Higher-genus GW theory of smooth CY hypersurfaces in weighted $\mathbb{P}^4$

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Goal: Say something about the behavior of the generating series of all $N_{g,d}$ when we fix the genus.

Our results

We are able to prove the following results for the threefolds $Z_6 \subset \mathbb{P}(1,1,1,1,2)$, $Z_8 \subset \mathbb{P}(1,1,1,1,4)$, and $Z_{10} \subset \mathbb{P}(1,1,1,2,5)$:

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1. If $F_g = \sum_d N_{g,d} Q^d$ is the generating series of genus g invariants, we prove that a normalized version P_g is a polynomial in five generators $A_1, B = B_1, B_2, B_3, X$ defined using only genus-zero data.

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2. More precisely, Coates-Corti-Lee-Tseng proved that the genus-0 GW theory of Z is controlled by the series

$$I(q,z) \coloneqq z \sum_{d \geq 0} q^d \frac{\prod_{m=1}^{6d}(6H + mz)}{\prod_{m=1}^{d}(H + mz)^4 \prod_{m=1}^{2d}(2H + mz)}$$
$$= I_0(q)z + I_1(q)H + I_2(q)\frac{H^2}{z} + I_3(q)\frac{H^3}{z^2}.$$

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3. Define $I_{11}(q) \coloneqq D\left(\frac{I_1(q)}{I_0(q)}\right)$, where $D = q\frac{\mathrm{d}}{\mathrm{d}q}$. Then define

$$A \coloneqq \frac{DI_{11}}{I_{11}}, \qquad B_k \coloneqq \frac{D^k I_0}{I_0}, \qquad X \coloneqq 1 - \frac{1}{1 - \frac{6^6}{2^2}q}.$$

Also define $Y \coloneqq 1 - X$.

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4. The first result, conjectured by Yamaguchi-Yau (2004), is that

$$P_{g,n} = \frac{(3Y)^{g-1} I_{11}^n}{I_0^{2g-2}} \left(Q \frac{\mathrm{d}}{\mathrm{d}Q}\right)^n F_g(Q) \in \mathbb{Q}[A, B, B_2, B_3, X]$$

after the substitution $Q = qe^{I_1/I_0}$.

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5. The second result is the equality

$$P_{1,1} = -\frac{1}{2}A - \frac{21}{2}B - \frac{1}{12}X - \frac{7}{4}.$$

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6. Consider the following Feynman rule (sum over *stable graphs*) defined by BCOV (1993) defining a power series $f_{g,m,n}$. First, introduce *propagators* $E_\psi = B$, $E_{\varphi\varphi} = A + 2B$, $E_{\varphi\psi} = -B_2$, and $E_{\psi\psi} = -B_3 + (B - X)B_2 - \frac{13}{36}BX$.

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7. Place $\varphi - E_\psi \psi$ at the first m vertices and ψ at the last n vertices, place

$$E_{\varphi\varphi}\varphi \otimes \varphi + E_{\varphi\psi}(\varphi \otimes \psi + \psi \otimes \varphi) + E_{\psi\psi}\psi \otimes \psi$$

at each edge, and place

$$\varphi^{\otimes m} \otimes \psi^{\otimes n} \mapsto P_{g,m,n} := \frac{(2g-2+m+n-1)!}{(2g-2+m-1)!} P_{g,m}$$

at each leg. Also, set $P_{1,0,1} = -\frac{19}{2}$.

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8. Our result is that the output $f_{g,m,n}$ is a polynomial of degree at most $3g-3+m$ in X. This implies the *modular anomaly equations*

$$-\partial_A P_g = \frac{1}{2}\left(P_{g-1,2} + \sum_{g_1+g_2=g} P_{g_1,1} P_{g_2,2}\right),$$
$$\left(-2\partial_A + \partial_B + (A+2B)\partial_{B_2} - \left((B-X)(A+2B) - B_2 - \frac{13}{36}X\right)\partial_{B_3}\right)P_g = 0.$$

Past work

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- Our genus-one formula was first proved for the quintic by Zinger (2007) and proved for complete intersections in projective space by Popa (2010).
- The modular anomaly equations were proved for the quintic independently by Chang-Guo-Li and Guo-Janda-Ruan in 2018.

Mixed-Spin-P fields

Our approach

We use the approach of Mixed-Spin-P fields, which were introduced by Chang-Li-Li-Liu (2015, 2016) and Chang-Guo-Li-Li (2018).

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The original geometric intuition was to implement the master space idea of Thaddeus, but in order to perform calculations we need to introduce a parameter N (which is a positive integer) to the theory.

MSP moduli

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$$\left[\begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & p & u_1 & \dots & u_N & v \\ \hline 1 & 1 & 1 & 1 & 2 & -6 & 1 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \end{array}\right].$$

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Let $W \coloneqq W_{g,n,\mathbf{d}}$ be the stack of commutative diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[drr, bend right, "\omega^{\log}"'] & {[\mathbb{C}^{N+7}/(\mathbb{C}^\times)^3]} \arrow[r] & B(\mathbb{C}^\times)^3 \arrow[d, "\mathrm{pr}_3"] \\ & & B\mathbb{C}^\times \end{tikzcd}$$

subject to a stability condition.

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Theorem (Chang-Kiem-Li, 2015)
Indexing the fixed loci by Γ*, we have*

$$[W]^{\mathrm{vir}} = \sum_{\Gamma} \frac{[W_\Gamma]^{\mathrm{vir}}}{e(N_\Gamma^{\mathrm{vir}})}.$$

Virtual localization II

Fixed loci in the MSP moduli space W are indexed by graphs Γ. There are three kinds (denoted by levels) of vertices:

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Definition
Define the MSP $[0, 1]$ theory by only considering the contributions of graphs without level ∞ vertices.

Calculations

Genus zero MSP theory

Lemma
Genus zero MSP invariants equal genus zero Gromov-Witten invariants of a degree 6 *hypersurface in* $\mathbb{P}(1, 1, 1, 1, 2, 1, \ldots, 1)$.

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This implies that all genus zero MSP invariants are recovered from the J-function

$$z\sum_{d\geq 0} q^d \frac{\prod_{m=1}^{6d}(6p+mz)}{\prod_{m=1}^{d}(p+mz)^4\prod_{m=1}^{2d}(2p+mz)\prod_{m=1}^{d}((p+mz)^N-t^N)}.$$

Here, we specialize the equivariant parameters to $t_\alpha = \zeta_N^\alpha t$ for $\alpha = 1,\ldots,N$.

MSP R-matrix

Define the R-matrix by the Birkhoff factorization

$$S^{\mathrm{MSP}}(z)\Delta = R(z)S^{\mathrm{loc}}(z),$$

where Δ comes from Quantum Riemann-Roch and S^{loc} is simply the direct sum of the S-matrix of Z and N copies of the S-matrix of a point.

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Theorem

The MSP $[0,1]$ *theory is a cohomological field theory given by the action*

$$R(z).\left(\Omega^Z \oplus \bigoplus_{\alpha=1}^{N} \Omega^{\mathrm{pt}_\alpha}\right)$$

of $R(z)$ *on the direct sum of the GW theory of* Z *and* N *copies of the GW theory of a point.*

Proof of polynomiality

Lemma
All entries of $R(z)$ *lie in* $\mathbb{Q}[A, B, B_2, B_3, X]$ *(possibly after normalization). There are also explicit degree bounds for the entries coming from the* N *points, which are polynomials in* X.

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Lemma
The MSP $[0,1]$ *correlator* $\langle p^{a_1}, \dots, p^{a_n} \rangle_{g,n}^{[0,1]}$ *is a polynomial in* q *of degree at most* $g - 1 + \frac{3g-3+\sum a_i}{N}$.

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Corollary

$$P_{g,n} = \frac{(3Y)^{g-1} I_{11}^n}{I_0^{2g-2}} \left(Q \frac{\mathrm{d}}{\mathrm{d}Q} \right)^n F_g(Q) \Big|_{Q = qe^{I_1/I_0}} \in \mathbb{Q}[A, B, B_2, B_3, X].$$

Feynman rule I

Edge contributions in the action of the MSP R-matrix look like the propagators $E_{\varphi\varphi}$, $E_{\varphi\psi}$, and $E_{\psi\psi}$. For example, the edge contribution between two level 0 vertices starts with

$$E_{\psi}(1 \otimes H^2 + H^2 \otimes 1) + \frac{1}{2}\left(E_{\varphi\varphi} + \frac{13}{36}X\right)H \otimes H$$

up to a prefactor.

Definition
Consider the factorization

$$R(z) = R^X(z)\begin{pmatrix} R^{\mathbf{A}}(z) & \\ & I_N \end{pmatrix},$$

where

$$R^{\mathbf{A}}(z)^{-1} = I - \begin{pmatrix} 0 & zE_\psi & z^2E_{\varphi\psi} & \cdots \\ & 0 & zE_{\varphi\varphi} & \cdots \\ & & 0 & zE_\psi \\ & & & 0 \end{pmatrix}.$$

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Corollary (MSP Feynman rule)
Let $f^{\mathbf{A}}_{g,m,n}$ *be the generating functions of the cohomological field theory*

$$R^{\mathbf{A}}(z).\Omega^Z.$$

Then $f^{\mathbf{A}}_{g,m,n}$ *is a polynomial in* X *of degree at most* $3g - 3 + m$.

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Corollary
We have the modular anomaly equations

$$-\partial_A P_g = \frac{1}{2}\left(P_{g-1,2} + \sum_{g_1+g_2=g} P_{g_1,1}P_{g_2,2}\right),$$

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Proof.
Differentiate the physics Feynman rule. □

Future work

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- Stay tuned in the future!