DAHA and Knot Homology Learning Seminar Fall 2021

Notes by Patrick Lei

Lectures by Various

Columbia University

Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

I patched the notes from Álvaro's lecture using his notes, which contain some material not covered during the lecture. The notes here only contain material covered during the lecture.

Seminar Website:
`https://math.columbia.edu/~samdehority/seminars/2021-fall-seminar-knot-homology`

Contents

1

Sebastian (Sep 14): Khovanov Homology

Recall that the *Kauffman bracket* of a link diagram D has axioms $\langle\emptyset\rangle = 1$, $\langle 0 \sqcup D\rangle = (q + q^{-1})\langle D\rangle$, and for any crossing

$$\langle\text{crossing}\rangle = q\,\langle\text{horizontal}\rangle + q^{-1}\,\langle\text{vertical}\rangle$$

If D has n_+ positive crossings and n_- negative crossings, then the *Jones polynomial* of the underlying link L is $J(L) = (-1)^n q^{n_+ - 2n_-}\langle D\rangle$.

We may have heard that Khovanov homology is a categorification of the Jones polynomial. To motivate this, there are quantum invariants like the Jones polynomial, the HOMFLY polynomial, WRT, and others. On the other side, there are gauge-theoretic invariants like instanton Floer homology and the Casson invariant. The quantum invariants generally come from representations of quantum groups are are combinatorial, but it is unclear what geometric information we obtain. On the other hand, the gauge theoretic invariants are more powerful, but of course harder to compute. We want to consider relations between the two types of invariants.

Some of the gauge-theoretic invariants end up being Euler characteristics; for example, the Casson invariant is twice the Euler characteristic of instanton Floer homology.

For a link diagram L, we will introduce a bigraded $\mathbb{Z}$-module $Kh^{*,*}(L)$ such that

$$J(L) = \sum_{i,j} (-1)^i q^j \operatorname{rk}_{\mathbb{Z}} Kh^{i,j}(L).$$

1.1 Category of Pictures

Given D, we will order its crossings. For each crossing we have a 0-resolution (horizontal) and a 1-resolution (vertical). If $n = n_+ + n_-$, then each element of $\{0,1\}^n$ gets a complete resolution, which is a collection of circles in the plane. For example, consider the Hopf link with resolution.

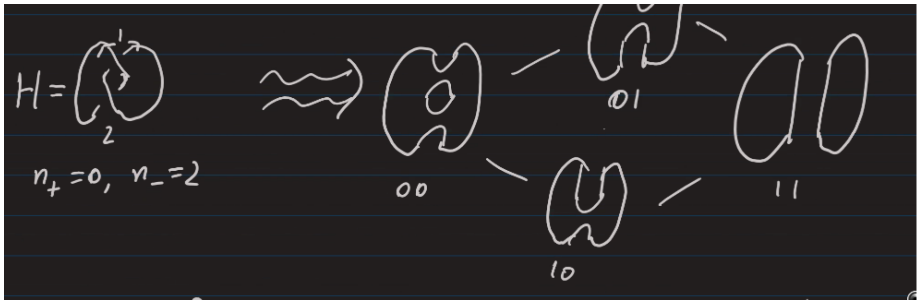

Figure 1.1: Hopf link and resolution

Then we have

$$\langle H \rangle = (q+q^{-1})^2 - q(q+q^{-1}) - q(q+q^{-1}) + q^2(q+q^{-1}) = q^4 + q^2 + 1 + q^{-2},$$

and thus

$$J(H) = (-1)^2 q^{-4} \langle H \rangle = 1 + q^{-2} + q^{-4} + q^{-6}.$$

Changing a 0-resolution to a 1-resolution is given by a saddle, or a cobordism.

Let $\overline{\mathrm{Cob}^3}$ be the category with objects collections of oriented circles in $\mathbb{R}^2$ and morphisms oriented cobordisms in $\mathbb{R}^2 \times [0,1]$. For each $\Sigma \in \mathrm{Hom}(O_1, O_2)$, we define $\deg \Sigma = \chi(\Sigma)$. Then we add objects $O\{m\}$ for each $O \in \mathrm{Obj}(\overline{\mathrm{Cob}^3})$ and $m \in \mathbb{Z}$. This gives a graded category Cob^3. Now let $\mathrm{Mat}(\mathrm{Cob}^3)$ denote the additive closure of Cob^3 and $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3))$ be its category of chain complexes.

Now for each $\alpha \in \{0,1\}^n$, we assign the object $O\{m\}$, where O is the resolution of L obtained from the α_i-resolution at the i-th crossing and $m = \sum \alpha_i$. For each edge, we will assign the saddle cobordism between objects at each edge.

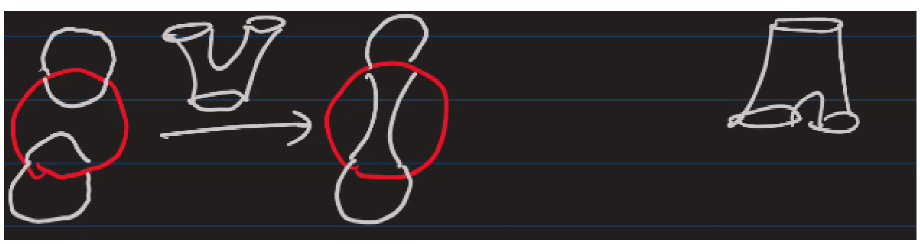

Figure 1.2: Saddle cobordism

To obtain $d^2 = 0$, we need that all square faces in the diagram anti-commute. The problem is that in the diagram, the two cobordisms are actually the same (we can reorder the saddles), so now we need to add signs to the edges such that each square face has an odd numbder of minus signs. In the example of the Hopf link, we just add a minus sign to the $10 \to 11$ edge.

Now we may construct the m-th chain group $[[L]]^m$ of $[[L]]$ to be

$$\bigoplus_{\alpha \mid \sum \alpha_i = m} (\text{resolution over } \alpha).$$

The chain homotopy type of $[[L]]$ is a link invariant after passing to a quotient Cob^3/ℓ of Cob^3 by the following relations:

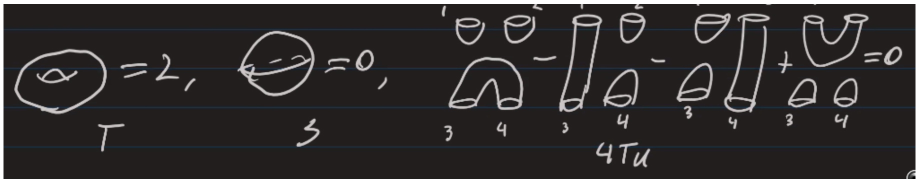

Figure 1.3: Relations for $\mathsf{Cob}^3//\ell$

Now we define the *Bar-Natan category* $\mathsf{BN} := \mathsf{Kom}(\mathsf{Mat}(\mathsf{Cob}^3/\ell))$. Of course, we now need to prove invariance under the Reidemeister moves. For the first Reidemeister move, we have the following diagram:

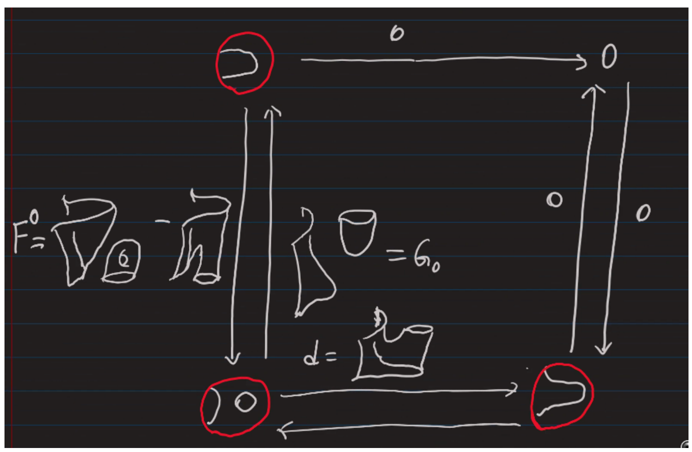

Figure 1.4: Diagram for R1

Then here hare the easy relations.

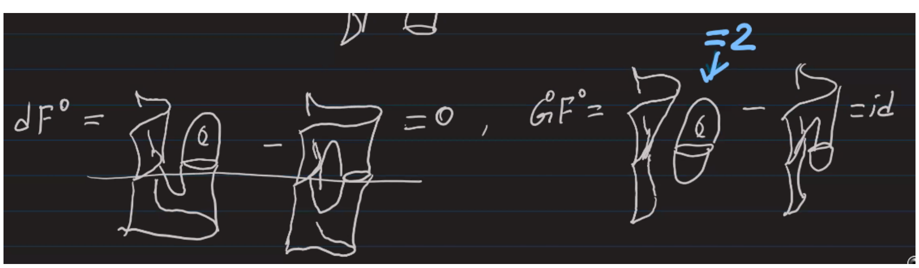

Figure 1.5: Easy relations

Finally, here we see that we have a homotopy:

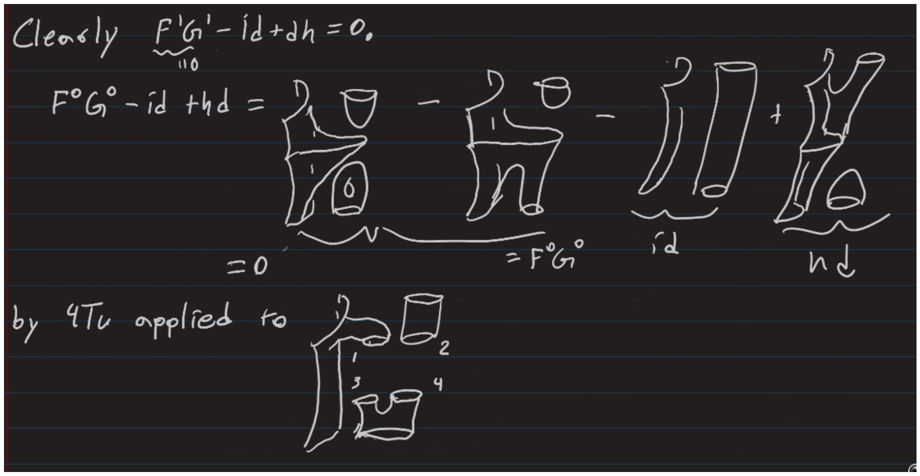

Figure 1.6: The homotopy really is a homotopy

The proofs of invariance under the other Reidemeister moves are similar, but significantly more complicated. But now any functor from Cob^3/ℓ to any abelian category will give us a knot invariant. Note that such a functor does not have to be a 2-dimensional TQFT.

1.2 Khovanov Homology

Returning to something more concrete, any cobordism in Cob^3/ℓ can be decomposed into pairs of pants and caps and cups, so to specify a functor, we only need to specify what it does to four morphisms. Here, the cobordisms correspond to

$$m\colon V \otimes V \to V \qquad \Delta\colon V \to V \otimes V \qquad \iota\colon V \to \mathbb{Z} \qquad \varepsilon\colon V \to \mathbb{Z},$$

where $V = \mathcal{F}(S^1)$ for $\mathcal{F}$ a TQFT. Then Khovanov homology is the functor where $V = \mathbb{Z}\{-1\} \oplus \mathbb{Z}\{1\}$, where m is given by

$$v_- \otimes v_- \mapsto 0 \qquad v_- \otimes v_+ \mapsto v_- \qquad v_+ \otimes v_+ \mapsto v_+ \qquad v_+ \otimes v_- \mapsto v_-,$$

Δ is given by

$$v_- \mapsto v_- \otimes v_- \qquad v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+,$$

ι is given by $1 \mapsto v_+$, and ε is given by $v_- \mapsto 1, v_+ \mapsto 0$. The grading on V is called the *Jones grading* because it records power of q in the Jones polynomial.

Now we are finally able to define the Khovanov homology

$$\mathrm{Kh}(L) := H^*(\mathcal{F}([[L]]))[-n_-]\{n_+ - 2n_-\}$$

Theorem 1.2.1. *We have the formula*

$$J(L) = \sum_{i,j} (-1)^i q^j \operatorname{rk}_{\mathbb{Z}} \mathrm{Kh}^{i,j}(L).$$

For the Hopf link, the chain complex is given by

$$(V \otimes V)\{-4\} \xrightarrow{m \oplus m} V\{-3\} \oplus V\{-3\} \xrightarrow{\Delta^1 - \Delta^2} (V \otimes V)\{-2\},$$

and thus we can compute the Khovanov homology.

Theorem 1.2.2 (Jacobsson, Bar-Natan)**.** *The map on* $[[L]]$ *associated to a cobordism in* $\mathbb{R}^3 \times [0,1]$ *is independent of its decomposition into elementary cobordisms up to a sign and cobordisms in the Bar-Natan category.*

2

Álvaro (Sep 21): HOMFLY-PT homology

If we already know about Hecke algebras, then we could discover the HOMFLY-PT polynomial with the unknot being 1 and the skein relation

$$t \cdot \text{under} + t^{-1} \cdot \text{over} = z \cdot \text{unlinked}.$$

Here, if we set $t = q^{2N}$ and $z = q + q^{-1}$, we recover the SL_2 link polynomial and if we set $N = 1$, we recover the Jones polynomial.

The reason that we could have discovered the HOMFLY-PT polynomial is using the representation theory of the braid group, which is generated by $s_1, \dots, s_{n-1}$ with relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ if $|i - j| \geqslant 2$.

First, note that every link is the closure of a braid. Then two braids give rise to isotopic links if they are related by Markov moves. Now we consider

$$\mathbb{Z}[q^{\pm}]\mathrm{Br}_n \twoheadrightarrow H(S_n) = \mathbb{Z}[q^{\pm}] \cdot \{\delta_w \mid w \in S_n\}/\delta_s^2 = \delta_s(q-1) + q.$$

Now we can define a trace map

$$\mathrm{Tr}\colon \bigcup_{n \geqslant 1} H(S_n) \to \mathbb{Z}[q^{\pm}, z^{\pm}]$$

with the condition that $\mathrm{Tr}(x s_n y) = z\,\mathrm{Tr}(xy)$ and $\mathrm{Tr}(1) = 1$. In particular, these imply that $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$, and this is called the *Oceanu trace*. If we normalize the s_i, then $\mathrm{Tr}(x s_i) = \mathrm{Tr}\left(x s_i^{-1}\right)$. Then under the correct normalization, we obtain the HOMFLY-PT polynomial.

2.1 Triply graded homology

This was discovered by Khovanov-Rozansky in 2004 using matrix factorizations and Khovanov in 2005 using Soergel bimodules. Then work of various subsets of Elias, Hogancamp, and Mellit computed HHH for torus (m, n)-links.

Consider the symmetric group $S_m = \langle s_1, \dots, s_{m-1} \rangle$. Then consider $R = \mathbb{Q}[x_1, \dots, x_m]$, where the x_i have degree 2. Then we have a natural action of S_m on R, and R^{S_i} are the invariants. Then we will define $B_{s_i} = R \otimes_{R^s} R(1)$ as an R-bimodule, and for $\underline{w} = s_{i_1} \cdots s_{i_n}$, we will define the *Bott-Samuelson bimodule* $\mathrm{BS}(\underline{w}) = B_{s_{i_1}} \otimes \cdots \otimes B_{s_{i_n}}$. Next, if we consider the Bott-Samuelson bimodules and add in all tensor products and direct sums, we obtain the Hecke category SBim, which categorifies the Hecke algebra.

In our running example, we will take $S_2 = \{1, s\}$ and $R = \mathbb{Q}[x_1, x_2]$. Here, we will take $R^s = \mathbb{Q}[x_1 + x_2, (x_1 - x_2)^2]$, and we will write $r = x_1 + x_2, t = x_1 - x_2$. Therefore, we have

$$\mathrm{BS}(s^2) = B_s \otimes_R B_s = R \otimes_{R^s} R \otimes_R R \otimes_{R^s} R(2) = R \otimes_{R^s} R \otimes_{R^s} R(2).$$

Theorem 2.1.1 (Soergel's categorification theorem). *Note that* $H(S_n)$ *has a KL-basis* $\{b_w \mid w \in S_n\}$. *Then there is a bijection between indecomposables in* SBim *and the KL-basis taking the tensor product to multiplication.*

Now if we write $b_{s_i} = \delta_{s_i} + q$, we have $b_{s_i}^2 = q b_{s_i} + q^{-1} b_{s_i}$. On the other side, we have

$$\begin{aligned} B_s^{\otimes 2} &= R \otimes_{R^2} \otimes R \otimes R(2) \\ &= R \otimes (\mathbb{Q}[r, t^2] \oplus t\mathbb{Q}[r, t^2]) \otimes_{R^2} R(2) \\ &= R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R \\ &= B_s(1) \oplus B_s(-1). \end{aligned}$$

Now we will categorify the braid group and construct HHH. We have $b_s = \delta_s + q$ and thus $\delta_s = b_s - q$, so in the categorified version we will have $q_i \colon B_{s_i} \to R$. Then we consider the complex

$$F_{s_i} = 0 \to \underline{B_{s_i}} \xrightarrow{q_i} R(1) \to 0 \to \cdots$$

and analogously

$$F_{s_i}^{-1} = R(-1) \xrightarrow{d_i} \underline{B_{s_i}} \to 0 \to \cdots$$

Theorem 2.1.2 (Rouquier). *The assignment* $\underline{w} \to F^\bullet(\underline{w}) = F_{s_1^\pm} \otimes \cdots \otimes F_{s_{i_n}^\pm}$ *is a well-defined homomorphism* $\mathrm{Br}_n \to K^b(\mathsf{SBim})$.

Now we apply the Hochschild homology HH_n to $F^\bullet(w)$, and we obtain the complex

$$HH_n(F^{-1}(w)) \to HH_n(F^0(w)) \to HH_n(F^1(w)),$$

and taking the homology, we obtain the HHH homology. The grading is given by the homological grading $HH_i(F^j(\underline{w}))$ that modifies j, the Hochschild grading, and the internal grading as R-graded bimodules.

2.2 Hochschild homology

Next we observe that Hochschild homology is easier for polynomial rings. Recall that if R is a k-algebra, then if we write $R^{env} = R \otimes_k R^{op}$, we have

$$HH_n(-) = \mathrm{Tor}_n^{R^{env}}(R, -).$$

Instead of the bar complex, we will consider the Koszul complex

$$K^\bullet \bigwedge^n V \otimes R^{env} \to \cdots \to V \otimes R^{env} \to R^{env} \to R \to 0,$$

which is a free resolution of R, where $V = \mathbb{Q}^m$. Then this complex computes the Hochschild homology. For example, if $R = \mathbb{Q}[x] = M$ where $\deg x = 2$, we have

$$K^\bullet = R^{env}(-2) \xrightarrow{(x \otimes 1 - 1 \otimes x)} R^{env}$$

and $R(-2) \xrightarrow{0} R$, and thus

$$HH_*(R) = \begin{cases} \mathbb{Q}[x] & * = 0 \\ \mathbb{Q}[x](-2) & * = 1. \end{cases}$$

for another example, when $R = \mathbb{Q}[x, y]$ and M is a graded R-bimodule, and $\deg x = \deg y\ 2$, we have

$$K^\bullet = R^{env}(-4) \to R^{env}(-2) \oplus R^{env}(-2) \to R^{env}$$

where the maps are given by $(x \otimes 1 - 1 \otimes x, 1 \otimes y - y \otimes 1)$ and $(y \otimes 1 - 1 \otimes y, x \otimes 1 - 1 \otimes x)$, and the Koszul resolution of M is similar.

2.3 Example of triply graded homology

Now the unknot is the closure of $1 \in \mathrm{Br}_1$, and so we want to compute its homology HHH_{***}. We know that

$$F^\bullet(1) = 0 \to \underline{R} \to 0,$$

and therefore

$$HH_*(R) = \begin{cases} \mathbb{Q}[x] & * = 0 \\ \mathbb{Q}[x](-2) & * = 1. \end{cases}$$

The Euler characteristic is

$$(1 + q^2 + q^4 + \cdots) + a(q^2 + q^4 + \cdots) = \frac{1 + aq^2}{1 - q^2},$$

and this recovers the unknot.

Now if we take the closure of $s \in \mathrm{Br}_2$, which is still the unknot, we have

$$F^\bullet(S) = \underline{B_s} \to R(1),$$

where $R = \mathbb{Q}[x_1, x_2]$. But now we need to compute the Hochschild homology

$$HH_*(B_s) \to HH_*(R(1))$$

in all degrees. For $R(1)$, the Koszul resolution is

$$R(-3) \xrightarrow{0} R(-1)^{\oplus 2} \xrightarrow{0} R(-1)$$

and for B_s it is

$$B_s(-4) \xrightarrow{(f,f)} B_s(-2) \oplus B_s(-2) \xrightarrow{(-f,f)} B_s,$$

where $R^s = \mathbb{Q}[s, t^2]$ and $f(m) = \frac{1}{2}tm - \frac{1}{2}mt$. Now we have $\ker(f) = \langle mt + tm \rangle, \mathrm{Im}(f) = mt - tm$. Now we compute $HH_0(B_s) = B_s/\mathrm{Im}\, f$, $HH_*(B_s) = R(-1) \oplus \ker(f)(-2)$, and $HH_2(B_s) = \ker(f)(-4)$. Then the three maps are the identity in degree 0, (id, i) in degree 1, and i in degree 2, where i is the counit. Finally we obtain the following table for the HHH:

Table 2.1: HHH homology

Hochschild degree	$HHH_{0**}(s_1)$	$HHH_{1**}(s_1)$
0	0	0
1	0	$\mathbb{Q}[r](-1)$
2	0	$\mathbb{Q}[r](-3)$

This is isomorphic to $\mathsf{HHH}(1)$ in the previous example after a shift.

Finally consider the Hopf link, which is the closure of s^2. Here $F^\bullet(s^2)$ is given by

$$\begin{tikzcd} B_s^{\otimes 2} \arrow[r] \arrow[d] & B_s(-1) \arrow[d] \\ B_s(1) \arrow[r] & R(2). \end{tikzcd}$$

This complex is homotopic to $B_s(-1) \to B_s(1) \to R(2)$.

2.4 Connections to geometry

Define the matrix $B_i(z)$ to have the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & z \end{smallmatrix}\right)$ in the i-th position. Then we have the identity

$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1).$$

For a positive braid $\beta = s_{i_1} \cdots s_{i_n}$, we assign the matrix $B_\beta(z_1, \dots, z_n) = B_{i_1}(z_1) \cdots B_{i_n}(z_n)$, and finally we consider the variety

$$X(\beta) = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid B_\beta(z_1, \dots, z_n) \text{ is upper triangular}\big\}.$$

For example, if β is the trefoil knot, then $X(\beta) = \{1 + z_1z_2 = 0\}$. There is a torus action on this variety, where if we have n strands there is an action of $(\mathbb{C}^\times)^{n-1}$. In this example, we have $(z_1, z_2, z_3) \mapsto (z_1 \cdot t, z_2 \cdot t^{-1}, z_3)$.

Theorem 2.4.1 (Trinh, 2021)**.** *We have* $H^T_{*,BM}(X(\beta))$ *has a nontrivial weight filtration whose associated graded module is isomorphic to* $\mathsf{HHH}_{0**}(\beta)$*.*

For the trefoil knot, we obtain a grading on $H_*(S^1)$.

Finally, relating this to Hilbert schemes we have the Oblomkov-Shende conjecture, where if C is an integral plane curve and p is a singularity, then

$$\text{HOMFLY}(\text{link}(p)) = \left(\frac{a}{q}\right)^{\mu-1} \sum_{\ell,m} q^{2\ell}(-a^2)^m \chi(C_p^{[\ell,\ell+m]}).$$

Next, there is the Oblomkov-Rasmussen-Shende conjecture, which says that replacing χ with H^*, we obtain HHH. Then there is Oblomkov-Rozansky homology, and finally a conjecture of Gorsky-Negut-Rasmussen, which says that if $X = \text{FHilb}_n^{dg}(\mathbb{A}^2)$ is a dg version of the flag Hilbert scheme, then there is a commutative diagram of monoidal functors

$$\begin{tikzcd} K^b(\mathsf{SBim}_n) \arrow[r, "L_*"] \arrow[d, "\mathsf{HHH}"] & D^b(\mathsf{Coh}_{\mathbb{C}^\times \times \mathbb{C}^\times}(X)) \\ \mathsf{Vect}_{\mathbb{Q}}^{\mathbb{Z}^{\oplus 3}}, & \end{tikzcd}$$

where $\mathsf{Vect}_{\mathbb{Q}}^{\mathbb{Z}^{\oplus 3}}$ is the category of $\mathbb{Z}^{\oplus 3}$-graded $\mathbb{Q}$-vector spaces.

3

Avi (Sep 28): Hilbert Schemes

3.1 Definitions

Let X be a projective variety over a field $k = \mathbb{C}$. For example, if $X = \mathbb{P}^n$, we in algebraic geometry are interested in projective varieties, which are closed subschemes of X. We are interested in classifying closed subschemes of X, and to do this we will construct a functor $\mathrm{Hilb}(X)$ given by

$$\mathrm{Hilb}(X)(T) = \{Z \subset T \times X \mid Z \to T \text{ flat}\}.$$

We are interested in the representability of this functor, and unfortunately this functor is not representable in any reasonable way.

For example, consider $X = \mathbb{P}^1$. If we consider subschemes with finite support, we obtain the space

$$\bigsqcup_n \mathrm{Sym}^n \mathbb{P}^1 = \bigsqcup_n \mathbb{P}^n.$$

This has infinitely many connected components and is infinite-dimensional in general. This tells us that we want to stratify our functor into pieces.

Now consider $Z \subset X = \mathbb{P}^n$. Then define graded coordinate ring $R = \bigoplus_i R_i$ of Z and the function $i \mapsto \dim_R R_i$. For $i \gg 0$, this function equals some polynomial in i. More generally, let $\mathcal{F}$ be a quasicoherent sheaf on X. Then we can consider the function

$$i \mapsto h^0(X, \mathcal{F}(i)),$$

and in particular, for $i \gg 0$, this function equals the Euler characteristic $\chi(\mathcal{F}(i))$ because for i sufficiently large, all twists have vanishing higher cohomology. Finally, $\chi(\mathcal{F}(i))$ is a polynomial, which we will call the *Hilbert polynomial* $P_{\mathcal{F}}(i)$.

For a subscheme $Z \subset X$ with ideal sheaf $\mathcal{I}_Z \subseteq \mathcal{O}_X$, we will define $P_Z = P_{\mathcal{I}_Z}$. Returning to $X = \mathbb{P}^1$, then if Z is a scheme which is n points, then $\dim H^0(X, \mathcal{I}_Z) = n$. For the subscheme X, we have $h^0(X, \mathcal{O}_X(i)) = i + 1$. We are now able to define the functor

$$\mathrm{Hilb}^f(X)(T) \coloneqq \{Z \subseteq T \times X \mid Z \to T \text{ flat}, P_{Z_t} = f\}.$$

In particular, we have a decomposition

$$\mathrm{Hilb}(X) = \bigsqcup_f \mathrm{Hilb}^f(X).$$

In our case, we will focus on 0-dimensional subschemes, and we will denote both the functor and the scheme that represents it as $X^{[n]} = \mathrm{Hilb}^n(X)$.

Example 3.1.1. Suppose $n = 0$. All subschemes of $T \times X$ with $P_{Z_t} = 0$ must have empty fibers over T and thus must be empty themselves.

If $n = 1$, then all ideals of colength 1 are maximal ideals, which are the same as closed points of X, so $X^{[1]} = X$.

When $n = 2$, we are looking for ideals $\mathcal{I}$ such that $\mathcal{O}_X/\mathcal{I}$ has length 2. Thus either this subscheme is supported at two distinct points or a single point. If the support of the subscheme is two distinct points $x_1 \neq x_2$, then at each x_i, the ring $\mathcal{O}_X/\mathcal{I}$ must have length 1. Thus we have a locus $(X \times X \setminus \Delta)/S_2$. In the second case, where $\mathcal{O}_X/\mathcal{I}$ is supported at a single point x, then we have a sequence of inclusions

$$\mathfrak{m}_x^2 \subseteq I \subset \Updownarrow_X \subset \mathcal{O}_X.$$

Thus we are looking for morphisms $\varphi\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ up to a scalar, so above the point x we have $\mathbb{P}T_{X,x}$, and therefore we have

$$X^{[2]} = (X \times X \setminus \Delta)/S_2 \sqcup \mathbb{P}T_X.$$

More generally, we have a morphism $\pi\colon X^{[n]} \to \mathrm{Sym}^n X$ sending a subscheme Z to its support $|Z|$, called the *Hilbert-Chow morphism*. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of n, we have a stratum $\mathrm{Sym}^\lambda X$ with fixed multiplicities given by λ. Thus we have

$$\mathrm{Sym}^n X = \bigsqcup_\lambda \mathrm{Sym}^\lambda X,$$

which induces a stratification of $X^{[n]}$ into pieces $X_\lambda^{[n]}$.

For example, when $\lambda = (n)$, we have the Hilbert-Chow morphism

$$X_{(n)}^{[n]} \xrightarrow{\pi} \mathrm{Sym}^{(n)} X = X,$$

and so the fibers are schemes $X_p^{[n]}$ parameterizing length n subschemes of X supported at p.

Example 3.1.2. When X is a smooth curve, π is actually an isomorphism. This is because all local rings $\mathcal{O}_{X,p}$ are discrete valuation rings, and in particular they have a unique idea of length n. Therefore all fibers of π must be points.

When X is a smooth surface, then $X^{[n]}$ is always smooth, but in fact Hilbert schemes of points are highly singular for $n > 2$ in higher dimension.

3.2 Hilbert schemes of smooth surfaces

Now suppose that X is a smooth surface. Suppose that $Z \subseteq X$ is supported at a point p with length n. In particular, we have a chain of inclusions

$$\mathfrak{m}_x^n \subseteq I \subseteq \mathfrak{m}_x^{n-1} \subset \mathcal{O}_{X,x},$$

and thus this data is equivalent to $I/\mathfrak{m}_x^n \subset \mathcal{O}_{X,x}/\mathfrak{m}_x^{n-1}$. Up to completion, all of this data is independent of both X and x, and in fact

$$\widehat{\mathcal{O}}_{X,x} = k[[T_1, T_2]].$$

Therefore, $X_x^{[n]}$ is independent of X, x, so we may consider $X = \mathbb{A}^2, x = 0$. We will see that $(\mathbb{A}^2)_0^{[n]}$ has dimension $n-1$. In general, we have

$$\begin{aligned}\dim X^{[n]}_{(\lambda_1,\dots,\lambda_r)} &= \dim X^{[\lambda_1]}_{(\lambda_1)} + \cdots + \dim X^{[\lambda_r]}_{(\lambda_r)} \\ &= \dim X + \dim X_p^{[\lambda_1]} + \cdots \\ &= (\lambda_1 + 1) + \cdots + (\lambda_r + 1) \\ &= n + r.\end{aligned}$$

In particular, $\dim X^{[n]} = 2n$.

Now we will compute the tangent space of $X^{[n]}$. If a subscheme Z is given by an ideal $\mathcal{I}$, we first show that the tangent space to $X^{[n]}$ at $\mathcal{I}$ is given by $\mathrm{Hom}(\mathcal{I}, \mathcal{O}_X/\mathcal{I})$, and then we can use Hirzebruch-Riemann-Roch. Alternatively, consider the action of $G_m \times G_m$ on $\mathbb{A}^2$. This lifts to an action on $(\mathbb{A}^2)^{[n]}$, and the fixed points are given by monomial ideals. Each monomial ideal corresponds to a Young diagram (just consider all monomials not in the ideal), and of course these are in bijection with partitions of n. If we do some allegedly very elegant combinatorics, we obtain the desired result.

We will now connect Hilbert schemes of surfaces to representation theory. We have the formula of Göttsche, which states that

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i(X^{[n]})t^i q^n = \prod_{m=1}^{\infty}\prod_{j=0}^{4}(1-(-1)^j t^{2m+j-2}q^m)^{-(-1)^i b_j(X)}$$

and another formula which states that

$$\sum_{n=0}^{\infty}\chi(X^{[n]})q^n = \prod_{j=0}^{4}\left(\prod_{m\geqslant 1}(1-q^m)\right)^{-(-1)^i b_j(X)}.$$

In particular, doing some generatingfunctionology gives us the Betti numbers.

These formulas are similar to those coming from representations of the Heisenberg algebra. In fact, the total cohomology of the Hilbert scheme is a representation of the Heisenberg algebra. We consider the incidence schemes

$$X^{[n,\ell]} \subset X^{[n]} \times X^{[n+\ell]}$$

which classifies subschemes $Z_1 \subset Z_2$ which differ at a single point x.

3.3 Plane curve singularities

There exists a similar formula to those of Göttsche for smooth curves due to Macdonald, which says that

$$\sum_{n=0}^{\infty}\chi(C^{[n]})q^n = (1-q)^{-\chi(C)}.$$

We would like to say something about the LHS, which we will call $W(C)$, in general when C is not smooth. Write $C = C_{sm} \cup \{p_i\}$, where the p_i are the singularities of C. Then we have

$$W(C) = W(C_{sm})\prod_{n=0}^{\infty}\chi(C_p^{[n]})q^n$$

by the cut-and-paste relation for the Euler characteristic. To make sense of what happens at the singular points, we will consider knot theory.

Near $p \subset C \subset \mathbb{C}^2$, we will consider $C \cap S^3$, which has real dimension 1 and is therefore a link L_p. We would hope that some knot invariant of L_p will recover the data of the fomula above, and according to the Oblomkov-Shende conjecture, this is true. The conjecture says that

$$\sum_{n=0}^{\infty} \chi(C_p^{[n]}) q^{2n} = \lim_{t \to 0} \left(\frac{q}{t}\right)^{\mu} P(L_p),$$

where P is the HOMFLY polynomial. To recover the entire Homfly polynomial, we have an upgraded version of the conjecture, which says that

$$P(L_p) = \left(\frac{t}{q}\right)^{\mu} (1 - q^2) \sum_{m,n} \chi(C_{p,m}^{[n]})(1 - t^2)^{m-1} q^{2n},$$

where $C_{p,m}^{[n]}$ parameterizes colength n ideals $\mathcal{I}$ generated by m elements. This conjecture is known for torus knots and in the special case when $t = 1$, among others.

4

Patrick (Oct 5 and Oct 12): The HOMFLY polynomial and enumerative geometry

Note: these are the speaker's notes. There are several mistakes, but I am too lazy to correct them.

4.1 Introduction

Let $C \subset \mathbb{C}^2$ be a reduced curve and suppose $p \in C$ is an (isolated) singularity. Then define a constructible function $m\colon C_p^{[n]} \to \mathbb{N}$ given by $m([Z])$ being the number of generators of the ideal $I_{Z,p} \subset \mathcal{O}_{C,p}$. Now consider the generating function

$$Z_{C,p}(\nu, w) = \sum_{n \geqslant 0} s^{2n} \int_{C_p^{[n]}} (1 - \nu^2) \, d\chi = \sum_{n \geqslant 0} s^{2n} \sum_k k\chi_{\mathrm{top}}(f^{-1}(k)).$$

Additionally, taking a small S^3 around $p \in \mathbb{C}^2$ and intersecting with C gives us a link $\mathcal{L}_{C,p}$. Recall the HOMFLY polynomial $P(\mathcal{L}; \nu, s) \in \mathbb{Z}[\nu^{\pm}, (s - s^{-1})^{\pm}]$. If μ is the Milnor number of the singularity (for example, the middle Betti number of the Milnor fiber), then the Oblomkov-Shende conjecture is

Theorem 4.1.1 (Maulik)**.**

$$P(\mathcal{L}_{C,p}; \nu, s) = \left(\frac{\nu}{w}\right)^{\mu - 1} Z_{C,p}(\nu, s).$$

The proof of this result relates both sides of the equality to enumerative geometry, and in particular Pandharipande-Thomas, or stable pairs, curve counting. In this lecture, we will (attempt to) sketch a proof of the Oblomkov-Shende conjecture, but first we will introduce PT theory to familiarize ourselves with the objects involved in the proof.

Remark 4.1.2. Maulik proves everything for a colored version of the HOMFLY polynomial, but here we will only work with uncolored data, which corresponds to all partitions being (1).

4.2 Introduction to PT Theory

In this part, we are following the papers *Curve counting via stable pairs in the derived category* and *Stable pairs and BPS invariants* by Pandharipande and Thomas.

Let X be a smooth threefold and $\beta \in H_2(X, \mathbb{Z})$. The *PT moduli space* $P_n(X, \beta)$ parameterizes two-term complexes

$$\mathcal{O}_X \xrightarrow{s} \mathcal{F},$$

where $\mathcal{F}$ is a pure 1-dimensional sheaf supported on a Cohen-Macaulay subcurve of X, s has 0-dimensional cokernel, $\chi(\mathcal{F}) = n$, and $[\operatorname{supp} F] = \beta$. The space $P_n(X, \beta)$ has a virtual fundamental class coming from the deformation theory of complexes in the derived category. Here, note that the deformation theory of $(\mathcal{F}, s)$ (really of the corresponding complex $\mathcal{I}^\bullet$) is given by

$$\operatorname{Ext}^0(\mathcal{I}^\bullet, \mathcal{F}) \to \operatorname{Ext}^1(\mathcal{I}^\bullet, \mathcal{F}) = \operatorname{Ext}^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \to \operatorname{Ext}^2(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0$$

and the virtual fundamental class lives in dimension

$$c_\beta \coloneqq \int_\beta c_1(X).$$

In particular, for a Calabi-Yau threefold, $c_\beta = 0$. Given this data, we may define the *PT invariant*

$$P_{n,\beta} \coloneqq \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1.$$

and the PT partition function

$$Z_{PT,\beta}(q) = \sum P_{n,\beta} q^n.$$

There is an alternative way to compute the PT invariants for X a projective Calabi-Yau threefold, due to Behrend (originally done for DT theory). In this case, the moduli space is actually a projective scheme, and if $P_n(X, \beta)$ is smooth everywhere, then

$$P_{n,\beta} = (-1)^{\dim P_n(X,\beta)} \chi_{\mathrm{top}}(P_n(X, \beta)).$$

Of course, moduli spaces are almost never smooth for nontrivial moduli problems, so instead, we have

Proposition 4.2.1 (Behrend). *There exists an integer-valued constructible function χ_B such that*

$$P_{n,\beta} = \int_{P_n(X,\beta)} \chi_B \coloneqq \sum_{n \in \mathbb{Z}} n \chi_{\mathrm{top}}((\chi_B)^{-1}(n)).$$

4.3 A computation in PT theory

This computation comes from *The 3-fold vertex via stable pairs* by Pandharipande and Thomas.

We will now compute the T-equivariant PT vertex of $\mathbb{C}^3$ (this is the local model for toric varieties) to give us all a feel for this enumerative theory. First, we need to define some combinatorial data. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a triple of partitions. Then there exists a unique minimal T-fixed subscheme C_μ with outgoing partitions the μ_i (simply take the curves C_{μ_i} given by the three partitions and take the union), whose ideal we will denote $\mathcal{I}_\mu$. Then we define

$$M = \bigoplus_{i=1}^{3} (\mathcal{O}_{C_{\mu_i}})_{x_i} \eqqcolon \bigoplus_{i=1}^{3} M_i.$$

Every T-invariant pair $(\mathcal{F}, s)$ on $\mathbb{C}^3$ corresponds to a finitely-generated T-invariant submodule

$$Q \subset M / \langle (1, 1, 1) \rangle,$$

and we will now give a combinatorial description of such submodules.

For each μ^i, we may consider the module M_i as an infinite cylinder $\mathrm{Cyl}_i \subset \mathbb{Z}^3$ (extending in both directions). Then for every $w \in \mathbb{Z}^3$, consider the vectors $\mathbf{1}_w, \mathbf{2}_w, \mathbf{3}_w$ representing w in each copy of $\mathbb{Z}^3$ (for each of the M_i). Clearly x_1 shifts w by $(1,0,0)$ and similarly for x_2, x_3. We will now consider the decomposition of the union of the Cyl_i into the following types:

- Type I^+ are those which have only nonnegative coordinates and lie in exactly one cylinder;
- Type $\mathbb{II}$ (resp $\mathbb{III}$) are those which lie in exactly 2 (resp 3) cylinders;
- Type I^- are those with at least one negative coordinate.

Clearly $M/\langle(1,1,1)\rangle$ is supported on types $\mathbb{II}, \mathbb{III}, \mathrm{I}^-$, and now we have three cases:

- If $w \in \mathrm{I}^-$, then clearly $\mathbb{C} \cdot \mathbf{i}_w \subset M/\mathcal{O}_{C_\mu}$;
- If $w \in \mathbb{II}$, then $\frac{\mathbb{C}\cdot\mathbf{i}_w \oplus \mathbb{C}\cdot\mathbf{j}_w}{\mathbb{C}\cdot(\mathbf{i}_w+\mathbf{j}_w)} \cong \mathbb{C}$;
- If $w \in \mathbb{III}$, then $\frac{\mathbb{C}\cdot\mathbf{1}_w \oplus \mathbb{C}\cdot\mathbf{2}_w \oplus \mathbb{C}\cdot\mathbf{3}_w}{\mathbb{C}\cdot(\mathbf{1}_w+\mathbf{2}_w+\mathbf{3}_w)} \cong \mathbb{C}^2$.

In particular, we need to consider *labelled box configurations*, where type $\mathbb{III}$ boxes may be labeled by an element of $\mathbb{P}^1$ (where $\mathbb{C}^2$ is identified with the vector space above) or unlabeled (corresponding to the inclusion of the entire $\mathbb{C}^2$ in Q). Now we will denote by $\mathcal{Q}_\mu$ the set of components of the moduli space of T-invariant submodules of $M/\mathcal{O}_{C_\mu}$.

We are now able to define the equivariant vertex. Let $\ell(\mathcal{Q})$ be the number of boxes in the labelled configuration associated to $\mathcal{Q} \in \mathcal{Q}_\mu$. Then let $|\mu|$ denote the *renormalized volume* of the partition π corresponding to $\mathcal{I}_{C_\mu}$, which is defined as

$$|\pi| = \#\left\{\pi \cap [0,\dots,N]^3\right\} - (N+1)\sum_1^3 \left|\mu^i\right|,$$

which is independent of a sufficiently large $N \gg 0$.

We need to define a few characters of T, which we will need to define the vertex and compute our example. Let P be the Poincaré polynomial of a free resolution of the universal complex $\mathbb{I}$ on $\mathcal{Q} \times \mathbb{C}^3$. Denote by F the character of $\mathcal{F}$. In particular, we have

$$F = \frac{1+P}{(1-t_1)(1-t_2)(1-t_3)}.$$

Then define a vertex character V by

$$V = \mathrm{tr}_{\mathcal{O}-\chi(\mathbb{I},\mathbb{I})} + \sum_{i=1}^3 \frac{G_{\alpha\beta_i}(t_i', t_i'')}{1-t_i},$$

where $G_{\alpha\beta}$ is a certain character defined by edge data. Now let

$$w(\mathcal{Q}) := \int_{\mathcal{Q}} e(T_{\mathcal{Q}}) \cdot e(-V) \in \mathbb{Q}[s_1, s_2, s_3]_{(s_1,s_2,s_3)} = (A_T^*)_{\mathrm{loc}}$$

be the contribution of V on $\mathcal{Q}$. Then the equivariant vertex is defined to be

$$W_\mu^P := \sum_{\mathcal{Q}\in\mathcal{Q}_\mu} w(\mathcal{Q}) q^{\ell(\mathcal{Q})+|\mu|} \in \mathbb{Q}(s_1, s_2, s_3)((q)).$$

Example 4.3.1. For $\mu = ((1), \emptyset, \emptyset)$, we have

$$W^P_\mu = (1+q)^{\frac{s_2+s_3}{s_1}}.$$

To see this, note that $\mathcal{Q}_\mu = \mathbb{Z}_{>0}$, where k corresponds to the length k box configuration in the negative x_1-direction.

Now we simply compute that

$$F_{\mathcal{Q}_k} = \frac{t_1^{-k}}{1-t_1}.$$

This implies that

$$V_{\mathcal{Q}_k} = \sum_{i=1}^{k} t_1^{-i} - \sum_{i=0}^{k-1} \frac{t_1^i}{t_2 t_3},$$

and therefore that

$$\begin{aligned} w(\mathcal{Q}_k) &= \int_{\mathcal{Q}_k} e(-V_{\mathcal{Q}^k}) \\ &= \frac{(-s_2-s_3)(s_1-s_2-s_3)\cdots((k-1)s_1-s_2-s_3)}{(-s_1)(-2s_1)\cdots(-ks_1)}, \end{aligned}$$

as desired.

4.4 Flop invariance of PT theory

We will now begin the proof of the Oblomkov-Shende conjecture. The first step is to understand what happens when we blow up C at p via flop invariance of PT partition functions.

Let Y be the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and Y_- be the threefold obtained via a flop of the zero section. The flop is some birational map $\phi\colon Y \dashrightarrow Y_-$. If $\pi\colon Y \to \mathbb{P}^1$ is the projection, choose an identification of $\mathbb{C}^2$ with $\pi^{-1}(0)$. Then the strict transform of $\pi^{-1}(0)$ with respect to ϕ is $\mathrm{Bl}_0 \mathbb{C}^2$ with exceptional fiber E_- which is the zero section of Y_- (isomorphic to Y). This implies that the strict transform of C with respect to ϕ is $\mathrm{Bl}_0 C$.

Definition 4.4.1. A stable pair $(\mathcal{F}, s)$ is C-*framed* if on $Y \setminus E$ if after restricting to $Y \setminus E$, we have an isomorphism $(\mathcal{F}, s) \simeq \mathcal{O}_Y \to \mathcal{O}_C$.

Given $r, n \in \mathbb{Z}$, we define the moduli space $P(Y, C, r, n)$ of C-framed stable pairs such that $\operatorname{supp} F$ has generic multiplicity r along E and for any projective compactification $\overline{Y}$ of Y, we have $\chi(\overline{\mathcal{F}}) = n + \chi(\mathcal{O}_{\overline{C}})$. This is a locally closed subscheme of the space of stable pairs on $\overline{Y}$ and is independent of the choice of $\overline{Y}$. Now we define the C-framed PT partition function

$$Z(Y, C; q, Q) := \sum_{r,n} q^n Q^r \chi_{\mathrm{top}}(P(Y, C, r, n)).$$

Proposition 4.4.2. *Let $m_1, \ldots, m_r$ be the multiplicities of the branches of C at p. We have the flop identity*

$$Q^{\sum m_r} Z'(Y, C; q, Q^{-1}) = q^\delta Z'(Y_-, C'; q, Q),$$

where

$$Z'(Y, C; q, Q) := \frac{Z(Y, C; q, Q)}{\prod_k (1+q^k Q)^k}$$

is the normalized PT partition function and $\delta = \binom{\sum m_i}{2}$.

4.5 Algebraic links

We will now study what happens to an algebraic link under blowup. Recall that if the singularity $(C, 0)$ is irreducible, then we can describe $\mathcal{L}_{C,0}$ as an iterated torus knot using the Puiseux series of C at 0. If the Puiseux series is

$$y(x) = x^{\frac{q_0}{p_0}}(a_0 + x^{\frac{q_1}{p_0 p_1}}(a_2 + \cdots)),$$

then $\mathcal{L}_{C,0}$ is simply the iterated torus knot with parameters $(q_0, p_0), \ldots, (q_s, p_s)$. To help us compute things, we will project all of our diagrams into the annulus and use skein theory. Before we do this, we need to review skein theory.

Definition 4.5.1. Let $F \subset \mathbb{R}^2$ be a surface with boundary and designated input and output points. The *framed Homfly skein* over $\Lambda := \mathbb{Z}[v^{\pm}, s^{\pm}, (s^r - s^{-r})^{-1} \mid r \geqslant 1]$ is the Λ-module generated by oriented diagrams in F up to isotopy, R1, R2, and the skein relations[1]

$$\text{over} - \text{under} = (s - s^{-1}) \cdot \text{resolved} \tag{4.1}$$

$$\text{R1 over/under} = v^{\mp} \cdot \text{resolved} \tag{4.2}$$

$$\text{unknot} = \frac{v^{-1} - v}{s - s^{-1}}. \tag{4.3}$$

If F is a rectangle with m inputs and outputs, then the skein $\mathcal{H}_m$ has a product given by stacking diagrams, and is isomorphic to the A_m Hecke algebra. If F is an annulus, the skein $\mathcal{C}$ is a commutative algebra with product obtained by placing one annulus inside another. There is a Λ-module morphism $\bigoplus \mathcal{H}_m \to \mathcal{C}$ given by sending a braid to its closure. The algebra $\mathcal{C}_+$ generated by the image is isomorphic to the ring of symmetric functions with coefficients in Λ, and we will denote by Q_λ the diagram associaated to the Schur function s_λ. Finally, if $F = \mathbb{R}^2$, then the skein is simply the ring Λ. This gives us a trace map $\langle\rangle : \mathcal{C} \to \Lambda$. Up to some monomial factor, the trace gives us the HOMFLY polynomial.

Now we will discuss a satellite construction. This is how we will turn algebraic links into diagrams in the annulus for computations.

Definition 4.5.2. Let $\mathcal{L}$ be a framed link with r components and $Q_1, \ldots, Q_r$ be diagrams in the annulus with counterclockwise orientation. The *satellite link* $\mathcal{L} * (Q_1, \ldots, Q_r)$ is obtained by drawing Q_i on the neighborhood of the i-th strand of L. If $\mathcal{L}$ comes from a counterclockwise-oriented diagram in the annulus, then this construction only depends on the equivalence classes of the decorations.

Remark 4.5.3. Using these terms, coloring a link $\mathcal{L}$ just means giving each component of $\mathcal{L}$ a partition and considering the link $\mathcal{L} * (Q_{\lambda_1}, \ldots, Q_{\lambda_r})$.

Now the iterated torus knot $\mathcal{L}_{C,0}$ with parameters (q_i, p_i) can be embedded as a diagram L_C in the annulus, where

$$L_C = T_{p_0}^{q_0} * (T_{p_1}^{q_1} * (\cdots * (T_{p_s}^{q_s}) \cdots))$$

and T_p^q is the diagram of the (q, p)-torus knot.

For the general case of an algebraic link $\mathcal{L}_{C,0}$ where $(C, 0)$ is **not** irreducible, there is a more complicated satellite construction for constructing a diagram in the annulus. This produces satellite operators $S_p^q * (-, -)$.

[1]Sorry there are no drawings. I have no idea how Maulik typeset the diagrams – I'm not a TeX expert, I just optimized my workflow to be able to type fast and look at TeX.SE efficiently.

Before we continue, we will construct several objects that we will need later. Recall that the diagram T_m^n is the n-th power of the diagram below:

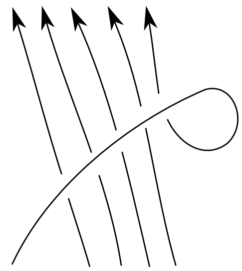

Figure 4.1: The diagram β_5

Next, we will consider the diagram σ_m below and denote its n-th power by S_m^n. These are required for the satellite construction for links, but we will not need it here.

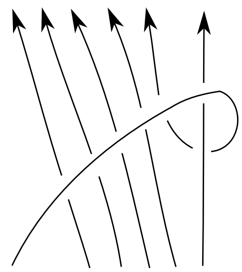

Figure 4.2: The diagram σ_5

To conclude this section, we will explain how to actually construct the link diagram in the annulus for a non-reducible singularity. Let $C_1, \dots, C_r$ be the branches of C at 0 and consider truncated Puiseux series

$$y_i(x) = x^{\frac{q^i}{p^i}}(a_i + z_i(x^{\frac{1}{p^i}})).$$

Write $\alpha_i = \frac{q^i}{p^i}$ and consider the pairs (α_i, a_i). It is always possible to find a finite truncation of the Puiseux series that does not affect the topological type of the link, so the inductive process we will define is actually finite. For each (α, a), let $\{i_0, \dots, i_n\}$ be the set of indices with associated pair (α, a). Assume we have an annulus diagram $L_{(\alpha,a)}$ and set

$$L_\alpha \coloneqq \prod_a L_{(\alpha,a)}.$$

We may assume that $\alpha_1 < \dots < \alpha_k$, and the link $\mathcal{L}_C$ can be represented by the annulus diagram

$$L_C \coloneqq S_{p^1}^{q^1} * (L_{\alpha_1}, S_{p^2}^{q^2} * (L_{\alpha_2}, \dots, S_{p^{k-1}} q^{k-1} * T_{p^k}^{q^k} * L_{\alpha_k})).$$

4.6 Some explicit calculations for the unknot and Hopf link

The following is already apparently known.

Proposition 4.6.1. *For any partition* λ*,*

$$\langle Q_\lambda \rangle = \prod_{\square \in \lambda} \frac{v^{-1} s^{c(\square)} - v s^{-c(\square)}}{s^{h(\square)} - s^{-h(\square)}}.$$

Let $X \in \mathcal{C}_+$ be a counterclockwise-oriented diagram. Define the *meridian operator* M_X on $\mathcal{C}_+$ by the construction below:

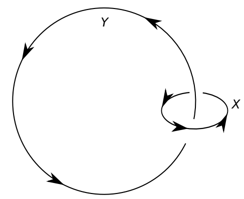

Figure 4.3: Meridian operator

For a partition μ, the Schur function Q_μ is an eigenvector for M_X with eigenvalue $t_\mu(X)$. Then the HOMFLY polynomial for the colored Hopf link decorated by μ, λ is simply $t_\mu(Q_\lambda)(t_\mu)$. In the remainder of this section, we will describe the operator t_μ, which is a ring homomorphism $\mathcal{C}_+ \to \Lambda$. Define the function

$$E_\mu(t) = \prod_{j=1}^{\ell(\mu)} \frac{1 + v^{-1} s^{\mu_j - 2j + 1} t}{1 + v^{-1} s^{-2j+1} t} \prod_{i \geqslant 0} \frac{1 + v s^{2i+1} t}{1 + v^{-1} s^{2i+1} t}.$$

Then we have $t_\mu(Q_\lambda) = s_\lambda(E_\mu(t))$, where for a power series $E(t) = \sum E_k t^k$, we write s_λ as a polynomial in the e_k and then substitute $e_k \leftarrow E_k$.

4.7 Behavior of links with respect to blowup

Having discussed the behavior of enumerative invariants with respect to blowups, we need to discuss the behavior of the link $\mathcal{L}_{C,0}$ with respect to blowing up the origin. Let $C_1, \ldots, C_r$ be the irreducible components of C through 0 and denote their strict transforms by C_i'. At each point $p_1, \ldots, p_e \in E$ (the exceptional divisor) let D_k denote the singularity of $C' \cup E$ at p_k and B_k be the singularity of C' at p_k. Choose (truncated) Puiseux expansions

$$y_i(x) = x^{\frac{q^i}{p^i}} (a_i + z_i(x^{\frac{1}{p^i}}))$$

For each of the branches C_i. We may also assume that $\frac{q^i}{p^i} \geqslant 1$ for all i. If we blow up at the origin, consider the chart with coordinates $(x, y = xw)$. Substitution, we obtain the new Puiseux expansion

$$w_i(x) = x^{\frac{q^i - p^i}{p^i}} (a_i + z_i(x^{\frac{1}{p^i}}))$$

for C_i' at p_k. In particular, we obtain the relation

$$[L_C] = \tau[L_{C'}] \qquad \tau(-) \coloneqq T_1^1 * (-).$$

If we perform deeper analysis, we obtain the following result:

Proposition 4.7.1. *If any* $\alpha_i = \frac{q_i}{p_i} > 1$*, then we have*

$$[L_C] = S_1^1 * (L_{B_1} \cdots L_{b_{e-1}}, \tau L_{B_e}).$$

Otherwise, we have

$$[L_C] = S_1^1 * (L_{b_1} \cdots L_{B_e}, \emptyset) = T_1^1 * (L_{B_1} \cdots L_{B_e}).$$

Now we will write down a blowup identity for links. The idea is to use the topological vertex (originally introduced by Aganagic-Klemm-Marino-Vafa) and its relationship with Chern-Sinons invariants of the unknot. For a partition μ, define

$$\begin{aligned} Z_\mu(q, Q) &= s_\mu(q^\rho) \prod_{\square \in \mu} (1 + Qq^{-c(\square)}) \\ &= q^{\kappa_\mu/4} \prod_{\square \in \mu} \frac{1 + Qq^{-c(\square)}}{q^{h(\square)/2} - q^{-h(\square)/2}}. \end{aligned}$$

Here, $\kappa_\mu = 2\sum_{\square \in \mu} c(\square)$. By the computation of the colored HOMFLY polynomial of the colored unknot (sorry), we obtain the identity

$$Z_\mu(q = s^2, Q = -\nu^2) = \nu^{|\mu|} \langle Q_\mu \rangle .$$

4.8 Relation between PT theory and HOMFLY

We will prove a relationship between the PT partition function for C-framed stable pairs in $Y = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and the HOMFLY polynomial for $\mathcal{L}_{C,0}$. We will focus on the simplest case of a node with two branches. The general case reduces to this one by careful checking of what happens on both sides under blowup of C at 0.

Proposition 4.8.1. *We have the identity (possibly up to monomials)*

$$Z'(Y, C; q, Q = 0) = (-1)^\varepsilon s^b \left\langle [L_C * Q_{(1)^t}] \right\rangle^{\text{low}},$$

where the superscript low means we take the lowest degree terms.

We only need to prove this for the unknot and the Hopf link.

Proof. This apparently follows from the fact that the topological vertex calculates both the $\nu = 0$ specialization of HOMFLY of the Hopf link and the stable pairs vertex. See the references to Maulik's paper for a reference. $\square$

5

Che (Oct 19): Braid varieties and Khovanov-Rozansky homology

5.1 Khovanov-Rozansky homology

Let $G = GL_n$ and $\mathfrak{h}$ be the Cartan. Write $R = \mathbb{C}[\mathfrak{h}] = \mathbb{C}[x_1, \ldots, x_n]$, and this has an action of the Weyl group permuting the coordinates. We will specify $\deg x_i = 2$. We may now define Soergel bimodules

$$B_i = R \otimes_{R^{s_i}} R(1).$$

Now define $T_i = [B_i(-1) \to R]$ by $1 \otimes 1 \mapsto 1$ and $T_i^{-1} = [R \to B_i(-1)]$ by $1 \mapsto x_i \otimes 1 - 1 \otimes x'_{i+1}$. We may now form the Rouquier complex for a braid $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$ as

$$T_\beta = T_{i_1} \otimes_R \cdots \otimes T_{i_k}.$$

For example, if $\beta = \sigma^2$, then we have

$$T_\beta = [B(-1) \otimes B(-1) \to B(-1) \oplus B(-1) \to R] = [B(-3) \to B(-1) \to R].$$

We may now take the Hochschild homology $HH^i(T_\beta)$. We will mostly be concerned with $HH^0(M) = \mathrm{Hom}(R, M)$. Continuing our example, we have

$$HH^0(T_\beta) = [R(-4) \xrightarrow{0} R(-2) \xrightarrow{x_1 - x_2} R].$$

We are finally able to define the *Khovanov-Rozansky homology*

$$HHH^*(\beta) := H^*(HH^*(\beta)).$$

This has gradings A for the Hochschild grading, Q for the internal degree in R, and T for the homological grading. Continuing our example again, we have

$$HHH^{A=0}(\beta) = \frac{1}{1 - Q^2} - \frac{Q^4 T^{-2}}{(1 - Q^2)^2}.$$

Remark 5.1.1. We will usually multiply our HHH by $1 - Q^2$ as a useful normalization. This is a consequence of us considering $G = GL_n$ instead of $G = SL_n$.

5.2 Braid varieties

Write $B_i(z)$ for the matrix with 1s on the diagonal except for a $\left(\begin{smallmatrix} 0 & 1 \\ 1 & z \end{smallmatrix}\right)$ block on the diagonal in the $i, i+1$ positions. Given $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$, we consider the matrix

$$B_\beta(z_1, \ldots, z_k) = B_{i_1}(z_1) \cdots B_{i_k}(z_k).$$

Then we may define the *braid variety*

$$X(\beta) \coloneqq \big\{(z_1, \ldots, z_k) \mid B_\beta(z_1, \ldots, z_k) \text{ is upper triangular}\big\}.$$

For example, if $\beta = \sigma^2$ for 2 strands, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix} = \begin{pmatrix} 1 & z_2 \\ z_1 & 1 + z_1 z_2 \end{pmatrix},$$

and so

$$X(\beta) = (z_1 = 0) \subseteq \mathbb{C}^2.$$

Similarly, for $\beta = \sigma^3$, we have the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_3 \end{pmatrix} = \begin{pmatrix} z_2 & 1 + z_2 z_3 \\ 1 + z_1 z_2 & z_1 + z_3 + z_1 z_2 z_3 \end{pmatrix},$$

and thus we have $X(\beta) = (1 + z_1 z_2 = 0) \times \mathbb{C}_{z_3}$. Similarly, we may compute

$$X(\sigma^4) = \{z_1 + z_3 + z_1 z_2 z_3 = 0\} \times \mathbb{C}_{z_4} = \{1 + z_2 z_2 \neq 0\} \times \mathbb{C}_{z_4}.$$

These braid varieties come with a torus action. We would like to write

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} B_i(z) = B_i(z') \begin{pmatrix} t_1' & & \\ & \ddots & \\ & & t_n' \end{pmatrix}.$$

For example, we have

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = \begin{pmatrix} 0 & t_1 \\ t_2 & t_2 z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \frac{t_2}{t_1} z \end{pmatrix}\begin{pmatrix} t_2 & \\ & t_1 \end{pmatrix}.$$

Doing this on $B_\beta(z_1, \ldots, z_k)$, we obtain an action of $T = (\mathbb{C}^\times)^n$ on $X(\beta)$. Continuing our example, we have

$$z_1 \mapsto \frac{t_2}{t_1} z_1, \qquad z_2 \mapsto \frac{t_1}{t_2} z_2, \qquad z_3 \mapsto \frac{t_2}{t_1} z_3, \qquad \ldots$$

Here are some facts:

1. Under mild assumptions, $X(\beta)$ is smooth.
2. If β closes to a knot, then the action of T is free.

We are now ready to state the main result.

Theorem 5.2.1. *We have an equality*

$$\mathrm{gr}_w H_T^*(X(\beta)) \cong \mathrm{HHH}^{A=n}(\beta)$$

up to a global shift in grading, where w is the weight filtration.

5.3 Mixed Hodge Structures

To understand the weight filtration, we will discuss mixed Hodge structures. For a smooth projective variety X of dimension n, recall the *Hodge decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^p(X, \Omega_X^q).$$

Of course, for general X which are not necessarily smooth or projective, this decomposition will fail. In this case, there is a *weight filtration*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2k} = H^k(X, \mathbb{C})$$

which morally measures how much the Hodge decomposition fails.

Theorem 5.3.1. *Let X be a smooth algebraic variety inside a smooth and projective $\overline{X}$ such that $D = \overline{X} \setminus X$ is a normal crossings divisor. Write $D = \bigcup_i D_i$ as a union of its irreducible components. Then there exists a spectral sequence*

$$E_1^{-p,q} = H^{q-2p}(D^{(p)}) \Rightarrow \mathrm{gr}_w^q H^{q-p}(X)$$

which collapses at the E_2 page. Here, $D^{(p)}$ is the locus of the p-fold intersection.

Note that our spectral sequence looks like

$$\begin{array}{ccccccccc}
H^0(D^{(2)}) & \longrightarrow & \vdots & & & & \vdots & & \vdots \\
0 & & \vdots & & & & \vdots & & \vdots \\
0 & & H^0(D^{(1)}) & \longrightarrow & H^2(\overline{X}) & \Longrightarrow & \mathrm{gr}_w^2 H^1 & & \mathrm{gr}_w^2 H^2 \\
0 & & 0 & & H^1(\overline{X}) & & 0 & & \mathrm{gr}_w^1 H^1 \\
0 & & 0 & & H^0(\overline{X}) & & 0 & & \mathrm{gr}_w^0 H^0.
\end{array}$$

For example, when $X = \mathbb{C}^\times$ and $\overline{X} = \mathbb{P}^1$, we have $D = 0 + \infty$, and so we obtain

$$\begin{array}{ccccccc}
2 & \longrightarrow & 1 & & 1 & & 0 \\
0 & \longrightarrow & 0 & \Longrightarrow & 0 & & 0 \\
0 & \longrightarrow & 1 & & 0 & & 1
\end{array}$$

and for example, $\mathrm{gr}_w^2 H^1(\mathbb{C}^\times) = \mathbb{C}$.

5.4 Sketch of proof of main theorem

What we will do is to construct a Bott-Semelson variety $\mathrm{BS}(\beta)$ from the Khovanov-Rozansky homology, then as a fiber over the flag variety we will consider the brick variety of β, which is a compactification of $X(\beta)$.

Definition 5.4.1. Given $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$, define the *Bott-Samelson variety*

$$\mathrm{BS}(\beta) = \left\{(F_0, \ldots, F_k) \mid F_i \in G/B, F_0 = F^{st}, F_j \neq F_{j+1} \text{ only in the } i_j\text{th position}\right\}.$$

For example, when $\beta = \sigma_1\sigma_2$, we have

$$F_0 = (e_1) \subset (e_1, e_2) \subset (e_1, e_2, e_3) F_1 = (e_2) \subset (e_1, e_2) \subset (e_1, e_2, e_3) F_2 = (e_2) \subset (e_2, e_3) \subset (e_1, e_2, e_3).$$

Then $(F_0, F_1, F_2) \in \mathrm{BS}(\beta)$. Now we define the open Bott-Samelson variety

$$\mathrm{OBS}(\beta) = \left\{(F_0, \ldots, F_k) \in \mathrm{BS}(\beta) \mid F_j \neq F_{j+1}\right\}.$$

We have the projection $\mathrm{pr}\colon \mathrm{BS}(\beta) \to G/B$ sending a tuple of flags to F_k. Then we define

$$\mathrm{brick}(\beta) = \mathrm{pr}^{-1}\delta(\beta)F^{st},$$

where $\delta(\beta) \in W$ is the Demazure product. Finally, define $\mathrm{brick}^0(\beta) = \mathrm{brick}(\beta) \cap \mathrm{OBS}(\beta)$.

Theorem 5.4.2.

1. $X(\reflectbox{$\beta$}; \delta(\beta)) \cong \mathrm{brick}^0(\beta)$, *where* $\reflectbox{$\beta$}$ *is* β *with the transpositions reversed.*
2. *The complement* $\mathrm{brick}(\beta) \setminus \mathrm{brick}^0(\beta)$ *is a normal crossings divisor.*
3. *The brick variety* $\mathrm{brick}(\beta)$ *has a stratification by* $\mathrm{brick}(\beta')$ *for subwords* β' *of* β *such that* $\delta(\beta) = \delta(\beta')$.

Remark 5.4.3. We define $X(\beta, w)$ for some $w \in W$ as

$$\left\{(z_1, \ldots, z_k) \mid B_\beta(z_1, \ldots, z_k)w \text{ is upper triangular}\right\}.$$

For example, we have $X(\sigma^3, (12)) = \{1 + z_1 z_2 \neq 0\}$. On the other hand, $\mathrm{brick}(\sigma^3) = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{brick}(\sigma^2) = \mathbb{P}^1$, and $\mathrm{brick}(\sigma) = \mathrm{pt}$.

Theorem 5.4.4. *As an* $R - R$ *bimodule, we have*

$$H_T^*(\mathrm{BS}(\sigma_{i_1} \cdots \sigma_{i_k})) \cong R \otimes_{R^{s_{i_1}}} \cdots \otimes_{R^{s_{i_n}}} R.$$