PACKAGING OF GROMOV-WITTEN INVARIANTS

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Abstract. The goal of this lecture is to explain, in increasing level of difficulty, how to package Gromov-Witten invariants.

Contents

1. Introduction

I apologize in advance if most of this talk is basic to the audience, but we do need to be on a common footing.

Let X be a smooth projective variety. Then for any $g, n \in \mathbb{Z}_{\geqslant 0}, \beta \in H_2(X, \mathbb{Z})$, there exists a moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ (Givental's notation is $X_{g,n,\beta}$) of *stable maps* $f\colon C \to X$ from genus-g, n-marked prestable curves to X with $f_*[C] = \beta$. It is well-known that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in A_\delta(\overline{\mathcal{M}}_{g,n}(X, \beta)), \qquad \delta = \int_\beta c_1(X) + (\dim X - 3)(1 - g) + 3.$$

In addition, there is a universal curve and sections

$$\mathcal{C} \underset{\sigma_i}{\overset{\pi}{\rightleftarrows}} \overline{\mathcal{M}}_{g,n}(X, \beta).$$

In this setup, there are tautological classes

$$\psi_i \coloneqq c_1(\sigma_i^* \omega_\pi) \in H^2(\overline{\mathcal{M}}_{g,n}(X, \beta)).$$

This allows us to define individual Gromov-Witten invariants by

$$\left\langle \tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle_{g,n,\beta}^X = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \phi_i \cdot \psi_i^{a_i}.$$

These invariants satisfy various relations. The first is the *string equation:*

$$\left\langle \tau_0(1)\tau_{a_1}(\phi_1) \cdots \tau_{a_n}(\phi_n) \right\rangle_{g,n+1,\beta}^X = \sum_{i=1}^n \left\langle \tau_{a_i - 1}(\phi_i) \prod_{j \neq i} \tau_{a_j}(\phi_j) \right\rangle_{g,n,\beta}^X.$$

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The next is the *dilaton equation:*

$$\left\langle \tau_1(1)\tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n+1,\beta} = (2g-2+n)\left\langle \tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n,\beta}.$$

Finally, we have the *divisor equation* when one insertion is a divisor $D \in H^2(X)$:

$$\begin{aligned}\left\langle \tau_0(D)\tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n+1,\beta} =& \left(\int_\beta D\right)\cdot\left\langle \tau_{a_1}(\phi_1)\cdots\tau_{a_n}(\phi_n)\right\rangle^X_{g,n,\beta} \\ &+ \sum_{i=1}^n \left\langle \tau_{a_i-1}(\phi_i\cdot D)\prod_{j\neq i}\tau_{a_j}(\phi_j)\right\rangle^X_{g,n,\beta}.\end{aligned}$$

It is often useful to package Gromov-Witten invariants into various generating series.

Definition 1.1. The *quantum cohomology* $QH^*(X)$ of X is defined by the formula

$$(a\star_t b, c) := \sum_{\beta,n}\frac{Q^\beta}{n!}\langle a,b,c,t,\dots,t\rangle^X_{0,3+n,\beta}$$

for any $t\in H^*(X)$. This is a commutative and associative product.

The *small quantum cohomology* is obtained by setting $t=0$ and the ordinary cohomology is obtained by further setting $Q=0$.

Remark 1.2. Convergence of the formula does not hold in general, so quantum cohomology needs to be treated as a formal object.

Definition 1.3. Let ϕ_i be a basis of $H^*(X)$ and ϕ^i be the dual basis. Then the J-*function* of X is the cohomology-valued function

$$J^X(t,z) := z+t+\sum_i\sum_{n,\beta}\frac{Q^\beta}{n!}\left\langle\frac{\phi_i}{z-\psi},t,\dots,t\right\rangle^X_{0,n+1,\beta}\phi^i.$$

Definition 1.4. The *genus-*0 *GW potential* of X is the (formal) function

$$\mathcal{F}^X(t(z)) = \sum_{\beta,n}\frac{Q^\beta}{n!}\langle t(\psi),\dots,t(\psi)\rangle^X_{0,n,\beta}.$$

The associativity of the quantum product is equivalent to the PDE

$$\sum_{e,f}\mathcal{F}^X_{abe}\eta^{ef}\mathcal{F}_{cdf} = \sum_{e,f}\mathcal{F}^X_{ade}\eta^{ef}\mathcal{F}^X_{bcf}$$

for any a,b,c,d, which are known as the *WDVV equations*. Here, we choose coordinates on $H^*(X)$.

2. Frobenius manifolds

A Frobenius manifold can be thought of as a formalization of the WDVV equations.

Definition 2.1. A *Frobenius manifold* is a complex manifold M with a flat symmetric bilinear form $\langle -,-\rangle$ (meaning that the Levi-Civita connection has zero curvature) on TM and a holomorphic system of (commutative, associative) products $\star_t$ on T_tM satisfying:

(1) The unit vector field $\mathbf{1}$ is flat: $\nabla\mathbf{1} = 0$;
(2) For any t and $a, b, c \in T_tM$, $\langle a \star_t b, c\rangle = \langle a, b \star_t c\rangle$;
(3) If $c(u, v, w) := \langle u \star_t v, w\rangle$, then the tensor $(\nabla_z c)(u, v, w)$ is symmetric in $u, v, w, z \in T_tM$.

If there exists a vector field E such that $\nabla\nabla E = 0$ and complex number d such that:

(1) $\nabla\nabla E = 0$;
(2) $\mathcal{L}_E(u \star v) - \mathcal{L}_E u \star v - u \star \mathcal{L}_E v = u \star v$ for all vector fields u, v;
(3) $\mathcal{L}_E\langle u, v\rangle - \langle \mathcal{L}_E u, v\rangle - \langle u, \mathcal{L}_E v\rangle = (2 - d)\langle u, v\rangle$ for all vector fields u, v,

then E is called an *Euler vector field* and the Frobenius manifold M is called *conformal*.

Example 2.2. Let X be a smooth projective variety. Then we can give $H^*(X)$ the structure of a Frobenius algebra with the Poincaré pairing and the quantum product. Note that the quantum product does not converge in general, so we must treat this as a formal object. The Euler vector field is given by

$$E_X = c_1(X) + \sum_i \left(1 - \frac{\deg \phi_i}{2}\right) t^i \phi_i,$$

where a general element of $H^*(X)$ is given by $t = \sum_i t^i \phi_i$. We will also impose that $\phi_1 = \mathbf{1}$. There is another very important structure, the *quantum connection*, which is given by the formula

$$\begin{aligned} \nabla_{t^i} &:= \partial_{t^i} + \frac{1}{z}\phi_i \star_t \\ \nabla_{z\frac{d}{dz}} &:= z\frac{d}{dz} - \frac{1}{z}E_X \star_t + \mu_X. \end{aligned}$$

Here, μ_X is the *grading operator*, defined for pure degree classes $\phi \in H^*(X)$ by

$$\mu_X(\phi) = \frac{\deg \phi - \dim X}{2}\phi.$$

Finally, in the direction of the Novikov variables, we have

$$\nabla_{\xi Q \partial_Q} = \xi Q\, \partial_Q + \frac{1}{z}\xi \star_t .$$

Definition 2.3. The *quantum* D*-module* of X is the module $H^*(X)[z][\![Q, t]\!]$ with the quantum connection defined above.

Remark 2.4. It is important to note that the quantum connection has a fundamental solution matrix $S^X(t, z)$ given by

$$S^X(t, z)\phi = \phi + \sum_i \sum_{n,\beta} \frac{Q^\beta}{n!}\phi^i \left\langle \frac{\phi_i}{z - \psi}, \phi, t, \ldots, t \right\rangle^X_{0,n+2,\beta}.$$

Using this formalism, the J-function is given by $S^X(t, z)\mathbf{1} = z^{-1}J^X(t, z)$.

3. Givental formalism

The Givental formalism is a geometric way to package enumerative (CohFT) invariants cleanly. We begin by defining the symplectic space

$$\mathcal{H} \coloneqq H^*(X, \Lambda)(\!(z^{-1})\!)$$

with the symplectic form

$$\Omega(f, g) \coloneqq \mathrm{Res}_{z=0}\, f(-z)g(z).$$

This has a polarization by Lagrangian subspaces

$$\mathcal{H}_+ \coloneqq H^*(X, \Lambda)[z], \qquad \mathcal{H}_- \coloneqq z^{-1}H^*(X, \Lambda)[\![z^{-1}]\!]$$

giving $\mathcal{H} \cong T^*\mathcal{H}_+$ as symplectic vector spaces.

Taking the *dilaton shift*

$$q(z) = t(z) - z = -z + t_0 + t_1 z + t_2 z^2 + \cdots,$$

we can now think of $\mathcal{F}^X$ has a formal function on $\mathcal{H}_+$ near $q = -z$. This convention is called the *dilaton shift*.