V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$
which satisfy the basic identity
$$\Omega_{0,3}(\mathbf{1}, u, v) = \eta(u, v)$$
and the following combinatorial identities (we will let e_μ be a basis of V such that $e_1 = \mathbf{1}$ and e^μ be the dual basis):
$$q^* \Omega_{g+h,n+m}(\mathbf{v}_1, \mathbf{v}_2) = \sum_\mu \Omega_{g,n+1}(\mathbf{v}_1, e_\mu) \cdot \Omega_{h,m+1}(\mathbf{v}_2, e^\mu);$$
$$s^* \Omega_{g,n}(\mathbf{v}) = \sum_\mu \Omega_{g,n+2}(\mathbf{v}, e_\mu, e^\mu).$$

If in addition the identity
$$\mathrm{pr}^* \Omega_{g,n}(\mathbf{v}) = \Omega_{g,n+1}(\mathbf{v}, \mathbf{1})$$
is satisfied, then we say the CohFT has a *flat unit* (or satisfies the *string equation*).

All of this can be generalized to super vector spaces, but for simplicity we will not deal with this case.

Example 3.2. The most important example of a CohFT is the Gromov-Witten theory of a smooth projective variety X. Here, recall that the source of a stable map is a **prestable** curve, so there is a stabilization morphism
$$\mathrm{st} \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$$

which forgets the map and stabilizes the curve. Then, working over the Novikov ring, we will set $V = H^*(X)$, η to be the Poincaré pairing, and $\mathbf{1}$ to be the fundamental class of X. The linear maps $\Omega_{g,n}^X$ are given by the formulae

$$\Omega_{g,n}^X(\boldsymbol{\tau}) \coloneqq \sum_\beta q^\beta \operatorname{st}_* \left(\prod_{i=1}^n \operatorname{ev}_i(\tau_i) \cap [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \right),$$

where $\operatorname{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ is the i-th evaluation map.

Example 3.3. Let $\pi \colon \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve. Then the *Hodge bundle* is the vector bundle

$$\mathbb{E}_{g,n} \coloneqq \pi_* \omega_\pi.$$

We then consider the vector space $V = \mathbb{C}$ with the usual pairing and define the CohFT $\Omega^{\mathbb{E}}$ by the formula

$$\Omega_{g,n}^{\mathbb{E}} \coloneqq c(\mathbb{E}_{g,n}).$$

The gluing axioms are satisfied by the equation $q^* \mathbb{E}_{g+h} = \mathbb{E}_g \oplus \mathbb{E}_h$ and the exact sequence

$$0 \to \mathbb{E}_{g-1} \to \mathbb{E}_g \to \mathcal{O} \to 0.$$

Here, because genus zero components do not contribute to global sections of ω and $\mathbb{E}$ does not depend on the marked points, this CohFT has a flat unit.

Remark 3.4. We will see later that this is related to the GW CohFT of a point by the quantum Riemann-Roch theorem [CG07].

Given a CohFT Ω, we may produce invariants by pairing the classes $\Omega_{g,n}(\mathbf{v})$ with various cohomology classes on $\overline{\mathcal{M}}_{g,n}$. The most important ones to consider are the classes

$$\bar{\psi}_i \coloneqq c_1(s_i^* \omega_\pi),$$

where $s_i \colon \overline{\mathcal{M}}_{g,n} \to \mathcal{C}_{g,n}$ is the i-th marked point.

Remark 3.5. In Gromov-Witten theory, these are called *ancestors*. There are also descendants, which are given by the same formula except using the moduli space $\mathfrak{M}_{g,n}$ of prestable curves instead of $\overline{\mathcal{M}}_{g,n}$, where we factor the morphism st defined above as

$$\overline{\mathcal{M}}_{g,n}(X, \beta) \xrightarrow{\text{forget map}} \mathfrak{M}_{g,n} \xrightarrow{\text{stabilize}} \overline{\mathcal{M}}_{g,n}.$$

The descendant classes are denoted by ψ_i.

Example 3.6. We will calculate an invariant which will appear in the GW theory of Calabi-Yau threefolds. For any CohFT Ω, consider the invariant

$$\langle \mathbf{1}\bar{\psi}_1 \rangle_{1,1}^\Omega \coloneqq \int_{\overline{\mathcal{M}}_{1,1}} \Omega_{1,1}(\mathbf{1})\bar{\psi}_1.$$

Using the second gluing equation, the degree zero part of $\Omega_{1,1}(\mathbf{1})$ is equal to

$$\sum_\mu \Omega_{0,3}(\mathbf{1}, e_\mu, e^\mu) = \sum_\mu \eta(e_\mu, e^\mu),$$

which yields the (graded) dimension $\chi(V)$ of V. Therefore, we obtain

$$\langle \mathbf{1}\bar{\psi}_1 \rangle_{1,1}^\Omega = \frac{\chi(V)}{24}.$$

Example 3.7. Consider the GW CohFT of a point. This is given by $V = \mathbb{C}$ with the usual pairing and the formula

$$\Omega_{g,n} = 1.$$

Then we define the invariants

$$\left\langle \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n} \right\rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n} \cdot \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n}.$$

Theorem 3.8 ([Kon92]). *The function*

$$\mathcal{D} := \exp\left(\sum_{g,n} \frac{\hbar^{g-1}}{n!} \sum_{a_1 + \cdots + a_n = n} \left\langle \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n} \right\rangle_{g,n} t_{a_1} \cdots t_{a_n} \right)$$

is annihilated by the operators

$$L_{-1} := -\frac{\partial}{\partial t_0} + \frac{\hbar^{-1}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i};$$

$$L_0 := -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16};$$

$$L_n := -\frac{(2n+3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1}} + \sum_{i=0}^{\infty} \frac{(2i+2n+1)!!}{(2i-1)!!2^{n+1}} t_i \frac{\partial}{\partial t_{i+n}}$$

$$+ \frac{\hbar}{2} \sum_{i=0}^{n-1} \frac{(2i+1)(2i-1)\cdots(2i+1-2n)}{2^{n+1}} \frac{\partial^2}{\partial t_i \, \partial t_{n-1-i}}.$$

Remark 3.9. This result (Virasoro constraints for a point) is equivalent to $\mathcal{D}$ being a tau-function for the KdV hierarchy and has been generalized by various authors.

3.2. **The genus-zero picture.** A CohFT in genus zero defines a Frobenius manifold [Dub96]. In particular, there is a product structure defined by the formula

$$\eta(\tau_1 \star_\tau \tau_2, \tau_3) := \sum_n \frac{1}{n!} p_* \langle \tau_1, \tau_2, \tau_3, \tau, \ldots, \tau \rangle_{0,3+n}^{\Omega}$$

and the quantum connection, which is defined by the formula

$$\nabla_\mu := \partial_{e_\mu} + \frac{1}{z} e_\mu \star_\tau .$$

The structure of a Frobenius manifold comes from a function $\mathcal{F}_0$, which is known as the *genus-zero descendant potential* and satisfies a set of PDEs, which are the string equation, dilaton equation, and an infinite set of topological recursion relations, where we write $\mathbf{v} = \sum_{\mu,n} t_n^\mu e_\mu z^n \in V[\![z]\!]$:

$$\frac{\partial}{\partial t_0^1} \mathcal{F}_0 = \frac{1}{2}(\mathbf{v}_0, \mathbf{v}_0) + \sum_{n=0}^{\infty} \sum_\mu t_{n+1}^\mu \frac{\partial}{\partial t_n^\mu} \mathcal{F}_0;$$

$$\frac{\partial}{\partial t_1^1} \mathcal{F}_0 = \sum_{n=0}^{\infty} \sum_\mu t_n^\mu \frac{\partial}{\partial t_n^\mu} \mathcal{F}_0 - 2\mathcal{F}_0;$$

$$\frac{\partial^3}{\partial t_{k+1}^\alpha \, \partial t_m^\beta \, \partial t_n^r} \mathcal{F}_0 = \sum_{\mu,\nu} \frac{\partial^2}{\partial t_k^\alpha \, \partial t_0^\mu} \mathcal{F}_0 \cdot \eta^{\mu\nu} \cdot \frac{\partial^3}{\partial t_0^\nu \, \partial t_m^\beta \, \partial t_n^r} \mathcal{F}_0.$$

To state the following result, we consider the infinite-dimensional vector space $V(\!(z^{-1})\!)$ (or a completion of this) with the symplectic form

$$(\mathbf{f}(z), \mathbf{g}(z)) := \operatorname{Res}_{z=0} \eta(\mathbf{f}(-z), \mathbf{g}(z)).$$

We also consider the new variable[6] $\mathbf{q} = \mathbf{v} - z$, so $\mathcal{F}_0$ is now a function near $\mathbf{q} = -z$.

Theorem 3.10 ([Giv04]). *$\mathcal{F}$ satisfies the above PDEs if and only if the graph $\mathcal{L}$ of $d\mathcal{F}$ is a Lagrangian cone with vertex at $\mathbf{q} = 0$ such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along $zL \subset L$.*

We may recover the Lagrangian cone $\mathcal{L}$ by the following procedure. We find a fundamental solution

$$S = \operatorname{Id} + S_1 z^{-1} + S_2 z^{-2} + \cdots$$

to the quantum connection, which satisfies the equation

$$z \frac{\partial}{\partial t^\mu} S = e_\mu \star S.$$

Then, setting $J := zS^*(z)\mathbf{1}$, we can then recover the Lagrangian cone by the following procedure:

- The derivatives $\frac{\partial}{\partial t^\mu} J$ form a basis of $L \cap V[\![z^{-1}]\!]$;
- This implies that $z \frac{\partial^2}{\partial t^\mu \partial t^\nu} \in L \cap V[\![z^{-1}]\!]$. Writing these in terms of the first derivatives and using the fact that J solves the quantum connection, we recover the Frobenius structure and hence the Lagrangian cone.

In Gromov-Witten theory, there is an explicit formula for the fundamental solution in terms of descendant invariants. It is given by

$$S_\tau(z)\phi := \phi + \sum_\mu \sum_{n,\beta} \frac{q^\beta}{n!} e^\mu \left\langle \frac{e_\mu}{z - \psi_1}, \phi, \tau, \ldots, \tau \right\rangle^X_{0,n+2,\beta}.$$

Example 3.11. Let X be the quintic threefold. We will use the genus-zero mirror theorem of Givental to compute the quantum product on $H^*(X)$. Let

$$I(q, z) := z \sum_{d \geq 0} q^d \frac{\prod_{m=1}^{5d}(5H + mz)}{\prod_{m=1}^d (H + mz)^5}$$

$$= zI_0(q) + I_1(q) \cdot H + I_2(q) \cdot \frac{H^2}{z} + I_3(q) \cdot \frac{H^3}{z^2}$$

be the (very small) I-function of X. Setting $Q(q) := qe^{\frac{I_1(q)}{I_0(q)}}$, the mirror theorem states that

$$J(0, Q(q), z) = \frac{I(q, z)}{I_0(q)}.$$

Because the mirror map $q \mapsto Q(q)$ corresponds to setting $\tau = \frac{I_1(q)}{I_0(q)} H$ by the divisor equation, we can use the mirror theorem to compute $\star_\tau$. Because of our nonstandard choice of the I-function, the quantum connection becomes the ODE

$$(H + zD)S_\tau(z)^* = S_\tau(z)^* \cdot I_{11} H \star_\tau,$$

---

[6]This is called the *dilaton shift*.

where $D := q\frac{\mathrm{d}}{\mathrm{d}q}$ (here, the coordinate is $\log q$) and $I_{11} := 1 + D\left(\frac{I_1(q)}{I_0(q)}\right)$. An explicit computation using the mirror theorem and the results of Zagier-Zinger [ZZ08] yields

$$I_{11}H + \cdots = S_\tau(z)^*(I_{11}H \star_\tau \mathbf{1})$$

$$\left(1 + D\left(\frac{\frac{I_1}{I_0} + D\left(\frac{I_2}{I_0}\right)}{I_{11}}\right)\right)H^2 + \cdots = S_\tau(z)^*(I_{11}H \star_\tau H)$$

$$I_{11}H^3 = S_\tau(z)^*(I_{11}H \star_\tau H^2).$$

Because S_τ begins with the identity, this computes $H\star_\tau$.

3.3. R-matrix action.

In the early 2000s, Givental [Giv01a; Giv04] discovered a remarkable property of axiomatic enumerative theories, namely that one can transform CohFTs by the action of matrices $R(z) = R_0 + R_1 z + R_2 z^2 + \cdots \in \mathrm{Hom}(V, V')[\![z]\!]$ which satisfy the property

$$R^*(-z)R(z) = \mathrm{Id}_V.$$

Traditionally, the literature [PPZ15] requires that $V = V'$ and $R_0 = \mathrm{Id}$, but the MSP group [CGL19] has removed this restriction and also allows $\dim V < \dim V'$.

In order to define the action of R on a CohFT Ω defined on V, we need to review some of the combinatorial structure of $\overline{\mathcal{M}}_{g,n}$. For a curve $C \in \overline{\mathcal{M}}_{g,n}(\mathbb{C})$, we may consider the dual graph of C, which has vertices, edges, and legs, which are defined as follows:

- The vertices correspond to irreducible components of C and are labelled by a non-negative integer, which is the genus;
- The edges correspond to nodes of C (in particular, we allow loops);
- The legs correspond to marked points.

Any graph which appears as the dual graph of a stable curve is called a *stable graph*.
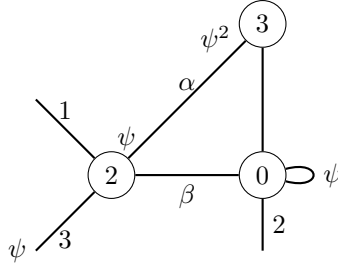
FIGURE 1. Example of a stable graph in $\overline{\mathcal{M}}_{7,3}$ and associated tautological class. This stable graph describes the image of a map $\overline{\mathcal{M}}_{2,4} \times \overline{\mathcal{M}}_{3,2} \times \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{7,3}$.

The first type of action on CohFTs is a translation action. Let $T \in V[\![z]\!]$. For a CohFT Ω, we define

$$(T\Omega)_{g,n}(\mathbf{v}) := \sum_m \frac{1}{m!} p_* \Omega_{g,n+m}(\mathbf{v}, T, \ldots, T)$$

whenever the infinite sum makes sense.

Example 3.12. If Ω^X is the GW CohFT of a smooth projective variety X and $\tau \in H^2(X)$ is a divisor class, then the infinite sum makes sense and we can define the shifted GW CohFT $\Omega^{X,\tau}$ of X. For example, if X is the quintic threefold and $\tau = \frac{I_1}{I_0}H$, then shifting to τ is the same as the mirror map $q \mapsto Q(q) = qe^{\frac{I_1}{I_0}}$.

The second type of action is the action of a matrix R as in the beginning of this section. Let $G_{g,n}$ define the set of all stable graphs of genus g with n legs. Then we define
$$(R\Omega)_{g,n} := \sum_{\Gamma \in G_{g,n}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\,\mathrm{Cont}_\Gamma,$$
where Cont_Γ is defined by the following construction:
- At the i-th leg, we place the map
$$R(-\bar\psi_i)^* \in \mathrm{Hom}(V', V)[\![\bar\psi_i]\!];$$
- At every edge, we place the bivector
$$V(\bar\psi_1, \bar\psi_2) := \frac{\sum_\mu e_\mu \otimes e^\mu - \sum_{\mu'} R(-\bar\psi_1)^* e_{\mu'} \otimes R(-\bar\psi_2)^* e^{\mu'}}{\bar\psi_1 + \bar\psi_2};$$
- At every vertex, we place the linear map
$$\Omega_{g_v, n_v} : V^{\otimes n_v} \to H^*(\overline{\mathcal{M}}_{g_v, n_v});$$
- Finally, we consider the pushforward in cohomology along the gluing morphism $\prod_v \overline{\mathcal{M}}_{g_v, n_v} \to \overline{\mathcal{M}}_{g,n}$.

Definition 3.13. Let R be as above. Then we define the translation
$$T_R := z(\mathbf{1} - R(-z)^*\mathbf{1}') \in zV'[\![z]\!].$$
Whenever it makes sense, we define
$$R.\Omega := RT\Omega.$$

Theorem 3.14 ([CGL19]). *Suppose we work with coefficients in $\mathbb{C}[\![q]\!]$. Then if*
$$T_R \in z^2 V[\![z]\!] + zqV[\![z]\!],$$
$R.\Omega$ is a well-defined CohFT. Moreover, if $\dim V = \dim V'$, $\mathbf{1}'$ is a unit for $R.\Omega$.

We would like to remark a bit more about the translation action when $R_0 \neq \mathrm{Id}$. For simplicity, we will assume that $V = V'$ and $R_0 \mathbf{1} = c \cdot \mathbf{1}$ for some constant c. Then we set $\tilde{T}_R := z(\mathbf{1} - cR(z)^{-1}\mathbf{1})$ and use the dilaton equation to compute

$$
\begin{aligned}
T_R\Omega_{g,n}(\mathbf{v}) &= \sum_{m=0}^\infty \frac{1}{m!} p_*\Omega_{g,n+m}(\mathbf{v}, T_R, \dots, T_R) \\
&= \sum_{k,\ell \geq 0} \frac{1}{k!\ell!} p_*\Omega_{g,n+k+\ell}(\mathbf{v}, ((1-c^{-1})\mathbf{1}\cdot\psi)^{\otimes k}, (c^{-1}\tilde{T}_R)^{\otimes \ell}) \\
&= \sum_{k,\ell \geq 0} \frac{(1-c^{-1})^k \cdot c^{-\ell}}{\ell!}\binom{2g-2+n+k+\ell-1}{k} p_*\Omega_{g,n+\ell}(\mathbf{v}, \tilde{T}_R^{\otimes \ell}) \\
&= \sum_{m=0}^\infty \frac{c^{2g-2+n}}{m!} p_*\Omega_{g,n+m}(\mathbf{v}, \tilde{T}_R, \dots, \tilde{T}_R)
\end{aligned}
$$

whenever this makes sense.

3.4. **Reconstruction theorem.** Recall that every CohFT defines a Frobenius algebra. We will call a CohFT *semisimple* if the corresponding Frobenius algebra is semisimple.

Theorem 3.15 ([Tel12])**.** *Let Ω be a semisimple CohFT with flat unit and ω be its topological part. If Ω is semisimple, there exists a unique*

$$R = \mathrm{Id} + R_1 z + \cdots \in \mathrm{End}(V)[\![z]\!]$$

such that

$$\Omega = R.\omega.$$

Example 3.16. Recall the Hodge bundle CohFT from before. Recall that it is given by the formula

$$\Omega_{g,n}^{\mathbb{E}} = c(\mathbb{E}) = 1 + \lambda_1 + \cdots + \lambda_g.$$

Taking the degree zero part, we see that $\omega^{\mathbb{E}}$ is the GW CohFT of a point. Using Mumford's computation

$$\mathrm{ch}(\mathbb{E}) = g + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\kappa_{2k-1} + \frac{1}{2}\iota_* \sum_{i=0}^{2k-2} \bar{\psi}_1^i \bar{\psi}_2^{2g-2+i}\right),$$

where $\kappa_m = p_* \bar{\psi}_{n+1}^{m+1}$, B_{2k} are the Bernoulli numbers, and ι is the inclusion of the boundary up to a $2:1$ étale cover, and the formula

$$c(E) = \exp\left(-\sum_k (-1)^k (k-1)! \, \mathrm{ch}_k(E)\right)$$

for any vector bundle E (or by just using the quantum Riemann-Roch theorem[7] directly), we obtain the R-matrix

$$R(z) = \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} z^{2k-1}\right) = 1 + \frac{1}{12}z + \cdots.$$

As a sanity check, we will compute $\Omega_{1,1}^{\mathbb{E}}$ using the R-matrix. First, we consider the stable graphs in Figure 2. The first graph Γ_1 gives us the contribution
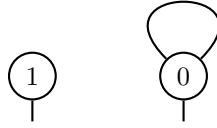
FIGURE 2. Stable graphs for $g = 1, n = 1$

$$\mathrm{Cont}_{\Gamma_1} = T\omega_{1,1}^{\mathbb{E}}(R(z)^{-1}\mathbf{1})$$
$$= \omega_{1,1}^{\mathbb{E}}(R(z)^{-1}\mathbf{1}) + p_*\omega_{1,1}^{\mathbb{E}}(R(z)^{-1}\mathbf{1}, T_R(z)).$$

Using the formula $B_2 = \frac{1}{6}$ and the fact that $\dim \overline{\mathcal{M}}_{1,1} = 1$, this becomes

$$1 - \frac{1}{12}\psi_1 + \frac{1}{12}\kappa_1.$$

[7]There is a sign error in Coates-Givental, which has propagated to the rest of the literature.

The second graph does not receive tail contributions because $\dim \overline{\mathcal{M}}_{0,3} = 0$. The constant term of the edge contribution is $\frac{1}{12}$, so considering the automorphism and pushing forward to $\overline{\mathcal{M}}_{1,1}$ gives us

$$\mathrm{Cont}_{\Gamma_2} = \frac{1}{12}\Delta,$$

where Δ denotes the boundary divisor (with the correct stack structure). Because $\psi_1 = \kappa_1$ in this case, we obtain

$$1 + \lambda_1 = 1 + \frac{1}{12}(\kappa_1 - \psi_1 + \Delta) = 1 + \frac{1}{12}\Delta.$$

3.5. **Operator formalism and geometric quantization.** We will return to the symplectic formalism of Givental. This is more convenient for certain computations, but it is in fact equal to what we have before, at least when we want to calculate generating functions. Recall that we had a Frobenius manifold structure on V and we considered the vector space $\mathcal{V} := V(\!(z^{-1})\!)$ with symplectic form

$$(\mathbf{f}(z), \mathbf{g}(z)) := \mathrm{Res}_{z=0} \eta(\mathbf{f}(-z), \mathbf{g}(z)).$$

If we consider the polarization given by $\mathcal{V}_+ = V[z]$ and $\mathcal{V}_- = z^{-1}V[\![z^{-1}]\!]$, then letting $\mathbf{q}$ be as before (with the dilaton shift), let $\mathbf{p}$ be coordinates on $\mathcal{V}_-$ such that $(\mathbf{q}, \mathbf{p})$ form a system of Darboux coordinates for $\mathcal{V}$.

Definition 3.17. We will call any formal function of the form

$$\mathcal{D} = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1}\mathcal{F}_g\right)$$

an *asymptotic element of the Fock space*.

Then given such an asymptotic element of the Fock space, we will quantize (infinitesimal) symplectic transformations on $\mathcal{V}$ by the following formulae:

$$\widehat{p_a p_b} := \hbar\, \partial_{q_a}\, \partial_{q_b}, \qquad \widehat{p_a q_b} := q_b\, \partial_{q_a}, \qquad \widehat{q_a q_b} = \frac{q_a q_b}{\hbar}.$$

In particular, this will allow us to understand expressions like $\hat{R}\mathcal{D}$. However, we need to be careful because our formulae will involve both the fundamental solution S_τ and the R-matrix, and S_τ is a power series in z^{-1}.[8]

Theorem 3.18 ([Giv01a])**.** *An operator of the form $S(z^{-1}) = \mathrm{Id} + S_1 z^{-1} + \cdots$ acts on (asymptotic) elements of the Fock space by the formula*

$$\hat{S}^{-1}\mathcal{D}(\mathbf{q}) = e^{\frac{1}{2\hbar}W(\mathbf{q},\mathbf{q})}\mathcal{D}([S\mathbf{q}]_+),$$

where $W = \sum \eta(W_{mn}q_m, q_n)$ is defined by the formula

$$\frac{S(w^{-1})^* S(z^{-1}) - \mathrm{Id}}{w^{-1} + z^{-1}} = \sum \frac{W_{mn}}{w^m z^n}.$$

Operators of the form $R(z) = \mathrm{Id} + R_1 z + \cdots$ act by the formula

$$\hat{R}\mathcal{D}(\mathbf{q}) = e^{\frac{\hbar}{2}V(\partial_{\mathbf{q}}, \partial_{\mathbf{q}})}\mathcal{D}(R^{-1}\mathbf{q}),$$

where $V = \sum \eta(p_m, V_{mn}p_n)$ is defined by

$$\frac{R(w)^* R(z) - \mathrm{Id}}{w + z} = \sum V_{mn}w^m z^n.$$

———————

[8]This is because we expand $\frac{1}{z-\psi} = \sum_{n=0}^{\infty} \psi^n z^{-n-1}$.

For a semisimple CohFT, there is a system of *canonical coordinates* u_α such that the 1-form du is a homomorphism of algebras $T_v V \to \mathbb{C}$. Then near a semisimple point, there is a asymptotic solution to the quantum connection of the form

$$\Psi \cdot R \cdot e^{\frac{U}{z}},$$

where Ψ switches from flat coordinates to canonical coordinates and $U = \mathrm{diag}(u_\alpha)$ is the matrix of canonical coordinates. Finally, define

$$C = \frac{1}{2} \int^u \sum R_1^{\alpha\alpha} \, du_\alpha.$$

A corollary of Teleman's reconstruction theorem is the formula

$$\mathcal{D}^X = e^{C(u)} \hat{S}_\tau^{-1} \hat{\Psi} \hat{R} \widehat{e^{\frac{U}{z}}} \prod_{i=1}^{\dim V} \mathcal{D}^{\mathrm{pt}}$$

for the Gromov-Witten theory of any smooth projective variety with semisimple quantum cohomology.

Example 3.19. As a final example, we will apply Teleman's theorem to compute F_1 of any semisimple CohFT. This result was first proved by Givental [Giv98] for the equivariant Gromov-Witten theory of GKM orbifolds and extended to the orbifold case by Zong [Zon16]. The argument we give is due to Guo-Ross [GR19]. Let e_μ denote an idempotent basis for the quantum product. We will compute

$$\int_{\overline{\mathcal{M}}_{1,1}} \Omega_{1,1}(e_\beta).$$

Using the reconstruction theorem, we obtain

$$\int_{\overline{\mathcal{M}}_{1,1}} \Omega_{1,1}(e_\beta) = \int_{\overline{\mathcal{M}}_{1,1}} T_R \omega_{1,1}(R(\bar{\psi})^{-1} e_\beta) + \frac{1}{2} T_R \omega_{0,3}(R(\bar{\psi})^{-1} e_\beta, V(\bar{\psi}_2, \bar{\psi}_3))$$

$$= \int_{\overline{\mathcal{M}}_{1,1}} T_R \omega_{1,1}(e_\beta) - \int_{\overline{\mathcal{M}}_{1,1}} T_R \omega_{1,1}(R_1 e_\beta)\bar{\psi}$$

$$+ \frac{1}{2} \sum_\mu \omega_{0,3}(e_\beta, R_1 e^\mu, e_\mu)$$

$$= \int_{\overline{\mathcal{M}}_{1,1}} (\omega_{1,1}(e_\beta) + p_*(\omega_{1,2}(e_\beta, R_1 \mathbf{1})\bar{\psi}_2^2))$$

$$- \int_{\overline{\mathcal{M}}_{1,1}} (\omega_{1,1}(R_1 e_\beta)\bar{\psi}_1 + p_*(\omega_{1,2}(R_1 e_\beta)\bar{\psi}_2^2)\bar{\psi}_1)$$

$$+ \frac{1}{2} \sum_\mu \eta(e_\beta \star e_\mu, R_1 e^\mu)$$

$$= \frac{1}{24} \sum_\mu (\omega_{0,4}(e_\mu, e^\mu, e_\beta, R_1 \mathbf{1}) - \omega_{0,3}(e_\mu, e^\mu, R_1 e_\beta))$$

$$+ \frac{1}{2} \eta(e_\beta, R_1 e^\beta)$$

$$= \frac{1}{24} \sum_\mu \sum_\nu \omega_{0,3}(e_\mu, e^\mu, e_\nu)\omega_{0,3}(e^\nu, e_\beta, R_1 \mathbf{1})$$

$$- \frac{1}{24} \sum_\mu \omega_{0,3}(e_\mu, e^\mu, R_1 e_\beta) + \frac{1}{2}(R_1)_{\beta\beta}$$

$$= \frac{1}{24}\left(\eta(e^\beta, R_1 \mathbf{1}) - \sum_\mu \eta(e^\mu, R_1 e_\beta)\right) + \frac{1}{2}(R_1)_{\beta\beta}.$$

Using the identities

$$\eta(R_1 \mathbf{1}, e^\beta) - \sum_\mu \eta(R_1 e_\beta, e^\mu) = \sum_\mu \Delta_\mu \int_{\overline{\mathcal{M}}_{0,4}} \Omega_{0,4}(e_\mu, e_\mu, e_\mu, e_\beta)$$

(2)
$$= -\frac{1}{2}\sum_\mu \Delta_\mu \, \partial_{u_\beta} \Delta_\mu^{-1}$$

$$= \frac{1}{2}\sum_\mu \partial_{u_\beta} \log \Delta_\mu,$$

where $\Delta_\mu := \eta(e_\mu, e_\mu)^{-1}$, we obtain the result

$$\int_{\overline{\mathcal{M}}_{1,1}} \Omega_{1,1} e_\beta = \frac{1}{48}\sum_\mu \partial_{u_\mu} \log \Delta_\mu + \frac{1}{2}(R_1)_{\beta\beta},$$

which can be placed in the suggestive form

$$dF_1^\Omega = \sum_\mu \frac{d\log \Delta_\mu}{48} + \frac{1}{2}(R_1)_{\mu\mu} \, du_\alpha.$$

We will now prove the identities in (2). For the first identity, we will compute $\Omega_{0,4}$ using the reconstruction theorem. There are two stable graphs, given in Figure 3. The first graph gives the contribution

FIGURE 3. Stable graphs for $g = 0, n = 4$

$$\text{Cont}_1 = T\omega_{0,4}(e_\mu, e_\mu, e_\mu, e_\beta) - 3T\omega_{0,4}(R_1 e_\mu, e_\mu, e_\mu, e_\beta)\bar{\psi}_1$$
$$- T\omega_{0,4}(e_\mu, e_\mu, e_\mu, R_1 e_\beta)\bar{\psi}_4$$
$$= \omega_{0,4}(e_\mu, e_\mu, e_\mu, e_\beta) + \omega_{0,5}(R_1 \mathbf{1}, e_\mu, e_\mu, e_\mu, e_\beta)\bar{\psi}_1$$
$$- 3\omega_{0,4}(R_1 e_\mu, e_\mu, e_\mu, e_\beta)\bar{\psi}_1 - \omega_{0,4}(e_\mu, e_\mu, e_\mu, R_1 e_\beta)\bar{\psi}_4$$
$$= \delta_{\beta\mu}\Delta_\mu + \delta_{\beta\mu}\eta(R_1 \mathbf{1}, e_\mu)\bar{\psi}_1 - 3\eta(R_1 e_\mu, e_\beta)\bar{\psi}_1 - \eta(e_\mu, R_1 e_\beta)\bar{\psi}_4.$$

The second graph gives the contribution

$$\text{Cont}_2 = \sum_\nu \omega_{0,3}(e_\mu, e_\mu, e^\nu)\omega_{0,3}(R_1 e_\nu, e_\mu, e_\beta)[\text{pt}]$$
$$= \eta(R_1 e_\mu, e_\beta)[\text{pt}].$$

This has multiplicity 3, so taking the sum and integrating over $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, we obtain

$$\int_{\overline{\mathcal{M}}_{0,4}} \Omega_{0,4}(e_\mu, e_\mu, e_\mu, e_\beta) = \delta_{\beta\mu}\eta(R_1 \mathbf{1}, e_\mu) - \eta(R_1 e_\beta, e_\mu)$$

$$= \Delta_\mu^{-1}(\delta_{\beta\mu}\eta(R_1\mathbf{1}, e^\mu) - \eta(R_1 e_\beta, e^\mu)),$$

where we have used the fact that

$$e^\mu = \frac{e_\mu}{\eta(e_\mu, e_\mu)} = \Delta_\mu e_\mu.$$

This proves the first identity.

To prove the second equality in (2), we note that

$$\Delta_\mu^{-1} = \Omega_{0,3}(\mathbf{1}, e_\mu, e_\mu) = \Omega_{0,3}(e_\mu, e_\mu, e_\mu)$$

because e_μ is idempotent. Differentiating with respect to u_β, we obtain

$$\partial_{u_\beta}\Delta_\mu^{-1} = 2\Omega_{0,3}(\mathbf{1}, e_\mu, \partial_{u_\beta}e_\mu) = 3\Omega_{0,3}(e_\mu, e_\mu, \partial_{u_\beta}e_\mu) + \int_{\overline{\mathcal{M}}_{0,4}} \Omega_{0,4}(e_\mu, e_\mu, e_\mu, e_\beta).$$

Using again the fact that e_μ is idempotent, we see that

$$\Omega_{0,3}(\mathbf{1}, e_\mu\,\partial_{u_\beta}e_\mu) = \Omega_{0,3}(e_\mu, e_\mu, \partial_{u_\beta}e_\mu).$$

This implies that

$$0 = \Omega_{0,3}(\mathbf{1}, e_\mu\,\partial_{u_\beta}e_\mu) + \int_{\overline{\mathcal{M}}_{0,4}} \Omega_{0,4}(e_\mu, e_\mu, e_\mu, e_\beta)$$

$$= \frac{1}{2}\,\partial_{u_\beta}\Delta_\mu^{-1} + \int_{\overline{\mathcal{M}}_{0,4}} \Omega_{0,4}(e_\mu, e_\mu, e_\mu, e_\beta),$$

which proves the identity.

Part 2. **Higher-genus computations via logarithmic GLSMs**

In algebraic geometry, one profitable way to deal with non-compactness issues is to forcibly compactify our spaces and then understand the contributions coming from the divisor at infinity. For example, this technique was used by Deligne to construct mixed Hodge structures [Del71]. In enumerative geometry, one profitable application of this technique is the capped topological vertex [OP10], which calculates relative invariants of a (partial compactification) of $\mathbb{C}^3$ relative to the divisor at infinity and is related to the usual topological vertex by a capping operator. While Jun Li's technique of expanded degenerations [Li01] allows us to calculate enumerative invariants relative to a smooth divisor, generalizing this to the case of a simple normal crossings divisor requires logarithmic geometry (at least if we want to do it using algebraic geometry). Now, Chen-Janda-Ruan [CJR21] have developed the theory of log GLSMs, which define stable map-type invariants for any GLSM-type theory defined over an arbitrary projective Deligne-Mumford stack (and resolves all compactness issues therein). This theory has been used by Guo-Janda-Ruan [GJR18] to prove finite generation, holomorphic anomaly equations, orbifold regularity, and the LG/CY correspondence for the quintic threefold and some other Calabi-Yau threefold complete intersections in projective space.

4. Geometry of log GLSM moduli spaces (Qile Chen and Felix Janda)

Our motivation is the quantum Lefschetz theorem. Here, we let $\mathcal{X}$ be a smooth projective variety or Deligne-Mumford stack over $\mathbb{C}$ and E be a vector bundle over $\mathcal{X}$. We are interested in computing the Gromov-Witten invariants of a smooth complete intersection $\mathcal{Z} \subset \mathcal{X}$ defined by a regular section of E. Because the ambient space

is generally easier to work with, it is desirable to find a way to compute the GW invariants of $\mathcal{Z}$ in terms of the data of $(\mathcal{X}, E)$.

Question 4.1. *Is there a way to compute the Gromov-Witten invariants of $\mathcal{Z}$ using the ambient data of $(\mathcal{X}, E)$, possibly involving some correction terms?*

While quantum Lefschetz holds in genus zero in the convex case, and the approach of desingularization of the moduli of stable maps to force quantum Lefschetz to hold works well in genus one, this is intractable in higher-genus. Instead, our approach will be to use GLSMs, which were introduced by Witten [Wit92] in the physics literature, and by Fan-Jarvis-Ruan and Kiem-Li, Chang-Li, Chang-Li-Li, and other authors in mathematics [FJR18; KL13; CL12a; CLL15]. We will combine this with the theory of punctured log maps due to Abramovich-Chen-Gross-Siebert [ACGS24] to obtain the theory of log GLSM.

4.1. R-maps and log targets.

Definition 4.2. An R-map is a commutative diagram

$$
\begin{array}{ccc}
& & \mathfrak{P} \\
& \nearrow{\scriptstyle f} & \downarrow \\
C & \xrightarrow{\omega_{\log}} & B\mathbb{C}_\omega^\times \\
\downarrow & & \\
S, & &
\end{array}
$$

where the morphism $\mathfrak{P} \to B\mathbb{C}_\omega^\times$ is a proper, log-smooth DM-type morphism. If the source is a twisted curve, this is the underlying R-map. If the source is a log curve, this is a *log R-map*, and if the source is a punctured curve, this is a *punctured R-map*.

Example 4.3. Consider the diagram

$$
\begin{array}{ccc}
\mathfrak{P}_{\mathbb{C}}^\circ = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n}(-d) \otimes \mathbb{C}_\omega) & \longrightarrow & \mathfrak{P}^\circ = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n}(-d) \boxtimes \mathcal{L}_\omega) \\
\downarrow & & \downarrow \\
\mathrm{Spec}\,\mathbb{C} \longrightarrow B\mathbb{C}_\omega^\times & \longleftarrow & \mathbb{P}^n \times B\mathbb{C}_\omega^\times.
\end{array}
$$

Taking the base change of the underlying R-map to C, consider the diagram

$$
\begin{array}{ccc}
\mathfrak{P}^\circ \times_{B\mathbb{C}_\omega^\times} C & \longrightarrow & \mathfrak{P}^\circ \\
\downarrow & & \downarrow \\
C & \longrightarrow & B\mathbb{C}_\omega^\times.
\end{array}
$$

Therefore, the data of an R-map to this target is equivalent to the data of a morphism $C \to \mathbb{P}^n$ and a section $\rho \in H^0(f^*\mathcal{O}(-d) \otimes \omega_{\log})$. In particular, if $n = 0$, we are left with the data of a differential $\rho \in H^0(\omega_{\log})$.

Example 4.4. We will now give some examples of some log targets. An easy way to compactify a vector bundle is to turn it into a weighted projective space bundle, so for example we may consider

$$
\underline{\mathfrak{P}}_{\mathsf{GW},\mathbb{C}} = \mathbb{P}^{(\tilde{r},1)}(\mathcal{O}_{\mathbb{P}^n}(-d) \otimes \mathbb{C}_\omega \oplus \mathcal{O}) \longrightarrow \underline{\mathfrak{P}}_{\mathsf{GW}} = \mathbb{P}^{(\tilde{r},1)}(\mathcal{O}_{\mathbb{P}^n}(-d) \oplus \mathcal{O}).
$$

We can give this the log structure of functions which vanish only on the boundary and call it $\mathfrak{P}_{\mathsf{GW}}$.

Another example is to consider the example of 5-spin curves. Here, $\mathfrak{X} = B\mathbb{G}_m$ with the degree 5 map to $B\mathbb{C}_\omega^\times$. Then we set

$$\underline{\mathfrak{P}}_{\mathrm{LG}} = \mathbb{P}(\mathcal{L}_{\mathfrak{X}}^5 \oplus \mathcal{O}).$$

Again, we give this the divisorial log structure and call it $\mathfrak{P}_{\mathsf{LG}}$.

A feature of this is that there is a geometric LG/CY correspondence, namely that $\infty_{\mathsf{LG}} \simeq \infty_{\mathsf{GW}}$ as log stacks.

Example 4.5. For the $(3,3)$ complete intersection in $\mathbb{P}^5$, we consider

$$\mathfrak{P}_{\mathsf{GW}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 2} \otimes \mathcal{L}_\omega \oplus \mathcal{O})$$

with the divisorial log structure.

Example 4.6. We will now consider a general hybrid target. The input is a smooth projective DM stack X, a vector bundle $E = \bigoplus E_i$ over X, a line bundle L over X, and $r \in \mathbb{Z}_{>0}$. The spin is the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathcal{L}_{\mathfrak{X}}} & B\mathbb{G}_m \\ \downarrow & & \downarrow{\scriptstyle r} \\ X \times B\mathbb{C}_\omega^\times & \xrightarrow{L^{-1} \otimes \mathcal{L}_\omega} & B\mathbb{G}_m \end{array}$$

The target is

$$\mathbb{P}\left(\bigoplus E_i^\vee \otimes \mathcal{L}_{\mathfrak{X}}^i \oplus \mathcal{O}\right)$$

with the divisorial log structure at infinity.

We will now turn our discussion to the discrete data. This consists of a genus, curve class, and twisted sector for each marking. We will now construct a dual graph, which has vertices and half-edges. There is an involution

$$\iota_G \colon V \cup H \to V \cup H$$

and a vertex map $v_G \colon H \to V$. Legs are the half-edges which are fixed by the involution, and we will label the legs by the marking map

$$m \colon L \to \{1, \dots, n\}.$$

The decorations are the curve classes, sectors, and degrees.

Consider a log map

$$\begin{array}{ccc} & & \mathfrak{P} \\ & \nearrow{\scriptstyle f} & \downarrow \\ P_h \subset C_{\omega_{\log}} & \longrightarrow & B\mathbb{C}_\omega^\times, \end{array}$$

where P_h is a special point. Because $\omega_{\log}|_{P_h}$ is the trivial bundle, we obtain a commutative diagram

$$\begin{array}{ccc} P_h & \longrightarrow & \mathfrak{P}_{\mathbb{C}} \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & \mathfrak{P}, \end{array}$$

which exhibits P_h as a gerbe over $\mathfrak{P}_{\mathbb{C}}$. Assuming that there are no orbifold points, theere is a sector map
$$\overline{r} \colon H(G) \to \{0_{\mathfrak{P}_{\mathbb{C}}}, \infty_{\mathfrak{P}_{\mathbb{C}}}, \mathfrak{P}_{\mathbb{C}}\}.$$
We require that $P_h \to \mathfrak{P}_{\mathbb{C}}$ factors through $\overline{r}_h$, and say that h is *compact type* if $\overline{r}_h = 0_{\mathfrak{P}}$.

Example 4.7. Consider again $X = \mathbb{P}^0$. Then $\mathfrak{P} = \mathbb{P}(\mathcal{L}_{\omega} \oplus \mathcal{O})$. If $g = 1$, then a log map is simply $\eta \in H^0(\omega_C)$, which cannot be stable. Adding a marked point, we obtain a section of $\omega_C^{\log}$, which we will force to vanish at the marked point.

The stability of underlying R-maps has two conditions:
(1) First, that the morphism is representable;
(2) Second, that
$$(\omega_{\log})^{1+\delta} \otimes f^* H^{\otimes k} \otimes f^* \mathcal{O}(\tilde{r}\infty) > 0,$$
where H is a polarization on the target. Here, $k \gg 1 \gg \delta > 0$.

Example 4.8. Again considering $X = \mathbb{P}^0$, we will let C be of genus 2. the limit $\lim_{\lambda \to \infty} \lambda \cdot \eta$ is the union of a genus-zero curve with a differential which vanishes at one point with multiplicity 2 connected to a genus 2 curve which sits entirely at infinity. Even though the rational component is not a priori stable, it is stabilized by its intersection with the zero section.

Remark 4.9. In the $\mathbb{P}^0$ case, stability is equivalent to the condition that
$$\omega^{\log} \otimes \eta^* \mathcal{O}(k \cdot 0_{\mathfrak{P}}) > 0$$
whenever $k \gg 1$.

4.2. **Log geometry and tropicalization.**

Definition 4.10. Let $\underline{Y}$ be a scheme or a stack. A *log structure* on $\underline{Y}$ is a sheaf of monoids $\mathcal{M}_Y$ on $\underline{Y}$, together with a morphism
$$\alpha \colon \mathcal{M}_Y \to (\mathcal{O}_{\underline{Y}}, \cdot)$$
inducing an isomorphism $\alpha^{-1} \mathcal{O}_{\underline{Y}}^{\times} \simeq \mathcal{O}_{\underline{Y}}^{\times}$. The pair $Y = (\underline{Y}, \mathcal{M}_Y)$ is called a *log scheme/stack* and the sheaf
$$\overline{\mathcal{M}}_Y / \mathcal{O}_{\underline{Y}}^{\times}$$
is called the *characteristic sheaf* or *ghost sheaf*.

Example 4.11. Any log GLSM target is a log stack.

Example 4.12. Another standard target is a toric variety with $\mathcal{M}_Y$ being all functions which vanish only on the toric boundary. This is usually called the divisorial log structure.

Example 4.13. For a toric monoid P, we can consider
$$S = \operatorname{Spec}(P \to \mathbb{C}) = (\operatorname{Spec}\mathbb{C}, P \times \mathcal{O}_{\operatorname{Spec}\mathbb{C}}^{\times}).$$

The tropical data we will consider is the category Cones of rational polyhedral cones (σ, N). This has a distinguished object $(\mathbb{R}_{\geq 0, \mathbb{Z}})$. Locally, we will define
$$\Sigma(\operatorname{Spec}(P \to \mathbb{C})) = \operatorname{Hom}(P, R_{\geq 0}) = P_{\mathbb{R}}^{\vee}$$
with the natural lattice structure $P^{\vee} := \operatorname{Hom}(P, \mathbb{N})$. In general, the tropicalization $\Sigma(X)$ of a log scheme X is the generalized cone complex given by gluing local pictures along face maps.

Example 4.14. Consider $\mathbb{A}^1$ with the log structure given by 0. Then

$$\Sigma(\mathbb{A}^1) = \mathbb{R}_{\geq 0}.$$

Here, there is only a log structure at 0, which is a copy of $\mathbb{N}$ measuring the vanishing order at the origin. The map

$$\overline{M}_{\mathbb{A}^1,0} \to \overline{M}_{\mathbb{A}^1,x}$$

sends 1 to 0, so induces $0 \hookrightarrow \mathbb{N}^\vee$.

Example 4.15. Consider $\mathbb{A}^2$ with the toric log structure. Then $\Sigma(\mathbb{A}^2)$ is simply $\mathbb{R}_{\geq 0}^2$, which is the same as the support of the toric fan. At the monoid level, there is an $\mathbb{N}^2$ at the origin, a copy of $\mathbb{N}$ on each axis, and 0 at a general point.

4.3. **Log curves.**

Definition 4.16. An *n-pointed log curve* over a log scheme S consists of

$$(\pi \colon C \to S, \{p_i\}_{i=1}^n),$$

such that

   (1) The underlying morphism of π is an n-pointed twisted curve;
   (2) π is log smooth and integral;
   (3) On the smooth locus of C, the log structure is given by $\overline{M}_S \oplus \bigoplus_{i=1}^n \mathbb{N} \cdot p_i$

Intuitively, the log structure is given by the log structure $\overline{M}_S$ at smooth unmarked points, $\overline{M}_S \oplus \mathbb{N}$ at marked points, and $\overline{M}_S \oplus_{\mathbb{N}} \mathbb{N}^2$ at the nodes, which means the pushout diagram

$$
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{\Delta} & \mathbb{N}^2 \\
\downarrow{\scriptstyle \ell} & & \downarrow \\
\overline{M}_S & \longrightarrow & \overline{M}_C|_q.
\end{array}
$$

Here, ℓ is the edge length parameter, and $(1,0)$ and $(0,1)$ are the two components.

Because tropicalization is functorial, the morphism $C \to S$ induces a morphism

$$\Sigma(C) \to \Sigma(S).$$

For example, if C has two components, then the node will correspond to the largest cone, each component corresponds to a copy of $\overline{M}_S$, and each marking will give a leg of infinite length. Above a point in $\Sigma(S)$, the distance between the two components is the edge length parameter evaluated at the point x. To see this more clearly, see Figure 4.

We will now consider punctured curves. We will consider a diagram

$$
\begin{array}{ccc}
p_i^\circ & \longrightarrow & p_i \\
\downarrow & & \downarrow \\
C^\circ & \xrightarrow{P} C & \longrightarrow S.
\end{array}
$$

Here, P is a morphism of log schemes which is an isomorphism away from P_i and $\underline{P}$ is an isomorphism. In addition, we will have an inclusion

$$\overline{M}_{C,p_i} = \overline{M}_S \oplus \mathbb{N}p_i \subset \overline{M}_{C^\circ,p_i} = \overline{M}_S \oplus \mathbb{Z}p_i.$$

This corresponds to allowing poles at p_i, and in the tropicalization makes edges finite-length. A picture is given in Figure 5.
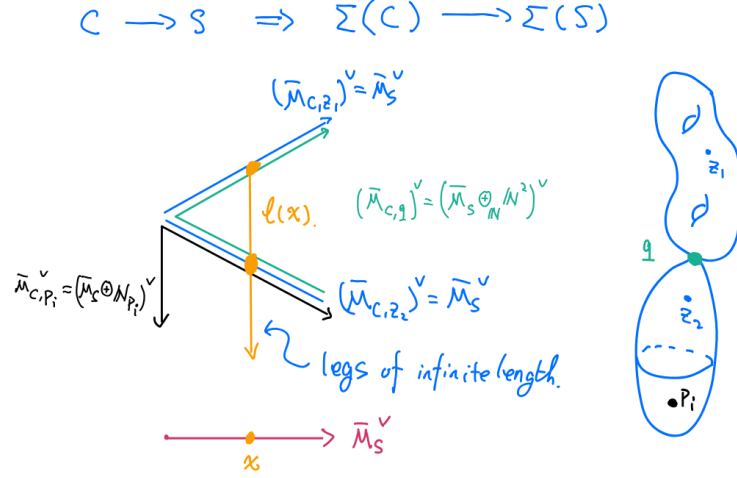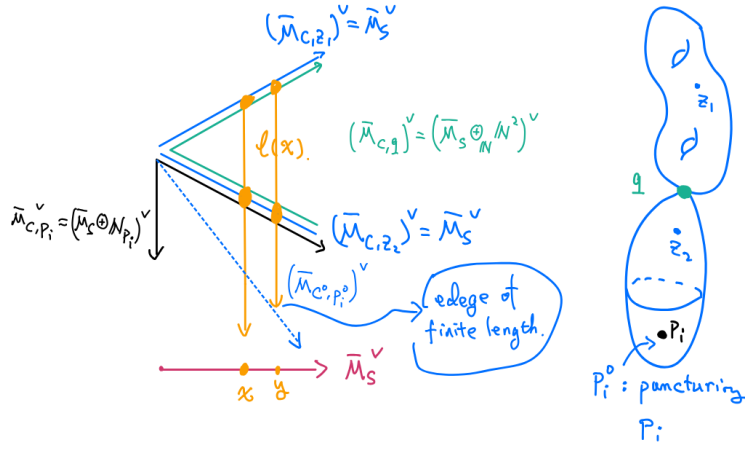
FIGURE 4. Tropicalization of a log curve.

FIGURE 5. Tropicalization of a punctured curve.

Definition 4.17. A *punctured curve* is a log curve with punctures

$$\mathbb{C}^\circ \to C \to S.$$

A *tropical punctured curve* is a tropical curve with the additional data of lengths of punctured legs.

4.4. **Superpotentials.**

Definition 4.18. A *superpotential* is a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{P}^\circ & \longrightarrow & \mathcal{L}_\omega \\
& \searrow & \downarrow \\
& & B\mathbb{C}^\times_\omega,
\end{array}
$$

where $\mathfrak{P}^\circ = \mathfrak{P} \setminus \infty$. We say that W has *proper critical locus* if

$$\operatorname{Crit} W \to B\mathbb{C}^\times_\omega$$

is proper.

Equivalently, a superpotential is a $\mathbb{C}^\times_\omega$-equivariant function

$$W_\mathbb{C} \colon \mathfrak{P}^\circ_\mathbb{C} \to \mathbb{C}_\omega.$$

In this formulation, W has proper critical locus if and only if $\operatorname{Crit} W_\mathbb{C}$ is proper (as a DM stack). This implies that the critical locus is contained in $0_\mathfrak{P}$.

Example 4.19. Let X be a smooth projective Deligne-Mumford stack and E be a vector bundle on X with section s. We will write

$$W_\mathbb{C} = \otimes(s \otimes 1_{\mathbb{C}_\omega}) \colon \mathfrak{P}^\circ_\mathbb{C} = E^\vee \otimes \mathbb{C}_\omega \to \mathbb{C}_\omega.$$

Then $\operatorname{Crit} W_\mathbb{C}$ is proper if and only if $Z = (s = 0)$ is smooth of codimension equal to the rank of E.

Example 4.20. Let $X = \mathbb{P}^0$ and $E = \mathbb{C}$. Let $s \in H^0(E)$ be nonzero. Now $W_\mathbb{C}$ is simply multiplication by s, so the critical locus is empty.

4.5. **Punctured R-maps.**

Definition 4.21. A *punctured R-map* is an R-map with domain a punctured R-map.

Definition 4.22. Let h be a half-edge. Then the *contact order* at h is defined by

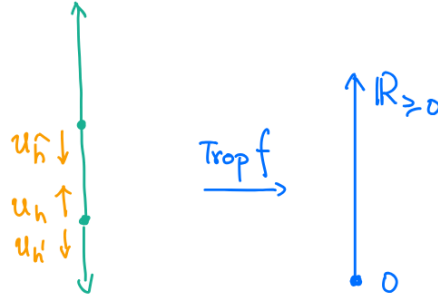$$c(h) := \frac{\partial \operatorname{Trop} f}{\partial u_h} \in \mathbb{Z}.$$

FIGURE 6. Contact orders. Note the last downward edge has finite length.

Example 4.23. Consider the genus 2 picture from before (target being $\mathbb{P}^0$) and consider the same stable limit as an R-map. If we calculate with $z = s^{-1}$, then

$$z^2 \, \mathrm{d}z = -s^{-3}\frac{\mathrm{d}s}{s},$$

so the pole order at the node must be 3. Tropically, if $\hat{h}$ is the half edge of the node attached to the $g = 2$ vertex, we have $c(\hat{h}) = -3$, so restricting to the $g = 2$ component, we obtain a punctured R-map.

The discrete data of punctured R-maps is

$$\boldsymbol{\tau} = (G, g, \beta, \deg, \overline{r}, \sigma\colon V \cup H \to \{0, R_{\geq 0}\}, c\colon H \to \mathbb{Z}).$$

Theorem 4.24. *The moduli stack $\mathcal{R}(\mathfrak{P}, \boldsymbol{\tau})$ is a proper log DM stack admitting a canonical perfect obstruction theory.*

Definition 4.25. We will consider $\boldsymbol{\tau}$ where $V(G) = \{\star\}$ and $L(G) = H(G)$. These are called *vertex-type moduli*. If $\sigma = 0$, then we will denote

$$\mathcal{R}_{g,\vec{c}}(\mathfrak{P}, \beta) \coloneqq \mathcal{R}(\mathfrak{P}, \boldsymbol{\tau})$$

for the stack of stable log R-maps, and if $\sigma = \mathbb{R}_{\geq 0}$, the we will denote

$$\mathcal{R}_{g,\vec{c}}(\infty, \beta) \coloneqq \mathcal{R}(\mathfrak{P}, \boldsymbol{\tau})$$

for the stack of punctured R-maps.

4.6. Obstruction theories.

Definition 4.26. Recall that we have a diagram

The *canonical perfect obstruction theory* is defined by

$$\varphi\colon \mathbb{T}_{\mathcal{R}(\mathfrak{P},\beta)/\mathfrak{M}(\boldsymbol{\tau})} \to R\pi_* f^* T_{\mathfrak{P}/B\mathbb{C}_\omega^\times}.$$

Because it is very complicated, we will not discuss $\mathfrak{M}(\boldsymbol{\tau})$.

Unfortunately, this is not the obstruction theory that we really want. We will consider the superpotential and then use the cosection localization technique of Kiem-Li [KL13]. Here, we have the diagram

whenever $\boldsymbol{\tau}$ is of compact type. The compact-type condition implies that

$$f^* \, \mathrm{d}W\colon f^* T_{\mathfrak{P}^\circ/B\mathbb{C}_\omega^\times} \to \omega_{\log}$$

factors through the sheaf ω of holomorphic differentials. Because $R^1\pi_*\omega \cong \mathcal{O}$, we obtain a cosection

$$\sigma_W \coloneqq R^1\pi_* f^* \, dW \colon R^1\pi_* f^* T_{\mathfrak{P}^\circ/B\mathbb{C}_\omega^\times} \to \mathcal{O}.$$

By the results of Kiem-Li, this gives us a virtual cycle

$$[\mathcal{R}_{g,\bar{c}}(\mathfrak{P}^\circ, \beta)]_{\sigma_W}^{\mathrm{vir}}$$

supported on R-maps to $\mathrm{Crit}\, W$, which coincides with the canonical virtual cycle after pushing forward to the stack of all R-maps.

Remark 4.27. In Gromov-Witten theory, the cosection localized virtual cycle satisfies the relation

$$[\mathcal{R}_{g,\bar{c}}(\mathfrak{P}^\circ, \beta)]_{\sigma_W}^{\mathrm{vir}} = \pm[\overline{\mathcal{M}}_{g,n}(Z, \beta)]^{\mathrm{vir}}$$

by work of Chang-Li, Chang-Li, Kim-Oh, Picciotto, and Chen-Janda-Webb [CL12a; CL20b; KO22; Pic21; CJW21].

Of course, we have the problem that $\mathcal{R}_{g,\mathfrak{c}}(\mathfrak{P}^\circ, \beta)$ is not proper, so we need to find a way to extend the cosection along the boundary

$$\Delta_{g,\bar{c}}(\mathfrak{P}, \beta) \coloneqq \mathcal{R}_{g,\bar{c}}(\mathfrak{P}, \beta) \setminus \mathcal{R}_{g,\bar{c}}(\mathfrak{P}^\circ, \beta).$$

We need to understand how to differentiate

$$W \colon \mathfrak{P} \dashrightarrow \mathcal{L}_\omega,$$

which will require compact-type legs and a principalization of the boundary.

Definition 4.28. We will say discrete data is *compact type* if for all $h \in L(G)$, either

- $c(h) = 0$ and $\bar{r} = 0_{\mathfrak{P}}$;
- $c(h) \leq -1$. In this case $\bar{r} = \infty_{\mathfrak{P}}$.

Example 4.29. Consider the $\mathbb{P}^0$ example in genus 1. If we consider $\eta \in H^0(\omega_C)$, we can view it as $\eta \in H^0(\omega_C^{\log})$. Then, at the marking, we see that $c(h) = 0$ and $\bar{r}_h = 0_{\mathfrak{P}}$, so the leg is of compact type. In the tropical picture, the entire infinite leg gets contracted to the origin.

Example 4.30. Now consider the same example but with $g = 2$. We will impose that there are no markings by $(\eta = 0) = 2p$. The stable limit as we scale η to infinity had a genus 0 component with a zero of order 2 and a genus 2 component mapping entirely to ∞. If we consider the half-edge emerging from the genus 0 vertex, it touches ∞ with contact order 3, so it is not compact type. On the other hand, the half-edge coming from the genus 2 vertex touches ∞ with contact order -3, so it is compact type (and in the tropical picture the leg has finite length).

Remark 4.31. The data of $\eta_0|_{C_2}$ is equivalent to a fixed isomorphism $\mathcal{O}_{C_2}(2p) \simeq \omega_{C_2}$, where C_2 is the genus 2 vertex.

Remark 4.32. Tropical curves with only compact-type legs have compact image in $\Sigma(\mathfrak{P}) = \mathbb{R}_{\geq 0}$. In this picture, they are either pointing downward or contracted to the origin.

Having defined compact type insertions, we will now define a modular principalization of the boundary. There is an edge length

$$\ell \colon E(G) \cup L^\circ(G) \to \overline{M}_S$$

and a degeneracy

$$e \colon V(G) \to \overline{M}_S$$

which intuitively records where v is sent in $\mathbb{R}_{\geq 0}$.

Definition 4.33. For a toric monoid P, define a partial order $a \leq b$ if there exists $c \in P$ such that $a + c = b$. If we consider a punctured R-map, consider the collection of degeneracies

$$\{e_V \mid v \in V(G)\}.$$

It has *uniform maximal degeneracy* if there is a unique maximum

$$e_{\max} = \max\{e_v \mid v \in V(G)\}.$$

Example 4.34. Let $\overline{M}_S = \mathbb{N}^2$. Suppose that there are v_1, v_2 and a vertex v_0 which is sent to $0 \in \mathbb{R}_{\geq 0}$. Then we compute

$$e_{v_1} = e_{v_0} + c_1 \cdot \ell_1$$
$$= \ell_1.$$

Similarly, $e_{v_2} = \ell_2$, where ℓ_1 and ℓ_2 are the generators of $\mathbb{N}^2$. These cannot be compared, so we do not have uniform maximal degeneracy.

To uniformize this, we consider a subdivision into three subcones. The first is when $\ell_2 > \ell_1$, the second is when $\ell_1 = \ell_2$, and the third is when $\ell_1 > \ell_2$. After this subdivision, each subcone has uniform maximal degeneracy.

Example 4.35. A family of tropical curves over $\bar{M}_S^\vee$ with compact-type legs and uniform maximal degeneracy $e_{\max} \in \overline{M}_S$ has images in $\mathbb{R}_{\geq 0}$ uniformly bounded from above by $e_{\max}$.

We now define a new stack

$$\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau})$$

to be the stack of punctured R-maps with discrete data $\boldsymbol{\tau}$ and having uniform maximal degeneracy. By removing the condition of uniform maximal degeneracy, we obtain a morphism

$$F \colon \mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}) \to \mathcal{R}(\mathfrak{P}, \boldsymbol{\tau}),$$

which satisfies the following:

- F is log étale, proper, and surjective;
- There is as canonical perfect obstruction theory;
- We have

$$F_*[\mathcal{U}_{g,\bar{c}}(\mathfrak{P}, \beta)]^{\mathrm{vir}} = [\mathcal{R}_{g,\bar{c}}(\mathfrak{P}, \beta)]^{\mathrm{vir}}_{\sigma_W}$$
$$F_*[\mathcal{U}_{g,\bar{c}}(\mathfrak{P}, \beta)]^{\mathrm{vir}} = [\mathcal{R}_{g,\bar{c}}(\infty, \beta)]^{\mathrm{vir}}_{\sigma_W}$$

  for vertex-type moduli.

The boundary $\Delta_{g,\bar{c}}(\mathfrak{P}, \beta)^\curlywedge$ defined by the Cartesian diagram

$$
\begin{array}{ccc}
\Delta^\curlywedge & \longrightarrow & \Delta_{\max} = 0 \\
\downarrow & & \downarrow \\
\mathcal{U} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]
\end{array}
$$

is a log Cartier divisor.

Theorem 4.36. *Assume all legs are of compact type and $W \colon \mathfrak{P}^\circ \to \mathcal{L}_\omega$ has proper critical locus. Then*

(1) $\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau})$ has a reduced perfect obstruction theory;

(2) In the log R-map case, we have

$$[\mathcal{U}_{g,\vec{c}}(\mathfrak{P}, \beta)]^{\mathrm{red}} = [\mathcal{R}_{g,\vec{c}}(\mathfrak{P}^\circ, \beta)]^{\mathrm{vir}}_{\sigma_W};$$

(3) The boundary $\Delta^{\curlywedge}_{g,\vec{c}}(\mathfrak{P}, \beta)$ also has a reduced perfect obstruction theory;

(4) There is the relation

$$[\mathcal{U}_{g,\vec{c}}(\mathfrak{P}, \beta)]^{\mathrm{red}} = [\mathcal{U}_{g,\vec{c}}(\mathfrak{P}, \beta)]^{\mathrm{vir}} - \tilde{r}[\Delta^{\curlywedge}_{g,\vec{c}}(\mathfrak{P}, \beta)]^{\mathrm{red}},$$

where $\tilde{r}$ is the pole order of W at infinity.

The reduced perfect obstruction theory for log R-maps is given by the triangle

$$\mathbb{E}^{\mathrm{red}} \to R\pi_* f^* T_{\mathfrak{P}/B\mathbb{C}^\times_\omega} \xrightarrow{\sigma} [\mathcal{O} \to \mathcal{O}(\tilde{r}\Delta_{\max})] \xrightarrow{[1]}$$

and for $e_{\max} > 0$ is given by

$$\mathbb{E}^{\mathrm{red}} \to R\pi_* f^* T_{\mathfrak{P}/B\mathbb{C}^\times_\omega} \xrightarrow{\sigma} \mathcal{O}(\tilde{r})[-1] \xrightarrow{[1]} .$$

The virtual components of $\Delta^{\curlywedge}_{g,\mathfrak{c}}$ are given by the formula

$$[\Delta^{\curlywedge}_{g,\vec{c}}(\mathfrak{P}, \beta)]^{\mathrm{red}} = \sum_{\boldsymbol{\tau}_\curlywedge} \frac{\mathrm{lcm}_{x \in E(G)} c(x)}{|\mathrm{Aut}\, \boldsymbol{\tau}_\curlywedge|} [\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge)]^{\mathrm{red}}$$

due to Abramovich-Chen-Gross-Siebert [ACGS24], where

$$\boldsymbol{\tau}_\curlywedge = (\boldsymbol{\tau}, V_{\max}(G))$$

is the tropical type of rigid tropical curves with uniform maximal degeneracy. Here, this implies that $\boldsymbol{\tau}_\curlywedge$ is bipartite and rigidity means there is no deformation fixing $\boldsymbol{\tau}$ and $V_{\max}(G)$ besides scaling $e_{\max}$

Decomposing this further, we have

$$[\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge)]^{\mathrm{red}}$$

$$= (-\tilde{r})^{|V_\infty(G)|-1} \frac{\prod_{E \in E(G)} c(E)}{\mathrm{lcm}_{E \in E(G)} c(E)} \Delta^!_{\boldsymbol{\tau}_\curlywedge} \left(\prod_{v \in V_\infty(G)} [\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_v)]^{\mathrm{red}} \times \prod_{v \in V_0(G)} [\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} \right).$$

This is not true if we replace everything with the canonical obstruction theory, so this is quite interesting. Putting these two formulae together, we obtain the tropical decomposition formula

$$[\mathcal{U}_{g,\vec{c}}(\mathfrak{P}, \beta)]^{\mathrm{red}} = \sum_{\boldsymbol{\tau}_\curlywedge} \frac{(-\tilde{r})^{|V_\infty(G)|}}{|\mathrm{Aut}\, \boldsymbol{\tau}_\curlywedge|} \cdot \prod_{E \in E(G)} c(E) \cdot$$

$$\cdot \Delta^!_{\boldsymbol{\tau}_\curlywedge} \left(\prod_{v \in V_\infty(G)} [\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_v)]^{\mathrm{red}} \times \prod_{v \in V_0(G)} [\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_v)]^{\mathrm{vir}} \right).$$
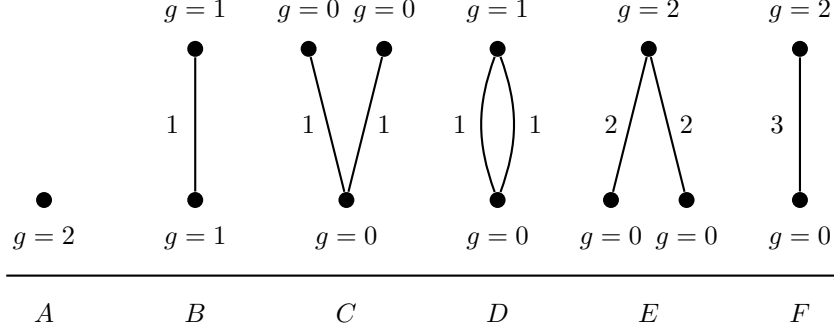
FIGURE 7. Graphs of tropical types for $X = \mathbb{P}^0$ when $g = 2$.

Example 4.37. Consider the $\mathbb{P}^0$ example again. When $g = 2$, we obtain six bipartite graphs, which are given in Figure 7. Choosing any nonzero superpotential, we obtain the tropical decomposition formula

$$0 = [\mathcal{U}_g(\mathfrak{P}, 0)]^{\mathrm{red}} = \sum_{\boldsymbol{\tau}_\curlywedge} \frac{(-1)^{|V_\infty(G)|}}{|\mathrm{Aut}\,\boldsymbol{\tau}_\curlywedge|} \cdot \prod_{E \in E(G)} c(E)$$

$$\cdot \Delta^!_{\boldsymbol{\tau}_\curlywedge} \left(\prod_{v \in V_\infty(G)} [\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_v)]^{\mathrm{red}} \times \prod_{v \in V_0(G)} [\mathcal{U}(\mathfrak{P}, \boldsymbol{\tau}_v)]^{\mathrm{vir}} \right).$$

Note here that the critical locus is empty, so the virtual cycle must be zero.

In GW theory, the tropical decomposition formula becomes

$$[\overline{\mathcal{M}}_{g,n}(Z, \beta)]^{\mathrm{vir}} = \pm \sum_{\boldsymbol{\tau}_\curlywedge} \frac{(-1)^{|V_\infty(G)|}}{|\mathrm{Aut}\,\boldsymbol{\tau}_\curlywedge|} \cdot \prod_{E \in E(G)} c(E) \cdot$$

$$\cdot \Delta^!_{\boldsymbol{\tau}_\curlywedge} \left(\prod_{v \in V_\infty(G)} [\mathcal{U}_{g_v, e_v}(\infty, \beta_v)]^{\mathrm{red}} \times \prod_{v \in V_0(G)} [\mathcal{R}_{g_v, e_v}(\mathfrak{P}, \beta_v)]^{\mathrm{vir}} \right).$$

4.7. **Effective invariants.** We will consider the reduced virtual cycle $[\mathcal{U}_{g,\mathcal{C}}(\infty, \beta)]^{\mathrm{red}}$. For each $h \in L$, the evaluation lands in $\underline{\infty}_\mathbb{C} = \mathbb{P}(E^\vee)$. Then effective invariants are given by

$$F_* \left(\prod_h \mathrm{ev}_h^* \alpha_h \cap [\mathcal{U}_{g,\mathcal{C}}(\infty, \beta)]^{\mathrm{red}} \right).$$

Remark 4.38. In the vertex case, we have the relation

$$[\mathcal{U}_{g,\bar{c}}(\mathfrak{P}, \beta)]^{\mathrm{vir}} = \tilde{r}\Delta_{\max} \cap [\mathcal{U}_{g,\bar{c}}(\mathfrak{P}, \beta)]^{\mathrm{red}}.$$

Before we continue, we will consider the geometry of $\mathcal{R}_{g,\bar{c}}(\infty, \beta)$ when $X = \mathbb{P}^N$. Consider the diagram

$$\begin{array}{ccc} & \infty & \longrightarrow \mathbb{P}^N \\ {\scriptstyle f} \nearrow & \downarrow & \\ C \xrightarrow{\ \omega_{\log}\ } & B\mathbb{C}^\times_\omega. & \end{array}$$

This induces a stable map $C \to \mathbb{P}^N$, so in fact it is equivalent to the data of $s \colon C \to \mathbb{P}^N$ and an isomorphism

$$f^* \mathcal{O}(d) \otimes \omega_{\log}^{-1} \simeq \mathcal{O}_C \Big(\sum c_h p_h \Big),$$

which is equivalent to an isomorphism

$$f^* \mathcal{O}(1)^d \simeq \omega \Big(\sum_h (c_h + 1) p_h \Big).$$

Therefore, when $N = 0$, $\mathcal{R}_{g,\mathbf{c}}(\infty, \beta)$ is the moduli of canonical divisors with specified zero orders, and for general N, we have the moduli of d-spin linear series of rank N.

We can now take a root of the log target ∞, which is given by the diagram

$$
\begin{array}{ccc}
\infty^{\frac{1}{\ell}} & \longrightarrow & \infty \\
\downarrow & & \downarrow \\
[\mathbb{A}^1/\mathbb{G}_m] & \overset{\ell}{\longrightarrow} & [\mathbb{A}^1/\mathbb{G}_m].
\end{array}
$$

On virtual cycles, we obtain

$$[\mathcal{R}_{g,\vec{c}}(\infty^{\frac{1}{\ell}}, \beta)]^\star = \ell^{-1}[\mathcal{R}_{g,\vec{c}}(\infty, \beta)]^\star,$$

where $\star$ is either "vir" or "red."

Geometrically, we must have a balancing condition

$$\deg f^* \mathcal{O}(\infty) = \sum_h c(h).$$

Example 4.39. Let $X = \mathbb{P}^N$ and suppose $E = \mathcal{O}(d)$ is a line bundle. Then we obtain

$$\sum_h c(h) = \deg f^* \mathcal{O}(\infty)$$

$$= \deg(f^* \mathcal{O}_{\mathbb{P}^N}(d) \otimes \omega_{\log}^{-1})$$

$$= \beta \cdot d - (2g - 2 + n).$$

This implies that

$$\frac{2g - 2}{d} \geq \beta \geq 0,$$

so β must be one of $0, 1, 2, \ldots, \left\lfloor \frac{2g-2}{d} \right\rfloor$.

When $g = 0$, the upper bound is negative, so there are no genus zero reduced invariants. When $g = 1$, then the upper bound is zero, so the only reduced invariant is in the case $\beta = 0$. This implies that

$$\sum_h (c(h) + 1) = 0,$$

so because the contact orders are all negative, they must equal -1.

Example 4.40. We will consider legs of contact order -1 in the $\mathbb{P}^0$ case. Let C be smooth of genus g and $\eta \in H^0(\omega_C)$. We now choose a marking $p \in C$, we obtain $\eta \in H^0(\omega_{\log})$. The stable limit as we scale η to infinity is a genus g component mapping entirely to ∞ and a genus 0 component with a zero at p. On $\mathbb{P}^1$, we see that

$$\eta_0|_{\mathbb{P}^1} = \mathrm{d}z = \mathrm{d}s^{-1} = -s^{-1}\frac{\mathrm{d}s}{s}.$$

This implies the contact order of the genus 0 component with ∞ is 1, so the genus g component has contact order -1.

Remark 4.41. The legs with contact order -1 are created by adding compact type markings outside of ∞ and should be viewed as the unit, denoted by $\mathbf{1}$.

4.8. **Log GLSM axioms.** Consider the diagram

$$\mathcal{U}_{g,\vec{c}+\mathbf{1}}(\infty,\beta) \xrightarrow{\ s\ } \mathcal{C}^\circ \xrightarrow{\ \pi\ } \mathcal{U}_{g,\vec{c}}(\infty,\beta),$$

with $F_{\mathbf{1}}$ labeling the composite arrow.

where $\pi\colon \mathcal{C}^\circ \to \mathcal{U}_{g,\vec{c}}(\infty,\beta)$ is the universal punctured curve.

Theorem 4.42. *We have the equations*

$$s_*[\mathcal{U}_{g,\vec{c}+\mathbf{1}}(\infty,\beta)]^{\mathrm{red}} = \pi^*[\mathcal{U}_{g,\vec{c}}(\infty,\beta)]^{\mathrm{red}}$$

and

$$F_{\mathbf{1},*}(\mathrm{ev}_{\mathbf{1}}^* D \cap [\mathcal{U}_{g,\vec{c}+\mathbf{1}}(\infty,\beta)]^{\mathrm{red}}) = \int_\beta D \cdot [\mathcal{U}_{g,\vec{c}}(\infty,\beta)]^{\mathrm{red}}$$

for any $D \in H^2(\underline{\infty})$. In addition, if we consider the diagram

$$
\begin{array}{ccc}
\mathcal{U}_{g,\vec{c}+1}(\infty,\beta) & \xrightarrow{\ F\ } & \overline{\mathcal{M}}_{g,n+1} \\
\downarrow{\scriptstyle F_{\mathbf{1}}} & & \downarrow{\scriptstyle \pi} \\
\mathcal{U}_{g,\vec{c}}(\infty,\beta) & \xrightarrow{\ F\ } & \overline{\mathcal{M}}_{g,n}
\end{array}
$$

then we have

$$F_*[\mathcal{U}_{g,\vec{c}+1}(\infty,\beta)]^{\mathrm{red}} = \pi^* F_*[\mathcal{U}_{g,\vec{c}}(\infty,\beta)]^{\mathrm{red}}.$$

Example 4.43. If we consider $Z_d \subset \mathbb{P}^N$ and let $g = 1$, then we only need to compute one non-ambient invariant.

The reduced virtual dimension of $\mathcal{U}_{g,\vec{c}}(\infty,\beta)$ is given by the formula

$$\chi(f^* T_{\mathfrak{P}/B\mathbb{C}_\omega^\times}) + 1 + \dim \mathfrak{M}_{g,\vec{c}}(\infty_{\mathcal{A}}),$$

which also happens to equal

$$\mathrm{virdim}\,\overline{\mathcal{M}}_{g,n}(Z,\beta) + (\mathrm{rk}\,E) \cdot \sum_h (c(h) + 1).$$

Therefore, when the virtual dimension of the stable map moduli is negative, then the reduced virtual dimension is zero.

Example 4.44. Consider $Z_5 \subset \mathbb{P}^4$. The reduced dimension of $\mathcal{U}_{g,\mathfrak{e}}(\infty,\beta)$ is simply

$$\mathrm{virdim}\,\overline{\mathcal{M}}_{g,n}(Z_5,\beta) + \sum_h (c(h) + 1) = \sum_h (c(h) + 2).$$

We can remove all legs of contact order -1, so we reduce to the case when $c(h) = -2$ for all h. Therefore, there are exactly $\lfloor \frac{2g-2}{5} \rfloor + 1$ reduced invariants.

Example 4.45. For a general semi-positive hypersurface in $\mathbb{P}^N$, the reduced virtual dimension when $g \geq 2$ is

$$(4 - N)(g - 1) - \beta \cdot (d - N - 1) + n + \sum_h (c(h) + 1).$$

Using the balancing condition, we obtain an upper bound of

$$(g-1)\left((2-N) + \frac{2N+2}{d}\right) + n + \sum_h (c(h)+1).$$

If $2 - N + \frac{2N+2}{d}$ is negative, then the reduced virtual dimension is structly less than

$$n + \sum_h \sum_h (c(h)+1).$$

After removing all legs of contact order -1, all effective cycles vanish! For example, we have this negativity whenever $d = 3$ and $N \geq 9$, when $d = 4$ and $N \geq 6$, and when $d \geq 5$ and $N \geq 5$. For complete intersections in other targets, we can run the same arguments, and they are governed by birational invariants.

4.9. Uniform minimal degeneracy.

Definition 4.46. A punctured R-map has *uniform minimal degeneracy* if there exists a unique

$$e_{\min} = \min\{e_v \mid v \in V(G)\}.$$

Sometimes, we will also need to consider disconnected graphs, so if we enforce uniform minimal degeneracy, we will obtain a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{U}(\infty, \boldsymbol{\tau}^{\curlyvee}) & \longrightarrow & \mathcal{R}(\infty, \boldsymbol{\tau}^{\curlyvee}) \\
\downarrow & & \downarrow \\
\mathcal{U}(\infty, \boldsymbol{\tau}) & \longrightarrow & \mathcal{R}(\infty, \boldsymbol{\tau})
\end{array}
$$

where both vertical arrows are log blowups and hence log étale and projective. These introduce two new tautological classes on $\mathcal{U}(\infty, \boldsymbol{\tau}^{\curlyvee})$, which are

$$\psi_{\max} = -[\Delta_{\max}]$$

coming from $e_{\max}$ and

$$\psi_{\min} = c_1(\mathcal{O}(-e_{\min}))$$

coming from $e_{\min}$. The class $\psi_{\min}$ is a key ingredient for the tropical decomposition and is needed for the virtual localization formula.

4.10. Some examples. Here, we will give a few exmples.

Example 4.47 (Quintic). Here, the target is $X_5 \subset \mathbb{P}^4$. We will consider the log target

$$\mathfrak{P}_{X_5} = [\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}(-5) \oplus \mathcal{O})/\mathbb{C}_{\times}^{\omega}]$$

with the infinity part

$$\infty_{X_5,\mathbb{C}} \cong \mathbb{P}^4.$$

Example 4.48 (Double cubic). Here, the target is $X_{3,3} \subset \mathbb{P}^5$. We will consider the log target

$$\mathfrak{P}_{X_{3,3}} = [\mathbb{P}_{\mathbb{P}^5}(\mathcal{O}(-3)^{\oplus 2} \oplus \mathcal{O})/\mathbb{C}_{\times}^{\omega}]$$

with the infinity part

$$\infty_{X_{3,3},\mathbb{C}} \cong \mathbb{P}(\mathcal{O}(-3)^{\oplus 2}) \cong \mathbb{P}^5 \times \mathbb{P}^1.$$

Example 4.49 (Quintic FJRW)**.** We will consider the log target

$$\mathfrak{P}_{\mathsf{LG}} = [\mathbb{P}^5/\mathbb{C}_R^\times] \to B\mathbb{C}_R^\times \xrightarrow{5} B\mathbb{C}_\omega^\times$$

with the infinity part

$$\infty_{\mathsf{LG},\mathbb{C}} \xrightarrow{\text{fifth root}} \infty_{X_5,\mathbb{C}}.$$

4.11. $\mathbb{C}_\omega^\times$ **action.** Our goal is now to proceed towards a virtual localization formula for log GLSM. Here, the two structural formulae are related as in Figure 8.
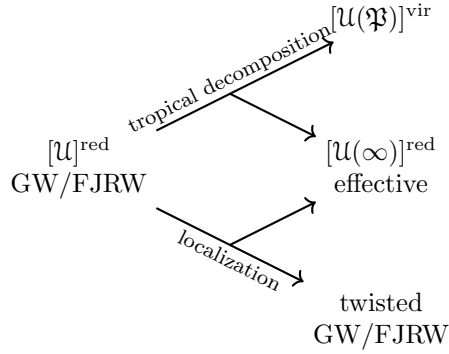

FIGURE 8. Relation between structural formulae and virtual cycles in log GLSM

There are two ways to think about this $\mathbb{C}_\omega^\times$-action. If we have an R-map

this was a morphism of **stacks**, so there is a 2-morphism making the diagram commute. Abstractly, the action simply scales the 2-morphism, which is an isomorphism of line bundles.

More concretely, consider the quintic example. Then an R-map is equivalent to the data of a stable map $f\colon C \to \mathbb{P}^4$ and a section of the projective bundle

$$\mathbb{P}_C(f^*\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C^{\log} \oplus \mathcal{O}_C).$$

Using this description, the $\mathbb{C}_\omega^\times$-action simply scales the p-field, where we scale the first summand and not the second.

We now have a $\mathbb{C}_\omega^\times$ action on all moduli spaces that we considered previously, for example $\mathcal{R}(\mathfrak{P}, \beta)$ or $\mathcal{U}(\mathfrak{P}, \beta)$.

Proposition 4.50. *The perfect obstruction theories $\mathbb{E}_\mathcal{R}$ and $\mathbb{E}_\mathcal{U}^{\mathrm{red}}$ are $\mathbb{C}_\omega^\times$-equivariant.*

Remark 4.51. A key input to this result is that the superpotential is $\mathbb{C}_\omega^\times$-equivariant. If it was in fact invariant, then there would be no need to develop the theory of log GLSM.

We may now apply the virtual localization theorem, proved in increasing strength by Graber-Pandharipande, Chang-Kiem-Li, and Aranha-Khan-Latyntsev-Park-Ravi [GP99; CKL17; Ara+24] to decompose the reduced virtual cycle as

$$[\mathcal{U}]^{\mathrm{red}} = \sum_F \iota_* \frac{[F]^{\mathrm{red}}}{e(N^{\mathrm{vir}}_{F/[U]})}.$$

4.12. **Fixed loci.** There are several kinds of fixed loci:
- There is a fixed locus $\mathcal{R}_{g,\vec{c}}(0_{\mathfrak{P}}, \beta)$ of all maps going into the zero-section.
- There is another fixed locus $\mathcal{R}_{g,\vec{c}}(\infty_{\mathfrak{P}}, \beta)$ of maps going into infinity.
- More general fixed loci may be described by decorated bipartite graphs, with vertices at either 0 or ∞ being decorated by a genus and curve class and edges being decorated by the contact order (which implies the degree of the edge).

From a bipartite graph, the R-maps which appear in the corresponding fixed locus arise from taking scaling limits of R-maps of the corresponding tropical type. From the fixed curves, we can obtain other curves of the same tropical type by smoothing nodes which appear at 0.

The moduli spaces corresponding to stable vertices are given as follows. When $v \in V_\infty$, the vertex moduli space is

$$\mathcal{R}_v = \mathcal{R}_{g(v),\vec{c}(v)}(\infty, \beta(v)),$$

where all contact orders are negative. When $v \in V_0$, then we have

$$\mathcal{R}_v = \mathcal{R}_{g(v),\vec{c}(v)}(0, \beta(V))$$
$$\cong \mathcal{M}_v = \overline{\mathcal{M}}_{g(v),n(v)}(X, \beta(v)),$$

where X was the ambient space.

4.13. **Virtual localization formula.**

Example 4.52. In the case of the quintic, note that $\infty \cong 0$. Then we have a stabilization morphism $\mathrm{st} \colon \mathcal{U} \to \overline{\mathcal{M}}_{g,n}(X, \beta)$. Then we will describe the virtual localization formula on a moduli space which is related to the fixed locus. It is given by the Cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_\Gamma & \longrightarrow & \prod_v \overline{\mathcal{M}}_v \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ (\mathbb{P}^4)^{|E|} & \stackrel{\Delta}{\longrightarrow} & (\mathbb{P}^4 \times \mathbb{P}^4)^{|E|}. \end{array}$$

We then obtain the virtual localization formula

$$\mathrm{st}_*[\mathcal{U}]^{\mathrm{red}} = \sum_\Gamma \frac{1}{|\mathrm{Aut}\,\Gamma|} \iota_{\Gamma,*} \Delta^! \cdot$$
$$\cdot \prod_{v \in V_0} \frac{[\mathcal{M}_v]^{\mathrm{vir}}}{e^{\mathbb{C}^\times}(R\pi_* \omega_C^{\log} \otimes f^* \mathcal{O}(-5) \otimes \mathbb{C}_\omega)} \prod_{h \in H_v} \frac{1}{\frac{t - \mathrm{ev}_h^*(5H)}{c_h} - \psi_h}$$
$$\cdot \prod_{v \in V_\infty} \mathrm{st}_* \frac{t[\mathcal{U}(\infty)]^{\mathrm{red}}}{-t - \psi_{\min}} \cdot \prod_e \cdots,$$

where t is the equivariant parameter and c_h is the contact order. The 0 part gives the twisted GW theory of $\mathbb{P}^4$ with special insertions and the ∞ part gives a more general version of effective invariants.

Example 4.53. In the case of the double cubic, note that $\infty_{\mathbb{C}} \cong \mathbb{P}^5 \times \mathbb{P}^1 \to \mathbb{P}^5 = 0_{\mathbb{C}}$. Then we have a stabilization morphism $\mathrm{st} \colon \mathcal{U} \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^5, \beta)$. For $v \in V_\infty$, we have

$$\overline{\mathcal{M}}_v = \overline{\mathcal{M}}_{g(v),n(v)}(\mathbb{P}^5, \beta(v)) \times_{(\mathbb{P}^5)^{n(v)}} (\mathbb{P}^5 \times \mathbb{P}^1)^{n(v)}.$$

Then we will describe the virtual localization formula on a moduli space which is related to the fixed locus. It is given by the Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{M}_\Gamma & \longrightarrow & \prod_v \overline{\mathcal{M}}_v \\
\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\
(\mathbb{P}^5 \times \mathbb{P}^1)^{|E|} & \xrightarrow{\ \Delta\ } & (\mathbb{P}^5 \times (\mathbb{P}^5 \times \mathbb{P}^1))^{|E|}.
\end{array}
$$

Here, Δ is composition the diagonal and deleting the $\mathbb{P}^1$ in the first factor. We then obtain the virtual localization formula

$$\mathrm{st}_*[\mathcal{U}]^{\mathrm{red}} = \sum_\Gamma \frac{1}{|\operatorname{Aut}\Gamma|} \iota_{\Gamma,*}\Delta^!\cdot$$

$$\cdot \prod_{v\in V_0} \frac{[\mathcal{M}_v]^{\mathrm{vir}}}{e^{\mathbb{C}^\times}(R\pi_*\omega_C^{\log}\otimes f^*\mathcal{O}(-3)^{\oplus 2}\otimes\mathbb{C}_\omega)} \prod_{h\in H_v} \frac{1}{\frac{t-\mathrm{ev}_h^*(3H+H_\infty)}{c_h}-\psi_h}$$

$$\cdot \prod_{v\in V_\infty} \mathrm{st}_* \frac{t[\mathcal{U}(\infty)]^{\mathrm{red}}}{-t-\psi_{\min}} \cdot \prod_e \cdots,$$

where t is the equivariant parameter, c_h is the contact order, and H_∞ is the hyperplane class on the $\mathbb{P}^1$ factor. The 0 part gives the twisted GW theory of $\mathbb{P}^5$ with special insertions and the ∞ part gives a more general version of effective invariants.

Example 4.54. In the case of the FJRW theory of the quintic, note that $\infty_C = \sqrt[5]{(\mathbb{P}^4, \mathcal{O}(1))} \to 0_{\mathbb{C}} = B\mu_5$. Then we have a stabilization morphism $\mathrm{st} \colon \mathcal{U} \to \overline{\mathcal{M}}_{g,\mathfrak{C}}^{\frac{1}{5}}$. We now set

$$\overline{\mathcal{M}}_v = \overline{\mathcal{M}}_{g(v),\mathfrak{C}(v)}^{\frac{1}{5}} \times_{(\bar{I}0_{\mathbb{C}})^{n(v)}} (\bar{I}\infty_v)^{n(v)}$$

Then we will describe the virtual localization formula on a moduli space which is related to the fixed locus. It is given by the Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{M}_\Gamma & \longrightarrow & \prod_v \overline{\mathcal{M}}_v \\
\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\
(\bar{I}\infty_{\mathbb{C}})^{|E|} & \xrightarrow{\ \Delta\ } & (\bar{I}0_{\mathbb{C}} \times \bar{I}\infty_{\mathbb{C}})^{|E|}.
\end{array}
$$

We then obtain the virtual localization formula

$$\mathrm{st}_*[\mathcal{U}]^{\mathrm{red}} = \sum_\Gamma \frac{1}{|\operatorname{Aut}\Gamma|} \iota_{\Gamma,*}\Delta^!\cdot$$

$$\cdot \prod_{v\in V_0} \frac{[\mathcal{M}_v]^{\mathrm{vir}}}{e^{\mathbb{C}^\times}(R\pi_*\mathcal{L}\otimes\mathbb{C}_\omega)} \prod_{h\in H_v} \frac{1}{\frac{t-\mathrm{ev}_h^*(H_\infty)}{c_h}-\psi_h}$$

$$\cdot \prod_{v \in V_\infty} \mathrm{st}_* \frac{5t[\mathcal{U}(\infty)]^{\mathrm{red}}}{-t - \psi_{\min}} \cdot \prod_e \cdots,$$

where t is the equivariant parameter and c_h is the contact order. The 0 part gives the twisted GW theory of $\mathbb{P}^4$ with special insertions and the ∞ part gives a more general version of effective invariants.

5. Applications to Gromov-Witten theory (Shuai Guo and Felix Janda)

5.1. **Genus two calculations.** The goal is to prove the formula

$$(3) \qquad F_2^{\mathsf{QM}}(Q) = \langle \ \rangle_2^{t,\mathsf{QM}} - \left\langle \frac{-\frac{5}{3}H^3 + \frac{5}{24}H^4 t^{-1}}{(t-5H)(t-5H-\psi)} \right\rangle_1^{t,\mathsf{QM}}$$

$$(4) \qquad + \frac{1}{2}\left\langle \frac{-\frac{5}{3}H^3 + \frac{5}{24}H^4 t^{-1}}{(t-5H)(t-5H-\psi)}, \frac{-\frac{5}{3}H^3 + \frac{5}{24}H^4 t^{-1}}{(t-5H)(t-5H-\psi)} \right\rangle_0^{t,\mathsf{QM}}$$

$$(5) \qquad + \frac{1}{2}\left\langle \Delta_*\left(\frac{\frac{5}{3}H^3 t^{-1} + \frac{65}{8}H^4 t^{-2}}{(t-5H)^2(t-5H-\psi_1)(t-5H-\psi_2)} \right) \right\rangle_0^{t,\mathsf{QM}}$$

$$(6) \qquad + F_2(Q=0)$$

for the quintic threefold.

We will first use the localization formula to compute

$$\deg[\mathcal{U}_2(\mathfrak{P}_{X_5}, \beta)]^{\mathrm{red}}.$$

There are many localization graphs, which may be obtained as modifications of the ones in Figure 7. The most important ones are displayed in Figure 9.
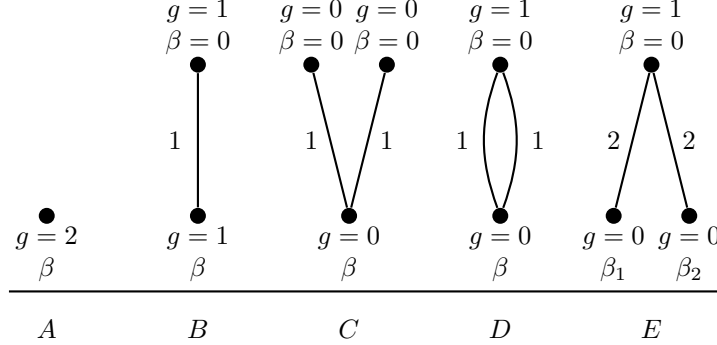

FIGURE 9. Localization graphs with nonzero contribution for $g = 2$.

The contributions will now be given. For graph A, the contribution is

$$\sum \deg \frac{[\overline{\mathcal{M}}_2(\mathbb{P}^4, \beta)]^{\mathrm{vir}}}{e(R\pi_* \omega_C^{\log} \otimes f^*[O](-5) \otimes \mathbb{C}_\omega)} = \langle \ \rangle_2^t,$$

which is the $\mathcal{O}(5)$-twisted Gromov-Witten potential of $\mathbb{P}^4$. Graph B contributes

$$\left\langle \frac{\mathrm{ev}_*\left(\frac{[\mathcal{U}_{1,(-1)}(\infty,0)]^{\mathrm{vir}}}{-t-\psi_{\min}} \right)}{(t-5H)(t-5H-\psi)} \right\rangle_1^t = \left\langle \frac{-\frac{5}{3}H^3 + \frac{5}{24}H^4 t^{-1}}{(t-5H)(t-5H-\psi)} \right\rangle_1^t.$$

Graph C contributes (4), and graph D contributes (5). Graph E will contribute

$$c^{\text{eff}} \cdot \left\langle \frac{\cdots}{\frac{t-5H}{2} - \psi} \right\rangle_0^t = F_2(Q = 0) \cdot P_1^2(Q),$$

where c^{eff} is the effective invariant.

To obtain a formula with five terms from all of our localization graphs, we will apply the shift $\mu = (1 - I_0)\psi + I_1 H$ which comes from quasimap wall-crossing. Under the shift, the contribution of graph E becomes the constant $F_2(Q = 0)$, while the contributions from all other graphs vanish.

5.2. Genus two mirror theorem.

For convenience, we will set $t = 5s$. This will make the equivariant parameters the same for both the Gromov-Witten and FJRW calculations. This gives us

$$
\begin{aligned}
F_2^{\mathsf{QM}}(Q) = \langle\ \rangle_2^{s,\mathsf{QM}} &- \left\langle \frac{-\frac{5}{3}H^3 + \frac{5}{24}H^4(5s)^{-1}}{(5s - 5H)(5s - 5H - \psi)} \right\rangle_1^{s,\mathsf{QM}} \\
&+ \frac{1}{2}\left\langle \frac{-\frac{5}{3}H^3 + \frac{5}{24}H^4(5s)^{-1}}{(5s - 5H)(5s - 5H - \psi)}, \frac{-\frac{5}{3}H^3 + \frac{5}{24}H^4(5s)^{-1}}{(5s - 5H)(5s - 5H - \psi)} \right\rangle_0^{s,\mathsf{QM}} \\
&+ \frac{1}{2}\left\langle \Delta_* \left(\frac{\frac{5}{3}H^3(5s)^{-1} + \frac{65}{8}H^4(5s)^{-2}}{(5s - 5H)^2(5s - 5H - \psi_1)(5s - 5H - \psi_2)} \right) \right\rangle_0^{s,\mathsf{QM}} \\
&+ F_2(Q = 0).
\end{aligned}
$$

Because the twisted theory is semisimple, we can compute it using the Givental-Teleman reconstruction theorem [Giv01b; Tel12]. For a graph $\Gamma \in \{A, B, C, D, E\}$, the first step is to write

$$\langle \alpha_1(\psi_1), \alpha_n(\psi_n) \rangle^s = \left\langle [S(\bar\psi_1)\alpha_1(\bar\psi_1)]_+, [S(\bar\psi_n)\alpha_n(\bar\psi_n)]_+ \right\rangle_{g,n}^s$$

using the descendant-ancestor correspondence of Kontsevich-Manin [KM98]. The second step is to use the Givental-Teleman reconstruction theorem to compute the ancestor invariants by a stable graph sum.

For simple reasons, we can rewrite $[S(\bar\psi)\alpha(\bar\psi)]_+ = S(5s - 5H)\alpha_1(\bar\psi)$. Both $S(5s - 5H)$ and

$$R^{-1}(z) = \sum R_k z^k$$

have entries in a ring

$$\tilde{\mathcal{R}} = \mathbb{Q}[\text{generators}]$$

where the generators are defined using genus-zero invariants. Using the fact that

$$I^{X_5,s}(q,z)|_{z=5s-5H} = 5s - 5H,$$

the mirror theorem, and $S^*1 = z^{-1}J$, we see that $S^*1|_{z=5s-5H} = 1$. To compute the other entries, we use the quantum connection

$$zq\frac{\mathrm{d}}{\mathrm{d}z}S^{\tau,s} = \dot\tau \star S^{\tau,s}.$$

Quantum multiplication by $\dot{\tau}$ is given by the matrix

$$
A = \begin{pmatrix}
* & * & & & \\
* & * & * & & \\
* & * & * & * & \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix},
$$

where the $*$ denote nonzero entries. However, the entries of A and their derivatives lie in

$$
\tilde{\mathcal{R}} = \mathbb{Q}[\text{9 generators}].
$$

Using a computer, we then get the exact formulae for

$$
\text{Cont}_\Gamma \in \tilde{R}.
$$

A miracle implies that

$$
\sum_\Gamma \text{Cont}_\Gamma \in \mathcal{R} = \mathbb{Q}[\text{5 generators}] \subset \tilde{\mathcal{R}}.
$$

If we recall that the physicists predicted that F_g is a polynomial in five generators (which are exactly the same as the five generators here).

Remark 5.1. A similar calculation for FJRW theory yields similar formulae. For example, graph B contributes

$$
\left\langle \frac{5\phi_{\frac{4}{5}}}{24s^2(s-\psi)} - \frac{200\phi_{\frac{4}{5}}}{24s(s-\psi)^2} \right\rangle_{1,1}^{s,\mathsf{QM}}
$$

and graph D contributes

$$
\left\langle \frac{\frac{65}{8}\phi_{\frac{4}{5}} \otimes \phi_{\frac{4}{5}}}{s^2(s-\psi_1)s^2(s-\psi_2)} + \frac{\frac{5}{3}\phi_{\frac{4}{5}} \otimes \phi_{\frac{4}{5}}}{s(s-\psi)^2 s^2(s-\psi_2)} + \frac{\frac{5}{3}\phi_{\frac{4}{5}} \otimes \phi_{\frac{4}{5}}}{s^2(s-\psi_1)s(s-\psi_2)^2} \right\rangle_{0,2}^{s,\mathsf{QM}}.
$$

5.3. **LG/CY correspondence for the quintic.** Write

$$
I^{\mathsf{GW}} = zI_0 + I_1 + I_2\frac{H^2}{z} + I_3\frac{H^3}{z^2}.
$$

Similarly, we can write

$$
zI_0\phi_{\frac{1}{5}} + I_1\phi_{\frac{2}{5}} + I_2\phi_{\frac{3}{5}}z^{-2} + I_3\phi_{\frac{4}{5}}z^{-3}.
$$

We can then construct the generators for the FJRW theory of the quintic similarly to the generators for the GW theory of the quintic.

Theorem 5.2. *Write*

$$
F_g^{\mathsf{GW}} = P_g^{\mathsf{CY}}(\text{9 generators})
$$
$$
F_g^{\mathsf{FJRW}} = P_g^{\mathsf{LG}}(\text{9 generators}).
$$

Then after identification of the generators, we have

$$
P_g^{\mathsf{CY}} = P_g^{\mathsf{LG}}.
$$

The proof of this result is a direct matching of contributions from graphs in the virtual localization formula. It relies on the fact that $A^{\mathsf{CY}} = A^{\mathsf{LG}}$ under a suitable choice of basis, which also matches the identification of the generators. The tropical input from log GLSM is that the effective invariants are equal in both the CY and LG phases. Finally, we need to identify the insertions in the two phases, and this follows from a direct computation of the specialized S-matrix.

5.4. Geometry of LG/CY correspondence. Let $\bullet \in \{\mathsf{LG}, \mathsf{CY}\}$. Recall that

$$\mathfrak{P}_{\mathsf{CY},\mathbb{C}} = \mathbb{P}^4 \times \mathbb{P}^1$$

and $\infty_{\mathsf{CY},\mathbb{C}} = \mathbb{P}^4$. The log target is $\mathfrak{P}_{\mathsf{LG},\mathbb{C}} = [\mathbb{P}^5/\mu_5]$, but the infinity-section is

$$\infty_{\mathsf{LG}} \cong \sqrt[5]{\mathbb{P}^4}.$$

Remark 5.3. There is a modification of the targets such that $\infty_{\mathsf{CY}} \cong \infty_{\mathsf{LG}}$.

The first step is to match the virtual localization formulae. In the LG phase, there are no curve classes at 0, so we compensate for this by using insertions. Also, note that we need to consider the inertia stack

$$IB\mu_5 = \bigsqcup_{i=0}^{5} B\mu_5,$$

and in fact the contact orders determine the twisted sectors at the nodes. Then there is a bijection between the LG and CY decorated graphs. For example, we match the graphs in Figure 10.
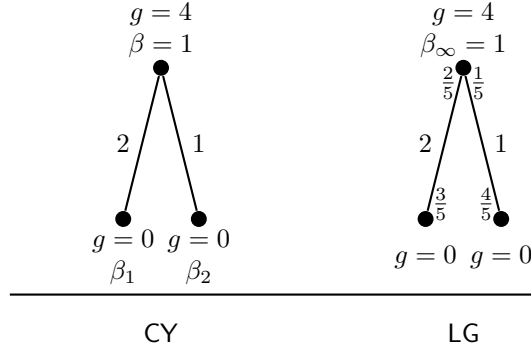

FIGURE 10. Corresponding graphs in CY and LG phases.

Now let

$$\tau^{\bullet} = \frac{I_1^{\bullet}}{I_0^{\bullet}} \in \mathcal{H}^{\bullet},$$

where the state spaces are given by

$$\mathcal{H}^{\mathsf{CY}} = H^*(\mathbb{P}^4) \qquad \text{and} \qquad \mathcal{H}^{\mathsf{LG}} = \mathbb{Q}(s)\Big\langle \phi_{\frac{0}{5}}, \phi_{\frac{1}{5}}, \phi_{\frac{2}{5}}, \phi_{\frac{3}{5}}, \phi_{\frac{4}{5}} \Big\rangle.$$

Defining

$$\mathrm{ev}_{\infty} \colon \prod_{v \in V_{\infty}} \mathcal{U}_{g(v),e(v)}(\infty_{\bullet}, \beta(v)) \to (\bar{I}\infty_{\bullet,\mathbb{C}})^{|E|} \to (\mathbb{P}^4)^{|E|},$$

we define
$$\mathrm{Cont}_{\Gamma,\bullet}^{\infty} = (\mathrm{ev}_{\infty})_* \left(\prod_{v \in V_{\infty}} \frac{5s[\mathcal{U}_v]^{\mathrm{red}}}{-5s - \psi_{\min}} \right).$$

We may also define the contribution $\mathrm{Cont}_{\Gamma}^{0,\tau} \colon (\mathcal{H}^{\bullet}[\![\bar{\psi}]\!])^{|E|} \to \mathbb{Q}$ of 0 by

$$\{T_e(\bar{\psi})\}_{e \in E} \mapsto \prod_{v \in V_0} \left\langle \{T_h(\bar{\psi})\}_{h \in H(v)} \right\rangle_{g(v),n(v)}^{s,\bullet,\tau}.$$

Finally, for any contact order $c \in \{1, 2\}$, let

$$\mathcal{S}_{c,\tau}^{\bullet} \colon H^*(\mathbb{P}^4) \to \mathcal{H}^{\bullet}$$

be defined by

$$\phi \mapsto \frac{\mathrm{edge\ factor}(c) \cdot \phi}{\frac{5s - 5H}{c} - \psi}$$

in the CY case and

$$\phi \mapsto \int_{\mathbb{P}^4} \frac{\mathrm{edge\ factor}(c) \cdot \phi}{\frac{5s - 5H_{\infty}}{c} - \psi} \cdot \phi_{\frac{5-c}{5}}$$

in the LG case.

Putting these ingredients together, we obtain the following formula:

Theorem 5.4. *We have the identity*

$$F_g^{\bullet,\tau} = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} \mathrm{Cont}_{\Gamma,\bullet}^{0,\tau} \left(\left(\bigotimes_e \mathcal{S}_{c(e),\tau}^{\bullet} \right) \mathrm{Cont}_{\Gamma,\bullet}^{\infty} \right).$$

The proof of this theorem follows the following steps:

(1) Virtual localization;
(2) Apply the shift coming from quasimap wall-crossing;
(3) Keep track of signs and powers of 5.

It now remains to match the contributions. Here, the contributions $\mathrm{Cont}_{\Gamma,\bullet}^{\infty}$ match because of the geometric isomorphism $\infty_{\mathsf{CY}} \cong \infty_{\mathsf{LG}}$. The matching of the $\mathcal{S}_{c,\tau}^{\bullet}$ follows from a direct computation.

5.5. Twisted theories.
Recall that the $\mathcal{O}(5)$-twisted GW theory of $\mathbb{P}^4$ is given by integrals against

$$e(R\pi_* f^* \mathcal{O}(5) \otimes \mathbb{C}_{5s}) \cap [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \beta)]^{\mathrm{vir}}.$$

This is not very hard to compute with, so we add a torus action of $(\mathbb{C}^{\times})^5$ on $\mathbb{P}^4$. We will denote the equivariant parameters on the base by $\lambda_0, \ldots, \lambda_4$. This gives us a general $\mathcal{O}(5)$-twisted theory.

Unfortunately, this is extremely complicated, but we consider the *formal quintic* theory, which is given by setting $s = 0$ and

$$\lambda_j = \zeta_j \lambda, \qquad \zeta = \exp\left(\frac{2\pi i}{5} \right).$$

Therefore, we have a theory with only one equivariant parameter λ. This theory (also called the *λ-twisted theory*) has several advantages:

- It has been studied before by Zinger, Kim-Lho, Lho-Pandharipande, and various other authors [Zin09; KL18; LP18];

- The λ-twisted I-function

$$I^\lambda = z \sum_{d \geq 0} q^d \frac{\prod_{i=1}^{5d}(5H + iz)}{\prod_{i=1}^d ((H + iz)^5 - \lambda^5)}$$

$$= I^{\mathsf{quintic}} + O(z^{-4})$$

is much nicer than the s-twisted I-function

$$I^s = z \sum_{d \geq 0} q^d \frac{\prod_{i=1}^{5d}(5H - 5s + iz)}{\prod_{i=1}^d (H + iz)^5}.$$

In particular, the appearance of $5s$ in the numerator of each term of I^s is the reason for the appearance of extra generators for the s-twisted theory.

Our goal now is to rewrite log GLSM invariants using the λ-twisted theory. Using the tropical decomposition formula, we obtain

$$\deg[\mathcal{U}(\mathfrak{P})]^{\mathrm{red}} = \sum_{\Gamma_{\mathsf{trop}}} \prod_{v \in V_0} \deg[\mathcal{U}(\mathfrak{P})]^{\mathrm{vir}} \prod_{v \in V_\infty} \mathsf{effective}.$$

Because the canonical virtual cycle does not depend on the equation of the quintic, we can compute

$$\deg[\mathcal{U}(\mathfrak{P})]^{\mathrm{vir}} = \deg[\mathcal{U}(\mathfrak{P})]^{\mathrm{vir},(\mathbb{C}^\times)^6}|_{s=0,\lambda_j=\zeta^j\lambda}.$$

Using the log GLSM localization formula, we now obtain

$$\sum_{\Gamma_{\mathsf{loc}}} \prod_{v \in V_0} \begin{matrix} \mathsf{general} \\ \mathsf{twisted} \end{matrix} \prod_{v \in V_\infty} \deg[\mathcal{U}(\infty)]^{\mathrm{vir},(\mathbb{C}^\times)^6}.$$

We will now apply operations to graphs as in Figure 11.

$$\Gamma_{\mathsf{trop}}$$

$$\Gamma_{\mathsf{loc}}$$

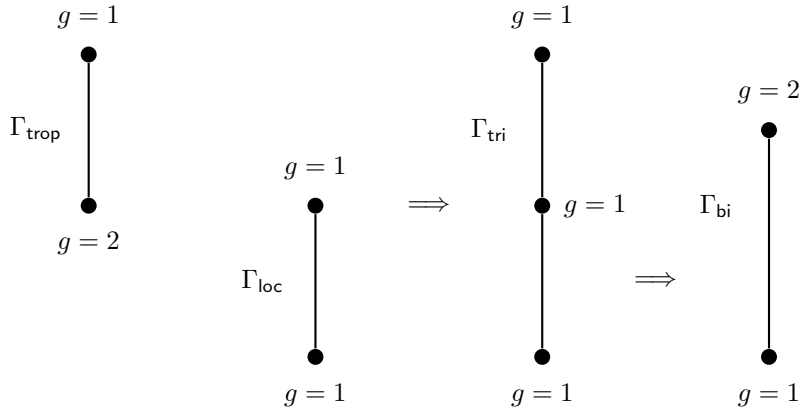$$\Gamma_{\mathsf{tri}}$$

$$\Gamma_{\mathsf{bi}}$$

FIGURE 11. Tropical graph, localization graph, and tripartite graph

We may then combine the two decompositions to form a tripartite graph. Using this, we then obtain

$$\mathsf{GW} = \sum_{\Gamma_{\mathsf{tri}}} \prod_{v \in V_\ell} \begin{matrix} \mathsf{general} \\ \mathsf{twisted} \end{matrix} \prod_{v \in V_m} \deg[\mathcal{U}(\infty)]^{\mathrm{vir},(\mathbb{C}^\times)^6} \cdot \prod_{v \in V_u} \mathsf{effective}$$

$$= \sum_{\Gamma_{\text{bi}}} \prod_{v \in V_0} \overset{\text{general}}{\underset{\text{twisted}}{}} \prod_{v \in V_\infty} (s, \lambda) - \text{effective}$$

$$= \left(\sum_{\Gamma_{\text{bi}}} \prod_{v \in V_0} \lambda - \text{twisted} \prod_{v \in V_\infty} \lambda - \text{effective} \right) \Bigg|_{\lambda = 0}.$$

Here, the second equality comes from combining the middle and upper layers of a tripartite graph.

Remark 5.5.

(1) It is not a priori clear that the specialization $s = 0, \lambda_j = \zeta^j \lambda$ makes sense. However, we note that $H^5 = \lambda^5$, so $H^{-1} = \frac{H^4}{\lambda^5}$, so the specialization

$$\frac{1}{5s - 5H - \psi_h} \to \frac{1}{-5H - \psi_h}$$

makes sense.

(2) The graph Γ_{bi} may have contact orders $c \geq 3$.

5.6. **Combinatorial structure theorem.** Recall that the R-matrix action was found by Givental as follows:

- We will stabilize a localization graph to a stable graph. This involves contracting rational tails and chains:
- The tails become the translation action;
- The chains become edge contributions $V(z, w)$;
- The ends become leg contributions $R(z)^{-1}$.

We will consider something similar in log GLSM. We will begin by stabilizing graphs Γ_{bi} as in Figure 12. Applying Givental-Teleman to the vertices at level 0, we

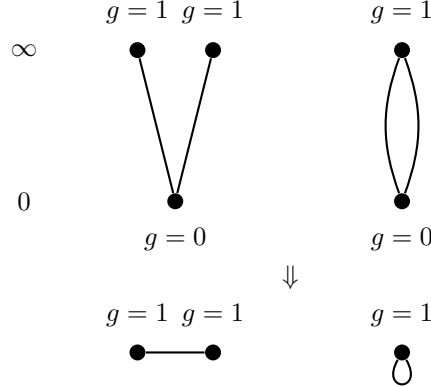FIGURE 12. Stabilization of Γ_{bi}

see that there are $\infty\infty$ edges, 0∞ edges, and 00 edges. There are also both 0 and ∞ legs. This implies that

$$F_g = \hat{\mathcal{R}}(\omega^\lambda \oplus \omega^\infty),$$

where $\mathcal{R}$ is generalized from the Givental formalism. It is given by

$$\mathcal{R} = \begin{cases} R^\lambda & \text{label } 0 \\ \mathcal{S} & \text{label } \infty \end{cases}$$

Also, the matrices R and S are determined from the quantum differential equation, which in the λ-twisted case are given by

$$A = \begin{pmatrix} 0 & I_{11} & & & \\ & 0 & I_{22} & & \\ & & 0 & I_{33} & \\ & & & 0 & I_{44} \\ I_0 & & & & 0 \end{pmatrix}.$$

5.7. GW theory of the quintic.

We are now able to apply the combinatorial structure theorem.

- We have a stable graph sum formula involving $\mathcal{R}$;
- The entries of $\mathcal{R}$ are polynomials in the five generators, which are determined by A;
- The generalized tail contribution

$$\mathcal{T} = \begin{cases} T & \text{label } 0 \\ J & \text{label } \infty \end{cases}$$

is a polynomial in the holomorphic generator $X = \frac{1}{1-5^5 q}$.

To prove the holomorphic anomaly equation, consider differentiating by $S \in \{A, B_1, B_2, B_3\}$. We look directly at the stable graph localization formula. Differentiation comes from cutting edges as in Figure 13.
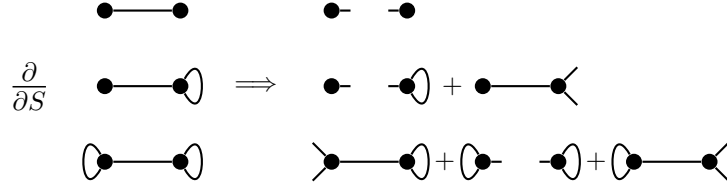

FIGURE 13. Cutting of edges.

If a matrix R satisfies the equation

$$\frac{\partial}{\partial S} R^{-1} = R^{-1} \cdot \Lambda_S,$$

then the V-tensor satisfies the equation

$$\frac{\partial}{\partial S} V = R^{-1}(z) \otimes R^{-1}(w)(\Delta_S),$$

where

$$\Delta_S = \frac{\Lambda_S(z) e^\alpha \otimes e_\alpha + e^\alpha \otimes \Lambda_S(w) e_\alpha}{z + w}.$$

Therefore, when we compute $\frac{\partial}{\partial S} F_g$, we place Δ_S at the nodes, and this will match the holomorphic anomaly equation.

Remark 5.6. This method also proves that for any extra generator $\mathcal{E}$, we have

$$\frac{\partial}{\partial \mathcal{E}} F_g = 0.$$

This implies that F_g is independent of $\mathcal{E}$, so we reduce the nine generators to five generators.

Remark 5.7. In fact, the holomorphic anomaly equations hold for arbitrary values of the effective invariants. In particular, if we define F_g^c by the formal theory and effective invariants, then

  (1) The generating function F_g^c satisfies the holomorphic anomaly equations;
  (2) F_g^c has orbifold regularity.

Orbifold regularity is proved by showing that every graph which appears in the graph sum for F_g satisfies orbifold regularity.

The meaning of orbifold regularity is that if we replace X, A_1, B_1, B_2, B_3 by the LG versions $X^{\mathsf{orb}}, A_1^{\mathsf{orb}}, B_1^{\mathsf{orb}}, B_2^{\mathsf{orb}}, B_3^{\mathsf{orb}}$, then F_g is a regular function near $q = \infty$.

Conjecture 5.8 (Conifold gap). *At the conifold point $q = 5^{-5}$, there is the conifold gap condition*

$$F_g^{\mathsf{con}} = \frac{B_{2g}}{\Delta^{2g-2}} + \mathsf{regular},$$

where Δ is the local flat coordinate near the conifold point.

The conifold gap condition has been checked for $g \leq 5$ by computer. Unfortunately, a systematic way of proving the conjecture is not known.

Part 3. See BCOV from the A-side: MSP fields

Using the master space construction introduced by Thaddeus and Dolgachev-Hu [Tha96; DH98] in the context of variation of GIT, Chang-Li-Li-Liu [CLLL19; CLLL22] defined the theory of *Mixed-Spin-P fields*, which connects the Gromov-Witten and FJRW theories of the quintic threefold. The moduli space has a torus action, and one type of fixed point gives the Gromov-Witten theory of the quintic, which led to hope that the theory could be used to compute GW invariants of the quintic. However, the calculations quickly proved intractable, and an insight of Guo, which led to the creation of *N-Mixed-Spin-P fields* [CGLL21], was required to resolve the difficulties. This led to the proof of the finite generation conjecture and holomorphic anomaly equations for the quintic threefold by Chang-Guo-Li [CGL21; CGL19], which was generalized to hypersurfaces in weighted projective space by the author [Lei24b; Lei24a]. More recently, a new stability condition for GLSMs was introduced by Chang-Guo-Li-Li-Zhou [Cha+23b], and this will enable the construction of MSP moduli spaces for other targets.

6. Geometry of MSP moduli spaces (Wei-Ping Li and Yang Zhou)

The genus-zero Gromov-Witten invariants of the quintic were computed by Givental, Lian-Liu-Yau, and various other authors [Giv96; LLY97]. Their computation uses the fact that the moduli space

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^4, \beta)$$

is smooth and the obstruction sheaf corresponding to $\mathcal{O}(5)$ is a vector bundle. This fails, however in higher genus, and in genus one, the work of Zinger, Vakil-Zinger, and

Li-Zinger [Zin09; VZ08; LZ09] on reduced Gromov-Witten invariants desingularizes the moduli spaces and enables computation of the genus one invariants. This result was also recovered by Kim-Loh [KL18] using quasimap wall-crossing [CK20; Zho22].

There is also the approach of Maulik-Pandharipande [MP06] using the degeneration formula, which in principle computes all of the invariants but is impossible to calculate with in practice. Finally, there is the approach of Fan-Lee [FL19].

6.1. P-**field reformulation of GW and FJRW theory.** Let $F = x_1^N + \cdots + x_r^N$ be an arbitrary Fermat polynomial. The setup works in full generality, but the calculations can only be conducted for the quintic threefold. The first step is to construct the moduli space of P-fields following the work of Chang-Li [CL12a], which comes from the work of Guffin-Sharpe [GS09] in physics.

Consider the vector space $\mathbb{C}^5 \times \mathbb{C}$ with the action of $\mathbb{C}^\times$ given by

$$t(x_1, \ldots, x_5, p) = (tx_1, \ldots, tx_5, t^{-5}p).$$

The GIT quotient with the positive stability chamber is $K_{\mathbb{P}^4}$. We will consider the superpotential

$$p(x_1^5 + \cdots + x_5^5).$$

A map $C \to K_{\mathbb{P}^4}$ is equivalent to the data

$$(C, \mathcal{L}, \varphi_1, \ldots, \varphi_5 \in H^0(\mathcal{L}), \rho \in H^0(\mathcal{L}^{-5})).$$

This is just a stable map, and the P-field is given by modifying ρ by ω_C to obtain

$$\xi = (C, \mathcal{L}, \varphi_1, \ldots, \varphi_5 \in H^0(\mathcal{L}), \rho \in H^0(\mathcal{L}^{-5} \otimes \omega_C)).$$

We will also require that $(\varphi_1, \ldots, \varphi_5)$ is nonvanishing, so it gives a stable map to $\mathbb{P}^4$.

The moduli space $\mathcal{M}^p$ of such ξ is noncompact. However, we can follow Kiem-Li [KL13] and use cosection localization for a cosection

$$\sigma \colon \mathsf{Ob} \to \mathcal{O}.$$

The degeneracy locus is in fact

$$\sigma^{-1}(0) = \overline{\mathcal{M}}_g(Z_5, \beta),$$

where $Z_5 \subset \mathbb{P}^4$ is the quintic threefold. This gives a virtual cycle[9]

$$[\mathcal{M}^p]_{\mathrm{loc}}^{\mathrm{vir}} = \pm [\overline{\mathcal{M}}_g(Z, \beta)]^{\mathrm{vir}}.$$

FJRW invariants were originally defined using analysis. There are algebraic reformulations by Polishchuk-Vaintrob and by Chiodo [PV16; Chi06]. We once again consider $\mathbb{C}^5 \times \mathbb{C}$ with an action of $\mathbb{C}^\times$ given by

$$t(x_1, \ldots, x_5, p) = (tx_1, \ldots, tx_5, t^{-5}p).$$

However, we use the negative stability chamber to obtain the quotient $[\mathbb{C}^5/\mu_5]$. The objects in our moduli space are given by

$$\eta = (C, \mathcal{L}, \varphi_1, \ldots, \varphi_5 \in H^0(\mathcal{L}), \rho \in H^0(\mathcal{L}^{-5} \otimes \omega_C))$$

such that ρ is nowhere zero. This implies that $\mathcal{L}^5 \cong \omega_C$. We will consider the same superpotential

$$p(x_1^5 + \cdots + x_5^5).$$

---

[9]The equality of virtual cycles was not proved by Chang-Li [CL12a] (who proved an equality of invariants only) as claimed in the talk. The first proof of the equality of virtual cycles is due to Kim-Oh [KO22]. See also [CL20b].

The moduli space $\mathcal{M}^{\frac{1}{5},5p}$ of such η is still noncompact, but cosection localization again provides a virtual cycle

$$[\mathcal{M}^{\frac{1}{5},5p}]^{\mathrm{vir}}_{\mathrm{loc}}$$

recovers the FJRW virtual cycle.

6.2. **Master space and MSP fields.** Following the work of Thaddeus [Tha96] and various other authors, we are interested in a geometric lift of a path connecting two GIT quotients. Consider the space $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1$ with the action of $\mathbb{C}^\times$ given by

$$t(x_1, \ldots, x_5, p, [u, v]) = (tx_1, \ldots, tx_5, t^{-5}p, [tu, v]).$$

Now define the master space

$$(\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1 \setminus \{(0, 0, 0, 0, p, [0, 1])\} \cup \{(x_1, \ldots, x_5, 0, [1, 0])\})/\mathbb{C}^\times.$$

Because $\mathbb{P}^1$ is a GIT quotient, we can reformulate the GLSM for this target as being given by the data of

$$\vartheta = (C, \mathcal{L}, \mathcal{N}, \varphi_1, \ldots, \varphi_5 \in H^0(\mathcal{L}), \rho \in H^0(\mathcal{L}^{-5} \otimes \omega_C), \mu \in H^0(\mathcal{L} \otimes \mathcal{N}), \nu \in H^0(\mathcal{N}))$$

such that $(\varphi_1, \ldots, \varphi_5, \mu)$ is nowhere zero, (ρ, ν) is nowhere zero, and (μ, ν) is nowhere zero. We also require that $\mathrm{Aut}\,\vartheta$ is finite.

Remark 6.1. Because the target is a Deligne-Mumford stack, we will need to require the source curve to be a twisted curve. For orbifold markings, we require that $\mathcal{L}|_\Sigma$ is a nontrivial automorphism of $\mathrm{Aut}\,\Sigma$. For scheme markings we will consider either $(1, \rho)$ markings where we force $\rho = 0$ or $(1, \varphi)$ markings were we force $\varphi = 0$. These are required to construct the cosection. These markings are called *narrow*, and all other markings are called *broad*.

6.3. **Cosection localization and virtual cycle.** The numerical data is $g = g(C)$, $d_0 = \deg \mathcal{L} \otimes \mathcal{N}$, and $d_\infty = \deg \mathcal{N}$. This gives the moduli space

$$\mathcal{W}_{g,\vec{d}} \to \mathcal{D}_g = \{(C, \mathcal{L}, \mathcal{N})\}$$

with a relative perfect obstruction theory

$$R\pi_*(\mathcal{L}^{\oplus 5} \oplus \mathcal{L}^{-5} \otimes \omega_C \oplus \mathcal{L} \otimes \mathcal{N} \oplus \mathcal{N}) \to \mathbb{L}^\bullet_{\mathcal{W}_{g,\vec{d}}/\mathcal{D}_g}.$$

Here, $\pi \colon \mathcal{C} \to \mathcal{W}_{g,\vec{d}}$ is the universal curve.

Fiberwise, the cosection is given by the following construction. Let [10]

$$(\dot{\varphi}, \dot{\rho}, \dot{\mu}, \dot{\nu}) \in H^1(\mathcal{L}^{\oplus 5}) \oplus H^1(\mathcal{L}^{-5} \otimes \omega_C) \oplus H^1(\mathcal{L} \otimes \mathcal{N}) \oplus H^1(\mathcal{N}).$$

Then the cosection is given by

$$\sigma = \dot{\rho}(\varphi_1^5 + \cdots + \varphi_5^5) + 5\rho(\varphi_1^4 \dot{\varphi}_1 + \cdots + \varphi_5^4 \dot{\varphi}_5)$$

and its image lies in $H^1(\omega_C) \cong \mathbb{C}$ (this isomorphism is canonical).

The degeneracy locus $\sigma^{-1}(0)$ is given by

$$\sigma^{-1}(0) = \{\rho\varphi_1^4 = \cdots = \rho\varphi_5^4 = \varphi_1^5 + \cdots + \varphi_5^5 = 0\}$$
$$= \{\rho = \varphi_1^5 + \cdots + \varphi_5^5 = 0\} \cup \{\varphi = 0\}.$$

The first difficult step is to prove the properness of the degeneracy locus.

Theorem 6.2. *The degeneracy locus $\sigma^{-1}(0)$ is proper in the narrow setting.*

---

[10]We are pretending to be physicists here.

6.4. **Virtual localization.** There is a $\mathbb{C}^\times$-action on $\mathcal{W}$ which actually preserves $\sigma^{-1}(0)$ given by

$$t \cdot (C, \mathcal{L}, \mathcal{N}, \varphi, \rho, \mu, \nu) = (C, \mathcal{L}, \mathcal{N}, \varphi, \rho, t\mu, \nu).$$

This allows us to compute MSP invariants using virtual localization.

We will now relate the fixed loci to the previous moduli spaces.

- Let $\vartheta \in \mathcal{W}$. If $\mu = 0$, the $\mathcal{N} = \mathcal{O}_C$ by the condition that (μ, ν) is nowhere vanishing. This reduces the fields to (φ, ρ), and we need $(\varphi_1, \ldots, \varphi_5)$ to be nowhere vanishing, so we obtain P-fields. We will call this level 0
- If we let $\nu = 0$, then the information of μ disappears and ρ must be nowhere zero. This reduces us to FJRW theory. This is called level ∞.
- Finally, if we set $\varphi = \rho = 0$, then both μ, ν are required to be nowhere vanishing, so we only have the data of a curve. When we perform virtual localization, this will give us Hodge integrals. This is called level 1.

Fixed loci $F \subset \mathcal{W}^{\mathbb{C}^\times}$ will be labelled by graphs. The problem is that there may be a situation of two edges $v_0 \to v_1 \to v_\infty$ where one edge has degree d and the other edge has degree $-d$. While there is no geometric smoothing of the node (it is banned by the stability condition), there are nontrivial infinitesimal deformations, so we cannot decompose the virtual cycle. Fortunately, there is a significant simplification, which was first discovered by Chang-Li [CL20a] for the quintic and should hold for all other Calabi-Yau threefolds.

Theorem 6.3 ([CL20a]). *These graphs contribute* 0 *to the MSP virtual localization formula.*

Let Λ be the set of regular graphs. Then the virtual localization theorem for cosection localized virtual cycles due to Chang-Kiem-Li [CKL17] gives

$$[\mathcal{W}]^{\mathrm{vir}}_{\mathrm{loc}} = \sum_{\Gamma \in \Lambda} (\iota_\Gamma)_* \frac{[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N_\Gamma^{\mathrm{vir}})}.$$

Because of irregular vanishing, we can decompose $[\mathcal{W}_\Gamma]^{\mathrm{vir}}_{\mathrm{loc}}$ along the vertices.

- At a level 0 vertex, we have

$$[\mathcal{W}_{v_0}]^{\mathrm{vir}}_{\mathrm{loc}} = \pm[\overline{\mathcal{M}}_{g_{v_0}, n_{v_0}}(Z, d_{v_0})]^{\mathrm{vir}},$$

  where Z is the quintic.
- At a level ∞ vertex, we have

$$[\mathcal{W}_{v_\infty}]^{\mathrm{vir}}_{\mathrm{loc}} = [\overline{\mathcal{M}}^{\frac{1}{5}, 5p}_{g_{v_\infty}, \gamma_{v_\infty}}]^{\mathrm{vir}}.$$

- At a level 1 vertex, we simply have

$$[\mathcal{W}_{v_1}]^{\mathrm{vir}}_{\mathrm{loc}} = [\overline{\mathcal{M}}_{g_{v_1}, n_{v_1}}].$$

6.5. **NMSP fields.** The setup can calculate the genus 1 Gromov-Witten invariants of the quintic, which was done by Chang-Guo-Li-Zhou [CGLZ20], but the calculations are too complicated to perform more computations. However, an insight of Guo is to replace μ with

$$\mu = (\mu_1, \ldots, \mu_N) \in H^0(\mathcal{L} \oplus \mathcal{N})^{\oplus N}$$

and consider an action of $(\mathbb{C}^\times)^N$ scaling μ diagonally.

6.6. **Reformulation of stability condition.** Recall that the original MSP master space is given by a GIT quotient of $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ with coordinates $x_1, \ldots, x_5, p, u, v$ by $(\mathbb{C}^\times)^2$ acting with weights

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and the polarization $t_1 t_2^2$. This is a change of coordinates from the standard presentation

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -5 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and polarization t_2.

Recall that a stable MSP field

$$\xi = (\mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi_1, \ldots, \varphi_5, \rho, \mu, \nu)$$

consists of the data of

- A pointed twisted curve $\mathcal{C}$ – the nodes are required to be balanced, which means that they locally look like

$$\left[\frac{\operatorname{Spec} \mathbb{C}[x, y]/(xy)}{(x, y) \mapsto (\zeta_r x, \zeta_r^{-1} y)} \right].$$

- $\mathcal{L}, \mathcal{N} \in \operatorname{Pic} \mathcal{C}$ are line bundles such that $\mathcal{L} \oplus \mathcal{N}$ is representable;
- $\varphi_1, \ldots, \varphi_5 \in H^0(\mathcal{L})$, $\rho \in H^0(\mathcal{L}^{-5} \otimes \omega_{\mathcal{C}}^{\log})$, $\mu \in H^0(\mathcal{L} \otimes \mathcal{N})$, and $\nu \in H^0(\mathcal{N})$ are sections such that
  - (φ, μ) is nonvanishing;
  - (ρ, ν) is nonvanishing;
  - (μ, ν) is nonvanishing;
  - $|\operatorname{Aut}(\xi)| < \infty$.

  We denote the nonvanishing condition by **MSP-Stab-I** and the finiteness of automorphisms by **MSP-Stab-II**.

There is a superpotential

$$p \sum_{i=1}^5 x_i^5 \colon [\mathbb{C}^8/\mathbb{C}^\times \times \mathbb{C}^\times] \to \mathbb{C}$$

with critical locus (see Figure 14)

$$\operatorname{Crit}\left(p \sum_{i=1}^5 x_i^5 \right) = \left(p = \sum_{i=1}^5 x_i^5 = 0 \right) \cup (x_1 = \cdots = x_5 = 0).$$
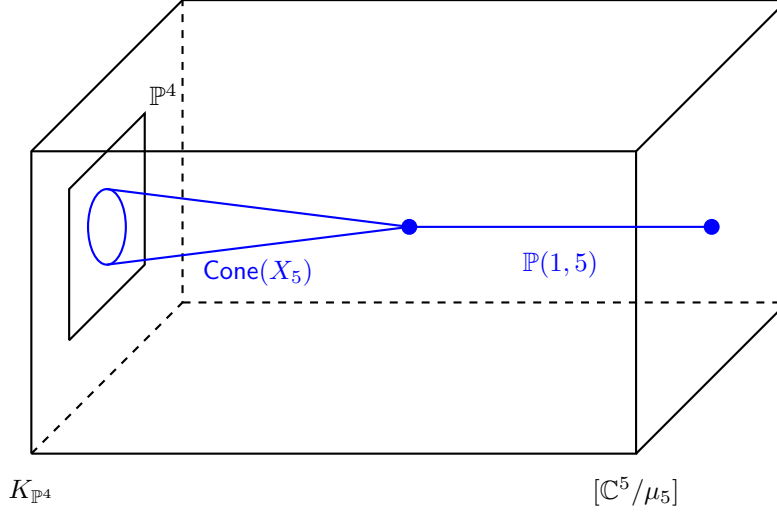
The semistable locus is

$$\mathsf{Cone}(X_5) \cup \mathbb{P}(1, 5).$$

The degeneracy locus of the MSP moduli space (with reduced stack structure) consists of all ξ such that $\mathcal{C}$ is mapped entirely into the critical locus. In other words, we require that

$$\left(p = \sum_{i=1}^5 x_i^5 \right) \cup (x_1 = \cdots = x_5 = 0) = \mathcal{C}.$$

The upshot of this is that the critical locus is proper.

FIGURE 14. Critical locus $\mathsf{Cone}(X_5) \cup \mathbb{P}(1,5)$

Theorem 6.4 ([CLLL19])**.** *Fixing the discrete data, the moduli of stable MSP fields is a separated DM stack of finite type and the degeneracy locus of the cosection is proper.*

Our goal is now to generalize the stability condition:

- Stability should come from GIT with the R-charge (which gives the $\omega^{\log}$);
- Properness should follow from properness of the target (or at least the critical locus).

Lemma 6.5. *Under* **MSP-Stab-I**, *the condition that* $|\operatorname{Aut}\xi| < \infty$ *is equivalent to the condition that*

$$(\mathcal{L} \otimes \mathcal{N}^2) \otimes (\omega_{\mathcal{C}}^{\log})^{\otimes A} > 0$$

as $A \to \frac{1}{5} + 0^+$.

Now note that the unstable locus is cut out by the functions

$$x_1 v^2, \dots, x_5 v^2, uv, u^{10} p.$$

This allows to reformulate **MSP-Stab-I** as the condition that the sections

$$\varphi_1 \nu^2, \dots, \varphi_5 \nu^2, \mu\nu \in H^0(\mathcal{L} \otimes \mathcal{N}^2) \qquad \text{and} \qquad \mu^{10}\rho \in H^0(\mathcal{L}^5 \otimes \mathcal{N}^{10} \otimes \omega_{\mathcal{C}}^{\log})$$

have no common zeros. This reformulation allows us to give easier proofs of basic facts about MSP fields.

Corollary 6.6. *Fixing the discrete data, stable MSP fields are bounded.*

Proof. We need to prove that

- There are finitely many irreducible components;
- The degrees $\deg \mathcal{L}_{\mathcal{C}'}$ and $\deg \mathcal{N}|_{\mathcal{C}'}$ are uniformly bounded for irreducible components $\mathcal{C}' \subset \mathcal{C}$.

The first condition follows from positivity of the line bundle

$$\mathcal{L} \otimes \mathcal{N}^2 \otimes (\omega_{\mathcal{C}}^{\log})^A.$$

For the second condition, suppose for simplicity that $\mathcal{C}' = \mathcal{C}$. Then we vary the polarization to obtain the inequalities

$$A_2 \leq \deg \mathcal{L} \otimes \mathcal{N}^2 \leq A_1;$$
$$A_3 \leq \deg \mathcal{L} \otimes \mathcal{N}^3;$$
$$A_4 \leq \deg \mathcal{L}^2 \otimes \mathcal{N}^3.$$

This gives a bounded region, and the denominators are bounded, so we are done. $\square$

Remark 6.7. In the nonabelian case, boundedness is much harder. See [Cha+23a] for more details.

6.7. **General setup.** The setup consists of the following data:
- An extension

$$1 \longrightarrow G \longrightarrow \Gamma \xrightarrow{\ \varpi\ } \mathbb{C}^\times \longrightarrow 1$$
$$\theta \searrow \quad \downarrow \vartheta$$
$$\mathbb{C}^\times$$

  of reductive groups;
- An affine scheme V (think a vector space or critical locus) with an action of Γ such that $V^s(\theta) = V^{ss}(\theta) \neq \emptyset$;

Remark 6.8. The lift ϑ of θ is not essential.

Definition 6.9. An *LG-quaasimap* is a tuple

$$\xi = (\mathcal{C}, u, \kappa),$$

where

(1) $\mathcal{C}$ is a pointed twisted curve;
(2) $u \colon \mathcal{C} \to [V/\Gamma]$ is representable such that the base locus $u^{-1}([V^{\mathrm{un}}(\theta)/\Gamma])$ is discrete and disjoint from the special points;
(3) $\kappa \colon u^* L_\varpi \xrightarrow{\cong} \omega_{\mathcal{C}}^{\log}$ is an isomorphism of line bundles.

Example 6.10. In the MSP case, either $V = \mathbb{C}^8$ or $\mathrm{Crit}\big(p \sum x_i^5\big)$. We also have $\Gamma = (\mathbb{C}^\times)^2 \times \mathbb{C}_R^\times$.
- A map $u \colon \mathcal{C} \to [V/\Gamma]$ is the data of $\mathcal{L}, \mathcal{N}, \mathcal{R} \in \mathrm{Pic}(\mathcal{C})$ and fields $\vec{\varphi}, \rho, \mu, \nu$. Here, note that $\rho \in H^0(\mathcal{L}^{-5} \otimes \mathcal{R})$.
- The data of κ is an isomorphism $u^* L_\varpi = \mathcal{R} \cong \omega_{\mathcal{C}}^{\log}$.

A fancier way to write an LG-quasimap is as a commutative diagram

$$
\begin{array}{ccc}
 & & \mathcal{C} \\
 & u \nearrow & \downarrow \\
[V/G] \longrightarrow & [V/\Gamma] \xrightarrow{\ \kappa\ } & \omega_{\mathcal{C}}^{\log} \\
\downarrow & \downarrow & \\
\mathrm{Spec}\,\mathbb{C} \longrightarrow & B\mathbb{C}_R^\times.
\end{array}
$$

6.8. Ω-stability. Define

$$R_k := \left\{ f \in \mathbb{C}[V] \mid g^* f = (\theta(g))^k f \text{ for all } g \in G \right\};$$
$$R_+ := \bigoplus_{k>0} R_k.$$

Then recall that we have a morphism

$$V /\!\!/_\theta G = \operatorname{Proj} \bigoplus_{k \geq 0} R_k \to \operatorname{Spec} R_0 = V /\!\!/_0 G.$$

Lemma 6.11 ([FJR18]). *Each R_k is $\Gamma - invariant$.*

Set

$$R_k = \bigoplus_{c \in \mathbb{Z}} R_{k,c},$$

where

$$R_{k,c} := \left\{ f \in \mathbb{C}[V] \mid \gamma^* f = (\vartheta(\gamma))^k (\varepsilon(\gamma))^c f \text{ for all } \gamma \in \Gamma \right\}.$$

Definition 6.12. An element $f \in R_\bullet = \bigoplus_{k \geq 0} R_k$ is *homogeneous* if $0 \neq f \in R_{k,c}$ for some k and c. We then define the *θ-weight* of f by $k =: \operatorname{wt}(f)$, the *$R$-charge* of f to be c, and the *slope* of f to be $\frac{c}{k}$.

Example 6.13. In the original MSP example, weights are given in Table 4. In this example, we have

$$u^* L_{k\vartheta + c\varpi} = (\mathcal{L} \otimes \mathcal{N}^2)^{\otimes k} \otimes (\omega_{\mathcal{C}}^{\log})^{\otimes c}.$$

TABLE 4. Weights.

	$x_i v^2$	uv	$u^{10} p$
Weight	1	1	5
R-charge	0	0	1.

In general, for $f \in R_{k,c}$ and a map $u \colon \mathcal{C} \to [V/\Gamma]$, we have

$$u^* f \in H^0(\mathcal{C}, u^* L_{k\vartheta + c\varpi}).$$

Now choose a package $\Omega = (S, A, \vartheta)$, where

- $S \subset R_+$ is a finite set of homogeneous elements;
- $A > \max_{f \in S} \{\operatorname{slope}(f)\}$.

Definition 6.14. A prestable LG-quasimap ξ is *Ω-stable* if

Ω-**1:** The base locus $u^{-1}([V(S)/\Gamma])$ is discrete and disjoint from the markings and nodes;

Ω-**2:** For all non-special points $x \in \mathcal{C}^{\operatorname{sm}}$, we have

$$\min_{f \in S} \left\{ \frac{1}{\operatorname{wt}(f)} \operatorname{ord}_x(u^* f) + \operatorname{slope}(f) \right\} \leq A;$$

Ω-**3:** We have the ampleness condition

$$u^* L_\vartheta \otimes (\omega_{\mathcal{C}}^{\log})^{\otimes A} > 0.$$

Remark 6.15.

(1) The A here corresponds roughly to $\frac{1}{\varepsilon}$ in the previous literature about quasimaps. Roughly, decreasing A means attempting to force more of the curve to go into the stable locus;
(2) $V(S)$ may be larger than the unstable locus $V^{\mathrm{un}} = V(R_+)$;
(3) If any irreducible component $\mathcal{C}' \subset \mathcal{C}$ satisfies $\omega_{\mathcal{C}}^{\log}|_{\mathcal{C}'} > 0$, then Ω-**3** is implied by either Ω-**1** or Ω-**2**;
(4) In some way, Ω-stability is independent of ϑ (up to changing the other data).

Theorem 6.16. *Fixing* $\deg u^* L_\vartheta = d$, *the genus* g, *and the number* k *of marked points, the stack*

$$\mathrm{LGQ}_{g,k}^{\Omega}(X, d)$$

is a separated DM stack of finite type.

Theorem 6.17. *Suppose that* $X /\!\!/_\theta G$ *is projective and* S *is full (which means that set-theoretically* $V(S) = V^{\mathrm{un}}(\theta)$*). Then* $\mathrm{LGQ}_{g,k}^{\Omega}(X, d)$ *is proper.*

Example 6.18. In the original MSP example, S is full. The element $u^{10}p \in S$ has slope $\frac{1}{5}$, so we need $A > \frac{1}{5}$. For any smooth point $q \in \mathcal{C}^{\mathrm{sm}}$, the conditions Ω-**1** and Ω-**2** are equivalent. In particular, they mean that at least one of the following conditions is satisfied:

$$\mathrm{ord}_q \varphi_i + 2 \, \mathrm{ord}_q \nu \le A, \qquad i = 1, \ldots, 5$$
$$\mathrm{ord}_q \mu + \mathrm{ord}_q \nu \le A$$
$$\frac{1}{5} \, \mathrm{ord}_q \rho + 2 \, \mathrm{ord}_q \mu \le A - \frac{1}{5}.$$

As $A \to \frac{1}{5} + 0^+$, we see that the first inequality holds if and only if $\mathrm{ord}_q \varphi_i = \mathrm{ord}_q \nu = 0$, which is equivalent to nonvanishing of $\varphi_i \nu^2$. This recovers the original nonvanishing condition **MSP-Stab-I**.

6.9. More examples.

Example 6.19. Consider $\mathbb{P}^N$ with an R-charge. This means that we take $\Gamma = \mathbb{C}^\times \times \mathbb{C}_R^\times$ and $V = \mathbb{C}^{N+1}$. We will have Γ act with weights

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ c_0 & c_1 & \cdots & c_N \end{bmatrix}.$$

For simplicity, assume that $c_0 \le \cdots \le c_N$. In this model, an LG quasimap is a package

$$\xi = (\mathcal{C}, \mathcal{L}, \varphi_i \in H^0(\mathcal{L} \otimes (\omega_{\mathcal{C}}^{\log})^{\otimes c_i})).$$

For the stability condition, we choose $S = \{x_0, \ldots, x_N\}$ and $A > c_N$.

The condition Ω-**2** becomes the condition

$$\min\{\mathrm{ord}_x(\varphi_i) + c_i\} \le A.$$

In general, we see that $\mathrm{ord}_x(\varphi_i)$ is allowed to be positive. The condition Ω-**3** becomes the positivity condition

$$\mathcal{L} \otimes (\omega_{\mathcal{C}}^{\log})^A > 0.$$

For example, if $N = 1$, $c_0 = 0$, and $c_1 = 1$, then as $A \to 1^+$, Ω-**2** becomes the condition that either $\varphi_1(x) \ne 0$ or $\mathrm{ord}_x(\varphi_0) \le 1$ at any non-special point $x \in \mathcal{C}$. In particular, it is impossible to completely remove basepoints.

Example 6.20. We will now consider the LG phase for a complete intersection

$$X_{d_1,d_2} \subset \mathbb{P}^N.$$

We will consider $V = \mathbb{C}^{N+1} \times \mathbb{C} \times \mathbb{C}$ and $\Gamma = \mathbb{C}^\times \times \mathbb{C}_R^\times$ acting on V with weights

$$\begin{bmatrix} 1 & \cdots & 1 & -d_1 & -d_2 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

We will choose the character $(t_1, t_2) \mapsto t_1^{-1}$, and the corresponding GIT quotient is

$$[V /\!\!/_\theta G] = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}(d_1,d_2)}(-1)^{\oplus N+1}).$$

An LG quasimap is given by the data

$$\xi = (\mathcal{C}, \mathcal{L}, \varphi_0, \dots, \varphi_N \in H^0(\mathcal{L}), \rho_1 \in H^0(\mathcal{L}^{-d_1} \otimes \omega_{\mathcal{C}}^{\log}), \rho_2 \in H^0(\mathcal{L}^{-d_2} \otimes \omega_{\mathcal{C}}^{\log})).$$

We choose $S = \{p_1, p_2\}$, and assuming that $d_1 \leq d_2$, we need to choose $A > \frac{1}{d_1}$.
The stability condition Ω-**2** becomes

$$\min\left\{ \frac{\mathrm{ord}_x(\rho_i)}{d_i} + \frac{1}{d_i} \right\} \leq A,$$

while Ω-**3** becomes the positivity condition

$$\mathcal{L}^{-1} \otimes (\omega_{\mathcal{C}}^{\log})^A > 0.$$

As $A \to \left(\frac{1}{d_1}\right)^+$, the condition Ω-**2** becomes the condition that either $\rho_1(x) \neq 0$ or $\mathrm{ord}_x(\rho_2) \leq \frac{d_2}{d_1} - 1$. Therefore, if $d_2 < 2d_1$, then ρ_1 and ρ_2 have no common zeroes.

Remark 6.21. If $d_1 = d_2$, this hybrid model admits a good lift in the sense of Fan-Jarvis-Ruan [FJR18] and the theory was studied by Clader [Cla17].

Example 6.22. We will now consider MSP theory for $X_{3,3} \subset \mathbb{P}^2 \times \mathbb{P}^2$. The Calabi-Yau phase is given by $K_{\mathbb{P}^2 \times \mathbb{P}^2}$, which is a GIT quotient of $\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}$ by $\mathbb{C}^\times \times \mathbb{C}^\times$ with weights

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 1 & -3 \end{bmatrix}.$$

There are three phases for this model, which are given in Figure 15. To construct
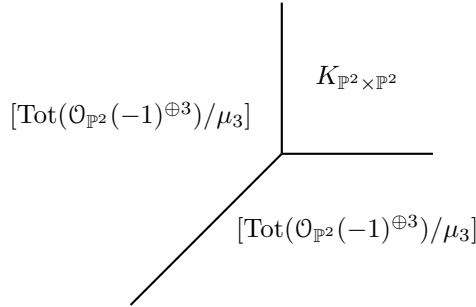

FIGURE 15. Phases of $X_{3,3} \subset \mathbb{P}^2 \times \mathbb{P}^2$.

the MSP moduli space involving all phases, consider the data

$$
\left[
\begin{array}{c|cccccccccc c}
 & x_1 & x_2 & x_3 & y_1 & y_2 & y_3 & p & u_1 & u_2 & v & \vartheta \\
\hline
\mathbb{C}^\times & 1 & 1 & 1 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 1 \\
\mathbb{C}^\times & 0 & 0 & 0 & 1 & 1 & 1 & -3 & 0 & 1 & 0 & 1 \\
\hline
\mathbb{C}^\times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\
\hline
\mathbb{C}^\times_R & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
\right].
$$

We will choose the set

$$
S = \left\{ xyv^3, xu_2v^2, yu_1v^2, u_1u_2v, y^9u_1^9p^2, x^9u_2^9p^2, u_1^9u_2^9p \right\}.
$$

The maximum slope is $\frac{2}{3}$, so we will choose $A = \frac{2}{3} + 0^+$. In particular, part of Ω-**2** becomes the condition

$$
\frac{1}{6}(9\operatorname{ord}(u_1) + 9\operatorname{ord}(u_2) + \operatorname{ord}(p)) + \frac{1}{6} \leq \frac{2}{3} + 0^+,
$$

which is equivalent to

$$
9\operatorname{ord}(u_1) + 9\operatorname{ord}(u_2) + \operatorname{ord}(p) \leq 3.
$$

This is equivalent to the condition that u_1 and u_2 are nonzero and $\operatorname{ord}(\rho) \leq 3$. Putting all of the conditions together, we obtain the following nonvanishing conditions:

- $(u_1, u_2, v) \neq 0$;
- $(y, u_2) \neq 0$;
- $(x, u_1) \neq 0$;
- Either $v \neq 0$ or $\operatorname{ord}(\rho) \leq 3$.

Unfortunately, it is not possible to remove the basepoints.

Consider the action of $\mathbb{C}^\times \times \mathbb{C}^\times$ given by scaling μ. A schematic diaagram of the the types of fixed loci is given in Figure 16.

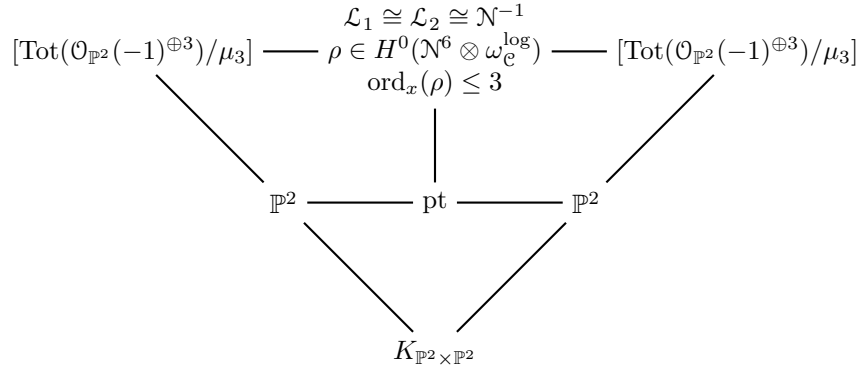FIGURE 16. Diagram of fixed loci for $X_{3,3} \in \mathbb{P}^2 \times \mathbb{P}^2$.

Remark 6.23. There is a wall-and-chamber structure as we vary A. If we consider $A \to \infty$, the condition Ω-**2** is always true, and Ω-**3** bans all rational tails. This gives a stability condition similar to stable quasimaps which was already studied by Fan-Jarvis-Ruan. In addition, there should be a wall-crossing structure as we

vary A (which is desirable because we want to separate the different kinds of fixed points).

Remark 6.24. Ω-stability exhibits a phenomenon which is close to being independent of S. If V is reduced and $V /\!\!/_\theta G$ is projective, then whenever S is full, we have

- The quantity
$$\max_{f \in S}\{\text{slope}(f)\}$$
  is independent of S;
- Fixing A and ϑ, $\Omega = (S, A, \vartheta)$-stability is independent of S.

6.10. **Proof of key properties of the moduli space.** The idea of the proof is as follows:

- First prove the $\mathbb{P}^N$ case;
- Compare the general case to $\mathbb{P}^N$.

To warm up, we will review the proof of properness for stable maps to $\mathbb{P}^1$. A stable map to $\mathbb{P}^1$ is equivalent to the data

$$(\mathcal{C}, \mathcal{L}, \varphi_1, \varphi_2 \in H^0(\mathcal{L}))$$

such that there are no common zeroes and $\mathcal{L}|_{\mathcal{C}'} > 0$ whenever $\omega_{\mathcal{C}}^{\log}|_{\mathcal{C}'} < 0$. We will use the valuative criteria.

Consider a family over a disk Δ_t. We have two divisors $(\varphi_1 = 0)$ and $(\varphi_2 = 0)$, but they may meet on the central fiber transversely with vanishing order 1. We can blow up the intersection point to separate the two divisors as in Figure 17. Denote the blowup morphism by $\pi \colon \tilde{\mathcal{C}} \to \mathcal{C}$. Then we actually have

$$\pi^*\varphi_1, \pi^*\varphi_2 \in H^0(\pi^*\mathcal{L}(-E)),$$

where E is the exceptional divisor. Then the stable limit is given by

$$(\tilde{\mathcal{C}}, \pi^*\mathcal{L}(-E), \pi^*\varphi_1, \pi^*\varphi_2).$$

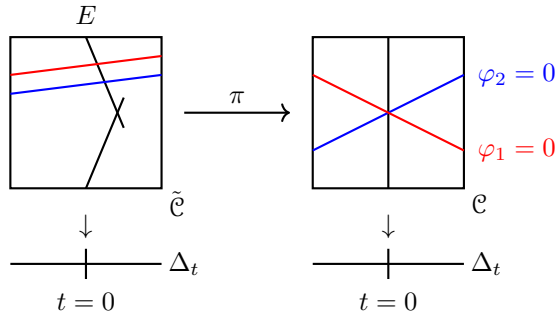If $A > 1$, we simply keep the base point (and disallow the rational tail).

FIGURE 17. Stable limit of family of stable maps.

If we introduce an R-charge $0, c$, an LG quasimap is given by

$$(\mathcal{C}, \mathcal{L}, \varphi_1 \in H^0(\mathcal{L}), \varphi_2 \in H^0(\mathcal{L} \otimes (\omega_{\mathcal{C}}^{\log})^{\otimes c})).$$

Now $\pi^*\varphi_2$ vanishes to order $1 + c$ on E. Blowing up again, the vanishing orders now become $(1, 2c) \sim (0, 2c - 1)$, which seems worse. If instead $(\varphi_1 = 0)$ has multiplicity

$k > c$, then on E the vanishing orders are $(k, 1 + c) \sim (k - 1 - c, 0)$. Then we blow up the node, which does not change $\omega_{\mathcal{C}}^{\log}$. Therefore, we cannot improve a length 1 zero of φ_1, but we can improve a length $> k$ zero of φ_1.
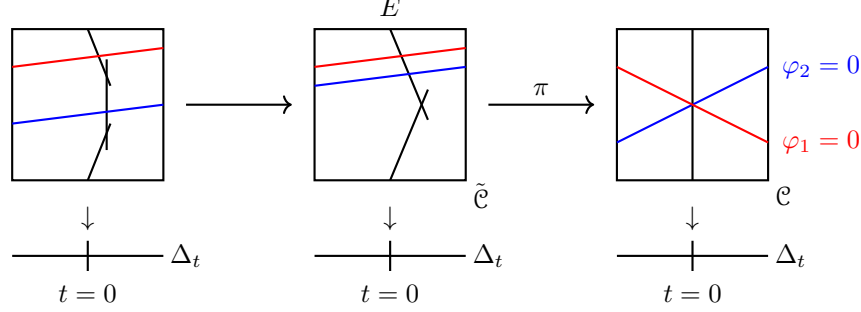

FIGURE 18. Stable limit of family of LG quasimaps.

We will now study the general case. We will assume that $S = \{f_0, \ldots, f_N\}$ all of weight 1. Then if we consider

$$
\begin{array}{ccc}
V & \xrightarrow{\ (f_0,\ldots,f_N)\ } & \mathbb{C}^{N+1} \\
\cup & & \cup \\
\Gamma & \xrightarrow{\ \vartheta \times \varepsilon\ } & \mathbb{C}^{\times} \times \mathbb{C}_R^{\times} \\
& \searrow{\scriptstyle \vartheta} \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\
& \mathbb{C}^{\times}, &
\end{array}
$$

This induces a cartesian diagram

$$
\begin{array}{ccc}
\mathrm{LGQ}(X) & \xrightarrow{\ \Phi\ } & \mathrm{LGQ}(\mathbb{P}^N) \\
\uparrow & & \uparrow \\
\mathrm{LGQ}^{\Omega}(X) & \longrightarrow & \mathrm{LGQ}^{\Omega}(\mathbb{P}^N).
\end{array}
$$

Properness reduces to proving properness of the bottom morphism. Using the valuative criterion, consider a curve $\mathcal{C}^{\circ} \to \Delta^{\circ}$ and an LG quasimap

$$
\xi^{\circ} = (\mathcal{C}^{\circ}, \ldots) \in \mathrm{LGQ}^{\Omega}(\mathcal{C}^{\circ}, X).
$$

If we have an extension of $\Phi(\xi^{\circ})$ to

$$
\underline{\xi} = (\mathcal{C}, \mathcal{L}, \varphi_0, \ldots, \varphi_N) \in \mathrm{LGQ}^{\Omega}(\underline{\mathcal{C}}, \mathbb{P}^N),
$$

we need a unique extension ξ of ξ° such that $\Phi(\xi) = \underline{\xi}$.

Because the line bundle $\omega_{\mathcal{C}}^{\log}$ comes from the coarse moduli space, we can trivialize it locally. Working locally, any LG quasimap becomes a usual quasimap. First assume that there are no base points (we will deal with base points using Hartogs's theorem). We already have an extension to the coarse moduli space $\underline{X}$.

Lemma 6.25 ([AV02])**.** *Locally, given $\underline{\mathcal{C}} \to \underline{X}$, there exists a unique commutative diagram*

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\underline{\mathcal{C}} & \longrightarrow & \underline{X}.
\end{array}
$$

This completes the proof of properness.

7. Applications to Gromov-Witten theory (Shuai Guo and Patrick Lei)

7.1. MSP invariants. We will consider MSP moduli spaces $\mathcal{W}_{g,n,d}$ with $d_0 = d$, $d_\infty = 0$, only $(1, \rho)$ insertions, and arbitrary values of N. We first note that the MSP virtual localization formula is given by

$$
\frac{1}{e(N_\Theta^{\mathrm{vir}})} = \prod_{v \in V_0} \prod_{\alpha=1}^{N} \frac{1}{e(R\pi_* f_v^* \mathcal{O}(1) \otimes t_\alpha)}
$$

$$
\cdot \prod_{\alpha=1}^{N} \prod_{v \in V_1^\alpha} \frac{5 t_\alpha \cdot e(\mathbb{E}^\vee \otimes (-t_\alpha))^5}{e(\mathbb{E} \otimes 5 t_\alpha) \cdot (-t_\alpha)^5} \frac{\prod_{\beta \neq \alpha} e(\mathbb{E}^\vee \otimes (t_\beta - t_\alpha))}{\prod_{\beta \neq \alpha} (t_\beta - t_\alpha)}
$$

$$
\cdot \left(\prod_{\mathfrak{a} \in \mathfrak{a}_\infty} \frac{1}{e(N_{\mathfrak{a}}^{\mathrm{vir}})} \right) \cdot \prod_{e \in E} A_e,
$$

where t_α are the equivariant variables and V_1^α denotes those vertices at level 1 where the curve satisfies $\mu_\alpha \neq 0$ and $\mu_{\beta \neq \alpha} = 0$. Also, $\mathfrak{a} \in \mathfrak{a}_\infty$ are connected components of the level ∞ part of Θ and A_e is the contribution of an edge. In particular, define

$$
[\overline{\mathcal{M}}_{g,n}(Z_5, d)]^{\mathrm{top}} = \frac{[\overline{\mathcal{M}}_{g,n}(Z_5, d)]^{\mathrm{vir}}}{e(R\pi_* f_v^* \mathcal{O}(1) \otimes t_\alpha)}
$$

$$
= (-t^N)^{d+1-g} [\overline{\mathcal{M}}_{g,n}(Z_5, d)]^{\mathrm{vir}}
$$

$$
[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{top}} = \left(\frac{1}{5} N (-t_\alpha)^{N+3} \right)^{g-1} [\overline{\mathcal{M}}_{g,n}].
$$

These are the top degree part of the contribution to the virtual localization formula coming from a vertex v. We will denote the full contribution at level 1 by $[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{tw}}$. From now on, we will specialize our equivariant variables to roots of unity as $t_\alpha = -\zeta_N^\alpha t$. For convenience, we will also specialize t such that $t^N = -1$.

We may define MSP invariants using virtual localization. Note that by the condition that ρ vanishes at the marked points, we have evaluation morphisms

$$
\mathrm{ev}_i \colon \mathcal{W}_{g,n,d} \to \mathbb{P}^{4+N},
$$

which restrict to

$$
\mathrm{ev}_i \colon \mathcal{W}_\Theta^- \to (x_1^5 + \cdots + x_5^5 = 0)^{(\mathbb{C}^\times)^N} = Z_5 \sqcup \bigsqcup_{\alpha=1}^{N} \mathrm{pt}_\alpha,
$$

where $\mathcal{W}_\Gamma^-$ is the degeneracy locus of $\mathcal{W}_\Gamma$. Therefore, we may define MSP invariants with insertions from the state space

$$
\mathcal{H} = H^*(Z_5) \oplus \bigoplus_{\alpha=1}^{N} H^*(\mathrm{pt}_\alpha).
$$

Using the vertex contributions to the virtual normal bundle, we define the pairing

$$(x, y)^M = \int_{Z_5} xy|_{Z_5} + \sum_\alpha \frac{5}{Nt_\alpha^3} xy|_{\mathrm{pt}_\alpha}.$$

The state space has several bases, which we will discuss now.

- Let $p = c_1(\mathcal{O}_{\mathbb{P}^{4+N}}(1))$ be the equivariant ambient hyperplane class. Then we have the basis $\phi_i = p^i$ for $i = 0, \ldots, N+3$;
- There is the basis $\{\mathbf{1}_{Z_5}, H, H^2, H^3\} \cup \{\mathbf{1}_\alpha\}_{\alpha=1}^N$;

The last kind of MSP invariant we need to define is the MSP $[0, 1]$ invariant. Here, we simply consider the class

$$[\mathcal{W}]^{[0,1]} = \sum_{\Theta \in \Lambda^{[0,1]}} \frac{[\mathcal{W}_\Theta]^{\mathrm{vir}}}{e(N_\Theta^{\mathrm{vir}})},$$

where $\Lambda^{[0,1]}$ denotes the set of all graphs without any level ∞ vertices.

7.2. Genus zero MSP theory.
In genus zero, the full MSP and the $[0, 1]$ theory are equal. This follows from the following lemma:

Lemma 7.1. *We have*

$$\mathcal{W}_{0,n,d} \cong \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+N}, d)$$

and an equality

$$[\mathcal{W}_{0,n,d}]^{\mathrm{vir}} = \pm e(R\pi_* f^* \mathcal{O}(5)) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+N}, d)]^{\mathrm{vir}}$$

of virtual cycles.

The lemma tells us that the genus-zero MSP invariants are the same as the GW invariants of a degree 5 hypersurface in $\mathbb{P}^{4+N}$, which is in particular Fano. In particular, the MSP I-function is given by the formula

$$I^M(q, z) = z \sum_{d \geq 0} q^d \frac{\prod_{m=1}^{5d}(5p + mz)}{\prod_{m=1}^d (p + mz)^5 \prod_{m=1}^d ((p + mz)^N - t^N)}.$$

This automatically implies the following result.

Lemma 7.2. *We have*

$$J^M(0, q, z) = I^M(q, z)$$

whenever $N \geq 2$.

The main result we need to know about genus zero MSP theory is the explicit form of the quantum connection. Let $D := q\frac{\mathrm{d}}{\mathrm{d}q}$.

Lemma 7.3. *The MSP S-matrix satisfies the differential equation*

$$(p + zD)S^M(z)^* = S^M(z) \cdot A^M,$$

where A^M is given by the matrix

$$\begin{bmatrix} 0 & & & & & & 120q & & & \\ 1 & 0 & & & & & & 770q & & \\ & 1 & 0 & & & & & & 1345q & \\ & & 1 & 0 & & & & & & 770q \\ & & & 1 & 0 & & & & & & 120q+t^N \\ & & & & 1 & 0 & & & & \\ & & & & & 1 & 0 & & & \\ & & & & & & 1 & 0 & & \\ & & & & & \cdots & & \cdots & & \\ & & & & & & & 1 & 0 & \\ & & & & & & & & 1 & 0 \\ & & & & & & & & & 1 & 0 \end{bmatrix}$$

in the basis $\{\phi_i\}$ for $N > 5$.

7.3. **MSP $[0, 1]$ CohFT.** Define the MSP R-matrix by the Birkhoff factorization

$$S^M(z)\begin{pmatrix} \Delta^1 & & & \\ & \ddots & & \\ & & \Delta^N & \\ & & & \mathrm{Id} \end{pmatrix} = R(z)\begin{pmatrix} S^{\mathrm{pt}_1} & & & \\ & \ddots & & \\ & & S^{\mathrm{pt}_N} & \\ & & & S^{Z_5} \end{pmatrix},$$

where

$$\Delta^\alpha(z)$$
$$:= \exp\left(\sum \frac{B_{2k}}{2k(2k-1)}\left(\frac{5}{(-t_\alpha)^{2k-1}} + \frac{1}{(5t_\alpha)^{2k-1}} + \sum_{\beta\neq\alpha}\frac{1}{(t_\beta - t_\alpha)^{2k-1}}\right)z^{2k-1}\right)$$

is defined using the quantum Riemann-Roch theorem. Here, we need to shift S^Z to the point $\tau_{Z_5} = \frac{I_1}{I_0}H$, and

$$S^{\mathrm{pt}_\alpha} = e^{\frac{\tau_\alpha}{z}},$$

where

$$\tau_\alpha = -t_\alpha\int_0^q (L(x) - 1)\frac{\mathrm{d}x}{x}.$$

Here, $L(q) := (1 - 5^5q)^{\frac{1}{N}}$.

Theorem 7.4. *The MSP $[0, 1]$ invariants come from a CohFT $\Omega^{[0,1]}$, which is defined by the formula*

$$\Omega^{[0,1]} = R.\left(\Omega^{Z_5} \oplus \bigoplus_{\alpha=1}^N \omega^{\mathrm{pt}_\alpha, \mathrm{top}}\right).$$

Remark 7.5. The normalized tail contribution at the isolated points is given by

$$\tilde{T}_\alpha(z) = z(\mathbf{1} - L^{\frac{N+3}{2}}R(z)^{-1}\mathbf{1})|_{\mathrm{pt}_\alpha} = O(z^2)$$

In addition, when $N \gg 3g - 3 + n$, there is no tail contribution at level 0.

7.4. **Degree bound for MSP theory.** In order to compute the invariants of a Calabi-Yau threefold using MSP theory, we need to control the MSP invariants. Our goal will be to control the MSP $[0, 1]$ invariants, but these are defined as a mysterious sum of virtual localization contributions. First, we will control the full MSP invariants.

Lemma 7.6. *The full MSP correlator*

$$\left\langle p^{a_1} \bar{\psi}_1^{k_1}, \ldots, p^{a_n} \bar{\psi}_n^{k_n} \right\rangle_{g,n}^M$$

is a polynomial in q of degree at most

$$g - 1 + \frac{3g - 3 + \sum a_i}{N}.$$

This follows from the fact that the virtual dimension of the MSP moduli space is $N(d + 1 - g) + n$. To obtain the same degree bound for the $[0, 1]$ correlators, we will need a decomposition formula for the full MSP theory in terms of the $[0, 1]$ theory and the remaining contributions. First, we will construct bipartite graphs from localization graphs, where vertices are either $[0, 1]$ vertices or ∞ vertices. For an example of this procedure, see Figure 19.
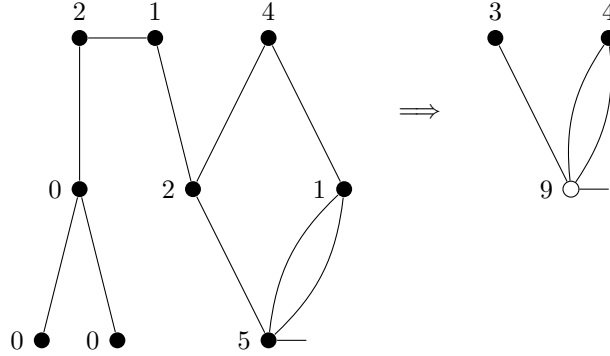
FIGURE 19. Obtaining a bipartite graph from a localization graph.

Lemma 7.7. *We have the MSP decomposition formula*

$$\left\langle \tau_1 \bar{\psi}_1^{a_1}, \ldots, \tau_n \bar{\psi}_n^{a_n} \right\rangle_{g,n}^M = \sum_{\Gamma \in \Lambda^{\text{bipartite}}} \frac{1}{|\text{Aut}\,\Gamma|} \prod_{v \in V_\infty} \text{Cont}_{[v]}^\infty \left(\bigotimes_{i \in L_v^\circ} \bar{\psi}_{c(i)}^{a_i} \right) \cdot$$

$$\cdot \prod_{v \in V_{[0,1]}} \left\langle \bigotimes_{i \in L_v} \tau_i \bigotimes_{i \in L_v^\circ} \bar{\psi}_{c(i)}^{a_i} \bigotimes_{e \in E_v} \frac{\mathbf{1}^{\alpha_e}}{\frac{5 t_{\alpha_e}}{a_e} - \psi_{(e,v)}} \right\rangle_{g_v, n_v}^{[0,1]}.$$

Here, the contribution $\text{Cont}_{[v]}^\infty$ of a vertex v at level ∞ is a generating series of FJRW-like invariants, which is a polynomial in q of degree at most

$$d_{\infty[v]} + \frac{1}{5}\left(2g_v - 2 - \sum_{e \in E_v}(a_e - 1)\right).$$

In addition, $\Lambda^{\mathsf{bipartite}}$ *is the set of* **stable** *bipartite graphs,* L_v° *is the set of legs which get contracted to* v *after stabilization, and* $c(i)$ *is the stable vertex that* i *gets contracted to after stabilization. For an example of stabilization, see Figure 20.*
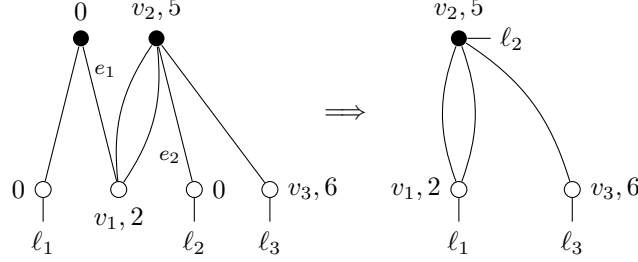

FIGURE 20. Stabilization of a bipartite graph. In this example, we have $c(\ell_1) = (v_1, e_1)$, $c(\ell_2) = (e_2, v_2)$, and $c(\ell_3) = \ell_3$.

This lemma is proved by directly applying the virtual localization formula and then analyzing the following two situations:

- What happens at a vertex at level ∞;
- What happens when we split a graph at a vertex at level 1.

By using the decomposition formula and a careful degree-counting argument, we obtain the following degree bound for the $[0, 1]$ theory.

Lemma 7.8. *The MSP* $[0, 1]$ *correlator*

$$\left\langle p^{a_1}\bar{\psi}_1^{k_1}, \ldots, p^{a_n}\bar{\psi}_n^{k_n} \right\rangle_{g,n}^{[0,1]}$$

is a polynomial in q *of degree at most*

$$g - 1 + \frac{3g - 3 + \sum a_i}{N}.$$

7.5. **Polynomiality.** We first introduce the ring of five generators. Let

$$I(q, z) := z \sum_{d \geq 0} q^d \frac{\prod_{m=1}^{5d}(5H + mz)}{\prod_{m=1}^{d}(H + mz)^5}$$

$$=: I_0 z + I_1 H + I_2 \frac{H^2}{z} + I_3 \frac{H^3}{z^2}$$

and define the following generators:

$$A_k := \frac{D^k I_{11}}{I_{11}}, \qquad B_k := \frac{D^k I_0}{I_0}, \qquad \text{and} \qquad Y = \frac{1}{1 - 5^5 q}.$$

Here, recall that $I_{11} = 1 + D\left(\frac{I_1}{I_0}\right)$.

Lemma 7.9 ([YY04]). *The ring*

$$\mathcal{R} := \mathbb{Q}[A_1, B_1, B_2, B_3, Y]$$

contains all A_k *and* B_k.

Theorem 7.10. *Introduce the series*

$$P_{g,n} := \frac{(5Y)^{g-1} I_{11}^n}{I_0^{2g-2}} \left(Q \frac{\mathrm{d}}{\mathrm{d}Q} \right)^n F_g(Q) \Bigg|_{Q=qe^{\frac{I_1}{I_0}}}.$$

Then $P_{g,n} \in \mathcal{R}$ for all g, n such that $2g - 2 + n > 0$.

If we want to prove this result using the results we have already proved, then we need to prove a polynomiality result for the the entries of the R-matrix. At level 0, we use the equation

$$(R(z)^{-1} x)|_{Z_5} = S^{Z_5}(q, z)(S^M(z)^{-1})|_{Z_5}$$

and the explicit forms of the MSP quantum connection and the quantum connection for the quintic to obtain

$$R(z)^* \mathbf{1}|_{Z_5} = I_0 + O(z^{N-3})$$

$$R(z)^* p|_{Z_5} = zD(I_0) + HI_0 I_{11} + O(z^{N-2}).$$

To simplify what follows, define the normalized basis

$$\varphi_b = I_0 I_{11} \cdots I_{bb} H^b,$$

where I_{22} was defined previously and $I_{33} = I_{11}$. If we define

$$(R_k)_j^b := (R_k \varphi^b, p^j)^M,$$

then the recursive formula

$$(R_k)_j^b = (D + C + b)(R_{k-1})_{j-1}^b + (R_k)_{j-1}^{b-1} - c_j q (R_k)_{j-N}^b,$$

where $C_b = D \log(I_0 \cdots I_{bb}) \in \mathcal{R}$ and $c_j = (0, \ldots, 0, 120, 770, 1345, 770)$, yields the following result:

Lemma 7.11. *If $j \not\equiv b + k \pmod{N}$, then $(R_k)_k^b = 0$. Otherwise, we have $(R_k)_{b+k}^b \in \mathcal{R}$ and $Y(R_k)_{b+N+k}^n \in \mathcal{R}$.*

At level 1, define the normalized basis $\bar{\mathbf{1}}_\alpha = L^{-\frac{N+3}{2}} \mathbf{1}_\alpha$. Then define

$$(R_k)_j^\alpha := L_\alpha^{-(j-k)}(R_k \bar{\mathbf{1}}^\alpha, p^j)^M,$$

where $L_\alpha = -t_\alpha L$.

Lemma 7.12. *The quantity $(R_k)_j^\alpha$ is independent of α and is a polynomial in Y of degree at most $k + \lfloor \frac{j}{N} \rfloor$.*

The lemma is proved as follows:

- Fix the case when $j = 0$ by using the Picard-Fuchs equation

$$D_{L_\alpha}^5 (D_{L_\alpha}^N - t^N) - q \prod_{m=1}^5 (D_{L_\alpha} + mz) R(z)^* \mathbf{1}|_{\mathrm{pt}_\alpha} = 0$$

and an oscillating integral. Solving the Picard-Fuchs equation expresses R_k as an antiderivative of some polynomial in $R_{m<k}$ and their derivatives. To prove that the constants of integration are zero, we use the fact that $R(z)\mathbf{1}|_{\mathrm{pt}_\alpha}$ can be computed from the asymptotic expansion of the oscillating integral

$$\int_{\gamma_\alpha} e^{\frac{W}{z}} \frac{\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_{N+3}}{x_0 \cdots x_{N+3}},$$

where

$$W = \sum_{i=0}^{N+3} (x_i - \lambda_i \log x_i) - \left(\frac{\prod_{i=0}^{N+3} x_i}{q} \right)^{\frac{1}{5}}$$

is the mirror superpotential and γ_α is the Lefschetz thimble near the critical point of W corresponding to pt_α.

- Use the recursion

$$(R_k)_j^\alpha = \left(D - \frac{1}{N} \left(\frac{N+3}{2} - j + k \right) (1 - Y) \right) (R_{k-1})_{j-1}^\alpha$$
$$+ (R_k)_{j-1}^\alpha + \frac{c_j}{5^5} (1 - Y)(R_k)_{j-N}^\alpha$$

to induct on j.

Proof of Theorem 7.10. First, note that we have the base cases $P_{0,3} = 1$ due to Zagier-Zinger [ZZ08] and

$$P_{1,1} = -\frac{1}{2}A_1 - \frac{31}{3}B_1 - \frac{1}{12}(1 - Y) - \frac{25}{12}$$

due to Zinger [Zin09]. The relation

$$P_{g,n+1} = (D + (g-1)(2B_1 + 1 - Y) - nA_1)P_{g,n}$$

implies that we only need to prove $P_{g \geq 2} \in \mathcal{R}$.

Consider the correlator $(5Y)^{g-1} \langle \ \rangle_{g,0}^{[0,1]}$, which is a polynomial in Y of degree at most $g - 1$. By the stable graph sum formula, we have

$$(5Y)^{g-1} \langle \ \rangle_{g,0}^{[0,1]} = P_g + \sum_\Gamma \mathrm{Cont}_\Gamma .$$

For all non-leading graphs, we use the relation $\sum_v (g_v - 1) + |E| = g - 1$ to assign powers of Y to all of the edges. Then the contributions from vertices are given as follows:

- At a level 0 vertex, the contributions are simply

$$Y^{g_v - 1} \left\langle \varphi_{b_1} \bar{\psi}_1^{a_1}, \ldots, \varphi_{b_{n_v}} \bar{\psi}_{n_v}^{a_{n_v}} \right\rangle_{g_v, n_v}^{Z_5},$$

  which reduces to $P_{g_v, m}$ by the string and dilaton equations.
- At a level 1 vertex, the contribution is

$$\sum_m \frac{L^{3(g_v - 1)}}{m!} \left\langle L_\alpha^{j_1 - k_1} \bar{\psi}_1^{k_1}, \ldots, L_\alpha^{j_{n_v} - k_{n_v}} \bar{\psi}_{n_v}^{k_{n_v}}, \tilde{T}_\alpha^m \right\rangle_{g_v, n_v + m}.$$

  After summing over all α, we see that this is nonzero only if the total power of t_α is a multiple of N (here, we may want N to be a prime number).

Using the fact that the contribution from an edge between two level 1 vertices satisfies a balancing condition, the total factor of the L_α for the various α becomes 1. This implies that $\mathrm{Cont}_\Gamma \in \mathcal{R}$ for any non-leading Γ, so we must have $P_g \in \mathcal{R}$. □

Remark 7.13. We can recover the genus one mirror theorem very quickly using the results we have already proved. If we consider the correlator

$$\langle p \rangle_{1,1}^{[0,1]} = \mathrm{const},$$

there are only two stable graphs. The contribution of the stable graph with a genus 1 vertex at the quintic is given by

$$\frac{1}{I_0}\big\langle R(z)^{-1}p|_{Z_5}\big\rangle_{1,1}^{Z_5} = \big\langle -B_1\bar{\psi}_1 + I_{11}H\big\rangle_{1,1}^{Z_5}$$
$$= P_{1,1} + \frac{200}{24}B_1.$$

The other graph contributes

$$\frac{1}{2}\Big(A_1 + 4B_1 + \frac{2}{5}(1-Y)\Big)$$

at level 0. Finally, we can prove that the total contribution from level 1 is a degree 1 polynomial in Y, so using the known values of $N_{1,1}$ and $\langle H\rangle_{1,1,0}^{Z}$ fixes the two coefficients of Y.

Remark 7.14. The formula

$$P_{1,1} = -\frac{1}{2}A_1 + \Big(\frac{\chi(Z)}{24} - 2\Big)B_1 - \frac{1}{12}(1-Y) - \frac{1}{24}\int_Z c_2(Z)\cdot H$$

is expected to hold for all of the examples in Table 2, where we denote the Calabi-Yau threefold by Z. It has been proven for hypersurfaces in weighted projective space by the author [Lei24b].

7.6. **Mirror symmetry picture.** Recall that mirror symmetry is for families

$$\mathcal{Z} = (Z,\omega) \qquad\qquad \mathcal{Z}^\vee = (Z^\vee, J)$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\mathcal{M} \subset \overline{\mathcal{M}} \ni Q \longleftrightarrow \mathcal{M}^\vee \subset \overline{\mathcal{M}}^\vee \ni q.$$

On the A-side, we have bases $\phi_i \in H^{1,1}(Z)$ mirror to $e_i \in H^{2,1}(Z^\vee)$. On the B-side, near the maximal unipotent monodromy point, we have periods

$$I_0 = \int_{\Gamma_0} \Omega(q) = 1 + O(q);$$
$$I_{1,i} = \int_{\Gamma_i} \Omega(q) = \log q_i + \cdots.$$

Then we will have the mirror map given by

$$Q_i = e^{\frac{I_{1,i}}{I_0}},$$

so we see that

$$\phi_j = \sum_i \frac{\partial \log Q_j}{\partial \log q_i} e_i.$$

Using the B-model basis, we define the Yukawa coupling

$$Y_{ijk} := I_0^2 \langle e_i, e_j, e_k\rangle_{0,3}^Z,$$

which is a rational function over $\overline{\mathcal{M}}$.

Remark 7.15. Physicists have more predictions about the Yukawa couplings, including pole orders.

In the case of the quintic $Z_5 \subset \mathbb{P}^4$, there is only one Kähler parameter, and the Yukawa coupling is simply

$$Y = \frac{1}{1 - 5^5 q}.$$

For $Z_{3,3} \subset \mathbb{P}^2 \times \mathbb{P}^2$, the Yukawa couplings are symmetric under permuting 1 and 2 and are given by

$$Y_{111} = \frac{3^4 q_1 (2 + 3^3 q_1 + 3^3 q_2)}{\Delta};$$
$$Y_{112} = \frac{(1 - 3^3 q_1 - 3^3 q_2)(1 + 2 \cdot 3^3 q_1 - 3^3 q_2)}{\Delta},$$

where the discriminant Δ is given by

$$\Delta = (1 - 3^3 q_1 - 3^3 q_2) - 3^9 q_1 q_2.$$

In higher genus, we define

$$P_{g,\vec{a}} := \frac{1}{I_0^{2g-2}} \langle e_a, \ldots, e_{a_n} \rangle_{g,n}^X.$$

Conjecture 7.16 ([BCOV94]). *Define*

$$f_{g,\vec{a}} := P_{g,\vec{a}} + \mathsf{lower\ order},$$

where the lower order terms come from a sum over stable graphs. This $f_{g,\vec{a}}$ is a rational function in q and after normalization by some power of the Yukawa couplings is a polynomial of degree at most $3g - 3 + n$.

Example 7.17. In the case when $g = 2$, the contributions to f_2 come from the graps in Figure 21. We put S^{ij} on each edge and at each vertex, we place
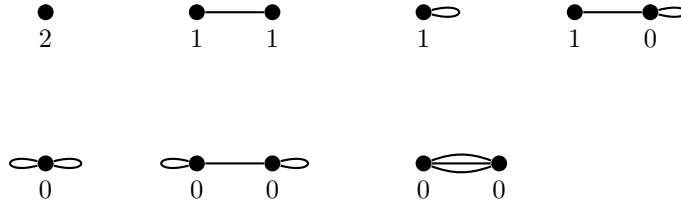
FIGURE 21. Stable graphs of genus 2 with no marked points.

$$\langle e_{i_1}, \ldots, e_{i_n} \rangle_{g,n}^B := I_0^{-(2g-2)} \langle e_{i_1}, \ldots, e_{i_n} \rangle_{g,n}^A - (n - 1)! \delta_{g,1} \delta_{\vec{i},\vec{0}},$$

where we replace all e_0 by $\bar{\psi}$ and there is a correction term in genus 1 with only ancestor insertions.

This Feynman graph sum can be realized using geometric quantization. In particular, consider the matrix

$$R = \begin{pmatrix} 1 & zK_i & z^2 S^i & z^2(S + S^i K_i) \\ & I & zS^{ij} & z^2(\tilde{S}^i)^T \\ & & U & zK_i^T \\ & & & 1 \end{pmatrix} = \begin{pmatrix} A & C \\ & D \end{pmatrix},$$

which is explicitly defined using BCOV's propagators [BCOV94]. This is symplectic with respect to the pairing

$$\langle f, g \rangle = \text{Res}_{z=0}(f(-z), g(z)).$$

Therefore, we may define its geometric quantization by

$$\hat{R}F(x) := \int_{\mathbb{R}^{h+1} \times \mathbb{R}^{h+1}} e^{\frac{1}{2}Q(x',p')-x'\cdot p'+F(x')} \, \mathrm{d}x \, \mathrm{d}p,$$

where Q is a quadratic form given by

$$Q = \langle p, D^{-1}x \rangle - \frac{1}{2}\langle p, D^{-1}Cp \rangle.$$

We will view $R \in \text{End}\, H_B$, where

$$H_B = \text{span}\{e^0 z^{-2}, e^i z^{-1}, e_i, e_0 z\}.$$

It naturally acts on Givental's symplectic vector space, but we can restrict it to a finite-dimensional vector space H_A (which is still larger than H_B). We will call it R^A when acting on H_A and R^B when acting on H^B.

Remark 7.18. Note that Givental's graph sum contains more information than BCOV's graph sum coming from the extra basis elements.

Theorem 7.19. *Here, we restrict to the case of the quintic. Let $\bullet$ denote either A or B. Denote*

$$f^\bullet(x, y) := \hat{R}^\bullet F^\bullet(x, y),$$

where we define

$$F^\bullet(x, y) := \sum \hbar^{g-1} F^\bullet_{g,m,n} \frac{x^m y^n}{m!n!}.$$

Then we have the identity

$$f^B_{g,m,n} = f^A_{g,m,n} - \delta_{g,1}\delta_{m,0}(n-1)!.$$

In other words, there is the commutative diagram

$$
\begin{array}{ccc}
f^A_{g,m,n} & \xrightarrow{\ -\delta_{g,1}\delta_{m,0}(n-1)!\ } & f^B_{g,m,n} \\
\hat{R}^A \Big\uparrow & & \hat{R}^B \Big\uparrow \\
F^A_{g,m,n} & \xrightarrow{\ -\delta_{g,1}\delta_{m,0}(n-1)!\ } & F^B_{g,m,n}.
\end{array}
$$

7.7. MSP realization of the Feynman rule. Recall that MSP for the quintic is given by the charges

$$
\left(
\begin{array}{c|cccccccc}
 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \rho & \mu & \nu \\
\hline
\mathcal{L} & 1 & 1 & 1 & 1 & 1 & -5 & 1 & \\
\omega^{\log} & & & & & & & 1 & \\
\mathcal{N} & & & & & & & 1 & 1
\end{array}
\right).
$$

Virtual localization gives us the fixed loci

$$W^{(\mathbb{C}^\times)^N} = Z_5 \sqcup N\text{pt},$$

where $W = (x_1^5 + \cdots + x_5^5 = 0) \subset \mathbb{P}^{4+N}$. At the level of CohFTs, recall that we had an identity

$$\Omega^{[0,1]} = R.(\Omega^{Z_5} \oplus \Omega^{N\text{pt}}).$$

One advantage of increasing N is that it makes the Dijkgraaf-Witten map τ (which sends $I(-z)$ onto the slice of $\mathcal{L}^{\mathrm{loc}}$ coming from $H^*(Z_5)\oplus H^*(N\mathrm{pt})$) very nice. The actual values of τ on components were given in Section 7.3. If we compute the edge contribution V^{00} connecting two vertices at level 0, this will be exactly the propagators as $N \to \infty$.

Our goal is to use the polynomiality of $\Omega^{[0,1]}$ to deduce polynomiality of the CohFT

$$R^A . \Omega^{Z_5}.$$

We will consider the factorization

$$R = \mathbb{G} \cdot \begin{pmatrix} R^A & \\ & I_{N\times N} \end{pmatrix},$$

which factorizes

$$\Omega^{[0,1]} = \mathbb{G}.(R^A.\Omega^{Z_5} \oplus \Omega^{N\mathrm{pt}}).$$

It now suffices to prove that $\mathbb{G}$ preserves polynomiality, and in fact we see that the entries of $\mathbb{G}$ lie in $\mathbb{Q}[Y]$ with explicit degree bounds.

7.8. **FJRW theory of the quintic.** The unfortunate issue with the MSP moduli space of the quintic is that there are N vertices at level ∞. Instead, we will consider N copies of the field ν. To calculate FJRW invariants, we will set $d_0 = 0$ and $d_\infty = d$. One nice feature of this is that the moduli space has no vertices at level 0, so we do not need to prove irregular vanishing.

The ambient space of the target is $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}(1,\ldots,1,5)}(-1)^{\oplus 5})$. Virtual localization will give N isolated points and the FJRW theory as fixed loci. When computing using MSP, there is a factorization $R = R^{\mathrm{loc}}\Delta$, where R^{loc} comes from virtual localization and Δ comes from Grothendieck-Riemann-Roch. In this model, neither step preserves the unit, but their composition does preserve the unit, and therefore we obtain

$$R.(\Omega^{\mathsf{FJRW}} \oplus \Omega^{N\mathrm{pt}}) = \Omega^{[1,\infty]}.$$

7.9. **Multi-parameter models.** We will consider only a 1-parameter deformation of the GLSM for $Z_{3,3} \subset \mathbb{P}^2 \times \mathbb{P}^2$. This is given by the charges

$$\begin{pmatrix} & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \rho & \mu & \nu \\ \hline \mathcal{L}_1 & 1 & 1 & 1 & & & & -3 & 1 & 0 \\ \mathcal{L}_2 & & & & 1 & 1 & 1 & -3 & 0 & 0 \\ \omega_{\log} & & & & & & & 1 & & \\ \mathcal{N} & & & & & & & & 1 & 1 \end{pmatrix}.$$

The fixed loci are now LG, $\mathbb{P}^2$, and $Z_{3,3}$. We then have the identities

$$\Omega_1^{[0,1]} = R_1.(\Omega^{Z_{3,3}} \oplus \Omega^{N\mathbb{P}^2});$$

$$\Omega_2^{[0,1]} = R_2.(\Omega^{Z_{3,3}} \oplus \Omega^{N\mathbb{P}^2}).$$

Note that $\Omega_1^{[0,1]}$ is polynomial only in q_1 and $\Omega_2^{[0,1]}$ is polynomial only in q_2. We then find that $R_1^A = R_2^A$, and so in fact we have

$$\Omega_1^{[0,1]} = \mathbb{G}_1.(R_A.\Omega^{Z_{3,3}} \oplus \Omega^{N\mathbb{P}^2});$$

$$\Omega_2^{[0,1]} = \mathbb{G}_2.(R_A.\Omega^{Z_{3,3}} \oplus \Omega^{N\mathbb{P}^2}),$$

where $\mathbb{G}_i$ is polynomial in q_i (with no bound in the other Kähler parameter). The first identity gives polynomiality in q_1 and the second gives polynomiality in q_2.
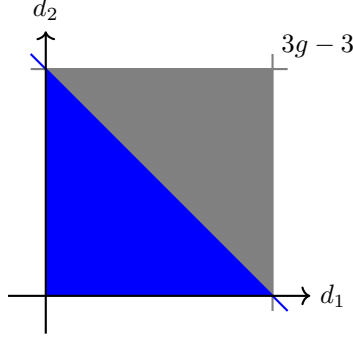

FIGURE 22. MSP bounds (gray) and sharp bounds (blue) for $Z_{3,3} \subset \mathbb{P}^2 \times \mathbb{P}^2$.

Remark 7.20. The bound obtained using MSP for this example is not sharp, for example see Figure 22.

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