DT/PT CORRESPONDENCE USING MOTIVIC HALL ALGEBRAS

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Abstract. A DT/PT correspondence for projective Calabi-Yau threefolds was proved by Bridgeland in 2011 and for orbifolds by Beentjes-Calabrese-Rennemo in 2018. We will state the necessary results about motivic Hall algebras and then discuss the DT/PT correspondence.

1. Introduction

Let $\mathcal{X}$ be a proper smooth Deligne-Mumford stack with $\omega_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$, $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$, and projective coarse moduli space X. Note here that X is Gorenstein, Calabi-Yau, and has at worst quotient singularities. We will also impose that $\mathcal{X}$ satisfies the hard Lefschetz condition, which says that age is invariant under the map $I\colon I\mathcal{X} \to I\mathcal{X}$ taking $(x, g) \mapsto (x, g^{-1})$.

Now let $N(\mathcal{X})$ denote the numerical K-theory of $\mathcal{X}$, that is $K(D(\mathcal{X}))$ modulo the Euler pairing

$$\chi(E, F) = \sum (-1)^i \dim \operatorname{Ext}^i_{\mathcal{X}}(E, F).$$

Then denote $N_{\leqslant 1}(\mathcal{X})$ denote the classes with support in dimension at most 1 and $N_0(\mathcal{X})$ be the classes with 0-dimensional support. Then choose[1] a splitting

$$N_{\leqslant 1}(\mathcal{X}) = N_1(\mathcal{X}) \oplus N_0(\mathcal{X}) \ni (\beta, c).$$

Finally, we call a class $\beta \in N_1(\mathcal{X})$ *effective* if it is represented by an actual sheaf $F \in \mathtt{Coh}_{\leqslant 1}(\mathcal{X})$.

Now let $Y = \mathrm{Quot}(\mathcal{O}_{\mathcal{X}}, [\mathcal{O}_x])$ for a non-stacky point $x \in \mathcal{X}$. Then Y is a smooth projective Calabi-Yau threefold and there is a map $f\colon Y \to X$, which is a crepant resolution. Then, if we consider the universal sheaf on $Y \times \mathcal{X}$ by $\mathcal{O}_{\mathcal{Z}}$, the Fourier-Mukai transform

$$\Phi = p_{\mathcal{X}*}(\mathcal{O}_{\mathcal{Z}} \otimes p_Y^*(-))\colon D(Y) \to D(\mathcal{X})$$

is a derived equivalence, called the *McKay correspondence*. Then there is a map $N_{\leqslant 1}(Y) \to N_{\leqslant 1}(\mathcal{X})$, and thus we obtain a splitting $N_{\leqslant 1}(Y) = N_{\text{n-exc}}(Y) \oplus N_{\text{exc}}(Y)$. Finally, write $N_{\text{mr}}(\mathcal{X}) = \Phi(N_{\leqslant 1}(Y))$ and $N_{1,\text{mr}}(\mathcal{X}) = \Phi(N_{\text{n-exc}}(Y))$. Also, write $\Psi \coloneqq \Phi^{-1}$.

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[1] For a manifold, ch_3 defines a canonical splitting, but this cannot always be done for orbifolds.

Then define following DT and PT generating series:

$$\mathrm{DT}(\mathcal{X})_\beta = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}(\mathcal{X})_{(\beta,c)} q^c;$$
$$\mathrm{DT}(\mathcal{X})_0 = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}(\mathcal{X})_{(0,c)} q^c;$$
$$\mathrm{PT}(\mathcal{X})_\beta = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{PT}(\mathcal{X})_{(\beta,c)} q^c.$$

Theorem 1.1 (DT/PT correspondence)**.** *Let $\beta \in \mathrm{N}_{1,\mathrm{mr}}(\mathcal{X})$. Then there is an equality of generating series*

$$\mathrm{PT}(\mathcal{X})_\beta = \frac{\mathrm{DT}(\mathcal{X})_\beta}{\mathrm{DT}(\mathcal{X})_0}.$$

2. Categorical preliminaries

Definition 2.1. Let $\mathtt{B}$ be an abelian category. Then a *torsion pair* is a pair of full subcategories $(\mathtt{T}, \mathtt{F})$ such that $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathtt{T}, F \in \mathtt{F}$ and every $E \in \mathtt{B}$ fits into a (unique) short exact sequence

$$0 \to T_E \to E \to F_E \to 0$$

with $T_E \in \mathtt{T}, F_E \in \mathtt{F}$.

This can be naturally generalized to the notion of a *torsion n-tuple* $(\mathtt{B}_1, \ldots, \mathtt{B}_n)$.

Example 2.2. Let $\mathtt{T} = \mathtt{Coh}_{\leqslant d}(\mathcal{X})$ and $\mathtt{F} = \mathtt{Coh}_{\geqslant d+1}(\mathcal{X})$ be the category of sheaves with no subsheaves of dimension $\leqslant d$. Then $(\mathtt{T}, \mathtt{F})$ is a torsion pair on $\mathtt{Coh}(\mathcal{X})$.

For a torsion pair $(\mathtt{T}, \mathtt{F})$, we can define a new abelian category

$$\mathtt{B}^\flat = \left\{ E \in \mathrm{D}^{[-1,0]}(\mathtt{B}) \mid H^{-1}(E) \in \mathtt{F}, H^0(E) \in \mathtt{T} \right\}.$$

This is called *tilting* at the torsion pair $(\mathtt{T}, \mathtt{F})$.

Example 2.3. Define the category $\mathtt{Coh}^\flat(\mathcal{X})$ by tilting at the torsion pair of the previous example for $d = 1$.

Unfortunately, this category is too large for our purposes, so we will define a new category. First, if $\mathcal{C}_1, \ldots, \mathcal{C}_n \subset \mathrm{D}(\mathcal{X})$ are full subcategories, let $\langle \mathcal{C}_1, \ldots, \mathcal{C}_n \rangle_{\mathrm{ex}} \subset \mathrm{D}(\mathcal{X})$ be the smallest full subcategory of $\mathrm{D}(\mathcal{X})$ containing all of the $\mathcal{C}_i$ and closed under extensions. Now we may define

$$\mathtt{A} = \left\langle \mathcal{O}_{\mathcal{X}}[1], \mathtt{Coh}_{\leqslant 1}(\mathcal{X}) \right\rangle_{\mathrm{ex}} \subset \mathtt{Coh}^\flat(\mathcal{X}).$$

This is a Noetherian abelian category with the inclusions $\mathtt{Coh}_{\leqslant 1}(\mathcal{X}) \subset \mathtt{A} \subset \mathrm{D}(\mathcal{X})$ being exact. In addition, $\mathtt{A}$ contains all ideal sheaves $I_C[1]$, PT stable pairs, and Bryan-Steinberg pairs.

Definition 2.4. Let $\mathtt{T}, \mathtt{F}$ be full subcategories of $\mathtt{Coh}_{\leqslant 1}(\mathcal{X})$. Write $\mathtt{Pair}(\mathtt{T}, \mathtt{F}) \subset \mathtt{A}$ for the full subcategory on objects E such that $\operatorname{rk} E = -1$, $\mathrm{Hom}(T, E) = 0$ for $T \in \mathtt{T}$, and $\mathrm{Hom}(E, F) = 0$ for $F \in \mathtt{F}$.

Example 2.5. For example, if $\mathtt{T}_{\mathrm{PT}} = \mathtt{Coh}_0(\mathcal{X})$ and $\mathtt{F}_{\mathrm{PT}} = \mathtt{Coh}_1(\mathtt{X})$, then a $(\mathtt{T}_{\mathrm{PT}}, \mathtt{F}_{\mathrm{PT}})$-pair is a PT stable pair.

For a pair of full subcategories as above, we may also define

$$\mathtt{V}(\mathtt{T}, \mathtt{F}) := \{E \in \mathtt{A} \mid \mathrm{Hom}(\mathtt{T}, E) = \mathrm{Hom}(E, \mathtt{F}) = 0\}.$$

If $(\mathtt{T}, \mathtt{F})$ and $(\widetilde{\mathtt{T}}, \widetilde{\mathtt{F}})$ are torsion pairs, then $(\mathtt{T}, \mathtt{V}(\mathtt{T}, \widetilde{\mathtt{F}}), \widetilde{\mathtt{F}})$ is a torsion triple on $\mathtt{A}$.

3. Moduli stack

There is a stack $\mathfrak{Mum}_Y$ parameterizing objects $E \in D(Y)$ such that $\mathrm{Ext}^{<0}(E, E) = 0$ which is an Artin stack locally of finite type, which was constructed by Lieblich in 2006.[2] Because Fourier-Mukai transforms behave well in families, by the McKay correspondence, we have a corresponding stack $\mathfrak{Mum}_{\mathcal{X}}$. In addition, there is a decomposition

$$\mathfrak{Mum}_{\mathcal{X}} = \bigsqcup_{\alpha \in N(\mathcal{X})} \mathfrak{Mum}_{\mathcal{X}, \alpha}$$

into open and closed substacks by the numerical K-theory class. If $\mathtt{C} \subset D(\mathcal{X})$ is a full subcategory of objects with vanishing self-$\mathrm{Ext}^{<0}$ and defines an open substack of $\mathfrak{Mum}_{\mathcal{X}}$, then we will call the corresponding open substack $\underline{\mathtt{C}} \subset \mathfrak{Mum}_{\mathcal{X}}$.

Definition 3.1. A torsion pair $(\mathtt{T}, \mathtt{F})$ on $\mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ is *open* if $\mathtt{T}, \mathtt{F}$ are open subcategories.

Lemma 3.2. *The following conditions define open substacks of $\mathfrak{Mum}_{\mathcal{X}}$ for full, open subcategories $\mathtt{T}, \mathtt{F}$ of $\mathtt{Coh}(\mathcal{X})$:*

(1) $H^0(E) \in \mathtt{T}$ *and* $E \in D^{[-1,0]}(\mathcal{X})$*;*

(2) $H^{-1}(E) \in \mathtt{F}$ *and* $E \in D^{[-1,0]}(\mathcal{X})$*.*

In particular, if $(\mathtt{T}, \mathtt{F})$ is an open torsion pair, then the objects of the tilt of $\mathtt{Coh}(\mathcal{X})$ at $(\mathtt{T}, \mathtt{F})$ form an open substack of $\mathfrak{Mum}_{\mathcal{X}}$

Proposition 3.3. *Let $(\mathtt{T}, \mathtt{F})$ be open torsion pairs on $\mathtt{Coh}_{\leqslant 1}(\mathcal{X})$. Suppose that $\mathtt{Coh}_0(\mathcal{X}) \subset \mathtt{T}$. Then the substack $\underline{\mathtt{Pair}}(\mathtt{T}, \widetilde{\mathtt{F}}) \subset \mathfrak{Mum}_{\mathtt{X}}$ is open.*

In particular, $\mathtt{Coh}^{\flat}(\mathcal{X})$ is an open subcategory. In addition, the category $\mathtt{A}_{\mathrm{rk} \geqslant -1} \subset \mathtt{Coh}^{\flat}(\mathcal{X})$ is open. This means that

$$\underline{\mathtt{A}}_{\mathrm{rk} \geqslant -1} \subset \underline{\mathtt{Coh}}^{\flat}(\mathcal{X}) \subset \mathfrak{Mum}_{\mathcal{X}}$$

are inclusions of open substacks, and in particular, the first two stacks are Artin stacks locally of finite type.

[2] As with all students of Johan, this was actually done in maximal generality for a proper morphism of algebraic spaces.

4. Motivic Hall algebras

Recall the Grothendieck ring of varieties $\mathrm{K}(\mathtt{Var}_{\mathbb{C}})$ (taken with $\mathbb{Q}$-coefficients), which is spanned by isomorphism classes of varieties with the relations $[X] = [Z] + [X \setminus Z]$ for a closed subvariety $Z \subset X$ and $[X] \cdot [Y] = [X \times Y]$. This will not be helpful when we consider stacks, so instead we will consider the following two relations.

Definition 4.1. Let Y be connected. Then a morphism $f\colon X \to Y$ is a *Zariski fibration* if there is an open cover $Y = \bigcup U_i$ such that $f^{-1}(U_i) = U_i \times F_i$.[3]

Definition 4.2. A morphism $f\colon X \to Y$ is a *geometric bijection* if f induces a bijection on $\mathbb{C}$-points.

Lemma 4.3. $\mathrm{K}(\mathtt{Var}_{\mathbb{C}})$ *is spanned by isomorphism classes of varieties with the following relations:*

(1) If $f\colon X \to Y$ is a geometric bijection, then $[X] = [Y]$;

(2) $[X_1] + [X_2] = [X_1 \sqcup X_2]$.

Now we say that a morphism $f\colon X \to Y$ of stacks is a Zariski fibration if for any scheme T, the base change $f_T\colon X \times_Y T \to T$ is a Zariski fibration of schemes. Of course, the definition of geometric bijection should be changed to that of inducing an equivalence on groupoids of $\mathbb{C}$-points.

Definition 4.4. The *Grothendieck ring of stacks* $\mathrm{K}(\mathtt{St}_{\mathbb{C}})$ is the $\mathbb{Q}$-vector space spanned by symbols $[X]$ for finite type Artin stacks with affine stabilizers X with the following relations:

(1) $[X \sqcup Y] = [X] + [Y]$;

(2) If $f\colon X \to Y$ is a geometric bijection, then $[X] = [Y]$;

(3) If $X_1, X_2 \to Y$ are Zariski fibrations with the same fibers, then $[X] = [Y]$.

For any Artin stack S locally of finite type with affine stabilizers, then there is a relative version $\mathrm{K}(\mathtt{St}_S)$, which is a module over $\mathrm{K}(\mathtt{St}_{\mathbb{C}})$. For our purposes, we will take $\mathtt{C} = \mathtt{Coh}^\flat(\mathcal{X})$ and let $S = \underline{\mathtt{C}}$.

Proposition 4.5. *There exists an Artin stack $\underline{\mathtt{C}}^{(2)}$ locally of finite type parameterizing short exact sequences in $\mathtt{C}$. This stack is equipped with maps*

$$\pi_i\colon \underline{\mathtt{C}}^{(2)} \to \underline{\mathtt{C}} \qquad [0 \to E_1 \to E_2 \to E_3 \to 0] \mapsto E_i$$

for $i = 1, 2, 3$. Then the morphism $(\pi_1, \pi_3)\colon \underline{\mathtt{C}}^{(2)} \to \underline{\mathtt{C}}$ is of finite type.

Now, for $[X_1 \to \underline{\mathtt{C}}], [X_2 \to \underline{\mathtt{C}}] \in \mathrm{K}(\mathtt{St}_{\underline{\mathtt{C}}})$, define the stack $X_1 * X_2$ as the fiber product

$$\begin{tikzcd}
X_1 * X_2 \arrow[r] \arrow[d] & \underline{\mathtt{C}}^{(2)} \arrow[r, "\pi_2"] \arrow[d, "(\pi_1,\pi_3)"] & \underline{\mathtt{C}} \\
X_1 \times X_2 \arrow[r] & \underline{\mathtt{C}} \times \underline{\mathtt{C}}.
\end{tikzcd}$$

[3]Given our assumptions, all F_i are isomorphic.

This defines an associative product on $\mathtt{K}(\mathtt{St}_{\underline{\mathtt{C}}})$, and we define the *motivic Hall algebra* $\mathrm{H}(\mathtt{C}) \coloneqq (\mathtt{K}(\mathtt{St}_{\underline{\mathtt{C}}}), *)$.

There is an inclusion $\mathtt{K}(\mathtt{Var}_{\mathbb{C}})[\mathbb{L}^{-1}, (1+\cdots+\mathrm{L}^{n})^{-1} \mid n \geqslant 1] \to \mathtt{K}(\mathtt{St}_{\mathbb{C}})$. Then we define the subalgebra of *regular elements* $\mathrm{H}_{\mathrm{reg}}(\mathtt{C})$ to be the $\mathtt{K}(\mathtt{Var}_{\mathbb{C}})[\mathbb{L}^{-1}, [\mathbb{P}^{n}]^{-1} \mid n \geqslant 1]$-submodule generated by schemes $[Z \to \underline{\mathtt{C}}]$.

Proposition 4.6. *$\mathrm{H}_{\mathrm{reg}}(\mathtt{C})$ is closed under $*$ and the quotiennt $\mathrm{H}_{\mathrm{reg}}(\mathtt{C})/(\mathbb{L}-1)\mathrm{H}_{\mathrm{reg}}(\mathtt{C})$ is commutative with Poisson bracket given by $\{f, g\} = \frac{f*g-g*f}{\mathbb{L}-1}$. This is called the semi-classical Hall algebra.*

Let $\mathbb{Q}[\mathrm{N}(\mathcal{X})]$ with product $t^{\alpha_1} \star t^{\alpha_2} = (-1)^{\chi(\alpha_1,\alpha_2)} t^{\alpha_1+\alpha_2}$ be the *Poisson torus*. The Poisson bracket is defined by $\left\{t^{\alpha}, t^{\beta}\right\} = (-1)^{\chi(\alpha,\beta)}\chi(\alpha,\beta)t^{\alpha+\beta}$. Then there is a Poisson algebra homomorphism

$$\mathrm{I}\colon \mathrm{H}_{\mathrm{sc}}(\mathtt{C}) \to \mathbb{Q}[\mathrm{N}(\mathcal{X})] \qquad [s\colon Y \to \mathtt{C}_{\alpha} \hookrightarrow \mathtt{C}] = e(Y, s^{-1}\nu),$$

where $\nu\colon \mathtt{C} \to \mathbb{Z}$ is the Behrend function.

We will not be considering the motivic Hall algebra but rather a completed version $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ which is analogous to the completion from Laurent polynomials to Laurent series. There is an algebra $\mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})$ of regular elements and a semi-classical algebra $\mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C})$, which comes with an integration morphism

$$\mathrm{I}\colon \mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C}) \to \mathbb{Q}\{\mathrm{N}(\mathcal{X})\} = \left\{ \sum_{\alpha \in \mathrm{N}(\mathcal{X})} n_{\alpha} t^{\alpha} \right\}.$$

5. Wall-crossing and the DT/PT correspondence

Definition 5.1. A torsion pair $(\mathtt{T}, \mathtt{F})$ on $\mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ is called *numerical* if whenever $[T \in \mathtt{T}] = [F \in \mathtt{F}]$ in $\mathrm{N}(\mathcal{X})$, then $T = F = 0$.

All of the torsion pairs we will consider are numerical.

Proposition 5.2. *Let $(\mathtt{T}, \mathtt{F})$ be an open numerical torsion pair and suppose that $\mathcal{M} \subset \underline{\mathtt{Pair}}(\mathtt{T}, \mathtt{F})$ is an open and finite type substack. Then $(\mathbb{L}-1)[\mathcal{M}] \in \mathrm{H}_{\mathrm{reg}}(\mathtt{C})$.*

Lemma 5.3. *Assume that $\mathtt{Pair}(\mathtt{T}_-, \mathtt{F}_-), \mathtt{Pair}(\mathtt{T}_+, \mathtt{F}_+), \mathtt{W}$ define elements of $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ and let $\alpha \in \mathrm{N}(\mathtt{A})$. The following are equivalent:*

1. *The stack $\underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_-)_{\alpha}$ is of finite type.*
2. *The stack $(\underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_+) * \underline{\mathtt{W}})_{\alpha}$ is of finite type.*
3. *The stack $(\underline{\mathtt{W}} * \underline{\mathtt{Pair}}(\mathtt{T}_-, \mathtt{F}_-))_{\alpha}$ is of finite type.*

Definition 5.4. A full subcategory $\mathtt{W} \subset \mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ is *log-able* if

1. $\mathtt{W}$ is closed under direct sums and summands;
2. $\mathtt{W}$ defines an element of $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$;
3. If $\alpha \in \mathrm{N}(\mathcal{X})$, there are only finitely many ways to write $\alpha = \alpha_1 + \cdots + \alpha_n$, where each α_i represents a nonzero element in $\mathtt{W}$.

Theorem 5.5 (No-poles)**.** *If* $\mathtt{W}$ *is log-able, then*

$$(\mathbb{L}-1)\log[\underline{\mathtt{W}}] \in \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C}).$$

We are finally ready to do wall-crossing, but we need one more definition.

Definition 5.6. Let $(\mathtt{T}_\pm, \mathtt{F}_\pm)$ be open torsion pairs on $\mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ such that $\mathtt{T}_+ \subset \mathtt{T}_-$. Let $\mathtt{W} = \mathtt{T}_- \cap \mathtt{F}_+$. These torsion pairs are *wall-crossing material* if $\mathtt{W}$ is log-able and the categories $\mathtt{Pair}(\mathtt{T}_+, \mathtt{F}_-), \mathtt{Pair}(\mathtt{T}_-, \mathtt{F}_+), \mathtt{Pair}(\mathtt{T}_+, \mathtt{F}_-)$ define elements of $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$.

Theorem 5.7. *Suppose* $(\mathtt{T}_\pm, \mathtt{F}_\pm)$ *are open torsion pairs with* $\mathtt{T}_+ \subset \mathtt{T}_-$ *which are wall-crossing material. Then* $w \coloneqq \mathrm{I}((\mathbb{L}-1)\log[\underline{\mathtt{W}}])$ *is well-defined and*

$$\mathrm{I}((\mathbb{L}-1)[\underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_+)]) = \exp(\{w, -\})\mathrm{I}((\mathbb{L}-1)[\underline{\mathtt{Pair}}(\mathtt{T}_-, \mathtt{F}_-)]).$$

Proof of DT/PT correspondence. Let $\mathtt{T}_{\mathrm{DT}} = 0, \mathtt{F}_{\mathrm{DT}} = \mathtt{Coh}_{\leqslant 1}(\mathcal{X}), \mathtt{W} = \mathtt{Coh}_0(\mathtt{X})$. Then the torsion pairs $(\mathtt{T}_{\mathrm{DT}}, \mathtt{F}_{\mathrm{DT}}), (\mathtt{T}_{\mathrm{PT}}, \mathtt{F}_{\mathrm{PT}})$ are wall-crossing material, so we can apply the previous theorem. Isolating the terms with class β and noting that $(\mathtt{T}_{\mathrm{DT}}, \mathtt{F}_{\mathrm{DT}})$-pairs are ideal sheaves $\mathrm{I}[1]$, we obtain

$$\mathrm{DT}(\mathcal{X})_\beta z^\beta t^{-[\mathcal{O}_\mathcal{X}]} = \exp(\{w, -\})\mathrm{PT}(\mathcal{X})_\beta z^\beta t^{-[\mathcal{O}_\mathcal{X}]}.$$

Now let $c \in \mathrm{N}_0(\mathcal{X})$. Applying the McKay equivalence, we have $\Psi(c) \in \mathrm{N}_{\leqslant 1}(\mathrm{Y})$. Because β is multi-regular, $\Psi(\beta, c') \in \mathrm{N}_{\leqslant 1}(\mathrm{Y})$ for all $c' \in \mathrm{N}_0(\mathcal{X})$. But then the Euler pairing is trivial on $\mathrm{N}_{\leqslant 1}(\mathrm{Y})$, so

$$\chi(c, (\beta, c')) = \chi(\Psi(c), \Psi(b, c')) = 0.$$

Next, write $w = \sum_{c \in \mathrm{N}_0(\mathcal{X})} w_c q^c$. We then have $\left\{w, z^\beta q^{c'}\right\} = 0$, so

$$\exp(\{w, -\})\mathrm{PT}(\mathcal{X})_\beta z^\beta t^{-[\mathcal{O}_\mathcal{X}]} = \mathsf{PT}(\mathcal{X})_\beta z^\beta \exp(\{w, -\})t^{-[\mathcal{O}_\mathcal{X}]}.$$

Combining this with the first DT-PT identity, we have

$$\frac{\mathrm{DT}(\mathcal{X})_\beta}{\mathrm{PT}(\mathcal{X})_\beta} = t^{[\mathcal{O}_\mathcal{X}]} \exp(\{w, -\})t^{-[\mathcal{O}_\mathcal{X}]}.$$

Finally, noting that $\mathrm{PT}(\mathcal{X})_0 = 1$ because $\mathcal{O}_\mathcal{X}[1]$ is the only stable pair with $\beta = 0$, we obtain

$$\frac{\mathrm{DT}(\mathcal{X})_\beta}{\mathrm{PT}(\mathcal{X})_\beta} = \frac{\mathrm{DT}(\mathcal{X})_0}{\mathrm{PT}(\mathcal{X})_0} = \mathrm{DT}(\mathcal{X})_0. \qquad \square$$