AIRY IDEALS AND TOPOLOGICAL RECURSION: AN INVESTIGATIVE TOOL FOR ENUMERATIVE GEOMETRY, VOAS, AND GAUGE THEORIES

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Abstract. The Chekhov-Eynard-Orantin topological recursion is an abstract framework that appears in many contexts, from enumerative geometry to mathematical physics. In these lectures I will introduce the concept of Airy ideals in the Rees Weyl algebra, which provides a clean reformulation of the topological recursion in the language of D-modules. After introducing the foundations of the theory, including the existence and uniqueness theorem originally proved by Kontsevich and Soibelman, I will focus on applications in enumerative geometry, VOAs and gauge theories. Such applications include the construction of Whittaker vectors for various W-algebras and Gaiotto vectors for supersymmetric gauge theories, ELSV-type formulae for Hurwitz numbers, W-constraints for various enumerative invariants, etc. The lectures are meant to be introductory; my hope is to convey why I believe that the formalism of topological recursion and Airy ideals should be in the toolbox of all geometers and mathematical physicists!

Topological recursion was invented by Eynard-Orantin in their work [EO07] on matrix models in 2007 and then reformulated in terms of Airy structures (or Airy ideals) by Kontsevich-Soibelman [KS18] in 2017. This is a correspondence between geometry, algebra, and differential equations.

1. Witten's conjecture

This story began with theories of 2d quantum gravity. Define the *partition function*

$$\mathcal{Z} = \exp\left(\sum_{\substack{g=0 \\ n=1 \\ 2g-2+n>0}}^{\infty} \frac{\hbar^{2g-2+n}}{n!} \sum_{k_1,\dots,k_n} F_{g,n}[k_1,\dots,k_n] t_{k_1}\cdots t_{k_n} \right),$$

where we define

$$F_{g,n}[k_1,\dots,k_n] = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}.$$

To simplify our notation, we will write

$$\mathcal{Z} = \exp\left(\sum_{k=1}^{\infty} \hbar^k q^{(k+2)}(t_A) \right).$$

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Theorem 1.1 ([Kon92]). *The function*

$$u(t_0, t_1, \dots) = \frac{\partial^2}{\partial t_0^2} \log \mathcal{Z}$$

is the unique solution to the KdV hierarchy with initial condition $u(t_0, 0, \dots) = t_0$.

1.1. **Reformulation as differential constraints for $\mathcal{Z}$.** For $k \geqslant -1$, define the operator

$$L_k = J_{k+1} - \frac{1}{2} \sum_{m+n=k-1} : J_m J_n : - \delta_{k,0} \frac{\hbar^2}{8},$$

where

$$J_m = \hbar \frac{\partial}{\partial t_m}, m \geqslant 0 \qquad J_{-m} = \hbar(2m-1)t_{m-1}, m \geqslant 1.$$

Now we can reformulate Theorem 1.1 as

Theorem. $\mathcal{Z}$ *is the unique solution to the constraints* $L_k \mathcal{Z} = 0$ *for all* $k \geqslant -1$*.*

Note that the L_k satisfy the relation

$$[L_i, L_j] = \hbar^2 (i-j) L_{i+j}.$$

Thus we have a representation of a subalgebra of the Virasoro algebra, which is defined by

$$[L_i, L_j] = \hbar^2 (i-j) L_{i+j} \frac{\hbar^4 c}{12} i^2 (i-1) \delta_{i,-j}$$

Note that L_k has the expansion

$$L_k = \hbar \frac{\partial}{\partial t_{k+1}} + O(\hbar^2),$$

which makes it easy to see uniqueness once we have existence.

1.2. **Potential generalizations.** The first possible generalization is the Virasoro conjecture [EHX97], where we remove uniqueness of the solution and preserve the geometry (using Gromov-Witten theory of projective varieties) and the Virasoro algebra. This has been proved in several cases, including toric varieties [Giv01; Iri07], flag varieties [JK20], Grassmannians [HV05], varieties with semisimple quantum cohomology [Tel12], Calabi-Yau varieties [Get99], and toric bundles over varieties satisfying the Virasoro conjecture [CGT15]. There is also progress on Virasoro constraints for sheaf-counting theories, see [MOOP20].

The second possible generalization is to keep existence and uniqueness of the solution $\mathcal{Z}$ to the constraints $H_i \mathcal{Z} = 0$. More precisely, we want to find constraints on H_i such that $H_i \mathcal{Z} = 0$ always has a unique solution of the form

$$\mathcal{Z} = \exp\left(\sum_{k=1}^{\infty} \hbar^k q^{(k+2)}(x_A) \right)$$

with the initial condition $\mathcal{Z}\_x_A = 0 = 1$. This is the generalization that we will take.

2. Airy ideals

2.1. **Weyl algebra and modules.** Let A be a (possibly infinite) indexing set. The *Weyl algebra* $\mathcal{D}_A$ is defined as

$$\mathcal{D}_A = \mathbb{C}[x_A]\langle \partial_A \rangle / ([\partial_i, x_j] = \delta_{ij}).$$

This is only a filtered algebra, so it is difficult to form power series. We will use the *Rees construction* to turn it into a graded algebra.

Here, we will use the filtration

$$\{0\} \subseteq F_0\mathcal{D}_A \subseteq F_1\mathcal{D}_A \subseteq \cdots \subseteq \mathcal{D}_A,$$

where

$$F_i\mathcal{D}_A = \left\{ \sum_{m+k \leqslant i} P^{(k)}_{a_1,\ldots,a_m}(x_A)\partial_{a_1}\cdots\partial_{a_m} \right\}.$$

For example, $F_2\mathcal{D}_A$ contains terms like x_1 and $x_1\partial_2$ but not terms like $x_1^2\partial_2$.

Now we may define the *Rees Weyl algebra*

$$\mathcal{D}^{\hbar}_A = \bigoplus_{k=0}^{\infty} \hbar^k F_k\mathcal{D}_A,$$

where $\deg \hbar = 1$, and the completion

$$\widehat{\mathcal{D}}^{\hbar}_A = \prod_{k=0}^{\infty} \hbar^k F_k\mathcal{D}_A.$$

This is in fact a good way to handle power series, and we will consider $H_i \in \widehat{\mathcal{D}}^{\hbar}_A$.

We will now consider the polynomial module $\mathcal{M}_A$, which has the following properties:

- $\mathcal{M}_A$ is generated by $1 \in \mathcal{M}_A$;
- The annihilator $\mathrm{Ann}_{\mathcal{D}_A}(1)$ is the left ideal generated by ∂_i;
- We recover
$$\mathcal{D}_A / \mathrm{Ann}_{\mathcal{D}_A}(1) \simeq \mathcal{M}_A.$$

We can now introduce $\hbar$ directly into our module, but we need a filtration on $\mathcal{M}$ such that

$$F_i\mathcal{D} \cdot F_j\mathcal{M} \subset F_{i+j}\mathcal{M}.$$

A natural choice is to let $F_i\mathcal{M}_A$ be polynomials of degree at most i. We can now repeat the Rees construction by considering

$$\mathcal{M}^{\hbar}_A = \bigoplus_{k=0}^{\infty} \hbar^k F_k\mathcal{M}_A, \qquad \widehat{\mathcal{M}}^{\hbar}_A = \prod_{k=0}^{\infty} \hbar^k F_k\mathcal{M}_A.$$

The partition function $\mathcal{Z}$ will live in a twist of $\widehat{\mathcal{M}}^{\hbar}_A$ rather than in the module itself. Some properties of $\hbar$ are as follows:

- $\widehat{\mathcal{M}}^{\hbar}_A$ is a cyclic left $\widehat{\mathcal{D}}^{\hbar}_A$-module generated by 1;
- $\mathrm{Ann}_{\widehat{\mathcal{D}}^{\hbar}_A}(1)$ is the left ideal $\mathcal{I}_{\mathrm{can}}$ generated by $\hbar\partial_i$;

- We can recover the module by
$$\widehat{\mathcal{D}}^{\hbar}_A/\mathcal{I}_{\mathrm{can}} = \widehat{\mathcal{M}}^{\hbar}_A.$$

We can now reformulate our problem as

Problem 2.1. *What conditions should a left ideal $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$ satisfy such that $\mathcal{I} \cdot \mathcal{Z}$ has a unique solution of the form*

$$\mathcal{Z} = \exp\left(\sum_{k=1}^{\infty} \hbar^k q^{(k+2)}(x_A)\right)? \tag{1}$$

2.2. **Characterization of ideals solving our problem.** From our expression for $\mathcal{Z}$, it is clear that the operators

$$\overline{H}_i = \hbar\partial_i - \sum_{k=1}^{\infty} \hbar^{k+1}\partial_i q^{(k+2)}(x_A)$$

satisfy $\overline{H}_i\mathcal{Z} = 0$. Denote the ideal generated by the $\overline{H}_i$ by $\overline{\mathcal{I}}$, which is in fact $\mathrm{Ann}_{\widehat{\mathcal{D}}^{\hbar}_A}(\mathcal{Z})$.

Definition 2.2. Define an automorphism $\Phi\colon \widehat{\mathcal{D}}^{\hbar}_A \to \widehat{\mathcal{D}}^{\hbar}_A$ by

$$\Phi\colon (\hbar, \hbar x_i, \hbar\partial_i) \mapsto \left(\hbar, \hbar x_i, \hbar\partial_i - \sum_{k=1}^{\infty} \hbar^{k+1}\partial_i q^{(k+2)}(x_A)\right).$$

Note that this is a valid construction because $[\overline{H}_i, \hbar x_j] = \hbar^2\delta_{ij}$ and $[\overline{H}_i, \overline{H}_j] = 0$.

Given the twist Φ, define the twisted module ${}^{\Phi}\widehat{\mathcal{M}}^{\hbar}_A$ by the twisted product

$$\widehat{\mathcal{D}}^{\hbar}_A \cdot^{\Phi} \widehat{\mathcal{M}}^{\hbar}_A \to \widehat{\mathcal{M}}^{\hbar}_A \qquad P \cdot^{\Phi} f = \Phi^{-1}(P) \cdot f.$$

This satisfies the same properties as the untwisted version. Because $\Phi(P) = \mathcal{Z} \cdot P \cdot \mathcal{Z}^{-1}$, we can define a *module of exponential type* by $\widehat{\mathcal{Z}\mathcal{M}^{\hbar}_A}$, which is cyclic and generated by $\mathcal{Z}$. Clearly we have

$$\mathrm{Ann}_{\widehat{\mathcal{D}}^{\hbar}_A}(\mathcal{Z}) = \Phi(\mathcal{I}_{\mathrm{can}}) = \overline{\mathcal{I}}.$$

Definition 2.3 (Airy ideal)**.** Let $\overline{\mathcal{I}} \subseteq \widehat{\mathcal{D}}^{\hbar}_A$ be a left ideal. It is *Airy* if it is generated by $(\overline{H}_i)_{i\in A}$.

Theorem 2.4. *The function $\mathcal{Z}$ defined in (1) is the unique solution to $\overline{\mathcal{I}} \cdot \mathcal{Z} = 0$ with $\mathcal{Z}|_{x_A=0} = 1$.*

Of course, any ideal can have many different generating sets. How can we recognize Airy ideals in a more intrinsic manner?

Theorem 2.5. *[KS18] Let $\mathcal{I} \subseteq \widehat{\mathcal{D}}^{\hbar}_A$ be a left ideal. Suppose that:*

(1) $\mathcal{I}$ is generated by $(H_i)_{i\in A}$ such that $H_i = \hbar\partial_i + O(\hbar^2)$;

(2) $[\mathcal{I},\mathcal{I}] \subset \hbar^2\mathcal{I}$.

Then $\mathcal{I}$ is Airy.

Remark 2.6. Clearly $[\mathcal{I},\mathcal{I}] \subseteq \mathcal{I}$ for any left ideal $\mathcal{I}$. In addition, in our situation we always have $[\mathcal{I},\mathcal{I}] \subseteq \hbar^2\widehat{\mathcal{D}}^{\hbar}_A$. However, in general we do **not** have $[\mathcal{I},\mathcal{I}] \subset \hbar^2\mathcal{I}$.

Example 2.7. Let $\mathcal{I} \subset \widehat{\mathcal{D}}^{\hbar}_A$ be the ideal generated by the Kontsevich-Witten differential operators

$$H_i = L_{i-1} = \hbar\partial_i + O(\hbar^2).$$

For the second condition, note the (shifted) Virasoro relations

$$[H_i, H_j] = \hbar^2(i-j)H_{i+j-1}.$$

Computing the full commutator of any $A, B \in \mathcal{I}$, we have

$$\left[\sum_i a_i H_i, \sum_j b_j H_j\right] = \\ = \sum_{i,j} \big(a_i[H_i, b_j]H_j + a_i b_i [H_i, H_j] + [a_i, b_j]H_i H_j + b_j[a_i, H_j]H_i\big),$$

and clearly this lands in $\hbar^2\mathcal{I}$.

Sketch of proof of Theorem 2.5. We need to prove that there exists a generating set $(\overline{H})_{i\in A}$ taking the desired form.

(1) First, we find $\overline{H}_i \in \mathcal{I}$ of the form

$$\overline{H}_i = \hbar\partial_i + \sum_{k=2}^{\infty} \hbar^k p_i^{(k)}(x_A).$$

(2) Next, if $\bar{\mathcal{I}}$ is the ideal generated by the $\overline{H}_i$, we need to show that $\mathcal{I} \subseteq \bar{\mathcal{I}}$. Therefore, $\mathcal{I} = \bar{\mathcal{I}}$.

(3) Finally, we prove that $[\overline{H}_i, \overline{H}_j] = 0$ for all i, j.

To complete the first step, start with the $H_i = \hbar\partial_i + O(\hbar^2)$. We can then replace

$$\hbar\partial_i \mapsto H_i + O(\hbar^2)$$

for the right-most derivative, and now we have

$$H_i = \hbar\partial_i + \sum_{k=1}^{\infty} \hbar^{k+1} p_i^{(k+1)}(x_A) + Q,$$

where $Q \in \mathcal{I}$. Simply set $\overline{H}_i$ to be the first two terms of H_i.

To complete the second step, consider the original generators $H_i = \hbar_i + O(\hbar^2)$. Then replace

$$\hbar\partial_i \mapsto \overline{H}_i + O(\hbar^2)$$

and so H_i is the sum of a polynomial and $\overline{Q}$, where $\overline{Q} \in \bar{I}$. The following lemma completes this step:

Lemma 2.8. *There are no nonzero polynomials in $\mathcal{I}$.*

To complete the third step, we use brute force to obtain

$$\begin{aligned}[\overline{H}_i, \overline{H}_j] &= \left[\hbar\partial_i + \sum_{k=1}^{\infty} \hbar^{k+1} p_i^{(k+1)}(x_A), \hbar\partial_i + \sum_{k=1}^{\infty} \hbar^{k+1} p_j^{(k+1)}(x_A)\right] \\ &= \hbar^2 \sum_{k=1}^{\infty} \hbar^k \left(\partial_i p_j^{(k+1)} - \partial_j p_i^{(k+1)}\right).\end{aligned}$$

By the lemma, we are finished. □

Trivially, we see that

- $\mathcal{I} = \Phi(\mathcal{I}_{\text{can}})$ for some transvection Φ;
- $\widehat{\mathcal{D}}_A^{\hbar}/\mathcal{I} \simeq {}^{\Phi}\mathcal{M}_A^{\hbar} \simeq \mathcal{M}_A^{\hbar}\mathcal{Z}$;
- $\mathcal{Z}$ is uniquely determined with $\mathcal{Z}|_{x_A=0} = 1$.

2.3. **Are Airy ideals interesting?** Returning to the algebra-geometry-integrable systems correspondence, we want either that $\mathcal{Z}$ is the τ-function for some integrable system or that $F_{g,n}$ are some kind of interesting enumerative invariants.

Definition 2.9. We say that an Airy ideal $\mathcal{I} \subseteq \widehat{\mathcal{D}}_A^{\hbar}$ is $\hbar$-*polynomial* if it is generated by $(H_i)_{i\in A}$ that are $\hbar$-polynomial. In addition, we say that $\mathcal{I}$ is $\hbar$-*finite* if there exists $N \in \mathbb{N}$ such that the $\hbar$-degree of all H_i is at most N.

Airy ideals are integrable when they are $\hbar$-finite. If

$$H_i = \hbar\partial_i + \sum_{k=1}^{N} \hbar^{k+1} p_i^{(k+1)}(x_A) + \text{derivatives},$$

then $H_i\mathcal{Z} = 0$ has the solution

$$\mathcal{Z} = \exp\left(\sum_{k=1}^{\infty} \hbar^k q^{(k+2)}(x_A)\right).$$

Example 2.10. Suppose that $\mathcal{I}$ has degree 2. Then we can write

$$H_i = \hbar\partial_i - \hbar^2\left[\frac{1}{2}A_{ijk}x_jx_k + B_{ijk}x_j\partial_k + \frac{1}{2}C_{ijk}\partial_j\partial_j + D_i\right].$$

Because $[\mathcal{I},\mathcal{I}] \subseteq \hbar^2\mathcal{I}$, we see that

$$[H_i, H_j] = \hbar^2 c_{ijk} H_k$$

and therefire the H_i are a representation of a Lie algebra. We can write an explicit recursive formula for $\mathcal{Z}$, which is of the form

$$\mathcal{Z} = \exp\left(\sum_{\substack{g=0\\ n=1\\ 2g-2+n>0}} \frac{\hbar^{2g-2+n}}{n!} \sum_{k_1,\dots,k_n} F_{g,n}[k_1,\dots,k_n] x_{k_1}\cdots x_{k_n}\right).$$

The $F_{g,n}$ satisfy a *topological recursion*, with the formula

$$F_{g,n}[k_1,\dots,k_n] = \sum_{m=2}^{\infty} B_{k_1 k_m a} F_{g,n-1}[a,k_2,\dots,\widehat{k}_m,\dots,k_n]$$
$$+\frac{1}{2}C_{k_1 ab}\left\{F_{g-1,n+2}[a,b,k_2,\dots,k_n] + \sum_{\substack{g_1+g_2=g \\ I\cup J=\{k_2,\dots,k_m\}}} F_{g_1,|I|+1}[a,I]F_{g_2,|J|+1}[b,J]\right\}.$$

Here, we impose the initial conditions

$$F_{0,3}[k_1,k_2,k_3] = A_{k_1k_2k_3} \qquad F_{1,1} = D_k.$$

2.4. **Where we can find Airy ideals.** Let $A = \{1,\dots,n\}$. We are interested in a classification of quadratic Airy ideals.

Example 2.11. Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ which is generated by E, H, F satisfying the relations

$$[H,E] = 2E, \qquad [H,F] = -2F \qquad ,[E,F] = H.$$

This does have a representation that is an Airy ideal, which is of the form

$$H = \hbar\partial_1 - \hbar^2(-)$$
$$E = \hbar\partial_2 - \hbar^2(-)$$
$$F = \hbar\partial_3 - \hbar^2(-).$$

See [ABCO17] for details.

Unfortunately, $\mathfrak{sl}_2$ is one of the rare simple Lie algebras admitting a representation that generates an Airy ideal in finitely many variables.

Proposition 2.12 ([ABCO17, Proposition 6.9])**.** *The following Lie algebras do **not** admit a representation in finitely many variables generating an Airy ideal:*

- A_n *for* $n \notin \{1,5\}$*;*
- C_n *for* $n \geqslant 6$*;*
- D_n *for* $n \geqslant 4$*;*
- E_6, E_7, E_8.

3. Applications

We will give some examples of Airy ideals that arise as representations of infinite-dimensional algebras and attempt to give them an interpretation in terms of enumerative geomtry. We will consider $\mathcal{W}$-algebras or vertex operator algebras as our algebras and then consider the algebra of modes. Then we want to realize our vertex operator algebras as sub-VOAs of the Heisenberg VOA. Roughly, a $\mathcal{W}$-algebra is a generalization of the Virasoro algebra, but we will not give a precise definition.

Example 3.1. $\mathcal{W}(\mathfrak{sl}_2)$ is the Virasoro algebra and is generated by a single field

$$W^2(z) = \sum_{n\in\mathbb{Z}} W^2_n z^{-n-2},$$

where $W^2_n = L_n$.

3.1. **Kontsevich-Witten partition function.**

Example 3.2. The algebra $\mathcal{W}(\mathfrak{gl}_2)$ is generated by two fields

$$W_1(z) = \sum W^1_n z^{-n-1}, \qquad W^2(z) = \sum W^2_n z^{-n-2}.$$

We would like to find expressions for our $\mathcal{W}$-algebra fields as differential operators, which will create Airy ideals for us. We would like to realize $\mathcal{W}(\mathfrak{gl}_2)$ as a subalgebra of the rank 2 Heisenberg algebra, which is defined by

$$[J_m, J_n] = m\delta_{m,-n}.$$

One basic representation is

$$J_m = \hbar\partial_m, \qquad J_{-m} = mx_m, \qquad J_0 = 0.$$

More precisely, we can start with any module for the Heisenberg algebra.

We begin with the Z_2-twisted representation for the rank 2 Heisenberg algebra. This gives us the following formulae for $W(\mathfrak{gl}_2)$:

$$\begin{aligned} W^1_k &= \hbar J_{2k} \\ W^2_k &= \hbar^2\left(-\frac{1}{2}\sum_{m+n=2k} :J_mJ_n: -\frac{1}{8}\delta_{k,0}\right). \end{aligned}$$

We now want to fix a subset of modes such that

$$[H_i, H_j] = \hbar^2 \sum_k c_{ijk} H_k$$

and then consider a dilaton shift, which breaks $\hbar$-homogeneity.

We will first choose modes W^1_k for $k \geqslant 1$ and W^2_k for $k \geqslant -1$. Recall that the W^2_k are Virasoro modes, so we have no problems there. To break homogeneity, we will shift

$$J_{-3} \mapsto J_{-3} - \frac{1}{\hbar},$$

and this gives us

$$W^2_k = \hbar J_{2k+3} + \hbar^2(\cdots).$$

If we compute the partition function $\mathcal{Z}$, we obtain the Kontsevich-Witten partition function, which is defined by

$$F_{g,n}[k_1, \ldots, k_n] = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n}.$$

Instead of choosing W^2_k for $k \geqslant -1$, we can take $k \geqslant 0$ (so we delete W^2_{-1}). The old dilaton shift does not work for indexing reasons, so we take

$$J_{-1} \mapsto J_{-1} - \frac{1}{\hbar}.$$

We then obtain

$$W^2_k = \hbar J_{2k+1} - \hbar^2(\cdots).$$

The resulting $\mathcal{Z}$ is the BGW τ-function for the KdV hierarchy, where our new initial condition is

$$u(x_1, 0, \ldots) = \frac{1}{8(1-x_1)^2},$$

where

$$u = \frac{\partial^2}{\partial x_1^2} \log \mathcal{Z}.$$

This $\mathcal{Z}$ is defined by

$$F_{g,n}[k_1, \ldots, k_n] = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

where $\Theta_{g,n}$ is the Norbury/Chiodo class [Nor17].

3.2. **The example of $\mathcal{W}(\mathfrak{gl}_r)$.** The algebra $\mathcal{W}(\mathfrak{gl}_r)$ is generated by $W^1(z), \ldots, W^r(z)$. There is a natural embedding

$$\mathcal{W}(\mathfrak{gl}_r) \subset \mathcal{H}_r$$

into the rank r Heisenberg algebra. We of course consider the $\mathbb{Z}_r$ twisted module for $\mathcal{H}_r$.

We want to classify subsets of modes such that there is a valid dilaton shift subject to including all non-negative modes. These are indexed by

$$\{s \in \{1, \ldots, r+1\} \mid r \equiv \pm 1 \pmod{s}\}$$

and were computed in [Bor+18]. For example, if $s = r+1$, we choose W^i_k for all $k \geqslant -i + 1 + \delta_{i,1}$ and if $s = 1$, we choose all W^i_k for $k \geqslant 0 + \delta_{i,1}$. Then our dilaton shift must be

$$J_{-s} \mapsto J_{-s} - \frac{1}{\hbar}.$$

When $s = r+1$, then $F_{g,n}$ are the r-spin intersection numbers and $\mathcal{Z}$ is the τ-function for the r-KdV hierarchy. This recovers the W-constraints for $\mathcal{Z}$. It is unclear what happens for other s. There is a natural candidate, which is called the Chiodo class. This is only known to work for $s = r-1$ by work of Chidambaram–Garcia-Falide–Giachetto [CGG22].

We may instead consider the permutation $\sigma = (1 \cdots r-1)(r)$ and consider the σ-twisted module for $\mathcal{H}_r$. We then obtain a choice of some $s \in \{1, \ldots, r\}$. When $r = 3$ and $s = r$, we recover the W_3 constraints of Alexandrov [Ale15] for open intersection numbers. Conjecturally, when $r > 3$ and $s = r$, we should recover the open r-spin intersection numbers. For other choices of s, it is unclear what we obtain.

3.3. **Other examples coming from Lie algebras.** For any $\mathfrak{g}$, we can construct the $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$. For some choices of embeddings into the Heisenberg algebra and twists, Airy structures have been constructed for $\mathcal{W}(\mathfrak{so}_{2n}), \mathcal{W}(\mathfrak{e}_k), \mathcal{W}(\mathfrak{sp}_{2n})$, see [Bor+18; BCJ22].

3.4. **Whittaker vectors.** We want to find a state such that

$$L_k |\wedge\rangle = \delta_{k,1}\Lambda |\wedge\rangle$$

for all $k \geqslant 1$. These states are called *Whittaker vectors*. These can be computed in our formalism (see [BBCC21]).

4. Spectral curve topological recursion

4.1. **Spectral curves.**

Definition 4.1. A *spectral curve* is a quadruple $S = (\Sigma, x, y, B)$ such that

- Σ is a Riemann surface;
- x, y are functions on Σ that are holomorphic except potentially at a finite number of points;
- B is a symmetric differential $\Sigma \times \Sigma$ with a double pole on the diagonal and biresidue 1.

For example, we can take $\Sigma = \mathbb{P}^1$ and

$$B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}.$$

Definition 4.2. A spectral curve S is *meromorphic* if x, y are meromorphic on Σ and is *compact* if Σ is compact.

Note that if x is meromorphic and Σ is compact, then x is a finite branched covering of $\mathbb{P}^1$. In general, we can consider infinite-degree covers. In any case, consider the ramification locus $R \subset \Sigma$. For any $a \in R$, we can write, we can write equations like

$$x = x(a) + \varphi^{r_a}, \qquad x = \frac{1}{\varphi^{r_a}}.$$

Definition 4.3. A spectral curve S is *admissible* if at all $a \in R$, $r_a = \pm 1 \pmod{s_a}$, where roughly we have the behavior

$$y \sim S^{s_a - r_a}$$

near each ramification point a. Here, $s_a \in \{1, \dots, r_a + 1\}$.

In general, suppose we have some equation $P(x, y) = 0$. Of course, if S is compact and meromorphic, then P is polynomial and S is a compactification of $P(x, y) = 0$.

Example 4.4. Let $\Sigma = \mathbb{P}^1$ and

$$x = z^r, \qquad y = z^{s-r},$$

where $s \in \{1, \dots, r+1\}$ such that $r = \pm 1 \pmod{s}$. Then

$$P(x, y) = \begin{cases} y^r - x & s = r+1 \\ x^{r-s} y^r - 1 & s < r. \end{cases}$$

These (r, s)-curves recover the (r, s)-partition functions discussed previously.

Example 4.5. Let $\Sigma = \mathbb{C} \setminus \text{cut}$ and

$$x = z + \frac{1}{z}, \qquad y = \log z.$$

Then

$$P(x, y) = x - e^{y} - e^{-y} = 0,$$

and this spectral curve recovers the stationary Gromov-Witten theory of $\mathbb{P}^1$.

Example 4.6. Let $\Sigma = \mathbb{C}$ and

$$x = ze^{-z^r}, \qquad y = e^{z^r}.$$

These satisfy the equation

$$P(x, y) = y - e^{x^r y^r} = 0.$$

This example reproduces the r-spin Hurwitz number and if $\Sigma = \mathbb{C}_\infty$, then we recover the Atlantes Hurwitz numbers.

Example 4.7. Let $f \in \mathbb{Z}$ and consider the curve

$$e^{x} + e^{fy} + e^{(f+1)y} = 0.$$

This produces the open Gromov-Witten theorey of $\mathbb{C}^3$. More generally, if we take the toric web diagram of a toric Calabi-Yau threefold X, we can produce the mirror spectral curve of X, which recovers the open Gromov-Witten theory of X.

4.2. **Topological recursion.** Topological recursion is a machine that produces symmetric differentials

$$\omega_{g,n}(z_1, \ldots, z_n)$$

on Σ^n from the initial data

$$\omega_{0,1}(z) = y\, dx\,, \qquad \omega_{0,2} = B.$$

This is done by performing local analysis around the ramification points.

For simplicity, we will assume that all $a \in R$ are have simple ramification, so $x = x(a) + \varphi^2$, and that Σ has genus 0. Then the formula becomes

$$\begin{aligned}
&\omega_{g,n+1}(z_0, \vec{z}) = \\
&\quad = \sum_{a \in R} \operatorname{Res}_{z=a} \Bigg[(dz_0\, z - z_0) \frac{1}{\omega_{0,1}(z) - \omega_{0,z}(\sigma_a(z))} \Bigg(\omega_{g-1,n+2}(z, \sigma_a(z), \vec{z}) \\
&\quad + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = \{z_1, \ldots, z_n\}}} \omega_{g_1,|I|+1}(z, I)\, \omega_{g_2,|J|+1}(\sigma_a(z), J) \Bigg) \Bigg].
\end{aligned}$$

This is related to our Airy structures as follows. Consider all $a \in R$ and attach a copy of $\mathcal{W}(\mathfrak{gl}_{r_a})$. We then consider the (r_a, s_a)-representation constructed previously, and the partition function $\mathcal{Z}$ of this Airy ideal is the same as the expresssion computed by topological recursion. Choose $a \in R$ and write

$$x = x(a) + \varphi^{r_a}.$$

Choose the vector space

$$V = \{\omega \in \mathbb{C}((\varphi))\, d\varphi \mid \operatorname{Res}_{\varphi=0} \omega(\varphi) = 0\}.$$

There is a positive subspace $V^+ \subseteq V$ with basis $d\xi_k = \varphi^{k-1}\, d\varphi$. The choice of V^- is dictated by B. There is a symplectic pairing

$$\Omega(df_1, df_2) = \operatorname{Res}_{\varphi=0} f_1\, df_2\,.$$

Thus we can choose $V^- \subset V$ with basis $d\xi_{-k}$ such that

$$\Omega(d\xi_e, d\xi_m) = \frac{1}{e}\delta_{e,-m}.$$

The simplest choice is the standard polarization

$$d\xi_{-e} = \varphi^{-e-1}\, d\varphi\,.$$

This can be deformed into

$$\begin{aligned} d\xi_{-e}^{(\varphi')} &= \operatorname{Res}_{\varphi=0}\left[\int_0^{\varphi} B(-,\varphi')\frac{d\varphi}{\varphi^{e+1}}\right] \\ &= \left(\frac{1}{\varphi^{e+1}} + \text{regular}\right) d\varphi\,. \end{aligned}$$

We can now write

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{\alpha_1,\ldots,\alpha_n=1}^{\infty} \sum_{k_1,\ldots,k_n=1}^{\infty} F_{g,n}\left[\begin{smallmatrix}\alpha_1 & \cdots & \alpha_n \\ k_1 & \cdots & k_n\end{smallmatrix}\right] d\xi_{-k_1}{}^{\alpha_1}(z_1)\cdots d\xi_{-k_n}^{\alpha_n}\,.$$

Theorem 4.8. *The partition function $\mathcal{Z}$ defined by the $F_{g,n}$ is a solution to the Airy ideal defined by the representations of $\mathcal{W}(\mathfrak{gl}_{r_i})$.*

4.3. **Enumerative applications.** Note that the Airy ideals only see the local structure of the spectral curve topological recursion. However, the differential forms are globally defined, and this gives interesting structure.

Example 4.9. Consider the example of the Gromov-Witten theory of $\mathbb{P}^1$. Then there is a mixing operator $\widehat{P}$ such that

$$\mathcal{Z} = \widehat{P}\left[\mathcal{Z}^{(2,3)} \cdot \mathcal{Z}^{(2,3)}\right].$$

The way to obtain the Gromov-Witten invariants of $\mathbb{P}^1$ is to consider expansions of $\omega_{g,n}$ at the punctures of Σ:

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{k_1,\ldots,k_n}^{\infty} H_{g,n}[k_1,\ldots,k_n]\frac{dx_1}{x_1^{h_1+1}}\cdots\frac{dx_n}{x_n^{h_n+1}}.$$

Then the Gromov-Witten theory of $\mathbb{P}^1$ will come from the punctures.

In general, expansions at the ramification points will give us integrals over $\overline{\mathcal{M}}_{g,n}$ (or r-spin versions), while expansions at the punctures will give enumerative invariants like the Gromov-Witten theory of $\mathbb{P}^1$, Gromov-Witten theory of of a toric Calabi-Yau threefold, or Hurwitz numbers.

4.4. **Some open questions.**

(1) Can we do spectral curve topological recursion for other Airy ideals? We only considered fully twisted representations of $\mathcal{W}(\mathfrak{gl}_r)$. The global structure contains more information, so we would like to find a global structure in some cases.

(2) Is there some way to globalize the notion of Airy structures? We would like to consider so-called "x-twisted modules" for the Heisenberg algebra, which will gives information about $\mathcal{W}(\mathfrak{gl}_r)$.

(3) If we begin with a spectral curve, can we construct a quantum version?

(4) There is a conjectural relationship between knot theory and topological recursion, where the Jones and HOMFLY polynomials should be constructed using topological recursion.

(5) If I have a family of spectral curves, what happens in the limit? For example, consider

$$x = z^r - \varepsilon z, \qquad y = z^{s-r}.$$

Is there some kind of limit

$$\lim_{\varepsilon \to 0} \omega_{g,n}[S_\varepsilon] \stackrel{?}{=} \omega_{g,n}[S_0]?$$

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