Chapter 4: Mathematical Expectation

Engr 0021

Review

- Sample space and events
- Probability measure for events
- Random Variables
- Discrete (pmf, CDF)
- Continuous (pdf, CDF)
- Joint random variables (marginal distribution)

Learning Objectives

- Mean of a random variable
- Variance of a random variable
- Variance and Covariance of random variables
- Mean and Variance of Linear Combinations of random variables

The Expected Value and Variance

- The probability distribution of a random variable provides a detailed description of the random variable.
- Sometimes, we would like to summarize that information.
- Two useful summary measures are:
    - Expected value (measure of location)
    - Variance (measure of dispersion)

The Expected Value

Let X be a discrete r.v. with set of possible values x and p.m.f./p.d.f. $f(x)$. The expected value (mean) of X, denoted by $E(X)$ or μ_X, is

$$\mu_X = E(X) = \sum_x x\, f(x) \qquad \text{if } X \text{ is discrete,}$$

and

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x\, f(x)\, dx \qquad \text{if } X \text{ is continuous.}$$

Example

Use the data below to find the expected number of textbooks the student will own each semester.

X = Number of textbooks.

x	4	5	6	7	8
$P(X = x)$	0.15	0.15	0.25	0.35	0.10

Example

Suppose that X is a continuous random variable whose probability distribution is

$$f(x) = \begin{cases} \frac{3}{2}\left(1 - x^2\right), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of X, i.e., $E(X)$.

Expectation of a Function of a Random Variable

Let X be a random variable with probability distribution $f(x)$. The expected value of $g(X)$ is

$$E[g(X)] = \sum_x g(x)\,f(x) \qquad \text{if } X \text{ is discrete,}$$

and

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)\,f(x)\,dx \qquad \text{if } X \text{ is continuous.}$$

Example

Let X represent the number of cylinders (4, 6, or 8) in the engine of the next car to be tuned up at a service center. The cost of a tune up is related to X by

$$g(X) = 20 + 3X + 0.5X^2.$$

The probability distribution of X is:

x	4	6	8
$f(x)$	0.5	0.3	0.2

Find $E[g(X)]$.

Expected Value for Joint Random Variables

Let X and Y be jointly distributed random variables with p.m.f. or p.d.f. $f(x, y)$. Then the expected value of $g(X, Y)$, denoted $E[g(X, Y)]$ (or $\mu_{g(X,Y)}$), is

$$E[g(X, Y)] = \sum_x \sum_y g(x, y)\, f(x, y) \qquad \text{if } X, Y \text{ are both discrete,}$$

and

$$E[g(X, Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y)\, f(x, y)\, dx\, dy \qquad \text{if } X, Y \text{ are both continuous.}$$

Example

An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X be the number of points earned on the first part and Y the number of points earned on the second part. The following table gives the joint probability distribution of (X, Y). Find $E(X + Y)$.

		X			
	$f(x, y)$	0	5	10	15
Y	0	0.02	0.06	0.02	0.10
	5	0.04	0.15	0.20	0.10
	10	0.01	0.15	0.14	0.01

Example

Example

Let X and Y be random variables with joint density function

$$f(x, y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

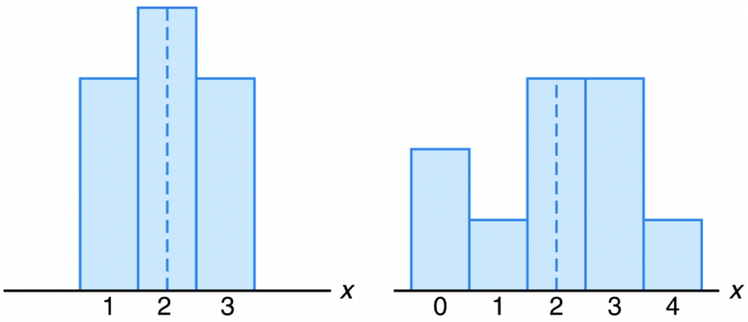
Find the expected value of

$$Z = \frac{Y}{X}.$$

Variance of Random Variables

- Useful summary measure
- Indication of dispersion about the expected value
- Denoted by $\mathrm{Var}(X)$ or σ_X^2

Distributions with Equal Means and Unequal Dispersions



The Variance

Let X have probability distribution $f(x)$ and mean μ. The variance is

$$\sigma_X^2 = \mathrm{Var}(X) = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 f(x) \qquad \text{if } X \text{ is discrete,}$$

and

$$\sigma_X^2 = \mathrm{Var}(X) = E\left[(X-\mu)^2\right] = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx \qquad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ_X, is called the *standard deviation*.

The Variance

An alternative formula for finding variance is

$$\sigma_X^2 = E(X^2) - \mu_X^2.$$

Example

Use the data below to find the variance of the number of textbooks the student will own each semester.

$$X = \text{Number of textbooks.}$$

x	4	5	6	7	8
$f(x)$	0.15	0.15	0.25	0.35	0.10

Example

Example

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} \frac{3}{2}\left(1 - x^2\right), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the variance of X, i.e., σ_X^2.

The Variance of a Function

Let X have probability distribution $f(x)$ and mean μ. The variance of $g(X)$ is

$$\sigma^2_{g(X)} = E\left[\left(g(X) - \mu_{g(X)}\right)^2\right] = \sum_x \left(g(x) - \mu_{g(X)}\right)^2 f(x), \quad \text{if } X \text{ is discrete,}$$

and

$$\sigma^2_{g(X)} = E\left[\left(g(X) - \mu_{g(X)}\right)^2\right] = \int_{-\infty}^{+\infty} \left(g(x) - \mu_{g(X)}\right)^2 f(x)\, dx, \quad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ, is called the standard deviation.

Example

Calculate the variance of

$$g(X) = 2X + 2.$$

x	4	5	6	7	8
$g(x)$	10	12	14	16	18
$f(x) = P(X = x)$	0.15	0.15	0.25	0.35	0.10

Example

Example

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} \frac{3}{2}\left(1 - x^2\right), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the variance of

$$h(X) = 2X + 3.$$

Example

Review

- Mean of a random variable
- Variance of a random variable

Learning Objectives

- Variance and Covariance of random variables
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Covariance of Two Random Variables

- Useful summary measure
- Degree of association between two variables, in terms of deviation from their means
- If covariance is:
  - **positive:** when one r.v. is larger than its mean, the other tends to be larger than its mean
  - **negative:** when one r.v. is larger than its mean, the other tends to be smaller than its mean

Covariance of Two Random Variables

- We like to know how strong two variables are related to one another if they are not independent.

Covariance of Two Random Variables

Let X and Y be random variables with joint probability distribution $f(x, y)$. The covariance of X and Y is

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)].$$

- If X and Y are discrete:

$$\sigma_{XY} = \sum_x \sum_y (x - \mu_X)(y - \mu_Y)\, f(x, y).$$

- If X and Y are continuous:

$$\sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)\, f(x, y)\, dx\, dy.$$

Short-cut Formula for Covariance

The covariance of two random variables X and Y with means μ_X and μ_Y is:

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = E(XY) - \mu_X \mu_Y.$$

Correlation Coefficient

Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y. The correlation coefficient is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

Correlation Coefficient

- A useful summary measure of association
- What does it mean when $\rho_{XY} = 1$ or -1?
- So, the correlation coefficient measures the degree of **linearity** between X and Y
- Qn: Why divide σ_{XY} by $\sigma_X \sigma_Y$?

Correlation Propositions

1. If a and c are either both positive or both negative,

$$\rho(aX + b,\ cY + d) = \rho(X, Y).$$

2. For any two r.v.s X and Y, $-1 \leq \rho_{XY} \leq 1$.
3. If X and Y are independent, then $\rho = 0$, but $\rho = 0$ does not imply independence.

Example

A restaurant serves three fixed-priced dinners costing \$7, \$9, and \$10. For a randomly selected couple, let

$$X = \text{man's dinner cost}, \qquad Y = \text{woman's dinner cost}.$$

Suppose the joint pmf $f(x, y)$ is:

	$f(x, y)$	X		
		7	9	10
Y	7	0.05	0.05	0.10
	9	0.05	0.10	0.35
	10	0	0.20	0.10

Example (Continued)

1. Compute the marginal probability distributions of X and Y.
2. What is $P(X \leq 9,\ Y \leq 9)$?
3. Are X and Y independent? Explain.
4. Find σ_{XY} and the correlation coefficient ρ.

Example (Solutions)

Rules of the Expected Value

If a and b are constants, then

$$E(aX + b) = aE(X) + b.$$

- Setting $b = 0$: $E(aX) = aE(X)$
- Setting $a = 0$: $E(b) = b$

Example

Calculate $E(g(X))$, where $g(X) = 2X + 2$.

x	4	5	6	7	8
$g(x)$	10	12	14	16	18
$f(x) = P(X = x)$	0.15	0.15	0.25	0.35	0.10

Independence

Let X and Y be two independent r.v.s. Then

$$E(XY) = E(X)\,E(Y).$$

Also, for independent X and Y,

$$\sigma_{XY} = 0.$$

Rules of Variance

If X and Y are random variables (with covariance σ_{XY}) and a, b, c are constants, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\,\sigma_{XY}.$$

- Setting $b = 0$: $\sigma^2_{aX+c} = a^2\sigma^2_X$
- Setting $a = 1$, $b = 0$: $\sigma^2_{X+c} = \sigma^2_X$
- Setting $b = 0$, $c = 0$: $\sigma^2_{aX} = a^2\sigma^2_X$

Rules of Variance and Independence

If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y, \qquad \sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

If $X_1, \ldots, X_n$ are independent random variables, then

$$\sigma^2_{a_1X_1+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + \cdots + a_n^2\sigma^2_{X_n}.$$