Probability

Chapter 2

Learning Objectives

- Sample Space
- Events
- Counting Sample Points
- Probability of an Event
- Additive Rules
- Conditional Probability, Independence, and the Product Rule
- Bayes' Rule

Definitions

- Probability refers to the study of *randomness* and *uncertainty*.

- Probability theory provides a basis with which we can make inferences about a population based on its distribution and it provides methods for quantifying the chances, or likelihoods, associated with various outcomes.

- Probability also helps to explain a lot of everyday occurrences and we actually discuss it frequently.

Definitions

Random Process - Any process whose possible results are known but whose *actual* results cannot be predicted with certainty in advance.

Outcome - Each possible result for a random process.

Experiment – any process that generates the set of data.

Sample space - the set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol *S.*

$S = \{1, 2, 3, 4, 5, 6\}$

Definitions

Event - any collection (or subset) of outcomes contained in the sample space.

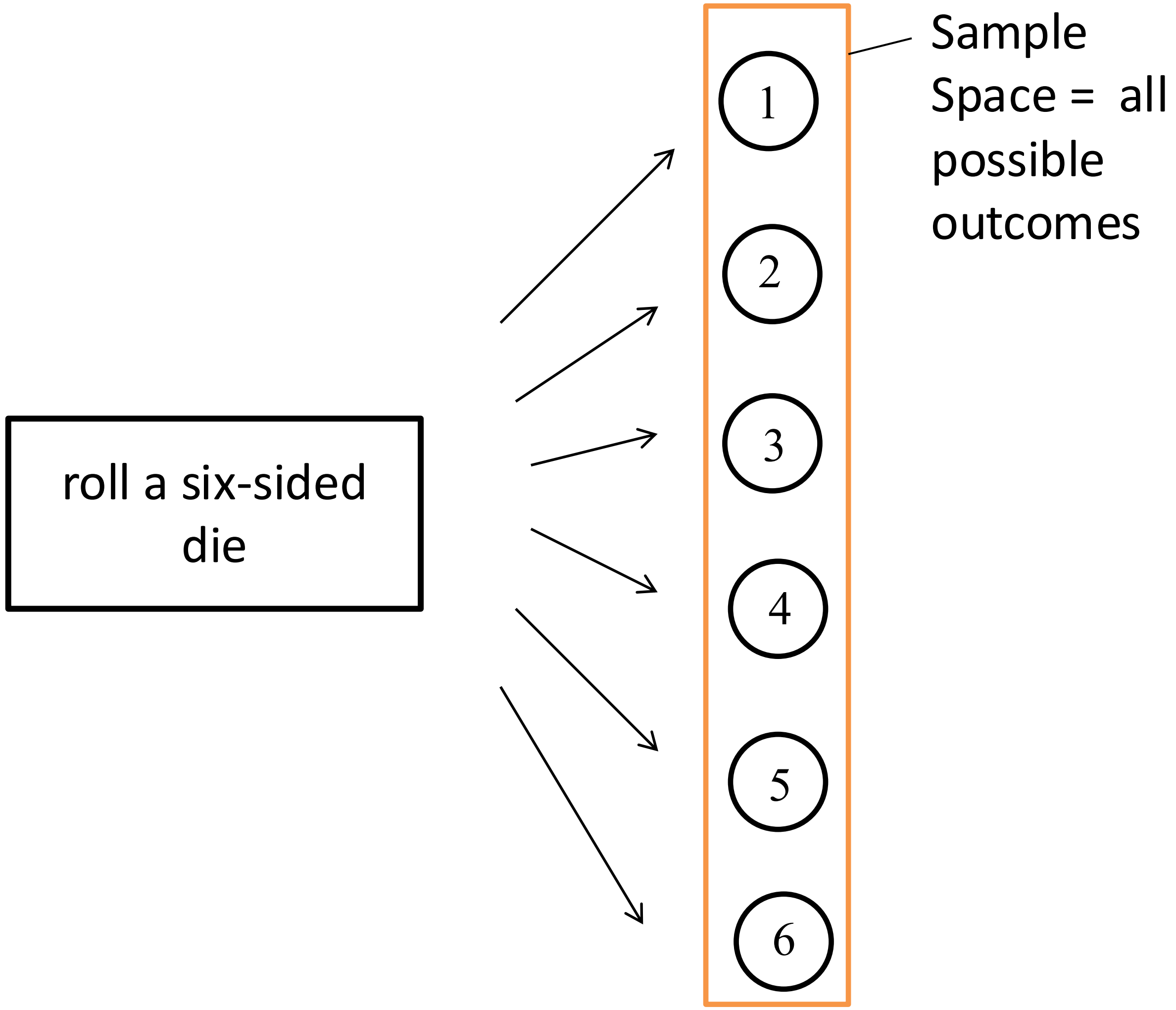
Simple Event - an event that cannot be decomposed, one outcome of the experiment or in the sample space.

Compound Event - collection of specified outcomes contained in the sample space.

Null Event ($\varnothing$): An event with *no* outcomes.

Example: Number of days with a high temperature over 150 degrees F.

Example

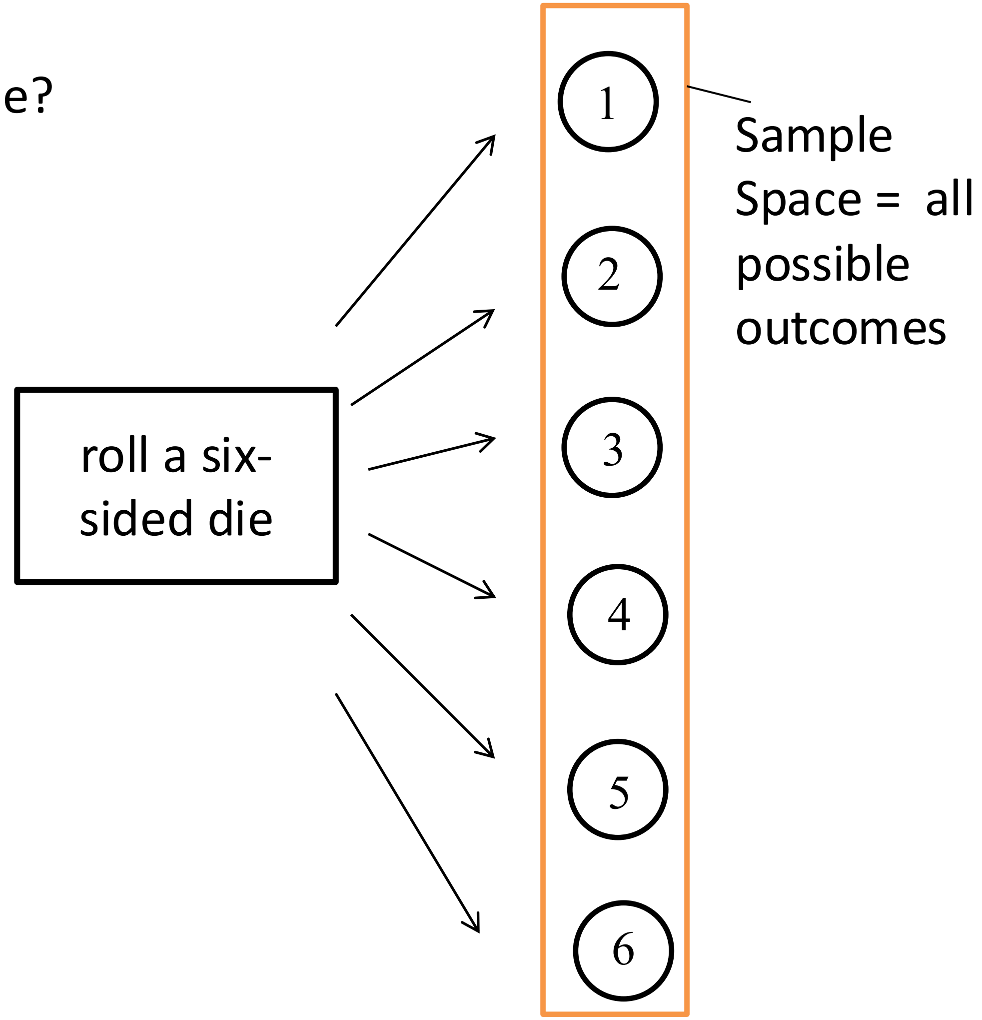


roll a six-sided die
1
2
3
4
5
6
Sample Space = all possible outcomes

Example

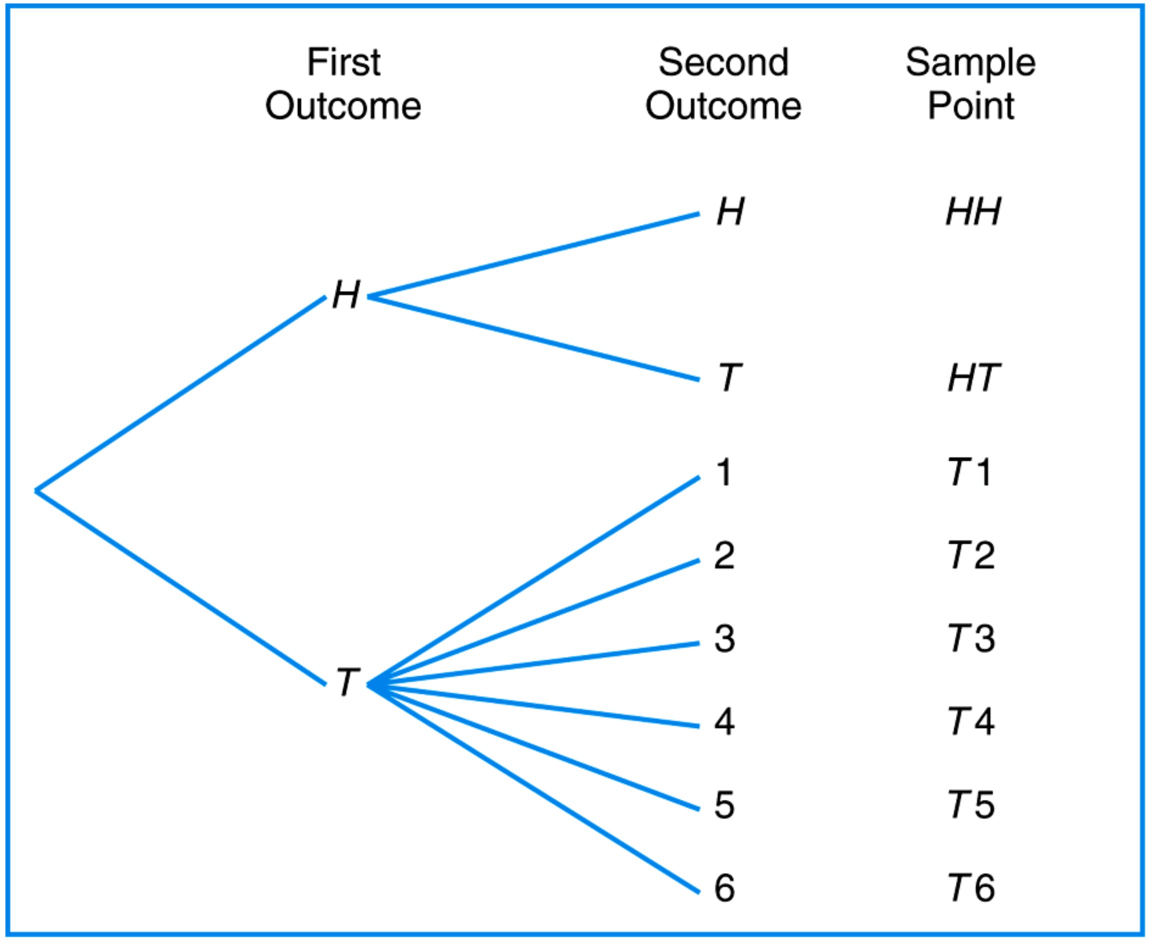
- What are possible collections of the outcome?
  - It can be 1, 2, 3, …, 6
  - It can be either odd or even
  - It can be greater than 4
  - …

- simple event:
  - observe a 2
- compound events:
  - observe an odd number
- null event
  - observe a number
    which is both even and odd



Tree diagram for Example 2.2

Experiment consists of flipping a coin and flipping it one more time if head occurs. If a tail occurs on the first flip, then a die is tossed once.



Example

Battery failures: If a new type D flashlight battery has a voltage that is outside certain limits, that battery is characterized as a failure (F); if the battery has a voltage within the prescribed limits, it is a success (S). Suppose the experiment consists of testing each battery as it comes off n assembly line until we first observe a success.

S = {S, FS, FFS, FFFS,…}

Definitions

The **Union** of events A and B, denoted by $A \cup B$ and read "A or B" is the event consisting of all outcomes that are either in A or in B or in both events (so that the union includes outcomes for which both A and B occur as well as outcomes for which exactly one of A or B occurs).

Example:

$A = \{1, 3, 5\}$, $B = \{4, 5, 6\}$

$A \cup B =$ $\{1, 3, 4, 5, 6\}$

Definitions

The **Intersection** of events A and B, denoted by $A \cap B$ and read "A and B", is the event consisting of all outcomes that are in both A and B.

Example:

$A = \{1, 3, 5\}$

$B = \{4, 5, 6\}$

$A \cap B = \{5\}$

Definitions

Complement - the complement of event A, denoted by A', is the set of all outcomes in the Sample Space that are not contained in A.

Example:

In the die example, what is the complement of following event?

$A = \{1, 3, 5\}$

$A' = \{2, 4, 6\}$

What is the union of A and A' ?

$A \cup A' = S = \{1, 2, 3, 4, 5, 6\}$

Definitions

If two events, *A* and *B* have no outcomes in common, they are said to be **mutually exclusive** or **disjoint** events. This means that if one of them occurs, the other cannot.

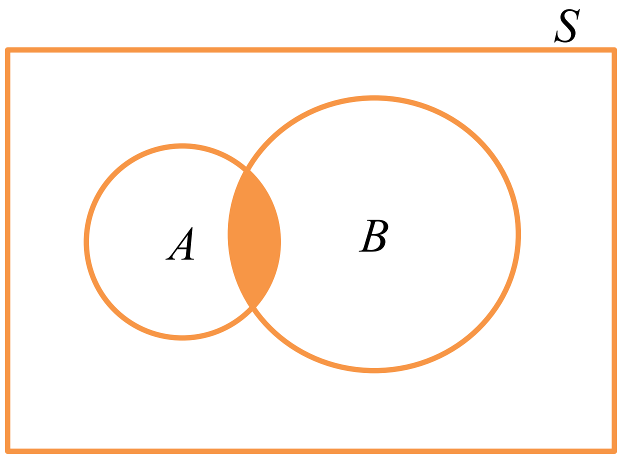
Example:

In our die example, define event *C* as observing an even number. What is the intersection of events *A* and *C*?

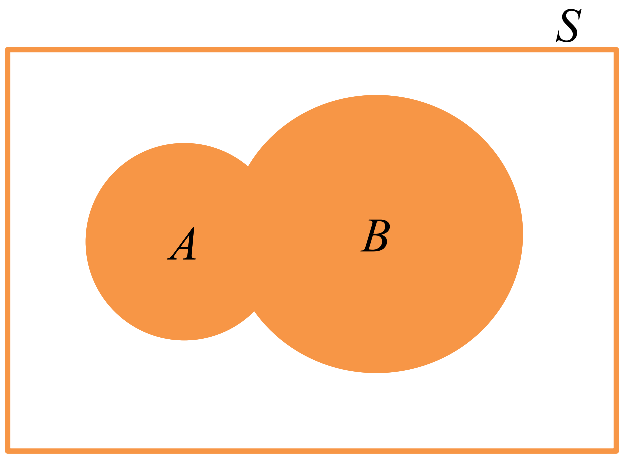
Remember that $A = \{1, 3, 5\}$

$A \cap C = \emptyset$

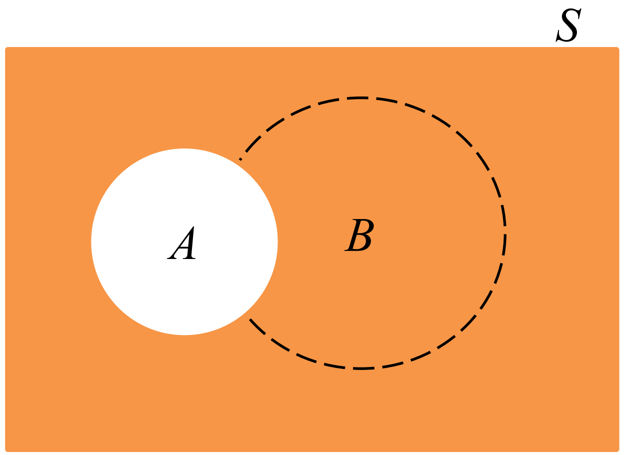
Theory of Sets and Venn Diagram



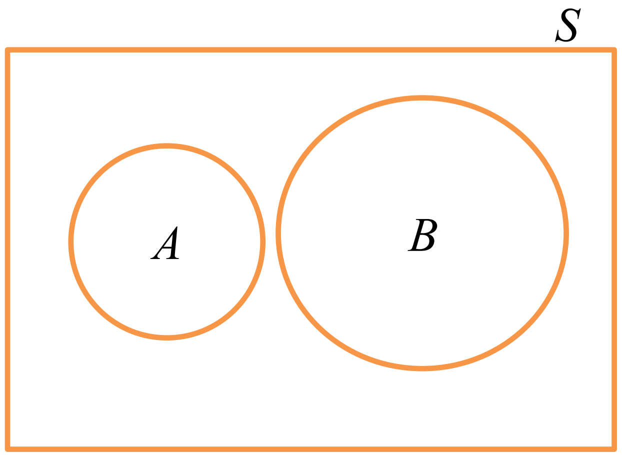
$A \cap B$



$A \cup B$



A'



$A \cap B = \varnothing$

Example

We roll 2 dice and read the 2 numbers facing up.

– What is the sample space?

36 events

– Let *A* be the event that both dice show the same number. How many members does *A* complement have?

$A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

$|A'| = 30$

Example

We roll 2 dice and read the 2 numbers facing up.

– Let B be the event that at least one of the dice shows a 1. What is the union of A and B?

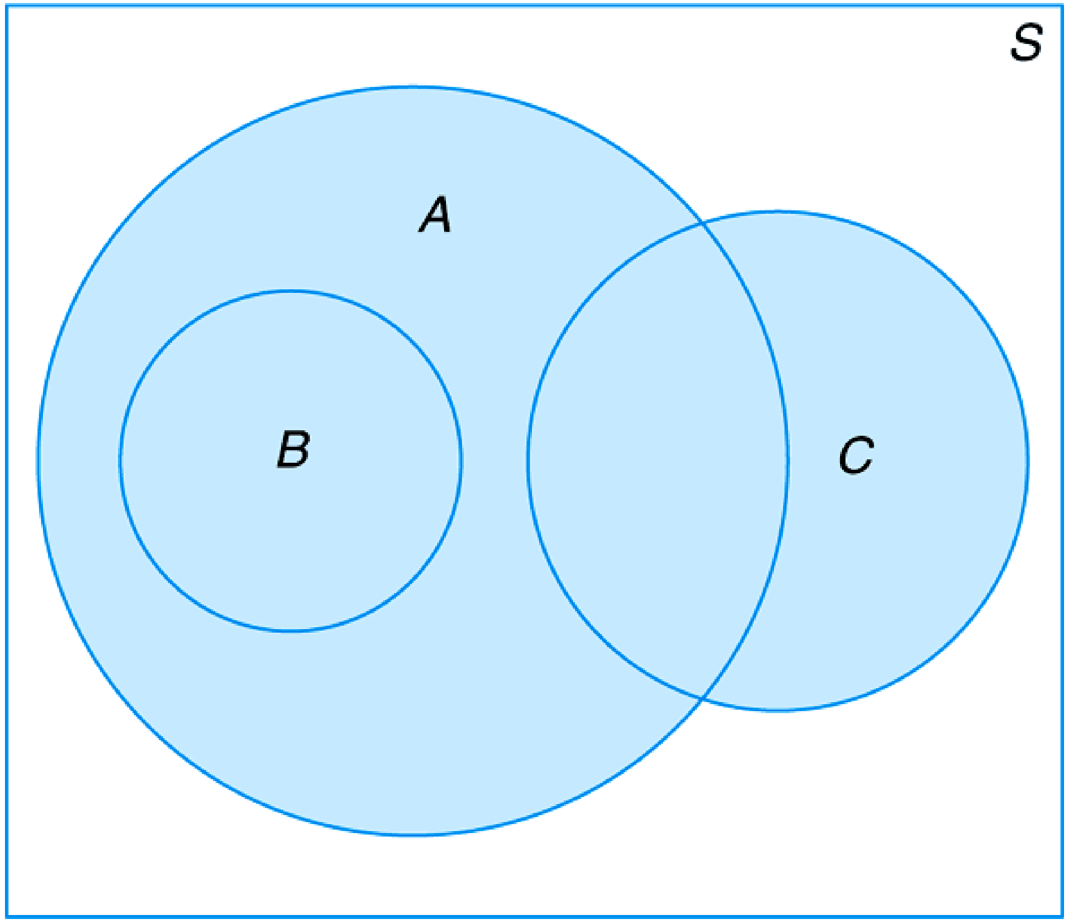
$B = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)\}$

$A \cup B = \{(1,1), (1,2), (1,3) \ldots (6,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

– What is the intersection of A and B?

$A \cap B = \{(1,1)\}$

Example: Events of the sample space *S*



Example

For the experiment in which the number of pumps in use at a single six-pump gas station is observed,

let $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{1, 3, 5\}$.

Then

$A' =$ $\{5, 6\}$

$A \cup B =$ S

$A \cup C =$ $\{0, 1, 2, 3, 4, 5\}$

$A \cap B =$ $\{3, 4\}$

$A \cap C =$ $\{1, 3\}$

$(A \cap C)' =$ $\{0, 2, 4, 5, 6\}$

Learning Objectives

- Sample Space
- Events
- Counting Sample Points
  - Permutations
  - Combinations
- Probability of an Event
- Additive Rules
- Conditional Probability, Independence, and the Product Rule
- Bayes' Rule

Counting Sample Points

- How many points are in the Sample Space?
  - List each possible outcome
  - If we roll 100 dice???
- When the outcomes are **equally likely**, can use:
  - Multiplication Rule
  - Permutations
  - Combinations

Multiplication Rule

- Multiplication Rule: If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.
  - Used when events consist of ordered pairs and we want to count the number of possible pairs

Generalized Multiplication Rule for *k*-tuples

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed *in* $n_1 n_2 \cdot \ldots \cdot n_k$ ways.

Generalized Multiplication Rule for *k*-tuples

- <u>Example:</u> How many license plates of 3 letters followed by 3 numbers are possible?

$26 \times 26 \times 26 \times 10 \times 10 \times 10$

Reverse Interpretation of Multiplication Rule

Suppose we know total number of outcomes N and operation i can be performed n_i ways. What would be one reasonable explanation of N/n_i

Permutations and Combinations

- If you have *n* distinct individuals or objects, how many ways can you select a subset of size *k* from the group?
  1. Permutation – order matters
  2. Combination – order doesn't matter

Permutations and Combinations

- Easy way to remember difference between Permutations and Combinations…
  - Example 1: Pick 3 buildings on campus to visit.
    - Permutation: order of visits matters because time is constrained, so Towers, WPU, Peterson is different from WPU, Towers, Peterson
    - Combination: order doesn't matter because you have all day, so Towers, WPU, Peterson is the same as WPU, Towers, Peterson

Difference between Permutations and Combinations

Example 2: 9 people participate in a competition.

In scenario one, three people are selected to receive the gold, silver and bronze medals. permutation

In scenario two, three people are selected to advance to the next round. combination

Q: Which scenario is a permutation and which scenario is a combination?

different ways can 9 people finish, $9!$

top three : $9 \times 8 \times 7$; ${}_nP_r = \frac{n!}{(n-r)!}$

order matters

top three order doesn't matter $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Permutations

- How do we determine the number of possibilities?
  - A **permutation** is an arrangement of all or part of a set of objects.
  - Can use the product rule for n-tuples

$$n(n-1)(n-2)\ldots$$

  - Can use a tree diagram

$n \quad n-1 \quad n-2 \quad \ldots$

Permutations

– The number of permutations of n objects is $n!$, which is called "n factorial" and is defined

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

Permutation Formula: The number of n distinct objects taken r at a time is ${}_nP_r = \frac{n!}{(n-r)!}$

Intuition?

$${}_9P_3 = \frac{9!}{(9-3)!} = 504$$

Circular Permutations

- <u>Circular Permutations:</u> Permutations that occur in a circle.
  - The number of permutations of n objects arranged in a circle is $(n\text{-}1)!$
- Example: Seating arrangements around a table.
  - If all people move one chair to the right – that is NOT a new arrangement.
  - If person 1 stays in same chair, and other people move then that is a new arrangement.

Permutations – Another Special Case

- What if all of our objects are not distinct?
  - Example: Number of ways to rearrange letters, banana

$$\frac{6!}{1!3!2!} = 60$$

- <u>Theorem 2.4:</u> The number of distinct permutations of n things of which n_1 are of one kind, n_2 are of a second kind, … , n_k of a k^{th} kind is $\frac{n!}{n_1!n_2!\cdots n_k!}$.

Example

- How many ways can one arrange the letters b,a,n,a,n,a?

$$\frac{n!}{n_1!\,n_2!\,n_3!}$$

Combinations

- When order doesn't matter, and we just want to split objects into different groups or **cells**, then we have a combination.
- The number of combinations of n distinct objects taken r at a time is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
  - Example: 9 people compete and 3 advance to next round.
    - Cell 1: 3 people that advance
    - Cell 2: 6 people that do not advance
    - # of Combinations:

$$\binom{9}{3} = \frac{9!}{3!(9-3)!} = 84$$

Combinations

- The number of combinations of n distinct objects taken r at a time is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
- <u>Example:</u> Suppose there are 8 men and 8 women. How many ways can we choose a committee that has 2 men and 2 women?

$$\binom{8}{2}\binom{8}{2} =$$

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  - Permutations
  - Combinations
- **Probability of an Event**
- **Additive Rules**
- Conditional Probability, Independence, and the Product Rule
- Bayes' Rule

How does counting possibilities help us determine probabilities?

→ We just learned how to count the number of possibilities, now we will learn how to use those numbers to determine probability values…

How does counting possibilities help us determine probabilities?

Example: You are an engineering manager with 30 engineers working for you. 18 are located in Pittsburgh and 12 are located in Chicago. You need to select 4 to attend the annual conference. To be fair, you draw names from a hat. What is the probability that you pick 2 employees from Pittsburgh and 2 employees from Chicago?

# of outcomes: $\binom{30}{4}$

favorable outcomes: $\binom{18}{2} \times \binom{12}{2}$

probability $\dfrac{\binom{18}{2} \times \binom{12}{2}}{\binom{30}{4}}$

Probability of an Event

- We want to assign probabilities or weights to different events.
  - In other words, we want to know the likelihood of occurrence of an event resulting from a statistical experiment.
  - Example: Toss a coin, we want to know what is the probability of getting Heads?
    - Already learned how to define the sample space
    - Learned how to count points in the sample space
    - Now will learn how to assign probabilities to each point in the sample space

$S = \{H, T\}$

$A = \{H\}$

$P(A) = .5$

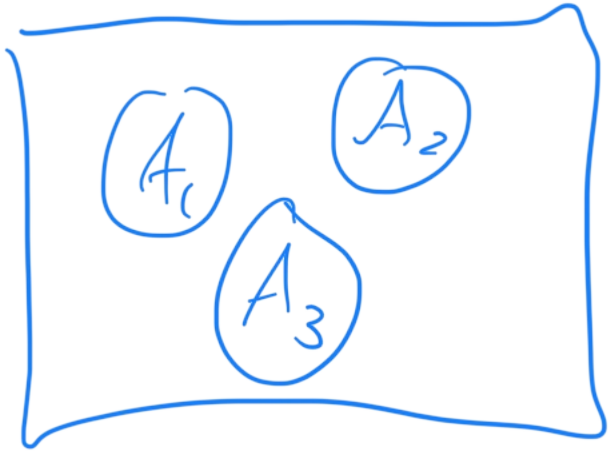
Basic Probability Properties

The probability of an event A is the sum of weights of all sample points in A. Therefore,

(1) $0 \leq P(A) \leq 1$

(2) $P(\emptyset) = 0$

(3) $P(S) = 1$



Furthermore, if A_1, A_2, A_3,… is a sequence of mutually exclusive events, then

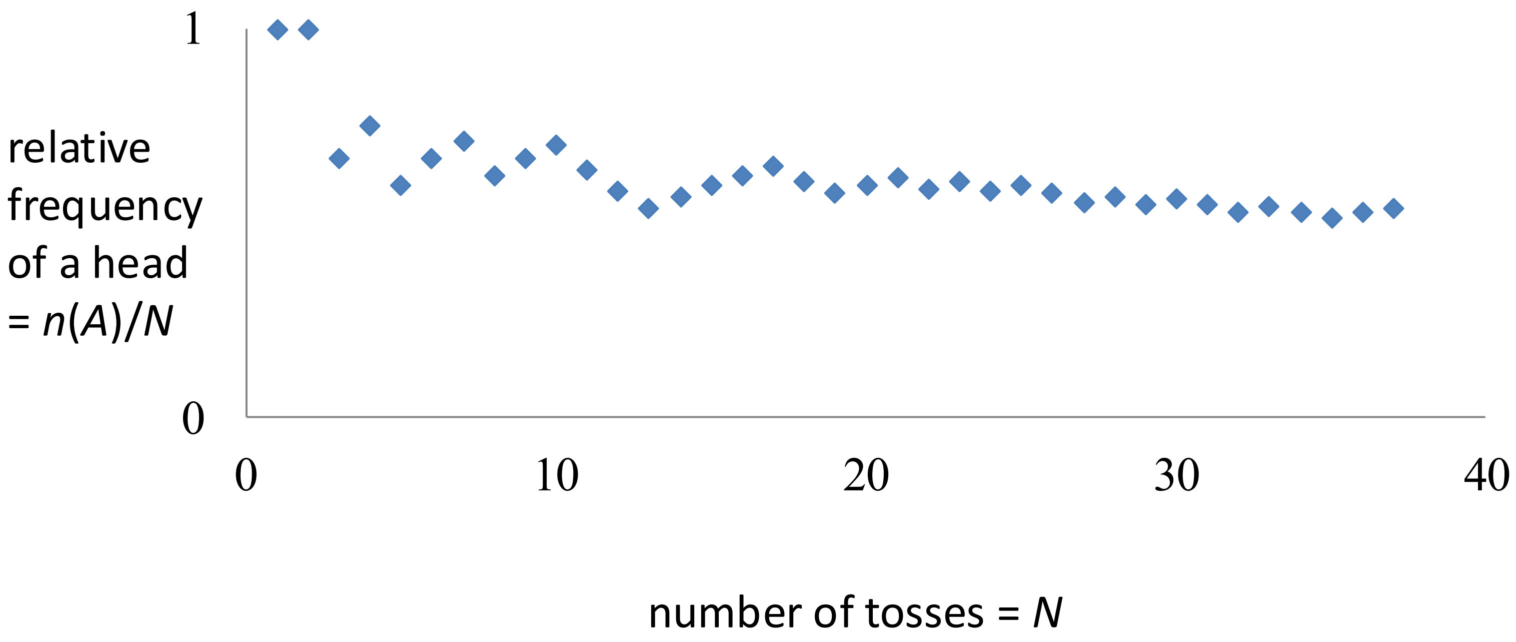
$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

Interpreting Probability

- The basic properties do not completely assign values to probabilities
- They simply rule out probability assignments that are not logical
- So how do we assign probabilities?
  - Relative frequency notion
  - Subjective judgments
  - Additive rules – use probabilities of other known events
  - Determine systematically
  - Equally likely outcomes

Defining Probabilities and Their Properties

- The objective of probability is to assign to each event, say *A*, a number *P*(*A*), called the probability of *A* which will give a precise measure of the chance that *A* will occur.
- This number, *P*(*A*), is the limiting relative frequency of event under certain conditions of the experiment.



Defining Probabilities and Their Properties

How do we usually assign probabilities to the experiments?

Typically, we assign probabilities based on our beliefs about the limiting relative frequencies.

- The probability of any event should equal n/N where:

  n = number of times we observe the event

  N = a very large number of trials

- Example:

  In the experiment of rolling a die, in the long run, what do we expect the probability of observing a 2 to equal?

$1/6$

Relative Frequency

<u>Rule 2.3:</u> If an experiment can result in any one of *N* different equally likely outcomes, and if exactly n of these outcomes correspond to event *A*, then the probability of event *A* is

$$P(A) = \frac{n}{N}$$

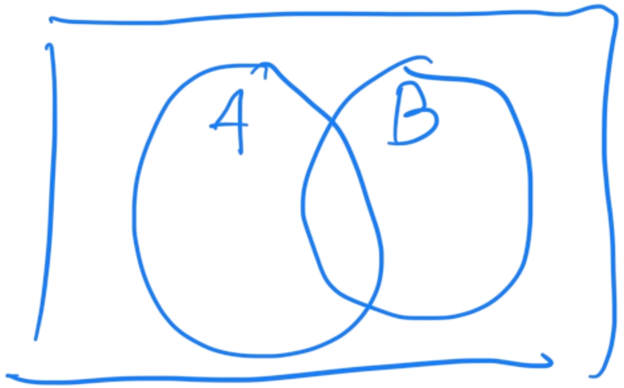
Subjective Judgments

- When situations are unrepeatable, probability assignments might differ based on individuals prior knowledge or opinions
  - Example: “Because Lebron has left the Cavaliers, I expect them to lose this game.”
  - Example: “The probability is 0.7 that I will pass my statistics exam.”
- We don’t study subjective judgments in this course

Additive Rules

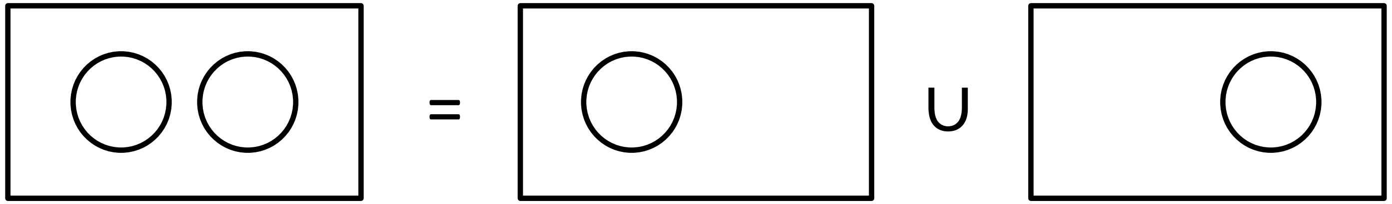
- Sometimes it is easiest to calculate the probabilities of events based on other known events.
- Additive Rule #1: If A and B are two events, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- Note that we are defining the probability of event A or event B or both with addition.



Additive Rules

- Corollary 2.1: When events A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$



Example:

$A = \{2, 4, 6\}$

$B = \{1, 3, 5\}$; $A \cap B \neq \phi$

$P(A \cup B) = P(A) + P(B) = \frac{3}{6} + \frac{3}{6} = 1$

Additive Rules

- Corollary 2.2: If A_1, A_2, …, A_n are mutually exclusive, then
$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

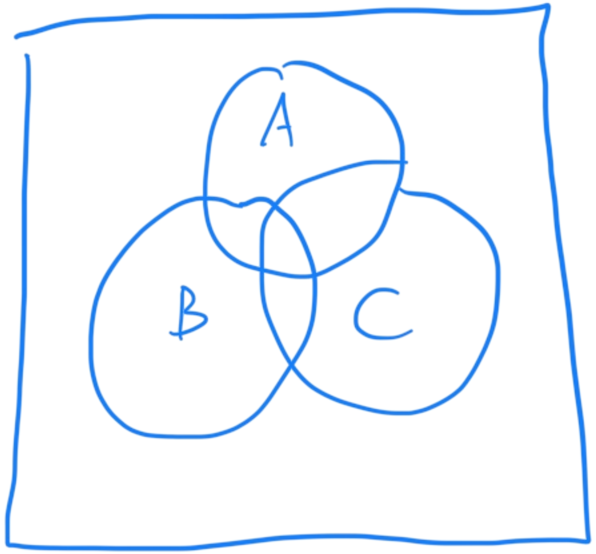
- Corollary 2.3: If A_1, A_2, …, A_n is a partition of a sample space S, then
$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
$$= P(S) = 1$$

Additive Rules

<u>Additive Rule #2:</u> For three events A, B and C,

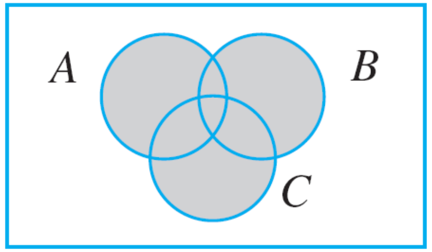
$$P(A \cup B \cup C)$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$

Example:



Additive Rules

- This can be verified by examining a Venn diagram of $A \cup B \cup C$.



$A \cup B \cup C$

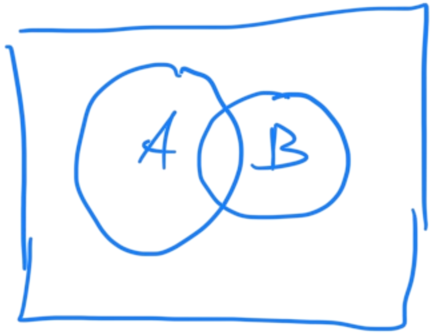
When $P(A)$, $P(B)$, and $P(C)$ are added, certain intersections are counted twice, so they must be subtracted out, but this results in $P(A \cap B \cap C)$ being subtracted once too often.

$P((A \cup B)') = 1 - P(A \cup B)$

Example

$P(A) = .4$
$P(B) = .3$
$P(A \cup B) = .5$

When I visit the local library, the probability that someone is reading the current issue of *Sports Illustrated* is .4, the probability that someone is reading *Time* is .3, and the probability that at least one of these two magazines is being read by someone is .5. What is the probability that



a.Both of the magazines are being read?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = .4 + .3 - .5 = .2$

b.Neither of the two is being read?

$P(\text{neither}) = 1 - P(A \cup B)$
$= 1 - .5 = .5$

c.Exactly one is being read?

$P(\text{exactly one}) = P(A \cup B) - P(A \cup B) = .5 - .2$

Additive Rules

<u>Additive Rule #3:</u> If A and A' are complementary events, then $P(A) + P(A') = 1$.

- Sometimes $P(A')$ is easier to calculate than $P(A)$
- <u>Example:</u> What is the probability that a student is 5'5'' or taller?

5'0" – 5'1"	5'2'' – 5'4''	5'5'' – 5'7''	5'8''–5'10''	5'11''–6'1''	6'2'' – 6'4''	>6'4''
2	6	7	14	10	6	0

$$P(\text{height} \geq 5'5'') = 1 - P(\text{height} < 5'5'')$$
$$= 1 - \frac{2+6}{45} = 0.82$$

Birthday Example

What's the chance that we (everybody in this classroom, say 39) all have different birthdays?

– number of possible ways to have different days

$$_{n}P_{r} = {}_{365}P_{39} = \frac{365!}{(365-39)!}$$

– total number of possibilities for our birthdays

$$365^{39}$$

– **probability they're all different**

$$P = \frac{\frac{365!}{(365-39)!}}{365^{39}}$$

Determining Probability Systematically

A commuter train has five cars. A commuter is twice as likely to select the middle car (#3) as to select either adjacent car (#2 or #4), and is twice as likely to select either adjacent car as to select either end car (#1 or #5).

1	2	3	4	5

$P(3) = 2 \times P(2) = 2P(4)$

$P(2) = 2 \times P(1) = 2 \times P(5) =$

$P(1) = x$

$P(1) + P(2) + P(3) + P(4) + P(5) = x + 2x + 4x + 2x + x$

$= 10x = 1$

$x = .1$; $P(1) = .1$, $P(2) = .2$, $P(3) = .4$, $P(4) = .2$; $P(5) = .1$

Example

1. A real estate agent is showing homes to a prospective buyer. There are ten homes in the desired price range listed in the area. The buyer has time to visit only three of them.

a. In how many ways could the three homes be chosen if the order of visiting is considered?

$$ {}_{10}P_3 = \frac{n!}{(n-r)!} = 10 \times 9 \times 8 $$

b. In how many ways could the three homes be chosen if the order is disregarded?

$$ \binom{10}{3} = \frac{n!}{r!(n-r)!} $$

a. If four of the homes are new and six have previously been occupied and if the three homes to visit are randomly chosen, what is the probability that all three are new (order is not considered)?

$$ P(\text{all new}) = \frac{\binom{4}{3}\cancel{\binom{6}{0}}^{\,1}}{\binom{10}{3}} $$

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Conditional Probability

- Returning to the die tossing experiment for example, let
  Event *A*: toss an odd number, and
  Event *B*: toss a number greater than or equal to four.
  What is *P*(*A* given that *B* has already occurred)?

$P(A|B)$

Conditional Probability

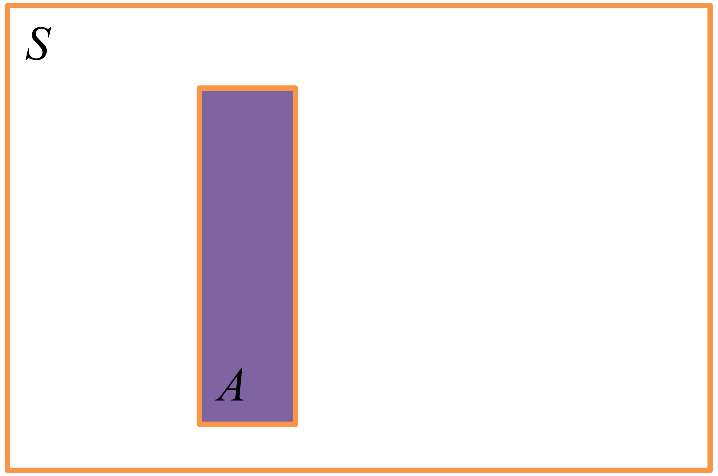
- The probabilities assigned to certain events often depend on what is known about the experimental situation when the assignment is made.

- Sometimes, the probability of an event will change if another event has already occurred or we have some extra information about the sample space.

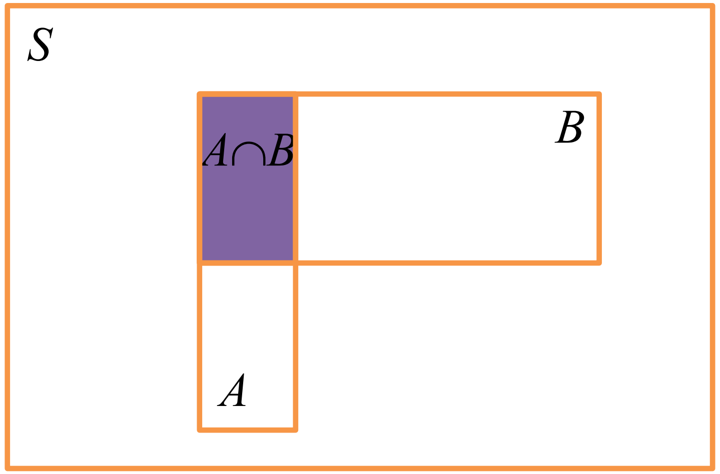
  For example: In manufacturing plant, the probability of a part being defective at one stage of the process may be dependent on whether it was produced on a particular production line or whether the raw material used was more likely to be defective itself.

- The conditional probability of A, given that B has already occurred, is denoted as $P(A|B)$.

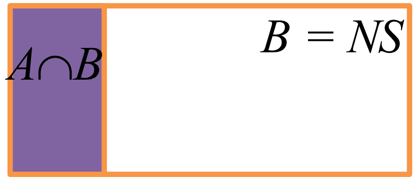
Conditional Probability



$P(A)$ is a number associated to the event A in a sample space, say S.



$P(A \cap B)$ is a number associated to the event $A \cap B$ in the same sample space, S.



$P(A|B)$ is a number associated to the event A, given that event B has already occurred. Therefore, $P(A|B)$ is obtained by calculating the probability of the event $A \cap B$ given that the event B makes the new sample space, say NS.

Conditional Probability

- Rather than listing the simple events (as we did in the last example) we can calculate conditional probability as follows if the $P(B) > 0$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$A \cap B = \{5\}$

$P(B) = 1/2$

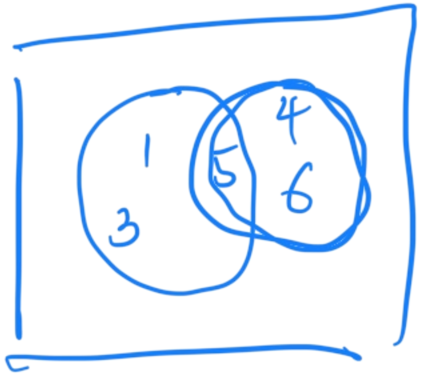
$P(A \cap B) = 1/6$

- Die tossing example:

$A = \{1, 3, 5\}$

$B = \{4, 5, 6\}$

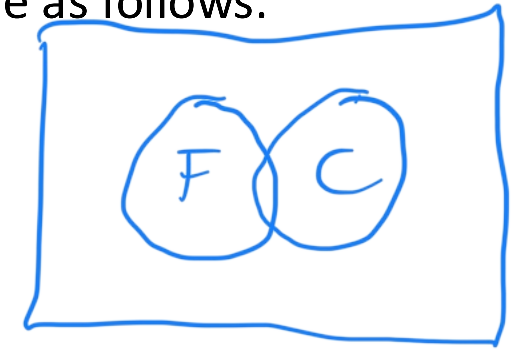
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/2} = 1/3$$



Example

Let F be the event that a randomly selected person watches *Food Network* from 10 to 11 pm on a Sunday night, and let C be the event that a randomly selected person watches *Comedy Central* on that night. Since many homes have DVRs, it is possible to record one of the shows on either of the channels and therefore, watch both shows eventually. Suppose the probabilities of the events are as follows:

$$P(F) = .25, \quad P(C) = .25, \text{ and } P(F \cap C) = .05$$



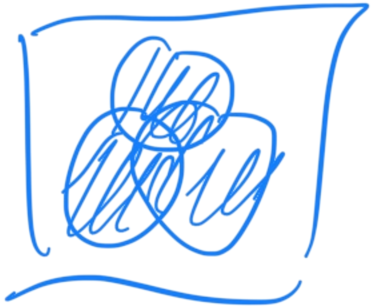
a. Are events F and C mutually exclusive?

No

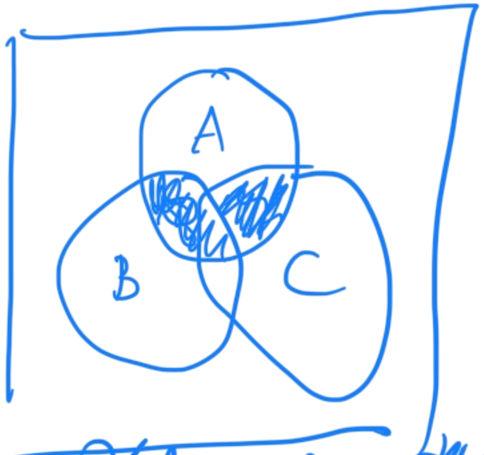
b. If a randomly selected person says that she watches *Comedy Central*, what is the probability that she also watches the recorded show of *Food Network* later?

$$P(F|C) = \frac{P(F \cap C)}{P(C)} = \frac{.05}{.25} = .2$$

Example



A news magazine published three columns “Art” (A), “Books” (B), and “Cinema” (C). Reading habits of a randomly selected reader with respect to these columns are $P(A) = .14$; $P(B)=.23$, $P(C)=.37$, $P(A\cap B) = .08$, $P(A\cap C) = .09$, $P(B\cap C)=.13$, $P(A \cap B \cap C)=.05$. Calculate:



- $P(A|B)$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{.08}{.23} \approx .35$$

- $P(A|B \cup C)$

$$P(A|B\cup C) = \frac{P(A\cap(B\cup C))}{P(B\cup C)} = \frac{P(A\cap B) + P(A\cap C) - P(A\cap B\cap C)}{P(B)+P(C)-P(B\cap C)} = \frac{.08+.09-.05}{.23+.37-.13} \approx .255$$

- $P(A|$ reads at least one$)$

$$\frac{P(A)}{P(A\cup B\cup C)} = \frac{.14}{.14+.23+.37-.08-.09-.13+.05} = .286$$

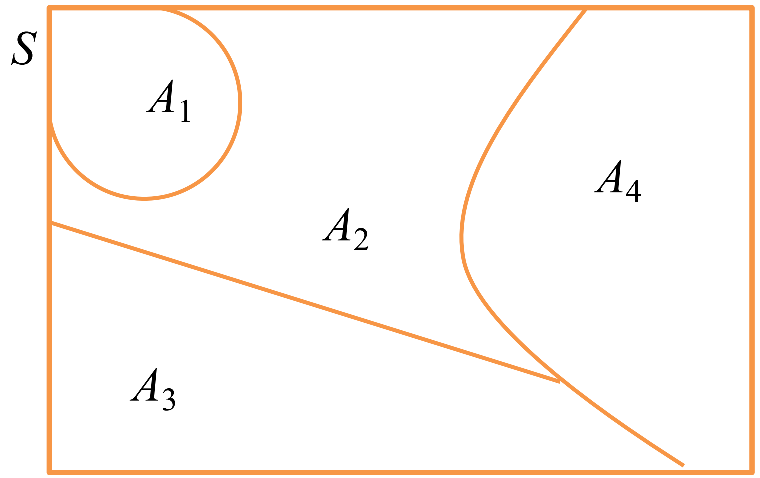
Mutually Exclusive and Exhaustive Events

- In a Venn diagram, show a set of mutually exclusive and exhaustive events A_1, ..., A_4 in the sample space, S.
  - **Mutually exclusive** events cannot occur at the same time, so have no common outcomes.
    - e.g., observing a head or a tail (but not both) after tossing a coin
  - **Exhaustive** events encompass the whole sample space, and hence, at least one of the events must occur.
    - e.g., observing an even number, a prime number, or a number smaller than 3 after rolling a die

Mutually Exclusive and Exhaustive Events

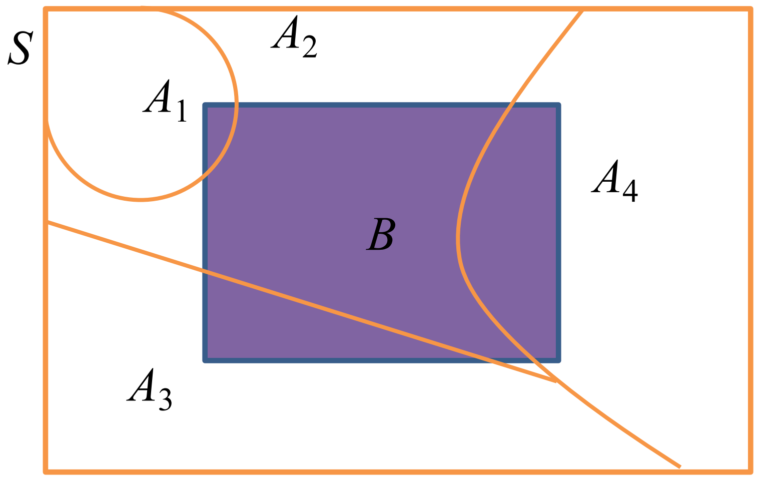
- In a Venn diagram, show a set of mutually exclusive and exhaustive events A_1, ..., A_4 in the sample space S.

  One possible answer:



The Law of Total Probability

- Consider the last example and let *B* be an event in the same sample space. Can we calculate the probability of the event B if we only know the probability of $A_i \cap B$ for i = 1, ..., 4?



$$P(B|A_i) = \frac{P(A_i \cap B)}{P(A_i)}$$

- What do we know about $A_1 \cap B$, $A_2 \cap B$, $A_3 \cap B$, and $A_4 \cap B$?

$$P(B) = \sum_{i=1}^{4} P(A_i \cap B) = \sum_{i=1}^{4} P(B|A_i)P(A_i)$$

The Law of Total Probability

- Remember the 3rd basic axiom of probability.
- Note that $A_1 \cap B$, $A_2 \cap B$, $A_3 \cap B$, and $A_4 \cap B$ are mutually exclusive.

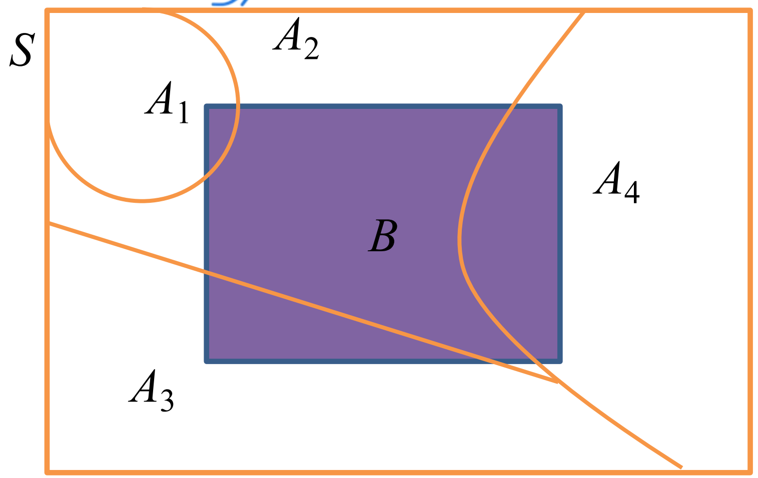
$P(B) =$ $P(B|A_1)$ $\cancel{P(B|A_2) + P(B|A_3) + P(B|A_4)}$

$P(A_1)$

$+ P(B|A_2)P(A_2) + P(B|A_3)P(A_3) + P(B|A_4)P(A_4)$

– Remember that

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$



The Law of Total Probability

If $A_1, \ldots, A_n$ are a set of mutually exclusive and exhaustive events, then for any other event B,

$$P(B) = P(B|A_1) \times P(A_1) + \ldots + P(B|A_n) \times P(A_n)$$
$$= \Sigma\, P(B|A_i) \times P(A_i)$$

– Mutually exclusive events cannot occur at the same time, so have no common outcomes.

– Exhaustive events encompass the whole sample space, and hence, at least one of the events must occur.

Bayes' Theorem

- Let $A_1, A_2, \ldots, A_n$ be a collection of n mutually exclusive and exhaustive events where $P(A_i) > 0$ for all i. Then, for any other event B for which $P(B) > 0$

$$P(A_k \mid B) = \frac{P(A_k \cap B)}{P(B)} = \frac{P(B \mid A_k) \times P(A_k)}{\sum_{i=1}^{n} P(B \mid A_i) \times P(A_i)}$$

  where $k = 1, \ldots, n$.

- We use Bayes' Theorem when we know one conditional probability but are actually interested in a different one!

D = event of having the disease

T = event of positive test

base rate fallacy

False Positives Example

1 in 1000 adults have a rare disease. There is a test for the disease. Let's say the test is 99% accurate. If a randomly selected individual tests positive for the disease, what is the probability that he or she actually has the disease?

$P(D) = .001$; $P(D') = .999$

$P(T|D) = .99$; $P(T|D') = .01$

$P(T) = P(T|D) \times P(D) + P(T|D') \times P(D')$

$= .99 \times .001 + .01 \times .999$

$= .01098$

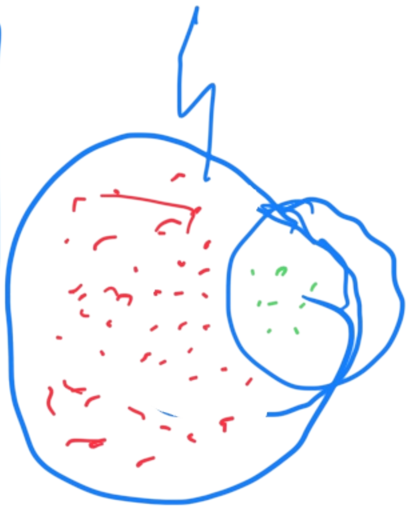
$P(D|T) = \dfrac{\cancel{P(T|D) \times P(D)}}{P(T)} = \dfrac{.99 \times .001}{.01098} \approx .0902$

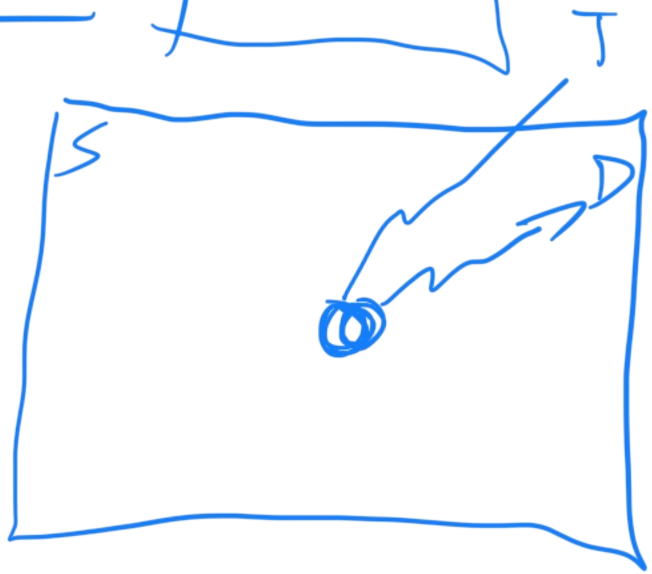
Table Solution

	T (positive)	T' (negative)	Total
D	(.99)(.001) = .00099	(.01)(.001) = .00001	.001
D'	(.01)(.999) = .00999	(.99)(.999) = .98901	.999
	.01098	.98902	

$$P(D \mid T) = \frac{.00099}{.01098}$$

.01





Example

1 in 1000 adults have a rare disease. There is a test for the disease. If you have the disease, the test is positive 99% of the time. If you **don't** have the disease, the test is positive 2% of the time. If a randomly selected individual tests positive for the disease, what is the probability that he or she actually has the disease?

About Physical Checkups from the Washington Post

Annual physical exam is probably unnecessary if you're generally healthy

For patients, the negatives include time from work and possibly unnecessary tests. "Getting a simply urinalysis could lead to a false positive, which could trigger a cascade of more tests, only to discover in the end that you had nothing wrong with you."

https://www.washingtonpost.com/national/health-science/annual-physical-exam-is-probably-unnecessary-if-youre-generally-healthy/2013/02/08/2c1e326a-5f2b-11e2-a389-ee565c81c565_story.html

Example

On a given night, LeBron James plays against one of three types of opponents:

A_1: weak defensive team

A_2: average defensive team

A_3: elite defensive team

with probabilities $P(A_1) = 0.4$, $P(A_2) = 0.35$, $P(A_3) = 0.25$.

Let B be the event that Lebron scores 30 or more points. Historical data shows $P(B|A_1) = 0.6$, $P(B|A_2) = 0.45$, $P(B|A_3) = 0.25$

a) What is the probability that Lebron plays an average defensive team and scores 30 or more points?

$$P(B \cap A_2) = P(B|A_2)\,P(A_2) = .45 \times .35 = .16$$

b) What is the probability that LeBron scores 30+ points on a random night?

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$$

$$= .46$$

Example (cont.)

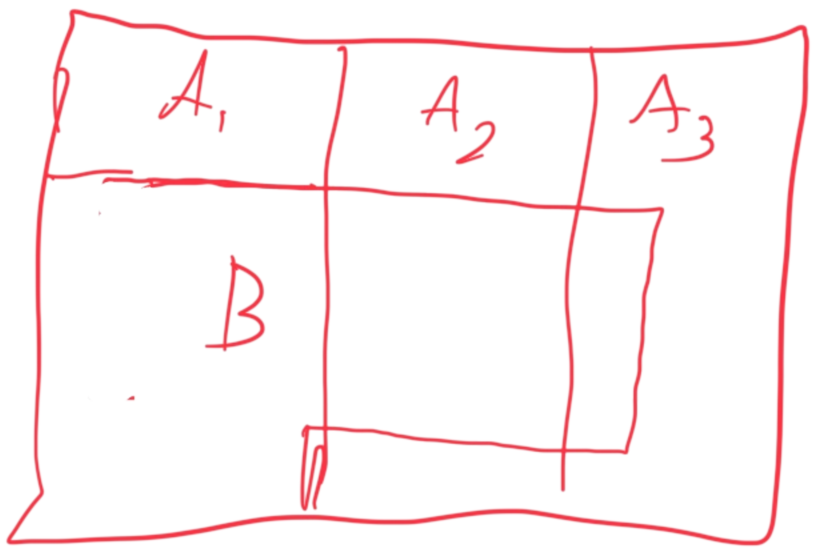
c) If LeBron scored 30+ points, what type of game was most likely?}

$$P(A_1|B) = P(B|A_1)P(A_1)/P(B) = .522$$

$$P(A_2|B) = P(B|A_2)P(A_2)/P(B) = .342$$

$$P(A_3|B) = P(B|A_3)P(A_3)/P(B) = .136$$

d) Draw a Venn diagram illustrating the various scenarios?



Independent Events

- We just studied conditional probability
  - It was common for P(B|A) to be different from P(B)
  - However, often times P(B|A) = P(B)
  - What does this mean?...
- The chance of B occurring is not affected by knowledge that A has occurred. In other words, events B and A are **independent**.

Two events A and B are **independent** if and only if P(B|A) = P(B) or P(A|B) = P(A), assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

Product Rule / Multiplicative Rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)}$$

- Often P(B|A) and P(A) are known and P(A∩B) is desired – use the product rule.

Product Rule / Multiplicative Rule

- Two events A and B are independent if and only if

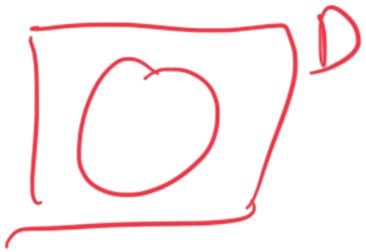
$$P(A \cap B) = P(A)P(B)$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Independent Events

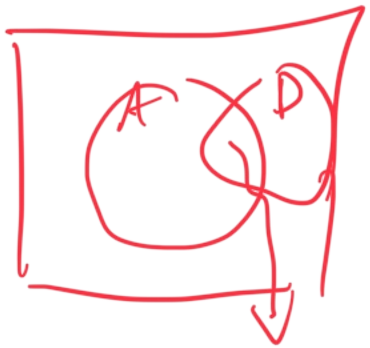
- If A and B are independent events, then
  - A' and B are independent events
  - A and B' are independent events
  - A' and B' are independent events

Independence



A card is selected at random from a deck.

Let A denote the event that it is an ace and D the event that it is a diamond. Check if events A and D are independent?



$1/52$

$P(A) = 1/13, \quad P(D) = 1/4$

$P(A \cap D) = \frac{1}{52}$

$P(A \cap D) = P(A)\,P(D) = 1/13 \times 1/4 = 1/52$

Example

30% of a company's washing machines require service while under warranty. 10% of dryers need service. If someone buys a washer and dryer, what is the probability that both machines will need service?

$$P(W \wedge D) = P(W)P(D)$$

$$= .3 \times .1$$

$$= .03$$

Example 2.39

An electrical system consists of four components as illustrated in Figure 2.9. The system works if components *A* and *B* work and either of the components *C* or *D* works. The reliability (probability of working) of each component is also shown in Figure 2.9. Find the probability that:

- (a) the entire system works.
- (b) the component *C* does not work, given that the entire system works. Assume that the four components work independently.

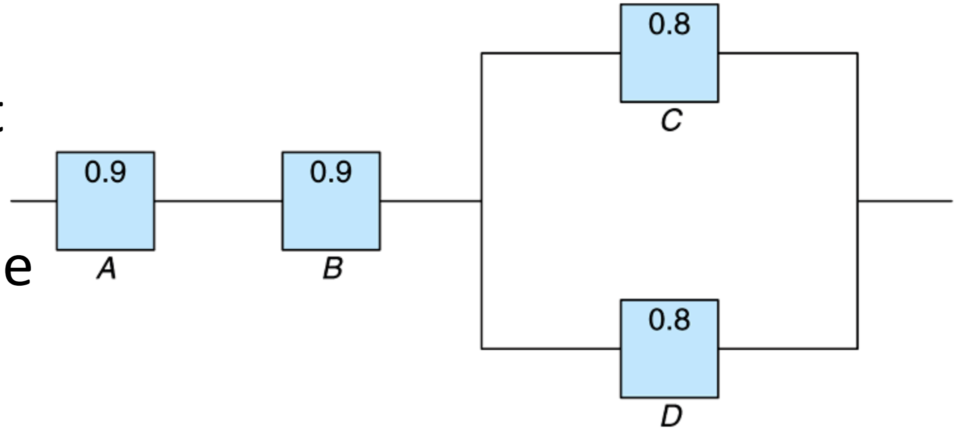


Figure 2.9 An electrical system

Example 2.39

(a) the entire system works.

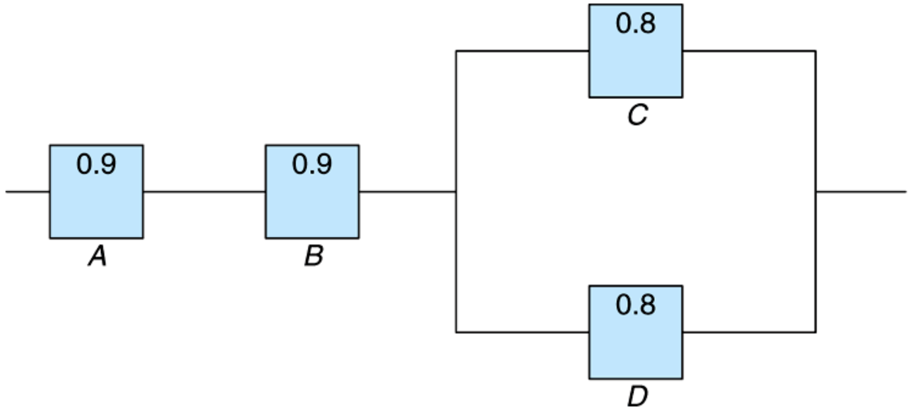


Figure 2.9 An electrical system

Example 2.39

(b) the component C does not work, given that the entire system works. Assume that the four components work independently.

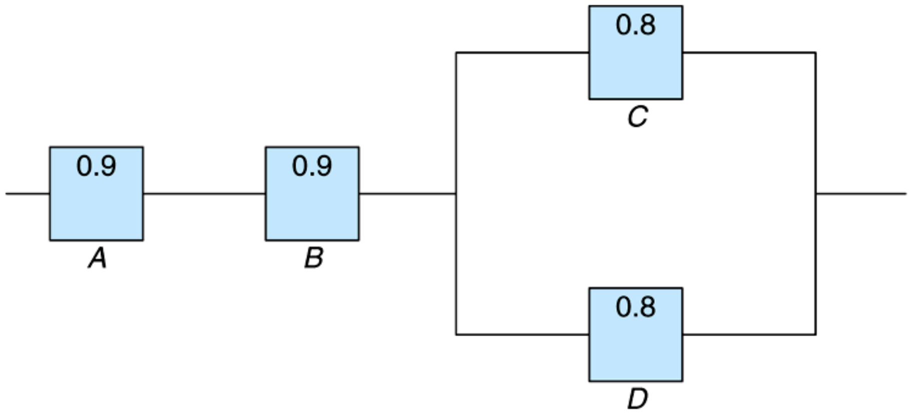


Figure 2.9 An electrical system

Example

If an aircraft is present (A), radar detects it with probability 0.99; otherwise, radar generates a false alarm with probability 0.10. An aircraft is present 5% of the time. What's the probability that an aircraft is present given the radar signals?

Prosecutor's fallacy: "A radar system detects an aircraft with 99% accuracy when an aircraft is present. Therefore, if the radar signals an aircraft, there is a 99% chance that an aircraft is actually present."