Idea 1: Varying θ_{Frogs}

Frogs

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1 Modelling and Data

1.1 Modelling

- $\theta_{Frogs}^{(0)} \sim \text{Po}(\lambda)$: initial number of frogs
- $\theta_{Frogs}^{(t)} = \theta_{Frogs}^{(t-1)} + R^{(t)}$: number of frogs at time t, $R^{(t)} \sim F_r$ is a random variable, can be Geometric, Possion, etc, with parameter r
- θ_{Love}, where $\theta_{Love} + 1 \sim \text{Geo}(p_{Love})$ (Can change dist): true love, (0 means don't want love, 1 means congwen, rest means frog number $\theta_{Love} - 1$), assume first send letter = more attractive
- $\theta_p \sim \text{U}(0,1)$: Pr(reply|is true love)
- Each Frogs have probability p_{send} to send letter, and they will only send one letter
- Congwen will send a letter in sats and suns

1.2 Data x_t

- $t = 0, \ldots, T$: time t
- $m_t \geq 0$: number of letters from congwen at time t
- $n_t \geq 0$: number of letters from other frogs at time t
- $Z_1, Z_2, \ldots, Z_{m_t} \in \{0,1\}$: whether the congwen's letter is replied
- $X_1, X_2, \ldots, X_{n_t} \in \{0,1\}$: whether the other frog's letter is replied
- $Z_t = X_t = 0$
- Truely random variables are n_t, X_t, Z_t

2 Computations

$$f(Z_1 = \ldots = Z_{m_t} = 0 | m_t, \theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p) = (1-\theta_p)^{m_t \mathbb{1}_{\{\theta_{Love}=1\}}}$$

$$f(X_1 = \ldots = X_{n_t} = 0 | n_t, \theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p) = (1-\theta_p)^{\mathbb{1}_{\{2 \leq \theta_{Love} \leq n_t+1\}}}$$

$$f(n_t | \theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p) = \binom{\theta_{Frogs}^{(t)}}{n_t} p_{send}^{n_t} (1-p_{send})^{\theta_{Frogs}^{(t)} - n_t} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t\}}$$

$$\begin{aligned} f(x_t | \theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p) &= \frac{\theta_{Frogs}^{(t)}!}{n_t! (\theta_{Frogs}^{(t)} - n_t)!} p_{send}^{n_t} (1-p_{send})^{\theta_{Frogs}^{(t)} - n_t} \\ &\quad \cdot (1-\theta_p)^{\mathbb{1}_{\{2 \leq \theta_{Love} \leq n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \geq n_t\}} \end{aligned}$$

$$\theta_{Frogs}^{(t)} = \theta_{Frogs}^{(0)} + \sum_{i=1}^{t} R^{(t)} \sim \text{Po}(\lambda + tr) \qquad R^{(t)} \sim \text{Po}(r)$$

$$\pi(\theta_{Frogs}^{(0)}, \theta_{Love}, \theta_p) = \frac{e^{-\lambda} \lambda^{\theta_{Frogs}^{(0)}}}{\theta_{Frogs}^{(0)}!} p_{Love} (1-p_{Love})^{\theta_{Love}} \mathbb{1}_{\{\theta_{Frogs}^{(t)}, \theta_{Love} \geq 0, 0 \leq \theta_p \leq 1\}}$$

$$\pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p) = \frac{e^{-(\lambda+tr)} (\lambda+tr)^{\theta_{Frogs}^{(t)}}}{\theta_{Frogs}^{(t)}!} p_{Love} (1-p_{Love})^{\theta_{Love}} \mathbb{1}_{\{\theta_{Frogs}^{(t)}, \theta_{Love} \geq 0, 0 \leq \theta_p \leq 1\}}$$

$$\pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p | x_t)$$

$$\propto f(x_t|\theta)\pi(\theta)$$

$$= \frac{\theta_{Frogs}^{(t)}!}{n_t!(\theta_{Frogs}^{(t)} - n_t)!} p_{send}^{n_t}(1-p_{send})^{\theta_{Frogs}^{(t)} - n_t}$$

$$\cdot (1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \ge n_t\}}$$

$$\cdot \frac{e^{-(\lambda+tr)}(\lambda+tr)^{\theta_{Frogs}^{(t)}}}{\theta_{Frogs}^{(t)}!} p_{Love}(1-p_{Love})^{\theta_{Love}} \mathbb{1}_{\{\theta_{Frogs}^{(t)}, \theta_{Love} \ge 0, 0 \le \theta_p \le 1\}}$$

$$\propto \frac{((1-p_{send})(\lambda+tr))^{\theta_{Frogs}^{(t)}}}{(\theta_{Frogs}^{(t)} - n_t)!}(1-p_{Love})^{\theta_{Love}}(1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \ge n_t, \theta_{Love} \ge 0, 0 \le \theta_p \le 1\}}$$

$$\sum_{\theta_{Frogs}^{(t)}=0}^{\infty} \frac{((1-p_{send})(\lambda+tr))^{\theta_{Frogs}^{(t)}}}{(\theta_{Frogs}^{(t)} - n_t)!} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \ge n_t\}}$$

$$= ((1-p_{send})(\lambda+tr))^{n_t} \sum_{x=0}^{\infty} \frac{((1-p_{send})(\lambda+tr))^x}{x!}$$

$$= e^{(1-p_{send})(\lambda+tr)}((1-p_{send})(\lambda+tr))^{n_t}$$

$$\int_{\mathbb{R}} \sum_{\theta_{Love}=0}^{\infty} (1-p_{Love})^{\theta_{Love}}(1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Love} \ge 0, 0 \le \theta_p \le 1\}} d\theta_p$$

$$= \int_0^1 \sum_{\theta_{Love}=0}^{\infty} (1-p_{Love})^{\theta_{Love}}(1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} d\theta_p$$

$$= \int_0^1 1 + (1-\theta_p)^{m_t}(1-p_{Love}) + (1-\theta_p) \sum_{\theta_{Love}=2}^{n_t+1} (1-p_{Love})^{\theta_{Love}} + \sum_{\theta_{Love}=n_t+2}^{\infty} (1-p_{Love})^{\theta_{Love}} d\theta_p$$

$$= \int_0^1 1 + (1-\theta_p)^{m_t}(1-p_{Love}) + \frac{(1-\theta_p)(1-p_{Love})^2}{p_{Love}}(1-(1-p_{Love})^{n_t}) + \frac{(1-p_{Love})^{n_t+2}}{p_{Love}} d\theta_p$$

$$= \int_0^1 1 + (1-\theta_p)^{m_t}(1-p_{Love}) + \frac{(1-\theta_p)(1-p_{Love})^2}{p_{Love}} + \frac{\theta_p(1-p_{Love})^{n_t+2}}{p_{Love}} d\theta_p$$

$$= 1 + \frac{1-p_{Love}}{m_t+1} + \frac{(1-p_{Love})^2 + (1-p_{Love})^{n_t+2}}{2p_{Love}}$$

$$C(x_t)$$

$$= \int_{\mathbb{R}} \sum_{\theta_{Love}=0}^{\infty} \sum_{\theta_{Frogs}^{(t)}=0}^{\infty} \frac{((1-p_{send})(\lambda+tr))^{\theta_{Frogs}^{(t)}}(1-p_{Love})^{\theta_{Love}}}{(\theta_{Frogs}^{(t)} - n_t)!}(1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}}$$

$$\cdot \mathbb{1}_{\{\theta_{Frogs}^{(t)} \ge n_t, \theta_{Love} \ge 0, 0 \le \theta_p \le 1\}} d\theta_p$$

$$= e^{(1-p_{send})(\lambda+tr)}((1-p_{send})(\lambda+tr))^{n_t} \left\{ 1 + \frac{1-p_{Love}}{m_t+1} + \frac{(1-p_{Love})^2 + (1-p_{Love})^{n_t+2}}{2p_{Love}} \right\}$$

$$\pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p | x_t)$$

$$= \frac{((1-p_{send})(\lambda+tr))^{\theta_{Frogs}^{(t)}}}{C(x_t)(\theta_{Frogs}^{(t)} - n_t)!}(1-p_{Love})^{\theta_{Love}}(1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \ge n_t, \theta_{Love} \ge 0, 0 \le \theta_p \le 1\}}$$

$$\pi(\theta_{Frogs}^{(0)} | \theta_{Frogs}^{(t)})$$

$$= \frac{\pi(\theta_{Frogs}^{(t)} | \theta_{Frogs}^{(0)})\pi(\theta_{Frogs}^{(0)})}{\pi(\theta_{Frogs}^{(t)})}$$

$$= \frac{e^{-tr}(tr)^{\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)}}}{(\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)})!} \cdot \frac{e^{-\lambda}\lambda^{\theta_{Frogs}^{(0)}}}{\theta_{Frogs}^{(0)}!} \cdot \frac{\theta_{Frogs}^{(t)}!}{e^{-(\lambda+tr)}(\lambda+tr)^{\theta_{Frogs}^{(t)}}}$$

$$= \frac{\theta_{Frogs}^{(t)}!}{\theta_{Frogs}^{(0)}!(\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)})!} \left(\frac{\lambda}{\lambda+tr}\right)^{\theta_{Frogs}^{(0)}} \left(\frac{tr}{\lambda+tr}\right)^{\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)}} \mathbb{1}_{\{\theta_{Frogs}^{(t)} \ge \theta_{Frogs}^{(0)} \ge 0\}}$$

$$
\begin{aligned}
&\pi(\theta_{Frogs}^{(0)}, \theta_{Love}, \theta_p | x_t) \\
&= \sum_{\theta_{Frogs}^{(t)}=0}^{\infty} \pi(\theta_{Frogs}^{(0)} | \theta_{Frogs}^{(t)}) \pi(\theta_{Frogs}^{(t)}, \theta_{Love}, \theta_p | x_t) \\
&= \frac{(\lambda/tr)^{\theta_{Frog}^{(0)}}}{C(x_t)\theta_{Frogs}^{(0)}!} (1-p_{Love})^{\theta_{Love}} (1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Love} \ge 0, 0 \le \theta_p \le 1\}} \\
&\quad \times \sum_{\theta_{Frogs}^{(t)}=\max\{n_t, \theta_{Frogs}^{(0)}\}}^{\infty} \frac{\theta_{Frogs}^{(t)}!((1-p_{send})tr)^{\theta_{Frogs}^{(t)}}}{(\theta_{Frogs}^{(t)} - \theta_{Frogs}^{(0)})!(\theta_{Frogs}^{(t)} - n_t)!} \mathbb{1}_{\{\theta_{Frogs}^{(0)} \ge 0\}} \\
&\approx \frac{(\lambda/tr)^{\theta_{Frog}^{(0)}}}{C(x_t)\theta_{Frogs}^{(0)}!} (1-p_{Love})^{\theta_{Love}} (1-\theta_p)^{\mathbb{1}_{\{2 \le \theta_{Love} \le n_t+1\}} + m_t \mathbb{1}_{\{\theta_{Love}=1\}}} \mathbb{1}_{\{\theta_{Frogs}^{(0)} \ge 0, \theta_{Love} \ge 0, 0 \le \theta_p \le 1\}} \\
&\quad \times \sum_{i=1}^{n} \frac{(P_i + \max\{n_t, \theta_{Frogs}^{(0)}\})!}{(P_i + |n_t - \theta_{Frogs}^{(0)}|)!} ((1-p_{send})tr)^{\max\{n_t, \theta_{Frogs}^{(0)}\}} e^{(1-p_{send})tr} \qquad P_{1:n} \sim \mathrm{Po}((1-p_{seed})tr)
\end{aligned}
$$

Interesting:

$$
(\theta_{Frogs}^{(t)} - n_t) | x_t \sim \mathrm{Po}((1-p_{send})(\lambda + tr))
$$

$$
\theta_{Frogs}^{(0)} | \theta_{Frogs}^{(t)} \sim \mathrm{Bin}\left(\theta_{Frogs}^{(t)}, \frac{\lambda}{\lambda + tr}\right)
$$