Category O Learning Seminar
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Lectures by Various

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Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Seminar Website:
`https://math.columbia.edu/~plei/f21-CO.html`

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Kevin (Sep 29): Review of semisimple Lie algebras and introduction to category $\mathcal{O}$

1.1 Review of semisimple Lie algebras

Throughout this lecture, we will work over $\mathbb{C}$.

Definition 1.1.1. A Lie algebra $\mathfrak{g}$ is *semisimple* if any of the following equivalent conditions hold:

1. $\mathfrak{g}$ is a direct sum of simple Lie algebras (those with no nonzero proper ideals).
2. The Killing form $\kappa(x, y) := \operatorname{tr}(\operatorname{ad}(x) \operatorname{ad}(y))$ is nondegenerate.
3. The radical (maximal solvable ideal) of $\mathfrak{g}$ is zero.

Some examples of semisimple Lie algebras include $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$, and in some sense (the classification of simple Lie algebras), these are essentially all semisimple Lie algebras.

Now given a semisimple Lie algebra $\mathfrak{g}$, we will fix a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$, which is just a maximal abelian subalgebra of semisimple elements. This gives us a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the subspace of $\mathfrak{g}$ where $\mathfrak{h}$ acts with weight α. Some important facts about these root systems are the following:

- For all α, we have $\dim \mathfrak{g}_\alpha = 1$.
- For all roots α, β, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
- If α is a root, so is $-\alpha$.

In addition, the α are required to form a (reduced) *root system* (denoted Φ), the precise definition of which is deliberately omitted. Given a choice of Borel subalgebra containing $\mathfrak{h}$, we obtain a set Φ^+ of positive roots and a set Δ of simple roots. In addition, given a root system Φ, there is a *dual root system* $\Phi^\vee$, whose roots are

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}, \alpha \in \Phi.$$

Now suppose that $\mathfrak{g}$ is a semisimple Lie algebra with root system Φ. For every $\alpha \in \Phi^+$, we may choose $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$, and these determine some $h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{h}$. This choice can be made such that $\alpha(h_\alpha) = 2$.

Recall that the Lie algebra $\mathfrak{sl}_2$ is spanned by the matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the choice of $x_\alpha, y_\alpha, h_\alpha$ gives an embedding $\mathfrak{sl}_2 \to \mathfrak{g}$. These maps, ranging over all α, cover all of $\mathfrak{g}$. Now a basis of $\mathfrak{g}$ is given by $x_\alpha, y_\alpha, \alpha \in \Phi$ and h_{α_i} for the **simple** roots α_i. Therefore, to specify $\mathfrak{g}$, we only need to give commutation relations for the basis elements.

Now suppose that Φ is some root system. We would like to construct a semisimple Lie algebra $\mathfrak{g}$ with root system Φ. We want to build a semisimple Lie algebra. To do this, choose a set of simple roots α_i, and consider the Lie algebra

$$\left\langle x_{\alpha_i}, y_{\alpha_i}, h_{\alpha_i} \right\rangle / \text{relations},$$

where the relations are as follows:

- $[h_{\alpha_i}, h_{\alpha_j}] = 0$.
- We have $[x_{\alpha_i}, y_{\alpha_j}] = h_{\alpha_i}$ if $i = j$ and this commutator vanishes otherwise.
- $[h_{\alpha_i}, x_{\alpha_j}] = \left\langle \alpha_j, \alpha_i^\vee \right\rangle x_{\alpha_j}$.
- $[h_{\alpha_i}, y_{\alpha_j}] = -\left\langle \alpha_j, \alpha_i^\vee \right\rangle y_{\alpha_j}$.
- $\mathrm{ad}(x_{\alpha_i})^{1-\left\langle \alpha_j, \alpha_i^\vee \right\rangle}(x_{\alpha_j}) = 0$ if $i \neq j$.
- $\mathrm{ad}(y_{\alpha_i})^{1-\left\langle \alpha_j, \alpha_i^\vee \right\rangle}(y_{\alpha_j}) = 0$ if $i \neq j$.

The first four relations are called the *Weyl relations* and the last two are called the *Serre relations*. Given this data, we end up with a semisimple Lie algebra $\mathfrak{g}_\Phi$ with root system Φ. In addition, if $\mathfrak{g}$ is any other semisimple Lie algebra with root system Φ, there is an isomorphism $\mathfrak{g}_\Phi \xrightarrow{\sim} \mathfrak{g}$. Moreover, we have a bijection between semisimple Lie algebras and reduced root systems, which restricts to a bijection between simple Lie algebras and irreducible root systems.

Table 1.1: Root systems and Lie algebras

Irreducible root systems	simple Lie algebras
A_n	$\mathfrak{sl}_{n+1}$
B_n	$\mathfrak{so}_{2n+1}$
C_n	$\mathfrak{sp}_{2n}$
D_n	$\mathfrak{so}_{2n}$
E_6, E_7, E_8, F_4, G_2	exceptional Lie algebras

We will now discuss the finite-dimensional representation theory of semisimple Lie algebras $\mathfrak{g}$.

Theorem 1.1.2 (Weyl's complete reducibility theorem)**.** *Any finite-dimensional representation of $\mathfrak{g}$ is decomposes as a direct sum of simple representations.*

Now suppose that M is a finite-dimensional $\mathfrak{g}$-representation. Then M has a weight decomposition

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda.$$

These λ are *integral weights*, which simply means that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all roots α. For any root α, $x_\alpha(M_\lambda) \subset M_{\lambda+\alpha}$ and $y_\alpha(M_\lambda) \subset M_{\lambda-\alpha}$. We would like to think that the x_α raise the weights and y_α lower the weights, so we introduce a partial order. We say that $\lambda \geqslant \mu$ if $\lambda - \mu \in \mathbb{Z}_{\geqslant 0}\Phi^+$.

By Weyl's complete reducibility theorem, it remains to classify the irreducible representations of $\mathfrak{g}$. These are in bijection with the *dominant* integral weights, which in particular means that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in \Phi^+$. For any dominant weight λ, there is a unique highest-weight representation $L(\lambda)$. Here, $L(\lambda)$ is generated by a single *maximal vector* v of weight λ. This means that for all positive roots α, $x_\alpha v = 0$.

1.2 Introduction to category $\mathcal{O}$

We would now like to study infinite dimensional representations of $\mathfrak{g}$. Of course, this is impossibly complicated in general, so we will impose some finiteness conditions on our representations.

Definition 1.2.1. The category $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-modules M satisfying:

1. M is finitely generated as a $U(\mathfrak{g})$-module.
2. M is $\mathfrak{h}$-semisimple and has a weight decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.
3. M is locally $\mathfrak{n}$-finite, where $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}$. Precisely, this means that the $U(\mathfrak{n})$ generated by any $v \in M$ is finite-dimensional.

Here are some facts about category $\mathcal{O}$, which are stated without proof.

- For all M in our category and weights λ, the weight space M_λ is finite-dimensional.
- $\mathcal{O}$ is a Noetherian (everything satisfies the descending chain condition) abelian category.

We will now describe some infinite-dimensional objects in category $\mathcal{O}$.

Definition 1.2.2. For any weight λ, the *Verma module* $M(\lambda)$ associated to λ is the module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is the Borel subalgebra associated to our choice of positive roots and $\mathbb{C}_\lambda$ is the $\mathfrak{b}$-module associated to the 1-dimensional representation of $\mathfrak{h}$ with weight λ and the identification $\mathfrak{b}/\mathfrak{n} = \mathfrak{h}$.