Chapter 4: Mathematical Expectation

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Review

- Sample space and events
- Probability measure for events
- Random Variables
- Discrete (pmf, CDF)
- Continuous (pdf, CDF)
- Joint random variables (marginal distribution)

Learning Objectives

- Mean of a random variable
- Variance of a random variable
- Variance and Covariance of random variables
- Mean and Variance of Linear Combinations of random variables

The Expected Value and Variance

- The probability distribution of a random variable provides a detailed description of the random variable.
- Sometimes, we would like to summarize that information.
- Two useful summary measures are:
  - Expected value (measure of location)
  - Variance (measure of dispersion)

The Expected Value

Let X be a discrete r.v. with set of possible values x and p.m.f./p.d.f. $f(x)$. The expected value (mean) of X, denoted by $E(X)$ or μ_X, is

$$\mu_X = E(X) = \sum_x x\, f(x) \qquad \text{if } X \text{ is discrete,}$$

and

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x\, f(x)\, dx \qquad \text{if } X \text{ is continuous.}$$

Example

Use the data below to find the expected number of textbooks the student will own each semester.

X = Number of textbooks.

x	4	5	6	7	8
$P(X = x)$	0.15	0.15	0.25	0.35	0.10

$$E[X] = \sum_x x P(y=x)$$

$$= 4 \times .15 + 5 \times .15 + 6 \times .25 + 7 \times .35 + 8 \times .1$$

$$= 6.1$$

Example

Suppose that X is a continuous random variable whose probability distribution is

$$f(x) = \begin{cases} \frac{3}{2}\left(1 - x^2\right), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of X, i.e., $E(X)$.

$$E(x) = \int_0^1 x \frac{3}{2}(1-x^2)\,dx$$
$$= \frac{3}{2}\int_0^1 (x - x^3)\,dx$$
$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{1} = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$

Expectation of a Function of a Random Variable

Let X be a random variable with probability distribution $f(x)$. The expected value of $g(X)$ is

$$E[g(X)] = \sum_{x} g(x)\, f(x) \qquad \text{if } X \text{ is discrete,}$$

and

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)\, f(x)\, dx \qquad \text{if } X \text{ is continuous.}$$

Example

Let X represent the number of cylinders (4, 6, or 8) in the engine of the next car to be tuned up at a service center. The cost of a tune up is related to X by

$$g(X) = 20 + 3X + 0.5X^2.$$

The probability distribution of X is:

x	4	6	8
$f(x)$	0.5	0.3	0.2

Find $E[g(X)]$.

$g(x)$ 40 56 76

$$E[g(x)] = \sum_x g(x)f(x) = 40 \times .5 + 56 \times .3 + 76 \times .2$$
$$= 52$$

Expected Value for Joint Random Variables

Let X and Y be jointly distributed random variables with p.m.f. or p.d.f. $f(x, y)$. Then the expected value of $g(X, Y)$, denoted $E[g(X, Y)]$ (or $\mu_{g(X,Y)}$), is

$$E[g(X, Y)] = \sum_x \sum_y g(x, y)\, f(x, y) \qquad \text{if } X, Y \text{ are both discrete,}$$

and

$$E[g(X, Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y)\, f(x, y)\, dx\, dy \qquad \text{if } X, Y \text{ are both continuous.}$$

Example

An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X be the number of points earned on the first part and Y the number of points earned on the second part. The following table gives the joint probability distribution of (X, Y). Find $E(X + Y)$.

$E(X)$; $E(Y)$

$E(X+Y)$

		X				
	$f(x, y)$	0	5	10	15	$h(y)$
Y	0	0.02	0.06	0.02	0.10	.2
	5	0.04	0.15	0.20	0.10	.49
	10	0.01	0.15	0.14	0.01	.31
	$g(x)$	.07	.36	.36	.21	

Example

$$E(X) = 0 \times .07 + 5 \times .36 + 10 \times .36 + 15 \times .21 = 8.55$$

$$E(Y) = 0 \times .2 + 5 \times .49 + 10 \times .31 = 5.55$$

$$E(X+Y) = E(X) + E(Y) = 14.1$$

Example

Let X and Y be random variables with joint density function

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of

$$Z = \frac{Y}{X}.$$

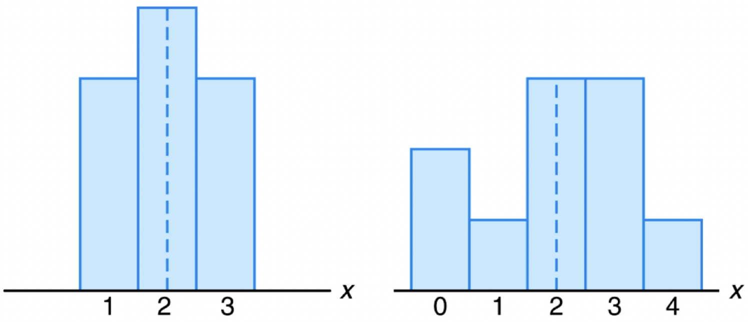
$$E(Z) = E\left(\frac{Y}{X}\right) = \int_0^2 \int_0^1 \frac{y}{\cancel{x}} \frac{\cancel{x}(1+3y^2)}{4}\, dy\, dx$$

$$\int_0^2 dx \int_0^1 \frac{y+3y^3}{4}\, dy = \frac{2}{4} \times \left[\frac{y^2}{2} + \frac{3y^4}{4}\right]_0^1 = \frac{5}{8}$$

Variance of Random Variables

- Useful summary measure
- Indication of dispersion about the expected value
- Denoted by $\mathrm{Var}(X)$ or σ_X^2

Distributions with Equal Means and Unequal Dispersions



The Variance

Let X have probability distribution $f(x)$ and mean μ. The variance is

$$\sigma_X^2 = \mathrm{Var}(X) = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 f(x) \qquad \text{if } X \text{ is discrete,}$$

and

$$\sigma_X^2 = \mathrm{Var}(X) = E\left[(X-\mu)^2\right] = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx \qquad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ_X, is called the *standard deviation*.

The Variance

An alternative formula for finding variance is

$$\sigma_X^2 = E(X^2) - \mu_X^2.$$

$$= E(x^2) - E(x)^2$$

Example

Use the data below to find the variance of the number of textbooks the student will own each semester.

$$X = \text{Number of textbooks.}$$

$E(x) = 6.1$

x	4	5	6	7	8
$f(x)$	0.15	0.15	0.25	0.35	0.10

$$\sigma_x^2 = E(X^2) - E(X)^2$$

$$E(X^2) = 4^2 \times .15 + 5^2 \times .15 + 6^2 \times .25 + 7^2 \times .35 + 8^2 \times .1 = 38.7$$

$$\sigma_x^2 = 38.7 - 6.1^2 = 1.49;\ \sigma_x = 1.22$$

Example

Example

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} \frac{3}{2}\left(1 - x^2\right), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the variance of X, i.e., σ_X^2.

$\frac{1}{3} - \frac{1}{5}$

$\frac{5}{15} - \frac{3}{15} = \frac{2}{15}$

$E(x) = 3/8$

$$E(x^2) = \int_0^1 x^2 \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\int_0^1 (x^2 - x^4)\,dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

$$\sigma_x^2 = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320}$$

The Variance of a Function

Let X have probability distribution $f(x)$ and mean μ. The variance of $g(X)$ is

$$\sigma^2_{g(X)} = E\left[\left(g(X) - \mu_{g(X)}\right)^2\right] = \sum_x \left(g(x) - \mu_{g(X)}\right)^2 f(x), \quad \text{if } X \text{ is discrete,}$$

and

$$\sigma^2_{g(X)} = E\left[\left(g(X) - \mu_{g(X)}\right)^2\right] = \int_{-\infty}^{+\infty} \left(g(x) - \mu_{g(X)}\right)^2 f(x)\, dx, \quad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ, is called the standard deviation.

Example

Calculate the variance of

$$g(X) = 2X + 2.$$

$$Var(aX+b)$$
$$= a^2 Var(X)$$

x	4	5	6	7	8
$g(x)$	10	12	14	16	18
$f(x) = P(X = x)$	0.15	0.15	0.25	0.35	0.10

$$Var(g(x)) = \cancel{Var}\ 4 \times Var(x) = 4 \times 1{,}49 = 5{,}96$$

Example

Example

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} \frac{3}{2}\left(1 - x^2\right), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$\sigma_x^2 = \frac{19}{320}$

Find the variance of

$$h(X) = 2X + 3.$$

$\sigma_h^2 = 4\sigma_x^2 = 4 \times \frac{19}{320} = \frac{19}{80}$

Example

Review

$$E(X+Y) = E(X) + E(Y)$$

$$E(aX+b) = aE(X) + b$$

- Mean of a random variable
- Variance of a random variable

$$\sigma_X^2 = E(X^2) - E(X)^2$$

$$\sigma_{aX+b}^2 = a^2 \sigma_X^2$$

Learning Objectives

- Variance and Covariance of random variables
- Mean and Variance of Linear Combinations of random variables

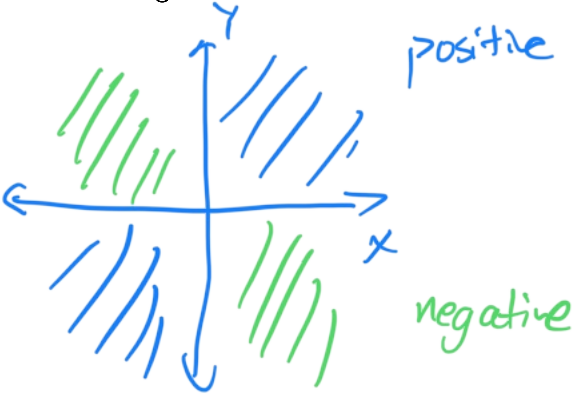
Covariance of Two Random Variables

- Useful summary measure
- Degree of association between two variables, in terms of deviation from their means
- If covariance is:
  - **positive:** when one r.v. is larger than its mean, the other tends to be larger than its mean
  - **negative:** when one r.v. is larger than its mean, the other tends to be smaller than its mean

Covariance units : (units of X) × (units of Y)

Covariance of Two Random Variables

- We like to know how strong two variables are related to one another if they are not independent.



Covariance of Two Random Variables

Let X and Y be random variables with joint probability distribution $f(x, y)$. The covariance of X and Y is

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)].$$

- If X and Y are discrete:

$$\sigma_{XY} = \sum_x \sum_y (x - \mu_X)(y - \mu_Y)\, f(x, y).$$

- If X and Y are continuous:

$$\sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)\, f(x, y)\, dx\, dy.$$

Short-cut Formula for Covariance

The covariance of two random variables X and Y with means μ_X and μ_Y is:

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = E(XY) - \mu_X \mu_Y.$$

If X and Y are independent

$E(XY) = E(X)E(Y)$, so $\sigma_{XY} = 0$

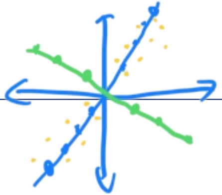
Correlation Coefficient

Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y. The correlation coefficient is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

$$-1 \leq \rho_{XY} \leq 1$$

Correlation Coefficient



- A useful summary measure of association
- What does it mean when $\rho_{XY} = 1$ or -1?
- So, the correlation coefficient measures the degree of **linearity** between X and Y
- Qn: Why divide σ_{XY} by $\sigma_X \sigma_Y$?

$\rho_{XY} = 1$, $Y = aX + b$ with $a > 0$

$\rho_{XY} = -1$, $Y = aX + b$ with $a < 0$

$$-1 \leq \rho_{XY} \leq 1$$

Correlation Propositions

1. If a and c are either both positive or both negative,

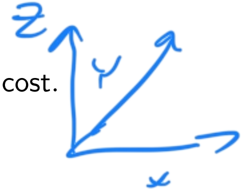
$$\rho(aX + b,\ cY + d) = \rho(X, Y).$$

2. For any two r.v.s X and Y, $-1 \leq \rho_{XY} \leq 1$.
3. If X and Y are independent, then $\rho = 0$, but $\rho = 0$ does not imply independence.

Example

A restaurant serves three fixed-priced dinners costing \$7, \$9, and \$10. For a randomly selected couple, let

$$X = \text{man's dinner cost}, \qquad Y = \text{woman's dinner cost}.$$



Suppose the joint pmf $f(x, y)$ is:

	$f(x,y)$	X			$h(y)$
		7	9	10	
Y	7	0.05	0.05	0.10	.2
	9	0.05	0.10	0.35	.5
	10	0	0.20	0.10	.3
	$g(x)$	.1	.35	.55	

$P(Y=7)=.2$
$P(Y=9)=.5$
$P(Y=10)=.3$

Example (Continued)

1. Compute the marginal probability distributions of X and Y.
2. What is $P(\underline{X \leq 9,\ Y \leq 9})$? $= .25$
3. Are X and Y independent? Explain.
4. Find σ_{XY} and the correlation coefficient ρ.

$f(x,y) \overset{?}{\neq} g(x)h(y)$ not independent

$P(X=7) = .1$, $P(Y=9) = .35$,

$P(X=10) = .55$

$\sigma_{XY} = E[XY] - E[X]E[Y]$

$\sigma_X = \sqrt{\sigma_X^2}$; $\sigma_Y = \sqrt{\sigma_Y^2}$; $\sigma_X^2 = E[X^2] - (E[X])^2$

$\sigma_Y^2 = E[Y^2] - (E[Y])^2$

Example (Solutions)

$\sigma_{XY} = .13$

$\rho = .14$

Rules of the Expected Value

If a and b are constants, then

$$E(aX + b) = aE(X) + b.$$

- Setting $b = 0$: $E(aX) = aE(X)$
- Setting $a = 0$: $E(b) = b$

Example

Calculate $E(g(X))$, where $g(X) = 2X + 2$.

x	4	5	6	7	8
$g(x)$	10	12	14	16	18
$f(x) = P(X = x)$	0.15	0.15	0.25	0.35	0.10

$E(x) = 6.1$

$E[g(x)] = 2 \times 6.1 + 2 = 14.2$

Independence

Let X and Y be two independent r.v.s. Then

$$E(XY) = E(X)\,E(Y).$$

Also, for independent X and Y,

$$\sigma_{XY} = 0.$$

If $\sigma_{XY} = 0$, X and Y are not necessarily independent

Rules of Variance

If X and Y are random variables (with covariance σ_{XY}) and a, b, c are constants, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\,\sigma_{XY}.$$

- Setting $b = 0$: $\sigma^2_{aX+c} = a^2\sigma^2_X$
- Setting $a = 1$, $b = 0$: $\sigma^2_{X+c} = \sigma^2_X$
- Setting $b = 0$, $c = 0$: $\sigma^2_{aX} = a^2\sigma^2_X$

Rules of Variance and Independence

If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y, \qquad \sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

If $X_1, \ldots, X_n$ are independent random variables, then

$$\sigma^2_{a_1X_1+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + \cdots + a_n^2\sigma^2_{X_n}.$$