# GOOD MODULI SPACES, POSITIVITY AND RATIONALITY

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ABSTRACT. This course will be an introduction to the theory of good moduli spaces (in the sense of Alper). We will generalize filtrations from a vector space to an algebraic variety, define theta-reductivity, S-completeness, and unpuncturedness of inertia. Examples will include good moduli spaces for Deligne-Mumford stable curves, and K-semistable Fanos, with the focus on the cases where there is a positive tautological line bundle and the good moduli space is particularly simple, i.e., rational.

## CONTENTS

## 1. MODULI OF RATIONAL CURVES

We will begin by interpolating between $\overline{M}_{0,n}$ (which is a scheme) and the stack $(\mathbb{P}^1)^n /\!\!/ \mathrm{PGL}(2)$ (which has a good moduli space by definition) via Hassett spaces $\overline{M}_{0,A}$. There is a single smooth rational curve, which is $\mathbb{P}^1 = \{x = 0\} \subseteq \mathbb{P}^2_{x,y,z}$.

Viewing the curve inside $\mathbb{P}^2$, we may (see Figure 1) consider the family

$$\{xy - t^2 z^2 = 0\} \;=\!=\!=\; \mathfrak{X} \longrightarrow \mathbb{P}^2$$
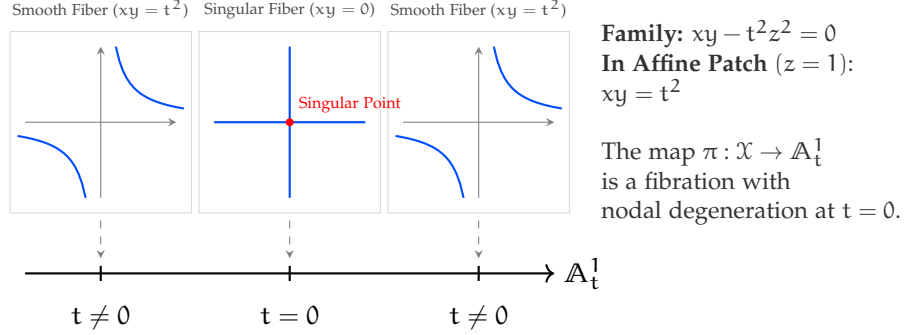$$\downarrow$$
$$\mathbb{A}^1_t$$

This morphism is a (flat) isotrivial degeneration, and will be the first example of a test configuration. What this means is that there is an action of $\mathbb{G}_m$ on $\mathfrak{X}$ by the formula

$$\lambda([x, y, z], t) = ([\lambda x, \lambda y, z], \lambda t)$$

such that $\mathfrak{X} \to \mathbb{A}^1$ is equivariant. More generally, any tree of rational curves joined by nodes is also a rational curve (as in it has genus 0).

*Remark* 1.1. A **reduced** rational curve is a "tree" of $\mathbb{P}^1$'s meeting in $m$-fold rational points, where each singularity is locally analytically isomorphic to the union of the coordinate axes in $\mathbb{A}^m$.

In fact, the definition of a rational curve tells about what kinds of singularities it has, and then there is a classification of curve singularities. Another viewpoint is that if we degenerate a rational normal curve in $\mathbb{P}^m$ to a cone over a hyperplace

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Smooth Fiber $(xy = t^2)$   Singular Fiber $(xy = 0)$   Smooth Fiber $(xy = t^2)$

Singular Point

$t \neq 0$            $t = 0$            $t \neq 0$

**Family:** $xy - t^2 z^2 = 0$
**In Affine Patch** $(z = 1)$:
$xy = t^2$

The map $\pi : \mathcal{X} \to \mathbb{A}^1_t$
is a fibration with
nodal degeneration at $t = 0$.

$\mathbb{A}^1_t$

FIGURE 1. The family $\{xy - t^2 z^2 = 0\}$ over $\mathbb{A}^1$

section, we get a union of $m$ lines meeting at a point in exactly the way described above.

We will now consider the functor $\widetilde{\mathcal{U}}_{0,n}$ of $n$-pointed smooth rational curves. In particular, a family $\mathcal{X} \to B$ must have $n$ sections $\sigma_1, \ldots, \sigma_n \colon B \to \mathcal{X}$, which are not required to be disjoint. We may now consider the larger functor $\widetilde{\mathcal{M}}_{0,n}$ of $n$-pointed reduced rational curves with sections $\sigma_i$ avoiding the singularities of the fibers. Unfortunately, this is extremely non-separated, so we will consider stability conditions.

Define the functor $\overline{M}_{0,n}$ of $n$-pointed rational curves with at worst nodal singularities such that all sections $\sigma_i(B)$ are disjoint and tha4t the relative dualizing sheaf

$$\omega_{\mathcal{X}/B}(\sigma_1 + \cdots + \sigma_n)$$

is relatively ample over $B$. One reason we consider this (and that it makes sense) is that the $m$-fold rational point is Gorenstein if and only if $m \leqslant 2$. This stability condition was introduced by Deligne and Mumford. In fact, the condition on the relative dualizing sheaf is equivalent to enforcing that the automorphism group of each fiber is finite, but is easier to modify for obvious reasons. From either perspective, it is clear that we must have a tree of $\mathbb{P}^1$'s with at least three special points on each component. One example can be seen in Figure 2.
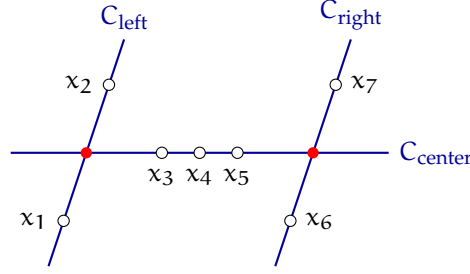
One way to turn a family with an $m$-fold rational point into a nodal one is to blow up the singular point. The remaining conditions are non-disjoint sections, which can be separated by finitely many blowups. Finally, the unstable components can be contracted by the basic theory of surfaces (they have self-intersection either $-1$ or $-2$). In particular, we have sketched a proof that $\overline{M}_{0,n}$ is universally closed. In fact, it is also separated, but we will omit the proof here.

Suppose we have a family $\pi \colon \mathcal{X} \to B$ in $\widetilde{\mathcal{M}}_{0,n}$. Then we have $n + 1$ line bundles on $B$ where $n$ of them are given by

$$\sigma_i^* \omega_{\mathcal{X}/B} = \sigma_i^*(N^\vee_{\sigma_i(B)/X})$$

(this is fine even in the non-Gorenstein case because the sections avoid the singularities) and the last is given by

$$\delta = \pi_*(c_2(\Omega^1_{\mathcal{X}/B})),$$

**Configuration in** $\overline{M}_{0,7}$
Dual Graph: Tree (Genus 0)
Stability: Each component has $\geqslant 3$
special points (nodes + marks).

FIGURE 2. An example of a stable curve of genus 0 with marked points

which counts singularities. For example, if B is a smooth projective curve, then

$$\deg \delta = \sum_m m \cdot (\#\{m\text{-fold singularities}\}).$$

**Theorem 1.2.** *The line bundle*

$$\psi - \delta = \left[\sum_{i=1}^n \psi_i\right] - \delta$$

*is ample on* $\overline{M}_{0,n}$.

*Remark* 1.3. Projectivity of $\overline{M}_{0,n}$ can be proved in more synthetic ways, for example via the explicit blowup construction of Keel.

There is a modification of this, which is the functor $\overline{M}_{0,A}$, where

$$A = (\alpha_1, \ldots, \alpha_n) \in (0,1] \cap \mathbb{Q}.$$

Here, we consider $n$-pointed rational curves which are at worst nodal, but where the sections $\sigma_{i_1}, \ldots, \sigma_{i_k}$ can meet if and only if

$$\alpha_{i_1} + \cdots + \alpha_{i_k} \leqslant 1$$

and the condition on the dualizing sheaf is that $\omega_{\mathcal{X}/B}(\sum_i \alpha_i \sigma_i)$ is ample. Projectivity of this is proven using similar methods to $\overline{M}_{0,n}$.

More specifically, the functor is given by

$$\overline{M}_{0,A}(T) = \left\{ \begin{array}{c} \mathcal{C} \\ \pi \downarrow \sigma_1, \ldots, \sigma_n \\ T \end{array} \middle| \begin{array}{c} \text{fibers at worst nodal} \\ \omega_{\mathcal{C}/T}(\sum \alpha_i \sigma_i) \text{ is } \pi\text{-ample} \\ \text{sections with sum of weights} > 1 \text{ cannot meet} \end{array} \right\}.$$

This functor is actually representable by a scheme because the three conditions imply that there are no automorphisms.

**Example 1.4.** If $A = (1, \ldots, 1)$ or more generally if the sum of any two weights is strictly greater than 1, then $\overline{M}_{0,A} = \overline{M}_{0,n}$.

**Example 1.5.** If $A = (1, \varepsilon, \ldots, \varepsilon)$ such that $1 - (n-1)\varepsilon > 2$ and $\varepsilon \leqslant \frac{1}{n-2}$, then the combinatorics imply that all marked points lie on the same component. If we call the heavy marked point $\infty$, then all of the light markings lie in $\mathbb{A}^1$. By translation invariance, we may assume that

$$x_1 + \cdots + x_{n-1} = 0,$$

and therefore we obtain

$$[x_1 : \cdots : x_{n-1}] \in \mathbb{C}^{n-2} \setminus \{0\}/\mathbb{C}^\times \simeq \mathbb{P}^{n-3}.$$

*Remark* 1.6. There exists a morphism $\overline{M}_{0,n} \to \overline{M}_{0,A}$ given (when the base is a curve) by changing the weights and contracting all of the $(-1)$-curves in the fibers where the total weight of the sections has weight at most 1.

**Example 1.7.** If we set $A = (\varepsilon, \ldots, \varepsilon)$ such that $\varepsilon \approx \frac{2}{n} + \varepsilon'$, then the curve must be a $\mathbb{P}^1$ with $n$ points such that we cannot have $\lceil \frac{2}{n} \rceil$ colliding. More concretely, if $n = 2m + 1$ and $\varepsilon = \frac{1}{m}$, then we have a $\mathbb{P}^1$ with $2m + 1$ marked points such that at most $m$ can collide. In particular, we have

$$\overline{M}_{0,A} = \frac{(\mathbb{P}^1)^{2m+1} \setminus \{(m+1)\text{-fold diagonals}\}}{\mathrm{PGL}(2)}$$

$$= ((\mathbb{P}^1)^{2m+1})^{\mathrm{ss}} /\!\!/ \mathrm{PGL}(2),$$

where the linearization is the tensor product of the $\mathcal{O}(1)$ on each factor. For an example when $m = 7$, see Figure 3.



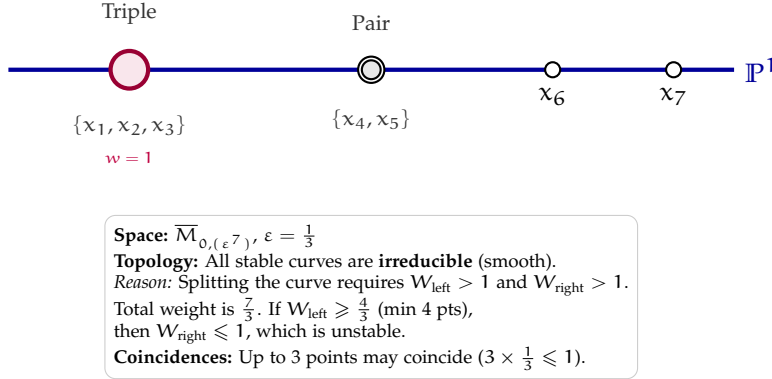**Space:** $\overline{M}_{0,(\varepsilon^7)}, \varepsilon = \frac{1}{3}$
**Topology:** All stable curves are **irreducible** (smooth).
*Reason:* Splitting the curve requires $W_{\mathrm{left}} > 1$ and $W_{\mathrm{right}} > 1$.
Total weight is $\frac{7}{3}$. If $W_{\mathrm{left}} \geqslant \frac{4}{3}$ (min 4 pts),
then $W_{\mathrm{right}} \leqslant 1$, which is unstable.
**Coincidences:** Up to 3 points may coincide ($3 \times \frac{1}{3} \leqslant 1$).

FIGURE 3. An object in $\overline{M}_{0,\left(\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}\right)}$

Now we will define a new moduli space

$$\overline{M}^*_{0,A}(T) = \left\{ \begin{array}{c} \mathcal{C} \\ \pi \left( \downarrow \uparrow \right) \sigma_1, \ldots, \sigma_n \\ T \end{array} \middle| \begin{array}{c} \text{fibers at worst nodal} \\ \omega_{\mathcal{C}/T}(\sum \alpha_i \sigma_i) \text{ is } \pi\text{-nef} \\ \text{sections with sum of weights} > 1 \text{ cannot meet} \end{array} \right\}$$

by relaxing the ampleness condition. This can happen if we have sections whose weights sum to exactly 1.

**Example 1.8.** If $n = 2m$ and $A = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$, then we have

$$\overline{M}_{0,A}^* = (\mathbb{P}^1)^{2m} /\!\!/ \operatorname{PGL}(2).$$

This is not a scheme and does not even have a coarse moduli space because the curve where the first $m$ markings collide and the last $m$ markings collide has an entire $\mathbb{G}_m$ of automorphisms. For an example when $n = 4$, see Figure 4.



**Configuration:** 4 points, weights $w_i = 1/2$.
**Geometry:** Smooth curve ($\mathbb{P}^1$) with 2 effective marked locations.
**Automorphisms:** Fixing 2 points on $\mathbb{P}^1$ leaves 1 degree of freedom (scaling), hence $\operatorname{Aut} \cong \mathbb{G}_m$.
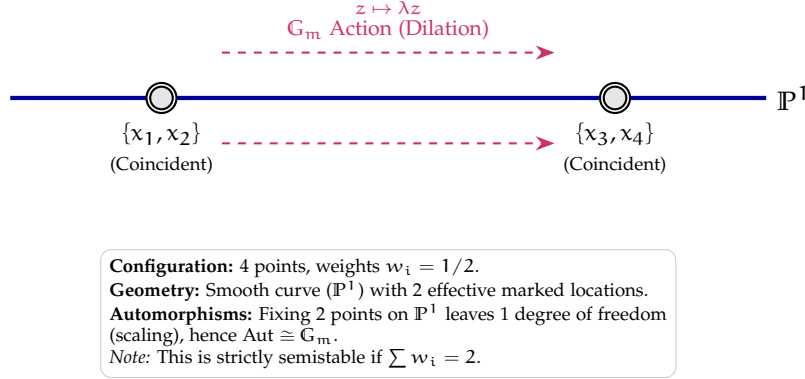*Note:* This is strictly semistable if $\sum w_i = 2$.

FIGURE 4. A strictly semistable object in $\overline{M}_{0,\frac{1}{2}}^*$.

The Hassett moduli spaces have a wall-and chamber-structure, where for example, if $A = (\alpha, \ldots, \alpha)$ is $S_n$-symmetric, then $\overline{M}_{0,A}$ is constant for $\alpha \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$, but on the walls we only have $\overline{M}_{0,A}^*$. The main observation is that if we write $\overline{M}_{0,\alpha}$ for $\overline{M}_{0,(\alpha,\ldots,\alpha)}$, then there are open immersions

$$\overline{M}_{0,\frac{1}{k}-\varepsilon} \subset \overline{M}_{0,\frac{1}{k}}^* \supset \overline{M}_{0,\frac{1}{k}+\varepsilon}.$$

If we consider the complement $\overline{M}_{0,\frac{1}{k}}^* \setminus \overline{M}_{0,\frac{1}{k}-\varepsilon}$, then it consists of curves which contain an irreducible component which is a tree in the dual graph and contains $k$ marked points, so this part of the moduli space has a factor which describes the rest of the curve and one which is $[\mathbb{A}^{k-2}/\mathbb{G}_m]$.

This actually allows us to construct a test configuration, where if we have $\mathbb{P}^1 \times \mathbb{A}^1$ with a bunch of sections, we can blow up a point on the central fiber and then contract the central $(-1)$-curve with all of the sections.

**Exercise 1.9.** Figure out what $\overline{M}_{0,\frac{1}{k}}^* \setminus \overline{M}_{0,\frac{1}{k}+\varepsilon}$ is.