THE DT CREPANT RESOLUTION CONJECTURE

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Abstract. We will prove the DT crepant transformation conjecture by crossing infinitely many walls in a finite amount of time.

1. Brief review

Recall that $\mathcal{X}$ is a projective Calabi-Yau 3-orbifold (where we require $H^1(\mathcal{O}_{\mathcal{X}} = 0)$), that X is the coarse moduli space (which is a projective, Gorenstein, Calabi-Yau variety with at worst quotient singularities), and Y was a distinguished crepant resolution of X. Also recall that we have derived equivalences $\Phi\colon D(Y) \leftrightarrow D(\mathcal{X})\colon \Psi$.

Let $\mathtt{C}$ be the category $\mathtt{Coh}(\mathcal{X})$ tilted at the torsion pair $(\mathtt{Coh}_{\leqslant 1}(\mathcal{X}), \mathtt{Coh}_{\geqslant 2}(\mathcal{X}))$. We consider the graded motivic Hall algebra $H_{\mathrm{gr}}(\mathtt{C})$, which is a module over $K(\mathtt{St}_{\mathbb{C}})$. Also, recall the category

$$\mathtt{A} = \left\langle \mathcal{O}_{\mathcal{X}}[1], \mathtt{Coh}_{\leqslant 1}(\mathcal{X}) \right\rangle.$$

Finally, recall the integration map $I\colon H_{\mathrm{gr,sc}}(\mathtt{C}) \to \left\{ \sum_{\alpha \in N(\mathcal{X})} n_\alpha q^\alpha \right\}$, where $H_{\mathrm{sc}}(\mathtt{C})$ is a quotient of the algebra

$$H_{\mathrm{reg}}(\mathtt{C}) = K(\mathtt{Var}_{\mathbb{C}})[\mathbb{L}^{-1}][[\mathbb{P}^n]^{-1} \mid n \geqslant 1] \cdot \{\text{schemes}\} \subset H(\mathtt{C}).$$

of regular elements.

2. Stability conditions

Fix an ample class $\omega \in N^1(Y)$ and an ample line bundle A on X.

Definition 2.1. A *stability condition* on $\mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ consists of a slope function $\mu\colon N_{\leqslant 1}(\mathcal{X}) \to S$ to a totally ordered set $(S, <)$ such that if

$$0 \to A \to B \to C \to 0,$$

then either $\mu(A) > \mu(B) > \mu(C)$ or $\mu(A) < \mu(B) < \mu(C)$ or $\mu(A) = \mu(B) = \mu(C)$ and every $F \in \mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ has a Harder-Narasimhan filtration $0 = F_0 \subset \cdots \subset F_n = F$.

We will now define a number of stability conditions. First, we fix a "generating" vector bundle V (where every coherent sheaf on $\mathcal{X}$) is locally a quotient of $V^{\oplus n}$ for some n. We can assume that $V = V^\vee$ (by taking $V \oplus V^\vee$). Now we define a modified Hilbert polynomial for a sheaf F by

$$p_F(k) = \chi(\mathcal{X}, V^\vee \otimes F(k)) = \ell(F)k + \deg(F).$$

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Definition 2.2. Define the *Nironi slope* of F to be

$$\nu(\mathsf{F}) \coloneqq \frac{\deg \mathsf{F}}{\ell(\mathsf{F})}$$

if $\mathsf{F} \notin \mathtt{Coh}_0(\mathcal{X})$ and $\nu(\mathsf{F}) = \infty$ otherwise. Also write $\nu_+(\mathsf{F}), \nu_-(\mathsf{F})$ for the slopes of the Harder-Narasimhan factor of F with largest (resp. smallest) slope.

Definition 2.3. Define the stability condition ζ on $N_1^{\mathrm{eff}}(\mathcal{X}) \setminus 0$ by

$$\zeta(\beta, c) = \left(-\frac{\deg_Y(\mathrm{ch}_2(\Psi(A\cdot\beta))\cdot\omega)}{\deg(A\cdot\beta)}, \nu(\beta, c)\right) \in (-\infty, \infty]^2$$

for $\beta = 0$ and $\zeta(0, c) = (\infty, \infty)$. Here, we use the lexicographical ordering on $(-\infty, \infty]$.

For a stability condition μ and $s \in S$, define a torsion pair by

$$\mathtt{T}_{\mu,s} \coloneqq \left\{\mathsf{T} \in \mathtt{Coh}_{\leqslant 1}(\mathcal{X}) \mid \mathsf{T} \twoheadrightarrow \mathsf{Q} \neq 0 \implies \mu(\mathsf{Q}) \geqslant s\right\};$$
$$\mathtt{F}_{\mu,s} \coloneqq \left\{\mathsf{F} \in \mathtt{Coh}_{\leqslant 1}(\mathcal{X}) \mid 0 \neq \mathsf{H} \hookrightarrow \mathsf{F} \implies \mu(\mathsf{H}) < s\right\}.$$

Also, call the category of $(\mathtt{T}_{\mu,s}, \mathtt{F}_{\mu,s})$-pairs $\mathtt{P}_{\mu,s}$. Finally, define the category of semistable sheaves of slope s by $\mathcal{M}_\mu^{\mathrm{ss}}(s)$.

In order to control DT-like invariants coming from stability conditions, we need our categories of semistable objects and of pairs to satisfy openness and boundedness properties.

Proposition 2.4.

(1) *For any $\delta \in \mathbb{R}$, the torsion pair $(\mathtt{T}_{\nu,\delta}, \mathtt{F}_{\nu,\delta})$ is open.*

(2) *For any $(\gamma, \eta) \in \mathbb{R}_{>0} \times \mathbb{R}$, the torsion pair $(\mathtt{T}_{\zeta,(\gamma,\eta)}, \mathtt{F}_{\zeta,(\gamma,\eta)})$ is open. In addition, the moduli stack $\underline{\mathcal{M}}_\zeta^{\mathrm{ss}}(a, b) \subset \underline{\mathtt{Coh}}_{\leqslant 1}(\mathcal{X})$ is open for any $(a, b) \in \mathbb{R}^2$.*

We will now discuss boundedness. For any real number $\gamma > 0$, define the function

$$L_\gamma \colon N_0(\mathcal{X}) \to \mathbb{R} \qquad c \mapsto \deg(c) + \gamma^{-1}\deg_Y(\mathrm{ch}_2(\Psi(c))\cdot\omega).$$

This will control the series expansion of the rational function $f_\beta(q)$, where $\mathrm{PT}(\mathcal{X})_\beta$ is the expansion of $f_\beta(q)$ in $\mathbb{Q}[N_0(\mathcal{X})]_{\deg}$ (this means roughly that degree is bounded below).

Definition 2.5. Let $S \subset N_0(\mathcal{X})$ and $L \colon N_0(\mathcal{X}) \to \mathbb{R}$ be a homomorphism. Then S is L-*bounded* if the set

$$S \cap \{c \in N_0(\mathcal{X}) \mid L(c) \leqslant M\}$$

is finite for every $M \in \mathbb{R}$. We also say that S is *weakly* L-*bounded* if the set

$$(S/\ker L) \cap \{c \in N_0(\mathcal{X})/L \mid L(c) \leqslant M\}$$

is finite for every $M \in \mathbb{R}$.

The main results about semistable sheaves and pairs are the following. Recall that a category $\mathtt{W}$ is log-able if $(\mathbb{L} - 1)\log[\mathtt{W}] \in H_{\mathrm{gr,reg}}(\mathtt{C})$.

Proposition 2.6. *Let $(a,b) \in \mathbb{R}^2$. The set*

$$\{c \in N_0(\mathcal{X}) \mid \underline{\mathcal{M}}^{ss}(a,b) \neq \emptyset\}$$

is L_γ-bounded. Moreover, the category $\mathcal{M}^{ss}_\gamma(a,b)$ is log-able.

Proposition 2.7. *For any $(\gamma,\eta) \in \mathbb{R}_{>0} \times \mathbb{R}$, the set*

$$\left\{c \in N_0(\mathcal{X}) \mid \mathtt{P}_{\zeta,(\gamma,\eta)}(\beta,c) \neq \emptyset\right\}$$

is L_γ-bounded. Moreover, the stack $\underline{\mathtt{P}}_{\zeta,(\gamma,\eta)}(\beta,c)$ is of finite type.

Corollary 2.8. *The category $\mathtt{P}_{\zeta,(\gamma,\eta)}$ defines an element of $H_{gr}(\mathtt{C})$.*

Finally, we will locate regions in which the notion of a $(\mathtt{T}_{\zeta,(\gamma,\eta)}, \mathtt{F}_{\zeta,(\gamma,\eta)})$-pair is constant.

Lemma 2.9. *Let $\beta \in N_1(\mathcal{X})$. $\mathtt{T}_{\gamma,(\gamma,\eta)} \cap \mathtt{Coh}_{\leqslant 1}(\mathcal{X})_{\leqslant\beta}$ and $\mathtt{F}_{\zeta,(\gamma,\eta)} \cap \mathtt{Coh}_{\leqslant 1}(\mathcal{X})_{\leqslant\beta}$ are constant on the components of $(\mathbb{R}_{>0} \times \mathbb{R}) \setminus (V_\beta \times \mathbb{R})$, where*

$$V_\beta = \left\{-\frac{\deg_Y(\mathrm{ch}_2(\Psi(A\cdot\beta')) \cdot \omega)}{\deg(A\cdot\beta')} \mid 0 < \beta' \leqslant \beta\right\} \cap \mathbb{R}_{>0}.$$

For $\gamma \in V_\beta$, the categories are locally constant on $\{\gamma\} \times \mathbb{R} \setminus W_\beta$, where $W_\beta = \frac{1}{\ell(\beta)!}\mathbb{Z}$.

3. DT-like invariants

We define DT-like invariants counting objects in the categories that we have defined. Once we do this, we will cross our infinitely many walls.

Recall that $\mathcal{M}^{ss}_\zeta(a,b)$ is log-able for any $(a,b) \in \mathbb{R}^2$. Therefore, we have an element

$$\eta_{\zeta,(a,b)} = (\mathbb{L}-1)\log[\underline{\mathcal{M}}^{ss}_\zeta(a,b)] \in H_{gr,reg}(\mathtt{C}).$$

Therefore, we can define DT-type (Joyce-Song) invariants by

$$\sum_{\zeta(\beta,c)=(a,b)} J^\zeta_{(\beta,c)} z^\beta q^c =: I\Big(\eta_{\zeta,(a,b)}\Big).$$

Now let $(\gamma,\eta) \in \mathbb{R}_{>0} \times \mathbb{R}$ be away from the walls. By a result of Abramovich-Corti-Vistoli, there is an element $(\mathbb{L}-1)[\underline{\mathtt{P}}_{\zeta,(\gamma,\eta)}(\beta,c)] \in H_{reg}(\mathtt{C})$. Then we can define DT-type invariants by

$$\mathrm{DT}^{\zeta,(\gamma,\eta)}_{(\beta,c)} z^\beta q^c t^{-[\mathcal{O}_\mathcal{X}]} \coloneqq I((\mathbb{L}-1)[\underline{\mathtt{P}}_{\zeta,(\gamma,\eta)}(\beta,c)]).$$

Finally, we can form generating series

$$\mathrm{DT}^{\zeta,(\gamma,\eta)}_\beta \coloneqq \sum_{c \in N_0(\mathcal{X})} \mathrm{DT}^{\zeta,(\gamma,\eta)}_{(\beta,c)} q^c \in \mathbb{Z}[N_0(\mathcal{X})]_{L_\gamma};$$

$$J^\zeta(a,b)_\beta \coloneqq \sum_{\substack{c \in N_0(\mathcal{X}) \\ \zeta(\beta,c)=(a,b)}} J^\zeta_{(\beta,c)} q^c \in \mathbb{Q}[N_0(\mathcal{X})]_{L_\gamma}.$$

Lemma 3.1. *Let $\beta \in N_1(\mathcal{X})$ and let $\gamma > \gamma'$ for all $\gamma' \in V_\beta$. An object $\mathtt{E} \in \mathtt{A}$ of class $(-1,\beta,c)$ is a $(\mathtt{T}_{\zeta,(\gamma,\eta)}, \mathtt{F}_{\zeta,(\gamma,\eta)})$-pair if and only if it is a PT stable pair.*

The following is a technical lemma whose proof requires the Hard Lefschetz condition.

Lemma 3.2. *For every $\gamma \in V_\beta$, there is a unique class $\beta_\gamma \in N_1(\mathcal{X})$ with $0 < \beta_\gamma \leqslant \beta$ such that $L_\gamma(A \cdot \beta_\gamma) = 0$. Class the class $c_\gamma \coloneqq A \cdot \beta_\gamma \in N_0(\mathcal{X})$.*

Now we will discuss wall-crossing. Once we cross all of the walls in V_β, we will have Bryan-Steinberg invariants, which we will define later. First, we need to understand what happens when we reach a wall $\gamma \in V_\beta$. Note that the set

$$S = \bigcup_{\beta' \leqslant \beta} \left\{ (\beta', c) \mid L_\gamma(c) \leqslant x, \underline{P}_{\zeta,(\gamma,\eta)}(\beta', c) \neq \emptyset \right\}$$

is finite for any $x \in \mathbb{R}$, so we can define

$$M^+_{\beta,\gamma,x} = \max_{(\beta',c) \in S} \deg(\beta', c) \qquad M^-_{\beta,\gamma,x} \coloneqq \min_{(\beta',c) \in S} \deg(\beta', c).$$

Lemma 3.3. *Let $\gamma \in V_\beta$, $E \in \mathtt{A}$ be of class $(-1, \beta, c)$, and let*

$$\eta^+_{\gamma,(\beta,c)} \coloneqq \max \left\{ 0, \deg(\beta, c) - M^-_{\beta,\gamma,L_\gamma(c)-K_\gamma} \right\};$$
$$\eta^-_{\gamma,(\beta,c)} \coloneqq \min \left\{ 0, \deg(\beta, c) - M^+_{\beta,\gamma,L_\gamma(c)-K_\gamma} \right\};$$

If $\eta > \eta^+_{\gamma,(\beta,c)}$, then E is a $(T_{\zeta,(\gamma,\eta)}, F_{\zeta,(\gamma,\eta)})$-pair iff E is a $(T_{\zeta,(\gamma+\varepsilon,\eta)}, F_{\zeta,(\gamma+\varepsilon,\eta)})$-pair. If $\eta < \eta^-_{\gamma,(\beta,c)}$, then E is a $(T_{\zeta,(\gamma,\eta)}, F_{\zeta,(\gamma,\eta)})$-pair iff E is a $(T_{\zeta,(\gamma-\varepsilon,\eta)}, F_{\zeta,(\gamma-\varepsilon,\eta)})$-pair.

This tells us that on each wall $\gamma \in V_\beta$, we can enter $\{\gamma\} \times \mathbb{R}$ at ∞ from the right and then leave the wall to the left at $-\infty$. Now we need to understand what happens at the walls W_β as we move from ∞ to $-\infty$.

Proposition 3.4. *Let $\beta \in N_1(\mathcal{X})$, $\gamma \in V_\beta$, and $\eta \in W_\beta$. Then*

$$\mathrm{DT}^{\zeta,(\gamma,\eta+\varepsilon)}_{\leqslant\beta} t^{-[\mathcal{O}_\mathcal{X}]} = \exp\left(\left\{ J^\zeta(\gamma,\eta)_{\leqslant\beta}, - \right\} \right) \mathrm{DT}^{\zeta,(\gamma,\eta-\varepsilon)}_{\leqslant\beta} t^{-[\mathcal{O}_\mathcal{X}]} \in \mathbb{Q}[N_1^{\mathrm{eff}}(\mathcal{X})]_{\leqslant\beta}.$$

Now define the series

$$\mathrm{DT}^{\zeta,(\gamma,\eta)}_{(\beta,c_0+\mathbb{Z}c_\gamma)} \coloneqq \sum_{k\in\mathbb{Z}} \mathrm{DT}^{\zeta,(\gamma,\eta)}_{(\beta,c_0+kc_\gamma)} q^{c_0+kc_\gamma}.$$

Lemma 3.5. *Let $\beta \in N_1(\mathcal{X})$, $c_0 \in N_0(\mathcal{X})$, $\gamma \in V_\beta$, and $\eta_0 \leqslant -\ell(\beta)$. Then*

$$\mathrm{DT}^{\zeta,(\gamma,\eta_0)}_{(\beta,c_0+\mathbb{Z}c_\gamma)} - \mathrm{DT}^{\zeta,(\gamma,\infty)}_{(\beta,c_0+\mathbb{Z}c_\gamma)}$$

is a rational function of degree less than $\deg(\beta, 0) + M^+_{\beta,\gamma,L_\gamma(c_0)} + n_0\ell(\beta) + \ell(\beta)^2$.

Taking $n_0 \to -\infty$, we obtain

Corollary 3.6. *Let β, c_0, γ be as above. Then $\mathrm{DT}^{\zeta,(\gamma,\infty)}_{(\beta,c_0+\mathbb{Z}c_\gamma)}$ and $\mathrm{DT}^{\zeta,(\gamma,-\infty)}_{(\beta,c_0+\mathbb{Z}c_\gamma)}$ are equal as rational functions.*

Theorem 3.7. *Let* $\beta \in N_1(\mathcal{X})$, $\gamma \in R_{>0} \setminus V_\beta$, $\eta \in \mathbb{R}$. *Then* $\mathrm{DT}_\beta^{\zeta,(\gamma,\eta)}$ *is the expansion of* $f_\beta(q)$ *in* $\mathbb{Z}[N_0(\mathcal{X})]_{L_\gamma}$.

4. Bryan-Steinberg invariants

In order to have a crepant resolution conjecture, we need some kind of enumerative invariants on Y. Define

$$\mathtt{T}_f = \left\{ F \in \mathtt{Coh}_{\leqslant 1}(Y) \mid Rf_* F \in \mathtt{Coh}_0(X) \right\}.$$

Then define $\mathtt{F}_f = \mathtt{T}_f^{\perp}$. We can define a Bryan-Steinberg pair (F, s) as a $(\mathtt{T}_f, \mathtt{F}_f)$-pair in $\mathtt{A}_Y$. Equivalently, we have

Definition 4.1. A *Bryan-Steinberg pair* (F, s) consists of $F \in \mathtt{Coh}_{\leqslant 1}(Y)$ and $s \in H^0(Y, F)$ such that $Rf_* \operatorname{coker}(s) \in \mathtt{Coh}_0(X)$ and F admits no maps from elements of $\mathtt{T}_f$.

For a class $(\beta, n) \in N_1(Y) \oplus \mathbb{Z}$, let $\underline{P}_{BS}(\beta, n)$ be the moduli stack of Bryan-Steinberg pairs of class $(-1, \beta, n) \in \mathbb{Z} \oplus N_{\leqslant 1}(Y)$. Then we can define the BS invariant $\mathrm{BS}(Y/X)_{(\beta,n)}$ via the Behrend function.

Before we continue, we will say a little but more about the McKay correspondence. Define the category $\mathtt{Per}(Y/X)$ to be the category of complexes $E \in D(Y)$ such that $Rf_*(E) \in \mathtt{Coh}(X)$ and such that for any $F \in \mathtt{Coh}(Y)$ with $Rf_* F = 0$, $\mathrm{Hom}(F[1], E) = 0$ (called *perverse coherent sheaves*).

Proposition 4.2. *The equivalence* $\Phi \colon D(Y) \to D(\mathcal{X})$ *restricts to an equivalence of abelian categories* $\mathtt{Per}(Y/X) \simeq \mathtt{Coh}(\mathcal{X})$.

We now need to relate Bryan-Steinberg pairs to objects living on $\mathcal{X}$. We first define a new torsion pair.

Definition 4.3. Let $\mathtt{T}_{\zeta,0} \subset \mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ denote the subcategory of sheaves T such that if $T \twoheadrightarrow Q$, then either $Q \in \mathtt{Coh}_0(\mathcal{X})$ or

$$\deg_Y(\mathrm{ch}_2(\Psi(Q \cdot A)) \cdot \omega) < 0.$$

Let $\mathtt{F}_{\zeta,0} \subset \mathtt{Coh}_{\leqslant 1}(\mathcal{X})$ be the full subcategory on sheaves F such that if $S \hookrightarrow F$, then S has pure dimension 1 and $\deg_Y(\mathrm{ch}_2(\Psi(S \cdot A)) \cdot \omega) \geqslant 0$.

The following result justifies the inclusion of $\zeta, 0$ in the subscript.

Lemma 4.4. *Let* $\beta \in N_1(\mathcal{X})$. *If* $0 < \gamma < \min_{\gamma' \in V_\beta} \gamma'$, *then for any* $\eta \in \mathbb{R}$ *an object* $E \in \mathtt{A}$ *of class* $(-1, \beta, c)$ *is a* $(\mathtt{T}_{\zeta,0}, \mathtt{F}_{\zeta,0})$*-pair if and only if it is a* $(\mathtt{T}_{\zeta,(\gamma,\eta)}, \mathtt{F}_{\zeta,(\gamma,\eta)})$*-pair.*

We should think of $(\mathtt{T}_{\zeta,0}, \mathtt{F}_{\zeta,0})$ as being the limit of $(\mathtt{T}_{\zeta,(\gamma,\eta)}, \mathtt{F}_{\zeta,(\gamma,\eta)})$-pairs as $\gamma \to 0$. Finally, we relate $(\mathtt{T}_{\zeta,0}, \mathtt{F}_{\zeta,0})$-pairs to $(\mathtt{T}_f, \mathtt{F}_f)$-pairs, and as a corollary, we can relate enumerative invariants on $\mathcal{X}$ to enumerative invariants on Y.

Lemma 4.5. *We have the following:*

$$\begin{aligned} \mathtt{T}_{\mathtt{f}} &= \Psi(\mathtt{Coh}_0(\mathcal{X})) \cap \mathtt{Coh}(Y); \\ \mathtt{T}_{\zeta,0} &= \left\langle \Phi(\mathtt{Per}_{\leqslant 1}(Y/X) \cap \mathtt{Coh}(Y)[1]), \Phi(\mathtt{T}_{\mathtt{f}}) \right\rangle_{\mathrm{ex}}; \\ \mathtt{F}_{\zeta,0} &= \Phi\left(\mathtt{Per}_{\leqslant 1}(Y/X) \cap \mathtt{Coh}(Y) \cap \mathtt{T}_{\mathtt{f}}^{\perp}\right). \end{aligned}$$

These give us the following:

Lemma 4.6. *If E is a $(\mathtt{T}_{\zeta,0}, \mathtt{F}_{\zeta,0})$-pair with $\beta_E \in N_{1,\mathrm{mr}}(\mathcal{X})$, then $\Psi(E)$ is an f-stable pair. On the other hand, if $E = (\mathcal{O}_Y \to F)$ is an f-stable pair, then ΦE is a $(\mathtt{T}_{\zeta,0}, \mathtt{F}_{\zeta,0})$-pair.*

This implies that $\underline{P}_{\mathrm{BS}}(\beta, n) \cong \underline{P}_{\zeta,(\gamma,n)}(\Phi(\beta, n))$ for $0 < \gamma < \min_{\gamma' \in V_\beta} \gamma'$, so we have proven

Theorem 4.7 (Crepant resolution conjecture)**.** *There exists a unique rational function $f_\beta \in \mathbb{Q}(N_0(\mathcal{X}))$ such that*

(1) The Laurent expansion of f_β with respect to deg *is the series $\mathrm{PT}(\mathcal{X})_\beta$;*

(2) The Laurent expansion of f_β with respect to L_γ for $0 < \gamma < \min_{\gamma' \in V_\beta} \gamma'$ is the series $\mathrm{BS}(Y/X)_\beta$.

Using results of Bryan-Steinberg and of the previous lecture, we have

Corollary 4.8 (Crepant resolution conjecture, original formulation)**.** *There is an equality of rational functions*

$$\frac{\mathrm{DT}(\mathcal{X})_\beta}{\mathrm{DT}(\mathcal{X})_0} = \frac{\mathrm{DT}(Y)_\beta}{\mathrm{DT}_{\mathrm{exc}}(Y)}.$$