THE HOMFLY POLYNOMIAL AND ENUMERATIVE GEOMETRY

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ABSTRACT. We will begin by introducing Pandharipande-Thomas theory and computing the equivariant vertex. After that, we will sketch a proof of the Oblomkov-Shende conjecture following Maulik. In particular, we will give a proof of invariance of certain PT invariants under flops.

1. Introduction

Let $C \subset \mathbb{C}^2$ be a reduced curve and suppose $p \in C$ is an (isolated) singularity. Then define a constructible function $m\colon C_p^{[n]} \to \mathbb{N}$ given by $m([Z])$ being the number of generators of the ideal $I_{Z,p} \subset \mathcal{O}_{C,p}$. Now consider the generating function

$$Z_{C,p}(v,w) = \sum_{n \geqslant 0} s^{2n} \int_{C_p^{[n]}} (1-v^2)\, d\chi = \sum_{n \geqslant 0} s^{2n} \sum_k k\chi_{\mathrm{top}}(f^{-1}(k)).$$

Additionally, taking a small S^3 around $p \in \mathbb{C}^2$ and intersecting with C gives us a link $\mathcal{L}_{C,p}$. Recall the HOMFLY polynomial $P(\mathcal{L}; v, s) \in \mathbb{Z}[v^{\pm}, (s-s^{-1})^{\pm}]$. If μ is the Milnor number of the singularity (for example, the middle Betti number of the Milnor fiber), then the Oblomkov-Shende conjecture is

Theorem 1.1 (Maulik)**.**

$$P(\mathcal{L}_{C,p}; v, s) = \left(\frac{v}{w}\right)^{\mu-1} Z_{C,p}(v,s).$$

The proof of this result relates both sides of the equality to enumerative geometry, and in particular Pandharipande-Thomas, or stable pairs, curve counting. In this lecture, we will (attempt to) sketch a proof of the Oblomkov-Shende conjecture, but first we will introduce PT theory to familiarize ourselves with the objects involved in the proof.

Remark 1.2. Maulik proves everything for a colored version of the HOMFLY polynomial, but here we will only work with uncolored data, which corresponds to all partitions being (1).

2. Introduction to PT Theory

In this part, we are following the papers *Curve counting via stable pairs in the derived category* and *Stable pairs and BPS invariants* by Pandharipande and Thomas.

Date: October 5, 2021.

Let X be a smooth threefold and $\beta \in H_2(X,\mathbb{Z})$. The *PT moduli space* $P_n(X,\beta)$ parameterizes two-term complexes

$$\mathcal{O}_X \xrightarrow{s} \mathcal{F},$$

where $\mathcal{F}$ is a pure 1-dimensional sheaf supported on a Cohen-Macaulay subcurve of X, s has 0-dimensional cokernel, $\chi(\mathcal{F}) = n$, and $[\operatorname{supp} F] = \beta$. The space $P_n(X,\beta)$ has a virtual fundamental class coming from the deformation theory of complexes in the derived category. Here, note that the deformation theory of $(\mathcal{F}, s)$ (really of the corresponding complex $\mathcal{I}^\bullet$) is given by

$$\operatorname{Ext}^0(\mathcal{I}^\bullet,\mathcal{F}) \to \operatorname{Ext}^1(\mathcal{I}^\bullet,\mathcal{F}) = \operatorname{Ext}^1(\mathcal{I}^\bullet,\mathcal{I}^\bullet)_0 \to \operatorname{Ext}^2(\mathcal{I}^\bullet,\mathcal{I}^\bullet)_0$$

and the virtual fundamental class lives in dimension

$$c_\beta \coloneqq \int_\beta c_1(X).$$

In particular, for a Calabi-Yau threefold, $c_\beta = 0$. Given this data, we may define the *PT invariant*

$$P_{n,\beta} \coloneqq \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1.$$

and the PT partition function

$$Z_{PT,\beta}(q) = \sum P_{n,\beta} q^n.$$

There is an alternative way to compute the PT invariants for X a projective Calabi-Yau threefold, due to Behrend (originally done for DT theory). In this case, the moduli space is actually a projective scheme, and if $P_n(X,\beta)$ is smooth everywhere, then

$$P_{n,\beta} = (-1)^{\dim P_n(X,\beta)} \chi_{\mathrm{top}}(P_n(X,\beta)).$$

Of course, moduli spaces are almost never smooth for nontrivial moduli problems, so instead, we have

Proposition 2.1 (Behrend)**.** *There exists an integer-valued constructible function χ_B such that*

$$P_{n,\beta} = \int_{P_n(X,\beta)} \chi_B \coloneqq \sum_{n\in\mathbb{Z}} n\chi_{\mathrm{top}}((\chi_B)^{-1}(n)).$$

3. A computation in PT theory

This computation comes from *The 3-fold vertex via stable pairs* by Pandharipande and Thomas.

We will now compute the T-equivariant PT vertex of $\mathbb{C}^3$ (this is the local model for toric varieties) to give us all a feel for this enumerative theory. First, we need to define some combinatorial data. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a triple of partitions. Then there exists a unique minimal T-fixed subscheme C_μ with outgoing partitions the μ_i (simply take the curves C_{μ_i} given by the three partitions and take the union), whose ideal we will denote $\mathcal{I}_\mu$. Then we define

$$M = \bigoplus_{i=1}^3 (\mathcal{O}_{C_{\mu_i}})_{x_i} \eqqcolon \bigoplus_{i=1}^3 M_i.$$

Every T-invariant pair $(\mathcal{F}, s)$ on $\mathbb{C}^3$ corresponds to a finitely-generated T-invariant submodule

$$Q \subset M/\langle(1,1,1)\rangle,$$

and we will now give a combinatorial description of such submodules.

For each μ^i, we may consider the module M_i as an infinite cylinder $\mathrm{Cyl}_i \subset \mathbb{Z}^3$ (extending in both directions). Then for every $w \in \mathbb{Z}^3$, consider the vectors $\mathbf{1}_w, \mathbf{2}_w, \mathbf{3}_w$ representing w in each copy of $\mathbb{Z}^3$ (for each of the M_i). Clearly x_1 shifts w by $(1,0,0)$ and similarly for x_2, x_3. We will now consider the decomposition of the union of the Cyl_i into the following types:

- Type I^+ are those which have only nonnegative coordinates and lie in exactly one cylinder;
- Type $\mathbb{I}$ (resp $\mathbb{III}$) are those which lie in exactly 2 (resp 3) cylinders;
- Type I^- are those with at least one negative coordinate.

Clearly $M/\langle(1,1,1)\rangle$ is supported on types $\mathbb{I}, \mathbb{III}, \mathrm{I}^-$, and now we have three cases:

- If $w \in \mathrm{I}^-$, then clearly $\mathbb{C}\cdot \mathbf{i}_w \subset M/\mathcal{O}_{C_\mu}$;
- If $w \in \mathbb{I}$, then $\frac{\mathbb{C}\cdot\mathbf{i}_w \oplus \mathbb{C}\cdot\mathbf{j}_w}{\mathbb{C}\cdot(\mathbf{i}_w+\mathbf{j}_w)} \cong \mathbb{C}$;
- If $w \in \mathbb{III}$, then $\frac{\mathbb{C}\cdot\mathbf{1}_w \oplus \mathbb{C}\cdot\mathbf{2}_w \oplus \mathbb{C}\cdot\mathbf{3}_w}{\mathbb{C}\cdot(\mathbf{1}_w+\mathbf{2}_w+\mathbf{3}_w)} \cong \mathbb{C}^2$.

In particular, we need to consider *labelled box configurations*, where type $\mathbb{III}$ boxes may be labeled by an element of $\mathbb{P}^1$ (where $\mathbb{C}^2$ is identified with the vector space above) or unlabeled (corresponding to the inclusion of the entire $\mathbb{C}^2$ in Q). Now we will denote by $\mathcal{Q}_\mu$ the set of components of the moduli space of T-invariant submodules of $M/\mathcal{O}_{C_\mu}$.

We are now able to define the equivariant vertex. Let $\ell(\mathcal{Q})$ be the number of boxes in the labelled configuration associated to $\mathcal{Q} \in \mathcal{Q}_\mu$. Then let $|\mu|$ denote the *renormalized volume* of the partition π corresponding to $\mathcal{I}_{C_\mu}$, which is defined as

$$|\pi| = \#\left\{\pi \cap [0,\dots,N]^3\right\} - (N+1)\sum_1^3 \left|\mu^i\right|,$$

which is independent of a sufficiently large $N \gg 0$.

We need to define a few characters of T, which we will need to define the vertex and compute our example. Let P be the Poincaré polynomial of a free resolution of the universal complex $\mathbb{I}$ on $\mathcal{Q} \times \mathbb{C}^3$. Denote by F the character of $\mathcal{F}$. In particular, we have

$$\mathsf{F} = \frac{1+\mathsf{P}}{(1-t_1)(1-t_2)(1-t_3)}.$$

Then define a vertex character V by

$$\mathsf{V} = \mathrm{tr}_{\mathcal{O}-\chi(\mathbb{I},\mathbb{I})} + \sum_{i=1}^{3} \frac{\mathsf{G}_{\alpha\beta_i}(t_i', t_i'')}{1-t_i},$$

where $G_{\alpha\beta}$ is a certain character defined by edge data. Now let

$$w(\mathcal{Q}) := \int_{\mathcal{Q}} e(T_{\mathcal{Q}}) \cdot e(-V) \in \mathbb{Q}[s_1, s_2, s_3]_{(s_1,s_2,s_3)} = (A_T^*)_{\mathrm{loc}}$$

be the contribution of V on $\mathcal{Q}$. Then the equivariant vertex is defined to be

$$W_\mu^P := \sum_{\mathcal{Q} \in \mathcal{Q}_\mu} w(\mathcal{Q}) q^{\ell(\mathcal{Q})+|\mu|} \in \mathbb{Q}(s_1, s_2, s_3)((q)).$$

Example 3.1. For $\mu = ((1), \emptyset, \emptyset)$, we have

$$W_\mu^P = (1+q)^{\frac{s_2+s_3}{s_1}}.$$

To see this, note that $\mathcal{Q}_\mu = \mathbb{Z}_{>0}$, where k corresponds to the length k box configuration in the negative x_1-direction.

Now we simply compute that

$$F_{\mathcal{Q}_k} = \frac{t_1^{-k}}{1-t_1}.$$

This implies that

$$V_{\mathcal{Q}_k} = \sum_{i=1}^{k} t_1^{-i} - \sum_{i=0}^{k-1} \frac{t_1^i}{t_2 t_3},$$

and therefore that

$$\begin{aligned} w(\mathcal{Q}_k) &= \int_{\mathcal{Q}_k} e(-V_{\mathcal{Q}^k}) \\ &= \frac{(-s_2 - s_3)(s_1 - s_2 - s_3) \cdots ((k-1)s_1 - s_2 - s_3)}{(-s_1)(-2s_1) \cdots (-ks_1)}, \end{aligned}$$

as desired.

4. Flop invariance of PT theory

We will now begin the proof of the Oblomkov-Shende conjecture. The first step is to understand what happens when we blow up C at p via flop invariance of PT partition functions.

Let Y be the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and Y_- be the threefold obtained via a flop of the zero section. The flop is some birational map $\phi \colon Y \dashrightarrow Y_-$. If $\pi \colon Y \to \mathbb{P}^1$ is the projection, choose an identification of $\mathbb{C}^2$ with $\pi^{-1}(0)$. Then the strict transform of $\pi^{-1}(0)$ with respect to ϕ is $\mathrm{Bl}_0 \mathbb{C}^2$ with exceptional fiber E_- which is the zero section of Y_- (isomorphic to Y). This implies that the strict transform of C with respect to ϕ is $\mathrm{Bl}_0 C$.

Definition 4.1. A stable pair $(\mathcal{F}, s)$ is C-*framed* if on $Y \setminus E$ if after restricting to $Y \setminus E$, we have an isomorphism $(\mathcal{F}, s) \simeq \mathcal{O}_Y \to \mathcal{O}_C$.

Given $r, n \in \mathbb{Z}$, we define the moduli space $P(Y, C, r, n)$ of C-framed stable pairs such that $\operatorname{supp} F$ has generic multiplicity r along E and for any projective compactification $\overline{Y}$ of Y, we have $\chi(\overline{\mathcal{F}}) = n + \chi(\mathcal{O}_{\overline{C}})$. This is a locally closed subscheme of the space of stable pairs on $\overline{Y}$ and is independent of the choice of $\overline{Y}$. Now we define the C-framed PT partition function

$$Z(Y, C; q, Q) \coloneqq \sum_{r,n} q^n Q^r \chi_{\mathrm{top}}(P(Y, C, r, n)).$$

Proposition 4.2. *Let $m_1, \ldots, m_r$ be the multiplicities of the branches of C at p. We have the flop identity*

$$Q^{\sum m_r} Z'(Y, C; q, Q^{-1}) = q^{\delta} Z'(Y_-, C'; q, Q),$$

where

$$Z'(Y, C; q, Q) \coloneqq \frac{Z(Y, C; q, Q)}{\prod_k (1 + q^k Q)^k}$$

is the normalized PT partition function and $\delta = \binom{\sum m_i}{2}$.

5. Algebraic links

We will now study what happens to an algebraic link under blowup. Recall that if the singularity $(C, 0)$ is irreducible, then we can describe $\mathcal{L}_{C,0}$ as an iterated torus knot using the Puiseux series of C at 0. If the Puiseux series is

$$y(x) = x^{\frac{q_0}{p_0}}(a_0 + x^{\frac{q_1}{p_0 p_1}}(a_2 + \cdots)),$$

then $\mathcal{L}_{C,0}$ is simply the iterated torus knot with parameters $(q_0, p_0), \ldots, (q_s, p_s)$. To help us compute things, we will project all of our diagrams into the annulus and use skein theory. Before we do this, we need to review skein theory.

Definition 5.1. Let $F \subset \mathbb{R}^2$ be a surface with boundary and designated input and output points. The *framed Homfly skein* over $\Lambda \coloneqq \mathbb{Z}[v^{\pm}, s^{\pm}, (s^r - s^{-r})^{-1} \mid r \geqslant 1]$ is the Λ-module generated by oriented diagrams in F up to isotopy, R1, R2, and the skein relations[1]

$$\text{over} - \text{under} = (s - s^{-1}) \cdot \text{resolved} \tag{1}$$

$$\text{R1 over/under} = v^{\mp} \cdot \text{resolved} \tag{2}$$

$$\text{unknot} = \frac{v^{-1} - v}{s - s^{-1}}. \tag{3}$$

If F is a rectangle with m inputs and outputs, then the skein $\mathcal{H}_m$ has a product given by stacking diagrams, and is isomorphic to the A_m Hecke algebra. If F is an annulus, the skein $\mathcal{C}$ is a commutative algebra with product obtained by placing one annulus inside another. There is a Λ-module morphism $\bigoplus \mathcal{H}_m \to \mathcal{C}$ given by sending a braid to its closure. The algebra $\mathcal{C}_+$ generated by the image is isomorphic to the ring of symmetric functions with coefficients in Λ, and we will denote by Q_λ the diagram assoociaated to the Schur function s_λ. Finally, if $F = \mathbb{R}^2$, then the skein is simply the ring Λ. This gives us a trace map $\langle\rangle : \mathcal{C} \to \Lambda$. Up to some monomial factor, the trace gives us the HOMFLY polynomial.

[1]Sorry there are no drawings. I have no idea how Maulik typeset the diagrams – I'm not a TeX expert, I just optimized my workflow to be able to type fast and look at TeX.SE efficiently.

Now we will discuss a satellite construction. This is how we will turn algebraic links into diagrams in the annulus for computations.

Definition 5.2. Let $\mathcal{L}$ be a framed link with r components and $Q_1, \ldots, Q_r$ be diagrams in the annulus with counterclockwise orientation. The *satellite link* $\mathcal{L} * (Q_1, \ldots, Q_r)$ is obtained by drawing Q_i on the neighborhood of the i-th strand of L. If $\mathcal{L}$ comes from a counterclockwise-oriented diagram in the annulus, then this construction only depends on the equivalence classes of the decorations.

Remark 5.3. Using these terms, coloring a link $\mathcal{L}$ just means giving each component of $\mathcal{L}$ a partition and considering the link $\mathcal{L} * (Q_{\lambda_1}, \ldots, Q_{\lambda_r})$.

Now the iterated torus knot $\mathcal{L}_{C,0}$ with parameters (q_i, p_i) can be embedded as a diagram L_C in the annulus, where

$$L_C = T_{p_0}^{q_0} * (T_{p_1}^{q_1} * (\cdots * (T_{p_s}^{q_s}) \cdots))$$

and T_p^q is the diagram of the (q, p)-torus knot.

For the general case of an algebraic link $\mathcal{L}_{C,0}$ where $(C, 0)$ is **not** irreducible, there is a more complicated satellite construction for constructing a diagram in the annulus. This produces satellite operators $S_p^q * (-, -)$.

Before we continue, we will construct several objects that we will need later. Recall that the diagram T_m^n is the n-th power of the diagram below:

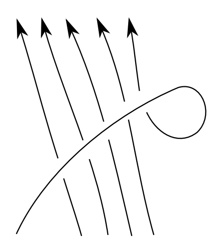

FIGURE 1. The diagram β_5

Next, we will consider the diagram σ_m below and denote its n-th power by S_m^n. These are required for the satellite construction for links, but we will not need it here.

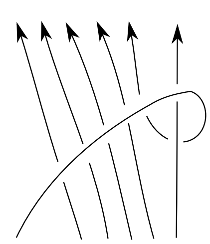

FIGURE 2. The diagram σ_5

To conclude this section, we will explain how to actually construct the link diagram in the annulus for a non-reducible singularity. Let $C_1, \dots, C_r$ be the branches of C at 0 and consider truncated Puiseux series

$$y_i(x) = x^{\frac{q^i}{p^i}}(a_i + z_i(x^{\frac{1}{p^i}})).$$

Write $\alpha_i = \frac{q^i}{p^i}$ and consider the pairs (α_i, a_i). It is always possible to find a finite truncation of the Puiseux series that does not affect the topological type of the link, so the inductive process we will define is actually finite. For each (α, a), let $\{i_0, \dots, i_n\}$ be the set of indices with associated pair (α, a). Assume we have an annulus diagram $L_{(\alpha,a)}$ and set

$$L_\alpha \coloneqq \prod_a L_{(\alpha,a)}.$$

We may assume that $\alpha_1 < \cdots < \alpha_k$, and the link $\mathcal{L}_C$ can be represented by the annulus diagram

$$L_C \coloneqq S_{p^1}^{q^1} * (L_{\alpha_1}, S_{p^2}^{q^2} * (L_{\alpha_2}, \dots, S_{p^{k-1}} q^{k-1} * T_{p^k}^{q^k} * L_{\alpha_k})).$$

6. Some explicit calculations for the unknot and Hopf link

The following is already apparently known.

Proposition 6.1. *For any partition* λ,

$$\langle Q_\lambda \rangle = \prod_{\square \in \lambda} \frac{v^{-1}s^{c(\square)} - vs^{-c(\square)}}{s^{h(\square)} - s^{-h(\square)}}.$$

Let $X \in \mathcal{C}_+$ be a counterclockwise-oriented diagram. Define the *meridian operator* M_X on $\mathcal{C}_+$ by the construction below:

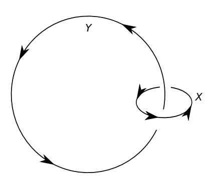

Figure 3. Meridian operator

For a partition μ, the Schur function Q_μ is an eigenvector for M_X with eigenvalue $t_\mu(X)$. Then the HOMFLY polynomial for the colored Hopf link decorated by μ, λ is simply $t_\mu(Q_\lambda)(t_\mu)$. In the remainder of this section, we will describe the operator t_μ, which is a ring homomorphism $\mathcal{C}_+ \to \Lambda$. Define the function

$$E_\mu(t) = \prod_{j=1}^{\ell(\mu)} \frac{1 + v^{-1}s^{\mu_j - 2j+1}t}{1 + v^{-1}s^{-2j+1}t} \prod_{i \geqslant 0} \frac{1 + vs^{2i+1}t}{1 + v^{-1}s^{2i+1}t}.$$

Then we have $t_\mu(Q_\lambda) = s_\lambda(E_\mu(t))$, where for a power series $E(t) = \sum E_k t^k$, we write s_λ as a polynomial in the e_k and then substitute $e_k \leftarrow E_k$.

7. Behavior of links with respect to blowup

Having discussed the behavior of enumerative invariants with respect to blowups, we need to discuss the behavior of the link $\mathcal{L}_{C,0}$ with respect to blowing up the origin. Let $C_1, \ldots, C_r$ be the irreducible components of C through 0 and denote their strict transforms by C_i'. At each point $p_1, \ldots, p_e \in E$ (the exceptional divisor) let D_k denote the singularity of $C' \cup E$ at p_k and B_k be the singularity of C' at p_k. Choose (truncated) Puiseux expansions

$$y_i(x) = x^{\frac{q^i}{p^i}}(a_i + z_i(x^{\frac{1}{p^i}}))$$

For each of the branches C_i. We may also assume that $\frac{q^i}{p^i} \geqslant 1$ for all i. If we blow up at the origin, consider the chart with coordinates $(x, y = xw)$. Substitution, we obtain the new Puiseux expansion

$$w_i(x) = x^{\frac{q^i - p^i}{p^i}}(a_i + z_i(x^{\frac{1}{p^i}}))$$

for C_i' at p_k. In particular, we obtain the relation

$$[L_C] = \tau[L_{C'}] \qquad \tau(-) \coloneqq T_1^1 * (-).$$

If we perform deeper analysis, we obtain the following result:

Proposition 7.1. *If any $\alpha_i = \frac{q_i}{p_i} > 1$, then we have*

$$[L_C] = S_1^1 * (L_{B_1} \cdots L_{b_{e-1}}, \tau L_{B_e}).$$

Otherwise, we have

$$[L_C] = S_1^1 * (L_{b_1} \cdots L_{B_e}, \emptyset) = T_1^1 * (L_{B_1} \cdots L_{B_e}).$$

Now we will write down a blowup identity for links. The idea is to use the topological vertex (originally introduced by Aganagic-Klemm-Marino-Vafa) and its relationship with Chern-Sinons invariants of the unknot. For a partition μ, define

$$\begin{aligned} Z_\mu(q, Q) &= s_\mu(q^\rho) \prod_{\square \in \mu} (1 + Qq^{-c(\square)}) \\ &= q^{\kappa_\mu/4} \prod_{\square \in \mu} \frac{1 + Qq^{-c(\square)}}{q^{h(\square)/2} - q^{-h(\square)/2}}. \end{aligned}$$

Here, $\kappa_\mu = 2\sum_{\square \in \mu} c(\square)$. By the computation of the colored HOMFLY polynomial of the colored unknot (sorry), we obtain the identity

$$Z_\mu(q = s^2, Q = -v^2) = v^{|\mu|} \langle Q_\mu \rangle.$$

8. Relation between PT theory and HOMFLY

We will prove a relationship between the PT partition function for C-framed stable pairs in $Y = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and the HOMFLY polynomial for $\mathcal{L}_{C,0}$. We will focus on the simplest case of a node with two branches. The general case reduces to this one by careful checking of what happens on both sides under blowup of C at 0.

Proposition 8.1. *We have the identity (possibly up to monomials)*

$$Z'(Y, C; q, Q = 0) = (-1)^{\varepsilon} s^{b} \left\langle [L_C * Q_{(1)^t}] \right\rangle^{\text{low}},$$

where the superscript low means we take the lowest degree terms.

We only need to prove this for the unknot and the Hopf link.

Proof. This apparently follows from the fact that the topological vertex calculates both the $\nu = 0$ specialization of HOMFLY of the Hopf link and the stable pairs vertex. See the references to Maulik's paper for a reference. □