Enumerative invariants and birational geometry Spring 2024

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Lectures by Various

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Disclaimer

These notes were taken during the lectures using `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. Also, notation may differe between lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

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Seminar Website: https://math.columbia.edu/~plei/s24-birat.html

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Preliminaries

1.1 Givental formalism (Patrick, Feb 01)

1.1.1 Introduction Let X be a smooth projective variety. Then for any $g, n \in \mathbb{Z}_{\geqslant 0}, \beta \in H_2(X, \mathbb{Z})$, there exists a moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ (Givental's notation is $X_{g,n,\beta}$) of *stable maps* $f \colon C \to X$ from genus-g, n-marked prestable curves to X with $f_*[C] = \beta$. It is well-known that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in A_\delta(\overline{\mathcal{M}}_{g,n}(X, \beta)), \qquad \delta = \int_\beta c_1(X) + (\dim X - 3)(1 - g) + 3.$$

In addition, there is a universal curve and sections

$$\mathcal{C} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} \overline{\mathcal{M}}_{g,n}(X, \beta).$$

In this setup, there are tautological classes

$$\psi_i \coloneqq c_1(\sigma_i^* \omega_\pi) \in H^2(\overline{\mathcal{M}}_{g,n}(X, \beta)).$$

This allows us to define individual Gromov-Witten invariants by

$$\left\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \, \phi_i \cdot \psi_i^{a_i}.$$

These invariants satisfy various relations. The first is the *string equation:*

$$\left\langle \tau_0(1) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n+1,\beta} = \sum_{i=1}^n \left\langle \tau_{a_i - 1}(\phi_i) \prod_{j \neq i} \tau_{a_j}(\phi_j) \right\rangle^X_{g,n,\beta}.$$

The next is the *dilaton equation:*

$$\left\langle \tau_1(1) \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n+1,\beta} = (2g - 2 + n) \left\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle^X_{g,n,\beta}.$$

Finally, we have the *divisor equation* when one insertion is a divisor $D \in H^2(X)$:

$$\left\langle \tau_0(D)\tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n+1,\beta} = \left(\int_\beta D\right)\cdot\left\langle \tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n,\beta} + \sum_{i=1}^{n}\left\langle \tau_{a_i-1}(\phi_i\cdot D)\prod_{j\neq i}\tau_{a_j}(\phi_j)\right\rangle^X_{g,n,\beta}.$$

It is often useful to package Gromov-Witten invariants into various generating series.

Definition 1.1.1. The *quantum cohomology* $QH^*(X)$ of X is defined by the formula

$$(a \star_t b, c) \coloneqq \sum_{\beta,n}\frac{Q^\beta}{n!}\langle a, b, c, t, \ldots, t\rangle^X_{0,3+n,\beta}$$

for any $t \in H^*(X)$. This is a commutative and associative product.

The *small quantum cohomology* is obtained by setting $t = 0$ and the ordinary cohomology is obtained by further setting $Q = 0$.

Remark 1.1.2. Convergence of the formula does not hold in general, so quantum cohomology needs to be treated as a formal object.

Definition 1.1.3. Let ϕ_i be a basis of $H^*(X)$ and ϕ^i be the dual basis. Then the J-*function* of X is the cohomology-valued function

$$J_X(t,z) \coloneqq z + t + \sum_i\sum_{n,\beta}\frac{Q^\beta}{n!}\left\langle \frac{\phi_i}{z-\psi}, t, \ldots, t\right\rangle^X_{0,n+1,\beta}\phi^i.$$

Definition 1.1.4. The *genus*-0 *GW potential* of X is the (formal) function

$$\mathcal{F}^X(t(z)) = \sum_{\beta,n}\frac{Q^\beta}{n!}\langle t(\psi), \ldots, t(\psi)\rangle^X_{0,n,\beta}.$$

The associativity of the quantum product is equivalent to the PDE

$$\sum_{e,f}\mathcal{F}^X_{abe}\eta^{ef}\mathcal{F}_{cdf} = \sum_{e,f}\mathcal{F}^X_{ade}\eta^{ef}\mathcal{F}^X_{bcf}$$

for any a, b, c, d, which are known as the *WDVV equations*. Here, we choose coordinates on $H^*(X)$ and set $z = 0$ (only consider primary insertions). In addition, set η_{ef} to be the components of the Poincaré pairing and let η^{ef} be the inverse matrix.

1.1.2 Frobenius manifolds A Frobenius manifold can be thought of as a formalization of the WDVV equations.

Definition 1.1.5. A *Frobenius manifold* is a complex manifold M with a flat symmetric bilinear form $\langle -, -\rangle$ (meaning that the Levi-Civita connection has zero curvature) on TM and a holomorphic system of (commutative, associative) products $\star_t$ on T_tM satisfying:

1. The unit vector field $\mathbf{1}$ is flat: $\nabla\mathbf{1} = 0$;

2. For any t and $a, b, c \in T_tM$, $\langle a \star_t b, c\rangle = \langle a, b \star_t c\rangle$;
3. If $c(u, v, w) := \langle u \star_t v, w\rangle$, then the tensor $(\nabla_z c)(u, v, w)$ is symmetric in $u, v, w, z \in T_tM$.

If there exists a vector field E such that $\nabla\nabla E = 0$ and complex number d such that:

1. $\nabla\nabla E = 0$;
2. $\mathcal{L}_E(u \star v) - \mathcal{L}_E u \star v - u \star \mathcal{L}_E v = u \star v$ for all vector fields u, v;
3. $\mathcal{L}_E\langle u, v\rangle - \langle \mathcal{L}_E u, v\rangle - \langle u, \mathcal{L}_E v\rangle = (2-d)\langle u, v\rangle$ for all vector fields u, v,

then E is called an *Euler vector field* and the Frobenius manifold M is called *conformal*.

Example 1.1.6. Let X be a smooth projective variety. Then we can give $H^*(X)$ the structure of a Frobenius algebra with the Poincaré pairing and the quantum product. Note that the quantum product does not converge in general, so we must treat this as a formal object. The Euler vector field is given by

$$E_X = c_1(X) + \sum_i \left(1 - \frac{\deg \phi_i}{2}\right) t^i \phi_i,$$

where a general element of $H^*(X)$ is given by $t = \sum_i t^i \phi_i$. We will also impose that $\phi_1 = \mathbf{1}$. There is another very important structure, the *quantum connection*, which is given by the formula

$$\nabla_{t^i} := \partial_{t^i} + \frac{1}{z}\phi_i \star_t$$
$$\nabla_{z\frac{d}{dz}} := z\frac{d}{dz} - \frac{1}{z}E_X \star_t + \mu_X.$$

Here, μ_X is the *grading operator*, defined for pure degree classes $\phi \in H^*(X)$ by

$$\mu_X(\phi) = \frac{\deg \phi - \dim X}{2}\phi.$$

Finally, in the direction of the Novikov variables, we have

$$\nabla_{\xi Q \partial_Q} = \xi Q\, \partial_Q + \frac{1}{z}\xi \star_t .$$

Remark 1.1.7. For a general conformal Frobenius manifold $(H, (-,-), \star, E)$, there is still a *deformed flat connection* or *Dubrovin connection* given by

$$\nabla_{t^i} := \frac{\partial}{\partial t^i} + \frac{1}{z}\phi_i \star$$
$$\nabla_{z\frac{d}{dz}} := z\frac{d}{dz} - \frac{1}{z}E \star .$$

Definition 1.1.8. The *quantum* D-*module* of X is the module $H^*(X)[z][\![Q, t]\!]$ with the quantum connection defined above.

Remark 1.1.9. It is important to note that the quantum connection has a fundamental solution matrix $S^X(t, z)$ given by

$$S_X(t, z)\phi = \phi + \sum_i \sum_{n,\beta} \frac{Q^\beta}{n!}\phi^i \left\langle \frac{\phi_i}{z - \psi}, \phi, t, \dots, t \right\rangle^X_{0,n+2,\beta}.$$

It satisfies the important equation

$$S_X^*(t, -z)S(t, z) = 1.$$

Using this formalism, the J-function is given by $S_X^*(t, z)\mathbf{1} = z^{-1}J_X(t, z)$.

1.1.3 Givental formalism The Givental formalism is a geometric way to package enumerative (CohFT) invariants cleanly. We begin by defining the symplectic space

$$\mathcal{H} := H^*(X, \Lambda)(\!(z^{-1})\!)$$

with the symplectic form

$$\Omega(f, g) := \mathrm{Res}_{z=0}\, f(-z)g(z).$$

This has a polarization by Lagrangian subspaces

$$\mathcal{H}_+ := H^*(X, \Lambda)[z], \qquad \mathcal{H}_- := z^{-1}H^*(X, \Lambda)[\![z^{-1}]\!]$$

giving $\mathcal{H} \cong T^*\mathcal{H}_+$ as symplectic vector spaces. Choose Darboux coordinates $\underline{p}, \underline{q}$ on $\mathcal{H}$. For example, there is a choice in Coates's thesis which gives a general element of $\mathcal{H}$ as

$$\sum_{k\geqslant 0} \sum_{i} q_k^i \phi_i z^k + \sum_{\ell \geqslant 0} \sum_{j} p_\ell^j \phi^j (-z)^{-\ell-1}.$$

Taking the *dilaton shift*

$$q(z) = t(z) - z = -z + t_0 + t_1 z + t_2 z^2 + \cdots,$$

we can now think of $\mathcal{F}^X$ has a formal function on $\mathcal{H}_+$ near $q = -z$. This convention is called the *dilaton shift*.

Before we continue, we need to recast the string and dilaton equations in terms of $\mathcal{F}^X$. Write $t_x = \sum t_k^i \phi_i$. Then the string equation becomes

$$\partial_0^1 \mathcal{F}(t) = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty} \sum_{j} t_{n+1}^j\, \partial_n^j \mathcal{F}(t)$$

and the dilaton equation becomes

$$\partial_1^1 \mathcal{F}(t) = \sum_{n=0}^{\infty} t_n^j\, \partial_n^j \mathcal{F}(t) - 2\mathcal{F}(t).$$

There are also an infinite series of topological recursion relations

$$\partial_{k+1}^i\, \partial_\ell^j\, \partial_m^k \mathcal{F}(t) = \sum_{a,b} \partial_k^i\, \partial_0^a \mathcal{F}(t) \eta^{ab}\, \partial_0^b\, \partial_\ell^j\, \partial_m^k \mathcal{F}(t).$$

We can make sense of these three relations for any (formal) function $\mathcal{F}$ on $\mathcal{H}_+$.

Now let

$$\mathcal{L} = \left\{ (\underline{p}, \underline{q}) \in \mathcal{H} \mid \underline{p} = d_{\underline{q}} \mathcal{F} \right\}$$

be the graph of $d\mathcal{F}$. This is a formal germ at $q = -z$ of a Lagrangian section of the cotangent bundle $T^*\mathcal{H}_+$ and is therefore a formal germ of a Lagrangian submanifold in $\mathcal{H}$.

Theorem 1.1.10. *The function $\mathcal{F}$ satisfies the string equation, dilaton equation, and topological recursion relations if and only if $\mathcal{L}$ is a Lagrangian cone with vertex at the origin $q = 0$ such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along zL.*

Because of this theorem, $\mathcal{L}$ is known as the *Lagrangian cone*. It can be recovered from the J-function by the following procedure. First consider $\mathcal{L} \cap (-z + z\mathcal{H}_-)$. Via the projection to $-z + H$ along $\mathcal{H}_-$, this can be considered as the graph of the J-function. Next, we consider the derivatives $\frac{\partial J}{\partial t^i}$, which form a basis of $L \cap z\mathcal{H}_-$, which is a complement to zL in L. Then we know that

$$z\frac{\partial J}{\partial t^i} \in zL \subset \mathcal{L},$$

so

$$z\frac{\partial^2 J}{\partial t^i\, \partial t^j} \in L \cap z\mathcal{H}_-.$$

Writing these in terms of the first derivatives $\frac{\partial J}{\partial t^i}$ and using the fact that J is a solution of the quantum connection, so we recover the Frobenius structure of quantum cohomology.

We will now express some classical results in this formalism. Let X be a toric variety with toric divisors $D_1, \dots, D_N$ such that $D_1, \dots, D_k$ form a basis of $H^2(X)$ and Picard rank k. Then define the I-*function*

$$I_X = ze^{\sum_{j=1}^k t_i D_i} \sum_\beta Q^\beta \frac{\prod_{j=1}^N \prod_{m=-\infty}^0 (D_j + mz)}{\prod_{j=1}^N \prod_{m=-\infty}^{\langle D_j, \beta\rangle} (D_j + mz)}.$$

Theorem 1.1.11 (Mirror theorem)**.** *The formal functions I_X and J_X coincide up to some change of variables, which if $c_1(X)$ is semi-positive is given by components of the I-function.*

Theorem 1.1.12 (Mirror theorem in this formalism)**.** *For any* t*, we have*

$$I_X(t, z) \in \mathcal{L}.$$

Another direction in Gromov-Witten theory is the Virasoro constraints. In the original formulation, these involved very complicated explicit differential operators, but in the Givental formalism, there is a very compact formulation.

Define $\ell^{-1} = z^{-1}$ and

$$\ell_0 = z\frac{d}{dz} + \frac{1}{2} + \mu + \frac{c_1(X) \cup -}{z}.$$

Then define

$$\ell_n = \ell_0(z\ell_0)^n.$$

Theorem 1.1.13 (Genus-0 Virasoro constraints)**.** *Suppose the vector field on $\mathcal{H}$ defined by ℓ_0 is tangent to $\mathcal{L}$. Then the same is true for the vector fields defined by ℓ_n for any $n \geqslant 1$.*

Proof. Let L be a tangent space to $\mathcal{L}$. Then if $f \in zL \subset \mathcal{L}$, the assumption gives us $\ell_0 f \in L$. But then $z\ell_0 f \in zL$, so $\ell_0 z\ell_0 f = \ell_1 f \in L$. Continuing, we obtain $\ell_n f \in L$ for all n. □

Next week, we will learn that the Quantum Riemann-Roch theorem can be stated in this formalism. Let $\mathcal{L}^{tw}$ be the twisted Lagrangian cone (where the twisted theory will be defined next week).

Theorem 1.1.14 (Quantum Riemann-Roch)**.** *For some explicit linear symplectic transformaiton Δ, we have $\mathcal{L}^{tw} = \Delta\mathcal{L}$.*

1.1.4 Quantization In the last part of the talk, we will briefly discuss the quantization formalism, which encodes the higher-genus theory. In Darboux coordinates p_a, q_b, we will quantize symplectic transformations by the standard rules

$$\widehat{q_a q_b} = \frac{q_a q_b}{\hbar}, \qquad \widehat{q_a p_b} = q_a \frac{\partial}{\partial q_b}, \qquad \widehat{p_a p_b} = \hbar \frac{\partial^2}{\partial q_a \, \partial q_b}.$$

This determines a differential operator acting on functions on $\mathcal{H}_+$.

We also need the genus-g potential

$$\mathcal{F}_g^X \coloneqq \sum_{\beta, n} \frac{Q^\beta}{n!} \langle t(\psi), \ldots, t(\psi) \rangle_{g,n,\beta}^X$$

and the *total descendent potential*

$$\mathcal{D} \coloneqq \exp\left(\sum_{g \geqslant 0} \hbar^{g-1} \mathcal{F}_g^X \right).$$

In this formalism, the Virasoro conjecture can be expressed as follows. Let $L_n = \widehat{\ell}_n + c_n$, where c_n is a carefully chosen constant.

Conjecture 1.1.15 (Virasoro conjecture)**.** *If* $L_{-1}\mathcal{D} = L_0\mathcal{D} = 0$, *then* $L_n\mathcal{D} = 0$ *for all* $n \geqslant 1$.

In this formalism, the higher-genus version of the Quantum Riemann-Roch theorem takes the very simple form

Theorem 1.1.16 (Quantum Riemann-Roch)**.** *Let* $\mathcal{D}^{\mathrm{tw}}$ *be the twisted descendent potential. Then*

$$\mathcal{D}^{\mathrm{tw}} = \widehat{\Delta}\mathcal{D}.$$