A KLEIMAN CRITERION FOR STACK QUOTIENTS

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1. INTRODUCTION

1.1. **Classical Kleiman criterion.** The Kleiman criterion determines when a divisor on a projective variety X is ample in terms of its intersections with curves in X.

Theorem 1.1 (Kleiman). *Let X be a projective variety over $\mathbb{C}$. Then let $\mathrm{NE}(X)$ be the cone spanned by effective curve classes in $N_1(X)_{\mathbb{R}}$. A divisor D on X is ample if and only if*

$$D \cdot \gamma > 0$$

for all $\gamma \in \overline{\mathrm{NE}(X)} \setminus 0$.

This theorem says that the ample cone is the interior of the the nef cone. If γ is represented by a smooth curve C, then $\mathcal{O}_X(D)$ is ample on C, so its higher cohomology vanishes, and then the Riemann-Roch formula reduces to

$$h^0(\mathcal{O}_C(D)) = \deg_C D - g(C) + 1 > 0,$$

and so D has positive degree on C.

1.2. **Variation of GIT.** Let G be a reductive group acting on X. If we want to construct a "quotient" of X by G as a scheme, we must fix a G-equivariant line bundle and consider the GIT quotient

$$X /\!\!/_L G \coloneqq X^{ss}(L) /\!\!/ G.$$

Clearly if L, L' have the same semistable locus, then the GIT quotient (schemes) are isomorphic, but the converse is not necessarily true.

Example 1.2. Consider the action of $(\mathbb{C}^\times)^2$ on $\mathbb{C}^3$ by

$$t \cdot x = (t_1^2 x_1, t_1 t_2 x_2, t_2 x_3).$$

Now let $\theta_1 = (4, 2)$ and $\theta_2 = (2, 4)$. In the first case, the semistable locus has either $(x, y) \neq (0, 0)$ or $(x, z) \neq (0, 0)$, so we remove the y and z axes and obtain the GIT quotient stack

$$[\mathbb{C}^3 /\!\!/_{\theta_1} (\mathbb{C}^\times)^2] = [\mathbb{P}^1/\mu_2].$$

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In the second case, the semistable locus has either $(x,z) \neq (0,0)$ or $(y,z) \neq (0,0)$, so we remove the x and y axes and obtain the GIT quotient stack

$$[\mathbb{C}^3 /\!\!/_{\theta_2} (\mathbb{C}^\times)^2] = \mathbb{P}(2,1).$$

Both stacks have coarse moduli space $\mathbb{P}^1$, so the GIT quotient varieties $\mathbb{C}^3 /\!\!/_{\theta_i} (\mathbb{C}^\times)^2$ are isomorphic for $i = 1, 2$.

We will now establish some notation. Recall that there is an equivariant first Chern class c_1^G which sits inside the exact sequence

$$0 \to \operatorname{Pic}_0(X) \to \operatorname{Pic}^G(X) \xrightarrow{c_1^G} H^2_G(X, \mathbb{Z})$$

and denote $\mathrm{NS}^G(X) \coloneqq \operatorname{Pic}^G(X)/\operatorname{Pic}_0(X)$. Note that if L and L' determine the same class in $\mathrm{NS}^G(X)$, then they have the same semistable locus, which can be seen using localization.

Denote the ample cone of G-linearized ample line bundles by $\mathrm{Amp}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{Q}}$ and then denote by $C^G(X)$ the cone containing G-linearized ample line bundles with $X^{ss}(L)$ nonempty. We will call these classes G-*ample*. Now, for $L \in C^G(X)$, define

$$C(L) \coloneqq \left\{ L' \in C^G(X) \mid X^{ss}(L) \subset X^{ss}(L') \right\}$$

and its relative interior (due to Ressayre)

$$C^\circ(L) \coloneqq \left\{ L' \in C^G(X) \mid X^{ss}(L) = X^{ss}(L) \right\}.$$

2. Quasimaps

Let $L \in \operatorname{Pic}^G(X)$ be ample.

Definition 2.1. An L-*stable quasimap* from a smooth curve C to $[X/G]$ is a morphism

$$f \colon C \to [X/G]$$

such that $f^{-1}([X^{ss}/L])$ is dense.

Given such a quasimap, for any G-equivariant line bundle N on X, we can consider the degree

$$\deg f^*([N/G]),$$

and the map

$$(N \mapsto \deg f^*([N/G])) \in \operatorname{Hom}(\mathrm{NS}^G(X), \mathbb{Z})$$

is called the *degree* of f. We will call the cone generated by all $\beta \in \operatorname{Hom}(\mathrm{NS}^G(X), \mathbb{Z})$ that can be realized as degrees of L-stable quasimaps

$$\mathrm{NE}(L) \subset \operatorname{Hom}(\mathrm{NS}^G(X), \mathbb{Q}).$$

Proposition 2.2. *If* $f \colon C \to [X/G]$ *is an* L*-stable quasimap, then* $\deg f^*[L/G] \geqslant 0$.

Proof. We can assume that C is connected. Then there exists $c \in C$ mapping to $[X^{ss}(L)/G]$. We can then lift $f(c)$ to $x \in X^{ss}(L)$. But now by definition of semistability, there exists some $m > 0$ and $s \in \Gamma(X, L^m)^G$ such that $s(x) \neq 0$. But now

$$f^*[L^m/G]$$

has a nonzero section $f^*(s)$, and thus it has non-negative degree on C. $\square$

Corollary 2.3. $C(L) \subset NE(L)^\vee$.

3. Main result

3.1. **For projective** X. Now we want to relate the weights $\mu^L(x, \lambda)$ to the degrees of quasimaps. Fix $\lambda\colon \mathbb{C}^\times \to G$ and a semistable point $x \in X^{ss}(L)$. Because X is proper, there exists a morphism $\overline{\lambda}\colon \mathbb{C} \to X$ extending $t \mapsto \lambda(t)x$. Now define the map

$$\widetilde{\phi}_{\lambda,x}\colon \mathbb{C}^2 \setminus \{0\} \to X \qquad (s,t) \mapsto \overline{\lambda}(t).$$

This is clearly $\mathbb{C}^\times$-equivariant (with respect to the scaling action on $\mathbb{C}^2$ and the action of λ on X), and so we obtain a stable quasimap

$$\phi_{\lambda,x}\colon \mathbb{P}^1 \to [X/G].$$

Lemma 3.1. *For any* $N \in \mathrm{Pic}^G(X)$,

$$\deg \phi^*_{\lambda,x}(N) = \mu^N(x, \lambda).$$

Proof. Note that $\phi_{\lambda,x}$ factors through the projection onto the second factor $\pi_2\colon \mathbb{C}^2 \setminus 0 \to \mathbb{C}$, and then we simply need to compute the weight of the action of $\mathbb{C}^\times$ on N at the origin of $[\mathbb{C}/\mathbb{C}^\times]$. $\square$

Proposition 3.2. *Suppose* $L, N \in \mathrm{Amp}^G(X)$ *such that* $N \notin C(L)$. *Then there exists an* L*-stable quasimap* $f\colon C \to [X/G]$ *such that* $\deg f^*[N/G] < 0$.

Proof. Consider the morphism $\phi_{\lambda,x}$ for some $x \in X^{ss}(L) \setminus X^{ss}(N)$. $\square$

Now we may state the main result, which is proven using the preceeding discussion.

Theorem 3.3. *Let* L *be a* G*-ample line bundle on* X*. Then*

$$C^\circ(L) = \mathrm{relint}(NE(L)^\vee) \cap \mathrm{Amp}^G(X).$$

3.2. **Quotients of vector spaces.** We can extend these results to quotients of vector spaces by embedding $V \subset \mathbb{P}(V \oplus \mathbb{C})$, where the extra copy of $\mathbb{C}$ carries the trivial representation of G. Then for a character $\theta \in \mathrm{Hom}(G, \mathbb{C}^\times)$, denote its GIT equivalence class by $A(\theta)$. The main result in the context of quotients of vector spaces is stated below.

Proposition 3.4. *Suppose that* $(\mathrm{Sym}^\bullet V^\vee)^G = \mathbb{C}$. *Then for any* $\theta \in \mathrm{Hom}(G, \mathbb{C}^\times)$,

$$A(\theta) = \mathrm{relint}(NE(\theta)^\vee).$$