Idea 2 (Improve on 1): Generalized Stephen's model

Frogs

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1 Modelling and Data

1.1 Modelling

- $\theta_{Frogs}^{(0)} \sim \text{Po}(\lambda)$: initial number of frogs
- $\theta_{Frogs}^{(t)} = \theta_{Frogs}^{(t-1)} + R^{(t)}$: number of frogs at time t, $R^{(t)} \sim F_r$ is a random variable, can be Geometric, Possion, etc, with parameter r
- $\theta_{Love}^{(i)} \sim \text{U}(0,1)$ is "likeness" of Frog i, or Congwen if $i=0$, or None of above is $i=-1$, this "likeness" encodes information of how much Peony loves the frog, $\theta_{Love}^{(-1)} \sim \pi$.
- $\Pr(\text{reply Frog } i) = f_{reply}(\theta_{Love}, \theta_p, i)$, which can be some function containing its own parameters θ_p.
- Each Frogs have probability p_{send} to send letter, and they will only send one letter
- Congwen will send a letter in sats and suns
- Stephen's model will be $\theta_3 = \theta_p$, with $\theta_2 = \arg\max_i \theta_{Love}^{(i)}$, $f_{reply}(\theta_{Love}, \theta_p, i) = \theta_3 \mathbb{1}_{\{i=\theta_2\}}$, with $r=0$.

1.2 Data x_t

- $t = 0, \ldots, T$: time t
- $m_t \geq 0$: number of letters from congwen at time t
- $n_t \geq 0$: number of letters from other frogs at time t
- $Z_1, Z_2, \ldots, Z_{m_t} \in \{0,1\}$: whether the congwen's letter is replied
- $X_1, X_2, \ldots, X_{n_t} \in \{0,1\}$: whether the other frog's letter is replied
- $Z_t = X_t = 0$
- Truely random variables are n_t, X_t, Z_t

2 Computations

$$f(Z_1 = \ldots = Z_{m_t} = 0 | n_t, \theta_{Love}, \theta_p) = (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t}$$

$$f(X_1 = \ldots = X_{n_t} = 0 | n_t, \theta_{Love}, \theta_p) = \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i))$$

$$f(n_t | \theta_{Frogs}^{(t)}) = \binom{\theta_{Frogs}^{(t)}}{n_t} p_{send}^{n_t} (1 - p_{send})^{\theta_{Frogs}^{(t)} - n_t} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t\}}$$

$$f(x_t | \theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p) = \frac{\theta_{Frogs}^{(t)}!}{n_t!(\theta_{Frogs}^{(t)} - n_t)!} p_{send}^{n_t} (1 - p_{send})^{\theta_{Frogs}^{(t)} - n_t} \cdot (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t} \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i)) \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t\}}$$

$$\theta_{Frogs}^{(t)} = \theta_{Frogs}^{(0)} + \sum_{i=1}^{t} R^{(t)} \sim \text{Po}(\lambda + tr) \qquad R^{(t)} \sim \text{Po}(r)$$

$$\pi(\theta_{Frogs}^{(0)}, \theta_{Love}, \theta_p) = \frac{e^{-\lambda} \lambda^{\theta_{Frogs}^{(0)}}}{\theta_{Frogs}^{(0)}!} \pi(\theta_p) \pi(\theta_{Love}) \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq 0\}}$$

$$\pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p) = \frac{e^{-(\lambda+tr)} (\lambda + tr)^{\theta_{Frogs}^{(t)}}}{\theta_{Frogs}^{(t)}!} \pi(\theta_p) \pi(\theta_{Love}^{(-1)}) \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq 0, 1 \geq \theta_{Love}^{(i)} \geq 0, \dim(\theta_{Love}) = \theta_{Frogs}^{(t)} + 2\}}$$

$$
\begin{aligned}
&\pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p | x_t) \\
&\propto f(x_t|\theta)\pi(\theta) \\
&\propto \frac{\theta_{Frogs}^{(t)}!}{(\theta_{Frogs}^{(t)} - n_t)!} (1 - p_{send})^{\theta_{Frogs}^{(t)} - n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t} \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i)) \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t\}} \\
&\quad \cdot \frac{e^{-(\lambda + tr)} (\lambda + tr)^{\theta_{Frogs}^{(t)}}}{\theta_{Frogs}^{(t)}!} \pi(\theta_p) \pi(\theta_{Love}^{(-1)}) \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq 0, 1 \geq \theta_{Love}^{(i)} \geq 0, \dim(\theta_{Love}) = \theta_{Frogs}^{(t)} + 2\}} \\
&\propto \frac{((1 - p_{send})(\lambda + tr))^{\theta_{Frogs}^{(t)} - n_t}}{(\theta_{Frogs}^{(t)} - n_t)!} \pi(\theta_p) \pi(\theta_{Love}^{(-1)}) (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t} \\
&\quad \cdot \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i)) \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t, 1 \geq \theta_{Love}^{(i)} \geq 0, \dim(\theta_{Love}) = \theta_{Frogs}^{(t)} + 2\}} \\
&\pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p | x_t) \\
&= \frac{e^{-(1 - p_{send})(\lambda + tr)} ((1 - p_{send})(\lambda + tr))^{\theta_{Frogs}^{(t)} - n_t}}{(\theta_{Frogs}^{(t)} - n_t)!} \cdot \frac{1}{C(x_t)} \pi(\theta_p) \pi(\theta_{Love}^{(-1)}) (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t} \\
&\quad \cdot \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i)) \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t, 1 \geq \theta_{Love}^{(i)} \geq 0, \dim(\theta_{Love}) = \theta_{Frogs}^{(t)} + 2\}} \\
&\pi(\theta_{Frogs}^{(0)} | \theta_{Frogs}^{(t)}) \\
&= \frac{\pi(\theta_{Frogs}^{(t)} | \theta_{Frogs}^{(0)}) \pi(\theta_{Frogs}^{(0)})}{\pi(\theta_{Frogs}^{(t)})} \\
&= \frac{e^{-tr} (tr)^{\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)}}}{(\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)})!} \cdot \frac{e^{-\lambda} \lambda^{\theta_{Frogs}^{(0)}}}{\theta_{Frogs}^{(0)}!} \cdot \frac{\theta_{Frogs}^{(t)}!}{e^{-(\lambda + tr)} (\lambda + tr)^{\theta_{Frogs}^{(t)}}} \\
&= \frac{\theta_{Frogs}^{(t)}!}{\theta_{Frogs}^{(0)}! (\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)})!} \left(\frac{\lambda}{\lambda + tr} \right)^{\theta_{Frogs}^{(0)}} \left(\frac{tr}{\lambda + tr} \right)^{\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)}} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq \theta_{Frogs}^{(0)} \geq 0\}} \\
&\pi(\theta_{Frogs}^{(0)}, \theta_{Love}, \theta_p | x_t) \\
&= \sum_{\theta_{Frogs}^{(t)} = 0}^{\infty} \pi(\theta_{Frogs}^{(0)} | \theta_{Frogs}^{(t)}) \pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p | x_t) \\
&= \frac{e^{-(1 - p_{send})(\lambda + tr)} (\lambda / tr)^{\theta_{Frogs}^{(0)}}}{\theta_{Frogs}^{(0)}! ((1 - p_{send})(\lambda + tr))^{n_t}} \cdot \pi(\theta_p) \mathbb{1}_{\{\theta_{Frogs}^{(0)} \geq 0, 1 \geq \theta_{Love}^{(i)} \geq 0\}} \\
&\quad \times \sum_{\theta_{Frogs}^{(t)} = \max\{n_t, \theta_{Frogs}^{(0)}\}}^{\infty} \frac{\theta_{Frogs}^{(t)}! ((1 - p_{send}) tr)^{\theta_{Frogs}^{(t)}}}{(\theta_{Frogs}^{(t)} - n_t)! (\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)})!} \cdot \frac{1}{C(x_t)} \pi(\theta_{Love}^{(-1)}) (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t} \\
&\qquad \cdot \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i)) \mathbb{1}_{\{\dim(\theta_{Love}) = \theta_{Frogs}^{(t)} + 2\}} \\
&\approx \frac{e^{-(1 - p_{send})\lambda} (\lambda / tr)^{\theta_{Frogs}^{(0)}}}{\theta_{Frogs}^{(0)}! ((1 - p_{send})(\lambda + tr))^{n_t}} \cdot \pi(\theta_p) \mathbb{1}_{\{\theta_{Frogs}^{(0)} \geq 0, 1 \geq \theta_{Love}^{(i)} \geq 0\}} \qquad P_{1:n} \sim \mathrm{Po}((1 - p_{send}) tr) \\
&\quad \times \sum_{i=1}^{n} \frac{(P_i + \max\{n_t, \theta_{Frogs}^{(0)}\})!}{(P_i + |n_t - \theta_{Frogs}^{(0)}|)!} ((1 - p_{send}) tr)^{\max\{n_t, \theta_{Frogs}^{(0)}\}} \mathbb{1}_{\{\dim(\theta_{Love}) = P_i + \max\{n_t, \theta_{Frogs}^{(0)}\} + 2\}} \\
&\qquad \cdot \frac{\pi(\theta_{Love}^{(-1)}) (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t}}{C(x_t)} \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i)) \\
&= \frac{e^{-(1 - p_{send})} (\lambda / tr)^{\theta_{Frogs}^{(0)}}}{\theta_{Frogs}^{(0)}! ((1 - p_{send})(\lambda + tr))^{n_t}} \cdot \pi(\theta_p) \mathbb{1}_{\{\theta_{Frogs}^{(0)} \geq 0, 1 \geq \theta_{Love}^{(i)} \geq 0\}} \cdot \frac{(\dim(\theta_{Love}) - 2)!}{(\dim(\theta_{Love}) - \min\{n_t, \theta_{Frogs}^{(0)}\} - 2)!} \\
&\quad \cdot ((1 - p_{send}) tr)^{\max\{n_t, \theta_{Frogs}^{(0)}\}} \cdot \frac{e^{-(1 - p_{send}) tr} ((1 - p_{send}) tr)^{\dim(\theta_{Love}) - \max\{n_t, \theta_{Frogs}^{(0)}\} - 2}}{(\dim(\theta_{Love}) - \max\{n_t, \theta_{Frogs}^{(0)}\} - 2)!} \\
&\quad \cdot \frac{\pi(\theta_{Love}^{(-1)}) (1 - f_{reply}(\theta_{Love}, \theta_p, 0))^{m_t}}{C(x_t)} \prod_{i=1}^{n_t} (1 - f_{reply}(\theta_{Love}, \theta_p, i)) \mathbb{1}_{\{\dim(\theta_{Love}) - 2 \geq \max\{n_t, \theta_{Frogs}^{(0)}\}\}}
\end{aligned}
$$

$\pi(\theta_{Frogs}^{(0)}, \theta_{Love}, \theta_p | x_t)$ Stephen's model

$$= \frac{e^{-(1-p_{send})}\lambda^{\theta_{Frogs}^{(0)}-n_t}}{C(x_t)\theta_{Frogs}^{(0)}!(1-p_{send})^{n_t}} \cdot \frac{(\dim(\theta_{Love})-2)!}{(\dim(\theta_{Love})-\min\{n_t,\theta_{Frogs}^{(0)}\}-2)!} \cdot (\dim(\theta_{Love})-1)(\theta_{Love}^{(-1)})^{\dim(\theta_{Love})-2}(1-p_{send})^{\theta_{Frogs}^{(0)}}$$

$$\cdot \left(1-\theta_p \mathbb{1}_{\{\arg\max_j \theta_{Love}^{(j)}=0\}}\right)^{m_t} \prod_{i=1}^{n_t} \left(1-\theta_p \mathbb{1}_{\{\arg\max_j \theta_{Love}^{(j)}=i\}}\right) \mathbb{1}_{\{\theta_{Frogs}^{(0)}=\dim(\theta_{Love})-2\geq n_t, 1\geq\theta_{Love}^{(i)},\theta_p\geq 0\}}$$

$$= \frac{e^{-(1-p_{send})}(\lambda(1-p_{send}))^{\theta_{Frogs}^{(0)}-n_t}(\theta_{Frogs}^{(0)}+1)(\theta_{Love}^{(-1)})^{\theta_{Frogs}^{(0)}}}{C(x_t)(\theta_{Frogs}^{(0)}-n_t)!} (1-\theta_p)^{m_t \mathbb{1}_{\{\arg\max_j \theta_{Love}^{(j)}=0\}}}$$

$$\cdot \prod_{i=1}^{n_t} (1-\theta_p)^{\mathbb{1}_{\{\arg\max_j \theta_{Love}^{(j)}=i\}}} \mathbb{1}_{\{\theta_{Frogs}^{(0)}=\dim(\theta_{Love})-2\geq n_t, 1\geq\theta_{Love}^{(i)},\theta_p\geq 0\}}$$

$$= \frac{e^{-(1-p_{send})}(\lambda(1-p_{send}))^{\theta_{Frogs}^{(0)}-n_t}(\theta_{Frogs}^{(0)}+1)(\theta_{Love}^{(-1)})^{\theta_{Frogs}^{(0)}}}{C(x_t)(\theta_{Frogs}^{(0)}-n_t)!}$$

$$\cdot (1-\theta_p)^{m_t \mathbb{1}_{\{\arg\max_j \theta_{Love}^{(j)}=0\}} + \mathbb{1}_{\{1\leq \arg\max_j \theta_{Love}^{(j)}\leq n_t\}}} \mathbb{1}_{\{\theta_{Frogs}^{(0)}=\dim(\theta_{Love})-2\geq n_t, 1\geq\theta_{Love}^{(i)},\theta_p\geq 0\}}$$

$\pi(\theta_1, \theta_2, \theta_3 | \theta_{Frogs}^{(0)}, \theta_{Love}, \theta_p)$

$$= \mathbb{1}_{\{\theta_1=\theta_{Frogs}^{(0)}, \theta_2=\arg\max_j \theta_{Love}^{(j)}, \theta_3=\theta_p\}}$$

$\pi(\theta_{Frogs}^{(0)}, \theta_{Love}, \theta_p, \theta_1, \theta_2, \theta_3 | x_t)$

$$= \frac{e^{-(1-p_{send})}(\lambda(1-p_{send}))^{\theta_{Frogs}^{(0)}-n_t}(\theta_{Frogs}^{(0)}+1)(\theta_{Love}^{(-1)})^{\theta_{Frogs}^{(0)}}}{C(x_t)(\theta_{Frogs}^{(0)}-n_t)!} \mathbb{1}_{\{\theta_1=\theta_{Frogs}^{(0)}, \theta_2=\arg\max_j \theta_{Love}^{(j)}, \theta_3=\theta_p\}}$$

$$\cdot (1-\theta_p)^{m_t \mathbb{1}_{\{\arg\max_j \theta_{Love}^{(j)}=0\}} + \mathbb{1}_{\{1\leq \arg\max_j \theta_{Love}^{(j)}\leq n_t\}}} \mathbb{1}_{\{\theta_{Frogs}^{(0)}=\dim(\theta_{Love})-2\geq n_t, 1\geq\theta_{Love}^{(i)},\theta_p\geq 0\}}$$

$$= \frac{e^{-(1-p_{send})}(\lambda(1-p_{send}))^{\theta_1-n_t}(\theta_1+1)(\theta_{Love}^{(-1)})^{\theta_1}}{C(x_t)(\theta_1-n_t)!} \mathbb{1}_{\{\theta_1=\theta_{Frogs}^{(0)}, \theta_2=\arg\max_j \theta_{Love}^{(j)}, \theta_3=\theta_p\}}$$

$$\cdot (1-\theta_3)^{m_t \mathbb{1}_{\{\theta_2=0\}} + \mathbb{1}_{\{1\leq\theta_2\leq n_t\}}} \mathbb{1}_{\{\theta_{Frogs}^{(0)}=\dim(\theta_{Love})-2\geq n_t, 1\geq\theta_{Love}^{(i)},\theta_p\geq 0\}}$$

$\pi(\theta_1, \theta_2, \theta_3 | x_t)$

$$= \frac{e^{-(1-p_{send})}(\lambda(1-p_{send}))^{\theta_1-n_t}}{C(x_t)(\theta_1-n_t)!} (1-\theta_3)^{m_t \mathbb{1}_{\{\theta_2=0\}} + \mathbb{1}_{\{1\leq\theta_2\leq n_t\}}} \mathbb{1}_{\{\theta_1\geq n_t, \theta_1\geq\theta_2\geq -1, 1\geq\theta_3\geq 0\}}$$

$$\cdot \int_{[0,1]^{\theta_1+2}} (\theta_1+1)(\theta_{Love}^{(-1)})^{\theta_1} \mathbb{1}_{\{\theta_2=\arg\max_j \theta_{Love}^{(j)}\}} d\theta_{Love}$$

$$= \frac{e^{-(1-p_{send})}(\lambda(1-p_{send}))^{\theta_1-n_t}}{4C(x_t)(\theta_1-n_t)!(\theta_1+1)^{\mathbb{1}_{\{0\leq\theta_2\leq\theta_1\}}}} (1-\theta_3)^{m_t \mathbb{1}_{\{\theta_2=0\}} + \mathbb{1}_{\{1\leq\theta_2\leq n_t\}}} \mathbb{1}_{\{\theta_1\geq n_t, \theta_1\geq\theta_2\geq -1, 1\geq\theta_3\geq 0\}}$$