FGA Explained Learning Seminar
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Lectures by Various

Columbia University

Disclaimer

These notes were taken during the seminar using the `vimtex` package of the editor `neovim`. Any errors are mine and not the speakers'. In addition, my notes are picture-free (but will include commutative diagrams) and are a mix of my mathematical style and that of the lecturers. If you find any errors, please contact me at `plei@math.columbia.edu`.

Seminar Website: `https://www.math.columbia.edu/~calebji/fga.html`

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Caleb (Oct 16): Representable Functors and Grothendieck Topologies

1.1 Representable Functors

We will always denote categories by C.

Definition 1.1.1. Given an object $x \in C$, define the functor $h_X \colon C^{\text{op}} \to \mathsf{Set}$ by $h_x = \text{Hom}(-, X)$.

Any morphism $f \colon x \to y$ induces a natural transformation $h_f \colon h_x \to h_y$. By the Yoneda lemma, this correspondence is bijective.

Lemma 1.1.2 (Yoneda Lemma)**.** *Let $x \in C$ and $F \colon C^{\text{op}} \to \mathsf{Set}$ be a functor. Then* $\text{Hom}(h_x, F) \simeq F(x)$.

Proof. Let $\theta \colon h_x \to F$. This gives a map $\theta_x \colon h_x(x) \to F(x)$, and we can consider $\text{id} \to \theta_x(\text{id})$. Now given $t \in F(x)$, we need $h_x(U) \to F(U)$. Given $U \to x$, then we have a map $F(x) \to F(U)$ and then $t \mapsto F_f(t)$. We can check that these are inverses. □

Definition 1.1.3. A functor $F \colon C^{\text{op}} \to \mathsf{Set}$ is *representable* if it is naturally isomorphic to h_x for some x.

Definition 1.1.4. If F is a presheaf, a *universal object* for F is a pair (X, ξ) such that $\xi \in FX$ and for any (U, σ) where $\sigma \in FU$, there exists a unique $f \colon U \to X$ such that $F_f(\xi) = \sigma$.

Note that representability is equivalent to having a universal object.

Example 1.1.5.

1. For the first example, consider $C = \mathsf{Sch}/R$ for some ring R. Then if $F = \Gamma(\mathcal{O})$, then clearly this is isomorphic to $h_{\mathbb{A}^1}$ and the universal object is $(\mathbb{A}^1, x)$.
2. Let $F(X) = \{\mathcal{L}, s_0, \ldots, s_n\}$ where $\mathcal{L}$ is a line bundle and $s_0, \ldots, s_n$ generate $\mathcal{L}$, then $(\mathbb{P}^n, x_0, \ldots, x_n)$ is a universal object.

1.2 Grothendieck Topologies

According to Wikipedia, this is supposed to be a pun on "Riemann surface." We want to generalize the idea of a topology because the Zariski topology is awful. Instead of open sets, we will consider suitable maps (coverings).

Definition 1.2.1. A *Grothendieck topology* on a category C is a specification of *coverings* $\{U_i \to U\}$ of U for each $U \in C$. Here are the axioms for coverings:

1. If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering.
2. If $\{U_i \to U\}$ is a covering, for all $V \to U$, the fiber products $\{U_i \times_U V \to V\}$ and form a covering of V.
3. If $\{U_i \to U\}$ is a covering and $\{V_{ij} \to U_i\}$ are coverings, then $\{V_{ij} \to U\}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

Example 1.2.2. Here are some topological examples. Let X be a topological space.

1. The site of X is the poset category of open subsets of X. The fiber product is just the intersection, and a covering is a normal open covering.
2. (Global classical topology) Let $C = \mathsf{Top}$. Here, the coverings are sets of open embeddings such that the union of the images covers the whole space.
3. (Global étale topology) Here, $C = \mathsf{Top}$ and the coverings are now local homeomorphisms.

Returning to schemes, we have several examples of Grothendieck topologies.

1. (Global Zariski Topology). Let $C = \mathsf{Sch}$. The coverings are jointly surjective open embeddings.
2. (Big étale site over S) The objects are schemes over S and the morphisms are S-morphisms that are étale and locally of finite presentation.
3. (Small étale site) This the same as the big étale site, but with the added requirement that $U \to S$ is also étale.
4. (fppf topology) This stands for the French *fidèlement plat et présentation finie*. The morphisms are $U_i \to U$ flat and locally of finite presentation. A covering is a set of jointly surjective morphisms such that the map $\bigsqcup U_i \to U$ is faithfully flat and of finite presentation. Note that flat and locally of finite presentation implies open.
5. (fpqc topology) This stands for the French *Fidèlement plat et quasi-compacte*. An *fpqc* morphism is a morphism $X \to Y$ that is faithfully flat and one of the following equivalent conditions:
   a) Every quasicompact open subset of Y is the image of a quasicompact open subset of X.
   b) There exists an affine open cover $\{V_i\}$ of Y such that V_i is the image of a quasicompact open subset of X.
   c) Given $x \in X$, there exists a neighborhood $U \ni x$ such that $f(U)$ is open in Y and $U \to f(U)$ is quasicompact.
   d) Given $x \in X$, there exists a quasicompact open neighborhood $U \ni x$ such that $f(U)$ is open and affine in Y.

   The fpqc topology is given by maps $\{U_i \to U\}$ such that $\bigsqcup U_i \to U$ is an fpqc morphism.

To check that this is a topology, we have to do a lot of work. However, we will list some properties of fpqc morphisms and coverings.

Proposition 1.2.3. *1. The composition of fpqc morphisms is fpqc.*

2. *Given $f\colon X \to Y$, if $f^{-1}(V_i) \to V_i$ is fpqc, then f is fpqc.*

3. *Open and faithfully flat implies fpqc. Moreover, faithfully flat and locally of finite presentation implies fpqc. This means that fppf implies fpqc.*

4. *Base change preserves fpqc morphsisms.*

5. *All fpqc morphsism are submersive. Thus $f^{-1}(V)$ is open if and only if V is open.*

Note that Zariski is coarser than étale is coarser than fppf is coarser than fpqc.

1.3 Sheaves on Sites

Recall that a presheaf on a space is a functor $X_{\mathrm{cl}}^{\mathrm{op}} \to \mathsf{Set}$. Similarly, if C is a site, then a presheaf is a functor $C^{\mathrm{op}} \to \mathsf{Set}$.

Definition 1.3.1. A presheaf on a site C is a *sheaf* if

1. Given a covering $\{U_i \to U\}$ and $a, b \in FU$ such that $p_i^* a = p_i^* b$, then $a = b$.

2. Given a covering $\{U_i \to U\}$ and $a_i \in FU_i$ such that $p_i^* a_j = p_j^* a_i$ (in the fiber product) for all i, j, there exists a unique $a \in FU$ such that $p_i^* a = a_i$.

An alternative definition of a sheaf is that $FU \to \prod FU_i \rightrightarrows F(U_i \times_U U_j)$ is an equalizer.

Theorem 1.3.2 (Grothendieck). *A representable functor on Sch/S is a sheaf in the fpqc topology.*

This means that given any fpqc cover $\{U_i \to U\}$, then applying h_X, if we have $f_i\colon U_i \to X$ that glue on $U_i \times_X U_j \to X$, then the sheaf condition says we can glue to a unique $f\colon U \to X$. In the Zariski topology, this is trivial. This also means that the fpqc topology is *subcanonical*, which means that h_X are all sheaves.

We will prove this result by reducing to the category of all schemes. Note that the topology on Sch/S comes from the topology on Sch. Then we can show that if C is subcanonical, then C/S is subcanonical. Then we use the following lemma.

Lemma 1.3.3. *Let S be a scheme and $F\colon \mathsf{Sch}/S^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. If F is a Zariski sheaf if $V \to U$ is a faithfully flat morphism of affine S-schemes, then $FU \to FV \rightrightarrows F(V \times_U V)$ is an equalizer, then F is an fpqc sheaf.*

Proof. Given $\{U_i \to U\}$ an fpqc covering, let $V = \coprod U_i$. Then consider the diagram

$$\begin{tikzcd}
FU \arrow[r] \arrow[d] & FV \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(V \times_U V) \arrow[d] \\
FU \arrow[r] & \prod FU_i \arrow[r, shift left] \arrow[r, shift right] & F(U_i \times_U U_j),
\end{tikzcd}$$

the columns are bijective, so it suffices to check this for single coverings.

Now if $\{U_i \to U\}$ are finite and all affine and the second assumption holds, we have the diagram

$$\begin{tikzcd}
FU \arrow[r] \arrow[d] & FV \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(V \times_U V) \\
\prod_i F(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_a FU_{ia} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_{ab} F(V_{ia} \times_U V_{ib}) \\
\prod_{ij} F(U_i \cap U_j) \arrow[r] & \prod_{ij} \prod_{ab} F(U_{ia} \cap U_{jb}).
\end{tikzcd}$$

Then the middle row is an equalizer. □

Proof of Theorem 1.3.2. If X, U, V are affine, then we know that $\operatorname{Hom}(R, -)$ is left exact, so the result follows from commutative algebra. Now it suffices to check the general case for single covers. If $X = \bigcup X_i$ is a union of affines, then separatedness follows by restricting to the X_i and using the affine case.

Please read the rest of this yourself. □

2

Caleb (Oct 23): Sieves and Fibered Categories

Recall that a Grothendieck topology on a site C is a collection of coverings $\{U_i \to U\}$ such that isomorphisms are coverings, pullbacks preserve coverings, and if $\{U_i \to U\}$ and $\{U_{ij} \to U_i\}$ are coverings, then $\{U_{ij} \to U_i\}$ is a covering.

Also recall from last time that two important examples of Grothendieck topologiyes are the global Zariski topology and the fpqc topology. We also defined a sheaf and proved that representable functors are sheaves in the fpqc topology.

2.1 Sieves

A sieve is a way to have "your barrel full and your wife drunk." The motivating question is:

Question 2.1.1. *When do two Grothendieck topologies give rise to the same sheaves?*

Definition 2.1.2. A *subfunctor* G of $F \colon C^{\mathrm{op}} \to \mathsf{Set}$ has $G(u) \subset F(u)$ and the morphisms are just restrictions.

Definition 2.1.3. A *sieve* on $U \in \mathrm{Ob}(C)$ is a subfunctor of h_U.

Example 2.1.4. Given a cover $\mathcal{U} = \{U_i \to U\}$, then $h_{\mathcal{U}}(T)$ are arrows that factor through the covering.

Definition 2.1.5. A sieve $S \subset h_U$ *belongs to a topology* T if $h_{\mathcal{U}} \subset S$ for some covering $\mathcal{U}$ in the topology T.

Definition 2.1.6. A covering $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if $h_{\mathcal{V}} \subset h_{\mathcal{U}}$.

This gives a poset structure on Grothendieck topologies, where $T \prec T'$ if every covering in T has a refinement that is a covering in T'. For schemes, we have

$$\text{Zariski} \prec \text{étale} \prec \text{fppf} \prec \text{fpqc}.$$

Then we know that T_1, T_2 are equivalent if $T_1 \prec T_2 \prec T_1$.

Proposition 2.1.7. *Equivalent topologies have the same sheaves.*

Proof. Note that $F \colon C^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf if and only for S belonging to T, the map $FU \simeq \mathrm{Hom}(h_U, F) \to \mathrm{Hom}(S, F)$ is bijective. $\square$

A real world application of this is that we can construct the sheafification of a functor $F\colon C^{\mathrm{op}} \to \mathsf{Set}$ by

$$F^a U = \varinjlim \operatorname{Hom}(S_i, F^s),$$

where S_i ranges over sieves belonging to T.

2.2 Fibered Categories

We will consider functors $p_F\colon F \to C$. Our notation will be

$$\begin{tikzcd} \xi \arrow[r] \arrow[d, mapsto, "p_F"] & \eta \arrow[d, mapsto, "p_F"] \\ U \arrow[r] & V. \end{tikzcd}$$

Definition 2.2.1. A morphism $\phi\colon \xi \to \eta$ in F is *Cartesian* if for all $\psi\colon \zeta \to \eta$ and $h\colon p_F\zeta \to p_F\xi$ in C with $p_F \circ h = p_F\psi$, then there exists a unique $\theta\colon \zeta \to \xi$ such that $p_f\theta = h$ and $\phi \circ \theta = \psi$.

If ϕ is a Cartesian arrow, then we say that ξ is a *pullback* of η. Pullbacks (with a fixed map in C) are unique up to isomorphism.

Definition 2.2.2. A *fibered category* is a functor $F \to C$ such that for any $f\colon U \to V$ in C and $\eta \in \operatorname{Ob}(F)$ with $p_F\eta = V$, then we have a cartesian arrow $\phi\colon \xi \to \eta$ with $p_F\phi = f$.

Definition 2.2.3. Note that the fiber $F(U)$ is a category. The objects are ξ such that $p_F(\xi) = U$ and morphisms are arrows h that map to the identity of U (i.e. $p_F h = \mathrm{id}_U$).

Definition 2.2.4. A morphism of fibered categories over C is a functor $H\colon F \to G$ such that $H_U\colon F(U) \to G(U)$ is a functor.

Definition 2.2.5. A *cleavage* of $F \to C$ is a class K of cartesian arrows such that for all $f\colon U \to V$ and $\eta \in F(U)$, there exists a unique morphism in K mapping to f.

Question 2.2.6. *Does a cleavage give a functor from C to the category of categories?*

Unfortunately, the answer is no. This is not a functor, but it does give a pseudofunctor, which is the same as a lax 2-functor. The idea is that id_U^* may not be the identity and f^*g^* may not be $(g \circ f)^*$. However, they are canonically isomorphic.

Definition 2.2.7. A *pseudofunctor* Φ on C (from C^{op} to Cat) is an assignment such that

1. For all objects U of C, ΦU is a category.
2. For each $f\colon U \to V$, $f^*\colon \Phi V \to \Phi U$ is a functor.
3. For all U of C, we have an isomorphism $\epsilon_U\colon \mathrm{id}_U^* \simeq \mathrm{id}_{\Phi U}$.
4. For all $U \to V \to W$ we have an isomorphism $\alpha_{f,g}\colon f^*g^* \simeq (gf)^*$ such that given $f\colon U \to V$ and $\eta \in \Phi U$, we have $\alpha_{\mathrm{id}_U,f}(\eta) = \epsilon_U(f^*\eta)$, and $\alpha_{f,\mathrm{id}_V}(\eta) = f^*\epsilon_U(\eta)$. In addition, we require that for all morphisms $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ and $\theta \in F(T)$, the diagram

$$\begin{tikzcd} f^*g^*h^*\theta \arrow[r, "\alpha_{f,g}(h^*\theta)"] \arrow[d, "f^*\alpha_{g,h}(\theta)"] & (gf)^*h^*\theta \arrow[d, "\alpha_{gh,f}(\theta)"] \\ f^*(hg)^*\theta \arrow[r, "\alpha_{f,hg}(\theta)"] & (hgf)^*\theta \end{tikzcd}$$

commutes.

Definition 2.2.8. A *cleavage* is a *splitting* if it gives an honest functor.

Apparently every fibered category is equivalent to a split fibered category.

2.3 Examples

1. If a category C has fiber products, then the category of arrows is a fibered category over C, and the Cartesian arrows are the Cartesian diagrams.

2. If G is a topological group, then there is a "classifying stack" given by principal G-bundles.

3. Consider the category of sheaves on a site C. Given X an object, denote sheaves on X as sheaves in C/X. Then given $f\colon X \to Y$, we have $f^*\colon \mathsf{Sh}Y \to \mathsf{Sh}X$.

4. Let $C = \mathsf{Sch}/S$. Then given $f\colon U \to V$, we have $f^*\colon \mathsf{QCoh} \to \mathsf{QCoh}(U)$. However, $(gh)^* \simeq f^*g^*$ is not an equality. In the affine case, if $f\colon A \to B$ is a morphism of rings, then recall that quasicoherent sheaves are modules. Then $f^*M = M \otimes_A B$. The problem is that $M \otimes_B B \simeq M$ but they are not the same object. However, note that $f_*g_* = (fg)_*$ and that f^*, f_* are adjoint.

3

Caleb (Oct 30): Étale morphisms and the étale fundamental group

Today we will take a break from the abstract nonsense.

3.1 Flat Morphisms

Definition 3.1.1. Let $f\colon X \to Y$ be a map of schemes. Then $f\colon X \to Y$ is *flat* at $x \in X$ if the map $f^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. Then f is flat globally if it is flat at all x.

Note if $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if $A \to B$ is flat. The geometric intuition behind flatness is that the fibers form a continuous family.

Example 3.1.2. Consider the map $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^3 - x^2)$ given by $x \mapsto t^2 - 1, y \mapsto t^3 - t$. This is the normalization of the nodal cubic, and we see that the integral closure of the second ring is given by adjoining the element $\frac{y}{x}$.

Outside $t = \pm 1$, the map of stalks is an isomorphism, but at 0, we see that

$$(x,y)k[t]_{(t)} = (t^2 - 1, t^3 - t)k[t]_{(t)} = k[t]_{(t)}$$

and thus the map is not flat.

Example 3.1.3. Let $A \subset B$ be integral domains with the same fraction field. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat if and only if $A = B$.

Proof. The idea is to use that for any ideal I of A, we have $IB \cap A = I$. Then go read your commutative algebra homework (or your favorite commutative algebra text). □

Remark 3.1.4. Normalization is flat if and only if it is an isomorphism.

Example 3.1.5. A closed embedding is flat if and only if it is also open. In particular, closed embeddings are generically not flat.

Proposition 3.1.6. *If $f\colon X \to Y$ is a flat map between irreducibles, then $f(U)$ is dense in Y for all nonempty $U \subset X$.*

Proof. Reduce to the case $\operatorname{Spec} B \to \operatorname{Spec} A$. We simply need $\eta = \sqrt{0}$ to be in the image of f. Writing the diagram

$$\begin{tikzcd} U_\eta \arrow[r] \arrow[d] & U \arrow[d] \\ \operatorname{Spec} k(\eta) \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

we see that $U_\eta = \operatorname{Spec} B \otimes_A k(\eta)$. This contains the ring $B/\eta B$ by flatness. If $B = \eta B$, then B is nilpotent and this implies $U = \emptyset$. $\square$

Theorem 3.1.7. *Let $f\colon X \to Y$ be a morphism of locally Noetherian schemes. Then*

$$\dim \mathcal{O}_{X_y,x} \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}$$

with equality if f is flat.

Sketch of Proof. By schematic nonsense, replace Y by $A = \operatorname{Spec} \mathcal{O}_{Y,n}$, so Y is Noetherian local. We can also assume Y is reduced. We will induct on $\dim Y$. Recall that if $t \in A$, then $\dim(A/tA) = \dim A - 1$ for t not a zero divisor or a unit. $\square$

Corollary 3.1.8. *If $f\colon X \to Y$ be a faithful flat map of algebraic varieties, then X_y is equidimensional and* $\dim X_y = \dim X - \dim Y$.

3.2 Étale morphisms

From now on we will assume $f\colon X \to Y$ is finite type and locally Noetherian.

Definition 3.2.1. A morphism f of schemes is *unramified* at x if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ satisfies $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)/k(y)$ is separable.

Definition 3.2.2. A morphism is *étale* if it is flat and unramified.

Example 3.2.3. Let L/K be an extension of fields. Then $\operatorname{Spec} L \to \operatorname{Spec} K$ is étale if and only if L/K is separable.

Example 3.2.4. Let L/K be a field extension. Then $f\colon \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ is flat and the separability condition holds. Then f is unramified (and hence étale) if and only if it is unramified in the sense of number theory. Here, we consider the product

$$\mathfrak{p}\mathcal{O}_L = \prod_{i=1}^{m} \mathfrak{q}_i^{e_i},$$

and f is unramified if all $e_i = 1$. For example, for the extension $\mathbb{Z} \to \mathbb{Z}[i]$, we see that $(2) = (1+i)^2$ and thus this map is not unramified.

Example 3.2.5. Consider $\operatorname{Spec} k[t]/(P) \to \operatorname{Spec} k$. Then prime ideals are irreducibles Q dividing P. Then f is étale at Q if and only if Q is separable and is a simple factor of P.

More generally, a *standard étale morphism* for a monic $P(T) \in A[T]$ and $b \in B = A[T]/P(T)$ such that $P'(T)$ is a is a unit in B_b, then $\phi_b\colon \operatorname{Spec} B_b \to \operatorname{Spec} A$ is standard étale.

Example 3.2.6. Consider $\operatorname{Spec} k[x,y]/(x^2 - y) \to \operatorname{Spec} k[y]$. This map is not étale at 0. To be explicit, note that $2x$ is not a unit in $k[x,y]/(x^2 - y)$, but it is a unit after localization at x, and thus $\operatorname{Spec} B_x \to \operatorname{Spec} k[y]$ is étale.

Theorem 3.2.7. *Every étale morphism is locally standard étale.*

The proof uses the following theorem:

Theorem 3.2.8 (Zariski's Main Theorem)**.** *Let $\phi\colon X \to Y$ be a quasi-finite morphism of finite type between Noetherian schemes. Then ϕ can be factored into $X \xrightarrow{f} X' \xrightarrow{g} Y$, where f is an open embedding and g is finite.*

Definition 3.2.9. A *quasi-finite* morphism is a morphism with finite fibers.

Remark 3.2.10. If $\phi\colon X \to Y$ is a finite-type morphism of schemes of finite type over l, then ϕ is étale if and only if $\widehat{\phi}_x\colon \widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ is an isomorphism. Geometrically, this means that the formal neighborhoods are isomorphic.

3.3 Étale fundamental group

Recall that if X is a nice topological space, there is an equivalence of categories

$$F\colon \{\text{Covering spaces of } X\} \longrightarrow \{\pi_1(X,x)\text{-sets}\}.$$

This functor is represented by the universal covering space $\widetilde{X} \to X$, and we have $\pi_1(X,x) = \mathrm{Aut}(\widetilde{X},\widetilde{X})$.

Now we want to make this work for schemes, so we take finite étale morphisms to be the covers. Then we define a functor

$$F\colon \{F \text{ étale over } X\} \to \mathsf{Set}$$

by $F(Y) = \mathrm{Hom}_X(\overline{x}, Y)$, where $\overline{X}$ is a geometric point of X. We see that F is not representable, but it is a projective limit of representables, so we define the étale fundamental group

$$\pi_1^{\mathrm{et}}(X,\overline{x}) = \varprojlim \mathrm{Aut}_X(X_i).$$

Example 3.3.1. Let $X = \operatorname{Spec} k$. Then $\pi_1(\operatorname{Spec} k, \overline{z}) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Similarly, $\pi_1(\operatorname{Spec}\mathbb{Z}) = \pi_1(\mathbb{A}^1_{\mathbb{C}}) = 1$.

If X is a variety over $\mathbb{C}$, then $\pi_1^{\mathrm{et}}(X) = \widehat{\pi_1(X^{\mathrm{an}})}$, where $\widehat{G}$ is the profinite completion. In particular, $\pi_1^{\mathrm{et}}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}) = \widehat{F}_2$. If $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$, then we have an exact sequence

$$1 \to \widehat{F}_2 \to \pi_1(X) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

4

Caleb (Nov 6): More on the étale fundamental group and fibered categories

Recall that étale is the same as flat and unramified. Consider the category of finite étale morphisms over X and choose a geometric point $\bar{x} \in X(\bar{k})$. Then consider the functor $F(Y) = \mathrm{Hom}_X(\bar{x}, Y)$. Unfortunately, the morphism $\operatorname{Spec} \bar{k} \to X$ is not étale, so this functor is not representable. If it was representable, it would be represented by the universal cover, which does not necessarily exist. We will now construct a Galois theory for schemes.

Definition 4.0.1. For a finite étale morphism $Y \to X$, define the *degree* $[Y : X] := |F(Y)|$. This is invariant of the choice of basepoint.

Proposition 4.0.2. *If Y is connected and we have a diagram*

$$\begin{tikzcd} Y \arrow[r, "\theta"] \arrow[dr, "\phi"] & Z \arrow[d] \\ & X \end{tikzcd}$$

then θ is determined by where it sends a single geometric point of Y. In particular, $|\mathrm{Aut}_X Y| \leq [Y : X]$.

Corollary 4.0.3. *There is a faithful action of $\mathrm{Aut}_X Y$ on $F(Y)$.*

Definition 4.0.4. A *Galois cover* $Y \to X$ is a finite étale morphism such that Y is connected and $|\mathrm{Aut}_X Y| = [Y : X]$. Equivalently, the action of $\mathrm{Aut}_Y X$ on $F(Y)$ is transitive.

Remark 4.0.5. A transitive action is the same as normality in both Galois theory and topological covering spaces.

Example 4.0.6. Let $X = \operatorname{Spec} K$. Then finite étale morphisms to X are schemes of the form $\bigcup \operatorname{Spec} L_i$ where L_i/K is finite separable. Then $\operatorname{Spec} L_i$ is Galois if and only if L_i/K is Galois. For a degree n extension, the action of $\mathrm{Aut}(L/K)$ on $F(\operatorname{Spec} L)$ is simply the action on the n embeddings $K \subset L \subset \overline{K}$.

Lemma 4.0.7. *For connected $Y \in \mathsf{F\acute{E}t}/X$, there exists Z Galois over X such that the diagram*

$$\begin{tikzcd} Z \arrow[r, two heads] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$

commutes. In fact, for any pair Y, Z, there exists a Galois W surjecting onto both.

4.1 Universal Cover

This universal cover is not a scheme, but is a projective limit of schemes. We will define the *universal cover* $\widetilde{X}$ of X with the following:

1. A poset I indexing Galois covers $X_i \to X$.
2. For any $X_i, X_j \in I$, there $k \in I$ such that $k > i$ and $k > j$.
3. If $i < j$, then we have $\phi_{ij} \colon X_j \to X_i$ compactible with geometric points and composition.

Example 4.1.1. Take all Galois covers and set $i < j$ if and only if there exists $X_j \to X_i$. To make everything work, we can adjust the map by an automorphism of X_i because the action of the Galois group is transitive.

Proposition 4.1.2. *Suppose Y is Galois over X and there is a finite étale map $Y \to Z$ over X. Then* $\operatorname{Aut}_Z Y \subset \operatorname{Aut}_X Y$ *is a subgroup and Z is Galois if and only if* $\operatorname{Aut}_Z Y$ *is a normal subgroup. In this case,* $\operatorname{Aut}_X Z \simeq \operatorname{Aut}_X Y / \operatorname{Aut}_Z Y$.

In particular, if Y, Z are both Galois, then we have a map $\operatorname{Aut}_X Y \to \operatorname{Aut}_X Z$.

Definition 4.1.3. Take $\widetilde{X}$ to be the universal cover of X. Then define the *étale fundamental group* by

$$\pi_1^{\text{ét}}(X, \overline{x}) := \varprojlim \operatorname{Aut}_X X_i.$$

Aternatively, recall the action of $\operatorname{Aut}_X(Y)$ on $F(Y)$. Then we obtain an action of $\pi_1(X, \overline{x})$ on

$$\operatorname{Hom}_X(\widetilde{X}, Y) := \varinjlim \operatorname{Hom}_X(X_i, Y) \simeq F(Y).$$

Proposition 4.1.4. *F is the direct limit of the functors* $\operatorname{Hom}_X(X_i, -)$.

Theorem 4.1.5. *The functor*

$$F \colon \mathsf{F\acute{E}t}/X \to \left\{ \begin{matrix} \textit{finite discrete} \\ \pi_1(X, \overline{x})\textit{-sets} \end{matrix} \right\}$$

is an equivalence of categories.

By Yoneda, we may define $\pi_1(X, \overline{x}) = \operatorname{Aut}(F)$. Here F is the "fiber functor."

1. Let $X = \operatorname{Spec} k$. Then if we take Galois coverings L_i/K whose union is k^{sep}. Then we have

   $$\pi_1^{\text{ét}}(\operatorname{Spec} k) = \varprojlim \operatorname{Gal}(L_i/K) = \operatorname{Gal}(k^{\text{sep}}/k).$$

2. Now consider a normal scheme X with function field $K(X)$. Then $K(X)^{\text{un}}$ is the composition of all finite extensions $K(Y)$ such that $Y \to X$ unramified. Then we have

   $$\pi_1^{\text{ét}}(X) \cong \operatorname{Gal}(K(X)^{\text{un}}/K(X)).$$

3. Because $\mathbb{Q}$ has no unramified extensions, we have $\pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{Z}) = 1$. By local class field theory, for any number field K, we have

   $$\pi_1^{\text{ab}}(\operatorname{Spec} \mathcal{O}_K) \simeq \operatorname{Gal}(H^\dagger/K) \simeq \operatorname{Gal}(H/K) \simeq I_K,$$

   where I_K is the ideal class group, $H^\dagger$ is the maximum abelian extension unramified at finite primes (narrow Hilbert class field), and H is the Hilbert class field, which is unramified at all primes.

4. If $X = \mathbb{A}^1_{\mathbb{C}}$, then we can prove using some differential form that $\pi_1^{\text{ét}}(X) = 1$. In fact, all finite étale covers are isomorphisms. However, this is **not true** in positive characteristic.

5. Consider the nodal cubic $\operatorname{Spec} k[x,y]/(y^2 - x^3 - x^2)$. Then finite étale morphisms over X are of the form $\operatorname{Spec} R_n$ with $\operatorname{Aut}_X R_n \simeq \mathbb{Z}/n\mathbb{Z}$ and thus $\pi_1^{\text{ét}}(X) = \widehat{\mathbb{Z}}$.

6. Let X_k be finite over k with $X_{\overline{k}}$ connected. Then we have an exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \xrightarrow{i} \pi_1(X_k) \xrightarrow{j} \operatorname{Gal}(k^{\text{sep}}/k) \to 1.$$

To prove this, first note that the composition of the two middle maps is trivial because the map $X_{\overline{k}} \to \operatorname{Spec} k$ factors through $\operatorname{Spec} \overline{k}$ (by definition of base change). Then because $X_{\overline{k}}$ is connected, we see that j is surjective by using the fact that for

$$\begin{tikzcd} X_L \arrow[r] \arrow[d] & X_k \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

we have $\operatorname{Aut}_{X_k}(X_L) \simeq \operatorname{Gal}(L/k)$. Finally, we use some magic to show that i is injective.

5

Caleb (Nov 13): Fibered categories

5.1 Review of Fibered Categories

Here is a motivating example for fibered categories. Recall that $F\colon \mathsf{Sch}/S^{\mathrm{op}} \to \mathsf{Set}$ sending a scheme X to the set of isomorphism classes of elliptic curves over X is not representable. Instead, we will replace this by a fibered category

$$p\colon M_{1,1} \to \mathsf{Sch}/S$$

with objects $(X, (E, e))$ being a scheme and an elliptic curve over it and morphisms are (f, g) such that

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

is a Cartesian diagram. This means that $E' \simeq E \times_X X'$. In particular, when $X = X'$, the morphisms are the automorphisms.

Recall the definition of a Cartesian arrow and a fibered category from the second lecture.

Example 5.1.1. Suppose a category C has fiber products. Then the arrow category of C is fibered over C under $(X \to Y) \mapsto Y$.

Definition 5.1.2. In a fibered category $F \to U$, the fiber $F(U)$ is the subcategory mapping down to U.

Recall that a cleavage is a class of Cartesian arrows such that there is one for every $U \to V$ and $\eta \in F(U)$. This does not give a functor from C to Cat, but it does give a pseudofunctor.

Example 5.1.3. Recall that a group is a category with one object. If $G \to H$ is a surjective morphism, then we can think about G as being fibered over H. Then the sequence $G \to H \to 1$ being split is equivalent to the existence of a split cleavage.

5.2 2-Yoneda Lemma

Recall that if $F\colon C^{\mathrm{op}} \to \mathsf{Set}$ is a functor, then $\mathrm{Hom}(h_X, F) \simeq F(X)$. This is the ordinary Yoneda lemma, and it tells us that we have an embedding $C \to \mathrm{Hom}(C^{\mathrm{op}}, \mathsf{Set})$. Now the 2-Yoneda lemma embeds $\mathrm{Hom}(C^{\mathrm{op}}, \mathsf{Set})$ into the 2-category of fibered categories over C.

Given $\Phi\colon C^{\mathrm{op}} \to \mathsf{Set}$, we will construct a fibered category $F_\Phi \to C$. The objects are pairs (U, ξ) where $U \in C$ and $\xi \in \Phi(U)$. This maps down to U. The morphisms $(U, \xi) \to (V, \eta)$ are maps $U \to V$ such that $\Phi_f(\eta) = \xi$.

If $X \in C$ is an object, then h_X is sent to the category C/X over X.

Lemma 5.2.1 (Weak 2-Yoneda)**.** *For $X, Y \in C$, we have* $\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(C/X, C/Y) \simeq \mathrm{Hom}(h_Y, h_X)$.

Lemma 5.2.2 (2-Yoneda lemma)**.** *Let $F \to C$ be a fibered category. Then* $\mathrm{Hom}(C/X, F) \simeq F(X)$.

5.3 Categories fibered in sets and groupoids

All moduli problems will be fibered in groupoids, and when they are representable by schemes, this is when the functor is fibered in sets.

Definition 5.3.1. A category is fibered in sets if $F(U)$ is a set, which means there are only identity morphisms.

Now recall that we have embeddings

$$C \hookrightarrow \mathrm{Hom}(C^{\mathrm{op}}, \mathsf{Set}) \hookrightarrow \{\text{Fibered categories over } C\} \qquad X \mapsto h_X \mapsto C/X.$$

Theorem 5.3.2. *The functor from presheaves to categories fibered in sets is an equivalence.*

Definition 5.3.3. A category is fibered in groupoids if $F(U)$ is a groupoid.

Definition 5.3.4. Recall the Grassmannian $\mathrm{Gr}(n, k)$ represents the functor given by

$$X \mapsto \{(X/S, q\colon \mathcal{O}_S^n \to f_*Q)\}$$

where Q is a quotient bundle over X that is free of rank $n - k$.

6

Caleb (Nov 20): Stacks

6.1 Pursuing Stacks

We have been discussing fibered categories, and now we will be discussing fibered categories over a site. Our slogan to keep in mind is that a stack is a fibered category over a site with descent. Alternatively, a stack is a 2-sheaf, or a sheaf taking values in categories.

Remark 6.1.1. Algebraic stacks are very special kinds of stacks and we have the following analogy: algebraic stacks are to stacks as schemes are to sheaves.

Example 6.1.2. The basic prototype of a stack is the arrow category of Top. Here, this is a fibered category with fiber $F(U) = \mathsf{Top}/U$. Next, it is easy to see that we can glue functions locally, and formally, this means that $\mathrm{Hom}_S(X, Y)$ is a sheaf. We can also construct spaces locally, and this is a cocycle condition. Given an open cover $\{U_i\}$ of U and maps $f_i \colon X_i \to U_i$ with transition maps $\phi_{ij} \colon f_j^{-1} U_{ij} \to f_i^{-1} U_{ij}$ with $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$, then there exists a space $f \colon X \to U$ such that there exist isomorphisms $f_i^{-1}(U_i) \simeq X_i$ and $\phi_{ij} = \phi_i \circ \phi_j^{-1}$.

Definition 6.1.3. We now define an *object with descent data*. This is a collection $(\{\varepsilon_i\}, \{\phi_{ij}\})$ for a cover $\{U_i \to U\}$ such that $\varepsilon_i \in F(U_i)$ and $\phi_{ij} \colon \mathrm{pr}_2^*(\varepsilon_j) \sim \mathrm{pr}_1^* \varepsilon_i$.

Remark 6.1.4. The choices of $\mathrm{pr}_1^*, \mathrm{pr}_2^*$ depend on the cleavage chosen. This can also be phrased in terms of sieves, where
$$\mathrm{pr}_{13}^* \phi_{ik} = \mathrm{pr}_{12}^* \phi_{ij} \circ \mathrm{pr}_{23}^* \phi_{jk}.$$
Here, we have the diagram

$$\begin{tikzcd}
 & U_{ijk} \arrow[rr, "\mathrm{pr}_{23}"] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dd, "\mathrm{pr}_{13}"] & & U_{jk} \arrow[dl] \arrow[dd] \\
U_{ij} \arrow[rr] \arrow[dd] & & U_j \arrow[dd] & \\
 & U_{ik} \arrow[rr] \arrow[dl] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & U &
\end{tikzcd}$$

Given a covering $\{U_i \to U\}$, define the category $F(\{U_i \to U\})$ with the following data:

Objects: These are simply objects with descent data.

Morphisms: These are morphisms $\alpha_i \colon \varepsilon_i \to \eta_i$ such that the diagram

$$\begin{tikzcd} \mathrm{pr}_2^*\varepsilon_j \arrow[r, "\mathrm{pr}_2^*\alpha_j"] \arrow[d, "\phi_{ij}"] & \mathrm{pr}_2^*\eta_j \arrow[d, "\psi_{ij}"] \\ \mathrm{pr}_1^*\varepsilon_i \arrow[r, "\mathrm{pr}_1^*\alpha_1"] & \mathrm{pr}_1^*\eta_i \end{tikzcd}$$

commutes.

Definition 6.1.5. A functor F is a *prestack* if for each covering $\{U_i \to U\}$ the functor $F(U) \to F(\{U_i \to U\})$ is fully faithful. If this functor is an equivalence, then F is a *stack*.

Note that this is a sheaf-like condition.

Definition 6.1.6. An object with descent data is *effective* if it is isomorphic to the image of an object in $F(U)$.

Thus by definition, stacks are prestacks such that all objects with descent data are effective.

Example 6.1.7. Let C be a site and $F \colon C^{\mathrm{op}} \to \mathsf{Set}$ a presheaf. Recall that F is a fibered category in Set. Then F is a sheaf if and only if F is a stack.

We now give some examples of stacks.

1. Let C be a site. Then $\mathsf{Sh}/C \to C$ is a stack. This example is called a *topos*. This was apparently one of Grothendieck's most profound ideas.

2. Let S be a scheme. Then the category $\mathsf{QCoh}/S \to \mathsf{Sch}/S$ is a stack in the fpqc topology. Apparently this is important.

3. The category $\mathsf{Aff}/S \to \mathsf{Sch}/S$ of affine arrows over S is a stack in the fpqc topology.

4. Because the speaker has been traumatized by combinatorics, we give the following: Let T be the category of trees with morphisms the embeddings and coverings the jointly surjective ones.[1] Then fiber products are just intersections. We will define $F(T)$ to be the category of proper colorings of T.

   At risk of further infecting these notes with combinatorics, consider the chromatic symmetric function
   $$X_T = \sum_{\substack{\text{proper} \\ \text{colorings}}} x_i^{\#(i \text{ appears})}.$$

   Then the conjecture is that X_T defines T. It is easy to see that the category defined above is a stack.

[1] I'm sure that Doron Zeilberger will love this.

7

Caleb (Dec 04): Jacobians I

7.1 Divisors and Picard Groups

Let X be Noetherian, integral, separated, and regular in codimension 1.

Definition 7.1.1. The group $\operatorname{Div} X$ of *WEil Divisors* is the group generated by codimension 1 closed integral subschemes.

Because X is integral, it has a function field K. If we choose $f \in K^*$, we can choose $Y \subset X$ of codimension 1. Then at η_Y, the stalk $\mathcal{O}_{\eta_Y}$ is a DVR with fraction field K. Then we can write $(f) = \sum_Y v_Y(f)Y$. This is a finite sum by the Noetherian condition. Divisors of this form are called *principal*.

Example 7.1.2. Let $X = \mathbb{P}^1$. Then Weil divisors of $\mathbb{P}^1$ are sums $\sum n_i P_i$, where P_i is a point. If $P_i = [2 : 1]$, then $V_{P_i}(f)$ is the order of the zero (or pole) at 2.

Definition 7.1.3. The *Class group* $\operatorname{Cl} X$ is the group $\operatorname{Div} X/K^*$, where we identify K^* with the principal divisors.

Definition 7.1.4. The principal divisors of $\mathbb{P}^1$ is all sums $\sum n_i P_i$ where $\sum n_i = 0$. Thus $\operatorname{Cl} \mathbb{P}^1 \simeq \mathbb{Z}$.

Now let X/k be a projective curve over an algebraically closed field k.

Definition 7.1.5. We define the *degree* of $\sum n_i P_i$ to be $\sum n_i [k(P_i) : k] = \sum n_i$.

Now if $f \in K(X)^*$, we know that $(f) = \varphi^*(\{0\} - \{\infty\})$ and thus $\deg(f) = 0$. Therefore we have a map $\operatorname{Cl} X \twoheadrightarrow \mathbb{Z}$, and the kernel is $\operatorname{Cl}^0 X$.

Example 7.1.6. Define $X = \operatorname{Proj} k[x, y, z]/(y^2 z - x^3 + zx^2)$. We will show that $\operatorname{Cl}^0 X$ is in bijection with the set of closed points. In one direction, let $p_0 = [0 : 1 : 0]$. Then $f(p) = p - p_0$. Then the line $(z = 0)$ intersects C with multiplicity 3 at p_0. Thus if $L \cap C = \{P, Q, R\}$, then $(L/Z) = P + Q + R - 3P_0$. On the other hand, if D is a degree 0 divisor, we can eventually replace it by $P_i - P_0$ because C is not rational.

Definition 7.1.7. The *Picard group* $\operatorname{Pic} X$ is defined to be the set of isomorphism classes of line bundles on X with group operation the tensor product.

If X is locally factorial, then $\operatorname{Cl} X \simeq \operatorname{Pic} X$.

Example 7.1.8. Let $X = \operatorname{Spec}\mathbb{Z}[\sqrt{-5}]$. This is not a UFD, so its class group is nontrivial. The line bundles are fractional ideals and we have $\operatorname{Pic} X \simeq C_2$ with generator $I := (2, 1+\sqrt{5})$. Over $D(2)$, we have $I = 1$ and over $D(3)$, I becomes generated by $1 + \sqrt{-5}$.

7.2 Jacobian Functor

Now we will always assume X is a smooth curve. Then we can define the *degree* of an invertible sheaf $L = \sum n_i [k(p_i) : k]$ by translation to Weil divisor language. Next, $\chi(C, L^n) = n \deg L + 1 - g$.

Definition 7.2.1. Define the functor $F(T)$ by

$$T \mapsto \{\text{line bundles on } C \times_k T \mid \deg 0 \text{ fiberwise}\}.$$

However, we don't want line bundles on T interfering, so we now define

$$P_C^0(T) = F(T)/q^* \operatorname{Pic} T.$$

Unfortunately, this is still not necessarily representable.

Theorem 7.2.2. *There exists an abelian variety J over k and a morphism of functors $f\colon P_C^0 \to J$ that is an isomorphism if $C(T)$ is nonempty. In particular, if $T = \operatorname{Spec} k$ and C has a rational point, then J represents P_C^0.*

Now let $G = \operatorname{Gal}(K'/K)$. We need the map $P_C^0(\operatorname{Spec} k) \to P_C^0(\operatorname{Spec} K')^G$ to be bijective for P_C^0 to be representable.

Proposition 7.2.3. *There exists an exact sequence*

$$0 \to \operatorname{Pic}(X) \to \operatorname{Pic}(X_{K'})^G \to \operatorname{Br}(k'/k) = H^2(G, k^*)$$

where $\operatorname{Br}(k'/k)$ *is the Brauer group.*

The idea of the proof is that to show injectivity, let $L, L' \in \operatorname{Pic} C$. If they become isomorphisc over k', we show the isomorphism descends to $\operatorname{Pic} X$. Choose $\sigma \in G$. If $i\colon L_{k'} \to L'_{k'}$ is an isomorphism, then we can apply σ to get another isomorphism between the two. This differs from i by $c_\sigma \in \operatorname{Aut}(L_{k'}) = (k')^\times$, and then $c_\sigma \in H^1(G, (k')^\times) = 0$ is a coboundary by Hilbert Theorem 90. Thus $c_0 = \frac{\sigma(\alpha)}{\alpha}$ for some $\alpha \in k'$. The upshow now is that $\alpha^{-1} i$ is Galois invariant, so this descends to an isomorphism $L \to L'$ over k.

Now suppose $L \in \operatorname{Pic}(X_{k'})^G$ that does not come from $\operatorname{Pic} X$. Here, for $\sigma, \tau \in G$, we have $i_{\tau\sigma} = c_{\tau\sigma}(\sigma^* i_\tau \circ i_\sigma)$, and thus $c_{\sigma\tau} \in H^2(G, (k')^\times)$.

Proposition 7.2.4. *Suppose $X(k) \neq \emptyset$, or equivalently X has a rational point. Then P_C^0 is an étale sheaf and it is representable.*

7.3 Relation to Complex Analysis

Consider $H^0(C, \Omega^1)$, which has dimension g. We have an integration pairing $H_1(C, \mathbb{Z}) \to H^0(C, \Omega^1)^\vee$. The quotient of this map is $J^{\text{an}} = H^0(C, \Omega^1)^\vee / H_1(C, \mathbb{Z})$ is a torus. Via Poincare duality, we have a pairing $H_1(C, \mathbb{Z}) \otimes H_1(C, \mathbb{Z}) \to \mathbb{Z}$ satisfying the Riemann bilinear equations.

These conditions are called a Riemann form on J^{an} and give it the structure of an abelian variety. Now we have a map $f\colon H^0(J, \Omega^1_J) \to H^0(C, \Omega^1_C)$. This gives a map

$$H^0(C, \Omega^1)^{\vee} \to H^0(J, \Omega^1)^{\vee} \simeq T_0 J \xrightarrow{\exp} J^{\mathrm{an}}$$

with kernel $H_1(C, \mathbb{Z})$ by Abel's theorem and Jacobi inversion. This gives an isomorphism between the algebraic and analytic constructions.

8

Caleb (Dec 11): Jacobians II

8.1 Construction of the Jacobian

Let X be a curve over k. Recall the functor

$$G(T) = \{L \in \operatorname{Pic}(X \times T) \mid \deg L_t = 0\}/q^* \operatorname{Pic} T.$$

If X has a k-point, then it is representable. Assume $x \in X(k)$. We will now construct the Jacobian representing the functor G. Recall that for an elliptic curve over $\mathbb{C}$, we have $\operatorname{Jac}(X) \cong X$. For a higher genus curve X, the Jacobian is birational to $X^{(g)} = X^g/S_g$.

We will use the correspondence between k'-points of $X^{(g)}$ and effective Cartier divisors of $X_{k'}$ of degree g. Here, a *Cartier divisor* is an element $\alpha \in \Gamma(X, K_x^*/\mathcal{O}_X)$ and a Cartier divisor is effective if we have charts U_i such that $\alpha_{u_i} \in \mathcal{O}_{U_i}$. For example, if we consider the Weil divisor $n[0]$, this is the Cartier divisor $(\mathbb{A}^1, x^n)$.

Recalling the Riemann-Roch formula $\ell(D) - \ell(K - D) = \deg D - g + 1$, chose effective divisors D, D' of degree g. Then $\ell(D + D' - g[x]) \geq 1$ and semicontinuity and an instruction gives that the set $U \subset X^{(g)} \times X^{(g)}$ such that equality holds is open and nonempty. Up to scaling, there exists a unique f such that $D'' = (f) + D + D' - g[x]$ is effective.

Now we may define the map $U \to X^{(g)}$ by $(D, D') \to D''$. This gives a rational map $X^{(g)} \times X^{(g)} \dashrightarrow X^{(g)}$. By general results of Weil, we can upgrade this to an abelian variety $X^{(g)} \to J$.

Theorem 8.1.1. *For any birational group V over k, there exists a unique group variety G over k and a birational $f \colon V \to G$ such that $f(ab) = f(a)f(b)$.*

In our case, we have $V = X^{(g)}$ and J is proper (or complete), so $X^{(g)} \to J$ is an honest morphism.

Theorem 8.1.2. *J represents the functor G.*

The idea of the proof is to define a map $f \colon \operatorname{Pic}^0(X) \to J$. Let $D \in \operatorname{Pic}^0(X)$ and suppose $D + g[x]$ is effective. Then if we consider the diagonal map $X \xrightarrow{\Delta} X^{(g)} \xrightarrow{h} J$, then we define $f(D) = h \circ \Delta(D + g[x])$. Otherwise, we can take D' such that $D + D' + g[x]$ and $D' + g[x]$ are effective, and we set $f(D) = f(D + D') - f(D')$.

8.2 Some results about the Jacobian

Here are some results about Jacobians. We are very far away from proving any of them.

Theorem 8.2.1. *For any abelian variety A over an infinitely field k, there exists a Jacobian J such that $J \twoheadrightarrow A$.*

Theorem 8.2.2 (Torelli)**.** *Let k be an algebraically closed field. Then a curve C over k is uniquely determined by its canonically polarized Jacobian (J, λ), where λ is a certain ample line bundle giving a map $J \to \widehat{J}$. Over $\mathbb{C}$, λ is given by the image of*

$$C^{g-1} \to J(C) \qquad (x_1, \ldots, x_{g-1}) \mapsto (\omega \mapsto \sum_i \int_q^{x_i} \omega).$$

The proof allegedly involves complicated combinatorial arguments. For arithmetic geometers, there is the Shafarevich conjecture:

Conjecture 8.2.3 (Shafarevich)**.** *Let K be a number field and S be a finite set of primes of K. Then there are finitely many isomorphism classes C/k of genus g with good reduction outside of S.*

Using various facts about Jacobians, this implies Faltings's theorem.

Theorem 8.2.4 (Faltings)**.** *Let X be a curve of genus $g \geq 2$ over $\mathbb{Q}$. Then X has finitely many rational points.*

Finally, there is a very roundabout way to prove the Riemann Hypothesis for curves. We can prove it for abelian varieties and then deduce it for a curve using the Jacobian.