THE HOMFLY POLYNOMIAL AND ENUMERATIVE GEOMETRY

PATRICK LEI

Abstract. We will begin by introducing Pandharipande-Thomas theory and computing the equivariant vertex. After that, we will sketch a proof of the Oblomkov-Shende conjecture following Maulik. In particular, we will give a proof of invariance of certain PT invariants under flops.

1. Introduction

Let $C \subset \mathbb{C}^2$ be a reduced curve and suppose $p \in C$ is an (isolated) singularity. Then define a constructible function $m \colon C_p^{[n]} \to \mathbb{N}$ given by $m([Z])$ being the number of generators of the ideal $I_{Z,p} \subset \mathcal{O}_{C,p}$. Now consider the generating function

$$Z_{C,p}(v,w) = \sum_{n \geqslant 0} s^{2n} \int_{C_p^{[n]}} (1-v^2)\, d\chi = \sum_{n \geqslant 0} s^{2n} \sum_k k\chi_{\mathrm{top}}(f^{-1}(k)).$$

Additionally, taking a small S^3 around $p \in \mathbb{C}^2$ and intersecting with C gives us a link $\mathcal{L}_{C,p}$. Recall the HOMFLY polynomial $P(\mathcal{L}; v, s) \in \mathbb{Z}[v^{\pm}, (s-s^{-1})^{\pm}]$. If μ is the Milnor number of the singularity (for example, the middle Betti number of the Milnor fiber), then the Oblomkov-Shende conjecture is

Theorem 1.1 (Maulik)**.**

$$P(\mathcal{L}_{C,p}; v, s) = \left(\frac{v}{w}\right)^{\mu-1} Z_{C,p}(v,s).$$

The proof of this result relates both sides of the equality to enumerative geometry, and in particular Pandharipande-Thomas, or stable pairs, curve counting. In this lecture, we will (attempt to) sketch a proof of the Oblomkov-Shende conjecture, but first we will introduce PT theory to familiarize ourselves with the objects involved in the proof.

Remark 1.2. Maulik proves everything for a colored version of the HOMFLY polynomial, but here we will only work with uncolored data, which corresponds to all partitions being (1).

2. Introduction to PT Theory

In this part, we are following the papers *Curve counting via stable pairs in the derived category* and *Stable pairs and BPS invariants* by Pandharipande and Thomas.

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Let X be a smooth threefold and $\beta \in H_2(X, \mathbb{Z})$. The *PT moduli space* $P_n(X, \beta)$ parameterizes two-term complexes

$$\mathcal{O}_X \xrightarrow{s} \mathcal{F},$$

where $\mathcal{F}$ is a pure 1-dimensional sheaf supported on a Cohen-Macaulay subcurve of X, s has 0-dimensional cokernel, $\chi(\mathcal{F}) = n$, and $[\operatorname{supp} F] = \beta$. The space $P_n(X, \beta)$ has a virtual fundamental class coming from the deformation theory of complexes in the derived category. Here, note that the deformation theory of $(\mathcal{F}, s)$ (really of the corresponding complex $\mathcal{I}^\bullet$) is given by

$$\operatorname{Ext}^0(\mathcal{I}^\bullet, \mathcal{F}) \to \operatorname{Ext}^1(\mathcal{I}^\bullet, \mathcal{F}) = \operatorname{Ext}^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \to \operatorname{Ext}^2(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0$$

and the virtual fundamental class lives in dimension

$$c_\beta \coloneqq \int_\beta c_1(X).$$

In particular, for a Calabi-Yau threefold, $c_\beta = 0$. Given this data, we may define the *PT invariant*

$$P_{n,\beta} \coloneqq \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1.$$

and the PT partition function

$$Z_{PT,\beta}(q) = \sum P_{n,\beta} q^n.$$

There is an alternative way to compute the PT invariants for X a projective Calabi-Yau threefold, due to Behrend (originally done for DT theory). In this case, the moduli space is actually a projective scheme, and if $P_n(X, \beta)$ is smooth everywhere, then

$$P_{n,\beta} = (-1)^{\dim P_n(X,\beta)} \chi_{\mathrm{top}}(P_n(X, \beta)).$$

Of course, moduli spaces are almost never smooth for nontrivial moduli problems, so instead, we have

Proposition 2.1 (Behrend)**.** *There exists an integer-valued constructible function χ_B such that*

$$P_{n,\beta} = \int_{P_n(X,\beta)} \chi_B \coloneqq \sum_{n \in \mathbb{Z}} n \chi_{\mathrm{top}}((\chi_B)^{-1}(n)).$$

3. A computation in PT theory

This computation comes from *The* 3*-fold vertex via stable pairs* by Pandharipande and Thomas.

We will now compute the T-equivariant PT vertex of $\mathbb{C}^3$ (this is the local model for toric varieties) to give us all a feel for this enumerative theory. First, we need to define some combinatorial data. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a triple of partitions. Then there exists a unique minimal T-fixed subscheme C_μ with outgoing partitions the μ_i (simply take the curves C_{μ_i} given by the three partitions and take the union), whose ideal we will denote $\mathcal{I}_\mu$. Then we define

$$M = \bigoplus_{i=1}^{3} (\mathcal{O}_{C_{\mu_i}})_{x_i} \eqqcolon \bigoplus_{i=1}^{3} M_i.$$

Every T-invariant pair $(\mathcal{F}, s)$ on $\mathbb{C}^3$ corresponds to a finitely-generated T-invariant submodule

$$Q \subset M/\langle(1,1,1)\rangle,$$

and we will now give a combinatorial description of such submodules.

For each μ^i, we may consider the module M_i as an infinite cylinder $\mathrm{Cyl}_i \subset \mathbb{Z}^3$ (extending in both directions). Then for every $w \in \mathbb{Z}^3$, consider the vectors $\mathbf{1}_w, \mathbf{2}_w, \mathbf{3}_w$ representing w in each copy of $\mathbb{Z}^3$ (for each of the M_i). Clearly x_1 shifts w by $(1,0,0)$ and similarly for x_2, x_3. We will now consider the decomposition of the union of the Cyl_i into the following types:

- Type I^+ are those which have only nonnegative coordinates and lie in exactly one cylinder;
- Type $\mathbb{II}$ (resp $\mathbb{III}$) are those which lie in exactly 2 (resp 3) cylinders;
- Type I^- are those with at least one negative coordinate.

Clearly $M/\langle(1,1,1)\rangle$ is supported on types $\mathbb{II}, \mathbb{III}, \mathrm{I}^-$, and now we have three cases:

- If $w \in \mathrm{I}^-$, then clearly $\mathbb{C} \cdot \mathbf{i}_w \subset M/\mathcal{O}_{C_\mu}$;
- If $w \in \mathbb{II}$, then $\frac{\mathbb{C}\cdot\mathbf{i}_w \oplus \mathbb{C}\cdot\mathbf{j}_w}{\mathbb{C}\cdot(\mathbf{i}_w+\mathbf{j}_w)} \cong \mathbb{C}$;
- If $w \in \mathbb{III}$, then $\frac{\mathbb{C}\cdot\mathbf{1}_w \oplus \mathbb{C}\cdot\mathbf{2}_w \oplus \mathbb{C}\cdot\mathbf{3}_w}{\mathbb{C}\cdot(\mathbf{1}_w+\mathbf{2}_w+\mathbf{3}_w)} \cong \mathbb{C}^2$.

In particular, we need to consider *labelled box configurations*, where type $\mathbb{III}$ boxes may be labeled by an element of $\mathbb{P}^1$ (where $\mathbb{C}^2$ is identified with the vector space above) or unlabeled (corresponding to the inclusion of the entire $\mathbb{C}^2$ in Q). Now we will denote by $\mathcal{Q}_\mu$ the set of components of the moduli space of T-invariant submodules of $M/\mathcal{O}_{C_\mu}$.

We are now able to define the equivariant vertex. Let $\ell(\mathcal{Q})$ be the number of boxes in the labelled configuration associated to $\mathcal{Q} \in \mathcal{Q}_\mu$. Then let $|\mu|$ denote the *renormalized volume* of the partition π corresponding to $\mathcal{I}_{C_\mu}$, which is defined as

$$|\pi| = \#\left\{\pi \cap [0,\ldots,N]^3\right\} - (N+1)\sum_{1}^{3}\left|\mu^i\right|,$$

which is independent of a sufficiently large $N \gg 0$.

We need to define a few characters of T, which we will need to define the vertex and compute our example. Let P be the Poincaré polynomial of a free resolution of the universal complex $\mathbb{I}$ on $\mathcal{Q} \times \mathbb{C}^3$. Denote by F the character of $\mathcal{F}$. In particular, we have

$$\mathsf{F} = \frac{1+\mathsf{P}}{(1-t_1)(1-t_2)(1-t_3)}.$$

Then define a vertex character V by

$$\mathsf{V} = \mathrm{tr}_{\mathcal{O}-\chi(\mathbb{I},\mathbb{I})} + \sum_{i=1}^{3}\frac{\mathsf{G}_{\alpha\beta_i}(t_i', t_i'')}{1-t_i},$$

where $G_{\alpha\beta}$ is a certain character defined by edge data. Now let

$$w(\mathcal{Q}) := \int_{\mathcal{Q}} e(T_{\mathcal{Q}}) \cdot e(-V) \in \mathbb{Q}[s_1, s_2, s_3]_{(s_1, s_2, s_3)} = (A_T^*)_{\mathrm{loc}}$$

be the contribution of V on $\mathcal{Q}$. Then the equivariant vertex is defined to be

$$W_\mu^P := \sum_{\mathcal{Q} \in \mathcal{Q}_\mu} w(\mathcal{Q}) q^{\ell(\mathcal{Q}) + |\mu|} \in \mathbb{Q}(s_1, s_2, s_3)((q)).$$

Example 3.1. For $\mu = ((1), \emptyset, \emptyset)$, we have

$$W_\mu^P = (1+q)^{\frac{s_2+s_3}{s_1}}.$$

To see this, note that $\mathcal{Q}_\mu = \mathbb{Z}_{>0}$, where k corresponds to the length k box configuration in the negative x_1-direction.

Now we simply compute that

$$F_{\mathcal{Q}_k} = \frac{t_1^{-k}}{1 - t_1}.$$

This implies that

$$V_{\mathcal{Q}_k} = \sum_{i=1}^{k} t_1^{-i} - \sum_{i=0}^{k-1} \frac{t_1^i}{t_2 t_3},$$

and therefore that

$$\begin{aligned} w(\mathcal{Q}_k) &= \int_{\mathcal{Q}_k} e(-V_{\mathcal{Q}^k}) \\ &= \frac{(-s_2 - s_3)(s_1 - s_2 - s_3) \cdots ((k-1)s_1 - s_2 - s_3)}{(-s_1)(-2s_1) \cdots (-ks_1)}, \end{aligned}$$

as desired.

4. Flop invariance of PT theory

We will now begin the proof of the Oblomkov-Shende conjecture. The first step is to understand what happens when we blow up C at p via flop invariance of PT partition functions.

Let Y be the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and Y_- be the threefold obtained via a flop of the zero section. The flop is some birational map $\phi \colon Y \dashrightarrow Y_-$. If $\pi \colon Y \to \mathbb{P}^1$ is the projection, choose an identification of $\mathbb{C}^2$ with $\pi^{-1}(0)$. Then the strict transform of $\pi^{-1}(0)$ with respect to ϕ is $\mathrm{Bl}_0 \mathbb{C}^2$ with exceptional fiber E_- which is the zero section of Y_- (isomorphic to Y). This implies that the strict transform of C with respect to ϕ is $\mathrm{Bl}_0 C$.

Definition 4.1. A stable pair $(\mathcal{F}, s)$ is C-*framed* if on $Y \setminus E$ if after restricting to $Y \setminus E$, we have an isomorphism $(\mathcal{F}, s) \simeq \mathcal{O}_Y \to \mathcal{O}_C$.

Given $r, n \in \mathbb{Z}$, we define the moduli space $P(Y, C, r, n)$ of C-framed stable pairs such that $\operatorname{supp} F$ has generic multiplicity r along E and for any projective compactification $\overline{Y}$ of Y, we have $\chi(\overline{\mathcal{F}}) = n + \chi(\mathcal{O}_{\overline{C}})$. This is a locally closed subscheme of the space of stable pairs on $\overline{Y}$ and is independent of the choice of $\overline{Y}$. Now we define the C-framed PT partition function

$$Z(Y, C; q, Q) \coloneqq \sum_{r,n} q^n Q^r \chi_{\mathrm{top}}(P(Y, C, r, n)).$$

Proposition 4.2. *Let $m_1, \ldots, m_r$ be the multiplicities of the branches of C at p. We have the flop identity*

$$Q^{\sum m_r} Z'(Y, C; q, Q^{-1}) = q^{\delta} Z'(Y_-, C'; q, Q),$$

where

$$Z'(Y, C; q, Q) \coloneqq \frac{Z(Y, C; q, Q)}{\prod_k (1 + q^k Q)^k}$$

is the normalized PT partition function and $\delta = \binom{\sum m_i}{2}$.

5. Algebraic links