Math 718: Lie Algebras

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Abstract

The goal of the course is the classification of semisimple Lie algebras over the complex numbers and an introduction to their representation theory. The classification is essentially the same as that for simply-connected compact Lie groups, which are central objects in mathematics and physics. Topics covered include representations of sl(2), Jordan decomposition, structure theorems, Weyl groups, roots systems, Dynkin diagrams, complete reducibility, finite-dimensional representations and their characters formulas.

CONTENTS

Disclaimer: Any errors in these notes are mine and not the instructor's. In addition, these notes are picture-free (but will contain commutative diagrams) and may not be exactly the same as

the lectures (I tend to leave out boring computations, streamline certain things, and use category theory).

1   LECTURE 1 (SEP 3)

We will be using the book by Humphreys in this course, but we will not be following the book too closely. Homework will be approximately weekly, with the first due in two weeks.

1.1   Overview   First we will discuss symmetry. Recall the platonic structures and their dual structure. We can consider the finite reflection group that preserves the object, which leads us to the following question:

Question 1. *What are the finite reflection groups?*

In the case of Lie Algebras, we will need our groups to preserve integer lattices (root systems), so we specialize the question:

Question 2. *What are the finite reflection groups that can be represented by integer matrices?*

In two dimensions, only the dihedral group for the triangle, square, and hexagon are representable by integer matrices. We will reduce the classification of semisimple Lie algebras to the above question.

Intimately related to Lie algebras are Lie groups, which are just group objects in **Diff**. We will not need Lie groups in this course. However, it is worth mentioning that Lie algebras arise as the tangent space at the identity of a Lie group G. We can also pass to algebraic groups, which are just group objects in a category of varieties. The advantage of this approach is that we can work over arbitrary fields.

1.2   Basic Notions   We will now define Lie algebras. First recall that a k-algebra A is a k-vector space A with a bilinear map $A \otimes A \to A$. If A is unital, we call it central. If A is associative, then A is a monoid in $\mathbf{Vect}_k$.

Example 3. An example of an associative algebra is a matrix algebra $M_n(k)$.

Definition 4. A *Lie Algebra* $\mathfrak{g}$ is a k-algebra with a multiplication

$$[-,-] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Example 5. From any associative algebra A, we can build a Lie algebra using the commutator as the Lie bracket. In particular, $M_n(k)$ can be turned into a Lie algebra $\mathfrak{gl}_n$.

The notion of a morphism of Lie algebras is entirely analogous to the case of rings, so we can define Lie subalgebras.

Example 6. Some subalgebras of $\mathfrak{gl}_n$ are the set of diagonal matrices, which is abelian, the set of strictly upper triangular matrices, and the set of upper triangular matrices $\mathfrak{b}$.[1].

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[1]For Borel

Definition 7. A Lie algebra is *abelian* if $[-,-] = 0$.

Example 8. An important subalgbebra of $\mathfrak{gl}_n$ is the set $\mathfrak{sl}_n$ of matrices of trace 0.

Remark 9. $\mathfrak{sl}_n$ is what is called a *simple* Lie algebra whose Weyl group is S_n.

2   LECTURE 2 (SEP 5)

2.1   Derivations   First we discuss derivations.

Definition 10. Let A be a k-algebra. Then $\delta \in \operatorname{End} A$ is a derivation if

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all $a, b \in A$.

We will denote the derivations of $\mathfrak{g}$ by $\operatorname{Der} \mathfrak{g}$.

Remark 11. $\operatorname{Der} \mathfrak{g}$ is a Lie subalgebra of $\operatorname{End} \mathfrak{g}$.

The Jacobi identity is related to derivations. Recall the adjoint representation of a Lie algebra $\mathfrak{g}$.

Definition 12. For $x \in \mathfrak{g}$, write ad_x for the *adjoint map*

$$\operatorname{ad}_x : y \mapsto [x, y].$$

Now observe that

$$\begin{aligned}
[x, [y, z]] &= -[y, [x, z]] - [z, [x, y]] \\
&= -[z, [x, y]] - [y, [z, x]] \\
&= [[x, y], z] + [y, [x, z]],
\end{aligned}$$

so we see that ad_x is a derivation.

2.2   Basic Examples   We will now expand our set of examples of Lie algebras.

Example 13. The *orthogonal group* $O(n)$ is the set of matrices that preserve a Hermitian inner product. Equivalently, $O(n)$ is defined by the equation $X^t X = I$.

Remark 14. On the Lie group side, we may define subgroups of GL_n via $X^t J X = J$ for invertible matrices J. For Lie algebras, this differentiates to $X^t J + J X = 0$.

Example 15. For $k = \mathbb{C}$, the orthogonal Lie algebra is denoted by $\mathfrak{o}(n)$ and is given by $J = I$. We may also define the symplectic Lie algebra $\mathfrak{sp}(2n)$ by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Remark 16. The orthogonal groups behave differently in even and ood dimension. In fact, they are given by different classes of Dynkin diagrams.

Remark 17. The four infinite families of simple Lie algebras are $\mathfrak{sp}_{2n}, \mathfrak{o}_{2n}, \mathfrak{o}_{2n+1}, \mathfrak{sl}_n$. There are five other exceptional simple Lie algebras.

2.3　More Basic Notions　We will now consider the basic structure of a Lie algebra.

Definition 18. The *derived subalgebra* $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is the span of all elements of the form $[x, y]$.

This is clearly a subalgebra of $\mathfrak{g}$, but in fact it is an ideal.

Definition 19. An *ideal I* of $\mathfrak{g}$ is a subalgebra satisfying $[\mathfrak{g}, I] \subset I$.

Example 20. The center of a Lie algebra is an ideal. To see this, note that $[A, X] = 0$ for all $A \in \mathfrak{g}$ and $X \in Z(\mathfrak{g})$. Also, it is easy to see that the derived subalgebra is an ideal.

Definition 21. A Lie algebra $\mathfrak{g}$ is *nilpotent* if the lower central series

$$\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]], \ldots$$

eventually becomes 0.

Example 22. Clearly all abelian Lie algebras are nilpotent because $[\mathfrak{g}, \mathfrak{g}] = 0$.

Definition 23. A Lie algebra $\mathfrak{g}$ is *solvable* if the upper central series

$$\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], \ldots$$

eventually becomes 0.

Example 24. We note that all nilpotent Lie algebras are solvable because the upper central series is contained in the lower central series.

Example 25. The set of all upper triangular matrices is solvable, and the set of strictly upper triangular matrices is nilpotent.

Definition 26. The *radical* $\operatorname{Rad} \mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$.

Definition 27. A Lie algebra $\mathfrak{g}$ is *semisimple* if $\operatorname{Rad} \mathfrak{g} = 0$.

Definition 28. A Lie algebra $\mathfrak{g}$ is *simple* if it has no nontrivial ideals and $\dim_k \mathfrak{g} \neq 1$.

Example 29. The simplest example of a simple Lie algebra is $\mathfrak{sl}_2$.

Homomorphisms and quotients are defined entirely analogously to the case of rings or groups. Checking this is well-defined is left to the reader.

Remark 30. The first isomorphism theorem holds for Lie algebras.

Recall that a representation of a finite group G is a morphism $G \to GL(V)$ for some vector space V. Alternately, this is a functor $\mathcal{B}G \to \mathbf{Vect}_k$.

Definition 31. A *representation* of a Lie algebra $\mathfrak{g}$ is a morphism $\mathfrak{g} \to \operatorname{End} V$ for some vector space V. Alternatively, we can define this as a $\mathfrak{g}$-module.[2]

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[2]Better, we can define this as a representation of the universal enveloping algebra, which allows us to define representations as functors $U_{\mathfrak{g}} \to \mathbf{Vect}_k$.

3   Lecture 3 (Sep 10)

Recall the adjoint representation of a Lie algebra $\mathfrak{g}$. This is a very important representation. For a simple Lie algebra, ad is an isomorphism onto its image. In general, we will write $\operatorname{ad}\mathfrak{g}$ for the image of the adjoint map. In general, the kernel of ad is the center.

Remark 32. When we define the E_8 algebra, which has dimension 248, we will have a 248-dimensional subalgebra of a 248^2-dimensional Lie algebra.

Remark 33. From now on, we will only consider finite-dimensional Lie algebras.

3.1   Nilpotent Lie Algebras

Notation 34. We will denote the set of all strictly upper triangular matrices by $\mathfrak{n}_n(k)$.

Eric then proceeded to spend time computing products and Lie brackets of matrices of the form e_{ij}.[3] It is then clear that $\mathfrak{n}$ is nilpotent because bracketing shifts nonzero entries towards the top right corner.

We now discuss some properties of nilpotent Lie algebras.

Proposition 35. *The following hold for nonzero nilpotent Lie algebras:*

1. *Subalgebras and quotients of nilpotent Lie algebras are nilpotent.*

2. *If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is $\mathfrak{g}$.*

3. *If $\mathfrak{g}$ is nilpotent, it has nonzero center.*

Proof of these is easy and is omitted.

Remark 36. The adjoint representation of a nilpotent Lie algebra consists entirely of nilpotent matrices.

Theorem 37 (Engel)**.** *If ad_x is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is a nilpotent Lie algebra.*

First, we need the following lemma:

Lemma 38. *Suppose $\mathfrak{g} \subset \mathfrak{gl}(V)$ consists of nilpotent endomorphisms. Then there exists $v \in V \smallsetminus 0$ such that $zv = 0$ for all $z \in \mathfrak{g}$.*

Proof of Engel's Theorem. Take the adjoint representation of $\mathfrak{g}$. By hypothesis, this consists of nilpotent endomorphisms. By Lemma 38, then we have a common eigenvector, which is in the center. Thus $\operatorname{ad}\mathfrak{g}$ has smaller dimension than $\mathfrak{g}$.

To complete the proof with induction, we need to show that nilpotent matrices are sent to milpotent matrices by ad. However, this is clear from the definition of commutator and nilpotence of the original matrix. Now, by induction, $\operatorname{ad}\mathfrak{g}$ is a nilpotent Lie algebra. Because $\operatorname{ad}\mathfrak{g} \simeq \mathfrak{g}/Z(\mathfrak{g})$, we use part 2 of Proposition 35 to conclude that $\mathfrak{g}$ is nilpotent. $\square$

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[3]This demonstrates the axiom that mathematicians are terrible at computing things.

Proof of Lemma 38. If $K \subset \mathfrak{g}$ is a proper maximal subalgebra, then the adjoint representation of K descends to $\mathfrak{g}/K$. Then $\operatorname{ad} z$ is nilpotent for $z \in K$. Therefore the image of K in $\mathfrak{gl}(\mathfrak{g}/K)$ consists of nilpotent endomorphisms. By induction on dimension, there is a common engenvector w for K.

Now consider $w = l + K$. Then $K + k \cdot l$ must generate all of $\mathfrak{g}$ with K an ideal of $\mathfrak{g}$. Viewing $K \subset \mathfrak{g} \subset \mathfrak{gl}(V)$, induction gives us an eigenvector in V for K. Define W to be the subspace spanned by the eigenvectors of K. Then it is easy to check that l preserves W. Now choose $v \in W$ an eigenvector of l, and then $K + k \cdot l$ annihlates v. Thus L annihlates v.   □

4   Lecture 4 (Sep 12)

We began class (30 minutes) by finishing the class of Lemma 38 from last time.

4.1   More Nilpotent Lie Algebras

Corollary 39. *Let $L \subset \mathfrak{gl}(V)$ consist of nilpotent endomorphisms. Then there exists a basis of V such that L is strictly upper triangular in this basis.*[4]

Proof. By Lemma 38, there exists a nonzero $v_1 \in V$ such that L annihlates v_1. Then L is nilpotent on $V/\langle v_1 \rangle$. Now induct.   □

Corollary 40. *If L is a nilpotent Lie algebra, and $0 \neq K \subset L$ is an ideal, then $K \cap Z(L) \neq 0$.*

4.2   Solvable and Semisimple Lie Algebras
We now discuss solvable Lie algebras. We know already that nilpotent implies solvable. Recall that $\operatorname{Rad} L$ is defined to be the maximal solvable ideal of L. This is unique and contains all solvable ideals.

Definition 41. L is semisimple if $\operatorname{Rad} L = 0$.[5]

Proposition 42. *Simple implies semisimple.*

Proof. $[L, L] = L$ because L is simple, so L cannot be solvable. Therefore there are no nonzero solvable ideals.   □

Now we will define a symmetric bilinear form $\kappa : S^2(L) \to k$ given by

$$\kappa(X, Y) = \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y)$$

known as the *Killing form*.[6]

From now on, we will work in an algebraically closed field of characteristic 0.

Theorem 43 (Killing). *L is semisimple if and only if κ is non-degenerate.*

Theorem 44 (Lie). *If $L \subset \mathfrak{gl}(V)$ is solvable, then there exists a common eigenvector for L.*

We concluded class with a linear algebra review. Specifically, observe the following:

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[4]This is the same as L fixing a flag $\mathcal{F}$. In fact, $L.F_i \subset F_{i-1}$.

[5]These are the main object of study in this course. Also, these will turn out to be sums of simple objects.

[6]This actually turns out to be a multiple of the normal trace form for the classical Lie algebras.

1. Diagonalizable is equivalent to semisimple;

2. If $[x, y] = 0$, then:

    - x, y nilpotent implies $x + y$ nilpotent;

    - x, y semisimple implies $x + y$ semisimple.

3. If $W \subset V$ and $x(W) \subset W$, then:

    - $x|_W$ is semisimple if x is semisimple.

    - $x|_W$ is nilpotent if x is nilpotent.

5   LECTURE 5 (SEP 17)

5.1   Jordan Decomposition Class began with a discussion of the Jordan canonical form. In particular, recall:

Proposition 45. $x \in \operatorname{End} V$ can be written $x = x_s + x_n$ where x_s is semisimple and x_n is nilpotent, and $[x_s, x_n] = 0$. In particular, they are both polynomials in x with constant term 0. Finally, $x(B) \subset A$ if and only if $x_s(B), x_n(B) \subset A$ for subspaces $A \subset B \subset V$.

Proof of this can be found in any standard algebra text. In addition, the decomposition is unique.

Now we would like to have the notion of Jordan decomposition in an arbitrary Lie algebra. This only has nice properties for semisimpe Lie algebras. In particular, we want:

1. If $L \subset \mathfrak{gl}(V)$, then the Jordan decomposition agrees with that of $\mathfrak{gl}(V)$.

2. The Jordan decomposition respects morphisms.

Definition 46. For a semisimple Lie algebra, an *abstract Jordan decomposition* is a Jordan decomposition of $\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$

Corollary 47. Let $x \in \mathfrak{gl}(V)$ with $x = x_s + x_n$. Then $(\operatorname{ad} x)_s = \operatorname{ad} x_s$ and $(\operatorname{ad} x)_n = \operatorname{ad} x_n$.

Proof. Note that if x is nilpotent, then $\operatorname{ad} x$ is also nilpotent. In addition, if x is semisimple, choose a basis of V such that x is diagonal. Then it is easy to see that $\operatorname{ad} x$ is semisimple. Finally, note that because x_s, x_n commute, so do their adjoint operators. $\square$

Recall the symmetric bilinear form $\langle -, - \rangle : M_n \otimes M_n \to \mathbb{F}$ given by $A \otimes B \mapsto \operatorname{tr}(AB)$. We can verify that this is associative with respect to the Lie bracket:

$$\langle [x.y], z \rangle = \langle x, [y, z] \rangle.$$

Also, it is easy to verify that this bilinear form is nondegenerate. Now recall the Killing form on L given by

$$L \otimes L \xrightarrow{\operatorname{ad} \otimes \operatorname{ad}} \mathfrak{gl}(L) \otimes \mathfrak{gl}(L) \xrightarrow{\langle -, - \rangle} \mathbb{F}.$$

This is an associative symmetric bilinear form, but in general, it is degenerate.

Next time, we will show that

Theorem 48. L is semisimple if and only if the Killing form is nondegenerate.

6   LECTURE 6 (SEP 19)

6.1   Proof of Theorem 48   Today we begin with the proof of Theorem 48.

Lemma 49. *Let $A \subset B$ be subspaces of $\mathfrak{gl}(V)$. Define*

$$M \coloneqq \{x \in \mathfrak{gl}(V) \mid [x, B] \subset A\}.$$

Suppose $x \in M$ satisfies $\operatorname{tr}(xy) = 0$ for all $y \in M$. Then x is nilpotent.

Proof of this fact is in Humphreys and uses the Jordan canonical form and the fact that our field has characteristic 0.

Theorem 50 (Cartan's Criterion)**.** *Let $L \subset \mathfrak{gl}(V)$. Suppose $\operatorname{tr}(xy) = 0$ for all $x \in [L, L], y \in L$. Then L is solvable.*

Proof. Note that it is enough to show that $x \in [L, L]$ is nilpotent. We will apply Lemma 49 to $A = [L, L], B = L$. Note that $L \subset M$. We know that $[L, L]$ is generated by $[y, z]$ for all $y, z \in L$. Let $m \in M$. We know that
$$\operatorname{tr}([y, z]m) = \operatorname{tr}(y[z, m]) = \operatorname{tr}([z, m]y) = 0$$
by hypothesis. Because $[y, z]$ is a generator of $[L, L]$, we know that $\operatorname{tr}(xm) = 0$ for all $x \in [L, L], m \in M$. By Lemma 49, x is nilpotent. $\qquad\square$

Corollary 51. *Let L be a Lie algebra such that $\operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y) = 0$ for all $x \in [L, L], y \in L$. Then L is solvable.*

Proof. $\operatorname{ad} L$ is solvable by the theorem. Then L is solvable because $L \simeq L/Z(L)$. $\qquad\square$

Lemma 52. *Let I be an ideal of L and κ_I be the Killing form for I as a Lie algebra. Then $\kappa_I = \kappa|_{I \otimes I}$.*

Proof of this amounts to the fact that the product respects the block form. To review some linear algebra, given a symmetric bilinear form $\beta : V \otimes V \to \mathbb{F}$, the radical is the kernel of the induced map $x \mapsto \beta(x, -)$. Then β is nondegenerate if the radical is trivial. Then note that if β is a symmetric bilinear associative form on a Lie algbera, the radical is an ideal.

Proof of Theorem 48. Let S be the radical of κ. Then it is easy to see that S is solvable by Corollary 51. Thus $S \subset \operatorname{Rad} L$. Thus if L is semisimple, $S = 0$.

In the other direction, note that being semisimple is equivalent to having no nonzero abelian ideals. Let I be a nonzero abelian ideal of L. Choose $x \in I, y \in L$. Then $L \xrightarrow{\operatorname{ad} y} L \xrightarrow{\operatorname{ad} x} I \xrightarrow{\operatorname{ad} y} I \xrightarrow{\operatorname{ad} x} 0$. Thus $\operatorname{ad}_y \operatorname{ad}_x$ is nilpotent, so it has trace 0. Thus $x \in S$ and thus $I \subset S$. In particular, if κ is nondegenerate, then $S = 0$, so every abelian ideal is zero. Therefore L is semisimple. $\qquad\square$

Theorem 53. *L is semisimple if and only if there exists $L_1, \ldots, L_t$ ideals of L which are simple such that $L = L_1 \oplus \cdots \oplus L_t$. Here the direct sum is just the coproduct.*

Also, note that every simple ideal in L coincides with some L_i and that the direct sum decomposition is unique.

Proof. Suppose L is semisimple. If L is simple, we are done. Otherwise, consider $I \subset L$ a proper nonzero ideal. Then it is easy to see that $I^\perp$ is an ideal and that $L = I \oplus I^\perp$. By induction, L is a direct sum of simple Lie algebras.

In the other direction, Note that $[L, L] = \oplus[L_i, L_i] = \oplus L_i = L$. Also note that $\operatorname{Rad} L = \oplus \operatorname{Rad} L_i = 0$. This gives semisimplicity. $\qquad\square$

7   Lecture 7 (Sep 24)

We began class by finishing the proof of Theorem 53. Everything is in the notes from last time.

Remark 54. If I is an ideal in $L_1 \oplus L_2$, then $I = I_1 \oplus I_2$.

Remark 55. Let L be semisimple and $I \subset L$ an ideal. Then $I = L_{i_1} \oplus \cdots \oplus L_{i_k}$. In particular, the L_i are the only simple ideals in L.

Thus our goal for this class is to classify the simple Lie algebras.

7.1   Some Classical Computations Now recall the definition of the orthogonal Lie algebra. We will take J to be the identity matrix. For $n = 2$, we have the set of matrices such that $A^t + A = 0$, which is one-dimensional. However, we will consider $n = 4$ with matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 56. $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Now we will compute the Killing form of $\mathfrak{sl}_2$ explicitly. We will use the standard basis e, h, f. By the standard relations, we have:

$$\operatorname{ad} e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad \operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}; \quad \operatorname{ad} f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Now we can see that

$$\kappa = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Now recall that a bilinear form is nondegenerate if and only if its matrix is nonsingular. In addition, a nondegenerate bilinear form gives an isomorphism to V^*. Thus we can construct a dual basis in V.

To motivate the dual basis, we can define a Casimir operator as follows: Choose a basis $\{x_i\}$ for L, and then choose a dual basis $\{y_i\}$. Then in the universal enveloping algebra U_L, consider $\Omega = \sum x_i y_i$. Then $\Omega \in Z(U_L)$.

Now recall Schur's Lemma:

Lemma 57 (Schur). *If V is an irreducible representation, then any nonzero equivariant endomorphism of V is a scalar.*

8   Lecture 8 (Sep 26)

8.1   Representations of Lie Algebras   Recall the definition of a representation of a Lie algebra L. Note that this is equivalent to the notion of a module over L. Then recall that a representation is *completely reducible* (or semisimple) if it is a direct sum of irreducible representations.

Theorem 58 (Weyl). *If L is semisimple, then any representation of V is semisimple.*

For finite groups, we have the analogous statement:

Theorem 59 (Maschke). *Let G be a finite group. If $\operatorname{char} k + |G|$, then any k-representation of G is semisimple.*

Remark 60. The original proof of Weyl's theorem was analytic and uses the unitary trick.

Note that if L is not semisimple, Weyl's theorem can fail. For example, consider the standard two-dimensional representation of the set of upper triangular matrices. Then the x-axis is a sub-representation, but there is no other common eigenvector.

Now we construct a representation structure on $\operatorname{Hom}(V,W)$ by $(x.f)(v) = x.f(v) - f(x.v)$, so $L-$**mod** is enriched over itself. In particular, the dual space of a representation is also a representation. Finally, we may construct a tensor product representation by $x.(v \otimes w) = (w.v) \otimes w + v \otimes (x.w)$.

Remark 61. The isomorphism $V^* \otimes W \simeq \operatorname{Hom}(V,W)$ holds in L-**mod**.

Remark 62. The tensor-hom adjunction holds for representations of Lie algebras, so L-**mod** is closed monoidal.

Proof of Schur's Lemma. Let $0 \neq \pi : V \to W$ be a morphism of irreducible representations. Consider the kernel. For a nonzero morphism, this must be the source. In addition, the image must be nonzero, so it is the entire target.

To show any nonzero endomorphism is a scalar, use algebraic closure to find an eigenvalue. Then the kernel of $\pi - \lambda I$ is nonzero, so it must be all of V. $\square$

Remark 63. When we prove Weyl's theorem, we use $\beta(x,y) = \operatorname{tr}(\phi(x)\phi(y))$. This is nondegenerate when the kernel is nonzero.

8.2   Jordan decomposition for Semisimple Lie Algebras   For an arbitrary Lie algebra L, call $x \in L$ semisimple if $\operatorname{ad} x$ is semisimple, and nilpotent if $\operatorname{ad} x$ is nilpotent. Then we can decompose $\operatorname{ad} x = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$. What we will show is that

Lemma 64. *If $\delta \in \operatorname{Der} L$, then $\delta_s, \delta_n \in \operatorname{Der} L$.*

Lemma 65. *If L is semisimple, then $\operatorname{Der} L = \operatorname{ad} L$.*

Now we can form an abstract Jordan decomposition of any semisimple Lie algebra.

Theorem 66. *Let ϕ be a representation of L and $x = s + n$ be the Jordan decomposition in L. Then $\phi(x) = \phi(x) + \phi(n)$ is the Jordan decomposition in $\mathfrak{gl}(V)$.*

9   Lecture 9 (Oct 1)

9.1   Jordan Decomposition Continued

Proof of Lemma 65. Recall that if L is semisimple, then $L \simeq \mathrm{ad}\, L$. Then we know that κ on $\mathrm{ad}\, L$ is non-degenerate. Also, $\mathrm{ad}\, L \subset \mathrm{Der}\, L$ is an ideal. Now take $(\mathrm{ad}\, L)^{\perp}$ and note that $\mathrm{ad}\, L \cap (\mathrm{ad}\, L)^{\perp} = 0$. Thus $\mathrm{Der}\, L = \mathrm{ad}\, L \oplus (\mathrm{ad}\, L)^{\perp}$. Then, by perpendicularity, the direct sub holds as Lie algebras.

We now recall that $[\delta, \mathrm{ad}\, x] = \mathrm{ad}\, \delta(x)$. Now for $\delta \in (\mathrm{ad}\, L)^{\perp}$, we have

$$0 = [\delta, \mathrm{ad}\, x] = \mathrm{ad}\, \delta(x).$$

However, L is semisimple, so $\delta(x) = 0$ for all x. The desired result follows.. $\qquad\square$

As a consequence, we can describe an abstract Jordan decomposition on a semisimple Lie algebra.

Corollary 67 (Jordan Decomposition). *Given $x \in L$, then the Jordan decomposition $\mathrm{ad}\, x = (\mathrm{ad}\, x)_s + (\mathrm{ad}\, x)_n$ restricts to*
$$\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$$
for some $x_s, x_n \in L$. In particular, $x = x_s + x_n$.

Now note that $[x_s, x_n] = 0$ because
$$\mathrm{ad}[x_s, x_n] = [\mathrm{ad}\, x_s, \mathrm{ad}\, x_n] = 0$$

by the usual Jordan decomposition and because L is semisimple.

Remark 68 (Eric's Research). Consider the group $G = SL_2(\mathbb{C})$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Now G acts on $\mathfrak{g}$ by conjugation. We consider the following question:

> What are the orbits of G on the set of nilpotent elements in $\mathfrak{g}$?

We find that there are two orbits of $\mathfrak{sl}_2$ and that the set of nilpotents is a closed subvariety of $\mathfrak{g}$.

Theorem 69. *Let $L \subset \mathfrak{gl}(V)$ be semisimple. Then the usual Jordan decomposition in $\mathfrak{gl}(V)$ agrees with the abstract Jordan decomposition in L.*

Corollary 70. *Let $\phi : L \to \mathfrak{gl}(V)$ be any representation of L. Let $x = x_s + x_n$ be the abstract Jordan decomposition. Then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the usual Jordan decomposition in $\mathfrak{gl}(V)$.*

9.2   Representations of $\mathfrak{sl}_2$

We will classify irreducible representations of $\mathfrak{sl}_2$. [7] Let V be some finite-dimensional representation of $\mathfrak{sl}_2$. Recall that $\mathfrak{sl}_2$ has standard basis e, h, f with the usual relations.

Now we note that h is semisimple in $\mathfrak{sl}_2$, so $\phi(h)$ is semisimple for any representation ϕ. Then we can take the eigenspace decomposition

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_{\lambda}.$$

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[7]This was done in Ivan's homework, and is an exercise in Etingof's book.

Lemma 71. *If $v \in V_\lambda$, then $e.v \in V_{\lambda+2}$ and $f.v \in V_{\lambda-2}$.*

Proof.
$$h.(e.v) = [h.e].v + e.(h.v) = (2e).v + e.(\lambda v) = (\lambda + 2)(e.v)$$
and
$$h.(f.v) = [h.f].v + f.(h.v) = -2f.v + f.(\lambda v) = (\lambda - 2)f.v.$$
$\square$

Next, because V has finite dimension, then there is an eigenspace V_λ that is killed by e. We will call any $v_0 \in V_\lambda$ a maximal vector. Now define $v_i = \frac{1}{i!}f^i.v_0$.

We will study the span of $\{v_m, \ldots, v_1, v_0\}$, where $f.v_m = 0$ but $v_m \neq 0$.

Example 72. Consider the usual 2-dimensional representation. Then e_1 has eigenvalue 1 and e_2 has eigenvalue -1. In addition, $f.e_1 = e_2$ and $e.e_2 = e_1$.

For the adjoint representation, f, h, e have weights $-2, 0, 1$.

We will show that $\langle v_m, \ldots, v_0 \rangle \subset V$ is a subrepresentation:

Proposition 73. *The following are true:*

1. $h.v_i = (\lambda - 2i).v_i$;
2. $f.v_i = (i+1)v_{i+1}$;
3. $e.v_i = (\lambda - i + 1)v_{i-1}$.

Proof. We only need to prove the third claim. Here, we note that

$$
\begin{aligned}
e.v_i &= \frac{1}{i}(e.(f.v_{i-1})) \\
&= \frac{1}{i}([e,f].v_{i-1} + f.(e.v_{i-1})) \\
&= \frac{1}{i}(h.v_{i-1} + f.(\lambda - i + 2)v_{i-2}) \\
&= \frac{1}{i}((\lambda - 2i + 2)v_{i-1} + (i-1)(\lambda - i + 2)v_{i-1}) \\
&= (\lambda - i + 1)v_{i-1}. \qquad\qquad \square
\end{aligned}
$$

Corollary 74. *If V is irreducible, then V_λ is one-dimensional. In particular, $V = \langle v_m, \ldots, v_0 \rangle$.*

Now all we need to do is classify the highest weights and the dimensions for irreducible representations. Here we observe that

$$0 = e.(f.v_m) = e.(m+1)v_{m+1} = (m+1)(\lambda - (m+1) + 1)v_m = (m+1)(\lambda - m)v_m.$$

Because m is nonnegative and v_m is nonzero, $m = \lambda$. Thus we have proved:

Theorem 75. *For an irreducible representation V, λ is a nonnegative integer and V has weights $\lambda, \lambda - 2, \ldots, -\lambda$.*

Next time we will construct a representation with these weights.

10   LECTURE 10 (OCT 3)

10.1   Representation Theory of $\mathfrak{sl}_2$ Continued

Last time, we proved the following result:

Theorem 76. *There is at most one irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of a given finite dimension. Furthermore, its weights are $m, m-2, \ldots, 2-m, -m$ where $m = \dim V - 1$. In addition, all weight spaces are 1-dimensional and there is a unique maximal weight space.*

This time, we will show:

Theorem 77. *There exists a unique irreducible representation of each positive dimension.*

Proof 1. We write down three matrices:

$$E = \begin{pmatrix} 0 & n & & & & \\ & 0 & n-1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 2 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, H = \begin{pmatrix} n & & & & & \\ & n-2 & & & & \\ & & n-4 & & & \\ & & & \ddots & & \\ & & & & 2-n & \\ & & & & & -n \end{pmatrix}, F = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & n-1 & 0 & \\ & & & & n & 0 \end{pmatrix}.$$

Checking this is left to the reader.[8] $\square$

Proof 2. Let $\mathbb{C}^2$ be the standard representation. Then note that $(\mathbb{C}^2)^* \simeq \mathbb{C}^2$. Thus we can consider the infinite-dimensional representation $S(V^*) = S(\mathbb{C}^2)$.[9]

Now we can take our representations to be $S^{k-1}(\mathbb{C}^2)$. Then we can check that

$$h.(x^i y^{k-1-i}) = \sum x^{i-1}(h.x)y^{k-1-i} + \sum_{k-1-i} x^i (h.y)(y^{k-2-i}) = (-k+1+2i)x^i y^{k-1-i}.$$

Therefore we have the correct weights, so we are done. $\square$

Now, Weyl tells us that a general representation of $\mathfrak{sl}_2$ is a sum of the form

$$V = \oplus_{m \in \mathbb{Z}} a_i V_i.$$

To find the multiplicities, just find the largest eigenvalue of h and take away weights of h.

10.2   Maximal Tori

Consider $\mathfrak{sl}_n$ and denote the set of diagonal elements by $\mathfrak{h}$. This is an abelian subalgebra, so we will find an analog for arbitrary semisimple Lie algebras.

Definition 78. A subalgebra of L consisting of semisimple elements is a *toral subalgebra.*.

Theorem 79. *Toral subalgebras are abelian.*

Proof. Let T be a toral subalgebra and choose $x \in T$. Because x is semisimple, $\mathrm{ad}\, x$ is semisimple as a map $T \to T$. Thus $\mathrm{ad}\, x|_T$ is semisimple.

Choosing an eigenvector $y \in T$, let y_i be a basis of eigenvectors for $\mathrm{ad}\, y$. Now we write $x = \sum c_i y_i$, we have

$$[y, x] = \sum c_i [y, y_i] = -ay,$$

which implies that $[x, y] = 0$. $\square$

---

[8]This is taken from Ivan's homework. Eric normalized the e, so we have a different f.
[9]This was on Ivan's homework, but for the group SL_2.

We consider the action of H on L. Because $\operatorname{ad} h$ are commuting semisimple operators, they are simultaneously diagonalizable.

Definition 80. Define the *root space*

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \text{ for } h \in H$$

where $\alpha \in H^*$.

Note that $H \subset L_0$. In particular, $L_0 = C_L(H)$ and

$$L = C_L(H) \oplus \bigoplus_{\alpha \neq 0} L_\alpha.$$

Example 81. In $\mathfrak{sl}_n$, $[h, e_{ij}] = \alpha_{ij}(t_i - t_j)e_{ij}$. Thus

$$\mathfrak{sl}_{\mathfrak{n}} = H \oplus \bigoplus_{i \neq j} L_{\alpha_{ij}}.$$

We conclude that H is a maximal torus and that $H = L_0$.

11   LECTURE 11 (OCT 08)

Recall our discussion of toral subalgebras from last time.

11.1   Tori in sl_n

Proposition 82. $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 2n \operatorname{tr}(xy)$ *for* $x, y \in \mathfrak{h}$.

Proof. Note $\operatorname{ad} x$ has eigenvalues $t_i - t_j$ and $\operatorname{ad} y$ has eigenvalues $u_i - u_j$. Then

$$\begin{aligned}
\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) &= \sum_{i,j}(t_i - t_j)(u_i - u_j) \\
&= \sum t_i u_i + \sum t_j u_j - \sum t_i u_j - \sum t_j u_i \\
&= 2n \sum t_i u_i \\
&= 2n \operatorname{tr}(xy).
\end{aligned}$$

$\square$

Corollary 83. κ *is nondegenerate on* H.

Proposition 84. $H = C_L(H)$.

Proof. Let $x = \sum a_{ij} e_{ij}$. Then

$$[h, x] = \sum a_{ij}(t_i - t_j)e_{ij}.$$

If $x \notin H$, then $a_{ij} \neq 0$ for some $i \neq j$. But then there exiss $h \in H$ such that $\alpha_{ij}(h) = t_i - t_j$. $\square$

Theorem 85. *In general,* $H = C_L(H)$ *for any maximal torus* H *in a semisimple Lie algebra* L.

11.2   The General Case Take $H \subset L$ to be a maximal torus.

Proposition 86.   *1. For all $\alpha, \beta \in H^*$, $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.*

   2. For $x \in L_\alpha$ with $\alpha \neq 0$, $\operatorname{ad} x$ is nilpotent.

   3. For $\alpha, \beta \in H^*$ with $\alpha + \beta \neq 0$, $L_\alpha \perp L_\beta$.

Proof.   1. $[h, [x_\alpha, x_\beta]] = [[h, x_\alpha], x_\beta] + [x_\alpha, [h, x_\beta]] = (\alpha(x) + \beta(x))[x_\alpha, x_\beta]$.

   2. This follows from finite dimensionality of L.

   3. $-\alpha(h)\kappa(x, y) = \kappa([x, h], y) = \kappa(x, [h, y]) = \beta(h)\kappa(x, y)$. $\square$

Corollary 87. *κ is nondegenerate on $C_L(H)$.*

Proof. If $\kappa(x, y) = 0$ for all $y \in C_L(H) = L_0$, then $\kappa(x, z) = 0$ for all $z \in L$. Thus $x = 0$. $\square$

Lemma 88. *If x is nilpotent and $[x, y] = 0$, then $\kappa(x, y) = 0$.*

Proof. $\operatorname{ad} x$ is nilpotent by definition, and because it commutes with $\operatorname{ad} y$, $\operatorname{ad} x \operatorname{ad} y$ must also be nilpotent. $\square$

Proof of Theorem 85. If $x \in C_L(H)$, then $\operatorname{ad}_X(H) = 0$. Therefore $\operatorname{ad} x_s(H) = \operatorname{ad} x_n(H) = 0$, so $x_s, x_n \in C$.

Now suppose $x \in C_L(H)$ is semisimple. Then $H + \mathbb{F}x$ is abelian and is toral because it consists of commuting semisimple elements. Thus $x \in H$.

Next we show κ is nondegenerate on H. Suppose $\kappa(h, h') = 0$ for all $h' \in H$. Then choose $z \in C_L(H)$. Because z_n commutes with H, $\kappa(h, z_n) = 0$ by Lemma 88. Thus $h = 0$ by Corollary 87.

We now show $C_L(H)$ is nilpotent. For $z \in C_L(H)$, $z_s \in H$, so $\operatorname{ad} z_s = 0$. On the other hand, $\operatorname{ad} z_n$ is nilpotent. Thus $\operatorname{ad} z$ is nilpotent because z_s, z_n commute. By Engel, $C_L(H)$ is nilpotent.

Next we show that $H \cap [C_L(H), C_L(H)] = 0$. Choosing $[x, y] \in [C_L(H), C_L(H)]$ and $h \in H$, we see $\kappa([x, y], h) = \kappa(x, [y, h]) = 0$, so $[C_L(H), C_L(H)] \perp H$. Because κ is nondegenerate on H, $H \cap [C, C] = 0$.

Next note that $Z(C_L(H)) \cap [C_L(H), C_L(H)] \neq 0$. Choose a nonzero $z \in Z(C_L(H)) \cap [C_L(H), C_L(H)]$. Thus $z_s, z_n \in Z(C_L(H))$. However, $[C_L(H), C_L(H)]$ is an ideal, so $z_s \in [C_L(H), C_L(H)] \cap H$. Thus $z_s = 0$. Therefore $\kappa(z, C_L(H)) = 0$. Thus $z = 0$. Therefore $C_L(H)$ is abelian.

Finally, we show that $H = C_L(H)$. Choose $z \in C_L(H)$. Then $z_n \in C_L(H)$, so $\kappa(z_n, C_L(H)) = 0$, so $z_n = 0$. Thus $z = z_s \in H$. $\square$

Corollary 89. *κ is nondegenerate on H.*

Now we may identify H with H^* using κ.[10] In addition, we may write $L = H \oplus \bigoplus_{\alpha \neq 0} L_\alpha$. We define the set of roots $\Phi = \{\alpha \in H^* \mid L_\alpha \neq 0, \alpha \neq 0.\}$.

---

[10]This will help us in our study of root systems.

12   LECTURE 12 (OCT 10)

Last time we proved that κ is nondegenerate on H and that $C_L(H) = H$. Thus we may associate $t_\alpha \in H$ to every $\alpha \in H^*$.

Proposition 90.   *1. Φ spans H^*.*

    2. If $\alpha \in H$, then $-\alpha \in \Phi$.

    3. If $\alpha \in \Phi$, $x \in L_\alpha, y \in L_{-\alpha}$, then $[x,y] = \kappa(x,y)t_\alpha$.

    4. If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is 1-dimensional.

    5. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$ for all $\alpha \in \Phi$.

    6. The span of $L_\alpha, L_{-\alpha}, t_\alpha$ is a copy of $\mathfrak{sl}_2$.

    7. $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ and $h_\alpha = h_{-\alpha}$.

Proof.   1. If not, then thee exists $h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. But this implies $h \in Z(L)$, which is a contradiction for nonzero h.

   2. Recall that $L_\alpha \perp L_\beta$ if $\alpha + \beta \neq 0$. Thus $\kappa(L_\alpha, L_{-\alpha}) \neq 0$, so $L_{-\alpha}$ is nonempty.

   3. Just evaluate $\kappa(h, [x,y])$. We have

$$\begin{aligned} \kappa(h, [x,y]) &= \kappa([h,x], y) \\ &= \kappa(\alpha(h)x, y) \\ &= \kappa(t_\alpha, h)\kappa(x, y) \\ &= \kappa(\kappa(x,y)t_\alpha, h). \end{aligned}$$

   4. Follows from the previous part.

   5. The first part follows from the definition. Then if the inner product is zero, $[t_\alpha, x] = 0 = [y, t_\alpha]$ for all $x \in L_\alpha, y \in L_{-\alpha}$. Choosing nonzero x, y such that $\kappa(x, y) \neq 0$, rescale so $\kappa(x, y) = 1$. Then $[x, y] = t_\alpha$. Then the subalgebra S spanned by x, t_α, y is solvable. By Lie's theorem, $[\mathrm{ad}_L S, \mathrm{ad}_L S]$ consists of nilpotent matrices. Then t_α is ad-nilpotent, so it must be 0, which is a contradiction.

   6. Choose h_α a multiple of t_α such that $[h_\alpha, x] = 2x$. Now let c be such that $h_\alpha = ct_\alpha$ and choose y_α such that $\kappa(y_\alpha, y_\alpha) = c$.

   7. $c^2\kappa(t_\alpha, t_\alpha) = \kappa(h_\alpha, h_\alpha) = c\kappa(h_\alpha, t_\alpha) = 2c$.

<div align="right">□</div>

We will call the h_α coroots and denote the set of coroots by $\Phi^\vee$.

Example 91. In $\mathfrak{sl}_n$, $\alpha(h) = t_i - t_j$. Then $h_\alpha = e_{ii} - e_{jj}$.

Now let S_α be the $\mathfrak{sl}_2$ associated to α. Then we consider L as an S_α-module. In particular, consider $M = H \oplus \bigoplus_c L_{c_\alpha}$. We can then write $M = \ker(\alpha) \oplus S_\alpha \oplus M'$, where M' has all of the odd weights. Also, this shows that the L_α are one-dimensional.

Now we can also observe that if $\alpha \in \Phi$, no other positive multiple of α is in Φ.

13   LECTURE 13 (OCT 17)

Recall our discussion of root systems from last time. We showed that all root spaces are one-dimensional and that if α is a root, no other positive multiple of α is a root. It is easy to show that:

Proposition 92.   1. $[L_\alpha, L_\beta] = L_{\beta+\alpha}$ if $\alpha + \beta$ is a root.

   2. $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$.

   3. The L_α generate L.

   4. Say $\beta + q\alpha \in \Phi$ but $\beta + q'\alpha \notin \Phi$ for $q' > q$. In addition, suppose $\beta - r\alpha \in \Phi$ but $\beta - r'\alpha \notin \Phi$ for $r' > r$. Then $\beta + i\alpha \in \Phi$ for all $-r \le i \le q$.

Proof. Consider the space $M = \oplus L_{\beta+i\alpha}$. Then M is an S_α-module. Then note the eigenvalues of h_α are $\beta(h_\alpha) + 2i$. Thus $\beta(h_\alpha) \in \mathbb{Z}$. In addition, the eigenvalues of h_α are distinct and have the same parity. Thus the last part is true. To complete the second part, note that $(\beta - \beta(h_\alpha)\alpha)(h_\alpha) = -\beta(h_\alpha)$. To do the first part, note that $\operatorname{ad} x_\alpha$ is surjective if and only if $\beta + \alpha$ is a root (because M is irreducible). Finally, the third part follows from the fact that the α span H^*.   □

13.1   An Example: The symplectic Lie algebra Use the standard symplectic form. Then a matrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic if and only if $A^T = -D, B^T = B, C^T = C$. Thus $\mathfrak{sp}_{2n}$ has dimension $2n^2 + n$. We can define H to be the set of diagonal matrices. Then everything simply restricts from $\mathfrak{sl}_{2n}$.

Restricting to $\mathfrak{sp}_4$, we have four roots: $t_1 - t_2, 2t_1, 2t_2, t_1 + t_2$. Note that H^* is an F-vector space. Now consider the $\mathbb{Q}$-vector space $E_{\mathbb{Q}}$ spanned by Φ.

Proposition 93. $\dim_{\mathbb{Q}} E_{\mathbb{Q}} = \dim_F H^*$.

Proof. Choose an F-basis of H^* from Φ. Then for $\beta \in \Phi$, we have $\beta = \sum c_i \alpha_i$. Then taking the inner product with β and using that $\beta(h_\alpha) \in \mathbb{Z}$, we are done.   □

Remark 94. The inner product on $E_{\mathbb{Q}}$ is positive-definite. Thus we can extend scalars to $\mathbb{R}$ to obtain a Euclidean space.

14   LECTURE 14 (OCT 22)

Last time, we defined the real Euclidean space associated to the root system Φ. Now we define the axioms of a root system.

14.1 Root Systems

Definition 95. A *root system* is a collection of finitely many nonzero vectors $\Phi \subset E$ in a Euclidean space that satisfies:

(R1): Φ spans E;

(R2): If $\alpha \in \Phi$ and $c\alpha \in \Phi$, then $c = \pm 1$.

(R3): For $\alpha \in \Phi$, $\sigma_\alpha(\Phi) = \Phi$, where σ_α is the reflection through α.

(R4): If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \coloneqq \frac{2(\beta, \alpha)}{\alpha, \alpha} \in \mathbb{Z}$.

Remark 96. If we relax (R2), then our definition is called "reduced." If we relax (R4), then our definition is called "crystallographic."

Proposition 97. $\Phi \subset E$ *attached to L is a root system.*

Proposition 98. *We showed last time that $\langle \beta, \alpha \rangle = \beta(h_\alpha) \in \mathbb{Z}$. Also we showed that the root strings are unbroken and that $\beta - \beta(h_\alpha)\alpha$ is a root. We already knew that the inner product was positive-definite and that (R1) and (R2) hold.*

Example 99. In $\mathfrak{sl}_{n+1}$, the roots are simply the $\varepsilon_i - \varepsilon_j$ and we use the usual Euclidean structure on $\mathbb{R}^n$ induced from $\mathfrak{gl}_n$. Checking the axioms is easy.[11]

14.2 Classifying Root Systems Note that (R4) limits the possibilities for $\langle \beta, \alpha \rangle$. In addition, we can analyze the Weyl group. We can also consider $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4 \cos^2 \theta_{\alpha, \beta}$. This constrains the angles between the roots to be $\pi/2, \pi/6, \pi/3, \pi/4, 3\pi/4, 2\pi/3, 5\pi/6, 0, \pi$.

Definition 100. A root system E, Φ is *reducible* if $E = E_1 \oplus E_2$ and all roots are contained in either E_1 or E_2.

Example 101. If $\dim E = 1$, then we only have one root system, A_1. If $\dim E = 2$, then one possibility is the root system $A_1 \oplus A_1$, which is just an orthonormal basis. In addition, we have the case where α, β have angle $2\pi/3$, which is the A_2 root system. If α, β have angle $3\pi/4$, then one has length $\sqrt{2}$ and the other has length 1. This is the root system B_2. Finally, if $\theta = 5\pi/6$, we have G_2.

Definition 102. The *Weyl group* is the group generated by the σ_α.

It is easy to see that $W \subset S_\Phi$. In particular, W is finite.

Example 103. The Weyl group of A_n is S_{n+1}.

15   LECTURE 15 (OCT 24)

Recall that the group SL_n acts transitively on flags in $\mathbb{C}^n$. Then the stabilizer of any flag is the group B of upper triangular matrices. Then the set of all flags can be identified with G/B. This has a left action on G/B, so we have

Theorem 104 (Bruhat). *The orbits of B on G/B are indexed by the Weyl group.*

---

[11]This was in Ivan's homework.

A standard fact is that the orbit X_w is an affine space of dimension $\ell(w)$. Here $\ell(w)$ is the smallest number adjacent transpositions needed to write w. In addition, the closure of X_w is no longer smooth, and we can study its singularities, where Kazhdan-Lusztig polynomials appear.[12]

15.1   Weyl Groups continued   Recall that the Weyl group preserves the inner product. Today we will discuss the analogue of the adjacent transpositions that allow us to define the length of an element in the Weyl group.

Remark 105. Any irreducible root system has a corresponding semisimple Lie algebra.

Definition 106. A set of *simple roots* $\Delta \subset \Phi$ is a set of roots such that every root $\gamma \in \Phi$ can be written $\gamma = \sum_\Delta c_\alpha \alpha$, where all c_α have the same sign. Also, Δ must span E.

Example 107. $\alpha, \alpha + \beta$ form a set of simple roots in A_2.

Proposition 108. *W acts transitively on the set of bases.*

Proposition 109. *Every root system has a base.*

Proof. Choose $\gamma \in E$ to not be orthogonal to any root. Now define $\Phi^+(\gamma)$ to be the set of roots with positive inner product with γ. Now call $\beta \in \Phi^+$ indecomposable if β cannot be written as $\alpha_1 + \alpha_2$ for positive roots α_i. Now choose $\Delta(\gamma)$ to be the set of indecomposable roots in $\Phi^+(\gamma)$.

Now we show that $\Delta(\gamma)$ is a base and every base is of this form. It is easy to see that any positive root not in Δ is decomposable, so then it must be a sum of $\beta_1 + \beta_2$. Then using the inner product and well-ordering, β must be a nonnegative sum.

Next we show that if $\alpha \neq \beta \in \Delta(\gamma)$, then $(\alpha, \beta \leq 0)$. This is because otherwise, $\alpha - \beta$ is a root. Now either $\alpha - \beta$ or $\beta - \alpha$ is a positive root, which implies that either α or β is decomposible.

Next, we show that $\Delta(\gamma)$ is linearly independent. If not, then we can take the inner product of some vector τ with itself to see that $\tau = 0$.

To show that every base is of this form, we can choose γ such that $(\alpha, \gamma) > 0$ for all $\alpha \in \Delta$. By construction, $\Delta \subset \Phi^+(\gamma)$. Now note that Φ^+ is the set of nonnegative combinations of Δ, so it is contained in $\Phi^+(\gamma)$. Also $\Phi^- \subset -\Phi^+(\gamma)$. By uniqueness of $\beta = \sum c_\alpha \alpha$, Δ consists of indecomposable elements. Because both $\Delta, \Delta(\gamma)$ are bases of E, they must be the same. $\square$

Now denote $\alpha^\perp$ by P_α. Then the connected components of $E \smallsetminus \cup P_\alpha$ are called *Weyl chambers*.

Theorem 110. *Let Δ be a base of Φ. Then W acts transitively on the set of Weyl chambers and on the set of bases. In addition, each base corresponds to a unique Weyl chamber. In fact, the action of W on the set of Weyl chambers is faithful.*

16   Lecture 16 (Oct 29)

Let Δ be a base of Φ. Then define $W' \subset W$ to be the subgroup generated by the simple reflections. Our goal is to show that $W' = W$. Then for any $w \in W'$, we define $\ell(w)$ to be the smallest t such that $w = \sigma_1 \cdots \sigma_t$.

---

[12]The Kazhdan-Lusztig conjectures relate these to representation theory, and were proven by Beilinson and Bernstein

Lemma 111. *For $\alpha \in \Delta$, σ_α permutes $\Phi^+ \smallsetminus \{\alpha\}$.*

Proof. Note that σ_α only changes the α coefficient of any positive root β. However, this coefficient remains nonnegative. $\qquad\square$

Lemma 112. *Let $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Then $\langle \delta, \alpha \rangle = 1$.*

Proof. Write $\delta = \frac{1}{2} \left(\sum_{\beta \neq \alpha} \beta + \alpha \right)$, apply σ_α, and then use the previous lemma. $\qquad\square$

Lemma 113. *Suppose $w = \sigma_{i_1} \cdots \sigma_{i_t}$ is reduced, then $\sigma_{i_1} \cdots \sigma_{i_{t-1}}(\alpha_t)$ is a negative root.*

Now we define a partial order on $\mathbb{Z}[\Phi]$ where $\alpha \leq \beta$ if $\beta - \alpha$ has positive coefficients in Δ. We write $\alpha > 0$ if α is a positive root.

Proposition 114. *W acts transitively on bases.*

Proof. First we show this for W'. Choose two bases $\Delta(\gamma), \Delta(\gamma')$. Then note that $\sigma(\Delta(\gamma)) = \Delta(\sigma(\gamma))$. Also, σ sends positive roots to positive roots.

Choose $\sigma \in W'$ such that $(\sigma(\gamma), \delta)$ is as large as possible. Then for $\alpha \in \Delta$, $(\sigma_\alpha \sigma(\gamma), \delta) \leq (\sigma(\gamma), \delta)$. This implies that $(\sigma(\gamma), \alpha) \geq 0$. Because γ is regular, so is $\sigma(\gamma)$. Therefore $\sigma(\gamma)$ is in the Weyl chamber associated to Δ, as desired. $\qquad\square$

Next, any root β is part of some base, so it can be obtained from some simple root by a series of simple reflections. Next, we want to show that $W' = W$. This follows from the fact that $g \sigma_v g^{-1} = \sigma_{g(v)}$ for any $g \in O(E)$.

Finally, we will show that W acts simply transitively on bases. Choosing a reduced expression for w, we have $w(\alpha_{i_t}) < 0$ if $t \geq 1$. However, $w(\Delta) = \Delta$, so $w(\alpha)_{i_t} > 0$. This is a contradiction, so $w = 1$.

This discussion defines a length function $W \to \mathbb{Z} \geq 0$. Then for $w \in W$, define $\Phi^+(w) = \{\alpha \in \Phi^+ \mid w(\alpha) < 0\}$. For example, $\Phi^+(1) = \varnothing$ and $\Phi^+(\sigma_\alpha) = \{\alpha\}$.

Proposition 115. *There exists a unique $w_0 \in W$ such that $\Phi^+(w_0) = \Phi^+$.*

Proof. If Δ is a base, then so is $-\Delta$. $\qquad\square$

Lemma 116. *$\ell(w) = \#\Phi^+(w)$.*

17   LECTURE 17 (OCT 31)

The projector is on and the screen is down over the board. I do not know what Eric plans to do with the computer.

Recall that last time we defined the length function. We need to prove Lemma 116.

Proof of Lemma 116. We use induction. It is true for $w = 1$ and for w a simple reflection, so then we use the fact that the expressions are reduced to write $\Phi^+(w') = \sigma_{i_t}(\Phi^+(w) \smallsetminus \{\alpha_{i_t}\})$. Checking this simply follows from applying σ_{i_t} to $\Phi^+(w')$. $\qquad\square$

Lemma 117. *Let C be a Weyl chamber for Δ. Then for $\lambda, \mu \in \overline{C}$, $w(\lambda) = \mu$ only if $\lambda = \mu$.*

Now given Δ, we may associate its *Dynkin diagram*. To each simple root, we associate a node. There is no connection if two roots are orthogonal, one line if $\langle \beta, \alpha \rangle, \langle \alpha, \beta \rangle$ are both -1, two lines if one is -2, and three lines if one is -3. The arrow points to the shorter root.

Example 118. Some Dynkin diagrams:

- A_n:
- B_n:
- C_n:
- D_n:
- G_2:
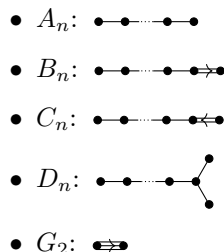
Now it is easy to see that if Φ is irreducible, then Δ can not be decomposed in the same way. Thus irreducibility is captured by the Dynkin diagram.

Proposition 119. *If Φ is irreducible, there are at most two lengths of roots.*

Proposition 120. *If Φ is irreducible, there exists a maximal positive root and it is long.*

Next we will classify the irreducible Dynkin diagrams and then show the existence of each possible Lie algebra. For now, we can assume that all roots have the same length (simply laced). We will consider the affine Dynkin diagram, formed by adjoining $-\theta$ to the diagram. Finally, we can decorate the affine diagram by the coefficients of $\Theta = \sum_{\alpha \in \Phi} K_\alpha \alpha$. Also, if $\beta_1, \ldots, \beta_k$ are the nodes connnected to α, we have $2K_\alpha = \sum K_{\beta_i}$.

18 Lecture 18 (Nov 5)

Recall the construction of the Dynkin diagrams from last time. Note that Φ is irreducible if and only if the Dynkin diagram is connected. Our goal is to classify irreducible root systems, which is equivalent to classifying connected Dynkin diagrams. Later, we will see that L is simple if and only if Φ is irreducible. Today, we will classify simply-laced (ADE) Dynkin diagrams.

Theorem 121. *The simply-laced connected Dynkin diagrams are the ADE.*

Proof. We will use the highest root θ, which is maximal in the partial order. We know that θ is unique, has positive inner product with all positive roots, and is long. We may adjoin $-\theta$ to the Dynkin diagram to obtain the affine Dynkin diagram.

By definition, we will attach a coefficient of 1 on $-\theta$. For the extended diagram, we have $2c_i - \sum_{i-j} c_j = 0$. We will use this fact to construct the decorated extended diagrams.

Starting from the coefficient of $-\theta$, note that we can either have a degree 2 vertex with label 1, where the other vertices are also 1, or a vertex with 1 attached to a vertex with label 2.

In the first case, it is easy to see that all vertices we attach have label 1, so to have a finite diagram, we must form a cycle. Then we have A_n.

In the second case, if we branch at the label 2 vertex, it is easy to see that we have D_n. If we do not branch, then we attach a vertex with label 3.

In this case, the next vertex can be either 2 or 4. If we have a 2, then we need a third vertex with label 2 and to each vertex of label 2, we attach a 1. This gives us E_6. If the fourth vertex is a 4, then the next vertex can be $2, 3, 5$. In the case of either 2 or 3, we have E_7. In the case of a 5, the next vertex is forced to be 6, which gives us E_8.   □

For non-simply laced root systems, we can show existence by writing down the Dynkin diagrams, writing down a list of vectors in an integral lattice of a given length, and showing that the inner product is integral. Then reflecting keeps us in the lattice.

Alternatively, we could use the computer to generate all roots from the Cartan matrix. This gives us a set of roots which is stable under simple reflections. Then this set is also stable under all reflections, and the integrality follows from this.

Proposition 122. *If L is simple, then Φ is irreducible. In ddition, if L is semisimple, then the decomposition into simples gives an orthogonal decomposition of Φ.*

Proof. Suppose $\Phi = \Phi_1 \times \Phi_2$ and that $\alpha \in \Phi_1, \beta \in \Phi_2$. If $\alpha + \beta$ is a root, it is in either Φ_1 or Φ_2. However, $\alpha + \beta$ has nonzero inner product with both α, β, so α, β commute. Thus the subalgebra K generated by Φ_1 commutes with β, so K is a proper subset of L. Then K must be an ideal, so L is not simple.

For the second part, consider the decomposition into irreducibles L_i. Then the maximal torus H in L is simply a direct sum of the H_i.   □

Corollary 123. Φ *is irreducible if and only if L is simple.*

19   Lecture 19 (Nov 7)

We know that each semisimple Lie algebra gives us a Dynkin diagram, but we have not shown whether each Dynkin diagram gives us a Lie algebra. Also, we need to show that all maximal tori are conjugate and that two Lie algebras are isomorphic iff they have the same root system.

Lemma 124. *All maximal Borels of L are conjugate via* $\mathrm{Aut}(L)$.

We know this to be true for $\mathfrak{sl}_n$ by Lie's theorem.

Lemma 125. *In a solvable subalgebra of L, all toral subalgebras are conjugate.*

To generate automorphisms of L, we can take $x \in L$ nilpotent. Then take $\exp(\mathrm{ad}\, x)$, which is an automorphism of L. Then we call the group $\mathrm{Int}(L)$ the group generated by $\exp(\mathrm{ad}\, x)$ for x nilpotent.

Proposition 126. *If $\sigma : L \to L'$ is an isomorphism that takes H to H', then it induces an isomorphism of Φ, Φ'.*

Proof. Let $h \in H$, $x \in L_\alpha$. Then $[h, x] = \alpha(h)x = \alpha \circ \sigma^{-1}(\sigma(h))\sigma(x)$. Therefore $\sigma(x) \in L_{\alpha \circ \sigma^{-1}}$. Thus $\alpha\sigma^{-1} \in \Phi'$. Then note that σ takes root strings to root strings, so we must have $\langle \beta, \alpha \rangle = \langle \beta', \alpha' \rangle$. Thus σ^* is an isomorphism of root systems. $\qquad\square$

Definition 127. $H \subset L$ is a *Cartan subalgebra* if H is nilpotent and $N_L(H) = H$.

Proposition 128. *If L is semisimple and H is maximal toral, then H is a Cartan subalgebra.*

Proof of this uses the decomposition of L into eigenspaces for H.

Theorem 129. *If L is semisimple and H is a Cartan subalgebra, then H is maximal toral.*

Define the group $E(L)$ to be the collection of $\exp(\operatorname{ad} x)$, where $\operatorname{ad} x$ is called strongly nilpotent. Here, strongly nilpotent means that $x \in L_a(\operatorname{ad} y)$ for some $y \in L$ and nonzero a.

Theorem 130. *If L is solvable, then all Cartans are conjugate under $E(L)$.*

Theorem 131. *If L is a general Lie algebra, then all Borels are conjugate under $E(L)$.*

Corollary 132. *If H is maximal toral, then $H = C_L(s)$ for some $s \in L$ semisimple.*

Definition 133. $s \in L$ is called *regular semisimple* if $C_L(s)$ has minimal dimension among all semisimple elements.

Example 134. In $\mathfrak{sl}_n$, the regular semisimple elements are the ones with all distinct eigenvalues.

Corollary 135. *All Borels containing H arise from some choice of positive roots Φ^+.*

Definition 136. A *parabolic subalgebra* is a subalgebra containing a Borel.

Theorem 137. *All parabolic subalgebra is conjugate under $E(L)$ to a standard parabolic (block upper-triangular).*

20   Lecture 20 (Nov 12)

From a Lie algebra L, we will construct its *universal enveloping algebra $U(L)$*. This is the left adjoint to the Lie functor from associative algebras to Lie algebras.[13]

Definition 138. A pair (i, U) is a *universal enveloping algebra* for L if U is an associative algebra over k with a representation $i : L \to U$ satisfying the following condition: for any associative algebra A, if $j : L \to A$ preserves the Lie bracket, then j factors uniquely through i.

Proposition 139. *The universal enveloping algebra is unique up to unique isomorphism.*

This follows directly from the universal property (equivalently from being the adjoint of the Lie functor).

Now we will show existence of the universal enveloping algebra. We will construct this by taking the quotient of the tensor algebra $T(V)$ by the relations $[x, y] = x \otimes y - y \otimes x$, recalling that the tensor algebra is the left adjoint of the forgetful functor from associative algebras to vector spaces. Note that the universal enveloping algebra is not graded.

Remark 140. Later, we will see that L is isomorphic to its image in $U(L)$ (the unit of the adjunction).

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[13]We can just show that the Lie functor preserves limits, but Eric turned down my shitposting request.

It is easy to see that $i([x,y]) = i(x)i(y) - i(y)i(x)$. This follows directly from the definition of $U(L)$.

Now we will show that (i, U) is universal. Note that the map $L \to A$ factors uniquely through $T(L)$. Then the map from $T(L)$ kills anything of the form $x \otimes y - y \otimes x - [x, y]$, so it factors uniquely through $U(L)$.

Now given any left ideal of $U(L)$, we may consider modules $U(L)/I$. There are also representations of L. We will use this technique to construct representations of semisimple Lie algebras.

Example 141. Let $L = \mathfrak{sl}_2$. Then we can construct $U(L)$, which is a filtered algebra.[14]. The filtration is given by
$$U_m := \pi(T^0 \oplus \cdots \oplus T^m).$$

Then the associated graded algebra is $G = U_0 \oplus U_1/U_0 \oplus \cdots$ with multiplication induced from $U(L)$. It is easy to see that G is commutative. In addition, it is easy to see that the Casimir element is $xy + \frac{1}{2}h^2 + yx$, which is contained in the center of $U(L)$.

Now we try to construct the two-dimensional representation as a quotient of $U(L)$. Consider the ideal $I = \langle x, h - 1, y^2 \rangle$. Then U/I is spanned by $1, y$.

21   Lecture 21 (Nov 14)

Note that by the universal property of the universal enveloping algebra, a representation of L is the same thing as a representation of $U(L)$. In fact, once we see that $L \subset U(L)$, then this gives an equivalence of categories between the two notions.

Theorem 142 (Poincare-Birkhoff-Witt). *Choose an ordered basis $x_1 < x_2 < \cdots$ for L. Then the products $x_{i(1)}x_{i(2)}\cdots x_{i(m)}$ where $1 \le i(1) \le i(2) \le \cdots \le i(m)$ form a basis for $U(L)$.*

Corollary 143.   *1. $L \subset U(L)$.*

   2. If $H \subset L$, then $U(H) \subset U(L)$. In addition, $U(L)$ is a free $U(H)$-module.

Now recall that a semisimple Lie algebra can be decomposed as $H \oplus N \oplus N^-$, where N is the sum of the weight spaces of the positive roots.

Also recall the symmetric and exterior algebras and their graded pieces. Next, if L is semisimple, $\wedge L$ splits as an an L-module. We may ask in what degrees the trivial representation appears because $\wedge^m L$ is an L-module. This allows us to construct a "Frobenius polynomial" by

$$\sum_{i=1}^{\dim L} \dim(\operatorname{Hom}(k, \overset{i}{\bigwedge} L))q^i.$$

By a result of Konstant, this polynomial is

$$\prod_{i=1}^{\dim H} \left(1 + q^{2m_i+1}\right)$$

where $m_1 \le \cdots \le m_n$.

---

[14]The associated graded algebra is a symmetric algebra by the PBW theorem

Example 144. $\bigwedge^2 \mathfrak{sl}_2$ is the irreducible representation with highest weight 2, so the Frobenius polynomial is $1 + q^3$.

Corollary 145. *The PBW theorem is equivalent to the following: the associated graded algebra for* $U(L)$ *is isomorphic to* $S(L)$.

After this, we went into a discussion about the infinite-dimensional exercise from the last homework. Now consider the ideal $I \subset U(\mathfrak{sl}_2)$ generated by $x, h - \lambda$. Then note U/I is generated by the image of 1. By construction, x kills 1 and $h.1 = \lambda$. Unfortunately, this is infinite-dimensional, so we need to kill some power of y. By PBW, all powers of y are linearly independent, so $U(L)/I_\lambda \simeq Z(\lambda)$.

Now let L be semisimple with Borel $H \oplus N$. Then N has a basis $\{x_\alpha\}$ given by the positive roots Φ^+ and H has a basis given by h_i for $\alpha_i \in \Delta$. Then for any $\lambda \in H^*$, we can define

$$I_\lambda = (\{x_\alpha \mid \alpha \in \Phi^+\}, \{h_i - \lambda(h_i)\}).$$

22   Lecture 22 (Nov 19)

Today we will begin the classification finite dimensional irreducible representations of L for L semisimple. Let V be a finite dimensional representation of L. Then restricting to H, we obtain that

$$V = \bigoplus_{\mu \in H^*} L_\mu.$$

We denote the μ-*weight space* of V by V_μ.

Now recall that the Borel is associated to a set of simple roots. It is easy to see that B is solvable. By Lie's theorem, there exists $v^+ \in V$ such that $B.v^+ \in F.v^+$. In particular, $N.v^+ = 0$ and $h.v^+ = \lambda(h).v^+$ for some $\lambda \in H^*$.

We will call v^+ a *maximal vector* and λ is called the *highest weight*. Now we consider what weights may occur as highest weights. If $\beta \in \Phi^+$, then a choice of nonzero $x_\beta \in L_\beta$ we have a unique $h_\beta \in H, y_{\beta \in L_{-\beta}}$ such that $x_\beta, h_\beta, y_\beta$ is the usual basis of $\mathfrak{sl}_2$. Now let S_i be the $\mathfrak{sl}_2$ associated to the simple root α_i. Then because v^+ is an eigenvector for h_i, $\lambda(h_i)$ must be a nonnegative integer. In addition, v^+ is maximal for S_i.

Definition 146.   1. Any $\lambda \in H^*$ is called a *weight*.

    2. If $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$ for all i, then λ is an *integral weight*.

    3. If $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{\geq 0}$, then λ is *dominant integral*. If $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{>0}$, then λ is *strictly dominant*.

We will call Λ the set of integral weights and Λ^+ the set of dominant integral weights. We focus on the integral weights, which must be rational combinations of the simple roots. We know that amongst the roots of L there are at most two dominant integral weights: the highest long root Θ and the short dominant root Θ_s.

It is also easy to see that $\Phi \subset \Lambda$, so we have $\mathbb{Z}[\Phi] \subset \Lambda \subset E \simeq \mathbb{R}^n$. We call $\mathbb{Z}[\Phi]$ the root lattice and Λ the weight lattice.

Example 147. In A_2, the dominant region is $0 < \theta < \pi/3$. An example of a dominant weight is $\frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$.

In addition, the dominant integral root is $\alpha_1 + \alpha_2$. Therefore $\mathfrak{sl}_3$ is a highest weight representation for $\Theta = \alpha_1 + \alpha_2$.

The standard representation has maximal vector $(1,0,0)$ and highest weight $\frac{1}{3}(\alpha_2 + 2\alpha_1)$.

This suggests

Proposition 148. *If V is a finite dimensional irreducible representation, then $\lambda \in \Lambda^+$.*

First recall that if R is a rint and L is some left ideal, then $M \simeq R/I$ is a left ideal. In adition, M is a cyclic module (generated by one element).

Definition 149. *V is a standard cyclic module if V is cyclic and $B.v^+ = \mathbb{F}v^+$, where v^+ is the highest weight vector.*

Note that any standard cyclic module is of the form $U(L)/I$, where I contains the span of $x_i, h_i - \lambda(h_i)$.

Theorem 150 (Theorem on standard cyclic modules of highest weight λ). *(a) V is spanned by $\prod_{\beta \in \Phi^+} y_\beta^{i_\beta}$ and V is a direct sum of its weight spaces.*

  (b) The weights are of the form $\mu = \lambda - \sum_{i=1}^n K_i \alpha_i$.

  (c) For all $\mu \in H^*$, V_μ is finite dimensional and $\dim V_\lambda = 1$.

Proof. Write $L = N^- \oplus B$. Then if $u \in U(L)$, by PBW, we have $\sum_I \prod y_\beta^{i_I} u_I$, where β runs over the positive roots and $u_I \in U(B)$. Then $u.v^+ = \sum_I \prod_\beta y_\beta^{i_I} c_I.v^+$. Because c_I is a constant we have the first part of (a).

Next note that $v^+ \in V_\lambda$ implies that w has eigenfunctional $\lambda - \sum i_k \beta_k$. This implies (a). For (b), just write β_i as a sum of the α_i.

Finally, fix $\mu \in H^*$. We need to solve $\mu = \lambda - \sum i_k \beta_k$ in nonnegative integers. This is the same as writing $\lambda - \mu = \sum i_k \beta_k$. Because the positive roots are nonnegative combinations of simple roots, there are a finite number of solutions. In addition, there is one solution for $\lambda - \mu$. $\qquad\square$

Remark 151. The number of solutions of $\lambda - \mu = \sum i_k \beta_k$ is called the *Kostant partition function*. This is given by
$$\prod_{\beta > 0} (1 - e^\beta)^{-1}.$$

We will prove additional parts of the theorem on standard cyclic modules.

Proposition 152. *Each submodule of V is a direct sum of its weight spaces.*

Proof. Let $W \subset V$ be a submodule. Then $w \in W$ is given by $w = v_1 + \cdots + v_n$. Then acting by h repeatedly, we obtain that $v_n \in W$. Then we use induction. $\qquad\square$

Proposition 153. *V is indecomposable with a unique proper maximal submodule.*

Proof. Note that any proper submodule $W \subset V$ lies in a sum of weight spaces not containing V_λ. Therefore V is indecomposable. In addition, the sum of all proper submodules is the unique maximal submodule. $\square$

Corollary 154. *Standard cyclic modules have a unique irreducible quotient.*

23   Lecture 23 (Nov 21)

Recall that V is standard cyclic if it has a maximal vector which generates V. Now let Φ be an irreducible root system. Then we define the *fundamental weights* to be the dual basis to $\frac{2\alpha_i}{(}\alpha_i, \alpha_i)$. We call the final weights λ_i (or ϖ). Then the λ_i are a $\mathbb{Z}$-basis of Λ and the $\mathbb{Z}_{\geq 0}$-monoid spanned by the λ_i is Λ^+.

In general, the λ_i can be written as $\sum d_j \alpha_j$, where $0 < d_j \in \mathbb{Q}$. As a consequence, we have

Proposition 155. *For $\lambda \in \Lambda^+$, the number of $\mu \in \Lambda^+$ with $\mu < \lambda$ is finite.*

Recall from last time that weight spaces in standard cyclic modules are finite-dimensional and that V has a maximal proper submodule. Therefore it has a unique irreducible quotient.

It turns out that the standard cyclic module $Z(\lambda)$ is isomorphic to $U(L) \otimes_{U(B)} D_\lambda$, where $x_\alpha.v = 0, h.v = \lambda(h)v$. This is called the *Verma module* of high weight λ. By PBW, we know $U(L)$ is free over $U(B)$.

It is easy to see that the map from $Z(\lambda)$ to the Verma module is an isomorphism. Then consider the unique irreducible quotient $V(\lambda)$.

Theorem 156. *If $\lambda \in \Lambda^+$, then $V(\lambda)$ is finite dimensional.*

Lemma 157. *Any $V(\lambda)$ has a unique maximal vector.*

Proof. If there are two maximal vectors with weights λ, μ, then it is easy to see that $\lambda = \mu$. We also know that V_λ is one-dimensional. $\square$

Lemma 158.   *1.* $[x_j, y_i^{k+1}] = 0$.

   2. $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_j^{k+1}$.

   3. $[x_i, y_i^{k+1}] = -(k+1)y_i^k(k - h_i)$.

Proof of Theorem 156. Choose $m_i = \langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{\geq 0}$. First set $w = y_i^{m_i+1}$. Then by Lemma 158, we have $x_j.w = x_i.w = 0$, so $N.w = 0$. Thus w is a maximal vector, but it has weight $\lambda - (m_i + 1)\alpha_i$, so $w = 0$.

This implies that $V(\lambda)$ contains a nonzero finite-dimensional S_i-module for each $i = 1, \ldots, n$. Next we observe that V is a sum of finite-dimensional S_i-modules. Now let $W \subset V(\lambda)$ be a finite dimensional S_i-submodule. Then consider the span of $x_\alpha W$ for all $\alpha \in \Phi$. This is finite-dimensional and is an S_i-submodule because $[x_i, x_\alpha]$ is either h_i or $x_{\alpha+\alpha_i}$.

Now let S' be the sum of finite-dimensional S_i-submodules. Then clearly S' is L-stable. Therefore because $V(\lambda)$ is irreducible, $S' = V(\lambda)$. Therefore, for each i, $V(\lambda)$ is a sum of finite-dimensional

S_i-submodules. However, each weight space is finite dimensional, so it is contained in some finite sum of finite-dimensional S_i-modules. Finaly, the sum

$$\bigoplus_{k \in \mathbb{Z}} V_{\mu + k\alpha_i}$$

is a finite-dimensional S_i-module. Thus we need to show that there are finitely many strings. By the standard theory of $\mathfrak{sl}_2$, we know that the string runs between $\mu, s_i(\mu)$, where s_i is the reflection across α_i. Also, $\dim V_{S_i(\mu)} = \dim V_\mu$. Also, there exists w such that $w(\mu) \in \Lambda^+$. Therefore all weights in $V(\lambda)$ are W-conjugate to a dominant weight μ^+, so there are only finitely many strings of the form given above. □

24   LECTURE 24 (DEC 3)

We begin by finishing the proof of Theorem 156. This is given at the end of the previous lecture.

Corollary 159. *Note that* $\dim V(\lambda)_\mu = \dim V(\lambda)_{w(\mu)}$ *for* $w \in W$.

Now it remains to compute the dimension of $V(\lambda)_\mu$ and the dimension of $V(\lambda)$. We have formulas of Weyl, Freudenthal, and Kostant.

Consider $G = \operatorname{Int} L$ the group of automorphisms of L generated by $e^{\operatorname{ad} x}$ for x nilpotent. Then construct the flag variety G/B and the line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is given by the character $x_i x_j^{-1}$.

Then we have:

Theorem 160 (Borel-Weil-Bott). *For any weight* λ, *we have*

$$H^0(G/B, \mathcal{L}_\lambda) = \begin{cases} 0 & \lambda \notin \Lambda^+ \\ V(\lambda) & \lambda \in \Lambda^+ \end{cases}.$$

In addition, if $w(\lambda + \delta)$ *is strictly dominant for some* δ, w, *then* $\mathcal{L}_\lambda$ *has a unique nonzero cohomology group*

$$H^{\ell(w)}(G/B, \mathcal{L}_\lambda) \simeq V(w(\lambda + \delta) - \delta).$$

Theorem 161 (Kostant's Formula). *For weights* λ, μ, *recall that* $\dim Z(\lambda)_\mu$ *is the number of ways to write* $\lambda - \mu$ *as a nonnegative sum of positive roots. Write* $p(\lambda - \mu) = \dim Z(\lambda)_\mu$. *We have*

$$\dim V(\lambda)_\mu = \sum_{w \in W} (-1)^{\ell(w)} p(w(\lambda + \delta) - (\mu + \delta)).$$

25   LECTURE 25 (DEC 5)

Note that $V(\lambda)$ is finite dimensional iff λ is dominant integral. Today we will compute $\dim V(\lambda)$ and $\dim V(\lambda)_\mu$. First, we will capture the action of H on V. Assume that V is a direct sum of weight spaces for H (this is always true if V is finite-dimensional). Consider the group ring $\mathbb{Z}[\Lambda]$ where $\mu \in \Lambda$ is represented by e^μ (so we do not confuse multiplication and addition).

Definition 162. The *character* of $V = \bigoplus_{\mu \in H^*} V_\mu$ is

$$\mathrm{ch}_V = \sum_{\mu \in \Lambda} (\dim V_\mu) e^\mu.$$

Then recall that $\dim Z(\lambda)_\mu = p(\lambda - \mu)$.

Example 163. The dominant weights (for A_2) less than $2\alpha_1 + 2\alpha_2$ are $2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2$. Then we obtain $p(2\alpha_1 + 2\alpha_2) = 3$. More generally, $p(a\alpha_1 + b\alpha_2) = \min(a, b) + 1$.

Example 164. Note that $V(\alpha_1 + \alpha_2) = L$. In the Verma module, we see that $\alpha_1 + \alpha_2, \alpha_1, \alpha_2, \alpha_1 - \alpha_2, \alpha_2 - \alpha_1$ have dimension 1, $0, -\alpha_1, -\alpha_2$ have dimension 2, and $-\alpha_1 - \alpha_2$ has dimension 3.

Now consider $V(2\alpha_1 + 2\alpha_2)$. Applying s_1, then we obtain $-\alpha_1 + 2\alpha_2$, and applying s_2, we obtain $2\alpha_1 - \alpha_2$. Then note that $2\alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2$ have dimension 1, and $\alpha_1 + \alpha_2$ has dimension 2, and 0 has dimension 3. Then we put this together to find that $\dim V(2\alpha_1 + 2\alpha_2) = 27$.

Recall the notion of a composition series for a finite group. We apply the same notion here for R-modules over any rings R. In addition, when composition series exist, a version of the Jordan-Hölder theorem holds.

Theorem 165. *Every Verma module has a composition series. In addition, simple quotients are of the form $V(\mu)$, where $\mu + \delta$ is w-conjugate to $\lambda + \delta$.*

Call $\mu \sim \lambda$ (linked) if there exists w such that $w(\lambda + \delta) = \mu + \delta$. Then define the set of $\mu \le \lambda$ that are linked to λ by $\Theta(\lambda)$. then we have

Corollary 166. $\mathrm{ch}\, Z(\lambda) = \sum_{\mu \in \Theta(\lambda)} c_\mu \mathrm{ch}(V(\mu))$, *where* $c_\mu \in \mathbb{N}$.

This formula comes from analyzing a short exact sequence. By some Möbius inversion-like process, we obtain $\mathrm{ch}(V(\lambda)) = \sum_{\mu \in \Theta(\lambda)} d_\mu^\lambda \mathrm{ch}(Z(\mu))$.

26   Lecture 26 (Dec 10)

We can define a q-analog of the Konstant partition function:

$$p_q(\mu) = \sum_{(k_\alpha \in S_\mu)} q^{\#\{\text{nonzero } k_\alpha\}}.$$

Then we may deine a q-analog of multiplicity. Then for dominant integral λ, μ, $m_\mu^\lambda(q)$ has nonzero coefficients and is a Kazhdan-Lusztig polynomial for the affine Weyl group of L.

Sketch of proof of Theorem 165. Consider $\mathcal{Z} = Z(U(L))$. In addition, consider a maximal vector v^+ of weight λ. Then $h.z.v^+ = z.h.v^+ = \lambda(h)z.v^+$ and $x_\alpha.z.v^+ = z.x_\alpha.v^+ = 0$, so $z.v^+$ is a scalar multiple of v^+. Then this scalar is $\chi_\lambda : \mathcal{Z} \to \mathbb{F}$. In fact, $z \in \mathcal{Z}$ acts by this scalar on all of $Z(\lambda)$.

Next choose α_i such that $\langle \lambda, \alpha_i \rangle = m \ge 0$. Then $y_{\alpha_i}^{m+1}.v^+$ is a maximal vector of weight $\lambda - (m+1)\alpha_i = s_{\alpha_i}(\lambda + \delta) - \delta$. Then, after taking the quotient, we can obtain that $\chi_{s_i \lambda} = \chi_\lambda$. Thus, χ_λ is invariant under W.

Now using Harish-Chandra, if $Z(\lambda)$ is not irreducible, then we have a proper submodule $V \subset Z(\lambda)$. Then choose v a maximal vector of some weight μ. Then we have a morphism $Z(\mu) \to U(L).v$, so

$\chi_\mu = \chi_\lambda$, and thus $\lambda \sim \mu$. There are only finitely many μ such that $\mu \sim \lambda$, and each $Z(\lambda)_\mu$ is finite dimensional, so the process terminates with a composition series. $\qquad\square$

Theorem 167 (Harish-Chandra). *For $\lambda \in H^*$, if $\chi_\lambda = \chi_\mu$, then $\lambda \sim \mu$.*

Remark 168. $\mathcal{Z} = U(L)^G$. In addition, by PBW, $\mathcal{Z} \simeq S(L)^G$. Finally, $S(L)^G \simeq S(H)^W$ and is a polynomial ring $\mathbb{C}[f_1, \ldots, f_n]$. In addition, $|W| = \prod \deg f_i$. We may also define a q-analog

$$\sum_{w \in W} q^{\ell(w)} = \prod \frac{q^{\deg f_i} - 1}{q - 1}.$$

Theorem 169 (Weyl Character Formula). *For λ dominant integral, we have*

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w \in W}(-1)^{\ell(w)} e^{w(\lambda+\delta)}}{\sum_{w \in W}(-1)^{\ell(2)} e^{w(\delta)}}.$$

Example 170. For $\mathfrak{sl}_2$, the Weyl character is

$$\frac{e^{m+1} - e^{-m-1}}{e^1 - e^{-1}} = e^m + e^{m-2} + \cdots + e^{-m+2} + e^{-m}.$$

Theorem 171 (Weyl Dimension Formula). *Choose an invariant inner product on H^*. Then we have*

$$\dim V(\lambda) = \frac{\prod_{\alpha>0}(\lambda + \delta, \alpha)}{\prod_{\alpha>0}(\delta, \alpha)}.$$