Chapter 6: Some Continuous Probability Distributions

ENGR 0021

Learning Objective

- **Uniform Distribution**
- **Exponential Distribution**
- Normal Distribution
  - Areas Under the Normal Curve
  - Applications of the Normal Distribution

Continuous Probability Distribution

- A random variable X is *continuous* if:
  - $P(X = c) = 0$ for any number c that is a possible value of X.
  - Possible values of X are either an interval on the real line or a union of disjoint intervals.
- In other words: any time a r.v. X has an *uncountable* (infinite) number of possibilities, it is continuous.
- Example: distance, length, time, etc.

Continuous Probability Distribution

- Discrete r.v.: split up the total probability of 1 over all possible values of X.
- Continuous r.v.: split up the total probability of 1 over all possible intervals of X.

Probability Distribution

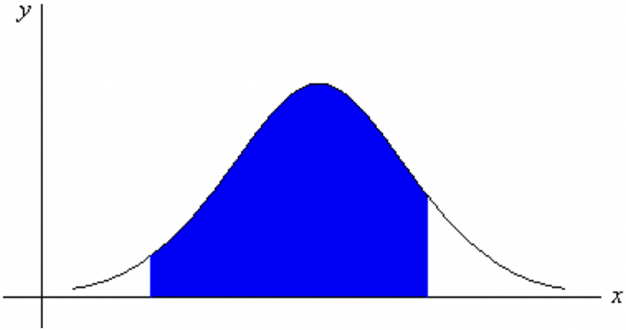
- Let X be a continuous r.v. A *probability density function* (pdf) of X is a function $f(x)$ such that for any two numbers a and b ($a < b$),

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

- The graph of f is the *density curve*.

Probability Density Function

- $P(a \leq X \leq b)$ is given by the area of the shaded region.



Probability Density Function

For $f(x)$ to be a pdf:

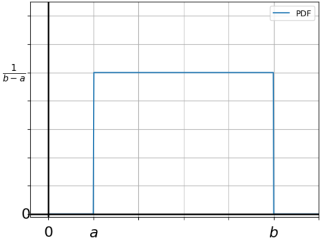
1. $f(x) \geq 0$ for all values of x.
2. The total area under f equals 1:

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1.$$

Uniform Distribution

- One of the simplest continuous distributions.
- The density function is "flat".
- Applications are based on the assumption that the probability of falling in an interval of fixed length is constant.

Uniform Distribution



A continuous random variable X has a uniform distribution on $[a, b]$, denoted

$$X \sim \mathrm{Unif}(a, b),$$

if the pdf of X is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

The Mean and Variance

If $X \sim \mathrm{Unif}(a, b)$:

- Mean:

$$E[X] = \frac{a+b}{2}.$$

- Variance:

$$\mathrm{Var}(X) = \frac{1}{12}(b-a)^2.$$

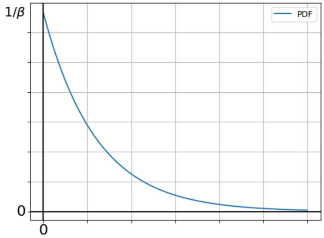
Example (Uniform)

A read/write head for a hard disk is supposed to locate a record, as the disk rotates once every 25 ms. Let $X =$ time it takes to locate the record. Assume X is uniformly distributed on $[0, 25]$.

- What is the probability density function?
- $P(10 \leq X \leq 20)$?
- What is the probability that it takes at least 10 ms to locate the record?

Example (Solution Sketch)

Exponential Distribution

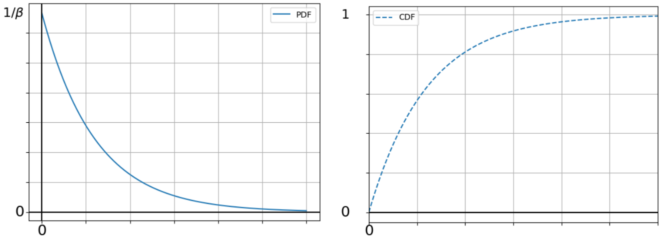


A continuous random variable X has an exponential distribution with parameter β if the pdf is

$$f(x;\beta) = \frac{1}{\beta} e^{-x/\beta} \quad (x \geq 0).$$

- Mean: $E[X] = \beta$
- Variance: $\mathrm{Var}(X) = \beta^2$

Exponential Distribution



$$f(x;\beta) = \frac{1}{\beta} e^{-x/\beta} \quad (x \geq 0).$$

$$F(x;\beta) = 1 - e^{-x/\beta} \quad (x \geq 0).$$

Example (Exponential)

Stress ranges in certain bridge connections can be modeled using an exponential distribution with mean value 6 MPa. What is the probability that the stress range is at most 10 MPa?

Exponential Distribution (Another Form)

A continuous random variable X has an exponential distribution with parameter λ if

$$f(x;\lambda) = \lambda e^{-\lambda x} \quad (x \geq 0).$$

- Mean: $E[X] = \dfrac{1}{\lambda}$
- Variance: $\mathrm{Var}(X) = \dfrac{1}{\lambda^2}$
- λ is called the *rate*.

Exponential Distribution vs. Poisson Distribution

- Exponential distribution has a "dual" relationship with the Poisson distribution.
- In the same counting process:
  - Poisson describes the *number* of appearances in a given time interval.
  - Exponential describes the *time between* appearances (inter-arrival time).
- Example: in a simple queueing system, arrival rate is λ.
  - Number of customers in a time window of length t: $\mathrm{Poisson}(\lambda t)$.
  - Time between consecutive arrivals: $\mathrm{Exp}(\lambda)$.
- Foundation of the simplest queueing systems.

Example (Waiting Time)

Suppose that on average, there are 2 hits per minute on a specific web page. Since there are 2 hits per minute, the average waiting time until a hit occurs is 0.5 minutes.

- What is the probability that we have to wait at most 40 seconds until we observe a hit?

Memoryless Property of the Exponential Distribution

$$P(X \geq t_0 + t \mid X \geq t_0) = P(X \geq t).$$

- Interpretation: the distribution of the remaining “waiting time” does not depend on how much time has already elapsed.
- Example: If you are waiting for a friend to go to a Steelers game with you, and you have waited for 1 hour, the waiting time until your friend shows up still follows the same exponential distribution as before.
- Memoryless property provides analytical simplicity for stochastic systems.
- Foundation of the Markov property for the simplest stochastic model, Markov chain model.

Learning Objective

- Uniform Distribution
- Exponential Distribution
- **Normal Distribution**
  - **Areas Under the Normal Curve**
  - **Applications of the Normal Distribution**

Normal Distribution

- The most important distribution, because many numerical populations have distributions that very closely fit the normal curve.
- Examples:
  - Heights
  - Weights
  - Test scores

Normal Distribution

- A continuous random variable X has a normal distribution with parameters μ and σ ($-\infty < \mu < \infty$, $\sigma > 0$), denoted $X \sim N(\mu, \sigma^2)$, if its pdf is

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad -\infty < x < \infty.$$

- Mean and variance:

$$E[X] = \mu, \qquad \mathrm{Var}(X) = \sigma^2.$$

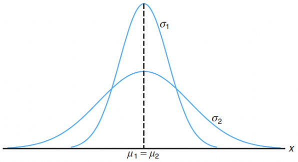
Normal Distribution

- What does the pdf graph look like? (Guess based on the function...)

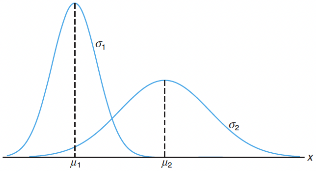
Normal Distribution

- Symmetrical and bell shaped
- Always symmetric about the mean, μ
- Different values of σ affect the shape

Normal Distribution



Normal curves with $\mu_1 = \mu_2$ and $\sigma_2 > \sigma_1$



Normal curves with $\mu_2 > \mu_1$ and $\sigma_2 > \sigma_1$

Areas under the Normal Curve

- How do you calculate the area under the normal pdf curve?
- In other words, how do you calculate the probability of normally distributed random variables?

$$P(a \leq X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

- Integration? Super hard!
- We want an easier way (normal tables).
- But we cannot have a separate table for every (μ, σ).
- Introduce a **standard normal distribution** with $\mu = 0$ and $\sigma = 1$.

Standardization and Normal Probabilities

Random variable X with mean μ and standard deviation σ.

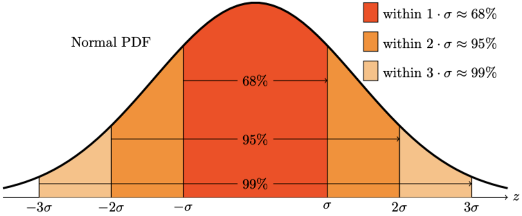
$$\text{Standardization:} \qquad Z = \frac{X - \mu}{\sigma}.$$

- If $X \sim N(\mu, \sigma^2)$, then
$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$
- Z has mean 0 and standard deviation 1.
- Standardizing any normal random variable produces the **standard normal**.
- This allows us to compute probabilities for X using tables/software for Z.
- For an observed value x, convert to
$$z = \frac{x - \mu}{\sigma}.$$

Standard Normal Distribution

- Standard normal ($\mu = 0, \sigma^2 = 1$) is rarely a direct model for naturally arising populations.
- It is a reference distribution used to help calculate probabilities for any normal distribution.
- Standard normal random variable: $Z \sim N(0, 1)$.
- pdf: $\varphi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$.
- CDF: $\Phi(z) = P(Z \leq z)$.
- Standard normal distribution is tabulated (Appendix Table A.3).

Concept Question: Standard Normal



1. $P(-1 < Z < 1)$ is:
   (a) .025 (b) .16 (c) .68 (d) .84 (e) .95
2. $P(Z > 2)$ is:
   (a) .025 (b) .16 (c) .68 (d) .84 (e) .95

Standard Normal Distribution- Z table

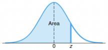


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Standard Normal Distribution- Z table

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example (Using the Z Table)

Use the standard normal table to determine:

- $P(Z \leq 1.26)$
- $P(Z > 2.44)$
- $P(0.08 < Z < 1.08)$

Example

If X is normal with $\mu = 5$ and $\sigma^2 = 4$, determine $P(1 < X < 8)$.

Areas under the Normal Curve

- In the previous example:
    - Given: value x
    - Step 1: transform to z
    - Step 2: use standard normal table to calculate probability / area under curve
- Now:
    - Given: probability / area under curve
    - Step 1: use standard normal table to find z value
    - Step 2: transform to x

Areas under the Normal Curve

- **Quantile***
- CDF: $F(x) = P(X \leq x)$
- Assume $F(q) = P(X \leq q) = p$, where p is known.
- Then q is the $p \times 100$ percent quantile (percentile).
- Example: for standard normal, $q = 0$ is the 50th percentile (median).

Areas under the Normal Curve

- Given: probability / area under curve
- Step 1: use standard normal table to find z value
- Step 2: transform to x
- Transformation:

$$X = \sigma Z + \mu$$

Example

Given a normal distribution with $\mu = 20$ and $\sigma^2 = 81$, find x that has:

- 19% of the area to the left
- 32% of the area to the right

Applications of the Normal Distribution

Suppose measurements of current in a strip of wire follow a normal distribution with mean 10 milliamperes and variance 4 milliamperes. What is the probability that a measurement will exceed 13 milliamperes?

Applications of the Normal Distribution

The average time to assemble a car in a certain plant is normal with mean 20 hours and standard deviation 2 hours. What is the probability that a car can be assembled in:

- less than 19.5 hours?
- between 20 and 22 hours?

Why the Normal Shows Up Everywhere: Law of Large Numbers

Informally:
An average of many measurements is more accurate than a single measurement.

Formally:
Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean μ and standard deviation σ.
Define the sample mean:

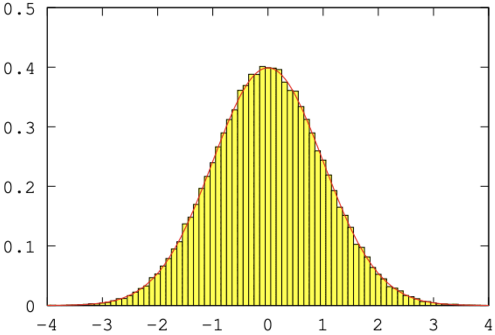
$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Then for any $a > 0$,

$$\lim_{n \to \infty} P\big(\, |\bar{X}_n - \mu| < a \,\big) = 1.$$

LoLN and histograms

Law of Large Numbers (LoLN) implies the **density histogram converges to the pdf**.



Histogram with bin width .1 showing 100000 draws from a standard normal distribution.
Standard normal pdf is overlaid in red.

Central Limit Theorem

Setting: $X_1, X_2, \ldots$ i.i.d. with mean μ and standard deviation σ.

Define

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S_n = \sum_{i=1}^{n} X_i.$$

Conclusion (for large n):

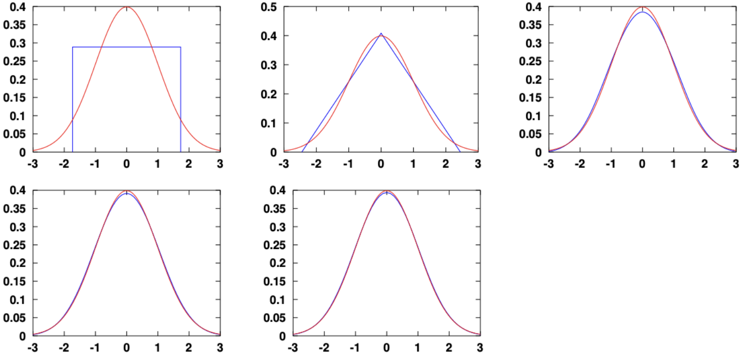
$$\overline{X}_n \approx \mathcal{N}\left(\mu,\ \frac{\sigma^2}{n}\right), \qquad S_n \approx \mathcal{N}\left(n\mu,\ n\sigma^2\right),$$

and the standardized form satisfies

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \ \approx\ \mathcal{N}(0,1).$$

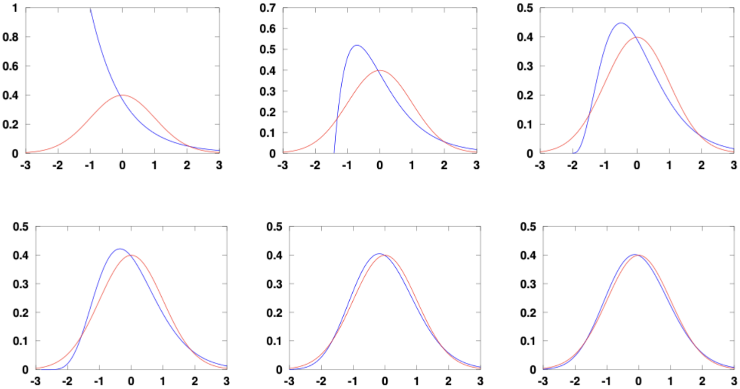
CLT: pictures

Standardized average of n independent and identically distributed uniform random variables, with $n = 1, 2, 4, 12$.



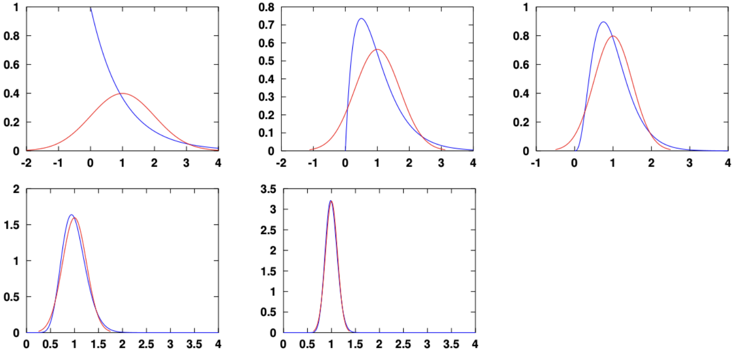
CLT: pictures 2

Show the standardize average of n independent and identically distributed exponential random variables, with $n = 1, 2, 8, 64$.



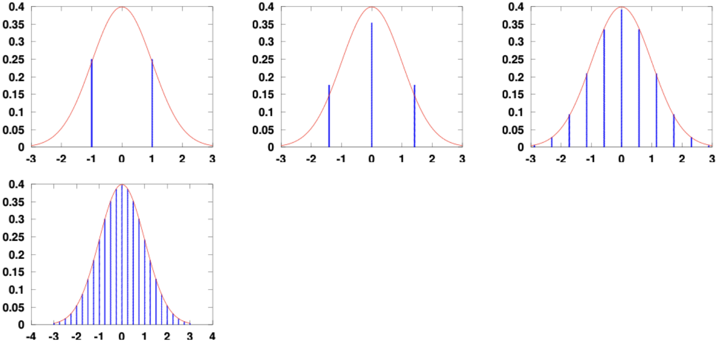
CLT: pictures 3

Show the non-standardized average of n independent and identically distributed exponential random variables, with $n = 1, 2, 8, 64$.



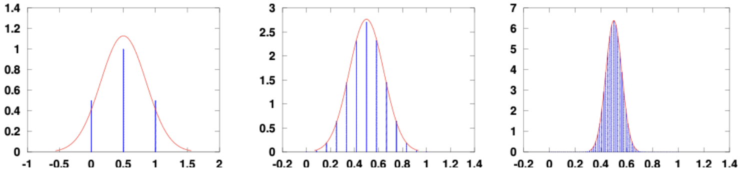
CLT: pictures 4

The standardized average of n i.i.d. Bernoulli(0.5) random variables, with $n = 1, 2, 12, 64$.



CLT: pictures 4

The (non-standardized) average of n Bernoulli(0.5) random variables, with $n = 4, 12, 64$.



Summary: Three Continuous Distributions

	Uniform	Exponential	Normal
Notation	$X \sim \mathrm{Unif}(a, b)$	$X \sim \mathrm{Exp}(\beta)$ or $\mathrm{Exp}(\lambda)$	$X \sim N(\mu, \sigma^2)$
Support	$a \leq x \leq b$	$x \geq 0$	$-\infty < x < \infty$
PDF	$f(x) = \dfrac{1}{b-a}$	$f(x) = \dfrac{1}{\beta} e^{-x/\beta}$ (equiv. $\lambda e^{-\lambda x}$)	$f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$
CDF	$F(x) = \dfrac{x-a}{b-a}$ (on $[a, b]$)	$F(x) = 1 - e^{-x/\beta}$	$F(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$
Mean	$E[X] = \dfrac{a+b}{2}$	$E[X] = \beta = \dfrac{1}{\lambda}$	$E[X] = \mu$
Variance	$\mathrm{Var}(X) = \dfrac{(b-a)^2}{12}$	$\mathrm{Var}(X) = \beta^2 = \dfrac{1}{\lambda^2}$	$\mathrm{Var}(X) = \sigma^2$