Intersection Theory Learning Seminar Spring 2021

Notes by Patrick Lei

Lectures by Various

Columbia University

Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Acknowledgements

I would like to thank Nicolás Vilches for pointing out that I was missing the accent in his name.

Seminar Website: `http://www.math.columbia.edu/~plei/s21-INT.html`

Contents

1

Caleb (Jan 22): Rational Equivalence

The idea of intersection theory is to define a well-defined intersection product on a scheme or variety that behaves like the intersection form in ordinary homology of a manifold.

Example 1.0.1. Let $f, g \in \mathbb{A}^2_k$ be two plane curves. Then we can define the *intersection multiplicity* by

$$i(I, F \cdot G) = \dim_k(k[x,y]/(f,g)),$$

which is the dimension of the intersection scheme. However, this does not generalize, so we will need to use Serre's formula, which involves the Tor functor.

1.1 Chow Groups

We want to define zeroes and poles for a singular variety. If V is codimension 1 in X for a nonsingular variety X, then consider $r \in K(X)^\times$, where $K(X) = \operatorname{Frac}(\mathcal{O}_{V,X})$, where this local ring is a discrete valuation ring. Then we can write $(r) = \sum \operatorname{ord}_V(r)[V]$, which is a Weil divisor. For a curve, we can compute $\operatorname{ord}_V(r) = \dim_k(\mathcal{O}_{V,X}/(r))$. In general, we can write $\operatorname{ord}_V(r) = \operatorname{len}_A(A/(r))$, where $A = \mathcal{O}_V, X$.

Example 1.1.1. If $X = \mathbb{A}^3$ and $r = \frac{y^2(x-2)(xy+1)}{zy+x}$, we see that along $y = 0$, we have $A = k[x,y,z]_{(y)}$ and thus $\operatorname{lem}_A(A/r) = 2$.

Proposition 1.1.2. *The map* $\operatorname{ord}_V \colon R(X)^* \to \mathbb{Z}$ *is a homomorphism.*

Proposition 1.1.3. *For any* r*, there are finitely many* V *such that* $\operatorname{ord}_V(r) \neq 0$.

Examples 1.1.4.

1. Let f be irreducible and f, g be plane curves. Then $i(P, F \cdot G) = \operatorname{ord}_P(\overline{g})$.

2. Let $\widetilde{X} \to X, \widetilde{V} \to V$ be normalizations. Then $\operatorname{ord}_V(r) = \sum \operatorname{ord}_{\widetilde{V}}(r)[K(\widetilde{V}) : K(V)]$.

Now we will define rational equivalence. Let W be a $(k+1)$-dimensional subvariety of X and $r \in R(W)^\times$. Then we may obtain cycles of the form $[\operatorname{div}(r)]$, and such cycles form the group $\operatorname{Rat}_k X$.

Definition 1.1.5. The *Chow groups* of X are defined by $A_k(X) \coloneqq Z_k(X)/\operatorname{Rat}_k X$.

Example 1.1.6 (Not Mayer-Vietoris)**.** Let X_1, X_2 be closed subschemes of X. Then the sequence

$$A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \to A_k(X_1 \cup X_2) \to 0$$

is exact.

1.2 Pushforward of Cycles

Let $f\colon X \to Y$ be a morphism. We want to define a pushforward $A_k(X) \to A_k(Y)$. This would imply that $\mathrm{Rat}_k(X) \to \mathrm{Rat}_k(Y)$. This is true when f is proper. Then $V \to f(V) = W$ is a map of cycles because proper implies universally closed. We obtain a field $K(W) \subseteq K(V)$, which is finite if V, W are of equal dimension. Then we define

$$\deg(V/W) = \begin{cases} [K(V) : K(W)] & \dim V = \dim W \\ 0 & \dim V > \dim W. \end{cases}$$

Definition 1.2.1. The *pushforward* of a cycle V is $f_*[V] = \deg(V/W)[W]$.

Theorem 1.2.2. *Let f is proper. If $\alpha \sim 0$, then $f_*\alpha \sim 0$.*

Proposition 1.2.3. *Let f be proper and surjective and let $r \in K(X)^\times$. Then*

1. $f_*[\mathrm{div}(r)] = 0$ *if* $\dim Y < \dim X$*;*
2. $f_*[\mathrm{div}(r)] = \mathrm{div}(N(r))$ *if* $\dim Y = \dim X$.

Proof. Consider the map $\mathbb{P}^1 \to \operatorname{Spec} k$. Because the order map is a homomorphism, let $r \in k[t] \subset k(t)$ be an irreducible polynomial, where p has degree n. Then $\mathrm{div}(r) = [P] - n[\infty]$ and then we have

$$f_*(\mathrm{div}(r)) = n[\mathrm{pt}] - n[\mathrm{pt}] = 0.$$

Now suppose f is finite. Given $W \subset Y$ of codimension 1, write $(A, \mathfrak{m}) = (\mathcal{O}_{W,Y}, \mathfrak{m}_{W,Y})$. Construct a domain B such that B/A is finite and $B \otimes_A K(Y) = K(X)$ such that $\{U_i\} \to W$ are obtained by $B_{\mathfrak{m}_i} = \mathcal{O}_{U_i,X}$. Here, we need to show that

$$\sum \mathrm{ord}_{V_i}(r)[K(V_i)\colon K(W)] = \mathrm{ord}_W(N(r)).$$

Now if $r \in B$, we want $\varphi\colon B \xrightarrow{r} B$, and then the LHS becomes $\ell_A(\operatorname{coker}\varphi) = \mathrm{ord}_W(\det \varphi_K)$. In general, we need to apply results from EGA III to show that B exists.

Now in the different-dimensional case, we replace Y with $\operatorname{Spec} K(Y)$ and reduce to the case of a curve over $\operatorname{Spec} K(Y)$. □

Corollary 1.2.4 (Bezout's Theorem)**.** *Let $k = \overline{k}$ and suppose f, g are plane curves of degree m, n with no common component. Then $\sum i(P, F \cdot G) = mn$.*

Proof. Assume f is irreducible and replace G with $G' = L^n$. Then $G/G' = r \in K(C_F)$ and therefore

$$\sum i(P, F \cdot G) - \sum i(P, F \cdot G') = \sum \mathrm{ord}_P(r) = 0,$$

so we may as well assume F, G are lines and then the result is obvious. □

Definition 1.2.5. We can define the *degree* $\deg(\alpha)$ to be $\deg(\alpha) = \int_X \alpha = \sum n_P[K(P) : K]$.

1.3 Alternative Definition of Rational Equivalence

Let X be a scheme with irreducible components $X_1, \dots, X_t$. Then let $m_i \coloneqq \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x})$. We define the fundamental class $[X] = \sum_{i=1}^{t} m_i [X_i]$.

Proposition 1.3.1. *A cycle* $\alpha \in Z_k(X)$ *is rationally equivalent to* 0 *if and only if there exist* $(k+1)$*-dimensional subvarieties* $V_1, \dots, V_t$ *of* $X \times \mathbb{P}^1$ *such that* $f_i \colon V_i \to \mathbb{P}^1$ *are dominant and* $\alpha = \sum [V_i(0)] - [V_i(\infty)]$.

Sketch of Proof. Suppose $\alpha = [\mathrm{div}(r)]$ for some $r \in K(W)^{\times}$. Now r defines $W \to \mathbb{P}^1$, so we will define $V = \overline{\Gamma(r)}$ to be the closure of the graph. This gives a dominant rational map $f \colon V \dashrightarrow \mathbb{P}^1$ and thus $[\mathrm{div}(r)] = p_*(\mathrm{div}(f)) = [V(0)] - [V(\infty)]$. The other direction is easy. $\square$

Remark 1.3.2. This discussion and more machinery allows us to show that if $f \colon X \to Y$ is flat of relative dimension n, then we can define pullbacks of cycles. Here, if $\alpha \sim 0$ is a k-cycle on Y, then $f^*\alpha \sim 0$ in $Z_{k+n}(X)$, so we have

$$f^* \colon A_k(Y) \to A_{k+n}(X).$$

2

Avi (Jan 29): Intersecting with divisors and the first Chern class

Note: These are the speaker's notes. Minor edits to the TEX source were made.

2.1 Cartier and Weil divisors

Let X be a variety of dimension n over a field k. We want to introduce two notions of divisors, one familiar from the last chapter.

Definition 2.1.1. A *Weil divisor* of X is an $n-1$-cycle on X, i.e. a finite formal linear combination of codimension 1 subvarieties of X. Thus the Weil divisors form a group $Z_{n-1}X$.

Definition 2.1.2. A *Cartier divisor* consists of the following data:

- an open cover $\{U_\alpha\}$ of X;
- for each α a nonzero rational function f_α on U_α, defined up to multiplication by a *unit* , i.e. a function without zeros or poles, such that for any α, β we have f_α/f_β a unit on $U_\alpha \cap U_\beta$.

Like the Weil divisors, the Cartier divisors form an abelian group: $(\{U_\alpha, f_\alpha\}) + (\{U_\alpha, g_\alpha\}) = (\{U_\alpha, f_\alpha g_\alpha\})$ (we can assume that the open covers are the same, since if not they refine to $\{U_\alpha \cap V_\beta\}$). We call this abelian group $\operatorname{Div} X$.

Given a Cartier divisor $D = (\{U_\alpha, f_\alpha\})$ and a codimension 1 subvariety V of X, we define

$$\operatorname{ord}_V D = \operatorname{ord}_V(f_\alpha)$$

for α such that $U_\alpha \cap V$ is nonempty; since each f_α is defined up to a unit, this order is well-defined. We define the *associated Weil divsior*

$$[D] = \sum_V \operatorname{ord}_V D \cdot [V].$$

This defines a homomorphism

$$\operatorname{Div} X \to Z_{n-1}X.$$

For any rational function f on X, we get a *principal* Cartier divisor $\operatorname{div}(f)$ by choosing any cover $\{U_\alpha\}$ and defining $f_\alpha = f|_{U_\alpha}$. It is immediate that the image $[\operatorname{div}(f)]$ of this divisor under the map to $Z_{n-1}X$ is the Weil principal divisor. Say that two Cartier divisors D and D' are *linearly* *equivalent* if $D - D' = \operatorname{div}(f)$ for some f; then we define $\operatorname{Pic} X$ to be the group of Cartier divisors modulo linear equivalence, and the above then shows that the map $\operatorname{Div} X \to Z_{n-1}X$ descends to a map $\operatorname{Pic} X \to A_{n-1}X$. This map is in general neither injective nor surjective.

Notice that the definition of a Cartier divisor yields that of a line bundle on X: given a divisor $D = (\{U_\alpha, f_\alpha\})$, define a line bundle $L = \mathcal{O}(D)$ to be trivialized on each U_α with transition functions f_α/f_β. Two Cartier divisors D and D' are linearly equivalent if and only if $\mathcal{O}(D) = \mathcal{O}(D')$, and so we get the alternate description of $\operatorname{Pic} X$ as the abelian group of line bundles on X with group operation given by the tensor product. Conversely, given a line bundle L, this determines a Cartier divisor $D(L)$ up to some additional data: a nonzero rational section s of L. Therefore we can also think of Cartier divisors as the data of a line bundle together with a nonzero rational section.

We define the *support* $\operatorname{supp} D$ or $|D|$ of a Cartier divisor D to be the union of codimension 1 subvarieties V of X such that f_α is not a unit for U_α nontrivially intersecting V, i.e. $\operatorname{ord}_V D$ is nonzero.

We say that a Cartier divisor $D = (\{U_\alpha, f_\alpha\})$ if all of the f_α are regular, i.e. have no poles.

2.2 Pseudo-divisors

In general, Cartier divisors are not well-behaved under pullbacks (although line bundles are). In particular, given the data of a line bundle L and a nonzero rational section s and a morphism $f\colon Y \to X$, there is no guarantee that the pullback f^*s is nonzero. Therefore we enlarge the notion to make it behave better: let L be a line bundle on X, $Z \subset X$ be a closed subset, and s be a nowhere vanishing section of L restricted to $X - Z$, or equivalently a trivialization of $L|_{X-Z}$. A *pseudo-divisor* on X consists of the data of such a triple (L, Z, s), up to the following equivalence: two triples (L, Z, s) and (L', Z', s') define the same pseudo-divisor if $Z = Z'$ and there exists an isomorphism $\sigma\colon L \to L'$ such that restricted to $X - Z$ we have $\sigma \circ s = s'$. Note that this is well-behaved under pullback.

Example 2.2.1. Let $D = (\{U_\alpha, f_\alpha\})$ be a Cartier divisor, with support $|D|$. Then each f_α away from $|D$ gives a local section of the associated line bundle $\mathcal{O}(D)$, and so these glue to a section s_D of $\mathcal{O}(D)$ on $X - |D|$; this makes $(\mathcal{O}(D), |D|, s_D)$ a pseudo-divisor.

We say that a Cartier divisor D *represents* a pseudo-divisor (L, Z, s) when $|D| \subseteq Z$ and there exists an isomorphism $\sigma\colon \mathcal{O}(D) \to L$ such that restricted to $X - Z$ we have $\sigma \circ s_D = s$, with notation as above.

Lemma 2.2.2. *If X is a variety, then every pseudo-divisor (L, Z, s) on X is represented by a Cartier divisor D. If $Z \subsetneq X$, then D is unique; if $Z = X$, then D is unique up to linear equivalence.*

Proof. If $Z = X$, then s is a section on $X - X = \{\}$, and so a pseudo-divisor is just a line bundle; and we saw in the previous section that the group of Cartier divisors up to linear equivalence is isomorphic to the group of line bundles, so L corresponds to a unique linear equivalence class of Cartier divisors.

If $Z \neq X$, let $U = X - Z$. As above, choose a Cartier divisor $D = (\{U_\alpha, f_\alpha\})$ with $\mathcal{O}(D) \simeq L$. The section s consists of a collection of functions s_α on $U \cap U_\alpha$ such that $s_\alpha = f_\alpha/f_\beta \cdot s_\beta$ on $U \cap U_\alpha \cap U_\beta$; thus $s_\alpha/f_\alpha = s_\beta/f_\beta$ on each intersection, i.e. there exists some rational function r such that $s_\alpha/f_\alpha = r$ on each $U \cap U_\alpha$. Then $D' \coloneqq D + \operatorname{div}(r)$ is the Cartier divisor $(\{U_\alpha, f_\alpha r\})$ and by definition $f_\alpha r = s_\alpha$ on each $U \cap U_\alpha$; therefore using the definition above $s_{D'} = s$. Since D' is linearly equivalent to D, it corresponds to the same line bundle, and since r is regular on each U_α the support of $\operatorname{div}(r)$ is contained in Z; therefore D' represents (L, Z, s).

For uniqueness, suppose that two Cartier divisors $D_1 = (\{U_\alpha, f_\alpha\})$ and $D_2 = (\{V_\beta, g_\beta\})$ both represent (L, Z, s). Then similarly there must exist some rational function r such that $rf_\alpha = rg_\beta$

on each $U_\alpha \cap V_\beta$. But since $s_{D_1} = s_{D_2} = s$, if $Z \neq X$, i.e. U is nonempty, then s_{D_1} and s_{D_2} must agree on every $U \cap U_\alpha \cap V_\beta$, and so r restricted to U must be 1; since f is rational it follows that $f = 1$ and $D_1 = D_2$. □

For any pseudo-divisor $D = (L, Z, s)$, as for Weil divisors we will write $\mathcal{O}(D) = L$, $|D| = Z$, and $s_D = s$.

If $D = (L, Z, s)$ and $D' = (L', Z', s')$ are two pseudo-divisors, we can define their sum

$$D + D' = (L \otimes L', Z \cup Z', s \otimes s').$$

This agrees with the sum on Cartier divisors, except that the supports may be larger in this case. Similarly defining

$$-D = (L^{-1}, Z, s^{-1})$$

makes the set of pseudo-divisors into an abelian group.

Given a pseudo-divisor D on a variety X of dimension X, we can define the Weil class divisor $[D]$ by taking $\tilde{D}$ to be the Cartier divisor which represents D and setting $[D] \coloneqq [\tilde{D}]$, the associated Weil divisor from the previous section. The above lemma shows that this yields a well-defined element of $A_{n-1}X$; this gives a homomorphism from the group of pseudo-divisors to $A_{n-1}X$.

2.3 Intersecting with divisors

Let X be a variety of dimension n, D be a pseudo-divisor on X, and V be a subvariety of dimension k. Let $j\colon V \hookrightarrow X$ be the inclusion of V into X; then the pullback j^*D is a pseudo-divisor on V with support $V \cap |D|$. We define the class $D \cdot [V]$ in $A_{k-1}(V \cap |D|)$ given by the Weil class divisor of j^*D:

$$D \cdot [V] = [j^*D].$$

For any closed subscheme $Y \subset X$ containing $V \cap |D|$, we can also view this as an element of $A_{k-1}Y$; we will also denote this by $D \cdot [V]$.

Let $\alpha = \sum_V n_V \cdot V$ be a k-cycle on X, with support $|\alpha$ the union of the subvarieties V such that n_V is nonzero. For a pseudo-divisor D on X, we define the *intersection class* $D \cdot \alpha$ in $A_{k-1}(V \cap |D|)$ by

$$D \cdot \alpha = \sum_V n_V \cdot (D \cdot [V]).$$

As above, we can also view this as an element of $A_{k-1}Y$ for any Y containing $|\alpha| \cap |D|$.

We will apply this in two main cases. First: f $|D| = X$, then the data of $D = (L, X, s)$ is just that of a line bundle as above; in this case the action of D on a k-cycle α is called that of the first Chern class, written $D \cdot \alpha = c_1(L) \cap \alpha$.

Second: if $i\colon |D| \hookrightarrow X$ is the inclusion of $|D|$ into X, then $D \cdot \alpha$ is called the Gysin pullback $i^*\alpha$.

Theorem 2.3.1. *Let X be a scheme, D be a pseudo-divisor on X, and α be a k-cycle on X.*

(a) Let α' be a k-cycle on X. Then

$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha'$$

in $A_{k-1}((|\alpha| \cup |\alpha'|) \cap |D|)$.

(b) Let D' be a pseudo-divisor on X. Then

$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha$$

in $A_{k-1}(|\alpha| \cap (|D| \cup |D'|))$.

(c) *Let* $f\colon Y \to X$ *be a proper morphism,* β *be a* k*-cycle on* Y*, and* $g\colon |\beta| \cap f^{-1}(|D|) \to f(|\beta|) \cap |D|$ *be the restriction of* f *to* $|\beta| \cap f^{-1}(|D|)$*. Then*

$$g_*(f^*D \cdot \beta) = D \cdot f_*\beta$$

in $A_{k-1}(f(|\beta|) \cap |D|)$*.*

(d) *Let* $f\colon Y \to X$ *be a flat morphism of relative dimension* n *and* $g\colon f^{-1}(|\alpha| \cap |D|) \to |\alpha| \cap |D|$ *be the restriction of* f *to* $f^{-1}(|\alpha| \cap |D|)$*. Then*

$$f^*D \cdot f^*\alpha = g^*(D \cdot \alpha)$$

in $A_{n+k-1}(f^{-1}(|\alpha| \cap |D|))$*.*

(e) *If the line bundle* $\mathcal{O}(D)$ *is trivial, then*

$$D \cdot \alpha = 0$$

in $A_{k-1}(|\alpha| \cap |D|)$*.*

Proof. Part (a) is immediate from the definition. Using part (a), then, we can assume by linearity that $\alpha = [V]$ for some k-dimensional subvariety $V \subset X$. Restricting to V, (b) is just the statement that taking the Weil class divisor is compatible with sums.

For part (c), we can likewise assume that $\beta = [W]$ for some k-dimensional subvariety $W \subset Y$; then $f^*D \cdot \beta$ is the restriction of the Cartier divisor $f^*\tilde{D}$ representing f^*D to W, and so we can assume that $Y = W$. Similarly on the right-hand side $D \cdot f_*\beta = D \cdot \deg(f(W)/W)[f(W)]$ and so concerns only the restriction of D to $f(W)$, and so we can assume that $f(W) = X$. In this case $g = f$ on the support of D and so the statement is

$$f_*(f^*[D]) = \deg(W/f(W))[D]$$

since $D \cdot [X] = [D]$ and $f^*D \cdot [Y] = f^*[D]$. If f is a map of degree d and $D = \operatorname{div}(r)$ for some function r on some open subset of $f(W)$, then from last time we know that locally

$$f_*[\operatorname{div}(f^*r)] = [\operatorname{div}(N(f^*r))] = d[\operatorname{div}(r)]$$

where N is the determinant map from functions on subsets of W to functions on their images, since $N(f^*r) = dr$ since f has degree d. But locally we can always assume that $[D]$ is principal, and so $f_*f^*[D] = d[D]$ as desired.

For (d), we can again assume that $\alpha = [V] = [X]$, so the statement similarly becomes

$$[f^*D] = f^*[D].$$

By linearity, we can assume $D = [W]$ for some subvariety W of $X = V$, at which point the statement is $f^*[W] = [f^{-1}(W)]$, which is true whenever f is flat.

Finally for (e) we can again assume $\alpha = [V] = [X]$, so that the statement is $[D] = 0$ in $A_{n-1}X$ whenever $\mathcal{O}(D)$ is trivial, where n is the dimension of $V = X$. Letting $\tilde{D}$ be the Cartier divisor representing D, we know from section 1 that $\mathcal{O}(D)$ is trivial precisely when $\tilde{D}$ is linearly equivalent to the trivial Cartier divisor $0 = (\{U_\alpha, 1\})$ for which every local function is a unit; and we know that the associated Weil divisor map $\operatorname{Div} X \to Z_{n-1}X$ descends to a map $\operatorname{Pic} X \to A_{n-1}X$, i.e. $[D] = [\tilde{D}] = 0$ whenever $\mathcal{O}(D)$ is trivial. $\square$

2.4 Commutativity

Suppose that we have two Cartier divisors D, D' on an n-dimensional variety X. Then they both determine associated Weil divisors $[D], [D'] \in Z_{n-1}X$ (and thus in $A_{n-1}X$), and so it is natural to consider the intersections

$$D \cdot [D'], \qquad D' \cdot [D].$$

Theorem 2.4.1. *In* $A_{n-2}(|D| \cap |D'|)$, *we have*

$$D \cdot [D'] = D' \cdot [D].$$

Corollary 2.4.2. *Let* D *be a pseudo-divisor on a scheme* X, *and* α *be a* k*-cycle on* X *rationally equivalent to* 0. *Then*

$$D \cdot \alpha = 0$$

in $A_{k-1}(|D|)$.

Proof. We can assume without loss of generality that $\alpha = [\mathrm{div}(f)]$ for some rational function f on a subvariety V of X. Then letting $\tilde{D}$ be the Cartier divisor representing D we can replace D with $\tilde{D}$ and X with V without changing the result; then we can apply Theorem 2.4.1 to get

$$D \cdot \alpha = \tilde{D} \cdot [\mathrm{div}(f)] = \mathrm{div}(f) \cdot [\tilde{D}].$$

But by part (e) of Theorem 2.3.1, we have $\mathrm{div}(f) \cdot [\tilde{D}] = 0$. □

Given a closed subscheme $Y \subset X$ and a k-cycle α on Y, we can construct its intersection $D \cdot \alpha \in A_{k-1}(Y \cap |D|)$ for any pseudo-divisor D on X. This gives a map

$$Z_k Y \to A_{k-1}(Y \cap |D|).$$

The above corollary shows that in fact this map descends to a map

$$A_k Y \to A_{k-1}(Y \cap |D|);$$

this is called *intersecting* with D.

Corollary 2.4.3. *For two pseudo-divisors* D, D' *on a scheme* X *and a* k*-cycle* α *on* X, *we have*

$$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$$

in $A_{k-2}(|\alpha| \cap |D| \cap |D'|)$.

Proof. We can assume without loss of generality that $\alpha = [V]$ for some subvariety $V \subseteq X$ of dimension k. Then we can restrict D and D' to V, so that $D' \cdot [V] = [\mathrm{id}^* D'] = [D']$ and similarly $D \cdot [V] = [D]$; and then applying Theorem 2.4.1 immediately gives the result. □

For pseudo-divisors $D_1, \ldots, D_n$ on X and a k-cycle α on X, we can then define inductively

$$D_1 \cdots D_n \cdot \alpha = D_1 \cdot (D_2 \cdots D_n \cdot \alpha)$$

in $A_{k-n}(|\alpha| \cap (|D_1| \cup \cdots \cup |D_n|))$. Theorem 2.4.1 implies that the order of the D_i is unimportant, and parts (a) and (b) of Theorem 2.3.1 implies that the action is linear in each D_i and in α. More generally if $p(t_1, \ldots, t_n)$ is a homogeneous polynomial of degree d and Z is a closed subscheme of X containing $|\alpha| \cap (|D_1| \cup \cdots \cup |D_n|)$, then we can define $p(D_1, \ldots, D_n) \cdot \alpha$ in $A_{k-d}(Z)$.

Definition 2.4.4. We say that an algebraic variety Y is *complete* if for any variety Z the projection $Y \times Z \to Y$ is a closed map.

For example, any projective variety is complete.

If $n = k$ and $Y = |\alpha| \cap (|D_1| \cup \cdots \cup |D_k|)$ is complete, then we can define the *intersection number*

$$(D_1 \cdots D_k \cdot \alpha)_X = \int_Y D_1 \cdots D_k \cdot \alpha.$$

Similarly if p is a homogeneous polynomial of degree k in k variables then we can define

$$(p(D_1, \ldots, D_k) \cdot \alpha)_X = \int_Y p(D_1, \ldots, D_n) \cdot \alpha.$$

For a subvariety V purely of dimension k, we will sometimes write simply V instead of $[V]$; similarly we will sometimes write D instead of $[D]$.

Example 2.4.5. Let X be the projective completion of the affine surface $X' \subset \mathbb{A}^3$ defined by $z^2 = xy$. Consider the Cartier divisor D on X defined everywhere by the equation x, corresponding to the subvariety cut out by $x = 0$. Define the lines ℓ, ℓ' by $x = z = 0$ and $y = z = 0$ respectively, and let P be the origin $(0, 0, 0)$. Along the subvariety $x = 0$, from the defining equation we also have $z = 0$ (in affine space), and so $[D] = \operatorname{ord}_\ell D \cdot [\ell]$; we have

$$\operatorname{ord}_\ell D = \operatorname{len}_A A/(x),$$

where (in the affine variety) $A = \mathcal{O}_{X,\ell} = K[x, y, z]/(z^2 - xy)$. Thus $A/(x) = K[x, y, z]/(z^2 - xy, x) = K[y, z]/(z^2)$ which has length 2, with maximal proper subsequence of modules given by $0 \subset K[y] = K[y, z]/(z) \subset K[y, z]/(z^2)$. Therefore $[D] = 2[\ell]$. We can compute

$$D \cdot [\ell'] = [j^*D] = [P]$$

where j is the inclusion of ℓ' into X, since restricted to the line $y = z = 0$ the equation $x = 0$ specifies only the point P with multiplicity 1. Therefore there cannot exist any Cartier divisor D' with $[D'] = [\ell']$, since if there were we would have

$$[P] = D \cdot [\ell'] = D \cdot [D'] = D' \cdot [D] = 2D' \cdot [\ell]$$

in either Z_1X or A_1X, by Theorem 2.4.1 and the above calculation. This proves our above claim that the maps $\operatorname{Div} X \to Z_{\dim X - 1}$ and $\operatorname{Pic} X \to A_{\dim X-1}X$ are not in general surjective.

2.5 The first Chern class

Let X be a scheme, $V \subseteq X$ a subvariety of dimension k, and L a line bundle on X. The restriction of L to V is a line bundle on V and so is isomorphic to $\mathcal{O}(C)$ for some Cartier divisor C on V, determined up to linear equivalence. This in turn defines a well-defined element $[C]$ of $A_{k-1}X$; we write $c_1(L) \cap [V] \coloneqq [C]$. More generally, if $\alpha = \sum_V n_V \cdot [V]$ is a k-cycle on X then define C_V for each V as above, and write

$$c_1(L) \cap \alpha \coloneqq \sum_V n_V \cdot [C_V].$$

If $L = \mathcal{O}(D)$ for some pseudo-divisor D, then if $j \colon V \hookrightarrow X$ is the inclusion then the Cartier divisor $\tilde{D}$ on V representing j^*D satisfies $\mathcal{O}(\tilde{D}) \simeq \mathcal{O}(D)$ by construction; by definition, this means that $[C_V] = [j^*D] = D \cdot [V]$ and so

$$c_1(L) \cap \alpha = D \cdot \alpha$$

in $A_{k-1}X$.

Theorem 2.5.1. *Let X be a scheme, L be a line bundle on X, and α be a k-cycle on X.*

(a) *If α is rationally equivalent to 0, then $c_1(L) \cap \alpha = 0$. Therefore there is an induced homomorphism $c_1(L) \cap - \colon A_k X \to A_{k-1} X$.*

(b) *If L' is a second line bundle on X, then*

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$$

in $A_{k-2} X$.

(c) *If $f \colon Y \to X$ is a proper morphism and β is a k-cycle on Y, then*

$$f_*(c_1(f^*L) \cap \beta) = c_1(L) \cap f_*\beta$$

in $A_{k-1} X$.

(d) *If $f \colon Y \to X$ is a flat morphism of relative dimension n, then*

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$$

in $A_{n+k-1} Y$.

(e) *If L' is a second line bundle on X, then*

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$$

and

$$c_1(L^{-1}) \cap \alpha = -c_1(L) \cap \alpha$$

in $A_{k-1} X$.

Proof. A line bundle on X defines a pseudo-divisor with support X, and so the analogous properties from Theorem 2.3.1 and its corollaries immediately imply these. □

2.6 The Gysin map

Fix an effective Cartier divisor D on a scheme X, with the inclusion given by $i \colon |D| \hookrightarrow X$. Then we define the "Gysin homomorphism"

$$i^*\alpha := D \cdot \alpha$$

for k-cycles α on X.

Proposition 2.6.1. *With notation as above:*

(a) *If α is rationally equivalent to 0, then $i^*\alpha = 0$, and so there is an induced homomorphism $i^* \colon A_k X \to A_{k-1}(|D|)$.*

(b) *We have*

$$i_* i^* \alpha = c_1(\mathcal{O}(D)) \cap \alpha.$$

(c) *If β is a k-cycle on $|D|$, then*

$$i^* i_* \beta = c_1(i^* \mathcal{O}(D)) \cap \beta.$$

(d) *If* X *is purely* n*-dimensional, then*

$$i^*[X] = [D]$$

in $A_{n-1}(|D|)$.

(e) *If* L *is a line bundle on* X*, then*

$$i^*(c_1(L) \cap \alpha) = c_1(i^*L) \cap i^*\alpha$$

in $A_{k-2}(|D|)$.

All of these follow immediately from the definitions and the results above.

3

Alex (Feb 05): Chern classes and Segre classes of vector bundles

3.1 Chern and Segre Classes

Let L be a line bundle over a scheme X. If $V^k \subseteq X$ is a subvariety, then there exists a Cartier divisor such that $L\big|_V = \mathcal{O}_V(C)$, which gives us a cycle $[C] \in A_{k-1}$.

Definition 3.1.1. We will define the *first Chern class* of L by $c_1(L) \cap [V] = [C]$. Therefore we obtain a map

$$c_1(L) \cap -\colon Z_k X \to Z_{k-1} X.$$

Proposition 3.1.2. *Here are some properties of Chern classes:*

1. *The Chern class gives a well-defined map* $A_k \to A_{k-1}$.
2. *We have* $c_1(L) \cap c_1(L') \cap - = c_1(L') \cap c_1(L) \cap -$.
3. *Let* $f\colon X \to Y$ *be proper and* $\alpha \in A_k X$. *For a line bundle* L *over* Y, *we have*
$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap \alpha.$$
4. *If* $f\colon X \to Y$ *is flat of relative dimension* n, L/Y, *and* $\alpha \in A_k Y$, *we have*
$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha).$$
5. *Let* L, L' *be line bundles. Then* $c_1(L \otimes L') \cap - = c_1(L) \cap - + c_1(L') \cap -$

In the smooth case, Segre classes agree with Chern classes, but they have a nice generalization to the singular case. Let E be a rank $e+1$ vector bundle over a scheme X and let $P = \mathbb{P}(E)$ be the associated projective bundle. Then we have a line bundle $\mathcal{O}_E(1)$, and so we define

$$s_i(E) \cap -\colon A_k X \to A_{k-i} X \qquad s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha).$$

If E is a line bundle, we see that $s_1(E) \cap \alpha = -c_1(E) \cap \alpha$.

Proposition 3.1.3. *Let* L *be a line bundle. Then*

$$s_P(E \otimes L) = \sum_{i=0}^{p} (-1)^{p-i} \binom{e+p}{e+i} \cdot s_i(E) \cdot c_1(L)^{p-i}.$$

Now we will define Chern classes for an arbitrary vector bundle E of rank $r = e + 1$. Set

$$s_t(E) = 1 + s_1(E)t + s_2(E)t^2 + \cdots$$

and set

$$c_t(E) = s_t(E)^{-1} = 1 - s_1(E)t + s_2(E)t^2 + \cdots$$

This tells us that $c_n(E) = -s_1 c_{n-1} - s_2 c_{n-2} - \cdots - s_n$. This gives us a map $c_i(E)\colon A_k \to A_{k-i}$. Here are some new properties of Chern classes:

- For all $i > r$, we have $c_i(E) = 0$.
- If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles, then $c_t(E) = c_t(E') \cdot c_t(E'')$.

Remark 3.1.4. The projection formula, flat pullback, and c_1 give us uniqueness for all of the Chern classes.

3.2 Splitting Principle

Let E be a vector bundle over X. Then there exists a flat $f\colon X' \to X$ such that

1. The pushforward $f_* A_k X' \to A_k X$ is injective.
2. There exists a filtration $E = E_r \supseteq E_{r-1} \supseteq \cdots \supseteq E_0 = 0$ such that $E_i/E_{i-1} = L_i$ for some line bundle L_i. In addition, we have
$$c_t(E) = \prod_i (1 + c_1(L_i)t).$$

This gives us a **splitting principle:**

> *To prove a universal formula for Chern clases of some vector bundles with certain relations, it suffices to show that the formula is true for filtrations with line bundle quotients and the relations are preserved under flat pullback.*

Here are some properties of Chern classes:

1. If $E^\vee$ is the dual of E, we have $c_i(E^\vee) = (-1)^i c_i(E)$ and $c_t(E^\vee) = \prod(1 - \alpha_i t)$, where α_i are the Chern roots of E.
2. Let E, F be vector bundles with Chern roots α_i, β_j. Then $\{\alpha_i + \beta_j\}_{i,j}$ are Chern roots for $E \otimes F$.
3. Let E be a rank r vector bundle. Then $\bigwedge^p E$ has Chern roots $\{\alpha_{i_1} + \cdots + \alpha_{i_p}\}$.

Example 3.2.1. Recall the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0.$$

From the fact that $c_t(\mathcal{O}_{\mathbb{P}^n}) = 1$ and $c_t(\mathcal{O}_{\mathbb{P}^n}(1)) = (1 + Ht)$, we can compute the Chern character of $T_{\mathbb{P}^n}$.

Let E be a vector bundle on a nonsingular variety X and suppose $p\colon \mathbb{P}(E) \to X$ is the projection. Then we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(E)} \to p^*E \otimes \mathcal{O}_E(1) \to T_{\mathbb{P}(E)} \to p^*T_X \to 0.$$

This allows us to calculate

$$c_t(T_{\mathbb{P}(E)}) = c_t(p^*T_X) \cdot c_t(p^*E \otimes \mathcal{O}_E(1)).$$

3.3 Rational equivalence of bundles

This generalizes the Gysin map from last week. Let E be a vector bundle of rank $r = e + 1$ and let $\pi\colon E \to X$ be the projection. Then we obtain a commutative diagram

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr, "\pi"'] & & \mathbb{P}(E \oplus \mathcal{O}) \arrow[dl, "q"] \\ & X. & \end{tikzcd}$$

Theorem 3.3.1.

1. *The map $\pi^*\colon A_{k-r} \to A_k E$ is an isomorphism for all k.*
2. *A cycle $\beta \in A_k \mathbb{P}(E)$ is uniquely expressible in the form*

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^* \alpha_i$$

where $\alpha_i \in A_{k+1-e} X$.

Definition 3.3.2. Let $s^*\colon A_k E \to A_{k-r} X$ be $(\pi^*)^{-1}$. This is the Gysin map.

Proposition 3.3.3. *Let $\beta \in A_k E$ and $\overline{\beta} \in A_k \mathbb{P}(E \oplus \mathcal{O})$ that restricts to β. Then $s^*(\beta) = q_*(c_r(\xi) \cap \overline{\beta})$ where ξ is the universal rank r quotient bundle of $q^*(E \oplus \mathcal{O})$. Here, we have an exact sequence*

$$0 \to ??? \to q^*(E \oplus \mathcal{O}) \to \xi \to 0.$$

4

Patrick (Feb 12): Cones: because not every coherent sheaf is locally free

Note: These are the speaker's notes.

4.1 Cones and Segre Classes

Our goal is to define a Segre class $s(X, Y)$ of a subvariety $X \subsetneq Y$ and study its properties.

4.1.1 Cones

Definition 4.1.1. Let $S^\bullet$ be a sheaf of graded $\mathcal{O}_X$-algebras such that $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\bullet$ is generated by S^1. Then any scheme of the form $C = \operatorname{Spec}_{\mathcal{O}_X}(S^\bullet)$ is called a *cone*.

If C is a cone, then $\mathbb{P}(C \oplus 1) = \operatorname{Proj}(S^\bullet[z])$ is the projective completion with projection $q \colon \mathbb{P}(C \oplus 1 \to X)$. Let $\mathcal{O}(1)$ be the canonical line bundle on $\mathbb{P}(C \oplus 1)$.

Definition 4.1.2. The *Segre class* $s(C) \in A_*X$ of C is defined as

$$s(C) \coloneqq q_*\left(\sum_{i \geqslant 0} c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus 1)]\right).$$

Proposition 4.1.3.

1. *If E is a vector bundle on X, then $s(E) = c(E)^{-1} \cap [X]$, where $c = 1 + c_1 + \cdots$ is the total Chern class.*

2. *Let $_1, \ldots, c_t$ by the irreducible components of C with geometric multiplicity m_i. Then*

$$s(C) = \sum_{i=1}^{t} m_i s(C_i).$$

Example 4.1.4. Let $\mathcal{F}, \mathcal{F}'$ be coherent sheaves and let $\mathcal{E}$ be locally free. Then we may define $C(\mathcal{F}) = \operatorname{Spec}(\operatorname{Sym} \mathcal{F})$. We may define $s(\mathcal{F}) = s(C(\mathcal{F}))$. Now if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{E} \to 0$$

is exact, then $s(\mathcal{F}') = c(E) \cap s(\mathcal{F})$.

4.1.2 Segre Class of a Subvariety Let X be a closed subscheme of Y defined by the ideal sheaf $\mathcal{I}$ and let

$$C = C_X Y = \operatorname{Spec}\left(\sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

be the normal cone. Note that if X is regularly embedded in Y, then $C_X Y$ is a vector bundle.

Definition 4.1.5. The *Segre class* of X in Y is defined by

$$s(X, Y) \coloneqq s(C_X Y) \in A_* X.$$

Lemma 4.1.6. *Let Y be a scheme of pure dimension m and let $Y_1, \dots, Y_r$ be the irreducible components of Y with multiplicity m_i. If X is a closed subscheme of Y and $X_i = X \cap Y_i$, then*

$$s(X, Y) = \sum m_i s(X_i, Y_i).$$

Proposition 4.1.7. *Let $f\colon Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subseteq Y$ a closed subscheme, and $g\colon X' = f^{-1}(X) \to X$ be the induced morphism.*

1. *If f is proper, Y is irreducible, and f maps each irreducible component of Y' onto Y, then*

$$g_*(s(X', Y')) = \deg(Y'/Y) \cdot s(X, Y).$$

2. *If f is flat, then $g^*(s(X, Y)) = s(X', Y')$.*

Remark 4.1.8. If f is birational, then $f_*(s(X', Y')) = s(X, Y)$. This says that Segre classes are unchanged by pushforward along birational modifications.

Corollary 4.1.9. *Let Y be a variety and $X \subseteq Y$ be a proper closed subsecheme. Then let $\widetilde{Y} = \operatorname{Bl}_X Y$ and $\widetilde{X} = \mathbb{P}(C)$ be the exceptional divisor with projection $\eta\colon \widetilde{X} \to X$. Then*

$$s(X, Y) = \sum_{k \geqslant 1} (-1)^{k-1} \eta_*(\widetilde{X}^k) = \sum_{i \geqslant 0} \eta_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C)]).$$

Example 4.1.10. Let A, B, D be effective Cartier divisors on a surface Y. Then let $A' = A + D, B' = B + D$, and let $X = A' \cap B'$. Suppose that A, B meet transversally at a single smooth point $P \in Y$. Then if $\widetilde{Y} = \operatorname{Bl}_P Y$ and $f\colon \widetilde{Y} \to Y$ is the blowup with exceptional divisor E, we see that $\widetilde{X} = f^{-1}(X) = f^*D + E$, so we have

$$\begin{aligned} s(X, Y) &= f_*[\widetilde{X}] - f_*(\widetilde{X} \cdot [\widetilde{X}]) \\ &= [D] - f_*(f^*D \cdot [f^*D] + 2f^*D \cdot [E] + E \cdot [E]) \\ &= [D] - D \cdot [D] + [P]. \end{aligned}$$

If A, B both have multiplicity m at P and no common tangents at P, then

$$s(X, Y) = [D] + (m^2[P] - D \cdot [D]).$$

In general, the answer is more complicated.

4.1.3 Multiplicity Let $X \subseteq Y$ be an (irreducible) subvariety. Then the coefficient of $[X]$ in the class $s(X, Y)$ is called the *algebraic multiplicity* of X on Y and is denoted $e_X Y$.

Suppose X has positive codimension n, $p\colon \mathbb{P}(C_X Y) \to X$ and $q\colon \mathbb{P}(C_X Y \oplus 1) \to X$ are the projections to X, and $\widetilde{Y} = \mathrm{Bl}_X Y$ with exceptional divisor $\widetilde{X} = \mathbb{P}(C)$. Then we have

$$\begin{aligned} e_X Y[X] &= q_*(c_1(\mathcal{O}(1))^n \cap [\mathbb{P}[C \oplus 1]]) \\ &= p_*(c_1(\mathcal{O}(1))^{n-1} \cap [\mathbb{P}(C)]) \\ &= (-1)^{n-1} p_*(\widetilde{X}^n). \end{aligned}$$

For example, if X is a point, then we have

$$e_P Y = \int_{\mathbb{P}(C)} c_1(\mathcal{O}(1))^{n-1} \cap [\mathbb{P}(C)] = \deg[P(C)].$$

Example 4.1.11. Let C be a smooth curve of genus g and $C^{(d)}$ be the d-th symmetric power of C. Then let $P_0 \in C$, $J = J(C)$ be the Jacobian, and $u_d\colon C^{(d)} \to J$ be given by $D \mapsto D - dP_0$. We know that the fibers of u_d are the linear systems $|D| \cong \mathbb{P}^r$; if $d > 2g-2$, then $u_d\colon C^{(d)} \to J$ is a projective bundle; and if $1 \leqslant d \leqslant g$, then μ_d is birational onto its image W_d. Now if $\deg D = d$ and $\dim |D| = r$, we have

$$s(D, C^{(d)}) = (1+K)^{g-d+r} \cap [|D|],$$

where $K = c_1(K_{|D|})$. When d is large, this follows from the second bullet, but if d is small, then we may embed

$$C^{(d)} \subset C^{(d+s)} \qquad E \mapsto E + sP_0$$

and then consider the normal bundle to this embedding restricted to $|D|$. Combined with Proposition 4.1.7, this gives us the *Riemann-Kempf formula*, which says that the multiplicity of W_d at $u_d(D)$ is given by $e_{\mu_d(D)} W_d = \binom{g-d+r}{r}$.

Remark 4.1.12. The previous example can be generalized to the Fano varieties of lines on a cubic threefold X. In particular if F is the Fano variety of lines on X, then there is a morphism of degree 6 from $F \times F$ to the theta divisor, and we can calculate (following Clemens-Griffiths) that

$$\int_F s_2(T_F) = \int_F c_1(T_F)^2 - c_2(T_f) = 45 - 27 = 18,$$

and then the theta divisor has a singular point of multiplicity 3.

4.1.4 Linear Systems Let L be a line bundle on a variety X (of dimension n) and let $V \subseteq |L|$ be a partial linear system of dimension $r+1$. Then let B be the base locus of V. Then if $\widetilde{X} = \mathrm{Bl}_B X$, we obtain a morphism $f\colon \widetilde{X} \to \mathbb{P}^r$ resolving the rational map $X \dashrightarrow \mathbb{P}^r$. By definition, we have $f^*\mathcal{O}(1) = \pi^*(L) \otimes \mathcal{O}(-E)$. Define $\deg_f \widetilde{X}$ to be the degree of $f_*[\widetilde{X}] \in A_n \mathbb{P}^r$.

Proposition 4.1.13. *We have the identity*

$$\deg_f \widetilde{X} = \int_X c_1(L)^n - \int_B c_1(L)^n \cap s(B, X).$$

Example 4.1.14. Let $B \subset \mathbb{P}^n$ be the rational normal curve. Then let $V \subset |\mathcal{O}(2)|$ be the linear system of quadrics containing B. If $\widetilde{P}^n = \mathrm{Bl}_B \mathbb{P}^n$, we see that

$$\deg_f \widetilde{\mathbb{P}}^n = 2^n - (n^2 - n + 2).$$

If $n = 4$, then $f(\widetilde{\mathbb{P}}^4) = \mathrm{Gr}(2,4) \subset \mathbb{P}^5$.

4.2 Deformation to the Normal Cone

Let $X \subseteq Y$ be a closed subscheme and $C = C_X Y$ be the normal cone. We will construct a scheme $M = M_X Y$ and a closed embedding $X \times \mathbb{P}^1 \subseteq M$ such that

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[rr, hook] \arrow[dr, "p_2"'] & & M \arrow[dl, "q"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

comutes and such that

1. Away from ∞, we have $q^{-1}(\mathbb{A}^1) = Y \times \mathbb{A}^1$ and the embedding is the trivial embedding $X \times \mathbb{A}^1 \subseteq Y \times \mathbb{A}^1$.
2. Over ∞, $M_\infty = \mathbb{P}(C \oplus 1) + \widetilde{Y}$ is a sum of two Cartier divisors, where $\widetilde{Y} = \mathrm{Bl}_X Y$. The embedding of X is given by $X \hookrightarrow C \hookrightarrow \mathbb{P}(C \oplus 1)$. We also have $\mathbb{P}(C \oplus 1) \cap \widetilde{Y} = \mathbb{P}(C)$, which is embedded as the hyperplane at ∞ in $\mathbb{P}(C \oplus 1)$ and as the exceptional divisor in $\widetilde{Y}$.

We will now construct this deformation. Let $M = \mathrm{Bl}_{X \times \infty} Y \times \mathbb{P}^1$. Clearly we have $C_{X \times \infty} Y \times \mathbb{P}^1 = C \oplus 1$. But now we can embed $X \times \mathbb{P}^1 \subseteq M$. The first property is obvious by the blowup construction, so now we need to show the second property.

We may assume $Y = \operatorname{Spec} A$ is affine and X is defined by the ideal I. Identify $\mathbb{P}^1 \setminus 0 = \mathbb{A}^1 = \operatorname{Spec} k[t]$. Then if we write $S^n = I, T^n$, then we see that $\mathrm{Bl}_{X \times 0} Y \times \mathbb{A}^1 = \operatorname{Proj} S^\bullet$. But now this is covered by affines

$$\left\{ \operatorname{Spec} S^\bullet_{(a)} \right\}_{a \in (I,T) \text{ generator}}.$$

Now for $a \in I$, we see that $\mathbb{P}(C \oplus 1) \subseteq \operatorname{Spec} S^\bullet_{(a)}$ is defined by the equation $a/1$, while $\widetilde{Y}$ is defined by T/a, and now we see that

$$M_\infty = V(T) = V\left(\frac{a}{1} \cdot \frac{t}{a}\right) = V(a) \cup V(T/a) = \mathbb{P}(C \oplus 1) + \widetilde{Y},$$

as desired.

Now this allows us to define a *specialization* morphism

$$\sigma\colon Z_k Y \to Z_k C \qquad [V] \mapsto [C_{V \cap X} V].$$

Proposition 4.2.1. *Specialization preserves rational equivalence. Therefore we have a specialization morphism*

$$\sigma\colon A_k Y \to A_k C.$$

Remark 4.2.2. Supposing that X, Y are smooth, then the embedding of $X \subset \mathbb{P}(N \oplus 1)$ is nicer than $X \subset Y$ in several ways:

1. There is a retraction $\mathbb{P}(N \oplus 1) \to X$.
2. There is a vector bundle ξ on $\mathbb{P}(N \oplus 1)$ or rank $\operatorname{codim}_Y X$ and a section $s \in \Gamma(\xi)$ such that $V(s) = X$. Therefore X is represented by the top Chern class of ξ.

5

Caleb (Feb 19): Chern classes and intersection products

5.1 Chern Classes

We will recall the notion of Chern classes in topology. Recall that characteristic classes are natural transformations from $\mathbb{C}-\mathsf{Vect}_n(-) \to H^*(-,\mathbb{Z})$. Because complex vector bundles of rank n are reprsented by $\mathrm{Gr}(n,\infty)$, the Yoneda lemma tells us that characteristic classes are given by $H^*(\mathrm{Gr}(n,\infty),\mathbb{Z})$. Now Chern classes satisfy the following properties:

1. $c(\mathbb{C}) = 1$, where $\mathbb{C}$ denotes the trivial bundle.
2. If E has rank n, then $c(E)$ is nonzero only in degrees $0,\dots,2n$.
3. $c(E \oplus E') = c(E)c(E')$.
4. $c(\mathcal{O}_{\mathbb{CP}^\infty}(-1))$ generates $H^2(\mathbb{CP}^\infty)$.

Remark 5.1.1. Chern classes can be computed using the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0,$$

where

$$c_1 \colon H^1(\mathcal{O}_X^\times) \to H^2(X,\mathbb{Z}) \qquad L \mapsto c_1(L)$$

under the isomorphism $H^1(\mathcal{O}_X^\times) \cong \operatorname{Pic} X$.

We now return to algebraic geometry. Let L be a line bundle and σ be a rational section. Then $c_1(L) = [\operatorname{div}\sigma] \in A_{n-1}(X)$.

Example 5.1.2. If X is a smooth variety over $\mathbb{C}$, then we can write $\omega_X = \bigwedge^n \mathcal{O}mega_X$, and $K_X = c_1(\omega_X)$. For example, if $X = \mathbb{P}^n$, then we can choose $\Theta = dx_1 \wedge \cdots \wedge dx_n$, and $K_X = (n+1)H$.

Here are some properties of Chern classes of vector bundles. Let E be a vector bundle on a smooth quasiprojective variety X. Then we define

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots \qquad c_i(E) \in A_{n-i}(E).$$

1. If L is a line bundle, then $c(L) = 1 + c_1(L)$.
2. If $\tau_0,\dots,\tau_{n-i}$ are global sections with degeneracy locus having codimension i, then $c_i(E) = [D] \in A_{n-i}(X)$

3. If $0 \to E \to F \to G \to 0$ is an exact sequence, then $c(F) = c(E)c(G)$.

4. If $\varphi\colon Y \to X$ is a morphism, then $\varphi^* c(E) = c(\varphi^* E)$.

This axiomatic construction of Chern classes was done first by Grothendieck in 1958.

Now we discuss degeneracy loci. We know that $c_1(L) = 0$ if and only if $L \simeq \mathcal{O}_X$, whic happens if and only if L has a nonvanishing section. Now if E has rank n and we have general global sections $\tau_0, \dots, \tau_{n-i}$, the degeneracy locus is where

$$\bigwedge^{n-i+1} \ni \tau_0 \wedge \cdots \tau_{n-i} = 0.$$

Now recall that we defined the map $c_1(L) \cap -\colon A_k \to A_{k-1}$ and the map $s_i(E) \cap -\colon A_k \to A_{k-i}$. Finally, we defined $c(E) = s(E)^{-1}$. We will now return to splitting. If $E \to X$ is a vector bundle of rank r, then there exists $f\colon Y \to X$ such that f_* is surjective and f^* is injective on A_* and f^*E has a filtration by line bundles. The proof is by induction and is thus omitted. This allows us to write

$$c(E) = \prod_{i=1}^{r} (1 + \alpha_i),$$

where $\alpha_i = c_1(L_i)$ are the *Chern roots*.

Example 5.1.3. If E has roots α_i, then $S^k E$ has roots $\left\{\alpha_{i_1} + \cdots + \alpha_{i_k}\right\}_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k}$.

Example 5.1.4. We will now compute the 27 lines on a cubic surface. Note that there is a 20-dimensional locus of cubic hypersurfaces in $\mathbb{P}^3$. On the other hand, there is a 4-dimensional space of cubic forms on a line L. Therefore, L lies on a cubic surface f if and only if $f\big|_L = 0$.

As L varies, the cubic forms give a vector bundle $S^3 E^\vee$. Now each f gives us a section, so the locus where L is contained in f is precisely the zero-locus of f. Now we have a vector bundle $S^3 E^\vee$ on the Grassmannian $G(2,4)$ of lines in $\mathbb{P}^3$. Now we may assume that

$$\int c_4(S^3 E^\vee)$$

(the number of lines on the cubic surface) is finite, so we want to calculate it. We know E has roots α_1, α_2. Then $c_1(E) = \alpha_1 + \alpha_2, c_2(E) = \alpha_1 \alpha_2$, so

$$\begin{aligned} c(S^3 E^\vee) &= (1 + 3\alpha_1)(1 + 2\alpha_1 + \alpha_2)(1 + \alpha_1 + 2\alpha_2)(1 + 3\alpha_2) \\ &= 1 + 6c_1 + 10c_2 + 11c_1^2 + 30c_1c_2 + 6c_1^3 + 9c_2^2 + 18c_1^2. \end{aligned}$$

Therefore we have $c_4 = 9c_2^2 + 18c_1^2 c_2$. But then $\int c_2(E^\vee)^2 = 1$ is the number of lines contained in two planes. On the other hand, $\int \left\{c_1(E^\vee)\right\}^2 c_2(E^\vee) = 1$ is the number of lines intersecting two lines in a given plane, so we see that $c_4 = 27$, as desired.

5.2 Intersection Products

Let $i\colon X \to Y$ be a regular closed embedding of codimension d with normal bundle $N_X Y$. Let V be purely k-dimensional with a map $f\colon C \to Y$. Our goal is to construct $X \cdot V \in A_{k-d}(X)$. If we want this to look like cohomology, use formal Poincare duality. We will use deformation to the normal cone for our construction. Consider the pullback

$$\begin{array}{ccc} f^{-1}(X) = W & \xrightarrow{p} & V \\ \Big\downarrow g & & \Big\downarrow f \\ X & \xrightarrow{i} & Y \end{array}$$

Now define $N = g^*N_XY$ be a rank d vector bundle on W and $\pi\colon N \to W$ be the projection. Then if I is the ideal sheaf of $i\colon X \to Y$, this generates the ideal sheaf J of $p\colon W \to V$. This means that

$$\bigoplus_n f^*(I^n/I^{n+1}) \twoheadrightarrow \bigoplus_n J^n/J^{n+1}$$

is surjective. This gives a closed embedding of the normal cone $C = C_WV \hookrightarrow N$. By a result in the appendix, C is purely k-dimensional. Now consider te cycle $[C]$ on N and let s be the zero-section of N. Now we will define

$$X \cdot V = s^*[C],$$

where $s^*\colon A_k(N) \to A_{n-d}(W)$ is the Gysin map.

Here are some properties of the intersection product:

1. The intersection product is compactible with proper pushforwards.
2. Intersection products are compatible with flat pullbacks.
3. The intersection product is commutative.
4. Functoriality.

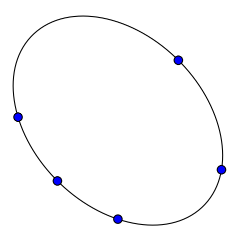
6

Nicolás (Mar 12): families of algebraic cycles

Note: these notes were adapted from the source for the `beamer` *provided by the speaker. Very light edits were made to the source.*

6.1 Introduction

"A typical problem in enumerative geometry is to find the number of geometric figures in a given family which satisfy certain conditions". One of the classical examples is that given five points in general position in $\mathbb{P}^2$, there exists a unique smooth conic passing through them.



The idea is that "conics are parametrized by $\mathbb{P}^5$, and passing through a point is a degree 1 equation in $\mathbb{P}^5$". But this might be dangerous:

- $\mathbb{P}^5$ parametrizes conics, not *smooth* conics.
- We need *transversal* intersections.

For instance, the argument *does not* work for smooth conics tangent to five lines. Each tangency is a degree 2 equation on $\mathbb{P}^5$, but they are not transversal (the conics of the form $\{L^2 = 0\}$ are "tangent" to all lines).

The correct number is 1, which may be seen by taking the *dual conic* (the set of tangent lines, as a subset of $(\mathbb{P}^2)^*$).

6.1.1 Conservation of number The classical principle is called *conservation of number*: if the problem has a finite numerical answer, this number is constant (or jumps to infinity).

Sadly, this does not work. Given four lines and a point in general position, there exists $1^4 \cdot 2 = 2$ smooth conics tangent to the lines and passing through the point (as one can show by taking the dual problem). But if the point lies in the diagonals of the quadrilateral given by the lines, then the number of smooth solutions decreases to 1 or 0.

Today we will discuss strong foundations for this principle, and some applications in enumerative geometry.

6.2 Families of cycle classes

During this section, T will denote an irreducible variety of dimension $m > 0$. We take $t \in T$ a regular closed point, and we denote

$$\{t\} = \operatorname{Spec} \kappa(t), \qquad t\colon \{t\} \to T$$

for the point and the inclusion.

We will use script letters (e.g. $\mathcal{X}, \mathcal{Y}$) for schemes over T, and the corresponding latin letters (e.g. X_t, Y_t) for the corresponding fibers over t (as schemes over $\{t\}$). If $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism, we denote $f_t\colon X_t \to Y_t$ the map on the fibers.

6.2.1 Specialization Let $p\colon \mathcal{Y} \to T, \alpha \in A_{k+m}\mathcal{Y}$. We define $\alpha_t \in A_k Y_t$ by

$$\alpha_t = t^!(\alpha)$$

where $t^!$ is the refined Gysin homomorphism induced by

$$\begin{array}{ccc} Y_t & \longrightarrow & \mathcal{Y} \\ \big\downarrow & & \big\downarrow p \\ \{t\} & \xrightarrow{t} & T. \end{array}$$

For instance, if $\alpha = [\mathcal{V}]$ and $\mathcal{V} \subseteq Y_t$, then $[\mathcal{V}]_t = 0$.

Here are some basic properties.

Proposition 6.2.1. *1. If $f\colon \mathcal{X} \to \mathcal{Y}$ is proper, $\alpha \in A_{k+m}\mathcal{X}$, then*

$$f_{t*}(\alpha_t) = (f_*(\alpha))_t \qquad \textit{in } A_k(Y_t).$$

2. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is flat of relative dimension n, $\alpha \in A_{k+m}\mathcal{Y}$*

$$f_t^*(\alpha_t) = (f^*(\alpha))_t \qquad \textit{in } A_{k+n}(X_t).$$

3. *If $i\colon \mathcal{X} \to \mathcal{Y}$ is a regular embedding of codimension d, such that $i_t\colon X_t \to Y_t$ is also a regular embedding of codimension d, $f\colon \mathcal{V} \to \mathcal{Y}$ a morphism, $\alpha \in A_{k+m}\mathcal{V}$, then*

$$i_t^!(\alpha_t) = (i^!(\alpha))_t \qquad \textit{in } A_{k-d}(W_t), \mathcal{W} = f^{-1}(\mathcal{X}).$$

4. *If E is a vector bundle over $\mathcal{Y}$, $\alpha \in A_{k+m}\mathcal{Y}$, then*

$$c_i(E_t) \cap \alpha_t = (c_i(E) \cap \alpha)_t \qquad \textit{in } A_{k-i}(Y_t).$$

The proof follows directly from similar statements for the refined Gysin homomorphism (see § 6.2–6.4).

We would now like to relate different fibers. Given a family $\mathcal{X} \to T$ and $\alpha \in A_{k+m}\mathcal{X}$, it is natural to compare $\alpha_t \in A_k(X_t)$ for different values of t. It is not obvious that such relation exists, even if $\mathcal{X} = Y \times T$ is the trivial family.

Example 6.2.2. Let $Y = T$ be a projective curve of genus $g \geqslant 2$, and $\Delta \subseteq Y \times T$ the diagonal. If $\alpha = [\Delta] \in A_1(Y \times T)$, then $\alpha_t = [t] \in A_0 Y$. But for $t_1 \neq t_2$, we have that α_{t_1} and α_{t_2} are not rationally equivalent.

This can be solved if we assume that $\mathcal{X} = Y \times T$, and if for every $t_1, t_2 \in T$, they can be connected by a chain of rational curves in T (see Example 10.1.7).

Here is a useful corollary.

Corollary 6.2.3. *Assume* T *is non-singular,* $t \in T$ *rational over the ground field,* $\mathcal{Y}$ *smooth over* T *with relative dimension* n*. If* $\alpha \in A_{k+m}(\mathcal{Y})$, $\beta \in A_{l+m}(\mathcal{Y})$*, then*

$$\alpha_t \cdot \beta_t = (\alpha \cdot \beta)_t \qquad \textit{in } A_{k+l-n}(Y_t).$$

This gives us a strategy to show that $a \cdot b = c$ in a non-singular variety Y. We construct a family $\mathcal{Y} \to T$ with $Y_t = Y$ for some t, and such that a, b, c can be lifted to α, β, γ. Then, it suffices to show that $\alpha \cdot \beta = \gamma$, which we can try to prove generically.

6.2.2 Sample Application

Let C be a non-singular curve, $C^{(n)}$ its n^{th} symmetric product (which points are effective divisors of degree n over C). If A is an effective divisor on C of degree $< n$, define

$$X_A = \{D \in C^{(n)} \mid D \geqslant A\}.$$

One can show that if A and B have disjoint support, then X_A and X_B intersect transversally, and so

$$[X_A] \cdot [X_B] = [X_{A+B}].$$

This is true even if A and B intersect, by using Corollary 6.2.3 and by “moving” A.

6.3 Conservation of number

We have seen that for $\alpha \in A_k \mathcal{Y}$, it is not clear that $\{\alpha_t\}_{t \in T}$ are related, even if $\mathcal{Y} = Y \times T$ is the trivial family. We have the following substitute.

Proposition 6.3.1 (Conservation of number)**.** *Let* $p\colon \mathcal{Y} \to T$ *be a proper morphism,* $\dim T = m$ *as before. Let* α *be an* m*-cycle on* $\mathcal{Y}$*. Then* $\alpha_t \in A_0(Y_t)$ *all have the same degree (which is obtained by* $p_{t*}(\alpha_t) = \deg \alpha_t \cdot [\{t\}]$*).*

The idea of the proof is write $p_*(\alpha) = N[T] \in A_m(T)$, for some $N \in \mathbb{Z}$. Then, by Proposition 6.2.1 we get

$$p_{t*}(\alpha_t) = (p_*(\alpha))_t = N[T]_t = N[\{t\}].$$

This proposition can be improved to compute the degree of intersections with Chern classes or some divisors (see § 10.2 for precise statements). We will need the following result.

Corollary 6.3.2. *Let* Y *be a scheme,* $\mathcal{H}_i \subseteq Y \times T$ *effective Cartier divisors which are flat over* T, $i = 1, \ldots, d$*. Let* a *be a* d*-cycle on* Y*. Assume that*

$$\mathcal{H}_1 \cap \cdots \cap \mathcal{H}_d \cap (\mathrm{Supp}(a) \times T)$$

is proper over T*. Then*

$$\deg((H_1)_t \cdots (H_d)_t \cdot a)$$

is independent of t*.*

6.4 An enumerative problem

Our main application of these techniques will be to solve the following problem.

Given an r-dimensional family of plane curves, and r curves in general position in the plane, how many curves in the family are tangent to the r given curves?

The answer will require to compute the *characteristics* $\mu^k\nu^{r-k}$ of the family, which are the number of curves in the family passing through k general points and tangent to $r-k$ general lines.

For instance, if we consider the family of smooth conics, then

$$\mu^5 = \nu^5 = 1, \quad \mu\nu^4 = \mu^4\nu = 2, \quad \mu^2\nu^3 = \mu^3\nu^2 = 4.$$

1. We will study the *incidence correspondence*

$$I = \{[x : y : z], [a : b : c] \mid ax + by + cz = 0\} \subseteq \mathbb{P}^2 \times \mathbb{P}^{2*}.$$

   This can be seen as a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. In fact, if E is the kernel of

$$1_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{(x,y,z)} \mathcal{O}_{\mathbb{P}^2}(1) \to 0,$$

   then $I = \mathbb{P}(E)$.

   This allows us to compute $A^\bullet(I)$ (see Example 8.3.4), with a basis

$$1, \lambda, \zeta, \lambda^2, \zeta^2, \lambda^2\zeta = \lambda\zeta^2,$$

   where $\lambda\zeta = \lambda^2 + \zeta^2, \lambda^3 = \zeta^3 = 0$, and λ, ζ the pullbacks of $c_1(\mathcal{O}_{\mathbb{P}^2}(1)), c_1(\mathcal{O}_{\mathbb{P}^{2*}}(1))$.

   Now, if M is a line and Q a point, consider

$$M' = \{(P, L) \in I \mid L = M\} \qquad Q' = \{(P, L) \in I \mid P = Q\}$$
$$M'' = \{(P, L) \in I \mid P \in M\} \qquad Q'' = \{(P, L) \in I \mid Q \in L\}.$$

   One can show that

$$\lambda = [M''], \quad \zeta = [Q''], \qquad \lambda^2 = [Q'], \quad \zeta^2 = [M'].$$

2. Let $D \subseteq \mathbb{P}^2$ be a curve without multiple components. Define $D' \subseteq I$ as the closure of

$$\{(P, L) \in I \mid P \text{ simple point of } D, L \text{ tangent at } P\}.$$

   We claim that

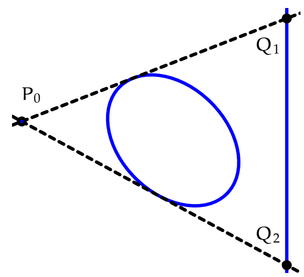
$$[D'] = n[M'] + m[Q'] = n\zeta^2 + m\lambda^2 \in A^2I,$$

   where n is the degree and m the *class* of D (the number of tangents from a general point to D). The idea is to compute

$$D' \cap M'' = \{(P_i, L_i) \mid P_i \in M \cap D, L_i \text{ tangent at } P_i\},$$

   which has generically $\#D' \cap M'' = n$ points.

   The equivalence $[D'] = m[M'] + n[Q']$ can be computed explicitely. Take P_0 a general point, M a general line, and let $Q_1, \dots, Q_m$ the intersections of M with the tangents from P_0.



The projection from P_0 to M gives a family $\mathcal{D} \to \mathbb{A}^1$ with $\mathcal{D}_1 = [D'], \mathcal{D}_0 = n[M'] + \sum[Q_i']$. (There is a explicit computation in § 10.4.)

3. Let $\mathcal{X} \subseteq \mathbb{P}^2 \times S$ be a flat family of plane curves, $\dim S = r$, S non-singular. Assume X_s has no multiple compontents for general s, and let $S^0 \subseteq S$ an open set with X_s reduced for $s \in S$.

   Let $\mathcal{X}(r) \subseteq I^r \times S^0$ given by $(P_1, L_1), \dots, (P_r, L_r), s$ such that P_i is a simple point of X_s, and L_i is tangent in P_i. Note that $\dim \mathcal{X}(r) = 2r$.

   Take $D_1, \dots, D_r \subseteq \mathbb{P}^2$ reduced curves, and consider

$$\begin{array}{ccc} W & \longrightarrow & D_1' \times \cdots \times D_r' \\ \downarrow & & \downarrow \\ \mathcal{X}(r) & \xrightarrow{\varphi} & I^r. \end{array}$$

   We can move $D_1, \dots, D_r$, so that the interseccion between $\mathcal{X}(r)$ and $D_1' \times \cdots \times D_r'$ is transversal (by taking a general element in $PGL(2)^r$). This way, W has N (reduced) points.

   Now, compactify $\overline{\mathcal{X}} \subseteq \mathbb{P}^2 \times \overline{S^0}$, and $\overline{\mathcal{X}(r)} \subseteq I^r \times \overline{S^0}$. If Z is a closed subsed of dimension less than 2r, which contains all $\overline{\mathcal{X}(r)} - \mathcal{X}(r)$, then the number N does not change after we remove Z.

4. We now degenerate each D_i to a multiple line (as we did for D). This gives a diagram

$$\begin{array}{ccccc} \mathcal{W} & \longrightarrow & \mathcal{D}_1' \times \cdots \times \mathcal{D}_r' & \longrightarrow & \mathbb{A}^r \\ \downarrow & & \downarrow & & \\ \overline{\mathcal{X}(r)} & \longrightarrow & I^r. & & \end{array}$$

   The space $\overline{\mathcal{X}(r)}$ is complete, so $\mathcal{W}$ is proper over $\mathbb{A}^r$. This way, we may take an open neighborhood T of $(1, \dots, 1)$ and $(0, \dots, 0)$, so that $\mathcal{W}$ is proper over T and disjoint from Z.

   Now, Corollary 6.2.3 applies, and so

$$\deg(\mathcal{X}(r) \cdot_\varphi (D_1' \times \cdots \times D_r')) = \deg(\mathcal{X}(r) \cdot_\varphi (E_1' \times \dots E_r')),$$

   where D_i', E_i' are the fibers over 1 and 0.

   The right hand side is just

$$\prod_{i=1}^{r}(m_i\mu + n_i\nu) = \sum_{k=0}^{r} N_k \mu^k \nu^{r-k},$$

   where each curve D_i has degree n_i and class m_i.

   The left hand side is the number of points N, provided that we take a *convenient* Z (which avoids technical difficulties such as bitangents).

6.4.1 A famous example The most famous example is the *Steiner's conic problem*, which tries to determine the number of conics tangent to five smooth conics in general position.

The natural family here is the family of smooth conics (as a subset of $\mathbb{P}^5$), which has characteristics
$$\mu^5 = \nu^5 = 1, \quad \mu^4\nu = \mu\nu^4 = 2, \mu^3\nu^2 = \mu^2\nu^3 = 4$$
(in characteristic zero!)

This way, the number of conics tangent to five non-singular curves of degree n in general position is

$$N = n^5((n-1)^5 + 10(n-1)^4 + 40(n-1)^3 + 40(n-1)^2 + 10(n-1) + 1),$$

which for $n = 2$ gives the famous number 3264.

7

Patrick (Mar 19): Doing Italian-style algebraic geometry rigorously

Note: these are the speaker's notes.

7.1 Intersection multiplicities

Consider a Cartesian square

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "g"] & V \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

where i is a regular embedding of codimension d and V has pure dimension k. Let $C = C_W V$ have components $C_1, \ldots, C_r$ with multiplicity m_i. Let Z_i be the support of C_i. We call the Z_i the *distinguished varieties* of the intersection.

Lemma 7.1.1.

(a) Every irreducible component of W *is distinguished.*

(b) For any distinguished variety Z*, we have* $k - d \leqslant \dim Z \leqslant k$*.*

Definition 7.1.2. An irreducible component Z of $W = f^{-1}(X)$ is a *proper component* of intersection of V by X if $\dim Z = k - d$. The *intersection multiplicity* of Z in $X \cdot V$, denoted $i(Z, X \cdot V; Y) = i(Z, X \cdot V) = i(Z)$ is the coefficient of Z in the class $X \cdot V \in A_{k-d}(W)$.

If N_Z is the pullback of $N_X Y$ to Z, then $i(Z, X \cdot V; Y)$ is the coefficient of N_Z in $[C]$. Now let $A = \mathcal{O}_{Z,V}$ and $J \subset A$ be the ideal of W. Then A/J has finite length when Z is an irreducible component of W.

Proposition 7.1.3. *Assume* Z *is a proper component of* W*.*

(a) If $\ell(A/J)$ *is the length of* A/J*, then* $1 \leqslant i(Z, X \cdot V; Y) \leqslant \ell(A/J)$*.*

(b) If J *is generated by a regular sequence of length* d*, then* $i(Z, X \cdot V; Y) = \ell(A/J)$*.*

If A *is Cohen-Macaulay, then local equations for* X *in* Y *give a regular sequence generating* J *and equality in (b) holds.*

Now suppose Z is a proper component of the intersection of V by X on Y. Let $A = \mathcal{O}_{V,Z}$ and J be the ideal in A generated by the ideal (sheaf) of X in Y, and let $\mathfrak{m}$ be the maximal ideal of A.

Proposition 7.1.4. *Suppose V is a variety. Then $i(Z, X \cdot V; Y) = 1$ if and only if A is regular and $J = \mathfrak{m}$.*

7.2 Nonsingular varieties

Let Y be a smooth variety of dimension n. Then $\Delta \subset Y \times Y$ is regularly embedded with codimension n. The global *intersection product* is the map

$$A_k(Y) \otimes A_\ell(Y) \to A_{k+\ell-n}(Y) \qquad x \otimes y \mapsto x \cdot y \Delta^*(x \times y),$$

where δ^* is the Gysin homomorphism.

More generally, let X be a scheme and $f\colon X \to Y$ a morphism to a smooth variety. Then Γ_f is regularly embedded in Y, so we can define a *cap product*

$$A_i(Y) \otimes A_j(X) \to A_{i+j-n}(X) \qquad y \otimes x \mapsto f^*(y) \cap x = \Gamma_f^*(x \times y).$$

If X is smooth, then we write $f^*y = f^*y \cap [X]$.

Remark 7.2.1. We may also replace the Gysin homomorphisms Δ^*, Γ_f^* with the refined Gysin homomorphisms $\Delta^!, \Gamma_f^!$.

Definition 7.2.2. Let $f\colon X \to Y$ be a morphism with Y smooth of dimension n. Let $p_X\colon X' \to X, p_Y\colon Y' \to Y$ be morphisms of schemes. Then form the square

$$\begin{tikzcd} X' \times_Y Y' \arrow[r] \arrow[d] & X' \times Y' \arrow[d, "p_X \times p_Y"] \\ X \arrow[r, "\Gamma_f"] & X \times Y. \end{tikzcd}$$

Now define the *refined intersection product* by

$$x \cdot_f y := \Gamma_f^!(x \times y) \in A_{k+\ell-n}(X' \times_Y Y')$$

for $x \in A_k(X'), Y \in A_\ell(Y')$. When $X' = X, Y' = Y$, this is the global product.

Proposition 7.2.3. *The refined products satisfy the following formal properties:*

(a) (Associativity) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ with Y, Z smooth, then

$$x \cdot_f (y \cdot_g z) = (x \cdot_f y) \cdot_{gf} z \in A_*(X' \times_Y Y' \times_Z Z').$$

(b) (Commutativity) If $f_i\colon X \to Y_i$ with Y_i smooth, then

$$(x \cdot_{f_1} y_1) \cdot_{f_2} y_2 = (x \cdot_{f_2} y_2) \cdot_{f_1} y_1 \in A_*(Y_1' \times_{Y_1} X' \times_{Y_2} Y_2').$$

(c) (Projection) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ with Z smooth. Let $f'\colon X' \to Y'$ be proper with $p_Y f' = f p_X$ and $f'' = f' \times_Z \mathrm{id}_Z$. Then

$$f''_*(x \times_{gf} z) = f'_*(x) \cdot_g z \in A_*(Y' \times_Z Z').$$

(d) (Compatibility) Let $f\colon X \to Y$ with Y smooth and $g\colon V' \to Y'$ be a regular embedding. Then

$$g^!(x \cdot_f y) = x \cdot_f g^! y \in A_*(X' \times_Y V').$$

Corollary 7.2.4. *Let Y be smooth and $j\colon V \to Y$ be a regular embedding. If x is a cycle on Y, then $x \cdot [V] = j^!(x) \in A_*(|x| \cap V)$.*

Corollary 7.2.5. *Let $f\colon X \to Y$ with X, Y smooth. Then $x \cdot_f y = (x \times y) \cdot [\Gamma_f] \in A_*(|x| \cap f^{-1}(|y|))$.*

Corollary 7.2.6. *Let $f\colon X \to Y$ with Y smooth and x a cycle on X. Then $x \cdot_f [Y] = x$.*

Definition 7.2.7. Let $f\colon X \to Y$ be a morphism with X purely m-dimensional and Y a smooth n-dimensional variety Y. For any morphism $g\colon Y' \to Y$, define a *refined Gysin homomorphism*

$$f^!\colon A_k(Y') \to A_{k+m-n}(X \times_Y Y') \qquad f^!(y) = [X] \cdot_f y.$$

Proposition 7.2.8.

(a) If f is flat, then $f^!(y) = f'^*(y)$, where $f'\colon X \times_Y Y' \to Y'$ is the base change.

(b) If f is a local complete intersection morphism, then $f^!$ agrees with the morphism constructed in Section 6.6 of Fulton.

Now let Y be a smooth variety of dimension n. Let V, W be closed subschemes of Y of pure dimension k, ℓ. Now a component $Z \subseteq V \cap W$ is a *proper component* if $\dim Z = k + \ell - n$. If Z is proper, then the coefficient of Z in $V \cdot W \in A_{k+\ell-n}(V \cap W)$ is called the *intersection multiplicity* $i(Z, V \cdot W; Y) = i(Z, \Delta_Y \cdot (V \times W); Y \times Y)$. If every component of $V \cap W$ is proper, then the intersection class is

$$V \cdot W = \sum_Z i(Z, V \cdot W; Y) \cdot [Z].$$

Proposition 7.2.9. *Assume Z is a proper component of $V \cap W$. Then*

(a) $1 \leqslant i(Z, V \cdot W; Y) \leqslant \mathcal{O}_{V \cap W, Z}$.

(b) If the local ring $\mathcal{O}_{V \cap W, Z}$ is Cohen-Macaulay, then $i(Z, V \cdot W; Y) = \ell(\mathcal{O}_{V \cap W, Z})$.

(c) If V, W are varieties, then $i(Z, V \cdot W; Y) = 1$ if and only if the maximal ideal of $\mathcal{O}_{Y,Z}$ is the sum of the prime ideals of V and W. In fact, $\mathcal{O}_{V,Z}, \mathcal{O}_{W,Z}$ are regular.

Now let Y be a smooth variety of dimension n. Set $A^p(Y) = A_{n-p}(Y)$. Now this indexing, the intersection product is now $A^p(Y) \otimes A^q(Y) \to A^{p+q}(Y)$. In addition, if $f\colon X \to Y$ is a morphism, the cap product is now $A^p(Y) \otimes A_q(X) \xrightarrow{\cap} A_{q-p}(X)$. If X is also smooth, the pullback now reads $f^*\colon A^p(Y) \to A^p(X)$.

Proposition 7.2.10.

(a) Suppose Y is a smooth variety. Then the intersection product makes $A^*(Y)$ into a commutative graded ring with unit $A^0 \ni 1 = [Y] \in A_n$. Then $Y \mapsto A^*(Y)$ is a contravariant functor from smooth varieties to rings.

(b) If $f\colon X \to Y$ is a morphism from a scheme X to a smooth variety Y, then A_*X is a A^*Y-module with action

$$A^p(Y) \otimes A_q(X) \xrightarrow{\cap} A_{q-p}(X).$$

(c) If $f\colon X \to Y$ is a proper morphism of smooth varieties, then

$$f_*(f^*y \cdot x) = y \cdot f_*(x)$$

for all classes x on X and y on Y.

7.3 Bézout's Theorem

We will now use this theory to discuss something classical. It is easy to see that $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ and is generated by $[L^k]$ for a linear subspace $L^k \subset \mathbb{P}^n$. If α is a k-cycle on $\mathbb{P}^n$, we define the *degree* $\deg(\alpha)$ to be the integer satisfying $\alpha = \deg(\alpha) \cdot [L^k]$. Equivalently, we may define

$$\deg(\alpha) = \int_{\mathbb{P}^n} c_1(\mathcal{O}(1))^k \cap \alpha.$$

Theorem 7.3.1 (Bézout)**.** *Let* $\alpha_i \in A^{d_i}(\mathbb{P}^n)$ *for* $i = 1, \ldots, r$. *If* $d_1 + \cdots + d_r \leqslant n$, *then*

$$\deg(\alpha_1 \cdots \alpha_r) = \deg(\alpha_1) \cdots \deg(\alpha_r).$$

Proof. We have an isomorphism $A^*(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1})$, where $h = [L^{n-1}]$. Thus $[L^{n-k}] = h^k$, and the desired result follows. □

Now if subschemes $V_1, \ldots, V_r \subseteq \mathbb{P}^n$ representing $\alpha_1, \ldots, \alpha_r$ meet properly, then

$$V_1 \cdots V_r = \sum_j i(Z_j, V_1 \cdots V_r; \mathbb{P}^n) \cdot [Z_j],$$

where Z_j are the components of $\bigcap V_i$. Then Bézout's theorem gives us the identity

$$\sum_j i(Z_j, V_1 \cdots V_r; \mathbb{P}^n) \cdot \deg(Z_j) = \prod \deg(V_i).$$

If $H_1, \ldots, H_n$ are hypersurfaces intersecting properly (so the intersection is a finite number of points), then consider the local ring $\mathcal{O}_{\bigcap H_i, P}$. Then complete intersections are Cohen-Macaulay, so

$$i(P, H_1 \cdots H_n; \mathbb{P}^n) = \ell(\mathcal{O}_{\bigcap H_i, P}).$$

Now $\dim_k \mathcal{O}_{\bigcap H_i, P} = \deg P \ell(\mathcal{O}_{\bigcap H_i, P})$, and thus we obtain

$$\sum_P \dim_k \mathcal{O}_{\bigcap H_i, P} = \prod \deg H_i.$$

This recovers the very classical Bézout's theorem.

Example 7.3.2. Let s be the hyperplane class on $\mathbb{P}^n$ and t be the hyperplane class on $\mathbb{P}^m$. Then

1. $A^*(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z}[s,t]/(s^{n+1}, t^{m+1})$.
2. If $H_1, \ldots, H_{n+m}$ are hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^m$ with bidegree (a_i, b_i), then
$$\int [H_1] \cdots [H_{n+m}] = \sum_{\substack{(i_1,\ldots,i_n,j_1,\ldots,j_m) \\ (n,m)\text{-shuffle}}} a_{i_1} \cdots a_{i_n} b_{j_1} \cdots b_{j_m}.$$
3. If Δ is the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$, then $[\Delta] = \sum_{i=0}^n s^i t^{n-i} \in A^n(\mathbb{P}^n \times \mathbb{P}^n)$. This formula follows from intersecting Δ with $[L_1 \times L_2]$, where L_1, L_2 are linear subspaces of complementary dimension.
4. Let $s\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ be the Segre embedding. It h is the hyperplane class on $\mathbb{P}^{nm+n+m}$, then $s^*u = s + t$. Also, the degree of $s(\mathbb{P}^n \times \mathbb{P}^m)$ is $\binom{n+m}{n}$.
5. If $\nu_m \colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{n}-1}$ is the Veronese embedding and s, h are the hyperplane classes on the source and target, then $\nu_m^* u = m \cdot s$. If V is a k-dimensional subvariety of $\mathbb{P}^n$ of degree d, then $\deg(\nu_m(V)) = d \cdot m^k$.

8

Morena (Mar 26): Grothendieck-Riemann-Roch

We will begin by reviewing some notions from algebraic topology.

1. Let E be a vector bundle over X with Chern roots $\alpha_1, \ldots, \alpha_r$. Then we may define the *Chern character*

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{\alpha_i} = \sum_{n=0}^{\infty} \frac{s_n(c_1, \ldots, c_n)}{n!},$$

   where s_n is the n-th Newton polynomial. Recall that if

$$0 \to E' \to E \to E'' \to 0$$

   is exact, then $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$ and $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$.

2. We may define the *Todd class*

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{-\alpha_i}}.$$

   Similarly to Chern classes, if

$$0 \to E' \to E \to E'' \to 0$$

   is exact, then $\mathrm{td}(E) = \mathrm{td}(E')\,\mathrm{td}(E'')$.

3. Now define the ring

$$K^0(X) = \bigoplus \mathbb{Z}[E]/(0 \to E' \to E \to E'' \to 0).$$

   This is well-behaved under pullback, but pushforward is not well-defined.

In order to fix the problems with pushforwards of vector bundles, we will attempt to define the Grothendieck group $K_0(X)$ of coherent sheaves. Unfortunately, pullbacks of coherent sheaves are only right exacts. Also, pushforwards do exist for proper morphisms, but this is only left exact. Therefore we need to consider the higher derived functors, and now we define

$$f_* \colon K_0(X) \to K_0(Y) \qquad [\mathcal{F}] \mapsto \sum_{i=0}^{\infty} (-1)^i [R^i f_* \mathcal{F}].$$

This is a finite sum because we assume all of our schemes are Noetherian.

For a smooth variety, there is always a map $K^0(X) \to K_0(X)$. In fact, there is an inverse, because we can resolve any coherent sheaf $\mathcal{F}$ by vector bundles. In this case, we will write $K_0(X) = K^0(X) = K(X)$. If $\mathcal{F}$ is resolved by

$$0 \to \mathcal{L}_n \to \cdots \to \mathcal{L}_0 \to \mathcal{F} \to 0,$$

we write $[\mathcal{F}] = \sum_{i=0}^{n} [\mathcal{L}_i]$ and $\operatorname{ch}(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \operatorname{ch}(\mathcal{L}_i)$.

8.1 Statement of the theorem

Recall the Chern character $\operatorname{ch}\colon K(X) \to A(X) \otimes \mathbb{Q}$. In topology, we need the Todd class to make the diagram

$$\begin{tikzcd} K(X) \arrow[r, "\lambda_E"] \arrow[d, "\operatorname{ch}"] & K(E) \arrow[d, "\operatorname{ch} \cdot (-1)^{\operatorname{rk}(E)} \operatorname{td}(E^*)"] \\ H^*(X, \mathbb{Q}) \arrow[r, "u_E"] & H^*(X, \mathbb{Q}) \end{tikzcd}$$

commute. Now we are able to state the Grothendieck-Riemann-Roch theorem:

Theorem 8.1.1 (Grothendieck-Riemann-Roch)**.** *Let $f\colon X \to Y$ be proper where X, Y are smooth quasiprojective varieties over k. Then the diagram*

$$\begin{tikzcd} K(X) \arrow[r, "f_*"] \arrow[d, "\operatorname{ch} \cdot \operatorname{td}(T_X)"] & K(Y) \arrow[d, "\operatorname{ch} \cdot \operatorname{td}(T_Y)"] \\ A(X)_{\mathbb{Q}} \arrow[r, "f_*"] & A(Y)_{\mathbb{Q}} \end{tikzcd}$$

commutes. In other words, $f_*(\operatorname{ch}(-) \cdot \operatorname{td}(T_X)) = \operatorname{ch}(f_*(-)) \cdot \operatorname{td}(T_Y)$.

First, we will see why this implies the classical (Hirzebruch)-Riemann-Roch theorem. First, let X be a curve, $Y = \mathrm{pt}$, and $\mathcal{L}$ be a line bundle. Then the right hand side is

$$\operatorname{ch}(f_*[\mathcal{L}]) = (-1)^i H^i(X, \mathcal{L}) = \chi(X, \mathcal{L}).$$

On the other side, we have

$$\begin{aligned} f_*(1 + c_1(\mathcal{L}))(1 + c_1(T_X)/2) &= f_*(c_1(\mathcal{L}) + c_1(\mathcal{T}_{\mathcal{X}})/2) \\ &= \deg(\mathcal{L}) - \frac{1}{2}\deg(K) \\ &= \deg(\mathcal{L}) - g + 1. \end{aligned}$$

This recovers the classical Riemann-Roch theorem.

Now let X be a smooth surface. Then our formula becomes

$$\begin{aligned} \chi(X, \mathcal{L}) &= f_*\left(\left(1 + c_1(\mathcal{L} + \frac{c_1^2(\mathcal{L})}{2})\right)\left(1 + \frac{c_1(T_X)}{2} + \frac{c_1^2(TX) + c_2(TX)}{2}\right)\right) \\ &= \deg\left(\frac{c_1^2(\mathcal{L})}{2} - \frac{c_1(\mathcal{L})c_1(\Omega_X)}{2} + \frac{c_1^2(\Omega_X) - c_2(\Omega_X)}{12}\right). \end{aligned}$$

This recovers the Hirzebruch-Riemann-Roch theorem.

8.2 Proof for $\mathbb{P}^m \to \mathrm{pt}$

First, we have the following theorem:

Theorem 8.2.1. $K(\mathbb{P}^m)$ *is generated by* $[\mathcal{O}_{\mathbb{P}^m}(n)]$ *for* $0 \leqslant n \leqslant m$.

To prove this, we need to find a resolution of a coherent sheaf $\mathcal{F}$ by line bundles, so we have

$$0 \to \bigoplus \mathcal{L}_{\rangle}^{(n)} \to \cdots \to \bigoplus \mathcal{L}_i^{(0)} \to \mathcal{F} \to 0.$$

Because $\mathcal{O}(1)$ is ample, $\mathcal{F} \otimes \mathcal{O}(N)$ is generated by global sections, so we have the exact sequence

$$0 \to \cdots \to \bigoplus \mathcal{O} \to \mathcal{F} \otimes \mathcal{O}(N) \to 0,$$

and we can tensor this by $\mathcal{O}(-N)$. Now if we set $V = \bigoplus^{m+1} \mathcal{O}(-1)$, we can take the Koszul complex

$$0 \to \bigwedge^{m+1} V \to \cdots \to \bigwedge^0 V \to 0.$$

Now we have the exact sequence

$$0 \to \mathcal{O}(-m-1) \to \cdots \to \bigoplus \mathcal{O}(-m-1+j) \to \cdots \to \mathcal{O} \to 0.$$

In particular, this means that we can generate $K(\mathbb{P}^m)$ by $\{\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(m)\}$.

Now it remains to prove Grothendieck-Riemann-Roch just for $\mathcal{O}(n), 0 \leqslant n \leqslant m$. Here, because Y is a point, we have $\operatorname{ch}(f_*(-)) \operatorname{td}(T_*) = \chi(X, -)$. In our case, we know that for $0 < i < m$, $H^i(\mathbb{P}^m, \mathcal{O}(n)) = 0$ and $H^m(\mathbb{P}^m, \mathcal{O}(n)) = H^0(\mathbb{P}^n, \mathcal{O}(-n-m-1))^{\vee} = 0$, so

$$\chi(\mathbb{P}^n, \mathcal{O}(n)) = h^n(\mathbb{P}^n, \mathcal{O}(n)) = \binom{n+m}{m}$$

by basic combinatorics.[1]

On the other hand, we want to compute $f_*(\operatorname{ch}(\mathcal{O}(n)) \operatorname{td}(T_{\mathbb{P}^m}))$. Recall the Euler exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus m+1} \to T_{\mathbb{P}^m} \to 0.$$

Therefore we have $\operatorname{td}(T_{\mathbb{P}^m}) = \operatorname{td}(\mathcal{O}_{\mathbb{P}^n})^{m+1}$. Also, $\operatorname{ch}(\mathcal{O}(n)) = \operatorname{ch}(\mathcal{O}(1))^n = e^{xn}$. Looking for the x^m-coefficient in $\frac{e^{xn} x^{m+1}}{(1-e^{-x})^{m+1}}$, we can multiply by x^{m+1} and compute the residue to obtain $\binom{m+n}{n}$.

Remark 8.2.2. Grothendieck wrote the following about this result:

> *Um dieser Aussage über* $f \colon X \to Y$ *einen approximativen Sunn zu geben, musste ich nahezu zwei Stunden lang die Geduld der Zuhörer missbrauchen. Schwartz auf weiss (in Springer's Lecture Notes) nimmt's wohl an die 400, 500 Seiten. Ein packendes Beispiel dafür, wie unser Wissens - und Entdeckungsdrang sich immer mehr in einem lebensentrücken logischen Delirium auslabt, während das Leben selbst auf tausendfache Art zum Teufel geht – und mit endgültiger Vernichtung bedroht ist. Höchste Zeit, unsern Kurs zu ändern!*

The English translation of this,[2] keeping sentence structure intact as much as possible, is

[1] If you are unsure of this, this is an exercise in Giulia's homework. If you need a hint, please email Caleb, who has several publications in combinatorics, at `calebji@math.columbia.edu`.

[2] Provided by the person who originally scanned the note and provided me with the transcription.

To give this statement about $f\colon X \to Y$ *an approximate meaning I'd have to abuse the patience of the listeners for nearly two hours. Black on white (in Springer's Lecture Notes) it should take about 400, 500 pages. A gripping example of how our thirst for knowledge and discovery indulges more and more in a life-divorced logical delirium, while life itself is going to hell in a thousand different ways – and is threatened with absolute extinction. High time to change our course!*

9

Patrick (Apr 02): Fine moduli memes for 1-categorical teens

Note: these are the speaker's notes. Throughout this lecture, we work over $\mathbb{C}$.

9.1 The moduli space

The space $\overline{M}_{0,n}$ parameterizes curves that look like this:

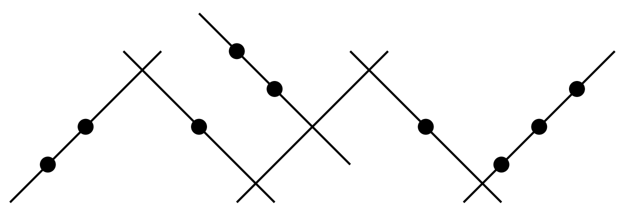


Figure 9.1: A stable curve

More precisely, these are reduced connected curves that are a tree of $\mathbb{P}^1$s such that each $\mathbb{P}^1$ has at least three marked points or nodes. In addition, at most two components meet at each node and we want $H^1(C, \mathcal{O}_C) = 0$. If all of these conditions are satisfied, we call our curve *stable*. More precisely, we want to represent the functor

$$S \mapsto \left\{ \begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] & S \arrow[l, bend left, "s_1,\dots,s_n"] \end{tikzcd} \text{ flat, proper } \middle| \begin{array}{c} s_1, \dots, s_n \text{ disjoint sections} \\ \text{geometric fibers are stable curves} \end{array} \right\}.$$

Theorem 9.1.1 (Knudsen)**.** *There exists a smooth complete variety $\overline{M}_{0,n}$ and universal curve $U_{0,n} \to \overline{M}_{0,n}$ with universal sections $s_1, \dots, s_n$ that is a fine moduli space for this functor. $\overline{M}_{0,n}$ also contains the space $M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$ as a dense open subset.*

In fact, Knudsen also shows that $U_{0,n} = \overline{M}_{0,n+1}$ and $U_{0,n+1}$ is a blowup of $\overline{M}_{0,n+1} \times_{\overline{M}_{0,n}} \overline{M}_{0,n+1}$ along some subscheme of the diagonal. In order to prove this, Knudsen introduces two operations that we can perform, called contraction and stabilization. Contraction happens when we delete a marked point and stabilization happens when we add a marked point.

Theorem 9.1.2 (Knudsen)**.** *Contraction and stabilization are functorial! Moreover, they commute with base change.*

Here are some pictorial depictions of our operations:

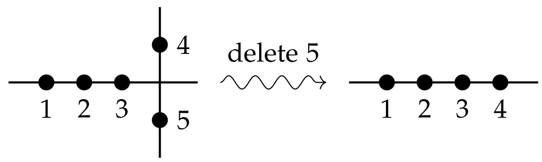


Figure 9.2: Contraction

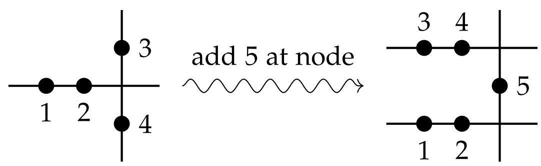


Figure 9.3: Stabilization (1)

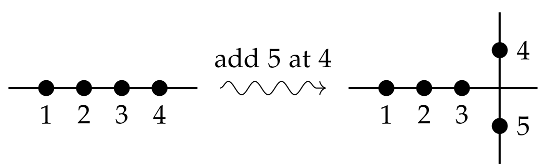
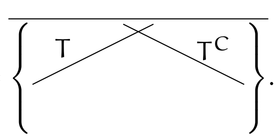


Figure 9.4: Stabilization (2)

9.2 Keel construction of $\overline{M}_{0,n}$

First, we will describe the boundary divisors of $\overline{M}_{0,n}$. Let $T \subseteq \{1, \ldots, n\} =: [n]$ satisfy $|T|, \left|T^C\right| \geqslant 2$. Then define

$D^T :=$



It is easy to see that $D^T = D^{(T^C)}$. Knudsen proves that D^T is a smooth divisor and that $D^T \cong \overline{M}_{|T|+1} \times \overline{M}_{|T^C|+1}$.

Now consider the map $\pi \colon \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ coming from the identification $\overline{M}_{0,n+1} = U_{0,n}$. Now Keel proves that we can factor π as

$$\overline{M}_{0,n+1} \xrightarrow{\pi_1=(\pi,\pi_{1,2,3,n+1})} \overline{M}_{0,n} \times \overline{M}_{0,4} \xrightarrow{p_1} \overline{M}_{0,n},$$

where $\pi_{1,2,3,n+1} \colon \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ forgets all sections besides $1, 2, 3, n+1$. Next, Keel shows that π_1 is a composition of blowups along smooth codimension 2 subvarieties using an inductive construction.

Set $B_1 = \overline{M}_{0,n} \times \overline{M}_{0,4}$. Then the universal sections $s_1, \ldots, s_n \colon \overline{M}_{0,n} \to \overline{M}_{0,n+1}$ induce sections $p \circ s_1, \ldots, p \circ s_n$. In fact, $D^T \cong p \circ s_i(D^T)$ and this is independent of i. We also have

Lemma 9.2.1 (Keel)**.** *The divisors $D^T \subset \overline{M}_{0,n+1}$ with $T \subset [n]$ are the exceptional divisors of π_1.*

Now we set B_2 to be the blowup of B_1 at $\bigcup_{|T^c|=2} D^T$. Inductively, we set B_{k+1} to be the blowup of B_k at $\bigcup_{|T^c|=k+1} D^T$. Now we summarize the main results as follows:

Theorem 9.2.2 (Keel)**.** *The map π_1 factors through B_k and $\overline{M}_{0,n+1} = B_{n-2}$.*

9.3 Intersection theory of $\overline{M}_{0,n}$

There are several major results about the intersection theory of $\overline{M}_{0,n}$. In fact, once we state these results, we will only be seven pages through Keel's paper, and the rest of the paper is dedicated to proving these results.

Theorem 9.3.1. *We have an isomorphism $A_*(\overline{M}_{0,n}) \to H_*(\overline{M}_{0,n})$. In particular, $\overline{M}_{0,n}$ has no odd homology and $A_*(\overline{M}_{0,n+1})$ is a finitely generated free abelian group. In fact, if a scheme Y satisfies $A^*(Y) = H^*(Y)$, then so does $Y \times \overline{M}_{0,n}$.*

Theorem 9.3.2. *For any scheme S, there is an isomorphism $A^*(\overline{M}_{0,n} \times S) = A^*(\overline{M}_{0,n}) \otimes A^*(S)$.*

Theorem 9.3.3. *For all k, we have an isomorphism*

$$A^k(\overline{M}_{0,n+1}) \cong A^k(\overline{M}_{0,n}) \oplus A^{k-1}(\overline{M}_{0,n}) \oplus \bigoplus_{\substack{T \subset [n] \\ |T \cap [3]| \leqslant 1}} A^{k-1}(D^T)$$

which is induced by the maps

$$A^k(\overline{M}_{0,n}) \xrightarrow{\pi^*} A^k(\overline{M}_{0,n+1})$$
$$A^{k-1}(\overline{M}_{0,n}) \xrightarrow{\pi^*} A^{k-1}(\overline{M}_{0,n+1}) \xrightarrow{\cup \pi^*_{1,2,3,n+1}(c_1(\mathcal{O}(1)))} A^k(\overline{M}_{0,n+1})$$
$$A^{k-1}(D^T) \xrightarrow{g^*} A^{k-1}(D^{T \subset [n+1]}) \xrightarrow{j_*} A^k(\overline{M}_{0,n+1}),$$

where g, j are as in the diagram

$$\begin{tikzcd}
D^{T \subset [n+1]} \arrow[r, hook, "j"] \arrow[d, "g"] & \overline{M}_{0,n+1} \arrow[d, "\pi"] \\
D^T \arrow[r, hook, "i"] & \overline{M}_{0,n}.
\end{tikzcd}$$

Theorem 9.3.4. *The Chow groups* $A^k(\overline{M}_{0,n})$ *are free abelian and the ranks* $a^k(n) = \mathrm{rk}(A^k(\overline{M}_{0,n}))$ *are given by the recursive formula*

$$a^k(n+1) = a^k(n) + a^{k-1}(n) + \frac{1}{2}\sum_{j=2}^{n-2}\binom{n}{j}\sum_{\ell=0}^{k-1} a^\ell(j+1)a^{k-1-\ell}(n-j-1).$$

In particular, we have the Picard rank $a^1(n) = 2^{n-1} - \binom{n}{2} - 1$.

Theorem 9.3.5. *The Chow ring* $A^*(\overline{M}_{0,n})$ *is the quotient of* $\mathbb{Z}[D^T \mid T \subset [n], |T|, |T^C| \geqslant 2]$ *by the relations*

1. $D^T = D^{(T^C)}$;
2. *For any distinct* $i, j, k, \ell \in [n]$, *we have the equality*
$$\sum_{\substack{i.j \in T \\ k,\ell \notin T}} D^T = \sum_{\substack{i,k \in T \\ j,\ell \notin T}} D^T = \sum_{\substack{i,\ell \in T \\ j,k \notin T}} D^T.$$
3. *For* $T_1, T_2 \subset [n]$, $D^{T_1}D^{T_2} = 0$ *unless one of* $T_1 \subset T_2, T_2 \subset T_1, T_1 \subset T_2^C, T_2 \subset T_1^C$ *holds.*

Remark 9.3.6. All of the relations encode geometric content:

1. As divisors, we already know that $D^T = D^{(T^C)}$.
2. If we consider the map $\pi_{i,j,k,\ell}\colon \overline{M}_{0,n} \to \overline{M}_{0,4}$, then the three sums are the pullbacks of the three boundary divisors $D^{i,j}, D^{i,k}, D^{i,\ell} \subset \overline{M}_{0,4} = \mathbb{P}^1$.
3. The final relation encodes the fact that $D^{T_1} \cap D^{T_2} = \emptyset$ unless one of the four inclusions holds. Pictorially, this is encoded in the diagram below:

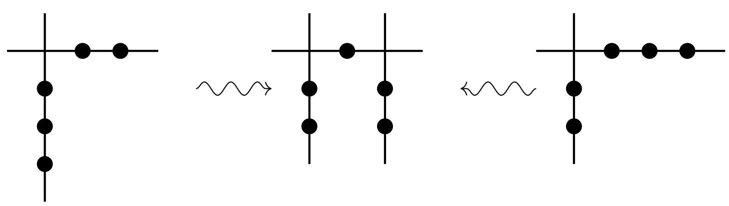


Figure 9.5: Degeneration to a common stable curve

9.4 Intersection theory of regular blowups

Let $i\colon X \subset Y$ be a regularly embedded subvariety, $\pi\colon \widetilde{Y} \to Y$ be the blowup along X, and $\widetilde{X}$ be the exceptional divisor. Let $g\colon \widetilde{X} \to X$ and $j\colon \widetilde{X} \to \widetilde{Y}$.

Theorem 9.4.1. *Suppose* i^* *is surjective. Then*

$$A^*(\widetilde{Y}) = \frac{A^*(Y)[T]}{(P(T), T \cdot \ker(i^*))},$$

where $P(T)$ *has constant term* $[X]$ *and* $i^*P(T) = T^d + T^{d-1}c_1(N_XY) + \cdots + c_d(N_XY)$, *where* d *is the codimension of* X *in* Y. *This is induced by* $-T = [\widetilde{X}]$.

Theorem 9.4.2. *A scheme* X *is HI if* $A_*(X) = H_*(X)$. *If* X, Y *are both HI, then so is* $\widetilde{Y}$.

Theorem 9.4.3. *The map*

$$A_k(Y) \oplus A_{k-1}(X) \xrightarrow{(\pi^*, j_* g^*)} A_k(\widetilde{Y})$$

is an isomorphism.

10

Patrick (Apr 09): Do we even need derived categories?

Note: these are the speaker's notes.

10.1 Serre formula

Recall the following from my March 19 lecture (this is Proposition 7.2.9 in my notes):

Proposition. *Let X be a smooth variety, $V, W \subset X$ be closed subschemes that intersect properly, and Z be an irreducible component of $V \cap W$. Then*

$$V \cdot W = \sum_Z i(Z, V \cdot W; X) \cdot [Z],$$

where $1 \leqslant i(Z, V \cdot W; X) \leqslant \ell(\mathcal{O}_{V \cap W, Z})$ is the intersection number and $i(Z, V \cdot W; X) = \ell(\mathcal{O}_{V \cap W, Z})$ if and only if the local ring is Cohen-Macaulay.

However, most rings are **not** Cohen-Macaulay, so we would like a formula to compute the intersection multiplicities in all cases. Serre gives a formula in terms of higher Tor functors (because $\mathcal{O}_{V \cap W, Z} = \mathcal{O}_{V,Z} \otimes \mathcal{O}_{W,Z}$). Before we state the formula, first we will state some results about the higher Tor functors.

First, if X is a locally ringed space and $\mathcal{F}, \mathcal{G}$ are modules on X, then

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x = \mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

This follows from the construction of the derived tensor product $\otimes^L$ in Stacks, which exposits derived categories much better than I ever could.

Lemma 10.1.1. *Let X be a locally Noetherian scheme. If $\mathcal{F}, \mathcal{G}$ are coherent, so is $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Also, if $L, K \in D^-_{\mathrm{coh}}(\mathcal{O}_X)$ (this means bounded above complexes of quasicoherent sheaves with coherent homology), then so is $L \otimes^L K$.*

Proof of this fact is pure homological algebra, and again can be found in the Stacks project.

Lemma 10.1.2. *Let X be a smooth variety and $\mathcal{F}, \mathcal{G}$ be coherent sheaves. Then $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is supported on $\mathrm{Supp}\,\mathcal{F} \cap \mathrm{Supp}\,\mathcal{G}$, and is nonzero only when $0 \leqslant i \leqslant \dim X$.*

Proof. The support condition is clear by looking at the stalks, so we need to consider when the stalks are nonzero. Here, we note that because X is smooth, the local rings $\mathcal{O}_{X,x}$ are regular local rings. By a result of Serre (Theorem 4.4.1 in my commutative algebra notes), $\mathcal{O}_{X,x}$ has finite global dimension $\dim \mathcal{O}_{X,x} \leqslant \dim X$. Here, the global dimension and the dimension are the same by Theorem 4.3.12 of my commutative algebra notes.[1] $\square$

Now we can compute the intersection multiplicities as (Stacks gives this as the definition of intersection multiplicity)

$$i(Z, V \cdot W; X) = \sum (-1)^i \ell(\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{V,Z}, \mathcal{O}_{W,Z})).$$

This formula is due to Serre, and Stacks writes the total intersection as

$$W \cdot V = \sum (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)].$$

Remark 10.1.3. Stacks writes the intersection multiplicity as $e(X, V \cdot W, Z)$. I am using the notation in Fulton's book.

Lemma 10.1.4. *Assume that $\ell(\mathcal{O}_{V \cap W, Z}) = 1$. Then $i(Z, V \cdot W; X) = 1$ and V, W are smooth in a general point of Z.*

Proof. Write $A = \mathcal{O}_{X,Z}$. Then $\dim A = \dim X - \dim Z$. Let I, J be the ideals of V, W. By Proposition 7.2.9 of the notes,[2] $I + J = \mathfrak{m}$. Thus there exists $f_1, \ldots, f_r \in I, g_1, \ldots, g_s \in J$ Forming a basis for $\mathfrak{m}/\mathfrak{m}^2$. But this is a regular sequence and a system of parameters, so $A/(f_1, \ldots, f_r)$ is a regular local ring of dimension $\dim X - \dim V$, so $I = (f_1, \ldots, f_r)$. Similarly, $J = (g_1, \ldots, g_s)$. Now by Corollary 4.4.3 of commutative algebra, the Koszul complex $K(f_1, \ldots, f_r, A)$ resolves A/I, so we obtain

$$\begin{aligned}\mathrm{Tor}_i^A(A/I, A/J) &= H_i(K(f_1, \ldots, f_r, A) \otimes A/J) \\ &= H_i(K(f_1, \ldots, f_r, A/J)).\end{aligned}$$

By Theorem 4.4.2 from commutative algebra, we only have $H_0 = k$. $\square$

Example 10.1.5. Suppose $V, W \subset X$ are closed subvarieties, $\dim X = 4$, $\widehat{\mathcal{O}}_{X,p} = \mathbb{C}[[x, y, z, w]]$ and $V = (xz, xw, yz, yw), W = (x - z, y - w)$. Then $\ell(\mathbb{C}[[x, y, z, w]]/(xz, xw, yz, yw, x - z, x - w)) = 3$, but the intersection multiplicity is 2 because V is locally a union $(x = y = 0) \cup (z = w = 0)$.

10.2 Some algebra

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. If M is a module and I is an ideal of definition, recall the Hilbert-Samuel polynomial $\varphi_{I,M}(n) = \ell(I^n M / I^{n+1} M)$. Similarly recall the function

$$\chi_{I,M}(n) = \ell(M/I^{n+1}M) = \sum_{i=0}^{n} \varphi_{I,M}(i).$$

Recall that $d(M) \coloneqq \deg \chi$ is independent of I and equals the dimension of the support of M (from the proof of Theorem 3.2.9 in my commutative algebta notes). Now write $\chi_{I,M}(n) = e_I(M, d)\frac{n^d}{d!} + O(n^{d-1})$.

[1] Originally there was an argument that the global dimension of a Noetherian local ring is the projective dimension of the residue field, which is Theorem 4.3.10 of the commutative algebra notes, and then by the Auslander-Buchsbaum formula this is the same as the depth, and finally regular implies Cohen-Macaulay, so depth equals dimension.

[2] This may be cheating, and a self-contained argument is given in Stacks

Definition 10.2.1. For $d = d(M)$ we write $e_I(M, d)$ as above, and for $d > d(M)$, we set $e_I(M, d) = 0$.

Lemma 10.2.2. *For all* I, M, *we have*

$$e_I(M, d) = \sum_{\dim A/\mathfrak{p} = d} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) e_I(A/\mathfrak{p}, d).$$

Lemma 10.2.3. *Let* P *be a polynomial of degree* r *with leading coefficient* a. *Then*

$$r!a = \sum_{i=0}^{r} (-1)^i \binom{r}{i} P(t - i)$$

for any t.

Proof. Write Δ for the operator taking a polynomial P to $P(t) - P(t-1)$. Then

$$\begin{aligned}
\Delta^{r+1}(P) &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} \Delta(P)(t - i) \\
&= \sum_{i=0}^{r} (-1)^i \binom{r}{i} (P(t - i) - P(t - i - 1)).
\end{aligned}$$

The desired claim follows from Pascal's identity. □

Theorem 10.2.4. *Let* A *be a Noetherian local ring and* $I = (f_1, \ldots, f_r)$ *be an ideal of definition. Then*

$$e_I(M, r) = \sum (-1)^i \ell(H_i(K(f_1, \ldots, f_r) \otimes M)).$$

There is a very long proof of this statement in Stacks using spectral sequences.

10.3 Computing intersection multiplicities without derived categories

We give some cases where intersection multiplicities can be computed without using derived categories.

Lemma 10.3.1. *Suppose* $\mathcal{O}_{V,Z}$ *and* $\mathcal{O}_{W,Z}$ *are Cohen-Macaulay. Then* $i(Z, V \cdot W; X) = \ell(\mathcal{O}_{V \cap W, Z})$.

Proof. Write $A = \mathcal{O}_{X,Z}, B = \mathcal{O}_{V,Z}, C = \mathcal{O}_{W,Z}$. Then by Auslander-Buchsbaum (exercise 4d of the final CA homework), we have a resolution $F_\bullet \to B$ of length $\operatorname{depth} A - \operatorname{depth} B = \dim A - \dim B = \dim C$. Then $F_\bullet \otimes C$ represents $B \otimes^L C$ and is supported in $\{\mathfrak{m}_A\}$, so by Lemma 10.108.2 in Stacks, it has nonzero cohomology only in degree 0. □

Lemma 10.3.2. *Let* A *be a Noetherian local ring and* $I = (f_1, \ldots, f_r)$ *is generated by a regular sequence. If* M *is a finite* A*-module with* $\dim \operatorname{Supp} M/IM = 0$, *then*

$$e_I(M, r) = \sum (-1)^i \ell(\operatorname{Tor}_i^A(A/I, M)).$$

In what follows, we will assume V is cut out in $\mathcal{O}_{X,Z}$ by a regular sequence $(f_1, \ldots, f_c)$.

Lemma 10.3.3. *In this case, we have* $i(Z, V \cdot W; X) = c!$. *This is the leading coefficient of the "Hilbert polynomial"* $n \mapsto \ell(\mathcal{O}_{W,Z}/(f_1, \ldots, f_c)^t)$.

Proof. By the previous lemma, $e(Z, V \cdot W; X) = e_{(f_1,\dots,f_c)}(\mathcal{O}_{W,Z}(c))$. Now we need to show that $\dim \mathcal{O}_{W,Z} = c$. But now if $\dim V = r, \dim W = s, \dim X = n$, $\dim Z = r+s-n$, so $k(Z)$ has transcendence degree $r+s-n$. Because $f_1, \dots, f_c$ is a regular sequence, $r+c = n$, so $\dim \mathcal{O}_{W,Z} = s-(r+s-n) = s-(n-c+s-n) = c$. □

Lemma 10.3.4. *Assume* $c = 1$ *(for example,* V *is an effeective Cartier divisor). Then* $i(Z, V \cdot W; X) = \ell(\mathcal{O}_{W,Z}/(f_1))$.

Proof. Note that $\mathcal{O}_{W,Z}$ is a Noetherian local domain of dimension 1. Then it is clear that $\ell(\mathcal{O}_{W,Z}/(f_1^t)) = t\ell(\mathcal{O}_{W,Z}/(f_1))$ for all $t \geqslant 1$. □

Lemma 10.3.5. *Asssume* $\mathcal{O}_{W,Z}$ *is Cohen-Macaulay. Then* $i(Z, V \cdot W; X) = \ell(\mathcal{O}_{W,Z}/(f_1, \dots, f_c))$.

Proof. Because $f_1, \dots, f_c$ is a regular sequence, it is also quasi-regular by Proposition 3.5.6 of my commutative algebra notes. Then

$$\ell(\mathcal{O}_{W,Z}/(f_1, \dots, f_c)^t) = \binom{c+t}{c}\ell(\mathcal{O}_{W,Z}/(f_1, \dots, f_c)).$$

Now take the leading coefficient. □

11

Caleb (Apr 16): Motivic cohomology

Note: these are the speaker's notes. No changes were made except to bring notational and stylistic conventions in line with the rest of the notes and to adapt pictures that were not available to me. Also, the quote of Grothendieck was changed to the original French.

These are my notes on motivic cohomology. Essentially everything here is based off of Voevodsky's lectures, now turned into a book by Mazza and Weibel.

11.1 Bloch's higher Chow groups

11.1.1 Some topological motivation

Recall the following exact sequence.

Proposition 11.1.1 (Fulton, Prop. 1.8)**.** *Let* Y *be a closed subscheme of a scheme* X*, and let* $U = X - Y$*. Let* $i : Y \to X, j : U \to X$ *be the inclusions. Then the sequence*

$$\mathrm{CH}^k Y \xrightarrow{i_*} \mathrm{CH}^k X \xrightarrow{j^*} \mathrm{CH}^k U \to 0$$

is exact for all k*.*

This is great, but if we want homology to continue to the left. We certainly cannot put these sequences together. In fact, the indexing is a bit misleading in this way – instead, we should put the ks together for a grading of $\mathrm{CH}^*(-)$. So, we really do need new groups if we want to continue this sequence. To construct these groups, we first recall the following definition of rational equivalence.

Definition 11.1.2 ($\mathrm{Rat}(X)$)**.** Let $Z(X)$ denote the cycles of a scheme X and let Φ be any subvariety of $X \times \mathbb{P}^1$. Then we define $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$[\Phi \cap (X \times \{0\})] - [\Phi \cap (X \times \{\infty\})].$$

Then rationally equivalent cycles are those which differ by something in $\mathrm{Rat}(X)$. We see that it looks like there is a homotopy between rationally equivalent cycles.

11.1.2 Definition

Motivated by algebraic topology, we define the algebraic simplex

$$\Delta^k = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0 + \cdots + x_k - 1).$$

Let $z^i(X, n)$ be the subgroup of $Z^i(X \times \Delta^n)$ that meet all faces properly. This gives both a simplicial abelian group $z^i(X, \bullet)$ and a chain complex $z^i(X, *)$.

Definition 11.1.3. The higher Chow groups $CH^i(X, m)$ are defined

$$CH^i(X, m) \coloneqq \pi_m(z^i(X, \bullet)) = H_m(z^i(X, *)).$$

11.1.3 Properties

1. Homotopy invariance:

   The projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism

   $$CH^i(X, m) \cong CH^i(X \times \mathbb{A}^1, m).$$

2. Long exact sequence:

   There is a distinguished triangle

   $$z_p(Y, *) \to z_p(X, *) \to z_p(U, *) \to z_p(Y, *)[1].$$

3. Isomorphism with rational K-theory:

   $$(K_i(X) \otimes \mathbb{Q})^{(q)} \cong CH^q(X, i) \otimes \mathbb{Q}.$$

We will see that $H^{p,q}(X; A) = CH^q(X, 2q - p; A)$. In particular, we have $H^{2q,q}(X, A) = CH^q(X) \otimes A$.

11.2 The category of correspondences

11.2.1 Correspondences Let $X, Y \in \mathsf{Sm}_k$ be smooth separated schemes of finite type over k. Very informally, one can think of $\mathsf{Cor}(X, Y)$ as a generalization of $\mathrm{Hom}(X, Y)$ to multivalued morphisms.

Definition 11.2.1. An *elementary correspondence* between a smooth connected scheme X/k to a separated scheme Y/k is an irreducible closed subset $W \subset X \times Y$ whose associated integral subscheme is finite and surjective over X.

If X is not connected, then an elementary correspondence refers to one that is one from a connected component of X to Y.

The group $\mathsf{Cor}(X, Y)$ of *finite correspondences* is the free abelian group generated by the elementary correspondences.

Then given a closed subscheme $Z \subset X \times Y$ finite and surjective over X, we can associate the finite correspondence $\sum n_i W_i$ where W_i are the irreducible components of the support of Z surjective over a component of X with generic points ξ_i and $n_i = \ell(O_{Z,\xi_i})$.

11.2.2 The category of correspondences We compose correspondences $V \in \mathsf{Cor}_k(X, Y)$ and $W \in \mathsf{Cor}_k(Y, Z)$ as follows. Construct the cycle $[T] = (V \times Z) \cdot (X \times W)$ on $X \times Y \times Z$. Then take its pushforward along the projection $p : X \times Y \times Z \to X \times Z$.

It is not difficult to check that Sm_k embeds into Cor_k as a subcategory, where $f : X \to Y$ becomes the graph $\Gamma_f \subset X \times Y$.

Furthermore, Cor_k is a symmetric monoidal category. Indeed, the tensor product is simply $X \otimes Y = X \times Y$. Given $V \in \mathsf{Cor}_k(X, X')$ and $W \in \mathsf{Cor}_k(Y, Y')$, we get the desired cycle $V \times W \in \mathsf{Cor}_k(X \otimes Y, X' \otimes Y')$.

11.2.3 Examples

1. $\mathsf{Cor}_k(\operatorname{Spec} k, X)$ is generated by the 0-cycles of X.
2. $\mathsf{Cor}_k(X, \operatorname{Spec} k)$ is generated by the irreducible components of X.
3. Take $W \in \mathsf{Cor}_k(\mathbb{A}^1, X)$ and two k-points $s, t : \operatorname{Spec} k \to \mathbb{A}^1$. Then the zero-cycles $W \circ \Gamma_s$ and $W \circ \Gamma_t$ are rationally equivalent.

11.3 Presheaves with transfers

11.3.1 Definition

Definition 11.3.1. A *presheaf with transfers* is a contravariant additive functor $F : \mathsf{Cor}_k \to \mathsf{Ab}$.

Additivity gives a map

$$\mathsf{Cor}_k(X, Y) \otimes F(Y) \to F(X).$$

Thus there are extra "transfer maps" $F(Y) \to F(X)$ coming from $\mathsf{Cor}_k(X, Y)$.

Theorem 11.3.2. $\mathsf{PST}(k)$ *is an abelian category with enough injectives and projectives.*

11.3.2 Examples

Example 11.3.3. The constant presheaf A on Sm_k can be extended to a pst.

For $W \in \mathsf{Cor}(X, Y)$ with X, Y connected, the corresponding homomorphism $A \to A$ is multiplication by the degree of W over X.

Example 11.3.4. $\mathcal{O}^*$ and $\mathcal{O}$, at least for X normal. Use the norm and trace maps.

$$\begin{tikzcd} \mathcal{O}^*(Y) \arrow[rr] \arrow[dr] & & \mathcal{O}^*(X) \\ & \mathcal{O}^*(W) \arrow[ur, "N"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}(Y) \arrow[rr] \arrow[dr] & & \mathcal{O}(X) \\ & \mathcal{O}(W) \arrow[ur, "\mathrm{Tr}"'] & \end{tikzcd}$$

Example 11.3.5. $\mathrm{CH}^i(-)$, the Chow groups.

Example 11.3.6. Representable functors: $h_X(-)$

11.3.3 Representable functors of $\mathrm{Cor}_k(X)$ Take $X \in \mathrm{Ob}(\mathrm{Cor}_k(X))$. We denote

$$\mathbb{Z}_{tr}(X) \coloneqq h_X(-).$$

By Yoneda, $\mathbb{Z}_{tr}(X)$ is a projective object in $\mathrm{PST}(k)$.

Note that $\mathbb{Z}_{tr}(\operatorname{Spec} k)$ is just the constant sheaf $\mathbb{Z}$ on Sm_k, with the transfer maps constructed in the example from the previous subsection. Let (X, x) be a pointed scheme. We define

$$\mathbb{Z}_{tr}(X, x) \coloneqq \operatorname{coker}[x_* : \mathbb{Z} \to \mathbb{Z}_{tr}(X)].$$

The structure map $X \to \operatorname{Spec} k$ provides a splitting, so

$$\mathbb{Z}_{tr}(X) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x).$$

Out of laziness we screenshot the following definitions from Voevodsky's lectures.

Definition 11.3.7. If (X_i, x_i) are pointed schemes for $i = 1, \ldots, n$ we define $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$, or $\mathbb{Z}_{tr}(X_1 \wedge \cdots X_n)$, to be:

$$\operatorname{coker}\left(\bigoplus_i \mathbb{Z}_{tr}(X_1 \times \cdots \widehat{X}_i \cdots \times X_n) \xrightarrow{id \times \cdots \times x_i \times \cdots \times id} \mathbb{Z}_{tr}(X_1 \times \cdots X_n)\right).$$

Lemma 11.3.8. *The presheaf $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ is a direct summand of $\mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)$. In particular, it is a projective object of* PST.

Moreover, the following sequence of presheaves with transfers is split-exact:

$$0 \to \mathbb{Z} \xrightarrow{x_i} \bigoplus_i \mathbb{Z}_{tr}(X_i) \to \bigoplus_{i,j} \mathbb{Z}_{tr}(X_i \times X_j) \times \cdots$$
$$\cdots \to \bigoplus_{i,j} \mathbb{Z}_{tr}(X_1 \times \cdots \widehat{X}_i \cdots \widehat{X}_j \cdots \times X_n) \to \bigoplus_i \mathbb{Z}_{tr}(X_1 \times \cdots \widehat{X}_i \cdots \times X_n) \to$$
$$\to \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n) \to \mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n) \to 0.$$

Consider the pointed scheme $(\mathbb{G}_m, 1)$. We will be interested in the presheaf with transfers $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$.

Before continuing, we recall our construction

$$\Delta^k = \operatorname{Spec} k[x_0, \ldots, x_n]/(x_0 + \cdots + x_k - 1).$$

Recall that a simplicial object of a category C is a functor $F : \Delta^{op} \to C$. Then if F is a presheaf of abelian groups on Sm_k, then $F(U \times \Delta^\bullet)$ is a simplicial abelian group. Then

$$C_\bullet F : U \mapsto F(U \times \Delta^\bullet)$$

is a simplicial presheaf with transfers. Similarly, $C_* F(U)$ gives the complex of abelian groups

$$\cdots \to F(U \times \Delta^2) \to F(U \times \Delta^1) \to F(U) \to 0.$$

11.3.4 Homotopy invariant presheaves

Definition 11.3.9. A presheaf F is *homotopy invariant* if for every X, the map $p^* : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism.

Note that this is equivalent to p^* being surjective. We can check that an equivalent condition is that for all X, we have

$$i_0^* = i_1^* : F(X \times \mathbb{A}^1) \to F(X).$$

Furthermore, if F is any presheaf, we have that $i_0^*, i_1^* : C_*F(X \times \mathbb{A}^1) \to C_*F(X)$ are chain homotopic. From this we deduce that if F is a presheaf, then the homology presheaves

$$H_n C_* F : X \mapsto H_n C_* F(X)$$

are homotopy invariant for all n.

Definition 11.3.10. Two finite correspondences from X to Y are $\mathbb{A}^1$-*homotopic* if they are the restrictions along $X \times 0$ and $X \times 1$ of an element of $\mathsf{Cor}(X \times \mathbb{A}^1, Y)$.

This is an equivalence relation on $\mathsf{Cor}(X, Y)$. Note that it is not one if we just look at morphisms of schemes! With this definition though, we define $f : X \to Y$ to be an $\mathbb{A}^1$-homotopy equivalence in the expected way.

11.4 Motivic cohomology

11.4.1 The motivic complex

Definition 11.4.1. For $q \in \mathbb{Z}_{\geqslant 0}$, the *motivic complex* $\mathbb{Z}(q)$ is defined as the following complex of presheaves with transfers.

$$\mathbb{Z}(q) \coloneqq C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q].$$

We can change coefficients to $A \in \mathsf{Ab}$ by setting $A(q) = \mathbb{Z}(q) \otimes A$.

These are actually complexes of sheaves with respect to the Zariski topology. In fact, they are also sheaves in the étale topology.

For example when $q = 0$, applying this to a scheme Y we just get

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

which is quasi-isomorphic to just $\mathbb{Z}$. When $q = 1$, the complex looks like

$$\cdots \to \mathsf{Cor}(Y \times \Delta^2, \mathbb{G}_m) \to \mathsf{Cor}(Y \times \Delta^1, \mathbb{G}_m) \to \mathsf{Cor}(Y, \mathbb{G}_m) \to 0.$$

11.4.2 Motivic cohomology groups

Definition 11.4.2. The *motivic cohomology groups* $H^{p,q}(X, \mathbb{Z})$ are defined to be the hypercohomology of the motivic complexes $\mathbb{Z}(q)$ with respect to the Zariski topology:

$$H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q)).$$

If A is any abelian group, we define

$$H^{p,q}(X, A) = \mathbb{H}^p_{Zar}(X, A(q)).$$

11.4.3 Weight 1 There is a quasi-isomorphism

$$\mathbb{Z}(1) \xrightarrow{\cong} \mathcal{O}^*[-1].$$

Thus we have the following table.

$H^{-2,2}$	$H^{-1,2}$	$H^{0,2}$	$H^{1,2}$	$H^{2,2}$	$H^{3,2}$
0	0	0	$\mathcal{O}^*(X)$	$\mathrm{Pic}(X)$	0
0	0	$\mathbb{Z}(X)$	0	0	0
0	0	0	0	0	0

Figure 11.1: Weight q motivic cohomology

11.5 Relation to other fields

11.5.1 Algebraic K-theory Atiyah-Hirzebruch:

$$E_2^{p,q} = H^p(X; K^q(*)) \Rightarrow K^{p+q}(X).$$

In the algebraic setting, it is much more difficult. Indeed, both algebraic K-theory and motivic cohomology are significantly harder to define than their topological counterparts. In 2002, Suslin and Friedlander built upon previous work of Bloch and Lichtenbaum to show the following spectral sequence.

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) = CH^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X).$$

11.5.2 Motives

> *C'est pour parvenir à exprimer cette intuition de "parenté" entre théories cohomologiques différentes, que j'ai dégagé la notion du "motif" associé à une variété algébrique. Par ce terme, j'entends suggérer qu'il s'agit du "motif commun" (or de la "raison commune") sous-jacent à cette multitude d'invariants cohomologiques différents associés à là variété, à l'aide de la multitude des toutes les théories cohomologiques possibles a priori.* Alexander Grothendieck

An rough English translation[1] of this is

> *In order to express this intuition of kinship between different cohomological, I the notion of "motive" associated to an algebraic variety. By this term, I hear the suggestion that it is the "common motive" (or "common reason") underlying the multitude of cohomological invariants associated to a variety, supporting all cohomological that are a priori possible.*

Grothendieck constructed Chow motives by replacing morphisms of schemes with correspondences (defined under rational equivalence, different from the correspondences discussed earlier), augmenting the category to look like an abelian category, and taking the opposite category. By construction, this works in that cohomology theories factor through it. However, to truly achieve what is desired from them, one must assume the standard conjectures on algebraic cycles (or some variants), which have been open for over 50 years!

Voevodsky used motivic cohomology to construct a triangulated category $\mathrm{DM}(k; R)$, which for all intents and purposes acts as the derived category of the desired category of motives. He studied mixed motives, which apply to all varieties (not just the smooth ones). These can be thought of as extensions of pure motives, and motivic cohomology studies these Ext groups.

11.5.3 Arithmetic geometry There's the Bloch-Kato conjecture and the Bloch-Kato conjectures, which are different!

The Bloch-Kato conjecture is now a theorem: the norm residue isomorphism theorem, proven by Voevodsky. Through proving it, Voevodsky developed motivic cohomology, motivic homotopy theory, motivic Steenrod algebra...

> *The norm residue isomorphism theorem (or Bloch-Kato conjecture) states that for a field k and an integer ℓ that is inverible in k, the norm residue map*
>
> $$\partial^n \colon K_n^M(k)/\ell \to H^n_{\acute{e}t}(k, \mu_\ell^{\otimes n})$$
>
> *from Milnor K-theory mod-ℓ to étale cohomology is an isomorphism. The case $\ell = 2$ is the Milnor conjecture, and the case $n = 2$ is the Murkurjev-Suslin theorem.* Wikipedia

The Bloch-Kato conjectures are on special values of L-functions.

Conjecture 11.5.1 (Soulé). *If X is regular and proper over $\operatorname{Spec} \mathbb{Z}$, then for an integer $n \in \mathbb{Z}$ we have*

$$\mathrm{ord}_{s=n} \zeta_X(s) = -\sum_{i \geqslant 0} (-1)^i \operatorname{rank}(H_i(X; \mathbb{Z}(n))).$$

Applied to elliptic curves, this implies (one of the two parts of) the Birch-Swinnerton Dyer conjecture!

[1] Provided by Patrick, whose French is not very good.