Category O Learning Seminar
Fall 2021

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Lectures by Various

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Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Seminar Website:
`https://math.columbia.edu/~plei/f21-CO.html`

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Kevin (Sep 29): Review of semisimple Lie algebras and introduction to category $\mathcal{O}$

1.1 Review of semisimple Lie algebras

Throughout this lecture, we will work over $\mathbb{C}$.

Definition 1.1.1. A Lie algebra $\mathfrak{g}$ is *semisimple* if any of the following equivalent conditions hold:

1. $\mathfrak{g}$ is a direct sum of simple Lie algebras (those with no nonzero proper ideals).
2. The Killing form $\kappa(x, y) := \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$ is nondegenerate.
3. The radical (maximal solvable ideal) of $\mathfrak{g}$ is zero.

Some examples of semisimple Lie algebras include $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$, and in some sense (the classification of simple Lie algebras), these are essentially all semisimple Lie algebras.

Now given a semisimple Lie algebra $\mathfrak{g}$, we will fix a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$, which is just a maximal abelian subalgebra of semisimple elements. This gives us a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the subspace of $\mathfrak{g}$ where $\mathfrak{h}$ acts with weight α. Some important facts about these root systems are the following:

- For all α, we have $\dim \mathfrak{g}_\alpha = 1$.
- For all roots α, β, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
- If α is a root, so is $-\alpha$.

In addition, the α are required to form a (reduced) *root system* (denoted Φ), the precise definition of which is deliberately omitted. Given a choice of Borel subalgebra containing $\mathfrak{h}$, we obtain a set Φ^+ of positive roots and a set Δ of simple roots. In addition, given a root system Φ, there is a *dual root system* $\Phi^\vee$, whose roots are

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}, \alpha \in \Phi.$$

Now suppose that $\mathfrak{g}$ is a semisimple Lie algebra with root system Φ. For every $\alpha \in \Phi^+$, we may choose $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$, and these determine some $h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{h}$. This choice can be made such that $\alpha(h_\alpha) = 2$.

Recall that the Lie algebra $\mathfrak{sl}_2$ is spanned by the matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the choice of $x_\alpha, y_\alpha, h_\alpha$ gives an embedding $\mathfrak{sl}_2 \to \mathfrak{g}$. These maps, ranging over all α, cover all of $\mathfrak{g}$. Now a basis of $\mathfrak{g}$ is given by $x_\alpha, y_\alpha, \alpha \in \Phi$ and h_{α_i} for the **simple** roots α_i. Therefore, to specify $\mathfrak{g}$, we only need to give commutation relations for the basis elements.

Now suppose that Φ is some root system. We would like to construct a semisimple Lie algebra $\mathfrak{g}$ with root system Φ. We want to build a semisimple Lie algebra. To do this, choose a set of simple roots α_i, and consider the Lie algebra

$$\left\langle x_{\alpha_i}, y_{\alpha_i}, h_{\alpha_i} \right\rangle / \text{relations},$$

where the relations are as follows:

- $[h_{\alpha_i}, h_{\alpha_j}] = 0$.
- We have $[x_{\alpha_i}, y_{\alpha_j}] = h_{\alpha_i}$ if $i = j$ and this commutator vanishes otherwise.
- $[h_{\alpha_i}, x_{\alpha_j}] = \left\langle \alpha_j, \alpha_i^\vee \right\rangle x_{\alpha_j}$.
- $[h_{\alpha_i}, y_{\alpha_j}] = -\left\langle \alpha_j, \alpha_i^\vee \right\rangle y_{\alpha_j}$.
- $\mathrm{ad}(x_{\alpha_i})^{1-\left\langle \alpha_j, \alpha_i^\vee \right\rangle}(x_{\alpha_j}) = 0$ if $i \neq j$.
- $\mathrm{ad}(y_{\alpha_i})^{1-\left\langle \alpha_j, \alpha_i^\vee \right\rangle}(y_{\alpha_j}) = 0$ if $i \neq j$.

The first four relations are called the *Weyl relations* and the last two are called the *Serre relations*. Given this data, we end up with a semisimple Lie algebra $\mathfrak{g}_\Phi$ with root system Φ. In addition, if $\mathfrak{g}$ is any other semisimple Lie algebra with root system Φ, there is an isomorphism $\mathfrak{g}_\Phi \xrightarrow{\sim} \mathfrak{g}$. Moreover, we have a bijection between semisimple Lie algebras and reduced root systems, which restricts to a bijection between simple Lie algebras and irreducible root systems.

Table 1.1: Root systems and Lie algebras

Irreducible root systems	simple Lie algebras
A_n	$\mathfrak{sl}_{n+1}$
B_n	$\mathfrak{so}_{2n+1}$
C_n	$\mathfrak{sp}_{2n}$
D_n	$\mathfrak{so}_{2n}$
E_6, E_7, E_8, F_4, G_2	exceptional Lie algebras

We will now discuss the finite-dimensional representation theory of semisimple Lie algebras $\mathfrak{g}$.

Theorem 1.1.2 (Weyl's complete reducibility theorem)**.** *Any finite-dimensional representation of $\mathfrak{g}$ is decomposes as a direct sum of simple representations.*

Now suppose that M is a finite-dimensional $\mathfrak{g}$-representation. Then M has a weight decomposition

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda.$$

These λ are *integral weights*, which simply means that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all roots α. For any root α, $x_\alpha(M_\lambda) \subset M_{\lambda+\alpha}$ and $y_\alpha(M_\lambda) \subset M_{\lambda-\alpha}$. We would like to think that the x_α raise the weights and y_α lower the weights, so we introduce a partial order. We say that $\lambda \geqslant \mu$ if $\lambda - \mu \in \mathbb{Z}_{\geqslant 0}\Phi^+$.

By Weyl's complete reducibility theorem, it remains to classify the irreducible representations of $\mathfrak{g}$. These are in bijection with the *dominant* integral weights, which in particular means that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in \Phi^+$. For any dominant weight λ, there is a unique highest-weight representation $L(\lambda)$. Here, $L(\lambda)$ is generated by a single *maximal vector* v of weight λ. This means that for all positive roots α, $x_\alpha v = 0$.

1.2 Introduction to category $\mathcal{O}$

We would now like to study infinite dimensional representations of $\mathfrak{g}$. Of course, this is impossibly complicated in general, so we will impose some finiteness conditions on our representations.

Definition 1.2.1. The category $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$-modules M satisfying:

1. M is finitely generated as a $U(\mathfrak{g})$-module.
2. M is $\mathfrak{h}$-semisimple and has a weight decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.
3. M is locally $\mathfrak{n}$-finite, where $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}$. Precisely, this means that the $U(\mathfrak{n})$ generated by any $v \in M$ is finite-dimensional.

Here are some facts about category $\mathcal{O}$, which are stated without proof.

- For all M in our category and weights λ, the weight space M_λ is finite-dimensional.
- $\mathcal{O}$ is a Noetherian (everything satisfies the descending chain condition) abelian category.

We will now describe some infinite-dimensional objects in category $\mathcal{O}$.

Definition 1.2.2. For any weight λ, the *Verma module* $M(\lambda)$ associated to λ is the module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is the Borel subalgebra associated to our choice of positive roots and $\mathbb{C}_\lambda$ is the $\mathfrak{b}$-module associated to the 1-dimensional representation of $\mathfrak{h}$ with weight λ and the identification $\mathfrak{b}/\mathfrak{n} = \mathfrak{h}$.

2

Fan (Oct 06): Beginnings in category $\mathcal{O}$: Vermas, central characters, and blocks

Recall the following fact ahout semisimple Lie algebras. If we have a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where

1. $\dim \mathfrak{g}_\alpha = 1$;
2. $\mathbb{Z}\Phi \subset \mathfrak{h}^*$ is a lattice of maximal rank;
3. $\alpha \in \Phi$ implies $-\alpha \in \Phi$;
4. $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$,

then $\mathfrak{g}$ is a semisimple Lie algebra.

Now recall that the Weyl group W is the group generated by the reflections s_α in the roots. For any $w \in W$, we define the *length*

$$\ell(w) = \#\{\alpha \in \Phi_+ \mid w(\alpha) \in \Phi_-\}.$$

Next, there is the Bruhat order, where if $w_2 = sw_1$ and $\ell(w_2) > \ell(w_1)$, we say $w_1 < w_2$.

Finally, throught this lecture, we will denote the weight lattice by Λ and the root lattice by Q. In addition, we will denote the λ-weight space of a module M by M^λ, and M_λ will be the Verma associated to λ. Also, we will need the notion of the universal enveloping algebra, which we will not write down here.

2.1 Definitions

Recall that $\mathcal{O}$ is the full subcategory of $\mathsf{Mod}(\mathcal{U}\mathfrak{g})$ of modules M such that

1. M is finitely generated over $\mathcal{U}\mathfrak{g}$.
2. M is $\mathfrak{h}$-semisimple.
3. M is locally $\mathfrak{n}$-finite.
4. $\dim M^\lambda < \infty$.

5. The set of weights of M is contained in some finite union of cones $\lambda - Q_+$.

Theorem 2.1.1. *The following properties hold for $\mathcal{O}$:*

1. *$\mathcal{O}$ is Noetherian.*

2. *$\mathcal{O}$ is closed under submodules, quotients, finite direct sums, and is abelian.*

3. *$\mathcal{O}$ is closed under tensoring with finite-dimensional representations (in fact, if tensoring with $\mathcal{O}$ is exact and lands in $\mathcal{O}$, then N must be finite-dimensional).*

4. *$\mathcal{M}$ is locally $Z\mathfrak{g}$-finite.*

5. *All $M \in \mathcal{O}$ are finitely generated over $\mathcal{U}\mathfrak{n}^-$.*

2.2 Highest weight modules

Definition 2.2.1. A vector v_+ is a *maximal vector* if $\mathfrak{n}^+ v_+ = 0$.

Definition 2.2.2. A module M is a *highest weight module* if there exists a maximal $v_+ \in M$ generateing M.

Definition 2.2.3. Let $\lambda \in \mathfrak{h}^*$ and consider the $\mathfrak{b}^+$-module $\mathbb{C}_\lambda$. Then the *Verma module* for λ is the module

$$M_\lambda \coloneqq \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda.$$

Note that we have the standard adjunction $\mathrm{Hom}_\mathfrak{g}(M_\lambda, -) = \mathrm{Hom}_\mathfrak{b}(\mathbb{C}_\lambda, -)$.

Theorem 2.2.4. *For any highest weight module M with highest weight λ,*

1. $M = \left\langle f_1^{n_1} \cdots f_{|\Phi_+|}^{n_{|\Phi|}} \right\rangle$*, and in particular M is $\mathfrak{h}$-semisimple.*

2. *All weights of M are at most λ.*

3. *For any $\mu < \lambda$, $\dim M^\mu < \infty$, and $\dim M^\lambda = 1$. In addition, $M \in \mathcal{O}$.*

4. *Any quotient of M is also a highest weight module with highest weight λ.*

5. *Any submodule of a highest weight module with weight $\mu < \lambda$ is a proper submodule. if M is simple, all maximal vectors have weight v_+^λ.*

6. *There exists a unique maximal submodule, and thus M has a unique simple quotient and thus is indecomposable.*

7. *All simple highest weight modules with highest weight λ are isomorphic, so $\dim \mathrm{End}\, M = 1$.*

Corollary 2.2.5. *Let $M \in \mathcal{O}$. Then M admits a filtration whose successive quotients are highest weight modules.*

2.3 Verma modules

Let M_λ be a verma module and L_λ be the unique simple quotient, and N_λ be the unique maximal submodule.

Theorem 2.3.1. *Any simple $L \in \mathcal{O}$ is isomorphic to L_λ for some λ.*

Proposition 2.3.2. *Let Σ be the set of simple roots and $\sigma \in \Sigma$. Let $\lambda \in \mathfrak{h}^*$ such that $\sigma^*(\lambda) \in \mathbb{N}$. Choose $v_+^\lambda \in M_\lambda$ a maximal vector. Then*

$$f_\sigma^{\sigma^*\lambda+1} v_+^\lambda = v_+^{\lambda-(\sigma^*\lambda+1)\sigma}.$$

In particular, there exists a nonzero morphism $M_{\lambda-(\sigma^*\lambda+1)\sigma} \hookrightarrow M_\lambda$.

Lemma 2.3.3. *We have the commutation relations*

$$[e_i, f_j^{k+1}] = 0, \qquad [e_i, f_i^{k+1}] = -(k+1) f_i^{k+1}(k - h_i), \qquad [h_i, f_j^{j+1}] = -(k+1)\alpha_j(h_i) f_j^{k+1}.$$

2.4 Examples

We will discuss the example of $\mathfrak{sl}_2$. Let ϕ_i be the operator that outputs the i-th diagonal of a matrix. Then let $\alpha = 2\phi_1 = \phi_1 - \phi_2$ be the root. Let $\alpha^\dagger$ be the matrix such that $\kappa(\alpha^\dagger, -) = \alpha(-)$. In particular, we have $\alpha^\dagger = \frac{1}{4}h$.

Then note that if we choose units so that $\phi_1 = 1$, then $\alpha = 2$. If $\lambda = n$, then the Verma module for λ has weights $n, n-2, \ldots$, and the simple module has weights $n, n-2, \ldots, -n$. For non-integral weights, we just have an infinite-dimensional representation. To see this, note that hitting any non-integral weight with $\mathfrak{n}^-$ will not reach another maximal vector.

2.5 Finite-dimensional modules

Theorem 2.5.1. *For any weight λ, $\dim L_\lambda < \infty$ if and only if $\lambda \in \Lambda_+$ is a dominant integral weight. This is equivalent to $\dim L_\lambda^\mu = \dim L_\lambda^{w(\mu)}$ for all $w \in W$.*

This result tells us that weights of L_λ are actually symmetric under the Weyl group.

Proof. First, if the span of $v \in M^\lambda$ is finite-dimensional for $\mathfrak{sl}_2$, then all of $\mathfrak{h}$ stabilizes $\mathrm{Span}_{\mathfrak{sl}_2} v$. This is because if $v^\mu \in N := \langle v \rangle$, then

$$h(e_i v^\mu) = e_i h v^\mu + \alpha_i(h) e_i v^\mu,$$

and thus $h(f_i v^\mu) \in \mathbb{C} f_i v^\mu$.

Next, if $\dim L_\lambda < \infty$, then after restriction to $\mathfrak{sl}_2$, we have $\lambda(h_i) = \alpha_i^*(\lambda) \in \mathbb{N}$, and thus $\lambda \in \Lambda_+$.

Now suppose that $\lambda \in \Lambda_+$. Then after restricting L_λ to $\mathfrak{sl}_2^i$, the span of v_+^λ is isomorphic to $L_{\alpha_i^*(\lambda)\cdot\phi_1}$, and in particular it is finite-dimensional. Next, we show that L_λ is a sum of finitely many $\mathfrak{sl}_2^i$-summands. To see this, consider the sum M of all $\mathfrak{sl}_2^i$-submodules of L_λ. But then if we denote a summand by N, we note that $\mathfrak{g} \otimes N$ is a finite-dimensional representation of $\mathfrak{sl}_2$, so the natural morphism

$$\mathfrak{g} \otimes N \to L_\lambda$$

lands inside M. But then $M = L_\lambda$ because M is a nonzero submodule.

Next, recall that for $M \in \mathrm{Rep}(\mathfrak{sl}_2)$, then the sets of weights are invariant under reflection across the origin, so we have isomorphisms

$$f^{\alpha(\lambda)}\colon M^\lambda \leftrightarrows M^{s(\lambda)}\colon e^{\alpha(\lambda)}.$$

Because L_λ is a sum of finite-dimensional representations of $\mathfrak{sl}_2$, for any $v \in L_\lambda^\mu$, consider the finite-dimensional $\mathfrak{sl}_2$-module containing it. If we add them up, we know that L_λ^μ is some finite-dimensional $\mathfrak{sl}_2$-module N^μ. But then the s_i generate W, and thus for all $w \in W$, $L_\lambda^\mu \cong L_\lambda^{w(\mu)}$.

Finally, for any orbit of W, there exists exactly one representative in the dominant weight lattice, and because there are only finitely many dominant integral weights less than λ, there must only be finitely many orbits, so L_λ is finite-dimensional. □

2.6 Central actions

Here, we will consider the action of $Z\mathfrak{g}$ on a module M. Suppose that M is a highest weight module for weight λ. Then we note that

$$h(z \cdot v_+^\lambda) = zhv_+^\lambda = \lambda(h)zv_+^\lambda,$$

and therefore $zv_+^\lambda = \vartheta_\lambda(z)v_+^\lambda$. Therefore z acts by $\mathcal{V}_\lambda(z)$ on any highest weight module of weight λ, and we call the function $\vartheta_\lambda\colon Z\mathfrak{g} \to \mathbb{C}$ a *central character*. In general, all algebra morphisms $Z\mathfrak{g} \to \mathbb{C}$ arise in this way. Then we have the decomposition

$$z \in \mathcal{U}\mathfrak{g} = \mathcal{U}\mathfrak{n}^- \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}^+,$$

and write $\pi_\mathfrak{h}\colon \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{h}$ for the morphism killing $\mathfrak{n}^\pm$. Then $\vartheta_\lambda(z) = \lambda(\pi_\mathfrak{h}(z))$, so $\pi_\mathfrak{h}\colon \mathbb{Z}\mathfrak{g} \to \mathcal{U}\mathfrak{h}$ is an algebra homomorphism, and we will call this $\varphi_{HC} =: \varpi$, the Harish-Chandra morphism. In particular, we obtain a morphism

$$\mathbb{A}^{\dim \mathfrak{h}} \to \mathrm{Spec}\, Z\mathfrak{g}.$$

Also, we will consider $\varpi \circ w\cdot$, where $w \circ \lambda = w(\lambda + \rho) - \rho$, where

$$\rho = \frac{1}{2} \sum_{\Phi^+} \alpha.$$

If we can identify the two morphisms on a Zariski-dense subset, they must agree in general. First, note that $\varpi(\lambda) = \vartheta_\lambda$, and now it suffices to show that $\vartheta_\lambda = \vartheta_{w\circ\lambda}$ for $\lambda \in \Lambda$.

To prove this, if there exists $\sigma \in \Sigma$ such that $\sigma^*(\lambda) \in \mathbb{N}$, then $M_{s_\sigma \circ \lambda} \subset M_\lambda$, and thus $\theta_\lambda = \theta_{s_\sigma\circ\lambda}$. In addition, if $\sigma^*(\lambda) = -1$, we have $s_\sigma \circ \lambda = \lambda$, and if $\sigma^*\lambda \leqslant -2$, we can reverse the roles of $\lambda, \sigma \circ \lambda$ because

$$\sigma^*(s_\sigma \circ \lambda) = \sigma^*\lambda - 2\sigma^*\lambda - 2 \geqslant 0.$$

2.7 More on Harish-Chandra

Consider the twisted Harish-Chandra morphism

$$Z\mathfrak{g} \xrightarrow{\varphi_{HC}} S\mathfrak{h} \xrightarrow{\lambda \mapsto \lambda - \rho} S\mathfrak{h}.$$

This gives us a morphism ψ_{HC}. In particular, we have

$$\vartheta_\lambda(z) = (\lambda + \rho)\psi_{HC}(z) = \lambda(\varphi_{HC}(z)).$$

Theorem 2.7.1. *The image of* ψ_{HC} *is contained in* $(S\mathfrak{h})^W$.

To see this, note that

$$\vartheta_{w\circ\lambda} = \psi_{\mathrm{HC}}(w\circ\lambda) + \rho = \psi_{\mathrm{HC}}(w(\lambda+\rho)) = \psi_{\mathrm{HC}}(\lambda+\rho).$$

Theorem 2.7.2.

1. ψ_{HC} *is an isomorphism* $Z\mathfrak{g} \to S\mathfrak{h}^W$;
2. *If* λ, μ *are linked, then* $\vartheta_\lambda = \vartheta_\mu$;
3. *Every element of* $\mathrm{Hom}_{\mathsf{Alg}}(Z\mathfrak{g}, \mathbb{C})$ *arises in this way.*

There is a simple way to see the last two parts of the theorem if we assume some algebraic geometry.

For a central character $\vartheta\colon Z\mathfrak{g} \to \mathbb{C}$, consider the module

$$M^\vartheta := \ker^\infty(\ker(\vartheta)) = \{v \in M \mid (z-\vartheta(z))^n v = 0 \text{ for all } z\},$$

where n depends on z. We have a decomposition

$$M^\mu = \bigoplus_\vartheta M^\mu \cap M^\vartheta,$$

which gives us

$$M = \bigoplus M^\vartheta.$$

Now we may define subcategories of $\mathcal{O}$ given by

$$\mathcal{O}^\vartheta := \left\{M \mid M = M^\vartheta\right\}.$$

Some examples are that all highest weight modules of weight λ are contained in $\mathcal{O}^{\vartheta_\lambda}$.

Proposition 2.7.3. *We have a decomposition*

$$\bigoplus_{\vartheta=\vartheta_\lambda} \mathcal{O}^\vartheta.$$

We will now consider blocks of category $\mathcal{O}$. We say that simple modules S_1, S_2 are in the same block if there is a nontrivial extension of S_2 by S_1. For general M, we know that M has a finite Jordan-Hölder decomposition because $\mathcal{O}$ is Artinian, so M is in some block if all of its Jordan-Hölder quotients are.

Proposition 2.7.4. *If* $\lambda \in \Lambda$, *then* $\mathcal{O}^{\vartheta_\lambda}$ *is a block of* $\mathcal{O}$.

To prove this, if $\mu < \lambda$ are linked, then we have the diagram

$$M_\mu \hookrightarrow N_\lambda \to M_\lambda$$

giving us an exact sequence

$$0 \to L_\mu \hookrightarrow N_\lambda / \operatorname{Im} N_\mu \twoheadrightarrow L_\lambda \to 0.$$

3

Che (Oct 13): Formal characters and applications to finite-dimensional modules

3.1 Weyl character and dimension formulas

Today we will see what Category $\mathcal{O}$ tells us about finite-dimensional modules. We will fix a semisimple Lie algebra $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

Definition 3.1.1. Given $M \in \mathcal{O}$, define the function $\operatorname{ch} M \colon \mathfrak{h}^* \to \mathbb{Z}^{\geqslant 0}$, where $\lambda \mapsto \dim M_\lambda$.

Also let e_λ be the characteristic function of λ. Now given $f, g \colon \mathfrak{h}^* \to \mathbb{Z}^{\geqslant 0}$, define the convolution product

$$f * g(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu)g(\nu).$$

For example, $e_\lambda * e_\mu = e_{\lambda+\mu}$. Here, we assume that f, g are supported on a finite union of things of the form $\lambda - \Gamma$, where Γ are the non-negative weights. We will call the set of such functions $\mathcal{X}$.

Proposition 3.1.2.

1. *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence, then* $\operatorname{ch}(M) = \operatorname{ch}(M') + \operatorname{ch}(M'')$.
2. *If* $M \in \mathcal{O}$ *and* L *is finite-dimensional, then* $\operatorname{ch}(L \otimes M) = \operatorname{ch}(M) * \operatorname{ch}(L)$.

Proof.

1. Note that $\dim M_\mu = \dim M'_\mu + \dim M''_\mu$ for any such exact sequence.
2. Note that $\dim (L \otimes M)_\lambda = \sum_{\mu+\nu=\lambda} \dim L_\mu \dim M_\nu$. □

Last time we considered central characters for a highest weight module of weight λ and highest weight vector v^+

$$\chi_\lambda \colon Z(\mathfrak{g}) \to \mathbb{C} \qquad z \mapsto \frac{z \cdot v^+}{v^+} = \lambda(\operatorname{pr}(z)).$$

If $L(\mu)$ is a subquotient of the Verma module $M(\lambda)$, then, then $\chi_\mu = \chi_\lambda$. Equivalently, μ and λ are linked by some element w of the Weyl group. Because $\mathcal{O}$ is Artinian, for all $M \in \mathcal{O}$ we have a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

where all M_{i+1}/M_i is simple, so they must be isomorphic to some $L(\lambda_i)$. But then we have

$$\begin{aligned}\operatorname{ch} M &= \sum_i \operatorname{ch} L(\lambda_i) \\ &= \sum_{w \in W} a(\lambda, w) L(w \circ \lambda).\end{aligned}$$

Our goal is now to compute the $a(\lambda, w)$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be weights linked to lambda arranged such that if $\lambda_i \geqslant \lambda_j$, then $i \geqslant j$. Then we should have some identity of the form

$$\begin{pmatrix} \operatorname{ch} M(\lambda_1) \\ \vdots \\ \operatorname{ch} M(\lambda_n) \end{pmatrix} = \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \operatorname{ch} L(\lambda_1) \\ \vdots \\ \operatorname{ch}(\lambda_n) \end{pmatrix}.$$

This implies that $\operatorname{ch} L(\lambda) = \sum_{w \in W} b(\lambda, w) \operatorname{ch} M(w \circ \lambda)$. Note that $b(\lambda, 1) = 1$.

Definition 3.1.3. Define the *Kostant function*

$$p \colon \mathfrak{h}^* \to \mathbb{Z}^{\geqslant 0} \qquad \nu \mapsto \#\Big\{ (c_\alpha)_{\alpha > 0} \in \mathbb{Z}^{\geqslant 0} \mid \sum c_\alpha \alpha = \nu \Big\}.$$

Proposition 3.1.4. $p = \operatorname{ch} M(0)$. *More generally,* $e_\lambda * 0 = \operatorname{ch} M(\lambda)$.

Proof. By the PBW theorem, we know that $M(0)$ is spanned by $\mathcal{U}(\mathfrak{n}_-)$. This is apparently equivalent to the definition of p. □

Definition 3.1.5. Define the function

$$q = \prod_{\alpha > 0} (e_{\alpha/2} - e_{-\alpha/2}).$$

Also define

$$f_\lambda = e_0 + e_{-\lambda} + \cdots = \begin{cases} 1 * \alpha & -k\lambda, k \in \mathbb{Z}^{\geqslant 0} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_\alpha * (1 - e_{-\alpha}) = 1$. Also note that

$$q * \prod_{\alpha > 0} f_\alpha = e_\rho \prod_{\alpha > 0} (1 - e_{-\alpha}) \prod_{\alpha > 0} f_\alpha = e_\rho.$$

Next, $p = \prod_{\alpha > 0} f_\alpha$ and if α is a simple root, then $s_\alpha \cdot q = -q$. The reason for this is that $s_\alpha(\alpha) = -\alpha$ but s_α fixes the other positive roots. This implies that $w \cdot q = (-1)^{\ell(w)} q$.

Theorem 3.1.6 (Weyl character formula)**.** *If* $\lambda \in \Lambda^+$, *then*

$$q * \operatorname{ch} L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e_{w \circ \lambda + \rho}.$$

Proof. If we apply $q * -$ to the formula

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} b(\lambda, w) \operatorname{ch} M(w \circ \lambda),$$

we obtain

$$\begin{aligned} q * \operatorname{ch} L(\lambda) &= \sum_{w \in W} q * \operatorname{ch}(w \circ \lambda) \\ &= \sum_{w \in W} b(\lambda, w) e_{w \circ \lambda + \rho}. \end{aligned}$$

Because $\lambda \in \Lambda^+$, we know that $L(\lambda)$ is finite-dimensional and all weight spaces are symmetric. If α is a simple root, then we apply s_α to both sides, and we obtain

$$-q \operatorname{ch} L(\lambda) = \sum_{w \in W} b(\lambda, w) s_\alpha(w \circ \lambda + \rho) = e_{s_\alpha w \circ \lambda + \rho}$$

because $s_\alpha(w \circ \lambda + \rho) = s_\alpha w(\lambda + \rho) = s_\alpha w \circ \lambda + \rho$. Therefore we see that $b(\lambda, s_\alpha w) = -b(\lambda, w)$, so $b(\lambda, w) = (-1)^{\ell(w)}$. $\square$

We would now like to compute $\dim L(\lambda)$ for dominant integral weights λ. We want something like

$$\operatorname{sum}(q) \cdot \dim L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)},$$

except that both sides here vanish, so this is too naïve. For example, if we consider $\mathfrak{sl}_2$, we have

$$(e_1 - e_{-1}) \operatorname{ch} L(\lambda) = e_{\lambda+1} - e_{-\lambda-1}.$$

If we divide, we actually obtain $\operatorname{ch} L(\lambda) = e_\lambda + e_{\lambda-2} + \cdots + e_{-\lambda}$.

In the general case, let $\mu \in \mathfrak{h}^*$ and $t \in \mathbb{R}$. Define $F_{\mu,t} \colon \mathcal{X} \to \mathbb{R}$ by extending $e_\lambda \mapsto e^{t(\lambda,\mu)}$ linearly. Applying $F_{\rho,t}$ to the Weyl charaacter formula, we obtain

$$\begin{aligned} e^{t(\rho,\rho)} \prod_{\alpha>0} (1 - e^{-t(\rho,\alpha)}) F_{\rho,t} \operatorname{ch} L(\lambda) &= \sum_{w \in W} (-1)^{\ell(w)} e^{t(\rho, w(\lambda+\rho))} \\ &= \sum_{w \in W} (-1)^{\ell(w)} e^{t(w^{-1}\rho, \lambda+\rho)} \\ &= \sum_{w \in W} (-1)^{\ell(w)} e^{t(w\rho, \lambda+\rho)} \\ &= F_{\lambda+\rho,t} \sum_{w \in W} (-1)^{\ell(w)} e_{w\rho} \\ &= F_{\lambda+\rho,t} \left(e_\rho \prod_{\alpha>0} (1 - e_{-\alpha}) \right) \\ &= e^{t(\rho,\rho+\lambda)} \prod_{\alpha>0} (1 - e^{-t(\alpha,\lambda+\rho)}). \end{aligned}$$

Note here that $F_{\rho,t}(e_\lambda * e_\mu) = F_{\rho,t}(e_\lambda) \cdot F_{\rho,t}(e_\mu)$. In the $t \to 0$ limit, we have $F_{\rho,t} \operatorname{ch} L(\lambda) \to \dim L(\lambda)$ and $e^{t(\rho,\rho)} \to 1$. Therefore we obtain

$$\begin{aligned} \dim L(\lambda) &= \lim_{t \to 0} \prod_{\alpha>0} \frac{1 - e^{-t(\alpha,\lambda+\rho)}}{1 - e^{-t(\alpha,\lambda)}} \\ &= \frac{\prod_{\alpha>0} (\alpha, \lambda+\rho)}{\prod_{\alpha>0} (\alpha, \lambda)}. \end{aligned}$$

This is called the *Weyl dimension formula*.

3.2 Maximal submodules of Verma modules

Theorem 3.2.1. *Let $\lambda \in \Lambda^+$ and $\alpha_1, \dots, \alpha_k$ be simple roots of $\mathfrak{g}$. Then*

$$\sum M(s_{\alpha_i} \circ \lambda)$$

is the maximal submodule of $M(\lambda)$.

Remark 3.2.2. Last time we saw that $M(s_{\alpha_i} \circ \lambda) \subset M(\lambda)$.

Lemma 3.2.3. *Let $a, b \in \mathcal{U}\mathfrak{g}$. Then*

$$[a^k, b] = k[a,b]a^{k-1} + \binom{k}{2}[a[a,b]]a^{k-2} + \cdots + [a, \cdots, [a,[a,b]]].$$

This is proved by induction, so like a certain Fields medalist, we omit the proof.

If x_α, x_β correspond to roots α, β, then note that eventually we will have $[x_\alpha, \cdots [x_\alpha, [x_\alpha, x_\beta]]] = 0$. In fact, four times is enough.

Lemma 3.2.4. *Let α be a simple root. Then for any $v \in M(\lambda)$, there exists $N \gg 0$ such that $y_\alpha^N \cdot v = 0$ in $M(\lambda)/\sum M(s_{\alpha_i} \circ \lambda)$.*

Proof. We proceed by induction. Suppose that $v = y_{i_1} y_{i_2} \cdots y_{i_t} v^+$. When $t = 0$, then $y_\alpha^{(\alpha,\lambda)+1} v^+ = 0$. For $t > 0$, we have

$$y_\alpha^N y_{i_1} \cdots y_{i_t} v^+ = y_{i_1} y_\alpha^N y_{i_2} \cdots y_{i_t} v^+ + [y_\alpha^N, y_{i_1}] y_{i_2} \cdots y_{i_t} v^+.$$

The first term on the right hand side vanishes by the inductive hypothesis, and the second term becomes $(-) \cdot y_\alpha^{N-3} y_{i_2} \cdots y_{i_+} = 0$ by the inductive hypothesis. $\square$

Proof of Theorem. By the discussion last time and the second lemma, we know that $M(\lambda)/\sum M(s_{\alpha_i} \circ \lambda)$ is finite-dimensional. This implies that $M(\lambda)/\sum M(s_{\lambda_i} \circ \lambda) = L(\lambda) \oplus M'$, but we know that the left hand side is a highest weight module, so $M' = 0$. $\square$

Remark 3.2.5. We have a resolution

$$\cdots \to \bigoplus_{\ell(w)=1} M(w \circ \lambda) \to M(\lambda) \to L(\lambda) \to 0.$$

This is called the *BGG resolution*.[1]

[1]This does not imply that the terms are projective.

4

Kevin (Oct 20): Duality and projectives in category $\mathcal{O}$

4.1 Duality

Recall from the finite-dimensional story that $\mathfrak{g}$-representation M have duals $M^\vee$ with $\mathfrak{g}$-action

$$(xf)(v) = -f(xv)$$

for $x \in \mathfrak{g}, f \in M^*, v \in M$. This is not well-behaved for infinite-dimensional representations (for example, $M^{**} \not\cong M$), so in this case we would like to construct a better-behaved duality functor.

Note that every semisimple Lie algebra $\mathfrak{g}$ has a transpose $\tau\colon \mathfrak{g} \to \mathfrak{g}$ (if $\mathfrak{g}$ is a matrix Lie algebra, this is literally the transpose) which is an anti-automorphism. Here, we have

$$\tau(x_\alpha) = y_\alpha, \qquad \tau(y_\alpha) = x_\alpha, \qquad , \tau(h_\alpha) = h_\alpha.$$

This allows us to define[1]

Definition 4.1.1. Let $M = \bigoplus_\lambda M_\lambda \in \mathcal{O}$. Then the *dual* of M is defined by

$$M^\vee = \bigoplus_\lambda M_\lambda^\vee \qquad (xf)(v) = f(\tau(x)v).$$

Proposition 4.1.2. $M^\vee \in \mathcal{O}$.

Proof. To prove finite generation, note that $M^\vee$ has finite length (here, $L(\lambda)^\vee = L(\lambda)$ because duality preserves formal characters and exchanges quotients and submodules). Clearly the weight spaces are finite-dimensional by assumption, and the weights lie in some union $\bigcup \lambda - \Lambda$ because formal characters are preserved, so we have local $\mathfrak{n}$-finiteness. $\square$

Here are some more facts about duality.

- Duality is a contravariant functor. This is obvious because everything is defined on the level of weight spaces.
- There is a natural isomorphism $M^{\vee\vee} \cong M$. This is clear because we are taking double duals of finite-dimensional things and adding them up, so in particular duality is an anti-equivalence of categories.

[1]Kevin is unsure how he is doing on time here.

- We have $L(\lambda)^\vee \cong L(\lambda)$. On the other hand, duality for $M(\lambda)$ is complicated.
- τ fixes $Z(\mathfrak{g})$ by an exercise in Humphreys.[2] In particular, this means that $(M^\chi)^\vee = (M^\vee)^\chi$.

4.2 Projectives

Recall that P is projective if $\mathrm{Hom}(P, -)$ is right exact.[3] Our goal is to prove that $\mathcal{O}$ has enough projectives (which will mean that we can do homological algebra). The first thing we will do is introduce dominance and antidominance.

Recall that for $\lambda \in \Lambda$, $W\lambda$ contains one dominant weight and one antidominant weight. This gives us two(!) good choices for representatives of $W\lambda$. Unfortunately, we care about nonintegral weights,[4] and we cannot choose representatives of $W\lambda$ for general $\lambda \in \mathfrak{h}^*$.

Remark 4.2.1. From now on we will use the $w \circ -$ action (because this is all we care about), and therefore our new notion of (anti)dominance will not restrict to the old notion of dominance.[5]

Definition 4.2.2. A weight $\lambda \in \mathfrak{h}^*$ is *dominant* if $\left\langle \lambda + \rho, \alpha^\vee \right\rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Phi^+$. A weight $\lambda \in \mathfrak{h}^*$ is *antidominant* if $\left\langle \lambda + \rho, \alpha^\vee \right\rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi^+$.

Note that this is **not** the same as the undotted definition. For example, $-\rho$ is dominant. Also, the set $W \circ \lambda$ can have multiple dominant and/or antidominant weights.

Definition 4.2.3. We define the subgroup

$$W_{[\lambda]} \coloneqq \{w \in W \mid w \circ \lambda - \lambda \in \Lambda_r\},$$

where Λ_r is the root lattice. We also define

$$\Phi_{[\lambda]} \coloneqq \left\{\alpha \in \Phi \mid \left\langle \lambda, \alpha^\vee \right\rangle \in \mathbb{Z}\right\}.$$

In fact, $W_{[\lambda]}$ is the Weyl group of $\Phi_{[\lambda]}$. We may similarly define $\Delta_{[\lambda]}$.

Proposition 4.2.4. *The following are equivalent:*

1. λ *is dominant.*
2. $\left\langle \lambda + \rho, \alpha^\vee \right\rangle \geqslant 0$ *for all* $\alpha \in \Delta_{[\lambda]}$.
3. $\lambda \geqslant s_\alpha \circ \lambda$ *for all* $\alpha \in \Delta_{[\lambda]}$.
4. $\lambda \geqslant w \circ \lambda$ *for all* $w \in W_{[\lambda]}$.

Proof. Clearly 1 implies 2, and 2 implies 1 because positive roots are sums of simple roots with nonnegative coefficients. To prove that 2 is equivalent to 3, note that

$$s_\alpha \circ \lambda = \lambda - \left\langle \lambda + \rho, \alpha^\vee \right\rangle \alpha.$$

[2] Professor Humphreys, I hope you don't descend upon us from heaven for not having done this exercise. Also please forgive me (the note taker) for never interacting with you when I was an undergrad.

[3] I (note taker) considered not putting this definition in the notes.

[4] Apparently Kevin is speaking for us all here.

[5] Said old definition has now been Stalined. Unfortunately, Humphreys just ignores the ambiguity.

Finally, to see that 3 is equivalent to 4, note that 4 implies 3 automatically. To prove that 3 implies 4, we induct on $\ell(w)$. If $w = w's_\alpha$ with $\ell(w') = \ell(w) - 1$, we see that

$$\lambda - w \circ \lambda = (\lambda - w' \circ \lambda) + w' \circ (\lambda - s_\alpha \circ \lambda).$$

It is clear that $\lambda - w' \circ \lambda \geqslant 0$ while $\lambda - s_\alpha \circ \lambda$ is a nonnegative multiple of α. Because of the length condition, we see that $w' \circ \alpha$ is positive. □

Corollary 4.2.5. *The orbit $W_{[\lambda]} \circ \lambda$ has a unique (anti)dominant weight.*

Proof. This is because 1 is equivalent to 4 in the proposition. □

Theorem 4.2.6.

1. *If λ is dominant, then $M(\lambda)$ is projective.*

2. *If $P \in \mathcal{O}$ is projective and $L \in \mathcal{O}$ is finite-dimensional, then $P \otimes L$ is projective.*

3. *$\mathcal{O}$ has enough projectives.*

Proof.

1. Consider $M \twoheadrightarrow N$ and suppose $\nu \in N$ is a maximal weight vector with weight λ (coming from a map $M(\lambda) \to N$). Assume that $M = M^\chi, N = N^\chi$. Our goal is to lift ν to a maximal weight vector in M, but because $M \to N$ is surjective, we can lift ν to $\nu' \in M_\lambda$. If ν' is maximal, then we are done, so suppose that ν' is not maximal.

   In this case, there exists $x \in \mathcal{U}\mathfrak{n}$ such that $x\nu'$ is a maximal vector with weight greater than λ. However, this weight must be linked to λ, so by dominance of λ, it cannot exist.

2. Here, we use the tensor-Hom adjunction

   $$\mathrm{Hom}_{\mathcal{O}}(P \otimes L, M) \cong \mathrm{Hom}_{\mathcal{O}}(P, L^* \otimes M).$$

   Because $L^* \otimes -$ is exact and $\mathrm{Hom}_{\mathcal{O}}(P, -)$ is exact, the functor $\mathrm{Hom}_{\mathcal{O}}(P \otimes L, -)$ is exact and thus $P \otimes L$ is projective.[6]

3. The first thing we want to do is to find projectives mapping onto $L(\lambda)$. For large n, $\lambda + n\rho$ is dominant. This implies that $M(\lambda + n\rho)$ is projective, but then $M(\lambda + r\rho) \otimes L(n\rho)$ is projective.

   In fact, there exists a surjection $M(\lambda + n\rho) \twoheadrightarrow L(n\rho)$. To see this, if M is a $\mathcal{U}\mathfrak{g}$-module and L is a $\mathcal{U}\mathfrak{b}$-module, then

   $$(\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} L) \otimes M \cong \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} (L \otimes M).$$

   This is known as the tensor identity and is apparently not obvious unless you have the arrogance level of a certain Chinese mathematician. Because $M(\lambda + n\rho) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_{\lambda+n\rho}$, we obtain

   $$M(\lambda + n\rho) \otimes L(n\rho) \cong \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} (\mathbb{C}_{\lambda+n\rho} \otimes L(n\rho)) \twoheadrightarrow \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda \cong M(\lambda).$$

   The surjection comes from the fact that the lowest weight of $L(n\rho)$ is $-n\rho$, so we can kill all of the higher weights.

[6] Apparently Stalinization is an invertible operation, although to be fair it is unclear what the history of the USSR says about this.

The rest of the proof is simply homological algebra. For a general $M \in \mathcal{O}$, because $\mathcal{O}$ is Artinian, we can induct on the length of M. First consider a short exact sequence

$$0 \to L(\lambda) \to M \to N \to 0.$$

By assumption, there exists a surjection $P \twoheadrightarrow N$, and this morphism lifts to $P \to M$. If P does not surject onto M, then $\mathrm{Im}(P \to M)$ cannot intersect $L(\lambda)$ (otherwise it would contain all of $L(\lambda)$ and thus surject onto M). This implies that $\mathrm{Im}(P \to M) \cong N$, which splits the exact sequence. □

By standard homological algebra, because $\mathcal{O}$ is Artinian and has enough projectives, then $\mathcal{O}$ has projective covers (i.e. unique minimal projectives surjecting onto M). If we define $P(\lambda)$ to be the projective cover of $L(\lambda)$, the $P(\lambda)$ are precisely the indecomposable projectives. Therefore every projective is a direct sum of $P(\lambda)$.

Theorem 4.2.7.

1. $P(\lambda)$ *has a standard filtration, which is a filtration with subquotients that are Verma modules.*

2. *(BGG reciprocity) The multiplicity of* $M(\mu)$ *in the composition series for* $P(\lambda)$ *is given by*

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)].$$