Chapter 3 – Random Variables and Probability Distributions (I)

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Review

- Chapter 1
  - Introduction
  - Measure of Location
  - Measure of Variability
- Chapter 2
  - Define sample space and measurable events
  - Assign probability measure

Learning Objectives

- Concept of a Random Variable
- Discrete Probability Distribution
- Continuous Probability Distribution
- Joint Probability Distribution
- Marginal Distribution
- Conditional Distribution

Concept of a Random Variable

- Statistical experiment: three electronic components are tested

$$S = \{NNN, NND, NDN, DNN, NDD, DND, DDN, DDD\}$$

- N: Non-defective
- D: Defective
- X – the number of defective items when 3 electronic components are tested

Random Variable

- For a given sample space S of some experiment, a random variable is any rule that associates a number with each outcome in S.
- Notation:
  - X: random variable
  - x: one realization or one observed value for r.v. X
- Example: die toss, number of defects in a sample of 100 productions, number of experimental trials before success, depth measurement of a lake, pH of a chemical compound, number of people who will vote for a particular candidate, weight of a coil of steel, length of a steel beam, diameter of a washer, etc.

Random Variable: Examples

- Consider an experiment that consists of tossing two coins.
- A die is thrown until 5 occurs.
- Components are arriving from a production line and they are stipulated to be defective or not defective.
- Record the successive speeders spotted by a radar unit.

Random Variable: Examples

- Consider an experiment that consists of tossing two coins.
  - Let X equal the number of heads observed.
- A die is thrown until 5 occurs.
  - Let X be the number of times the die is thrown before 5.
- Components are arriving from a production line and they are stipulated to be defective or not defective.
  - Let X be the number of defective parts.
- Record the successive speeders spotted by a radar unit.
  - Let X be a random variable defined by waiting time, in hours between successive speeders spotted by a radar unit.

Discrete and Continuous Sample Space

- If a sample space contains a finite number of possibilities or an unending (infinite) sequence with as many elements as there are whole integers, it is called **discrete sample space**.
- If a sample space contain an infinite number of possibilities equal to the number of points on a line segment, it is called a **continuous sample space**.

Types of Random Variables

- A **discrete** random variable is a random variable whose possible values either constitute a finite set or else can be listed in an infinite sequence in which there is a first element, a second element, and so on ("countable" infinite).
- A random variable is **continuous** if both of the following apply:
  - Its set of possible values consists either of all numbers in a single interval on the number line (possibly infinite in extent, e.g., from $-\infty$ to ∞) or all numbers in a disjoint union of such intervals (e.g., $[0, 10] \cup [20, 30]$).
  - No possible value of the variable has positive probability, i.e., $P(X = c) = 0$ for all possible value c.

Examples: Discrete or Continuous?

Are the following random variables discrete or continuous?

- The total number of points scored in a basketball game
- The shelf life of a particular drug
- The length of a two-year old black bass
- The number of aircraft near-collisions last year

Answers: Basketball points: **discrete**.
Shelf life: **continuous**.
Length of bass: **continuous**.
Near-collisions: **discrete**. Comment: Counts are discrete; measurements on a continuum are continuous.

Probability Distribution

- The Probability Distribution for a random variable X tells us how the total probability of 1 for the sample space S is distributed among each of the mutually exclusive simple events (or outcomes for X) that describe the sample space.

Discrete Probability Distribution Example

- Consider an experiment that consists of tossing two coins and let X equal the number of heads observed. Find the probability distribution for X.

Simple Event	Coin 1	Coin 2	Outcome	$p(E_i)$	x
E_1	H	H	HH	$1/4$	2
E_2	H	T	HT	$1/4$	1
E_3	T	H	TH	$1/4$	1
E_4	T	T	TT	$1/4$	0

X	$p(x)$
0	$1/4$
1	$1/2$
2	$1/4$

Discrete Probability Distribution

The set of ordered pairs $(x, f(x))$ is a probability function, probability mass function, or probability distribution of the discrete random variable X if, for each possible outcome x, denoted as pmf:

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$
3. $P(X = x) = f(x)$

Example

- A probability distribution for a random variable X:

x	-8	-3	-1	0	1	4	6
$P(X=x)$	0.13	0.15	0.17	0.20	0.15	0.11	0.09

- Find
  - $P(X \leq 0)$
  - $P(-3 \leq X \leq 1)$

Answers:

$$P(X \leq 0) = 0.13 + 0.15 + 0.17 + 0.20 = 0.65.$$

$$P(-3 \leq X \leq 1) = 0.15 + 0.17 + 0.20 + 0.15 = 0.67.$$

Example

A shipment of 7 television sets contain 2 defective sets. A hotel makes a random purchase of 3 of the sets. If X is the number of defective sets purchased by the hotel, find the probability distribution of X. **Answer (Hypergeometric):**

$$P(X = x) = \frac{\binom{2}{x}\binom{5}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2.$$

$$P(X = 0) = \frac{10}{35} = \frac{2}{7}, \quad P(X = 1) = \frac{20}{35} = \frac{4}{7}, \quad P(X = 2) = \frac{5}{35} = \frac{1}{7}.$$

Comment: Sampling *without replacement* $\Rightarrow$ hypergeometric.

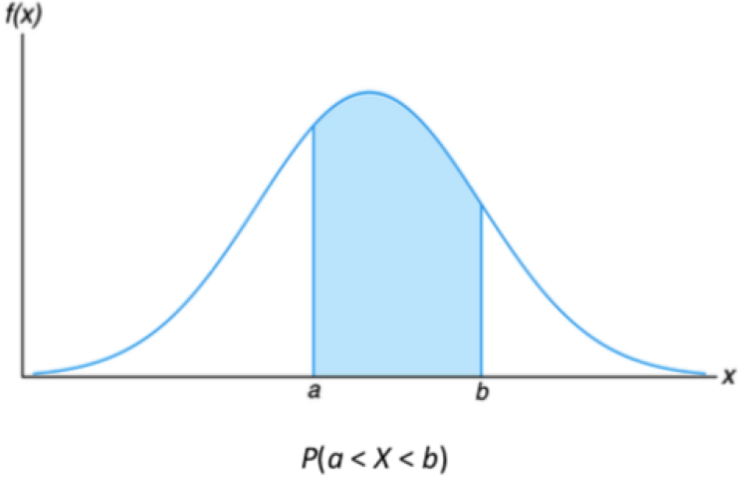
Example

A shipment of 20 laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives. **Answer (Hypergeometric):**

$$P(X = x) = \frac{\binom{3}{x}\binom{17}{2-x}}{\binom{20}{2}}, \qquad x = 0, 1, 2.$$

$$P(X = 0) = \frac{136}{190} \approx 0.716, \quad P(X = 1) = \frac{51}{190} \approx 0.268, \quad P(X = 2) = \frac{3}{190} \approx 0.0158.$$

Probability mass function plot



The Cumulative Distribution Function (Discrete)

- The cumulative distribution function $F(x)$ of discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \qquad -\infty < x < \infty.$$

Example

- A probability distribution for a random variable X:

x	-8	-3	-1	0	1	4	6
$P(X = x)$	0.13	0.15	0.17	0.20	0.15	0.11	0.09

- Write the CDF of X

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < -8, \\ 0.13, & -8 \leq x < -3, \\ 0.28, & -3 \leq x < -1, \\ 0.45, & -1 \leq x < 0, \\ 0.65, & 0 \leq x < 1, \\ 0.80, & 1 \leq x < 4, \\ 0.91, & 4 \leq x < 6, \\ 1, & x \geq 6. \end{cases}$$

Proposition

- For any two number a and b with $a \leq b$,

$$P(a \leq X \leq b) = F(b) - F(a^-)$$

- 'a^-' represents the largest possible X value that is strictly less than a
- Note: for integers (as possible values for x):

$$P(a \leq X \leq b) = F(b) - F(a-1)$$

Example

- Suppose a random variable has the following distribution

x	0	1	2	3	4
$f(x)$	0.4	0.1	0.1	0.1	0.3

- Find the CDF
- Find $P(X \leq 2)$
- Find $P(X < 2)$
- Find $P(2 \leq X \leq 4)$
- Find $P(X \neq 0)$

$$F(x) = \begin{cases} 0 & x < 0, \\ 0.4 & 0 \leq x < 1, \\ 0.5 & 1 \leq x < 2, \\ 0.6 & 2 \leq x < 3, \\ 0.7 & 3 \leq x < 4, \\ 1 & x \geq 4. \end{cases}$$

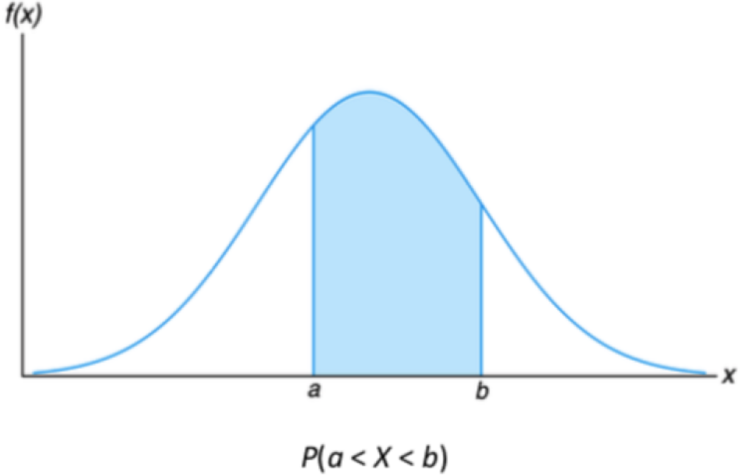
$$P(X \leq 2) = 0.6,\ P(X < 2) = 0.5,\ P(2 \leq X \leq 4) = 0.5,\ P(X \neq 0) = 0.6.$$

Continuous Random Variable

- Continuous random variable can assume infinitely many value corresponding to points in a particular interval.
- A r.v. X is continuous if its set of possible values is an entire interval of number (If $A < B$, then any number between A and B is possible.)
- If the measurement scale of X can be subdivided to any extent desired, then a variable is continuous; if it cannot, the variable is discrete.

Continuous Probability Distribution

- X – random variable whose values are heights of all people over 21 years of age.



Continuous Probability Distribution

- The function $f(x)$ is a probability density function, denoted as pdf, for the continuous random variable X, defined over the set of real numbers, if
  1. $f(x) \geq 0$, for all $x \in \mathbb{R}$
  2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$
  3. $P(a < X < b) = \int_{a}^{b} f(x)\, dx$

The Cumulative Distribution Function (Continuous)

- The cumulative distribution function $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt, \qquad -\infty < x < \infty$$

Example

- For the density function

$$f(x) = \begin{cases} 0.5x, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

- Find $F(x)$, and use it to evaluate $P(0 < X \le 1)$

Answer:

$$F(x) = \begin{cases} 0, & x < 0, \\ \int_0^x 0.5t\,dt = 0.25x^2, & 0 \le x \le 2, \\ 1, & x \ge 2, \end{cases} \qquad P(0 < X \le 1) = F(1) - F(0) = 0.25.$$

Example

- Suppose the pdf of the magnitude X of a dynamic load on a bridge (in newtons) is

$$f(x) = \begin{cases} \dfrac{1}{8}x + \dfrac{3}{8}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- Find $F(x) = \int_{-\infty}^{x} f(y)\,dy$

Answer:

$$F(x) = \begin{cases} 0, & x < 0, \\ \int_0^x \left(\frac{1}{8}y + \frac{3}{8}\right) dy = \frac{1}{16}x^2 + \frac{3}{8}x, & 0 \leq x \leq 2, \\ 1, & x \geq 2. \end{cases}$$

Example

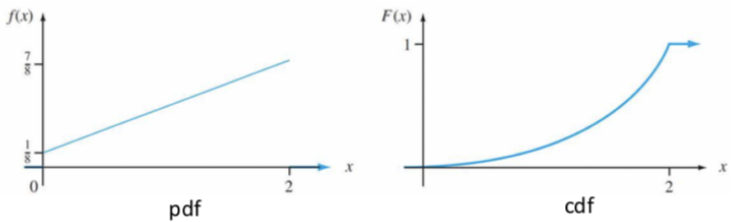


Figure: pdf and CDF

- Find the probability that the load is between 1 and 1.5
- Find the probability that the load exceeds 1

$$P(1 < X < 1.5) = F(1.5) - F(1) = \left(\frac{1}{16}(1.5)^2 + \frac{3}{8}(1.5)\right) - \left(\frac{1}{16}(1)^2 + \frac{3}{8}(1)\right) = 0.265625.$$

$$P(X > 1) = 1 - F(1) = 1 - \left(\frac{1}{16} + \frac{3}{8}\right) = 0.5625.$$

Example

- Consider the density function

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- Evaluate k
- Find $F(x)$ and use it to evaluate $P(0.3 < X < 0.6)$

Answer:

$$1 = \int_0^1 \frac{k}{\sqrt{x}} dx = k\left[2\sqrt{x}\right]_0^1 = 2k \;\Rightarrow\; k = \frac{1}{2}.$$

$$F(x) = \begin{cases} 0, & x \leq 0, \\ \int_0^x \frac{1}{2\sqrt{t}} dt = \sqrt{x}, & 0 < x < 1, \\ 1, & x \geq 1, \end{cases} \qquad P(0.3 < X < 0.6) = \sqrt{0.6} - \sqrt{0.3} \approx 0.227.$$

Obtaining $f(x)$ from $F(x)$

- For X discrete, the pmf is obtained from the CDF by taking the difference between two $F(x)$ values. The continuous analog of a difference is a derivative.
- The following result is a consequence of the Fundamental Theorem of Calculus.

Proposition: If X is a continuous r.v. with pdf $f(x)$ and CDF $F(x)$, then at every x at which the derivative $F'(x)$ exists,

$$F'(x) = f(x).$$

Review

- Concept of a random variable
- Focus on univariate random variable so far
- Discrete probability distributions
  - pmf probability mass function
  - CDF cumulative distribution function
- Continuous distributions
  - pdf probability distribution function
  - CDF

This Lecture

- Joint Probability Distribution (Multiple random variables)
- Marginal Distribution
- Conditional Distribution
- Statistical Independence

Joint Probability Distributions

- Interested in the probability multiple random variables take on particular values
- Examples:
  - demand for three different products next period
  - number of different birthdays for the women and men in the room
  - height and weight of an animal one year from now
- There are many experimental situations in which more than one random variable is of interest
- Joint distributions can be discrete or continuous (or neither); we focus on discrete and continuous cases

Example: Joint pmf Table (Insurance Deductibles)

- X = homeowner's deductible, Y = auto deductible.
- Joint probability table $f(x, y)$:

$f(x,y)$	$y = 0$	$y = 100$	$y = 200$
$x = 100$	0.20	0.10	0.20
$x = 250$	0.05	0.15	0.30

Joint Probability Distribution (Discrete)

- The function $f(x, y)$ is a **joint probability distribution** (joint pmf) of discrete random variables X and Y if:
  1. $f(x, y) \geq 0$ for all x, y
  2. $\sum_x \sum_y f(x, y) = 1$
  3. $P(X = x,\ Y = y) = f(x, y)$
- For any region A in the xy-plane:

$$P\big((X, Y) \in A\big) = \sum \sum_{(x,y)\in A} f(x, y).$$

Example 3.14

- Two ballpoint pens are selected at random from a box with:

$$3 \text{ blue}, \ 2 \text{ red}, \ 3 \text{ green} \quad (8 \text{ total}).$$

- X = number of blue pens selected, Y = number of red pens selected.
- Find the joint pmf $f(x, y)$.

Example 3.14 (Joint pmf)

$$\text{Total equally likely outcomes} = \binom{8}{2} = 28.$$

The joint pmf becomes

$$f(x,y) = P(X = x, Y = y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

subject to

$$x \geq 0, \quad y \geq 0, \quad 2 - x - y \geq 0.$$

$f(x,y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	$\frac{\binom{3}{2}}{28} = \frac{3}{28}$	$\frac{\binom{2}{1}\binom{3}{1}}{28} = \frac{6}{28} = \frac{3}{14}$	$\frac{\binom{2}{2}}{28} = \frac{1}{28}$
$x = 1$	$\frac{\binom{3}{1}\binom{3}{1}}{28} = \frac{9}{28}$	$\frac{\binom{3}{1}\binom{2}{1}}{28} = \frac{6}{28} = \frac{3}{14}$	0
$x = 2$	$\frac{\binom{3}{2}}{28} = \frac{3}{28}$	0	0

Joint Density Function (Continuous)

- The function $f(x, y)$ is a **joint density** (joint pdf) of continuous r.v.s X and Y if:
  1. $f(x, y) \geq 0$
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$
  3. For any region A:
  $$P\big((X, Y) \in A\big) = \iint_A f(x, y)\, dx\, dy$$

Example (Joint pdf)

- Let X be reaction time (s) and Y be temperature (°F).
- Joint density:

$$f(x, y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- Find $P\big(0 \leq X \leq \frac{1}{2} \text{ and } \frac{1}{4} \leq Y \leq \frac{1}{2}\big)$.

Example (Solution)

$$\begin{aligned}
P\left(0 \le X \le \tfrac{1}{2},\ \tfrac{1}{4} \le Y \le \tfrac{1}{2}\right) &= \int_0^{1/2} \int_{1/4}^{1/2} 4xy \, dy \, dx \\
&= 4 \left(\int_0^{1/2} x \, dx \right) \left(\int_{1/4}^{1/2} y \, dy \right) \\
&= 4 \left(\frac{1}{8} \right) \left(\frac{3}{32} \right) = \frac{3}{64} \approx 0.0469.
\end{aligned}$$

Problem 3.61: Find k

- Find k so that $f(x, y)$ is a valid joint pdf:

$$f(x,y) = \begin{cases} kxy, & 0 < x < 1,\ 0 < y < 1,\ x + y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\begin{aligned} 1 &= \int_0^1 \int_0^{1-x} kxy \, dy \, dx \\ &= k \int_0^1 x \left[\frac{(1-x)^2}{2} \right] dx = \frac{k}{2} \int_0^1 \left(x - 2x^2 + x^3 \right) dx \qquad \Rightarrow \qquad \boxed{k = 24}. \\ &= \frac{k}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{k}{2} \left(\frac{1}{12} \right) = \frac{k}{24}. \end{aligned}$$

Marginal Distributions

- The marginal distributions of X alone and Y alone are:
  - Discrete:

$$g(x) = \sum_{y} f(x,y), \qquad h(y) = \sum_{x} f(x,y)$$

  - Continuous:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \qquad h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Marginal Distributions: Discrete Case Example (Figure 2)

$f(x,y)$	$x=0$	$x=1$	$x=2$	Row total ($h(y)$)
$y=0$	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
$y=1$	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
$y=2$	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Col total ($g(x)$)	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

- Marginal pmf of X: $g(0)=\frac{5}{14}$, $g(1)=\frac{15}{28}$, $g(2)=\frac{3}{28}$.
- Marginal pmf of Y: $h(0)=\frac{15}{28}$, $h(1)=\frac{3}{7}$, $h(2)=\frac{1}{28}$.

Marginal Distributions: Discrete Case Example (Insurance Table)

$f(x,y)$	$y=0$	$y=100$	$y=200$
$x=100$	0.20	0.10	0.20
$x=250$	0.05	0.15	0.30

- Marginal pmf of X: $g(100)=0.50,\ g(250)=0.50.$
- Marginal pmf of Y: $h(0)=0.25,\ h(100)=0.25,\ h(200)=0.50.$

Marginal Distributions: Continuous Case Example

$$f(x, y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- Marginal pdf of X: $g(x) = \int_0^1 4xy\, dy = 2x, \quad 0 < x < 1.$
- Marginal pdf of Y: $h(y) = \int_0^1 4xy\, dx = 2y, \quad 0 < y < 1.$

Independence

- Let X and Y have joint distribution $f(x, y)$ and marginals $g(x)$ and $h(y)$.
- X and Y are **statistically independent** iff

$$f(x, y) = g(x)\, h(y)$$

for all (x, y) in their range.

Independence: Discrete Case (Show NOT independent)

$$g(100) = 0.5,\ g(250) = 0.5, \qquad h(0) = 0.25,\ h(100) = 0.25,\ h(200) = 0.5.$$

Check one cell:

$$g(100)h(0) = (0.5)(0.25) = 0.125 \neq f(100, 0) = 0.20.$$

Therefore, $f(x, y) \neq g(x)h(y)$, so X and Y are **not independent**.

Independence: Continuous Case

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

We found:

$$g(x) = 2x, \quad 0 < x < 1, \qquad h(y) = 2y, \quad 0 < y < 1.$$

Then

$$g(x)h(y) = (2x)(2y) = 4xy = f(x,y).$$

So X and Y are **independent**.

Statistical Independence (Multiple Variables)

- Let $X_1, X_2, \ldots, X_n$ have joint distribution $f(x_1, \ldots, x_n)$ and marginals $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$.
- They are **mutually independent** iff

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

  for all $(x_1, \ldots, x_n)$ in their range.