Idea 1: Varying θ_{Frogs}

Frogs

November 13, 2024

1 Modelling and Data

1.1 Modelling

- $\theta_{Frogs}^{(0)} \sim \text{Po}(\lambda)$: initial number of frogs
- $\theta_{Frogs}^{(t)} = \theta_{Frogs}^{(t-1)} + R^{(t)}$: number of frogs at time t, $R^{(t)} \sim F_r$ is a random variable, can be Geometric, Possion, etc, with parameter r
- θ_{Love}, where $\theta_{Love} + 1 \sim \text{Geo}(p_{Love})$ (Can change dist): true love, (0 means don't want love, 1 means congwen, rest means frog number $\theta_{Love} - 1$), assume first send letter = more attractive
- $\theta_p \sim \text{U}(0, 1)$: Pr(reply|is true love)
- Each Frogs have probability p_{send} to send letter, and they will only send one letter
- Congwen will send a letter in sats and suns

1.2 Data x_t

- $t = 0, \ldots, T$: time t
- $m_t \geq 0$: number of letters from congwen at time t
- $n_t \geq 0$: number of letters from other frogs at time t
- $Z_1, Z_2, \ldots, Z_{m_t} \in \{0, 1\}$: whether the congwen's letter is replied
- $X_1, X_2, \ldots, X_{n_t} \in \{0, 1\}$: whether the other frog's letter is replied
- $Z_t = X_t = 0$
- Truely random variables are n_t, X_t, Z_t

2 Computations

$$f(Z_1 = \ldots = Z_{m_t} = 0 | m_t, \theta) = (1 - \theta_p)^{m_t \mathbb{1}_{\{\theta_{Love}=1\}}}$$

$$f(X_1 = \ldots = X_{n_t} = 0 | n_t, \theta) = (1 - \theta_p)^{\mathbb{1}_{\{2 \leq \theta_{Love} \leq n_t+1\}}}$$

$$f(n_t|\theta) = \binom{\theta_{Frogs}^{(t)}}{n_t} p_{send}^{n_t} (1 - p_{send})^{\theta_{Frogs}^{(t)} - n_t} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t\}}$$

$$\begin{aligned} f(x_t|\theta) &= \frac{\theta_{Frogs}^{(t)}!}{n_t!(\theta_{Frogs}^{(t)} - n_t)!} p_{send}^{n_t} (1 - p_{send})^{\theta_{Frogs}^{(t)} - n_t} \\ &\quad \cdot (1 - \theta_p)^{\mathbb{1}_{\{2 \leq \theta_{Love} \leq n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t\}} \end{aligned}$$

$$\theta_{Frogs}^{(t)} = \theta_{Frogs}^{(0)} + \sum_{i=1}^{t} R^{(t)} \sim \text{Po}(\lambda + tr) \qquad R^{(t)} \sim \text{Po}(r)$$

$$\pi(\theta) = \frac{e^{-(\lambda+tr)} (\lambda + tr)^{\theta_{Frogs}^{(t)}}}{\theta_{Frogs}^{(t)}!} p_{Love} (1 - p_{Love})^{\theta_{Love}} \mathbb{1}_{\{\theta_{Frogs}^{(t)}, \theta_{Love} \geq 0, 0 \leq \theta_p \leq 1\}}$$

$$
\begin{aligned}
&\pi(\theta|x_t) \\
&\propto f(x_t|\theta)\pi(\theta) \\
&= \frac{\theta^{(t)}_{Frogs}!}{n_t!(\theta^{(t)}_{Frogs}-n_t)!} p_{send}^{n_t}(1-p_{send})^{\theta^{(t)}_{Frogs}-n_t} \\
&\quad \cdot (1-\theta_p)^{\mathbb{1}_{\{2\le\theta_{Love}\le n_t+1\}}+m_t\mathbb{1}_{\{\theta_{Love}=1\}}}\mathbb{1}_{\{\theta^{(t)}_{Frogs}\ge n_t\}} \\
&\quad \cdot \frac{e^{-(\lambda+tr)}(\lambda+tr)^{\theta^{(t)}_{Frogs}}}{\theta^{(t)}_{Frogs}!} p_{Love}(1-p_{Love})^{\theta_{Love}}\mathbb{1}_{\{\theta^{(t)}_{Frogs},\theta_{Love}\ge 0, 0\le\theta_p\le 1\}} \\
&\propto \frac{((1-p_{send})(\lambda+tr))^{\theta^{(t)}_{Frogs}}(1-p_{Love})^{\theta_{Love}}}{(\theta^{(t)}_{Frogs}-n_t)!}(1-\theta_p)^{\mathbb{1}_{\{2\le\theta_{Love}\le n_t+1\}}+m_t\mathbb{1}_{\{\theta_{Love}=1\}}}\mathbb{1}_{\{\theta^{(t)}_{Frogs}\ge n_t,\theta_{Love}\ge 0,0\le\theta_p\le 1\}}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\theta^{(t)}_{Frogs}=0}^{\infty}\frac{((1-p_{send})(\lambda+tr))^{\theta^{(t)}_{Frogs}}}{(\theta^{(t)}_{Frogs}-n_t)!}\mathbb{1}_{\{\theta^{(t)}_{Frogs}\ge n_t\}} \\
&= ((1-p_{send})(\lambda+tr))^{n_t}\sum_{x=0}^{\infty}\frac{((1-p_{send})(\lambda+tr))^{x}}{x!} \\
&= e^{(1-p_{send})(\lambda+tr)}((1-p_{send})(\lambda+tr))^{n_t}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}}\sum_{\theta_{Love}=0}^{\infty}(1-p_{Love})^{\theta_{Love}}(1-\theta_p)^{\mathbb{1}_{\{2\le\theta_{Love}\le n_t+1\}}+m_t\mathbb{1}_{\{\theta_{Love}=1\}}}\mathbb{1}_{\{\theta_{Love}\ge 0,0\le\theta_p\le 1\}}d\theta_p \\
&= \int_0^1\sum_{\theta_{Love}=0}^{\infty}(1-p_{Love})^{\theta_{Love}}(1-\theta_p)^{\mathbb{1}_{\{2\le\theta_{Love}\le n_t+1\}}+m_t\mathbb{1}_{\{\theta_{Love}=1\}}}d\theta_p \\
&= \int_0^1 1+(1-\theta_p)^{m_t}(1-p_{Love})+(1-\theta_p)\sum_{\theta_{Love}=2}^{n_t+1}(1-p_{Love})^{\theta_{Love}}+\sum_{\theta_{Love}=n_t+2}^{\infty}(1-p_{Love})^{\theta_{Love}}d\theta_p \\
&= \int_0^1 1+(1-\theta_p)^{m_t}(1-p_{Love})+\frac{(1-\theta_p)(1-p_{Love})^2}{p_{Love}}(1-(1-p_{Love})^{n_t})+\frac{(1-p_{Love})^{n_t+2}}{p_{Love}}d\theta_p \\
&= \int_0^1 1+(1-\theta_p)^{m_t}(1-p_{Love})+\frac{(1-\theta_p)(1-p_{Love})^2}{p_{Love}}+\frac{\theta_p(1-p_{Love})^{n_t+2}}{p_{Love}}d\theta_p \\
&= 1+\frac{1-p_{Love}}{m_t+1}+\frac{(1-p_{Love})^2+(1-p_{Love})^{n_t+2}}{2p_{Love}}
\end{aligned}
$$

$$
\begin{aligned}
C(x_t) &= \int_{\mathbb{R}}\sum_{\theta_{Love}=0}^{\infty}\sum_{\theta^{(t)}_{Frogs}=0}^{\infty}\frac{((1-p_{send})(\lambda+tr))^{\theta^{(t)}_{Frogs}}(1-p_{Love})^{\theta_{Love}}}{(\theta^{(t)}_{Frogs}-n_t)!}(1-\theta_p)^{\mathbb{1}_{\{2\le\theta_{Love}\le n_t+1\}}+m_t\mathbb{1}_{\{\theta_{Love}=1\}}} \\
&\quad \cdot\mathbb{1}_{\{\theta^{(t)}_{Frogs}\ge n_t,\theta_{Love}\ge 0,0\le\theta_p\le 1\}}d\theta_p \\
&= e^{(1-p_{send})(\lambda+tr)}((1-p_{send})(\lambda+tr))^{n_t}\left\{1+\frac{1-p_{Love}}{m_t+1}+\frac{(1-p_{Love})^2+(1-p_{Love})^{n_t+2}}{2p_{Love}}\right\}
\end{aligned}
$$

$$
\pi(\theta|x_t) = \frac{((1-p_{send})(\lambda+tr))^{\theta^{(t)}_{Frogs}}(1-p_{Love})^{\theta_{Love}}}{C(x_t)(\theta^{(t)}_{Frogs}-n_t)!}(1-\theta_p)^{\mathbb{1}_{\{2\le\theta_{Love}\le n_t+1\}}+m_t\mathbb{1}_{\{\theta_{Love}=1\}}}\mathbb{1}_{\{\theta^{(t)}_{Frogs}\ge n_t,\theta_{Love}\ge 0,0\le\theta_p\le 1\}}
$$

Interesting:

$$
(\theta^{(t)}_{Frogs}-n_t)|x_t \sim \mathrm{Po}((1-p_{send})(\lambda+tr))
$$