Symplectic Topology
Math 705

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Disclaimer

These notes were taken during lecture using the `vimtex` package of the editor `neovim`. Any errors are mine and not the instructor's. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style (omit lengthy computations, use category theory) and that of the instructor. If you find any errors, please contact me at `plei@umass.edu`.

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1.1 Course Description

This is an introductory course on symplectic topology, along with its connections to differential, complex algebraic, and contact geometry and topology.

1.2 Organization

İnanç passed around a syllabus. Prerequisites for this course are smooth manifolds, some algebraic topology (cohomology), and complex analysis. Because this is a second year graduate course, we are expected to read any missing background on our own. Grading will be based on homework and a final presentation. We will cover the first four topics on the syllabus and do some of the last four. Finally, İnanç may add more geometry to the course.

1.2.1 Notational conventions We will denote finite dimensional real vector spaces over $\mathbb{R}$ by V and smooth manifolds by M.

1.3 Basic Notions

Definition 1.1. A *symplectic form* (or a *symplectic* structure) on a vector space V is a nondegenerate alternating bilinear form $V \otimes V \to \mathbb{R}$.

Definition 1.2. A *symplectic form* (or symplectic structure) on a smooth manifold M is a differential form $\omega \in \Omega^2 M$ which is closed and everywhere nondegenerate.

Remark 1.3. A fundamental question to ask is when a manifold admits a symplectic structure. We will see that symplectic structures exist only on even-dimensional manifolds. Saying more is an extremely difficult problem, although we can say that symplectic manifolds admit an almost complex structure and are orientable. Uniqueness up to both symplectomorphism and deformation is also very difficult.

Example 1.4. Let $V = \mathbb{R}^{2n}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. Then we can define a form

$$\omega_0(u, v) = \sum_{i=0}^{n} (x_i y_i' - x_i' y_i) = -u^T J_0 v,$$

where

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Checking that ω_0 is a nondegenerate alternating bilinear form is easy. Later we will see that this is the only symplectic vector space.

Example 1.5. Consider $M = \mathbb{R}^{2n}$ with coordinates $x_1, y_1, \dots, x_n, y_n$. Now consider the form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Checking that this form is closed and nondegenerate is easy. We can also define the form on $\mathbb{C}^n$, where the form becomes

$$\frac{i}{2} \sum_{i=j}^{n} dz_j \wedge d\overline{z_j}.$$

Note that the requirement that symplectic forms are closed is a subtle condition. If we multiply by a general function, the resulting form will not be closed. Recall that smooth manifolds are locally Euclidean. We can require our transitions to lie in the symplectic group, and later we will show that this definition of a symplectic manifold is equivalent to the one we gave today.

For a general manifold M, we want to associate linear spaces to it.

Definition 1.6. A vector bundle E over M is a *symplectic vector bundle* if there exists a smooth section ω of $E^* \wedge E^*$ such that (E_x, ω_x) is a symplectic vector space for all $x \in M$.

Example 1.7. For a symplectic manifold M, the tangent bundle TM is a symplectic vector bundle.

Remark 1.8. Observe that if the tangent bundle is symplectic, then the manifold is not necessarily symplectic.

1.4 Symplectic Linear Algebra

Definition 1.9. For (V_i, ω_i) symplectic vector spaces, a *linear symplectomorphism* $\phi : (V_1, \omega_1) \to (V_2, \omega_2)$ is an isomorphism of vector spaces such that $\phi^* \omega_2 = \omega_1$. They (V_1, ω_1) and (V_2, ω_2) are *symplectomorphic.*

The usual definition of orthogonal complements carries over and will be denoted W^ω for a subspace $W \subset V$.

Lemma 1.10. *Let (V, ω) be a symplectic vector space.*

1. *$v \mapsto \omega(v, -)$ is an isomorphism $V \to V^*$.*
2. *$\dim W + \dim W^\omega = \dim V$.*
3. *$(W^\omega)^\omega = W$.*
4. *The following are equivalent:*
    a) *W is a symplectic subspace of V;*
    b) *W^ω is symplectic;*
    c) *$W \cap W^\omega = \{0\}$;*
    d) *$W \oplus W^\omega = V$.*

Proof.

1. This part is equivalent to nondegeneracy.
2. Use a rank-nullity argument to note that W^ω maps to the annihlator of W under the isomorphism $V \to V^*$.
3. Clearly $W \subset (W^\omega)^\omega$. Then use the dimension result.
4. Clearly a) implies c) and c) and d) are equivalent. Finally, it is easy to see that d) implies a). Finally equivalence of b) to the rest is easy.

□

Theorem 1.11 (Symplectic Basis). *For any symplectic vector space* (V, ω)*, there exists a basis* $u_1, \dots, u_n, v_1, \dots, v_n$ *such that*

$$\omega(u_i, u_j) = 0, \omega(v_i, v_j) = 0, \omega(u_i, v_j) = \delta_{ij},$$

called a symplectic basis for (V, ω).

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2.1 More Basic Linear Algebra

Last time we stated the following:

Theorem. *For any symplectic vector space (V, ω), there exists a basis $u_1, \dots, u_n, v_1, \dots, v_n$ such that $\omega(u_i, v_i) = 1$ and any other pairing is zero.*

Proof. We will induct on the dimension of the vector space. For $n = 1$, take any nonzero u. Then by nondegeneracy, we can find a desired v. Then we simply use Lemma 1.4.2, note that u_1, v_1 span a symplectic subspace W of V, and apply the inductive hypothesis to W^ω. Because u_1, v_1 are orthogonal to the symplectic basis for W^ω, we are done. ☐

Corollary 2.1. *Any symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.*

Proof. Observe that

$$\omega = \sum_{i=1}^{n} u_i^* \wedge v_i^*.$$

Define our morphism in the obvious way:

$$(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum x_i u_i + y_i v_i.$$

Then it is easy to check that

$$\phi^*\omega(x, x') = \omega(\phi x, \phi x') = \left(\sum u_i^* \wedge v_i^*\right)\left(\sum x_i u_i + y_i v_i, \sum x_i' u_i + y_i' v_i\right) = \sum x_i y_i' - x_i' y_i = \omega_0(x, x').$$

☐

Corollary 2.2. *A skew-symmetric ω on V is symplectic iff $\omega^n \neq 0$.*

Proof. If ω is symplectic, then there exists a symplectic basis $\{u_i, v_i\}$, and $\omega^n(u_1, v_1, \dots, u_n, v_n) \neq 0$. In the other direction, if ω is degenerate, there exists $u \in 0$ such that $\omega(u, v) = 0$ for all $v \in V$. Then we can complete u to a basis by $u_2, \dots, u_{2n}$, and here $\omega^n(u, u_2, \dots, u_{2n}) = 0$, so $\omega^n = 0$. ☐

2.2 Compatible Complex Structures and Inner Products

Tailoring the class to the audience, İnanc will return to the theme of the three geometries: Riemannian, symplectic, and complex. We will see that symplectic geometry lies between the other two geometries (every manifold admits a metric, complex algebraic structures are very rare).

The group of linear symplectomorphisms of (V,ω) is denoted by $Sp(V,\omega)$.

Definition 2.3. A real matrix $A \in GL_{2n}(\mathbb{R})$ is *symplectic* if $A^T J_0 A = J_0$, where J_0 was defined in Example 1.3.4. The group of such matrices is $Sp_{2n}(\mathbb{R})$.

By Corollary 2.1.1, we see that $Sp(V,\omega) \simeq Sp_{2n}(\mathbb{R})$.

Definition 2.4. A *complex structure* on a real vector space V is an automorphism $J : V \to V$ such that $J^2 = iI$. Then (V,J) is a complex vector space with the definition $(x+iy)v = xv + yJv$.

Then $\mathrm{Aut}(V,J) \simeq GL_n(\mathbb{C})$.

Definition 2.5. If (V,ω) is a symplectic vector space, a complex structure J is called *ω-compatible* if $\omega(Ju,Jv) = \omega(u,v)$ for all $u,v \in V$ and $\omega(v,Jv) > 0$ for all nonzero $v \in V$.

Let $\mathcal{J}(V,\omega)$ be the space of ω-compatible complex structures on (V,ω) with topology inherited from $\mathrm{End}\, V$ (here endomorphisms are taken in **Diff**). This space will turn out to be contractible, but first we need to show that it is nonempty. This is true because J_0 is compatible with the standard form. The taming property is also easy to check.

Recall that an inner product on a real vector space is a nondegenerate symmetric positive-definite bilinear form g. Here, $\mathrm{Aut}(V,g) \simeq O(2n,\mathbb{R})$.

Definition 2.6. A *Hermitian structure* on (V,J) is an inner product g on V such that $g(Ju,Jv) = g(u,v)$ for all $u,v \in V$.

Remark 2.7. If $\tilde{g}$ is any inner product on V, then $g(u,v) = \tilde{g}(u,v) + \tilde{g}(Ju,Jv)$ is Hermitian.

Remark 2.8. $J \in \mathcal{J}(V,\omega)$ if and only if $g_J(u,v) := \omega(u,Jv)$ is a Hermitian inner product.

Proof. Note that $\omega(Ju,Jv) = \omega(u,v)$ iff $\omega(Ju,-v) = \omega(u,Jv)$ iff $g_J(Ju,Jv) = g_J(u,v)$. In addition, $\omega(v,Jv) > 0$ iff $g_J(v,v) = 0$ clearly. □

Example 2.9. The standard symplectic form, the matrix J_0, and the standard inner product on $(\mathbb{R}^{2n},\omega)$ is a compatible triple.

Theorem 2.10. *$\mathcal{J}(V,\omega)$ is contractible.*

Proof is left to the next lecture because it will take too long for the rest of this lecture.

Remark 2.11. The analogous result for almost complex structures on symplectic manifolds will allow us to discuss Chern classes on symplectic manifolds.

Now let $\mathcal{G}(V)$ be the space of inner products on V. We can define $r_t : \mathcal{G}(V_t) \to \mathcal{J}(V_t,\omega_t)$ varying smoothly in t.[1]

Remark 2.12. Given a complex vector space (V,J) and a compatible inner product g, we can derive a symplectic structure ω on V for which J is ω-compatible by $\omega(u,v) = g(Ju,v)$.

Proof of this fact is left to the reader. Exercises will be assigned nect time.

[1]This will be a key ingredient in our proof because the space of inner products is convex.

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3.1 A Big Theorem

Last time we stated the following:

Theorem. *$\mathscr{J}(V, \omega)$ is contractible.*

Proof. Let $\mathscr{G}(V)$ be the space of inner products on V with topology given by choosing a unitary basis for V. We identify $\mathscr{G}(V)$ with the set of symmetric positive-definite matrices. Under this identification, $\mathscr{G}(V)$ becomes a smooth manifold. We will show that $\mathscr{G}(V)$ retracts onto $\mathscr{J}(V, \omega)$. Because $\mathscr{G}(V)$ is convex, it is contractible.

Consider the automorphism $A : V \to V$ given by $A = \mu_g^{-1} \circ \mu_\omega$ Now observe that $g(Au, v) = g(u, -Av)$. Therefore the adjoint of A is $-A$. Then we write $P := -A^2 = AA^T$, which is symmetric and positive definite. We now have

$$g(AA^T v, v) = g(A^T v, -Av) = g(-Av, -Av) = g(Av, Av) > 0.$$

Then we can write $Q = \sqrt{P}$, which is symmetric and positive definite. Now set $J = AQ^{-1}$.[1]

We will show that $J \in \mathscr{J}(V, \omega)$. First note that A and Q commute (because A and P commute). Therefore A preserves the eigenspaces of Q. This gives

$$J^2 = AQ^{-1}AQ^{-1} = A^2Q^{-2} = A^2P^{-1} = -PP^{-1} = -\mathrm{id}_V.$$

Moreover, Q is self-adjoint and A is skew-adjoint, so J is skew-adjoint. Also J commutes with A.

Now we show ω-compatibility. The first condition is just

$$\omega(Ju, Jv) = g(AJu, Jv) = g(JAu, Jv) = g(Au, -J(Jv)) = g(Au, V) = \omega(u, v).$$

The taming condition is simply

$$\omega(u, Ju) = g(Au, Ju) = g(u, -AJu) = g(u, Qu) > 0$$

because all eigenvalues of Q are positive. This gives us a map $r : \mathscr{G}(V) \to \mathscr{J}(V, \omega)$ which can be checked to be smooth. We can define the right inverse by $J \mapsto \omega(-, J-)$. This can also be checked to be smooth. It is easy to see that if $A = J$, then the polar decomposition gives $Q = \sqrt{-J^2} = \mathrm{id}_V$, so r is a retraction. Therefore $\mathscr{G}(V)$ is homotopy equivalent to $\mathscr{J}(V, \omega)$. $\square$

[1] $A = JQ$ is a *polar decomposition* into unitary and symmetric positive definite matrices.

Lemma 3.1. *Let A be symmetric. Then the following are equivalent:*

1. *A is positive definite.*
2. *All eigenvalues of A are positive.*
3. *$A = BDB^{-1}$, where B is orthogonal and D is diagonal with positive entries.*
4. *$A = BB^T$ for some nonsingular B.*

3.2 More Compatibility

Definition 3.2. We call ω, J, g on V a *compatible triple* if $g(u, v) = \omega(u, Jv)$.

Observe that any two determine the third. Moreover, we have seen that we can complete two of them to a compatible triple if:

- For ω, J, J is ω-compatible.
- For g, J, g is J-compatible.
- There are no conditions on ω, g to obtain a J.

Exercise 3.3. Let (V, ω, J, g) be a compatible triple on V with $\dim V = 2n$. Show that

(a) $\omega^n = n!\mathrm{vol}_g$.

(b) For any subspace $W \subset V$, $JW^\omega = W^\perp$.

Note that for standard $(\mathbb{R}^{2n}, \omega_0, J_0, g_0)$, the automorphism groups overlap as $GL_n(\mathbb{C}) \cap O(2n) \cap Sp(2n) = U(n)$.

Theorem 3.4. *Any two of the above groups intersect as $U(n)$.*

Proof. Recall that

1. $A \in Sp(2n)$ iff $A^T J_0 A = J_0$;
2. $A \in GL_n(\mathbb{C})$ iff $AJ_0 = J_0 A$;
3. $A \in O(2n)$ iff $A^T A = I$.

It is not hard to see that any two imply the third. For example, if we have the first two, then

$$A^T A = A^T J_0 J_0^{-1} A = J_0 A^{-1} J_0^{-1} A = J_0 A^{-1} A J_0^{-1} = I.$$

□

How about the spaces of these structures? We denote the space of symplectic structures by Ω, the space of complex structures by J, and the space of inner products by $\mathscr{G}$. Note that $G = GL_{2n}(\mathbb{R})$ acts transitively on $\Omega, J, \mathscr{G}$ with stabilizers $Sp(2n), GL_n(\mathbb{C}), O(2n)$. Because G is a Lie group acting transitively on a smooth manifold, every stabilizer is closed. Then a quotient of a Lie group by a closed subgroup is a smooth manifold.

Therefore we may consider the spaces $GL_{2n}(\mathbb{R})/Sp(2n), GL_{2n}(\mathbb{R})/GL_n(\mathbb{C}), GL_{2n}(\mathbb{R})/O(2n)$. These spaces are exactly $\Omega, J, \mathscr{G}$, respectively. Thus they are smooth manifolds.

Exercise 3.5. Compute their dimensions. Also compute their π_0 for $n = 1, 2$.

Theorem 3.6. $\mathcal{J}(\mathbb{R}^{2n}, \omega_0) = Sp(2n)/U(n)$.

Proof. If J is ω_0-compatible, let $g_J(u, v) = \omega(u, Jv)$. This is a hermitian structure. For a unitary basis $u_1, \dots, u_n$ of $\mathbb{C}^n$, we have a symplectic basis $u_1, \dots, u_n, Ju_1, \dots, Ju_n$. Then define the map

$$A_J(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i=1}^{n} x_i u_i + y_i J u_i,$$

which is a symplectomorphism. Set $\mathcal{J}(\mathbb{R}^{2n}, \omega_0) \to Sp(2n)/U(n)$ by $J \mapsto [A_J]$. This is an isomorphism. □

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4.1 Homework Exercises

İnanç will post homework later, and it will be due in approximately two and a half weeks.

Exercise 4.1. Find $J \in J(\mathbb{R}^{2n}) \setminus J(\mathbb{R}^{2n}, \omega_0)$. In other words, find a complex structure not compatible with the standard symplectic form.

Exercise 4.2. Find $A \in SL_{2n}(\mathbb{R}) \setminus Sp_{2n}(\mathbb{R})$. In other words, find a matrix which is not symplectic.

Here are some facts that will be helpful on the homework.

Proposition 4.3. *Let $A \in Sp_{2n}$. Then*

1. *$A \in SL_{2n}$;*
2. *$A^T \in Sp_{2n}$;*
3. *λ is an eigenvalue with multiplicity m iff $1/\lambda$ is as well;*
4. *If ± 1 is an eigenvalue of A, then it has even multiplicity;*
5. *If v_1, v_2 are eigenvectors for λ_1, λ_2 with $\lambda_1\lambda_2 \neq 1$, then $\omega_0(v_1, v_2) = 0$.*
6. *If A is symmetric and positive-definite, then $A^\alpha \in Sp_{2n}$ for any $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof.

1. Note that A preserves the symplectic form ω_0. Therefore it preserves the volume form and the orientation.
2. We see $(A^T)^T J_0 A^T = AJ_0J_0A^{-1}J_0^{-1} = -AA^{-1}(-J_0) = J_0$.
3. Because $A^TJ_0A = J_0$, then $A^T = J_0A^{-1}J_0^{-1}$, so A, A^{-1} have the same eigenvalues. Then if λ if an eigenvalue of A with multiplicity m, we have $Av = \lambda v$, so $\lambda^{-1}v = A^{-1}v$. Thus λ^{-1} is an eigenvalue of A^{-1} and therefore of A.
4. -1 must have even multiplicity to ensure that A has determinant 1.
5. Note that $\omega_0(v_1, v_2) = \omega_0(Av_1, Av_2) = \omega_0(\lambda_1 v_1, \lambda_2 v_2) = \lambda_1\lambda_2\omega_0(v_1, v_2)$. Because $\lambda_1\lambda_2 \neq 1$, we must have $\omega_0(v_1, v_2) = 0$.
6. Let V_λ be an eigenspace of A. This is an eigenspace for λ^α under A^α. By the above, we if $\lambda_1\lambda_2 \neq 1$, then $V_{\lambda_1}, V_{\lambda_2}$ are orthogonal under ω_0. In addition, by the previous, it is easy to see that A^α preserves ω_0 on the eigenbasis. □

4.2 Subspaces of Symplectic Vector Spaces

For a given symplectic manifold, there are four kinds of submanifolds we will describe. Two of them are important, and the other two will allow us to reduce to the first two.

Let W be a subspace of the symplectic vector space (V, ω). Recall that W is *symplectic* if $W \cap W^\omega = 0$.

Definition 4.4. W is

- *isotropic* if $W \subset W^\omega$;
- *coisotropic* if $W \supset W^\omega$;
- *Lagrangian* if $W = W^\omega$.

Example 4.5. Consider $\mathbb{R}^{2n}$ with the standard form with basis u_1, v_1, u_2, v_2. Then

- The spaces $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle$ are symplectic;
- The spaces $\langle u_1\rangle, \langle u_2\rangle, \langle v_1\rangle, \langle v_2\rangle$ are isotropic;
- The spaces $\langle u_1, u_2, v_2\rangle, \langle u_1, v_1, v_2\rangle, \langle u_1, u_2, v_1\rangle, \langle u_2, v_1, v_2\rangle$ are coisotropic;
- The spaces $\langle u_1, u_2\rangle, \langle v_1, v_2\rangle, \langle u_1, v_2\rangle, \langle u_2, v_1\rangle$ are Lagrangian.

Proposition 4.6. *Let (V, ω) be a synplectic vector space of dimension $2n$.*

1. *Any line is isotropic;*
2. *Any hyperplane is coisotropic;*
3. *Any isotropic subspace is contained in some Lagrangian subspace;*
4. *Any coisotropic subspace contains some Lagrabgian subspace;*
5. *$Sp(V, \omega)$ preserves the four types of subspaces.*

Proof. 1. ω is alternating.

2. W^ω is a line, so it is isotropic. Thus $W^\omega \subset (W^\omega)^\omega = W$.
3. If W is isotropic but not Lagrangian, then there exists $0 \neq v \in W^\omega \setminus W$. Then set $W_1 = \langle W, v\rangle$. This is clearly also isotropic. Repeat until W has dimension n.
4. W^ω is isotropic, so use the above to find $W^\omega \subset L$. Then $W \supset L^\omega = L$.
5. We show that $\phi(W^\omega) = \phi(W)^\omega$. If $v \in W^\omega$, then $\omega(-, v)|_W = 0$, so $\omega(\phi(-), \phi(v))|_W = 0$. Therefore, $\omega(-, \phi(v))|_{\phi(W)} = 0$, so $\phi(v) \in \phi(W)^\omega$. Thus $\phi(W^\omega) \subset \phi(W)^\omega$ Because the two spaces have the same dimension, they are equal. ☐

Proposition 4.7 (Symplectic Reduction). *Let $W \subset (V, \omega)$. If W is isotropic (resp. coisotropic), then W^ω/W (resp. W/W^ω) is symplectic.*

Proof. Let $v_1, v_2 \in W^\omega$. Then $\omega(v_1 + w_1, v_2 + w_2) = \omega(v_1, v_2)$ for any w_1, w_2. Therefore ω is defined on equivalence classes modulo W. Therefore we only need to check nondegeneracy. If $v_1 \in W^\omega$ is such that $\omega(v_0, v) = 0$ for all $v \in W^\omega$, then $v_0 \in W$. Therefore ω is nondegenerate on W^ω/W. ☐

Note that symplectic and Lagrangian subspaces are good for constructing new symplectic spaces and defining symplectic invariants (Floer homology, Gromov-Witten invariants). When we switch to discussing manifolds, we will see that Lagrangian submanifolds are hard to find, while symplectic submanifolds are easy to find. We will only want to consider some symplectic submanifolds.

Define $\mathscr{L}(V,\omega)$ to be the space of Lagrangian subspaces of (V,ω) and $\mathscr{L}(n)$ to be the space of Lagrangian subspaces of $\mathbb{R}^{\omega_0}$, the *Lagrangian Grassmannian.*

Proposition 4.8. $\mathscr{L}(n) \cong U(n)/O(n)$.

Sketch. For $\Lambda \in \mathscr{L}(n)$, choose an orthonormal basis $u_1, \dots, u_n$ with respect to g_0. Set

$$A = \begin{pmatrix} u_1 & \dots & u_n & J_0 u_1 & \dots & J_0 u_n \end{pmatrix}.$$

Therefore, $\Lambda = A\Lambda_h$, where Λ_h is the span of the first n standard vectors. Also, note A is unitary. Conversely, for all $A \in O(n) \subset Sp(2n)$, clearly $\Lambda = A\Lambda_h$ is also Lagrangian. Set $\mathscr{L}(n) \to U(n)/O(n)$ by $\Lambda \mapsto [A]$. We can check that this is an isomorphism. Next time we will say some things about $\pi_1(\mathscr{L}(n))$. □

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February 4

5.1 Linear Algebra, Conclusion

Last time we discussed the Lagrangian Grassmannian $\mathscr{L}(n) = U(n)/O(n)$.

Remark 5.1. 1. $\pi_1(\mathscr{L}(n)) = \mathbb{Z}$. In addition, the determinant map $\mathscr{L}(n) \to S^1$ is a fibration with fiber $SU(n)/SO(n)$. Then, the projection onto the first colume $SU(n) \to S^{2n-1}$ is a fibration with fiber $SU(n-1)$, so by the homotopy long exact sequence, we have an exact sequence

$$0 = \pi_2(S^{2n-1}) \to \pi_1(SU(n-1)) \to \pi_1(SU(n)) \to \pi_1(S^{2n-1}) = 0.$$

This tells us that $\pi_1(SU(n)) = 1$. In particular, $\pi_1(SU(n)/SO(n)) = 1$. Now going back to $\mathscr{L}(n)$, then we use the homotopy LES to obtain

$$0 = \pi_1(SU(n)/SO(n)) \to \pi_1(\mathscr{L}(n)) \to \pi_1(S^1) \to \pi_0(SU(n)/SO(n)) = 0,$$

so $\pi_1(\mathscr{L}(n)) = \mathbb{Z}$.

2. By the universal coefficient theorem, $H^1(\mathscr{L}(n), \mathbb{Z}) \simeq \operatorname{Fr} H_1(\mathscr{L}(n), \mathbb{Z}) = \mathbb{Z}$. The generator is called the *Maslow index* M_n. For a loop λ in $\mathscr{L}(n)$, we then define the Maslow index of λ as $M_n([\lambda])$.

5.2 Symplectic Vector Bundles

Recall that a *symplectic vector bundle* over a manifold M is a real vector bundle $E \to M$ with a C^∞ section ω of $E^* \wedge E^*$ such that E_x, ω_x is a symplectic vector space for all $x \in M$. Then two symplectic vector bundles (E_i, ω_i) are isomorphic if there exists an isomorphism $\phi : E_1 \to E_2$ such that $\phi^*\omega_2 = \omega_1$.

Example 5.2. If (M, ω) is symplectic, then (TM, ω) is a symplectic vector bundle. In addition, the pushforward of a symplectomorphism is an isomorphism of symplectic vector bundles.

Remark 5.3. Given any vector bundle $E \to M$, we can define a symplectic vector bundle on the Whitney sum $E \oplus E^* \to M$ by

$$\omega((v, \eta), (v', \eta')) := \eta(v') - \eta'(v).$$

This is clearly bilinear, antisymmetric, and non-degenerate.

Definition 5.4. A *complex vector bundle* over a manifold M is a real vector bundle $E \to M$ with a C^∞ section J of $\operatorname{End}(E)$ such that (E_x, J_x) is a complex vector space for all $x \in M$.

We say two complex vector bundles (E_i, J_i) are isomorphic if there exists a vector bundle isomorphism $\phi : E_1 \to E_2$ such that $\phi(J_1 v) = J_2\phi(V)$.

Example 5.5. Let M be a complex manifold of dimension n. Then define the usual multiplication by i on T_pM. Now we need to see if this is globally defined. To do this, we simply compute on two charts and then use the Cauchy-Riemann equations. This is left to the reader.

Remark 5.6. Given any vector bundle $E \to M$, we can define a complex vector bundle on $E \otimes \mathbb{C}$.

Definition 5.7. If $(E, omega)$ is a symplectic vector bundle over M, a complex structure J on E is called ω-compatible if $J_x \in \mathcal{J}(E_x, \omega_x)$ for all $x \in M$.

Denote by $\mathcal{J}(E, \omega)$ the space of ω-compatible complex structures on (E, ω) with the topology inherited from End E.

Definition 5.8. A *Hermitian structure* on (E, J) is a J-compatible inner product g (at every $x \in M$).

Remark 5.9.

1. We can construct a Hermitian inner product from any inner product as before.
2. The space of Hermitian structures is convex.
3. $J \in \mathcal{J}(E, \omega)$ if and only if $g_J(u, v) := \omega(u, Jv)$ is Hermitian.

Theorem 5.10. *$\mathcal{J}(E, \omega)$ is nonempty and contractible.*

Proof. Define the retract by using the map in Theorem 2.10 pointwise. In a local trivialization of E, we can see that $r : G(E) \to \mathcal{J}(E, \omega)$ is smooth. □

Now we will consider chart transitions for $(E, \omega), (E, J), (E, g)$. Denote the chart for the local trivializations by $\{U_i\}$ with trivializations ϕ_i.

Type	Group
E	$GL(2n)$
(E, ω)	$Sp(2n)$
(E, J)	$GL(n, \mathbb{C})$
(E, g)	$O(2n)$

In each special case, the structure of the bundle can be reduced. Note here that we also have the two out of three property for reduction to $U(n)$ as before. We will illustrate this by checking the (E, ω) case. We simply note that $E|_{U_i} \simeq U_i \times \mathbb{R}^{2n}$, which admits a symplectic basis. Therefore we have a symplectic trivialization of (E, ω). Therefore, locally we have an isomorphicm to $U_i \times \mathbb{R}^{2n}, \omega_0$. Therefore we have reduction of the structure to $Sp(2n)$.

Now suppose the structure of (E, ω) can be reduced to $U(n)$. Then the corresponding J is ω-compatible.

Theorem 5.11.

1. *Let (E, ω) be a symplectic vector bundle and $J_1, J_2 \in \mathcal{J}(E, \omega)$. Then $(E, J_1) \simeq (E, J_2)$.*
2. *Let (E_i, ω_i) be symplectic vector bundles and $J_i \in \mathcal{J}(E_i, \omega_i)$. Then $(E_1, \omega_1) \simeq (E_2, \omega_2)$ if and only if $(E_1, J_1) \simeq (E_2, J_2)$.*

6

February 6

6.1 Proof of Theorem 5.11

İnanç will post the homework tonight. Last time we stated Theorem 5.11, which is reproduced below.

Theorem.

1. *Let (E, ω) be a symplectic vector bundle and $J_1, J_2 \in \mathcal{J}(E, \omega)$. Then $(E, J_1) \simeq (E, J_2)$.*
2. *Let (E_i, ω_i) be symplectic vector bundles and $J_i \in \mathcal{J}(E_i, \omega_i)$. Then $(E_1, \omega_1) \simeq (E_2, \omega_2)$ if and only if $(E_1, J_1) \simeq (E_2, J_2)$.*

Proof.

1. First, note that for any structure group G of a real vector bundle, there exists a classifying space BG with a contractible universal G-bundle EG. In particular, G-bundles E/M are classified by homotopy classes of maps $M \to BG$. In other words, BG represents the functor $M \mapsto \{\text{principle } G\text{-bundles on } M\}$ on the homotopy category.

   Now use the homotopy LES of the fibration $U(n) \to Sp(2n) \to Sp(2n)/U(n) \simeq \text{pt}$ to obtain isomorphisms $\pi_k(U(n)) \simeq \pi_k(Sp(2n))$. In particular, the inclusion $U(n) \to Sp(2n)$ induces an isomorphism on all π_k. By Whitehead, this is a homotopy equivalence. Therefore $BU(n) \simeq BSp(2n)$. Therefore for any (E, ω), this determines $f : M \to BSp(2n)$, which can be homotoped to $f' : M \to BU(n)$, which reduces (E, ω) uniquely to a $U(n)$-bundle.

2. By the previous, isomorphism as $Sp(2n)$-bundles implies isomorphism as $U(n)$-bundles. Now consider the inclusion $U(n) \to GL_n(\mathbb{C})$. This is a homotopy equivalence (use Gram-Schmidt to deform any loop $GL_n(\mathbb{C})$ into $U(n)$). By similar arguments as above, two $U(n)$-bundles are isomorphic iff they are isomorphic as $GL_n(\mathbb{C})$-bundles. This concludes the proof. □

Remark 6.1. Note that this allows us to determine whether two symplectic vector bundles are isomorphic by comparing their Chern classes.

6.2 Vector Bundles, Continued

Let F be a subbundle of the symplectic vector bundle (E, ω).

Definition 6.2. The *symplectic complement* of F is

$$F^\omega := \bigcup_{x \in M} F_x^\omega.$$

Definition 6.3. We define the vector subbundle F to be

- *symplectic* if $F \cap F^\omega$ is the zero section of E;
- *isotropic* if $F \subset F^\omega$;
- *coisotropic* if $F \supset F^\omega$;
- *Lagrangian* if $F = F^\omega$.

Many previous results on subspaces carry over to subbundles.

Proposition 6.4. *Let F be a subbundle of the symplectic vector bundle (E, ω).*

1. *If F is symplectic, $J_1 \in \mathcal{J}(F, \omega|_F)$, then there exists $J \in \mathcal{J}(E, \omega)$ extending J_1.*
2. *If F is Lagrangian, $J \in \mathcal{J}(E, \omega)$, then $(E, J) \cong F \otimes \mathbb{C}$.*

Proof.

1. If F is symplectic, then so if F^ω. Therefore there exists $J_2 \in \mathcal{J}(F^\omega, \omega|_{F^\omega})$. Then we have $E = F \oplus F^\omega$, so we write $J = J_1 \oplus J_2$. All properties of an ω-compatible complex structure follow from the orthogonal decomposition of E.
2. Let g be a compatible inner product for ω, J. Then $E = F \oplus F^{\perp_g} = F \oplus JF^\omega = F \oplus JF \cong F \otimes \mathbb{C}$. To see this, note that $J(u + Jv) = -v + Ju$, which is the same complex structure as $F \otimes \mathbb{C}$. □

6.3 Compatible Triples on Manifolds

Let $E = TM$ with symplectic structure ω, complex structure J, and inner product g. When only defined on TM, then ω is an *almost symplectic structure* and J is an *almost complex structure*. In this course, we will focus on the case when ω is a true symplectic structure on M. We define the following spaces:

- $\Omega(M)$ the space of symplectic structures on M;
- $\mathcal{J}(M)$ the space of almost complex structures on M;
- $\mathcal{G}(M)$ the space of almost complex structures on M;
- $\mathcal{J}(M, \omega)$ the space of compatible almost complex structures.

Definition 6.5. Now let S be a submanifold of (M, ω). Then TS is a subbundle of $(TM|_S, \omega)$, and we call S

- *symplectic* if $TS \subset TM|_S$ is symplectic;
- *isotropic* if $TS \subset TM|_S$ is isotropic;
- *coisotropic* if $TS \subset TM|_S$ is coisotropic;
- *Lagrangian* if $TS \subset TM|_S$ is Lagrangian;

Note that S is symplectic iff $(S, \omega|_S)$ is a symplectic manifold. In addition, note that S must be even dimensional and the nomal bundle $\nu S = (TS)^\omega$. Also, note that S is Lagrangian iff $\omega|_S = 0$.

Definition 6.6. For S a submanifold of (M, J), we call S, we call S *J-holomorphic* if $J(TS) \subset TS$, which happens iff $J|_S$ is an almost complex structure on S.

Remark 6.7. Our results about symplectic vector bundles carry over to (TM, ω). In particular, $\mathcal{J}(M, \omega)$ is nonempty and contractible.

Proposition 6.8. *Let S be a submanifold of (M, ω).*

1. *If S is symplectic, then S is J-holomorphic for some $J \in \mathcal{J}(M, \omega)$;*

2. *If S is J-holomorphic for any $J \in \mathcal{J}(M, \omega)$, then S is symplectic.*

Proof. 1. This is (mostly) just the first part of Proposition 6.4. However, we must extend J from $TM|_S$ to all of M. First we take a compatible metric, extend it by a partition of unity, and then use the retract to find a compatible J'.

2. Note $J(TS \cap (TS)^{\omega}) \subset JTS \cap J(TS)^{\omega} = TS \cap (TS)^{\perp_g} = \{0\}$. Therefore $TS \cap (TS)^{\omega} = \{0\}$, so S is symplectic. □

Exercise 6.9. Find a symplectic submanifold of $\mathbb{R}^{2n}$ that is not J_0-holomorphic.

7

February 11

7.1 Obtaining Compatible Triples

Input	Condition	Output	Integrable?
ω, J	J is ω-compatible	$g(u,v) = \omega(u, Jv)$ metric	g flat?
J, g	g is Hermitian	$\omega(u,v) = g(Ju, v)$ almost symplectic	ω closed?
ω, g	none	$r(J) \sim \mu_g^{-1}\mu_\omega$ almost complex	J integrable?

1. This is the same as local flatness. By a theorem of Bieberbach, all compact flat manifolds are finitely covered by tori.

2. This is the same as $d\omega = 0$. If J is a complex structure and g is Hermitian, then ω is a fundamental 2-form. The Goldberg conjecture states that if g is Einstein, then $d\omega = 0$.

3. By a theorem of Newlander and Nirenberg, there exists a complex structure on M inducing J if and only if the Nijenhuis tensor $N_J = 0$.

Definition 7.1. The *Nijenhuis tensor* of J is defined by

$$N_J(u,v) = [Ju, Jv] - J(u, Jv) - J(Ju, v) - [u, v].$$

Exercise 7.2. 1. If the map $- \mapsto [-, v]$ is J-holomorphic for all v, then $N_J = 0$.

2. Check that N_J is a tensor.

3. If M has dimension 2, then $N_J = 0$.

7.2 Complex Structures

Definition 7.3. (M, ω, J, g) is a *Kähler manifold* when J is integrable.

Example 7.4. Any $M = \Sigma_h$ a closed oriented surface of genus h is a Kähler manifold. To see this, take $\omega = \mathrm{vol}_g$. Then it is easy to see that ω is bilinear, skew-symmetric, nondegenerate, and closed. Now let $J = r(g)$, so (ω, J, g) is a compatible triple on Σ. But any almost complex structure on a surface is integrable by Exercise 7.2, so we are done.

Now let J be an almost complex structure on M. Take the complexified vector bundle $TM \otimes \mathbb{C}$. Then J extends linearly to $TM \otimes \mathbb{C}$ as

$$J(v \otimes c) := Jv \otimes c.$$

Then we can see that $J^2 = -I$, so the eigenvalues are $\pm i$. Then we can define $T_{1,0}, T_{0,1}$ as the eigenspaces of $\pm i$. Therefore we obtain an isomorphism

$$TM \otimes \mathbb{C} \cong T_{0,1} \oplus T_{1,0}.$$

In addition, we can write $T^*M \otimes \mathbb{C} \cong T^{0,1} \oplus T^{1,0}$. Then we have

$$\bigwedge^k (T^*M \otimes \mathbb{C}) \simeq \bigwedge^k (T^{1,0} \oplus T^{0,1}) = \bigoplus_{\ell+m=k} \bigwedge^{\ell,m}(M).$$

Therefore we have

$$\Omega^k(M, \mathbb{C}) \cong \bigoplus_{\ell+M=k} \Omega^{\ell,m}(M).$$

We want to define a differential that describes this structure. We can define $\partial : \Omega^{\ell,m}(M) \to \Omega^{\ell+1,m}$ by

$$\partial = \pi^{\ell+1,m} \circ d$$

and then define $\overline{\partial}$ analogously.

Now if $d = \partial + \overline{\partial}$, then we must have $\partial^2 = 0, \overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0$. This allows us to define *Dolbeaut cohomology*.

Suppose J is a complex structure on M. Then in a local chart U, we have coordinates z_j and we can write $T_{1,0} = \left\langle \frac{\partial}{\partial z_j} \right\rangle, T_{0,1} = \left\langle \frac{\partial}{\partial \overline{z}_j} \right\rangle$. In addition, $T^{1,0} = \langle dz_j \rangle, T^{0,1} = \langle d\overline{z}_j \rangle$. Writing a form β locally, we will see that $d = \partial + \overline{\partial}$.

Remark 7.5. For any almost complex structure J on M, if $\overline{\partial}^2 = 0$, then J is integrable.

Theorem 7.6 (Newlander-Nirenberg). *Let J be an almost complex structure on M. Then the following are equivalent:*

1. *J is induced by a complex structure on M;*
2. *$N_J = 0$;*
3. *$d = \partial + \overline{\partial}$;*
4. *$\overline{\partial}^2 = 0$.*

Now suppose (M, ω, J, g) be Kähler. Then what can we say about ω?

- ω is a closed form over $\mathbb{C}$;
- If $\omega = \sum a_{jk} dz_j \wedge dz_k + \sum b_{jk} dz_j \wedge d\overline{z}_k + \sum c_{jk} d\overline{z}_j \wedge d\overline{z}_k$, then $a_{jk} = c_{jk}$ and $b_{jk} = -\overline{b}_{kj}$;
- Computing $J^*\omega$, we will see that ω is a $(1,1)$-form.
- $\overline{\partial}\omega = 0$.

8

February 13

Recall that there will be no class next Tuesday. Last time we began the discussion of Kähler forms.

8.1 Kähler Forms Continued

Recall from last time that Kähler forms are of type $(1,1)$. Also recall that locally we have

$$\omega = \sum b_{jk}\, dz_j \wedge d\overline{z}_k.$$

In order to express the coefficients as a metric, we can rewrite

$$\omega = \frac{i}{2} \sum h_{jk}\, dz_j \wedge d\overline{z}_k. \tag{8.1}$$

Then the matrix (h_{jk}) is a Hermitian matrix. In particular, the volume form is locally

$$\omega^n = \left(\frac{i}{2}\right)^n n!\, \det(H)\, dz_1 \wedge d\overline{z}_1 \cdots \wedge dz_n \wedge d\overline{z}_n.$$

Proposition 8.1. *Now let M be a compact complex manifold of dimension n with complex structure J. Then the following are equivalent:*

1. *ω is a Kähler form on M;*
2. *ω is a ∂ and $\overline{\partial}$-closed $(1,1)$-form locally given by (8.1) with coefficient matrix given by a Hermitian positive-definite matrix.*

Proof. We did one of the directions above. In the other direction, observe that $g(u,v) := \omega(u, Jv)$ is locally given by $g(u,v) = u^T H v$ for $H = (h_{jk})$ a Hermitian positive definite matrix. Thus ω is nondegenerate. In addition, ω is clearly J-compatible because it has type $(1,1)$. □

Example 8.2. $(\mathbb{C}^n, \omega_0)$ is Kähler with complex structure J_0.

Example 8.3. Consider $\mathbb{CP}^n = \mathbb{C}^{n+1}/\{0\}/\mathbb{C}^*$. Recall the standard complex atlas for $\mathbb{CP}^n$. Let J_0 be the induced complex structure on $\mathbb{CP}^n$. Then there exists a symplectic form ω_{FS}, the Fubini-Study metric, on $\mathbb{CP}^n$ compatible with J_0.

Proposition 8.4. *Any complex submanifold of a Kähler manifold is Kähler.*

First Proof. Let M be a complex manifold of dimension n and S be a complex manifold of dimension m. Then locally at $p \in S \subset M$, we have charts for M such that $S = \{z_{m+1} = \cdots = z_n = 0\}$. If ω is a Kähler form on M, then locally we have

$$\iota^* \omega = \frac{i}{2} \sum_{j,k \leq m} h_{jk}\, dz_j \wedge d\bar{z}_k.$$

In particular, it is easy to see that the coefficient matrix is Hermitian and positive definite. Finally, ω is a $\partial, \bar{\partial}$-closed $(1,1)$-form, so $\iota^*\omega$ is as well. □

Second Proof. Note that S is a J-holomorphic submanifold of M. By Proposition 6.8, S is also a symplectic submanifold of M. Therefore $\omega|_S$ is symplectic and compatible with $J|_S$, which is already integrable. □

8.2 Some Algebraic Geometry

Thus we have proven the following corollary, which tells us that any smooth affine or projective complex algebraic variety is Kähler.

Theorem 8.5. *Any complex submanifold of $\mathbb{C}^n$ or $\mathbb{CP}^n$ is Kähler.*

Remark 8.6. We will state some facts from complex algebraic geometry in context for culture.

1. Many complex submanifolds of $\mathbb{C}^n$ are not affine, e.g. $\mathbb{C}^2 \setminus \{0\}$.
2. By Chow's theorem (1949), any closed complex submanifold of $\mathbb{CP}^n$ is algebraic.
3. The Kodaira embedding theorem (1960s) states that a closed Kähler manifold (M, ω) is algebraic if and only if ω is integral.

Proposition 8.7. *In particular, any compact complex curve Σ is projective.*

Proof. On Σ equipped with complex structure J, take any J-compatible metric g. Then set $\omega(u, v) := g(Ju, v)$. Then ω is symplectic. Therefore ω is a Kähler form on Σ compatible with J. Now suppose that

$$\int_\Sigma \omega = \alpha \in \mathbb{R}^+.$$

Because any nonzero real multiple of a Kähler form is still Kähler, then $\frac{1}{\alpha}\omega$ is the desired integral form. □

Remark 8.8. Not all compact Kähler surfaces are projective. In complex dimension 2, Kähler manifolds become projective after deforming the complex structure. In dimension at least 4, there exist compact Kähler manifolds (due to Voisin) that are not even homotopy equivalent to a projective variety.

Theorem 8.9 (Gromov's Embedding Theorem). *Let (M, ω) be a closed symplectic manifold. If ω is integral, then there is a symplectic embedding of M into $(\mathbb{CP}^N, \omega_{FS})$ for some high enough N.*

Remark 8.10. İnanç says this theorem is conceptually nice, but he does not know any use for the result.

Example 8.11. Let $C_d = Z\left(\sum z_i^d\right) \subset \mathbb{CP}^2$. Then $C_1 \cong C_2 \cong \mathbb{CP}^1, C_3 \cong T^2, C_4 \cong \Sigma_6$.

Now let S_d be a hypersurface of degree d in $\mathbb{CP}^3$. Then $S_1 = \mathbb{CP}^2, S_1 = \mathbb{P}^1 \times \mathbb{P}^1, S_3 = \mathrm{Bl}_6\, \mathbb{P}^2$, and S_4 is a K3 surface (use the adjunction formula).

8.3 Stein Manifolds

Let M be a complex manifold of dimension n and let $\rho \in C^\infty(M, \mathbb{R})$ be a proper, strictly plurisubharmonic function on M.

Definition 8.12. A function ρ is structly plurisubhamonic if on each complex chart, $H_{\partial\bar{\partial}}\rho$ is positive-definite.

In this case,

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho$$

is a Kähler form on M, and we say (M, ω) is a *Stein manifold*, where ρ is the *Kähler potential*.

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February 20

9.1 Stein Manifolds Continued

Today's lecture will be slower than usual. Last time we defined Stein manifolds (M, ρ), where ρ is strictly plurisubharmonic. It is easy to see that $\omega = \frac{i}{2}\partial\overline{\partial}\rho$ is closed and real. Also, $J^*\omega = \omega$.

Example 9.1. Choose $M = \mathbb{C}^n$ and $\rho = \sum |z_j|^2$. Then we can check that $H_{\partial\overline{\partial}}$ is the identity matrix. Then we can also see that

$$\omega = \frac{i}{2}\sum \delta_{jk} dz_j \wedge d\overline{z}_k = \sum dx_j \wedge dy_j$$

is the standard form.

Thus every complex submanifold of $\mathbb{C}^n$ is a Stein manifold.

Theorem 9.2 (Remmert Embedding Theorem (1956)). *Let (M, ω) be a Stein manifold. Then there exists a proper holomorphic embedding of M into $\mathbb{C}^N$ for some N.*

Corollary 9.3. *Stein manifolds do not have compact complex submanifolds of positive dimension. In particular, they are never compact.*

Theorem 9.4 (Behnke-Stein (1958)). *Every connected non-compact complex curve is Stein.*

9.2 Topological Properties of Kähler Manifolds

Theorem 9.5 (Hodge). *On a compact Kähler manifold (M, ω), the Dolbeaut cohomology groups satisfy the Hodge decomposition*

$$H^k(M, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M).$$

In addition, we have an isomorphism

$$H^{p,q}(M) \simeq \overline{H^{q,p}(M)}.$$

Theorem 9.6 (Serre Duality). *For any complex manifold M, we have $H^{p,q}(M) \simeq H^{n-p,n-q}(M)$.*

Remark 9.7. We obtain the Hodge numbers $h^{p,q} = \dim H^{p,q}$, which are traditionally arranged in the *Hodge diamond*, which has symmetry given by the Hodge symmetry and Serre duality.

Example 9.8. Consider a curve of genus g. Then the Hodge diamond is

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}.$$

Example 9.9. The Hodge diamond of $\mathbb{P}^2$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

Example 9.10. The Hodge diamond of $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

Example 9.11. The Hodge diamond of $\mathbb{P}^1 \times \Sigma_g$ is

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ 0 & & 2 & & 0 \\ & g & & g & \\ & & 1 & & \end{array}.$$

Example 9.12. Consider the Hopf surface, which is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the action of $\mathbb{Z}$ acting by powers of 2. This is a biholomorphic, free, and properly discontinuous action. Up to diffeomorphism, our surface is $M = \mathbb{C}^2 \setminus \{0\}/(z_1, z_2) \sim (2z_1, 2z_2) \simeq S^3 \times S^1$. Note that the second cohomology vanishes, so the surface is not Kähler. The Hodge diamond of the Hopf surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}.$$

Remark 9.13. The odd Betti numbers of compact Kähler manifolds are even. The even Betti numbers are positive because ω^k is closed but not exact. In fact, any symplectic manifold has nonvanishing even cohomology. In particular, $h^{p,p}$ is always positive.

9.3 Complex and Symplectic Structures on 4-Manifolds

Let X be a closed, connected, oriented, smooth 4-manifold. The equivalences will be orientation-preserving diffeomorphisms. We will see that for complex surfaces, Hodge numbers depend only on the topological type of the oriented manifold.

Recall the Euler characteristic $e = \sum(-1)^{i+j}h^{i,j}$ and the signature $\sigma = \sum(-1)^j h^{i,j}$. The signature of the manifold is the signature of the quadratic form Q_X, called the intersection form.

Remark 9.14. If $X_1 \simeq X_2$ are homotopy equivalent, then they have the same intersection form. Also, if $\overline{X}$ is X with the opposite orientation, then $Q_{\overline{X}} = -Q_X$. Third, if $X = X_1 \# X_2$, then $Q_X = Q_{X_1} \oplus Q_{X_2}$.

Theorem 9.15 (Whitehead). *If X_1, X_2 are simply-connected and $Q_{X_1} \cong Q_{X_2}$, then $X_1 \simeq X_2$.*

Theorem 9.16 (Freedman). *If X_1, X_2 are simply connected, are either both smoothable or both not smoothable, and $Q_{X_1} \cong Q_{X_2}$, then $X_1 \cong X_2$.*[1]

Example 9.17. The intersection form of $\mathbb{P}^2$ is $Q_{\mathbb{P}^2} = (1)$ and the intersection form of $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$Q_{\mathbb{P}^1 \times \mathbb{P}^1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[1]This implies the topological Poincare conjecture in dimension 4 and won Freedman the Fields Medal.

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February 25

10.1 Complex Structures in Dimension 4

Recall that for Kähler surfaces, the Hodge numbers depend only on the top type of the oriented manifold.

Theorem 10.1 (Kodaira-Siu (1981)). *Let X be a compact complex surface. Then X is Kähler if and only if b_1X is even.*

Remark 10.2. The existence of an almost complex structure is a purely homotopy-theoretic problem.

Theorem 10.3 (Wu's criterion). *Let X be a closed oriented 4-manifold. Then X admits an almost complex structure J if and only if $c \in H^2(X, \mathbb{Z})$ such that $c^2 = 2e + 3\sigma$ and $c \cdot \alpha \equiv \alpha \cdot \alpha \mod 2$.*[1]

Example 10.4. On S^4, the second cohomology vanishes, so $c = 0$. This implies $c^2 = 0$, but $2e + 3\sigma = 4$. Therefore S^4 admits no almost complex structure.

Example 10.5. On $\mathbb{P}^2$, note that $2e+3\sigma = 9$, so $c = \pm 3H$. Clearly c satisfies the second part of Wu's criterion, so $\mathbb{P}^2$ admits a complex structure. However, the two almost complex structures are not equivalent.

Example 10.6. Consider $\overline{\mathbb{P}}^2$. Here, if $c = mH$, then $c^2 = -m^2$. However, $2e + 3\sigma = 3$, so there is no almost complex structure.

Example 10.7. Now consider the manifold $\mathbb{P}^2 \# \mathbb{P}^2$. Then we have $H^2 = \mathbb{Z}^2$, so if we write $c = (m, n)$, then $c^2 = m^2 + n^2$. However, $2e + 3\sigma = 14$, which is not a sum of two squares. Therefore there is no almost complex structure.

Remark 10.8. We see that orientation reversal and connected sum are not almost complex operations. Therefore, they are also not symplectic operations.

Example 10.9. Consider the connected sum $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$. Here we see that $2e + 3\sigma = 19 = 3^2 + 3^2 + 1^2$. In addition, $3, 3, 1$ are all odd, so this manifold does admit an almost complex structure.

[1] Here, $c = c_1(X, J)$.

10.2 Symplectic Structures in Dimension 4

10.2.1 Seiberg-Witten Invariants These come from a certain system of differential equations on X which come from physics. The solutions yield a very nice moduli space $\mathscr{M}_{X,c}$ under facorable conditions for each $c \in H^2(X)$: If $b^+(X) > 0$ and X admits an almost complex structure, then $\mathscr{M}_{X,c} = \emptyset$ for all but finitely many c. Then $\mathscr{M}_{X,c}$ is a compact oriented manifold with dimension

$$\dim \mathscr{M}_{X,c} = \frac{c^2 - (2e(X) + 3\sigma(X))}{4}.$$

For c with $\dim \mathscr{M}_{X,c} = 0$, then we can assign the signed count of points in $\mathscr{M}_{X,c}$. If $b^+(X) = 1$, we need an additional choice. Then this defines the Seiberg-Witten invariant $SW_X : H^2(X, \mathbb{Z}) \to \mathbb{Z}$. Then for any α with $SW_X(\alpha) \neq 0$ is called a Seiberg-Witten basic class.

Remark 10.10. Here are some fundamental results:

1. The Seiberg-Witten invariant is invariant under orientation preserving diffeomorphisms.
2. (Vanishing Theorem). If $X = X_1\#X_2$, with $b^+X_i > 0$ for $i = 1, 2$, then $SW_X \equiv 0$.
3. (Taubes). If X is symplectic with $b^+X > 0$, then there exists a particular $c \in H^2(X, \mathbb{Z})$ such that $SW_X(c) = 1$. In fact, $c = c_1(X, J)$ for any ω-compatible J.

Theorem 10.11 (Diversity). *There exist infinitely many closed, connected, oriented maniflds of dimension $2n \geq 4$. Smooth manifolds in each subclass are depicted below.*

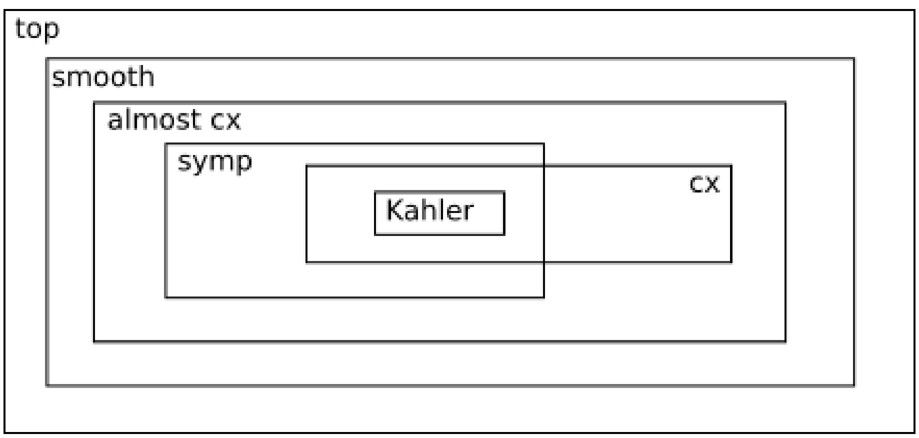

Figure 10.1: Classes of manifolds.

Proof. First, Freedman's E_8-manifold is not smoothable. Then S^4 has no almost complex structure. Third, $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$ is not symplectic by Taubes and not complex by Kodaira-Siu. Fourth, $S^1 \times S^3$ admits a complex structure, but is not symplectic or Kähler. Fifth, the Kodaira-Thurston manifold is complex and symplectic, but not Kähler. □

Remark 10.12. Infinitely many examples in each subclass can be obtained by blowups $X\#\overline{\mathbb{P}}^2$.

Example 10.13 (Kodaira-Thurston Manifold). Let $G = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle$ act in $\mathbb{R}^4$ with coordinates x_1, y_1, x_2, y_2, where γ_i increases the ith component by 1 for $i = 1, 3, 4$, while $\gamma_2 = (x_1, y_1 + 1, x_2 + y_2, y_2)$. This is a free and properly dicontinuous action, so $X = \mathbb{R}^4/G$ is a smooth manifold. Then $q : \mathbb{R}^4 \to X$ is the universal cover of X. Then we see that all γ_i commute exept for $[\gamma_2, \gamma_3]$. Therefore $H_1 X = G/[G, G]$, so $b_1 X = 3$. Therefore X cannot be Kähler.

11

February 27

Exercise 11.1. Let $f : X \to Y$ be a finite unbranched cover. Show that if Y is symplectic, almost complex, complex, Kähler, projective, or Stein, then so is X.

11.1 Kodaira-Thurston Manifold

Recall the definition of the Kodaira-Thurston manifold X as a quotient of $\mathbb{R}^4$ by a discrete group G from last time.

Remark 11.2. We may obtain X as a T^2-bundle over the torus.

To see this, note that we can project X onto the first two coordinates. This is a map $f : X \to \mathbb{R}^2/\mathbb{Z}^2 = T^2$, where $\mathbb{Z}^2$ acts by shifts. To compute the fiber, note that $f^{-1}(0,0) = \mathbb{R}^2/\sim$, where $\sim$ identifies $(x_2, y_2) \sim (x_2 + 1, y_2)$ and $(x_2, y_2) \sim (x_2, y_2 + 1)$. Thus the fiber is a torus. The action of γ_2 becomes nontrivial monodromy.

To see the monodromy, note that if we move in the x_1-direction in the base, γ_1 identifies $(0, 0, x_2, y_2)$ with $(1, 0, x_2, y_2)$. However, in the y_1-direction, γ_2 identifies $(0, 0, x_2, y_2)$ with $(0, 1, x_2 + y_2, y_2)$.

This gives us a description of the monodromy by the matrices $A = I_2, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$.[1]

Theorem 11.3 (Thurston (1976)). *The manifold X is symplectic.*

Proof. Choose the standard symplectic form $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Next we can see that $\gamma_i^* \omega_0 = \omega_0$ for all $i = 1, \dots, 4$. The only one we need to check is γ_2, and we see that

$$\gamma_2^* \omega_0 = dx_1 \wedge d(y_1 + 1) + d(x_2 + y_2) \wedge dy_2 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

because $dy_2 \wedge dy_2 = 0$. Therefore each γ_i is a symplectomorphism, so G is a symplectomorphism of $(\mathbb{R}^4, \omega_0)$. Thus $\mathbb{R}^4 \to X$ is a covering with symplectic deck transformations. Thus there is an induced symplectic form ω on X. ☐

Theorem 11.4 (Kodaira (1969)). *The Kodaira-Thurston manifold is complex.*

Proof. Let $\alpha = dx_1, \beta = dy_1, \gamma = dx_2 - y_1 dy_2, \delta = dy_2$. Then this gives a G-invariant basis for $\Omega^1(\mathbb{R}^4)$. The only one we need to check is γ with γ_2, and we see that

$$\gamma_2^* \gamma = d(x_2 + y_2) - (y_1 + 1) d(y_2 + 1) = dx_2 - y_1 dy_2.$$

[1] $SL_2(\mathbb{Z})$ is the mapping class group of the torus, which is the group of orientation preserving diffeomorphisms modulo isotopy.

Then note that α, β, δ are all closed, but $d\gamma = -dy_1 \wedge dy_2 = -\beta \wedge \delta = \delta \wedge \beta$. Then the dual basis

$$a = \frac{\partial}{\partial x_1}, b = \frac{\partial}{\partial y_1}, c = \frac{\partial}{\partial x_2}, d = -y_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}$$

is a G-invariant basis for the Lie algebra of vector fields on $\mathbb{R}^4$. To see this, we only need to check γ_2, d, and we see

$$\begin{aligned}(\gamma_2)_* d &= -(y_1+1)\left(\frac{\partial x_1}{\partial x_2}\frac{\partial}{\partial x_1} + \frac{\partial(y_1+1)}{\partial x_2}\frac{\partial}{\partial y_1} + \frac{\partial(x_2+y_2)}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial y_2}{\partial x_2}\frac{\partial}{\partial y_2}\right) \\ &+ \left(\frac{\partial x_1}{\partial y_2}\frac{\partial}{\partial x_1} + \frac{\partial(y_1+1)}{\partial x_y}\frac{\partial}{\partial y_1} + \frac{\partial(x_2+y_2)}{\partial y_2}\frac{\partial}{\partial x_2} + \frac{\partial y_2}{\partial y_2}\frac{\partial}{\partial y_2}\right) \\ &= -(y_1+1)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2},\end{aligned}$$

as desired. Now define $Ja = c, Jc = -a, Jb = d, Jd = -b$. Then we see that

$$\begin{aligned}N_J(a,b) &= [Ja, Jb] - J[a, Jb] - J[Ja, b] - [a, b] \\ &= [c, d] - J[a, d] - J[c, b] - [a, b] \\ &= 0.\end{aligned}$$

Similarly, note that

$$\begin{aligned}N_J(b,d) &= [Jb, Jd] - J[b, Jd] - J[Db, d] - [b, d] \\ &= [d, -b] - J[b, -b] - J[d, d] - [b, d] \\ &= 0.\end{aligned}$$

Here we use the fact that all commutators vanish besides $[b, d] = -c$. Therefore J is a complex structure on $\mathbb{R}^4$. In addition, the G-invariant basis for vector fields on $\mathbb{R}^4$ descends to a basis for vector fields on X. Also, J descends to an integrable almost complex structure on X. □

Corollary 11.5. *The Kodaira-Thurston manifold is complex and symplectic but not Kähler.*

Remark 11.6. The G-invariant basis $\alpha, \beta, \gamma, \delta$ descends to $\Omega^1(X)$. In addition, the symplectic form ω_0 descends to the form

$$\omega = \overline{\alpha} \wedge \overline{\beta} + \overline{\gamma} \wedge \overline{\delta}.$$

11.2 Algebraic Topology of X

For simplicity, we will drop the bars. Recall that $d\alpha = d\beta = d\delta = 0$, while $d\gamma = \delta \wedge \beta$. Also, note that $\alpha \wedge \alpha = \beta \wedge \beta = \delta \wedge \delta = 0$. Then recall that $\dim H^1(X, \mathbb{R}) = 3$ and note that $e(X) = 0$. Therefore, we obtain $b_2 = 4$.

We will write a basis for each cohomology and write the intersection form. We have

$$H^1(X, \mathbb{R}) = \langle [\alpha], [\beta], [\gamma] \rangle$$

and

$$H^2(X, \mathbb{R}) = \langle [\alpha \wedge \beta], [\alpha \wedge \delta], [\gamma \wedge \beta], [\gamma \wedge \delta] \rangle.$$

Next, we see that $H^3(X, \mathbb{R}) = \langle [\beta \wedge \gamma \wedge \delta], [\alpha \wedge \gamma \wedge \delta], [\alpha \wedge \beta \wedge \gamma] \rangle$ and $H^4(X, \mathbb{R}) = \langle [\alpha \wedge \beta \wedge \gamma \wedge \delta] \rangle$.

Finally, we can see the intersection form is given by

$$Q_X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

From this, we see that $b^+X = B^-X = 2$, so $\sigma(X) = 0$.

12

March 3

Note: I was not here on this day. Notes were provided by Arthur Wang.

12.1 Topology of Complex Manifolds

We will answer the following question:

Question 12.1. *Are there any differential topological constraints for a symplectic 4-manifold to admit a complex structure?*

Recall that all symplectic manifolds have an almost complex structure. Also, they have positive b^+.

Theorem 12.2 (Kodaira-Siu). *If X is a compact complex surface, X is Kähler if and only if $b_1 X$ is even.*

We can consider X with even first Betti number. Then if X is complex, then it is Kähler. However, there are no constraints that come from Hodge theory. There are, however, more constraints from differential topology.

Theorem 12.3 (Hard Lefschetz). *For ω the Kähler form, $L_\omega^k : H^{n-k}(M, \mathbb{C}) \to H^{n+k}(M, \mathbb{C})$ is an isomorphism for all k. Here L_ω^k sends α to $\alpha \cup \omega^k$.*

This gives us Serre duality on Dolbeault cohomology.

Remark 12.4. All *triple Massey products* on Kähler M are zero: Let $a_1, a_2, a_3 \in H^*(M, \mathbb{R})$ with $a_1 \cup a_2 = 0 = a_2 \cup a_3$. Then

$$\langle a_1, a_2, a_3 \rangle \in H^*(M, \mathbb{R})/a_1 \cup H^*M + H^*M \cup a_3$$

is defined as follows:

Take $\alpha_i \in \Omega^* M$ with $a_i = [\alpha_i]$, so $\alpha_1 \wedge \alpha_2 = d\eta_{12}$ and $\alpha_2 \wedge \alpha_3 = d\eta_{23}$. Then

$$\langle a_1, a_2, a_3 \rangle = [\eta_{12} \wedge \alpha_3 - (-1)^{|a_1|} \alpha_1 \wedge \eta_{23}].$$

There is a constraint from algebraic topology. Not all finitely presented groups are fundamental groups of Kähler manifolds. For example, if the abelianization of π_1 has odd rank, then M is not Kähler. Also, π_1 that are nontrivial free products cannot be realized.

Remark 12.5. In dimension 4, we can also use the Enriques-Kodaira classification of complex surfaces.

12.2 Chern Classes

We will define the canonical class of a complex or symplectic manifold. First, however, we need to define Chern classes. The k-th *Chern class* of a complex vector bundle E is a cohomology class $c_k(E) \in H^{2k}(M, \mathbb{Z})$ and the total Chern class of E is the sum $c(E) = c_0(E) + c_1(E) + \cdots$.

These uniquely satisfy the following axioms:

1. $c_0(E) = 1$;
2. (Naturality) For all $f : N \to M$, $c_k(f^*E) = f^*c_k(E)$.
3. (Additivity) $c(E \oplus F) = c(E) \cup c(F)$.
4. (Normalization) For the tautological line bundle over $\mathbb{P}^k$, we have $c = 1-h$, where h is the hyperplane class.

Remark 12.6. The existence and uniqueness of the Chern classes follows from the theory of classifying spaces. Also, $c_k(E) = 0$ for all $k > \dim E$. The top Chern class is the Euler class, and for $E = TM$, $e(E)[M] = e(M)$, the Euler characteristic. Finally, Chern classes are invariant under isomorphism.

Note that if M is symplectic, then the space of compatible J is contractible and nonempty, so we can define Chern classes uniquely.

Definition 12.7. The *canonical class* of a (symplectic, complex) manifold M is $K := -c_1(M)$.[1]

Now specialize again to the 4-dimensional case. Then we have two cohomological invariants: The class of the symplectic form and the canonical class.

Definition 12.8. We define X to be minimal if X is not a connected sum of another closed manifold and $\overline{\mathbb{P}}^2$.

For minimal X, the we can define the *symplectic Kodaira dimension* of (X, ω) as:

$$\kappa(X) := \begin{cases} -\infty & K \cdot [\omega] < 0 \text{ or } K^2 < 0 \\ 0 & K \cdot [\omega] = 0 \text{ and } K^2 = 0 \\ 1 & K \cdot [\omega] > 0 \text{ but } K^2 = 0 \\ 2 & K \cdot [\omega] > 0 \text{ and } K^2 > 0 \end{cases}.$$

For non-minimal manifolds, then we define the Kodaira dimension to be the Kodaira dimension of a minimal model.

Remark 12.9. There is a *minimal madel program* for symplectic 4-manifolds. One can always find a minimal symplectic X such that $X' = X \# m\overline{\mathbb{P}}^2$.

Remark 12.10. By Taube's work on Seiberg-Witten invariants of symplectic manifolds, no other combinations for $K \cdot [\omega]$ and K^2 can occur.

This implies that $\kappa(X)$ is well-defined.

Remark 12.11. If $\kappa(X) = -\infty$, then X is either rational or ruled.

Remark 12.12. When X is Kähler, then its algebriac and symplectic Kodaira dimensions agree.

The general problem we want to solve is:

Question 12.13. *For a given class of manifolds, what are the constraints on their algebraic topology? Which values can be realized as invariants of such manifolds?*

[1]In algebraic geometry, this is defined as $\det T^*M$.

13

March 5

13.1 Geography Problem

Today we will discuss problems of realizing certain algebraic invariants with a manifold admitting a certain structure. Let X be a closed connected oriented smooth 4-manifold. Which pairs of integers $(x, y) \in \mathbb{Z}^2$ can be realized as

1. $(e(X), \sigma(X))$?
2. $(c_1^2(X), c_2(X))$?
3. $(c_1^2(X), \chi_h(X))$?

Note that $c_1^2 = 2e + 3\sigma$ and $\chi_h = \frac{e+\sigma}{4}$ (Noether's formula). We will ask that X is a minimal (irreducible smooth, almost complex, complex, symplectic, Kähler, projective) manifold. We will also fix $G = \pi_1 X$, which we will usually take to be trivial. We can also fix the type for Q_X.

Remark 13.1. Any pair determines the other pairs. Moreover, if $G = 1$, then by results of Serre, ..., Donaldson, then e, σ, t determine Q_X. Then Freedman tells us that Q_X determines the homeomorphism of X.

13.2 Geography of Compact Complex Surfaces

Let $X^4 = S$ be a minimal complex surface. Then we have the following:

Kodaira If S is not Class VII, rational, or ruled, then $c_1^2 \geq 0$ and $c_2 \geq 0$.

Noether If S is minimal and has Kodaira dimension 2, then $c_1^2 \geq 2\chi_h - 6$.

Bogolomov-Minyaoka-Yau If S is not rational, ruled, or Class VII, then $c_1^2 \geq 9\chi_h$.

Yau If $c_1^2 = 9\chi_h$, then X is a complex ball quotient (if X is not $\mathbb{P}^2$).

Moreover, we have a restricted *complete* list for:

- If $\kappa = -\infty$, then X is rational or ruled.
- If $\kappa = 0$, then X is Enriques, K3, T^4, or bielliptic.
- If $\kappa = 1$, then X is elliptic.

Theorem 13.2 (Hirokawa, Xiao, Persson-Peters,...). *All lattice points with $9\chi_h \geq c_1^2 \geq 2\chi_h - 2$ and $c_1^2 > 0$ are realized by minimal Kähler surfaces (mostly with π_1 trivial).*

Example 13.3. Note that T^2 and bielliptic surfaces lie at the origin. The ruled surfaces lie on $y = 8x$ in the third quadrant. $\mathbb{P}^2$ lies at the point $(1, 9)$. The Class VII and Hopf surfaces lie on the negative y-axis. The elliptic surfaces lie on the positive x-axis, with the first two being Enriques and K3.

Theorem 13.4 (Kodaira, Hirzebruch, Sommest, Persson, Moishezon-Teicher, Roulleau-Urzua). *Several lattice points with $9\chi_h > c_1^2 > 8\chi_h$ can be realized arbitrary close to the BMY line with $\pi_1 = 1$.*

Theorem 13.5 (Mumford, Ishida-Cato, Keum, Prasad-Yeung, Cartwright-Steger). *There are exactly $1+50+1$ surfaces the point $(1, 9)$. These are fake $\mathbb{P}^2$ when $b_1 = 0$.*

13.2.1 Geography of Kähler Surfaces

We take the compact complex picture and simply remove the Class VII and Hopf surfaces.

13.3 Geography of Simply-Connected Symplectic 4-Manifolds

Theorem 13.6 (Taubes). *If X is not rational or ruled, then $c_1^2 \geq 0$.*

Remark 13.7. The Noether inequality fails in the symplectic case. This is due to Gompf, Fintushel-Stern, J. Park, who showed that all lattice points with $2\chi_h - 6 > c_1^2 > 0$ are realized by minimal simply-connected symplectic X.

Remark 13.8. There are some restricted cases.

- If $\kappa = -\infty$, then X is rational or ruled.
- If $\kappa = 0$, then the known examples are Enriques, K3, T^4, bielliptic, or T^2-bundles over T^2. We also know that any manifold with $\kappa = 0$ has the same rational homology as one of these (Li-Bauer).
- For $\kappa = 1$, Gompf and Fintuschel-Stern showed that there are infinitely more minimal symplectic 4-manifolds than elliptic surfaces. In addition, by a result of Gompf, J. Park, Stipsicz, Akhmedov-Boldridge-Baykur-Kirk-Park, there are infinitely many sumplectic manifolds on almost all lattice points with $8\chi_h \geq c_1^2 \geq 0$.

13.3.1 Five Fundamental Problems

1. Does every symplectic 4-manifold which is not a ruled surface have $c_2 = e \geq 0$?
2. (Symplectic BMY) Are there any symplectic 4-manifolds that are not ruled that violate BMY?
3. (Symplectic Yau) Are there any symplectic 4-manifolds with $c_1^2 = 9c_2$ Kähler?
4. (Symplectic Poincare Conjecture) Every symplectic 4-manifold homeomorphic to $\mathbb{P}^2$ is diffeomorphic to it.
5. (Symplectic Calabi-Yau Conjecture) Every symplectic 4-manifold with $c_1 = 0$ is diffeomorphic to a K3 or a T^2-bundle over T^2.

Solution to any of these problems will give us an A for the course regardless of what else happens during the semester. İnanç have us five problems so we wouldn't be fighting over them.

14

March 10

Note: I was away on this day; notes were provided by Arthur Wang.

14.1 Darboux-Moser-Weinstein Local Theory

Let $\psi : M \times \mathbb{R} \to M$ be a smooth isotopy. This means that $\psi(-,t)$ is a diffeomorphism for all t and $\psi(-,0)$ is the identity on M. From this flow we may obtain a time-dependent vector field V_t such that

$$V_t = \frac{d}{ds}\psi_s(\psi_t^{-1}(x))\bigg|_{s=t}.$$

Equivalently, we have

$$V_t \circ \psi_t = \frac{d}{dt}\psi_t$$

or

$$(\psi_t)^* V_t = \frac{d}{dt}\psi_t.$$

From any time-dependent vector field V_t we can find a ψ_t solving the above ODE locally. If V_t is compactly supported on M (for example if M is compact), then we can obtain ψ_t globally.

14.1.1 Moser's Method

We will construct an isotopy to match symplectic forms by constructing a time-dependent flow in an analogous fashion. Let $\omega_t \in \Omega^2 M$ be a family of symplectic forms. Assume that

$$\frac{d}{dt}\omega_t = d\sigma_t$$

for some $\sigma_t \in \Omega^1 M$. We want to conclude that there exists an isotopy ψ_t of M such that

$$\psi_t^* \omega_t = \omega_0. \tag{14.1}$$

This will imply that (M, ω_t) is symplectomorphic to (M, ω_0). If M is compact, then it suffices to construct a flow satisfying (14.1).

Differentiating and integrating with respect to t, we see that (14.1) is equivalent to

$$\begin{aligned}
\frac{d}{dt}\psi_t^*\omega_t = 0 &\Leftrightarrow \psi_t^*\left(L_{V_t}\omega_t + \frac{d}{dt}\omega_t\right) \equiv 0 \\
&\Leftrightarrow L_{V_t}\omega_t + \frac{d}{dt}\omega_t \equiv 0 \\
&\Leftrightarrow i_{V_t} d\omega_t + d i_{V_t}\omega_t + d\sigma_t \equiv 0 \\
&\Leftrightarrow d(i v_t \omega_t + \sigma_t) \equiv 0,
\end{aligned}$$

by Cartan's magic formula, which means we must have

$$i_{v_t}\omega_t + \sigma_t \equiv 0.$$

We need this equation to have a solution for all t, but it does because ω_t is nondegenerate. Therefore, we can take $V_t := \mu_{\omega_t}^{-1}(-\sigma_t)$.

Lemma 14.1 (Moser's Isotopy). *Let M be a $2n$-dimensional manifold and $S \subset M$ be a compact submanifold. Let $\omega_0, \omega_1 \in \Omega^2 M$ be closed forms such that their restrictions to $T_S M$ are equal and nondegenerate on S. Then there exist open neighborhoods $N_i \supset S$ and a diffeomorphism $\psi : N_0 \to N_1$ such that $\psi^*\omega_1 = \omega_0$ and $\psi|_S = \mathrm{id}_S$.*

Proof. Due to Moser's method, it suffices to show that there exists an open neighborhood $N_0 \supset S$ and $\sigma \in \Omega^1 N_0$ such that $\omega_1 - \omega_0 = d\sigma$ and $\sigma|_{T_S M} \equiv 0$.

Using this, we can take $\omega_t = (1-t)\omega_0 + t\omega_1$ on N_0. It is easy to see that ω_t is closed for all t. Then because nondegeneracy is an open condition, we can shrink N_0 to a smaller open neighborhood of S to ensure nondegeneracy of ω_t. Therefore we have a family of symplectic forms ω_t on N_0 such that $\omega_t = \omega_0 + td\sigma$. Therefore we have a vector field V_t whose flow is an isotopy ψ_t of N_0, where we shrink N_0 further if needed so that $\psi_t^*\omega_t = \omega_0$. Because σ is identically zero on $\sigma|_{T_S M}$, we have $V_t|_S \equiv 0$ and thus $\psi_t|_S = \mathrm{id}_S$.

To show the existence of N_0, fix a Riemannian metric g on M and identify the normal bundle ν_S with $TS^\perp$. Consider the restriction of the exponential map $TS^\perp \to M$ around the open neighborhood of the zero-section:

$$U_\varepsilon = \{(s,u) \in TM \mid s \in S, v \in T_s S^\perp, |v| < \varepsilon\} \subset TS^\perp.$$

Because S is compact, $\exp|_{U_\varepsilon}$ is a diffeomorphism for small enough ε. Set $N_0 = \exp(U_\varepsilon)$. Define $\phi_t : N_0 \to N_0$ by $\phi_t(\exp(s,v)) = \exp(s,tv)$ so it is a diffeomorphism for $t > 0$ and $\phi_0(N_0) \subset S$. We also have $\phi_1 = \mathrm{id}_{N_0}$ and $\phi_t|_S = \mathrm{id}_S$.

Therefore, for $\tau = \omega_1 - \omega_0^*$, we have $\phi_0^*\tau = 0$ and $\phi_1^*\tau = \tau$. Because ϕ_t is a diffeomorphism for $t > 0$, there exists a vector field

$$V_t := \frac{d}{dt}\phi_t(\phi_t^{-1})$$

whose flow is ϕ_t. Therefore, for $\delta > 0$, we have

$$\begin{aligned}
\phi_1^*\tau - \phi_\delta^*\tau &= \int_\delta^1 \frac{d}{dt}\phi_t^*\tau\, dt \\
&= \int_\delta^1 \phi_t^*\left(\mathscr{L}_{V_t}\tau + \frac{d}{dt}\tau\right) dt \\
&= \int_\delta^1 \phi_t^*(i_{V_t}d\tau + di_{V_t}\tau)\, dt \\
&= \int_{\delta^1 d\phi_t^*(i_{V_t}\tau)} dt \\
&= d\int_\delta^1 \phi_t^*(i_{V_t}\tau)\, dt \\
&= d\sigma_\delta.
\end{aligned}$$

Therefore, as $\delta \to 0^+$, $\sigma_{\delta\to\sigma}$, so $\omega_1 - \omega_0 = \tau = \phi_1^*\tau = \phi_1^*\tau - \phi_0^*\tau = d\sigma$. Thus $\omega_1 - \omega_0 = d\sigma$ and

$$\sigma|_{T_S M} = \int_0^1 \phi_t^*(i_{V_t}\tau)\, dt\Big|_{T_S M} = \int_0^1 i_{V_t}\tau\, dt = \int_0^1 0\, dt = 0.$$

$\square$

Theorem 14.2 (Darboux). *Let (M, ω) be a symplectic manifold. Around any point $p \in M$, there exists a local coordinate chart $(U, \{x_i, y_i\})$ such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

It follows that chart transitions for M lie in $Sp(2n)$.

Proof. Using any symplectic basis for $T_pM, \omega|_p$, construct coordinates centered at p and defined in some neighborhood U' of p such that $\omega = \sum dx_i' \wedge dy_i'$. Then apply Moser's lemma for $S = \{p\}, \omega_0 = \omega$, and $\omega_1 = dx_i \wedge dy_i$. Then there exist neighborhoods U_0, U_1 of p and a diffeomorphism ψ such that $\psi^*\omega_1 = \omega_0$ and $\psi(p) = p$. Then

$$\omega|_{U_0} = \psi^* \left(\sum dx_i' \wedge dy_i' \right) = \sum d(x_i' \circ \psi) \wedge d(y_i' \circ \psi) = \sum dx_i \wedge dy_i.$$

□

15

March 12

Note: I was away on this day; notes were provided by Arthur Wang.

Exercise 15.1. Let X be a closed, connected, oriented almost complex 4-manifold.

1. Show that $X \# \overline{\mathbb{P}}^2$ is also almost complex;
2. Show that any lattice point can be realized by a non-minimal simply-connected almost-complex manifold.

Exercise 15.2. Let Σ be a closed symplectic surface in a closed symplectic 4-manifold (X, ω).

1. Show that any symplectic form on Σ is determined up to isotopy by $\int_\sigma \omega$;
2. Any symplectic neighborhood of Σ is determined by $\Sigma \cdot \Sigma$ and $\int_\Sigma \omega$.

We know there always exists a standard symplectic neighborhood of $S = \text{pt}$. Generally, the goal is when $S \subset (M, \omega)$ is symplectic, Lagrangian, or (co)isotropic given some data on νS, the normal bundle of S. We will obtain a standard form for ω on a small tubular neighborhood of S.

15.1 S is Symplectic

Recall that there exists $J \in \mathcal{J}(M, \omega)$ such that S is J-holomorphic. Then $\nu S = (TS)^\omega = (TS)^\perp$, so νS is a symplectic vector bundle on S. Then recall that the isomorphism class is determined by the isomorphism class of the complex vector bundle $(\nu S, J)$. We will show that a neighborhood of S is completely determined by $\omega|_S$ and the isomorphism class of $(\nu S, \omega)$.

Theorem 15.3 (Weinstein, Symplectic neighborhood theorem). *For $j = 0, 1$, let (M_j, ω_j) be symplectic manifolds with compact symplectic submanifolds $S_j \subset M_j$ such that there exists a vector bundle isomorphism $\Phi : (\nu S_0, \omega_0) \to (\nu S_1, \omega_1)$ commuting with a symplectomorphism $\phi : S_0 \to S_1$. Then ϕ extends to a symplectomorphism ψ of neighborhoods $N_i \supset S_i$ with $d\psi = \Phi$.*

Proof. There exists $J_j \in \mathcal{J}(M_j, \omega_j)$ and compativle g_j for which S_j is J_j-holomorphic and $\nu S_j = TS_j^\perp$. Under these identifications, let $\varphi_j : \nu S_j \to M_j$ be the exponential maps. Then

$$\varphi = \varphi_1 \circ \varphi_0^{-1}$$

is a diffeomorphism from a neighborhood of S_0 to a neighborhood of S_1 where

$$\varphi^* \omega_1|_{T_S M} = \omega_0|_{T_S M}.$$

Then use the Moser isotopy lemma for $S_0, \varphi^*\omega_1, \omega_0$. $\square$

15.2 L is Lagrangian

In this case, $\dim L = n = \dim M/2$, and we will see that the symplectomorphism class of a tubular neighborhood of L is completely determined by the diffeomorphism class of L. Observe that if $L \subset (V, \omega)$ is a Lagrangian subspace of a symplectic vector space, then ω gives a canonical identification of V/L with L^*.

In the manifold case, if $L \subset (M, \omega)$ is Lagrangian, then $\nu L = T^*L$. Therefore a neighborhood of L in M is diffeomorphic to a neighborhood of the zero-section of the cotangent bundle T^*L.

Example 15.4 (Canonical Symplectic Structure on the Cotangent Bundle). Let L be any n-dimensional smooth manifold and $M = T^*L$. Then we will define the *tautological* 1*-form* λ on M. We have the projection $\pi : M \to L$. For any $v \in M = T^*L$, v is pulled back by π to $\pi^* v \in T_v^* M$. Then we define $\lambda \in \Omega^1 M$ by

$$\lambda_v := \pi^*(v) \in T_v^* M.$$

The canonical symplectic structure ω on M is $\omega = -d\lambda$, an exact form.

In local coordinates, if p_j are the coordinates on L and q_j are the cotangent coordinates, then we have

$$\lambda = \sum p_j dq_j.$$

In fact λ is characterized by the property that for all $\sigma \in \Omega^1 L$, $\sigma^* \lambda = \sigma$. By the local characterization, ω is symplectic. Finally, it is relatively easy to check that L is Lagrangian in T^*L (just use the local description).

Theorem 15.5 (Weinstein, Lagrangian neighborhood theorem). *Let (M, ω) be a symplectic manifold and $L \subset (M, \omega)$ be a compact Lagrangian submanifold. Then there exist neighborhoods U of the zero section in T^*L and V of L in M and a diffeomorphism $\phi : U \to V$ such that $\phi^*\omega = -d\lambda$ that is the identity on L.*

Example 15.6. Let L be a closed Lagrangian surface in a closed symplectic 4-manifold (X, ω). Since $\nu L \simeq T^*L \simeq TL$ as bundles over L, then they have the same Euler class. Then we see that $e_{TL}[L] = e_{\nu L}[L]$, so $L \cdot L = e(L)$. Thus the isomorphism class of νL is determined by the diffeomorphism type of L.

By Theorem 15.5 there exists a Weinstein neighborhood $N(L) \subset (X, \omega)$ such that

$$N(L) \simeq \{(q, p) \in T^*L \mid q \in L, |p| < \varepsilon\}.$$

Observe that any radial push-off of L in $N(L)$ is also Lagrangian.

For $L = T^2$, note that $L \cdot L = e(L) = 0$ and T^*L is trivial, so $N(L) \simeq T^2 \times B_\varepsilon(0)$.

Exercise 15.7. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\omega = \pi_1^* \omega_{FS} + \pi_2^* \omega_{FS}$. Then recall that $\mathbb{RP}^1$ is the fixed locus of complex conjugation on $\mathbb{P}^1$. Then $L = \mathbb{RP}^1 \times \mathbb{RP}^1$ is Lagrangian. How about a Lagrangian Klein bottle?