

Two interpolation methods for vector fields that conserve flux and line integrals

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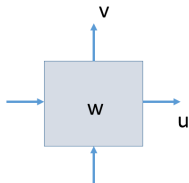


New Zealand eScience Infrastructure

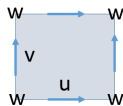
Motivation: want answers to

How to interpolate vector fields with staggered components?

- Arakawa C/D grids
- Components are on cell faces or edges
- Arises in computational fluid dynamics and electromagnetics



Arakawa C-grid



Arakawa D-grid

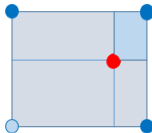
Currently used interpolation methods in earth sciences

Is there hope to unify these?

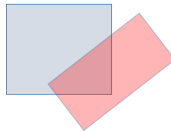
- Linear
- Conservative or area weighted, used in climate studies to enforce conservation of total mass, energy



Babylonian tablet (1700BC)



Bilinear

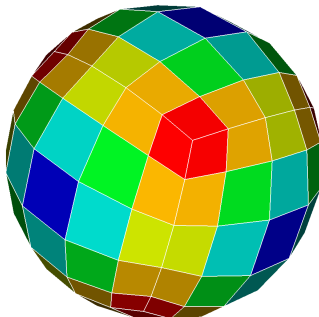


Conservative (used since the 1990s)

Earth science grids are curvilinear

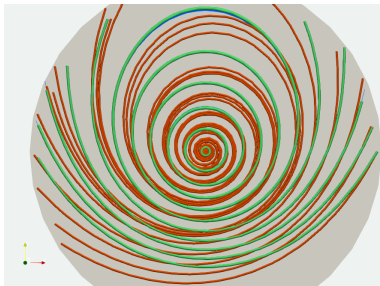
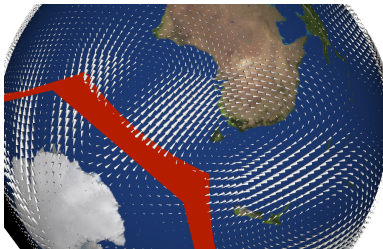
How to handle non-orthogonal cells? Example: cubed-sphere grid

- Project grids on the surface of a cube onto a sphere
- Six logically rectangular grids (cannot be represented as a single structured grid)
- No pole-like singularity but some distortion where three tiles meet



Where interpolation is used

- computing fluxes across an area
- advecting fields
- visualising streamlines



Four types of fields - one derivative

Exterior calculus tells us:

- **0-form**: just a function of space, one component
 - invariant under coordinate change
 - Example: temperature
- **1-form**: vector field, 3 components in 3D
 - Examples: electric field E , induction H and velocity
- **2-form**: pseudo-vector field, 3 components in 3D
 - Examples: magnetic field B , displacement field D and vorticity
- **3-form**: pseudo-scalar, 1 component
 - Example: density

Discretized fields corresponding to the 1-3 forms

Association of form with cell elements

- 0-form: **on nodes**
 - $\int \alpha = \alpha$ (integral is a no op)
- 1-form: **on edges**
 - $\int \beta$ is a line integral
- 2-form: **on faces**
 - $\int \gamma$ is a surface integral
- 3-form: **cell centred**
 - $\int \omega$ is a volume integral

Differential forms like to be integrated

Generalizing “interpolation”

Making interpolation work for nodal, edge, face and cell fields

$$\int f = \sum_i f_i \int_T \phi_i$$

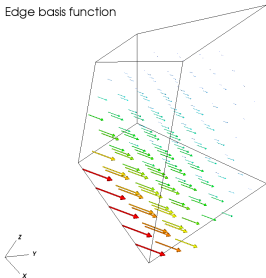
- ϕ_i is basis k -form, $k = 0, 1, 2$ or 3
- T is target (point, line, area or volume)
- f_i is field integral over cell element k (node, edge, face or cell)
- $\int_T \phi_i$ is the **interpolation weight**
- i index runs over all the degrees of freedom (points, edges, faces etc., as appropriate)

Basis functions satisfy orthogonality condition

$$\int_i \phi_j = \delta_{ij}$$

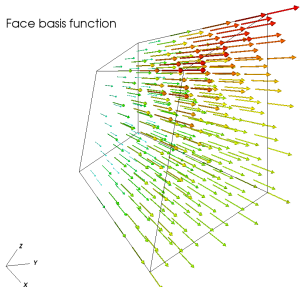
i is cell element (node, edge, face, cell), j is basis function index

Edge basis function



Edge basis is perpendicular
to neighbouring edges

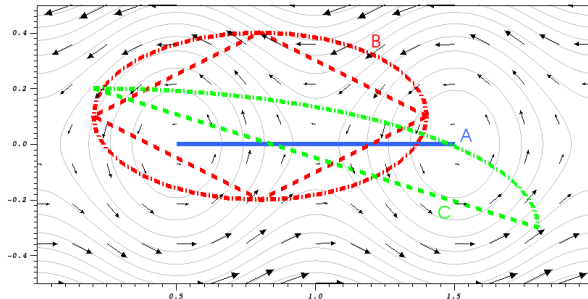
Face basis function



Face basis is tangent
to neighbouring faces

Result 1: divergence-free field $v = dz \wedge d\psi$

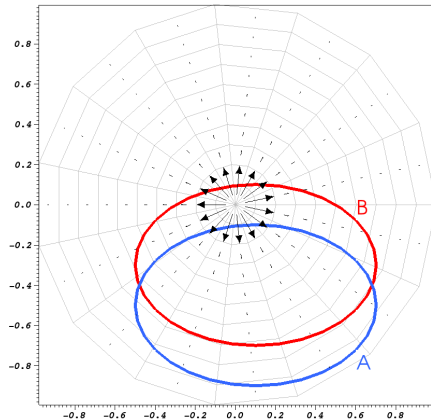
Flux integral depends only on distance of endpoints to nearest grid node



Closed loop integral is exact!

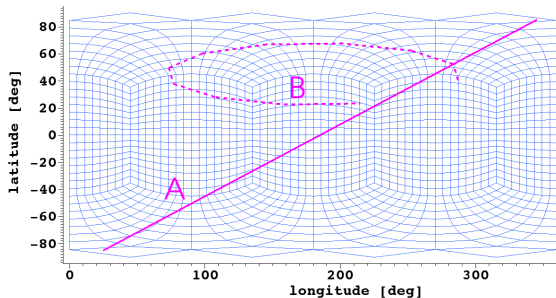
Result 2: vector field $v = \frac{xdx+dy}{2\pi(x^2+y^2)}$ with singularity

Numerical loop integral A gives 0, B gives 1

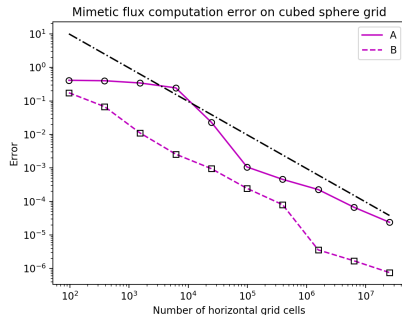


Result 3: flux on the cubed sphere $v = d\psi \wedge dr$

Edge/face interpolation works for highly distorted cells



Integration path/surface



Error is $\sim 1/N^2$

Summary

Different types of field \leftrightarrow different staggerings

- use bilinear for nodal (scalar) field
- use **edge** for vector field
- use **face** for pseudo-vector field
- use cell for pseudo-scalar fields

Masking and partially valid cells?

Ok if taking account of partial cell, faces, edges when setting cell, face and edge integrals. Done!

Summary (2)

What about tetrahedra?

Similar approach except that the basis functions are Whitney's

Higher order basis functions?

Initial work indicates that higher order basis functions can be used. These also satisfy the orthogonality condition $\int_i \phi_j = \delta_{ij}$ on sub-cell edges, faces and cells. Quadratic elements effectively split each cell into 8 sub-cells, each face into 4 sub-faces and each edge into 2 sub-edges.

The time is ripe to treat interpolation with the same rigour as modelling

“Mimetic Interpolation of Vector Fields on Arakawa C/D Grids”:

<https://journals.ametsoc.org/doi/10.1175/MWR-D-18-0146.1>