FINITE GENERATION OF GW POTENTIAL OF SMOOTH CY HYPERSURFACES IN (WEIGHTED) $\mathbb{P}^4$

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ABSTRACT. I will explain the proof of the Yamaguchi-Yau finite generation conjecture for the Gromov-Witten theory of $Z_5 \subset \mathbb{P}^4$, $Z_6 \subset \mathbb{P}(1,1,1,1,2)$, $Z_8 \subset \mathbb{P}(1,1,1,1,4)$, and $Z_{10} \subset \mathbb{P}(1,1,1,2,5)$. The proof is due to Chang-Guo-Li in the case of the quintic and to the author in the other three examples.

CONTENTS

1. MSP INVARIANTS

We will consider MSP moduli spaces $\mathcal{W}_{g,n,d}$ with $d_0 = d$, $d_\infty = 0$, only $(1,\rho)$ insertions, and arbitrary values of N. We first note that the MSP virtual localization formula is given by

$$\begin{aligned}\frac{1}{e(N_\Theta^{\mathrm{vir}})} = &\prod_{v \in V_0} \prod_{\alpha=1}^{N} \frac{1}{e(R\pi_* f_v^* \mathcal{O}(1) \otimes t_\alpha)} \\ &\cdot \prod_{\alpha=1}^{N} \prod_{v \in V_1^\alpha} \frac{5t_\alpha \cdot e(\mathbb{E}^\vee \otimes (-t_\alpha))^5}{e(\mathbb{E} \otimes 5t_\alpha) \cdot (-t_\alpha)^5} \frac{\prod_{\beta \neq \alpha} e(\mathbb{E}^\vee \otimes (t_\beta - t_\alpha))}{\prod_{\beta \neq \alpha}(t_\beta - t_\alpha)} \\ &\cdot \left(\prod_{v \in V_\infty} \cdots\right) \cdot \prod_{e \in E} \cdots,\end{aligned}$$

where t_α are the equivariant variables and V_1^α denotes those vertices at level 1 where the curve satisfies $\mu_\alpha \neq 0$ and $\mu_{\beta \neq \alpha} = 0$. In particular, define

$$\begin{aligned}[\overline{\mathcal{M}}_{g,n}(Z,d)]^{\mathrm{top}} &= \frac{[\overline{\mathcal{M}}_{g,n}(Z,d)]^{\mathrm{vir}}}{e(R\pi_* f_v^* \mathcal{O}(1) \otimes t_\alpha)} \\ &= (-t^N)^{d+1-g} [\overline{\mathcal{M}}_{g,n}(Z,d)]^{\mathrm{vir}}\end{aligned}$$

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$$[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{top}} = \left(\frac{1}{5}N(-t_\alpha)^{N+3}\right)^{g-1}[\overline{\mathcal{M}}_{g,n}].$$

These are the top degree part of the contribution to the virtual localization formula coming from a vertex v. We will denote the full contribution at level 1 by $[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{tw}}$. From now on, we will specialize our equivariant variables to roots of unity as $t_\alpha = -\zeta_N^\alpha t$. For convenience, we will also specialize t such that $t^N = -1$.

We may define MSP invariants using virtual localization. Note that by the condition that ρ vanishes at the marked points, we have evaluation morphisms

$$\mathrm{ev}_i \colon \mathcal{W}_{g,n,d} \to \mathbb{P}^{4+N},$$

which restrict to

$$\mathrm{ev}_i \colon \mathcal{W}_\Theta^- \to (x_1^5 + \cdots + x_5^5 = 0)^{(\mathbb{C}^\times)^N} = Z \sqcup \bigsqcup_{\alpha=1}^N \mathrm{pt}_\alpha,$$

where $\mathcal{W}_\Gamma^-$ is the degeneracy locus of $\mathcal{W}_\Gamma$. Therefore, we may define MSP invariants with insertions from the state space

$$\mathcal{H} = H^*(Z) \oplus \bigoplus_{\alpha=1}^N H^*(\mathrm{pt}_\alpha).$$

Using the vertex contributions to the virtual normal bundle, we define the pairing

$$(x,y)^M = \int_Z xy|_Z + \sum_\alpha \frac{5}{Nt_\alpha^3} xy|_{\mathrm{pt}_\alpha}.$$

The state space has several bases, which we will discuss now.

- Let $p = c_1(\mathcal{O}_{\mathbb{P}^{4+N}}(1))$ be the equivariant ambient hyperplane class. Then we have the basis $\phi_i = p^i$ for $i = 0, \ldots, N+3$;
- There is the basis $\{\mathbf{1}_Z, H, H^2, H^3\} \cup \{\mathbf{1}_\alpha\}_{\alpha=1}^N$;

The last kind of MSP invariant we need to define is the MSP $[0,1]$ invariant. Here, we simply consider the class

$$[\mathcal{W}]^{[0,1]} = \sum_{\Theta \in \Lambda^{[0,1]}} \frac{[\mathcal{W}_\Theta]^{\mathrm{vir}}}{e(N_\Theta^{\mathrm{vir}})},$$

where $\Lambda^{[0,1]}$ denotes the set of all graphs without any level ∞ vertices.

2. Genus zero MSP theory

In genus zero, the full MSP and the $[0,1]$ theory are equal. This follows from the following lemma:

Lemma 2.1. *We have*

$$\mathcal{W}_{0,n,d} \cong \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+N}, d)$$

and an equality

$$[\mathcal{W}_{0,n,d}]^{\mathrm{vir}} = \pm e(R\pi_* f^* \mathcal{O}(5)) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+N}, d)]^{\mathrm{vir}}$$

of virtual cycles.

The lemma tells us that the genus-zero MSP invariants are the same as the GW invariants of a degree 5 hypersurface in $\mathbb{P}^{4+N}$, which is in particular Fano. In particular, the MSP I-function is given by the formula

$$I^M(q,z) = z\sum_{d\geqslant 0} q^d \frac{\prod_{m=1}^{5d}(5p+mz)}{\prod_{m=1}^{d}(p+mz)^5\prod_{m=1}^{d}((p+mz)^N - t^N)}.$$

This automatically implies the following result.

Lemma 2.2. *We have*

$$J^M(0,q,z) = I^M(q,z)$$

whenever $N \geqslant 2$.

The main result we need to know about genus zero MSP theory is the explicit form of the quantum connection. Let $D \coloneqq q\frac{d}{dq}$.

Lemma 2.3. *The MSP* S*-matrix satisfies the differential equation*

$$(p+zD)S^M(z)^* = S^M(z)\cdot A^M,$$

where A^M *is given by the matrix*

$$\begin{bmatrix}
0 & & & & & & & 120q & & & & \\
1 & 0 & & & & & & & 770q & & & \\
 & 1 & 0 & & & & & & & 1345q & & \\
 & & 1 & 0 & & & & & & & 770q & \\
 & & & 1 & 0 & & & & & & & 120q+t^N \\
 & & & & 1 & 0 & & & & & & \\
 & & & & & 1 & 0 & & & & & \\
 & & & & & & 1 & 0 & & & & \\
 & & & & & & & \cdots & \cdots & & & \\
 & & & & & & & & 1 & 0 & & \\
 & & & & & & & & & 1 & 0 & \\
 & & & & & & & & & & 1 & 0
\end{bmatrix}$$

in the basis $\{\phi_i\}$ *for* $N > 5$.

3. MSP $[0,1]$ CohFT

Define the MSP R-matrix by the Birkhoff factorization

$$S^M(z)\begin{pmatrix}\Delta^1 & & & \\ & \ddots & & \\ & & \Delta^N & \\ & & & \mathrm{Id}\end{pmatrix} = R(z)\begin{pmatrix} S^{pt_1} & & & \\ & \ddots & & \\ & & S^{pt_N} & \\ & & & S^Z\end{pmatrix},$$

where

$$\Delta^\alpha(z)$$

$$\coloneqq \exp\left(\sum \frac{B_{2k}}{2k(2k-1)}\left(\frac{5}{(-t_\alpha)^{2k-1}} + \frac{1}{(5t_\alpha)^{2k-1}} + \sum_{\beta\neq\alpha}\frac{1}{(t_\beta - t_\alpha)^{2k-1}}\right)z^{2k-1}\right)$$

is defined using the quantum Riemann-Roch theorem. Here, we need to shift S^Z to the point $\tau_Z = \frac{I_1}{I_0}H$, and

$$S^{\mathrm{pt}_\alpha} = e^{\frac{\tau_\alpha}{z}},$$

where

$$\tau_\alpha = -t_\alpha \int_0^q (L(x)-1)\frac{dx}{x}.$$

Theorem 3.1. *The MSP* $[0,1]$ *invariants come from a CohFT* $\Omega^{[0,1]}$*, which is defined by the formula*

$$\Omega^{[0,1]} = R.\left(\Omega^Z \oplus \bigoplus_{\alpha=1}^N \omega^{\mathrm{pt}_\alpha,\mathrm{top}}\right).$$

Remark 3.2. The normalized tail contribution at the isolated points is given by

$$\tilde{T}_\alpha(z) = z(\mathbf{1} - L^{\frac{N+3}{2}}R(z)^{-1}\mathbf{1})|_{\mathrm{pt}_\alpha} = O(z^2),$$

where $L = (1-5^5q)^{\frac{1}{N}}$. In addition, when $N \gg 3g-3+n$, there is no tail contribution at level 0.

4. Degree bound for MSP theory

In order to compute the invariants of a Calabi-Yau threefold using MSP theory, we need to control the MSP invariants. Our goal will be to control the MSP $[0,1]$ invariants, but these are defined as a mysterious sum of virtual localization contributions. First, we will control the full MSP invariants.

Lemma 4.1. *The full MSP correlator*

$$\langle p^{a_1}\bar{\psi}_1^{a_1},\dots,p^{a_n}\bar{\psi}_n^{a_n}\rangle_{0,n}^M$$

is a polynomial in q *of degree at most*

$$g-1+\frac{3g-3+\sum a_i}{N}.$$

This follows from the fact that the virtual dimension of the MSP moduli space is $N(d+1-g)+n$. To obtain the same degree bound for the $[0,1]$ correlators, we will need a decomposition formula for the full MSP theory in terms of the $[0,1]$ theory and the remaining contributions.

Lemma 4.2. *We have the MSP decomposition formula*

$$\begin{aligned}\langle \tau_1\bar{\psi}_1^{a_1},\dots,\tau_n\bar{\psi}_n^{a_n}\rangle_{g,n}^M = &\sum_{\Gamma\in\Lambda^{\mathrm{bipartite}}}\frac{1}{|\mathrm{Aut}\,\Gamma|}\prod_{v\in V_\infty}\mathrm{Cont}_{[v]}^\infty\left(\bigotimes_{i\in L_v^\circ}\bar{\psi}_{c(i)}^{a_i}\right)\cdot\\ &\cdot\prod_{v\in V_{[0,1]}}\left\langle\bigotimes_{i\in L_v}\tau_i\bigotimes_{i\in L_v^\circ}\bar{\psi}_{c(i)}^{a_i}\bigotimes_{e\in E_v}\frac{\mathbf{1}^{\alpha_e}}{\frac{5t_{\alpha_e}}{a_e}-\psi_{(e,v)}}\right\rangle_{g_v,n_v}^{[0,1]}.\end{aligned}$$

Here, the contribution $\mathrm{Cont}_{[v]}^\infty$ *of a vertex* v *at level* ∞ *is a generating series of FJRW-like invariants, which is a polynomial in* q *of degree at most*

$$d_{\infty[v]}+\frac{1}{5}\left(2g_v-2-\sum_{e\in E_v}(a_e-1)\right).$$

*In addition, $\Lambda^{\mathrm{bipartite}}$ is the set of **stable** bipartite graphs, L_v° is the set of legs which get contracted to v after stabilization, and $c(i)$ is the stable vertex that i gets contracted to after stabilization.*

This lemma is proved by directly applying the virtual localization formula and then analyzing the following two situations:

- What happens at a vertex at level ∞;
- What happens when we split a graph at a vertex at level 1.

By using the decomposition formula and a careful degree-counting argument, we obtain the following degree bound for the $[0,1]$ theory.

Lemma 4.3. *The MSP $[0,1]$ correlator*

$$\langle p^{a_1}\bar{\psi}_1^{a_1},\dots,p^{a_n}\bar{\psi}_n^{a_n}\rangle_{0,n}^{[0,1]}$$

is a polynomial in q of degree at most

$$g-1+\frac{3g-3+\sum a_i}{N}.$$

5. Polynomiality

We first introduce the ring of five generators. Let

$$\begin{aligned} I(q,z) &\coloneqq z\sum_{d\geqslant 0} q^d\frac{\prod_{m=1}^{5d}(5H+mz)}{\prod_{m=1}^{d}(H+mz)^5}\\ &\eqqcolon I_0 z+I_1 H+I_2\frac{H^2}{z}+I_3\frac{H^3}{z^2} \end{aligned}$$

and define the following generators:

$$A_k \coloneqq \frac{D^k I_{11}}{I_{11}},\qquad B_k\coloneqq\frac{D^k I_0}{I_0},\qquad\text{and}\qquad Y=\frac{1}{1-5^5q}.$$

Here, recall that $I_{11}=1+D\left(\frac{I_1}{I_0}\right)$.

Lemma 5.1 (Yamaguchi-Yau)**.** *The ring*

$$\mathcal{R}\coloneqq\mathbb{Q}[A_1,B_1,B_2,B_3,Y]$$

contains all A_k and B_k.

Theorem 5.2. *Introduce the series*

$$P_{g,n}\coloneqq\frac{(5Y)^{g-1}I_{11}^n}{I_0^{2g-2}}\left(Q\frac{d}{dQ}\right)^n F_g(Q)\bigg|_{Q=qe^{\frac{I_1}{I_0}}}.$$

Then $P_{g,n}\in\mathcal{R}$ for all g,n such that $2g-2+n>0$.

If we want to prove this result using the results we have already proved, then we need to prove a polynomiality result for the the entries of the R-matrix. At level 0, we use the equation

$$(R(z)^{-1}x)|_Z=S^Z(q,z)(S^M(z)^{-1})|_Z$$

and the explicit forms of the MSP quantum connection and the quantum connection for the quintic to obtain

$$R(z)^*\mathbf{1}|_Z=I_0+O(z^{N-3})$$

$$R(z)^*p|_Z = zD(I_0) + HI_0I_{11} + O(z^{N-2}).$$

To simplify what follows, define the normalized basis

$$\varphi_b = I_0I_{11}\cdots I_{bb}H^b,$$

where I_{22} was defined previously and $I_{33} = I_{11}$. If we define

$$(R_k)_j^b \coloneqq (R_k\varphi^b, p^j)^M,$$

then the recursive formula

$$(R_k)_j^b = (D + C + b)(R_{k-1})_{j-1}^b + (R_k)_{j-1}^{b-1} - c_j q (R_k)_{j-N}^b,$$

where $C_b = D\log(I_0\cdots I_{bb}) \in \mathcal{R}$ and $c_j = (0,\dots,0,120,770,1345,770)$, yields the following result:

Lemma 5.3. *If* $j \not\equiv b + k \pmod N$, *then* $(R_k)_k^b = 0$. *Otherwise, we have* $(R_k)_{b+k}^b \in \mathcal{R}$ *and* $Y(R_k)_{b+N+k}^n \in \mathcal{R}$.

At level 1, define the normalized basis $\bar{\mathbf{1}}_\alpha = L^{-\frac{N+3}{2}}\mathbf{1}_\alpha$. Then define

$$(R_k)_j^\alpha \coloneqq L_\alpha^{-(j-k)}(R_k\bar{\mathbf{1}}^\alpha, p^j)^M,$$

where $L_\alpha = -t_\alpha L$.

Lemma 5.4. *The quantity* $(R_k)_j^\alpha$ *is independent of* α *and is a polynomial in* Y *of degree at most* $k + \left\lfloor \frac{j}{N} \right\rfloor$.

The lemma is proved as follows:

- Fix the case when $j = 0$ by using the Picard-Fuchs equation and an oscillating integral;
- Use the recursion
$$\begin{aligned}(R_k)_j^\alpha = \left(D - \frac{1}{N}\left(\frac{N+3}{2} - j + k\right)(1-Y)\right)(R_{k-1})_{j-1}^\alpha \\ + (R_k)_{j-1}^\alpha + \frac{c_j}{5^5}(1-Y)(R_k)_{j-N}^\alpha\end{aligned}$$
to induct on j.

Proof of Theorem 5.2. First, note that we have the base cases $P_{0,3} = 1$ due to Zagier-Zinger and

$$P_{1,1} = -\frac{1}{2}A_1 - \frac{31}{3}B_1 - \frac{1}{12}(1-Y) - \frac{25}{12}$$

due to Zinger. The relation

$$P_{g,n+1} = (D + (g-1)(2B_1 + 1 - Y) - nA_1)P_{g,n}$$

implies that we only need to prove $P_{g\geqslant 2} \in \mathcal{R}$.

Consider the correlator $(5Y)^{g-1}\langle\,\rangle_{g,0}^{[0,1]}$, which is a polynomial in Y of degree at most $g-1$. By the stable graph sum formula, we have

$$(5Y)^{g-1}\langle\,\rangle_{g,0}^{[0,1]} = P_g + \sum_\Gamma \mathrm{Cont}_\Gamma\,.$$

For all non-leading graphs, we use the relation $\sum_v (g_v - 1) + |E| = g - 1$ to assign powers of Y to all of the edges. Then the contributions from vertices are given as follows:

- At a level 0 vertex, the contributions are simply
$$Y^{g_v-1}\left\langle \varphi_{b_1}\bar{\psi}_1^{a_1},\ldots,\varphi_{b_{n_v}}\bar{\psi}_{n_v}^{a_{n_v}}\right\rangle_{g_v,n_v}^{Z},$$
which reduces to $P_{g_v,m}$ by the string and dilaton equations.
- At a level 1 vertex, the contribution is
$$\sum_m \frac{L^{3(g_v-1)}}{m!}\left\langle L_\alpha^{j_1-k_1}\bar{\psi}_1^{k_1},\ldots,L_\alpha^{j_{n_v}-k_{n_v}}\bar{\psi}_{n_v}^{k_{n_v}},\tilde{T}_\alpha^m\right\rangle_{g_v,n_v+m}.$$
After summing over all α, we see that this is nonzero only if the total power of t_α is a multiple of N (here, we may want N to be a prime number).

Using the fact that the contribution from an edge between two level 1 vertices satisfies a balancing condition, the total factor of the L_α for the various α becomes 1. This implies that $\mathrm{Cont}_\Gamma \in \mathcal{R}$ for any non-leading Γ, so we must have $P_g \in \mathcal{R}$. □

Remark 5.5. We can recover the genus one mirror theorem very quickly using the results we have already proved. If we consider the correlator
$$\langle p\rangle_{1,1}^{[0,1]} = \text{const},$$
there are only two stable graphs. The contribution of the stable graph with a genus 1 vertex at the quintic is given by
$$\begin{aligned}\frac{1}{I_0}\left\langle R(z)^{-1}p|_Z\right\rangle_{1,1} &= \left\langle -B_1\bar{\psi}_1 + I_{11}H\right\rangle_{1,1}\\ &= P_{1,1} + \frac{200}{24}B.\end{aligned}$$
The other graph contributes
$$\frac{1}{2}(A+4B+\frac{2}{5}(1-Y))$$
at level 0. Finally, we can prove that the total contribution from level 1 is a degree 1 polynomial in Y, so using the known values of $N_{1,1}$ and $\langle H\rangle_{1,1,0}$ fixes the two coefficients of Y.