Two interpolation methods for vector fields that conserve flux and line integrals

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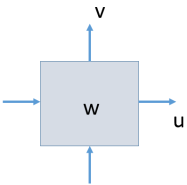
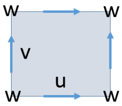
Motivation: want answers to

How to interpolate vector fields with staggered components?

- Arakawa C/D grids
- Components are on cell faces or edges
- Arises in computational fluid dynamics and electromagnetics



Arakawa C-grid



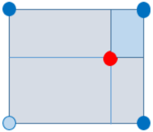
Arakawa D-grid

Currently used interpolation methods in earth sciences

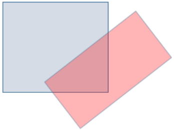
Is there hope to unify these?

- Linear
- Conservative or area weighted, used in climate studies to enforce conservation of total mass, energy

Babylonian tablet (1700BC)



Bilinear

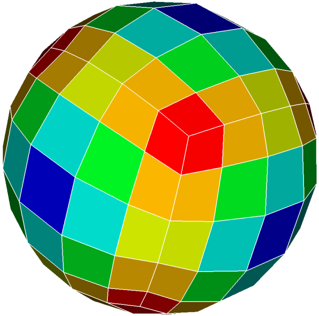


Conservative (used since the 1990s)

Earth science grids are curvilinear

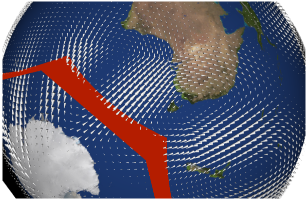
Example: cubed-sphere grid employed by exascale weather prediction/climate modelling system LFRic developed at UK Met Office

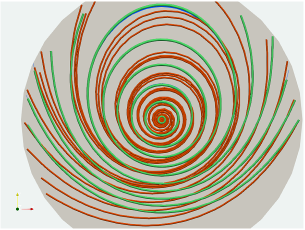
- Six logically rectangular grids (cannot be represented as a single structured grid)
- No pole-like singularity but some distortion where three tiles meet



Interpolation is required for

- regridding/remapping fields deom one grid to another
- computing fluxes across an area
- advecting fields
- visualising streamlines





Four types of fields - one derivative

Exterior calculus tells us:

- 0-form: just a function of space, one component
  - invariant under coordinate change
  - Example: temperature
- 1-form: vector field, 3 components in 3D
  - Examples: electric field E, induction H and velocity
- 2-form: pseudo-vector field, 3 components in 3D
  - Examples: magnetic field B, displacement field D and vorticity
- 3-form: pseudo-scalar, 1 component
  - Example: density

Discretized fields corresponding to the 1-3 forms

Association of form with cell elements

- 0-form: on nodes
  - $\int \alpha = \alpha$ (integral is a no op)
- 1-form: on edges
  - $\int \beta$ is a line integral
- 2-form: on faces
  - $\int \gamma$ is a surface integral
- 3-form: cell centred
  - $\int \omega$ is a volume integral

Differential forms like to be integrated

Generalizing "interpolation" to work for nodal, edge, face and cell fields

One formula for all cases

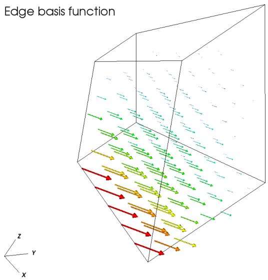
$\int f = \sum_i f_i \int_T \phi_i$

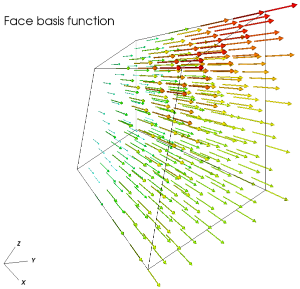
- ϕ_i is **basis** k-form, $k = 0$, 1, 2 or 3
- T is **target** (point, line, area or volume)
- f_i is **field integral** over cell element k (node, edge, face or cell)
- $\int_T \phi_i \equiv$ **interpolation weight**
- i index runs over all the degrees of freedom (points, edges, faces etc., as appropriate)

Basis functions ϕ_j satisfy orthogonality condition

$$\int_i \phi_j = \delta_{ij}$$

i is cell element (node, edge, face, cell), j is basis function index



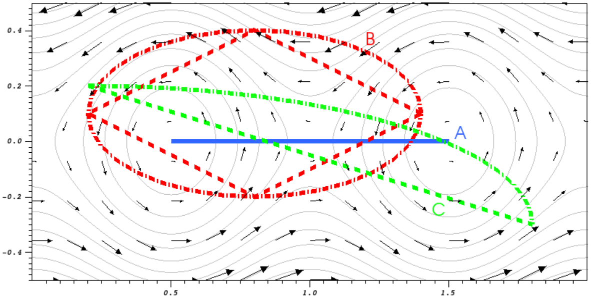


Edge basis is perpendicular to neighbouring edges

Face basis is tangent to neighbouring faces

Result 1: divergence-free field $v = dz \wedge d\psi$

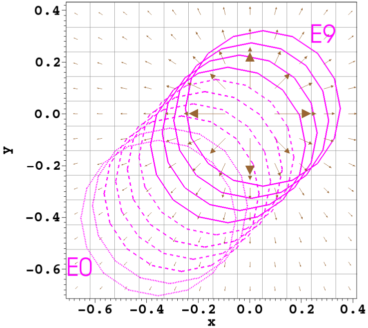
Flux integral depends only on distance of endpoints to nearest grid node



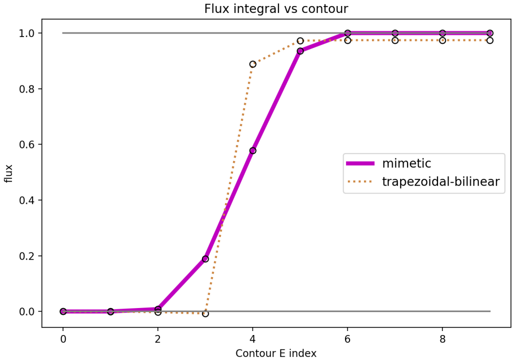
Closed loop integral is exact!

Result 2: Singular vector field $v = \frac{xdx+ydy}{2\pi(x^2+y^2)}$

Loop intergral is 1 if singularity is inside contour, 0 otherwise. Getting exact 0 for $E0$ and $E1$, exact 1 for $E6$-$E9$ and in between values for contours that intersect the cell containing (0,0)



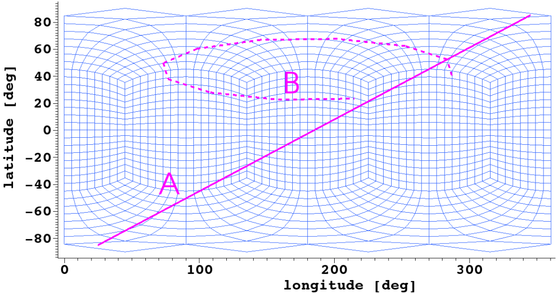
Loop integrals

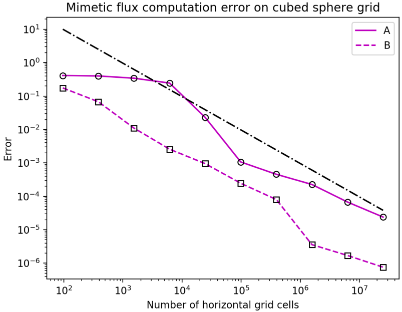


Numerical value of integral

Result 3: flux on the cubed sphere $v = d\psi \wedge dr$

Edge/face interpolation works for highly distorted cells. Zero error start/end points fall onto grid nodes.





Integration path/surface

Error is $\sim 1/N^2$

Summary

Type of field $\rightarrow$ discretised field staggering $\rightarrow$ basis functions $\rightarrow$ interpolation method

- use bilinear for nodal (scalar) field
- use edge for vector field - conserves line integrals
- use face for pseudo-vector field - conserves flux integrals
- use cell for pseudo-scalar fields - conserves volume integrals

Masking and partially valid cells?

Ok if taking account of partial cell, faces, edges when setting cell, face and edge integrals. Done!

Summary (2)

What about tetrahedra?

Similar approach except that the basis functions are Whitney's bases (1957)

Higher order basis functions?

Initial work indicates that higher order basis functions can be used. These also satisfy the orthogonality condition $\int_i \phi_j = \delta_{ij}$ on sub-cell edges, faces and cells. Quadratic elements effectively split each cell into 8 sub-cells, each face into 4 sub-faces and each edge into 2 sub-edges.

The time is ripe to treat interpolation with the same rigour as modelling

"Mimetic Interpolation of Vector Fields on Arakawa C/D Grids":
https://journals.ametsoc.org/doi/10.1175/MWR-D-18-0146.1