DECOMPOSITION THEOREM AND RELATIVE HARD LEFSCHETZ THEOREM

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Abstract. I will explain the decomposition theorem and the relative Hard Lefschetz theorem and then discuss some applications.

1. Introduction

In this talk, we will work with sheaves with coefficients in a field $\Bbbk$ of characteristic 0. X will be an algebraic variety over $\mathbb{C}$. Recall that the derived category $D^b_c(X)$ of constructible sheaves has a *perverse t-structure* given by

$$
\begin{aligned}
{}^pD^b_c(X)^{\leq 0} &= \left\{\mathcal{F} \in D^b_c(X) \mid \dim \operatorname{supp} \mathsf{H}^i(\mathcal{F}) \leq -i \text{ for all } i\right\},\\
{}^pD^b_c(X)^{\geq 0} &= \left\{\mathcal{F} \in D^b_c(X) \mid \dim \operatorname{supp} \mathsf{H}^i(\mathbb{D}\mathcal{F}) \leq -i \text{ for all } i\right\},
\end{aligned}
$$

where the Verdier duality functor is given by

$$\mathbb{D}\mathcal{F} := \mathsf{Hom}(\mathcal{F}, \omega_X).$$

Here, the *dualizing complex* ω_X is given by

$$\omega_X := (X \to \mathrm{pt})^! \underline{\Bbbk}.$$

The category $\mathsf{Perv}(X)$ of perverse sheaves is defined to be the heart of this t-structure.

We will now introduce intersection cohomology (IC) complexes.

Definition 1.1. Let $j \colon Y \hookrightarrow X$ be a locally closed embedding. Then the *intermediate extension* functor is given by

$$j_{!*} \colon \mathsf{Perv}(Y) \to \mathsf{Perv}(X), \qquad \mathcal{F} \mapsto \operatorname{Im}({}^p\mathsf{H}^0(j_!\mathcal{F}) \to {}^p\mathsf{H}^0(j_*\mathcal{F})).$$

Definition 1.2. Let $j \colon Y \subset X$ be a smooth, locally closed subvariety and $\mathcal{L}$ be a local system on Y. Then the *intersection cohomology complex* of $(Y, \mathcal{L})$ is the sheaf

$$\mathrm{IC}(Y, \mathcal{L}) := j_{!*}\mathcal{L}[\dim Y].$$

If $Y = X^{\mathsf{sm}}$ is the smooth locus of X and $\mathcal{L} = \underline{\Bbbk}$ is the constant sheaf, then we abbreviate $\mathrm{IC}(X) := \mathrm{IC}(X^{\mathsf{sm}}, \underline{\Bbbk})$.

We are now ready to state the main results. Recall that simple objects in $\mathsf{Perv}(X)$ are IC complexes.

Theorem 1.3 (Decomposition theorem). *Let $f \colon X \to Y$ be proper. Then if $\mathcal{F} \in \mathsf{Perv}(X)$ is semisimple, we have the following:*

(1) The pushforward $f_*\mathcal{F}$ decomposes as a direct sum

$$f_*\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathsf{H}^i(f_*\mathcal{F})[-i];$$

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(2) Each summand decomposes as a direct sum

$$
{}^p\mathsf{H}^i(f_*\mathcal{F}) \cong \bigoplus_\beta \mathrm{IC}(X_\beta, \mathcal{L}_\beta)
$$

of IC sheaves.

This theorem was originally proved by Beilinson-Bernstein-Deligne(-Gabber) using an argument involving reduction to positive characteristic. However, they require that $\mathcal{F}$ satisfies a technical condition called being *of geometric origin*. There is another proof given by Saito using mixed Hodge modules. Because these proofs are far beyond my mathematical ability, we will not discuss them.

2. Relative Hard Lefschetz

Recall that for a compact Kähler manifold X, the *hard Lefschetz theorem* states that

$$
\omega^k \colon H^{n-k}(X) \to H^{n+k}(X)
$$

is an isomorphism, where ω is the Kähler form and n is the complex dimension of X. We will generalize this to projective morphisms of algebraic varieties. Recall that if $f\colon X \to Y$ is projective, it factors as

$$
X \hookrightarrow Y \times \mathbb{P}^N \to Y.
$$

Then let $\eta \coloneqq c_1(\mathcal{O}(1)) \in H^2(Y \times \mathbb{P}^N, \mathbb{Q})(1)$. By definition, this is a morphism $\eta\colon \underline{\mathbb{Q}}_X \to \underline{\mathbb{Q}}_X[2](1)$, so tensoring with $\mathcal{F}$ and pushing forward gives us a morphism

$$
f_*(\eta \otimes \mathrm{id}_{\mathcal{F}})\colon f_*\mathcal{F} \to f_*\mathcal{F}[2](1),
$$

which by abuse of notation we will also call η.

Theorem 2.1 (Relative Hard Lefschetz)**.** *For any non-negative $k \geq 0$, the map*

$$
\eta^k\colon {}^p\mathsf{H}^{-k}(f_*\mathcal{F}) \to {}^p\mathsf{H}^k(f_*\mathcal{F})(k)
$$

is an isomorphism.

Sketch of proof. We will outline how to deduce this result from the decomposition theorem. The first simplification we make is that by using the projection formula for the pushforward along $g\colon X \hookrightarrow Y \times \mathbb{P}^N$, we obtain

$$
g_*(\eta \otimes \mathrm{id}_{\mathcal{F}}) = \eta \otimes \mathrm{id}_{g_*\mathcal{F}}.
$$

Then define varieties

$$
\begin{aligned}
F &\coloneqq \left\{(L, L') \in \mathbb{P}^N \times (\mathbb{P}^N)^\vee \mid L' \text{ annihilates } L\right\},\\
S &\coloneqq (\mathbb{P}^N \times (\mathbb{P}^N)^\vee) \setminus F.
\end{aligned}
$$

Now consider the following diagram:

$$
\begin{tikzcd}
Y \times F \arrow[r, "i"] & Y \times \mathbb{P}^N \times (\mathbb{P}^N)^\vee \arrow[dl, "p'"'] \arrow[dr, "u'"] & Y \times S \arrow[l, "j"'] \arrow[d, "q_1"] \arrow[dll, "q_2"] \\
Y \times (\mathbb{P}^N)^\vee \arrow[dr, "u"] & & X = Y \times \mathbb{P}^N \arrow[dl, "f"] \\
& Y. &
\end{tikzcd}
$$

(1) Consider η as a class in $H^2(Y \times \mathbb{P}^N \times (\mathbb{P}^N)^\vee, \mathbb{Q})(1)$. Alternatively, consider the divisor class
$$\theta = \mathrm{cl}(Y \times F) \in H^2(Y \times \mathbb{P}^N \times (\mathbb{P}^N)^\vee, \mathbb{Q})(1).$$
Then for any $\mathcal{G} \in D^b_c(Y \times \mathbb{P}^N \times (\mathbb{P}^N)^\vee)$, we have $\eta = \theta$ as maps
$${}^p\mathsf{H}^k(p'_*\mathcal{G}) \to {}^p\mathsf{H}^{k+2}(p'_*\mathcal{G})[2](1).$$
From now on, we will replace $\mathcal{F}$ with $(u')^*\mathcal{F}[N]$, which is fine because
$$u^* f_* \mathcal{F} \cong p'_*(u')^*\mathcal{F}$$
by smooth base change.

(2) Consider the diagram
$$\begin{tikzcd}
\mathcal{F} \arrow[rr] \arrow[dd, "\theta"] & & i_*i^*\mathcal{F} \arrow[dl, "\sim"] \arrow[dd, "\theta"] \\
& i_*i^!\mathcal{F}[2](1) \arrow[dl] & \\
\mathcal{F}[2](1) \arrow[rr] & & i_*i^*\mathcal{F}[2](1),
\end{tikzcd}$$
where all unlabelled arrows come from adjunctions. The fact that the first diagonal morphism exists and is an isomorphism is because the morphism $Y \times F \to X$ is smooth, and so in this case there is a natural isomorphism $i^*\mathcal{F} \to i^!\mathcal{F}[2](1)$. Because the cycle class θ is defined by
$$\underline{\mathbb{Q}} \to i_*i^*\underline{\mathbb{Q}} \cong i_*i^!\underline{\mathbb{Q}}[2](1) \to \underline{\mathbb{Q}}[2](1),$$
we see that the top left triangle commutes. Applying adjunction twice to see
$$\mathrm{Hom}(i_*i^*\mathcal{F}, i_*i^*\mathcal{F}[2](1)) \cong \mathrm{Hom}(i^*\mathcal{F}, i^*\mathcal{F}[2](1)) \cong \mathrm{Hom}(\mathcal{F}, i_*i^*\mathcal{F}[2](1)),$$
we see that the bottom right triangle commutes. Then apply p'_* and obtain the diagram
$$\begin{tikzcd}
p'_*\mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\theta"] & p'_*i_*i^*\mathcal{F} \arrow[dl, "\beta"] \arrow[d, "\theta"] \\
p'_*\mathcal{F}[2](1) \arrow[r, "\alpha"] & p'_*i_*i^*\mathcal{F}[2](1).
\end{tikzcd}$$

(3) Applying $p'_* = p'_!$ (here p' is proper) to the distiguished triangle
$$j_!j^* \to \mathrm{id} \to i_*i^* \xrightarrow{[1]}$$
yields the distinguished triangle
$$(q_2)_!j^*\mathcal{F} \to p'_*\mathcal{F} \to p'_*i_*i^*\mathcal{F} \xrightarrow{[1]} .$$
Using the fact that S is affine over both $\mathbb{P}^N$ and $(\mathbb{P}^N)^\vee$ and the fact that $j^*\mathcal{F}$ is perverse (compactly supported pushforward along affine morphisms is left exact in the perverse t-structure), the long exact sequence
$$\cdots \to {}^p\mathsf{H}^{-1}((q_2)_!j^*\mathcal{F}) \to {}^p\mathsf{H}^{-1}(p'_*\mathcal{F}) \to {}^p\mathsf{H}^{-1}(p'_*i_*i^*\mathcal{F}) \to {}^p\mathsf{H}^0((q_2)_!j^*\mathcal{F}) \to \cdots$$
tells us that ${}^p\mathsf{H}^{-k}(\alpha)$ is an isomorphism for $k > 1$ and injective when $k = 1$. In fact, it identifies ${}^p\mathsf{H}^{-1}(p'_*\mathcal{F})$ with the largest subobject of ${}^p\mathsf{H}^{-1}(p'_*i_*i^*\mathcal{F})$ which lies in the essential image of $u^*[N]$.

(4) By a similar argument, the map

$$ {}^p\mathsf{H}^{-1}(\beta)\colon\ {}^p\mathsf{H}^{-1}(p'_* i_* i^* \mathcal{F}) \to {}^p\mathsf{H}^1(p'_*\mathcal{F})(1) $$

is surjective and identifies ${}^p\mathsf{H}^1(p'_*\mathcal{F})(1)$ with the largest quotient of ${}^p\mathsf{H}^{-1}(p'_* i_* i^*\mathcal{F})$ which lies in the essential image of $u^*[N]$.

(5) We need to prove that

$$ \eta^k\colon\ {}^p\mathsf{H}^{-k}(f_*\mathcal{F}) \to {}^p\mathsf{H}^k(f_*\mathcal{F})(k) $$

is an isomorphism. First note that $i^*\mathcal{F}[-1] = (u' \circ i)^*\mathcal{F}_X[N-1]$ is a semisimple perverse sheaf, so $i_* i^* \mathcal{F}[-1] \in \mathsf{Perv}(Y \times \mathbb{P}^N \times (\mathbb{P}^N)^\vee)$ is also semisimple. We will now induct on k. When $k = 0$, η^0 is the identity, so there is nothing to do. When $k = 1$, note that ${}^p\mathsf{H}^{-1}(p'_* i_* i^* \mathcal{F})$ is semisimple. But this identifies the images of ${}^p\mathsf{H}^{-1}(\alpha)$ and ${}^p\mathsf{H}^{-1}(\beta)$, so the desired result follows from the fact that $\theta = \beta \circ \alpha$. Finally, factorizing

$$ \begin{aligned} \theta^k_{\mathcal{F}} &= \beta[2k-2] \circ \alpha[2k-2] \circ \cdots \circ \beta[2] \circ \alpha[2] \circ \beta \circ \alpha \\ &= \beta[2k-2] \circ \theta_{i_* i^* \mathcal{F}} \circ \alpha \end{aligned} $$

and using the inductive hypothesis gives the desired result. □

3. Example: toric varieties

Definition 3.1. If P is a simplicial polytope of dimension n with f_k faces of dimension k for all $k < n$, define the *h-polynomial* $h(P, t) = \sum h_k(P) t^k$ of P by

$$ h(P, t) \coloneqq (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}. $$

Also define the *g-polynomial* of P by

$$ g(P, t) \coloneqq h_0 + \sum_{k=1}^{\lfloor n/2 \rfloor} (h_k - h_{k-1}) t^{k-1}. $$

A well-known result in combinatorics, proved in Fulton's book, is the following.

Theorem 3.2. *Let X_P (which is simplicial) denote the toric variety corresponding to P. Then $h^{2k}(X_P) = h_k(P)$ for all $0 \le k \le n$.*

As a corollary, the g-polynomial completely determines the h-polynomial and has positive coefficients. Here, use the fact that intersection cohomology and usual cohomology coincide for simplicial toric varieties and then use Poincaré duality and hard Lefschetz.

Definition 3.3. For an arbitrary convex polytope, define $h(P, t)$ inductively by the formula

$$ h(P, t) \coloneqq \sum_{F < P} g(F, t)(t-1)^{n-1-\dim F}, $$

where we define $g(\emptyset, t) = h(\emptyset, t) = 1$ and the set of proper faces $F < P$ includes the empty set.

Theorem 3.4. *If P is any rational convex polytope, we have*

$$ h_k(P) = \dim \mathrm{IH}^{2k}(X_P, \mathbb{Q}) $$

for $0 \le k \le n$.

This result can be proved by using the decomposition theorem for a simplicial subdivision of P, but we will just illustrate it with an example. Let Δ be the three-dimensional cube and let X be the corresponding toric variety, which has six singular points. We will consider the resolution $\tilde{X}$ of X given by adding a new vertex at the center of each facet of Δ and taking the stellar subdivision. Call the resulting polytope $\tilde{\Delta}$. First note that

$$\begin{aligned}
h(\Delta, t) &= g(\emptyset, t)(t-1)^3 + 8g(\mathrm{pt}, t)(t-1)^2 + 12g([0,1], t)(t-1) + g([0,1]^2, t) \\
&= (t-1)^3 + 8(t-1)^2 + 12(t-1) + 6(t+1) \\
&= t^3 + 5t^2 + 5t + 1.
\end{aligned}$$

On the other hand, for $\tilde{\Delta}$, we have $f_0 = 14$, $f_1 = 36$, and $f_2 = 24$. This implies that

$$h(\tilde{\Delta}, t) = (t-1)^3 + 14(t-1)^2 + 36(t-1) + 24 = t^3 + 11t^2 + 11t + 1.$$

Geometrically, $f\colon \tilde{X} \to X$ blows up the six singular points of X and replaces them with a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ each. This implies that

$$f_*\underline{\mathbb{Q}}[3] = \bigoplus_{k=-1}^{1} {}^p\mathsf{H}^k(f_*\underline{\mathbb{Q}}[3]) = \mathrm{IC}(X) \oplus \bigoplus_{i=1}^{6} \underline{\mathbb{Q}}_{p_i}[1] \oplus \bigoplus_{i=1}^{6} \underline{\mathbb{Q}}_{p_i}[-1].$$

This implies that $h^k(\tilde{\Delta}) = h^k(\Delta) + 6$ for $k = 2, 4$, which is exactly what we calculated previously.