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Bayes' Theorem Review (Chapter 2)

Let $A_1, A_2, ..., A_n$ be a collection of n mutually exclusive and exhaustive events where $P(A_i) > 0$ for all i. For any other event B for which $P(B) > 0$, Bayes' Theorem is as follows:

$$P(A_k|B) = \frac{P(A_k \cap B)}{P(B)} = \frac{P(B|A_k) \cdot P(A_k)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)} \quad \text{where } k = 1, \ldots, n$$

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How do we get Bayes' Theorem?

$\rightarrow$ Bayes' Theorem is just the rewritten conditional probability formula in the numerator and the Total Law of Probability in the denominator!

Chapter 2, Slide 83 (Example)

If an aircraft is present (A), radar detects it with probability 0.99; otherwise, radar raises a false alarm with probability 0.10. An aircraft is present 5% of the time. What is the probability that an aircraft is present given the radar signals?

Prosecutor's Fallacy: "A radar system detects an aircraft with 99% accuracy when an aircraft is present. Therefore, if the radar signals an aircraft, there is a 99% chance that an aircraft is actually present." Is this true? Let's see.

Chapter 2, Slide 83 (Solution)

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What exactly are we trying to find?

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What exactly are we trying to find?

- We want to find the probability that an aircraft is present given the radar signals $\rightarrow P(A|S)$

Chapter 2, Slide 83 (Solution Continued)

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How do we write Bayes' Rule for the probabilities we have?

$$P(A|S) = \frac{P(S|A) \cdot P(A)}{P(S|A) \cdot P(A) + P(S|A') \cdot P(A')}$$

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$$P(A|S) = \frac{P(S|A) \cdot P(A)}{P(S|A) \cdot P(A) + P(S|A') \cdot P(A')}$$

Now, we can just plug in what we have:

$$P(A|S) = \frac{P(S|A) \cdot P(A)}{P(S|A) \cdot P(A) + P(S|A') \cdot P(A')}$$

$$= \frac{0.99 \cdot 0.05}{0.99 \cdot 0.05 + 0.10 \cdot 0.95} = 0.3426$$

Chapter 2, Slide 83 (Solution Continued)

Prosecutor's Fallacy: "A radar system detects an aircraft with 99% accuracy when an aircraft is present. Therefore, if the radar signals an aircraft, there is a 99% chance that an aircraft is actually present."

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Prosecutor's Fallacy: "A radar system detects an aircraft with 99% accuracy when an aircraft is present. Therefore, if the radar signals an aircraft, there is a 99% chance that an aircraft is actually present."

This is not true. We just determined that the probability that an aircraft is present given the radar signals is actually 34.26% in this case, even when the radar system detects an aircraft with 99% accuracy when an aircraft is present.

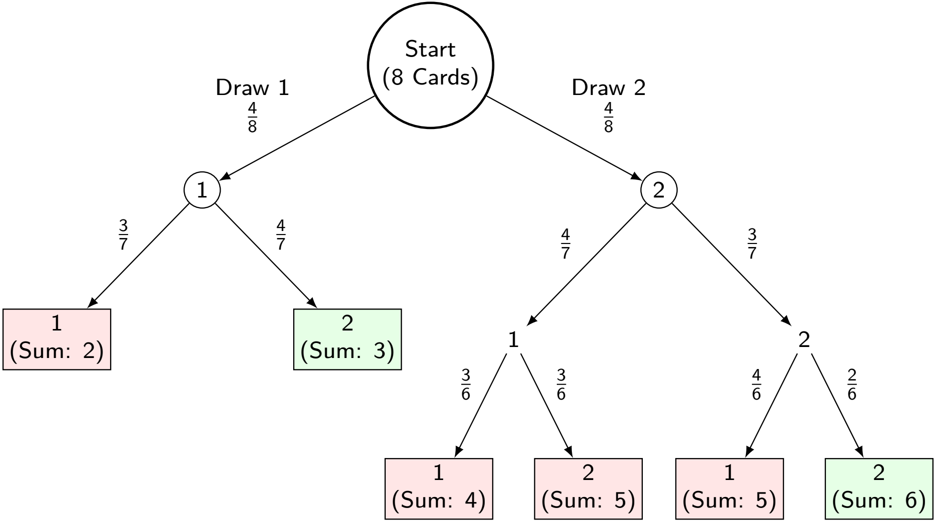
Assignment 3, Question 2

There are 8 cards in a hat: $1\heartsuit$, $1\spadesuit$, $1\diamondsuit$, $1\clubsuit$, $2\heartsuit$, $2\spadesuit$, $2\diamondsuit$, $2\clubsuit$. You draw one card at random. If its rank (number) is 1, you draw one more card; if its rank (number) is two, you draw two more cards. The cards are drawn without replacement. *Hint: Use a tree diagram.*

(a) What is the probability that the sum of all the cards that you draw is 3?

(b) What is the probability that the sum of all the cards that you draw is 6?

Assignment 3, Question 2: Tree Diagram



Assignment 3, Question 2 (Solution)

Part A:

Let C_i be the rank of the i^{th} card. Let S be the sum of all the cards we have drawn.

Let's use the tree diagram we constructed to answer the first part. The question was: What is the probability that the sum of all the cards that you draw is 3?

There is only one outcome where we get a total sum of 3:

Assignment 3, Question 2 (Solution)

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Part A:

Let C_i be the rank of the i^{th} card. Let S be the sum of all the cards we have drawn.

Let's use the tree diagram we constructed to answer the first part. The question was: What is the probability that the sum of all the cards that you draw is 3?

There is only one outcome where we get a total sum of 3: $C_1 = 1, C_2 = 2$.

Therefore, we can calculate the probability of this occurring as follows:

$$\begin{aligned} P(S = 3) &= P((C_1 = 1) \cap (C_2 = 2)) \\ &= P(C_1 = 1) \cdot P(C_2 = 2 | C_1 = 1) \\ &= \frac{4}{8} \cdot \frac{4}{7} = \frac{2}{7} \end{aligned}$$

Assignment 3, Question 2 (Solution Continued)

Part B:

Now, let's use the tree diagram we constructed again to answer the second part. The question was: What is the probability that the sum of all the cards that you draw is 6?

There is only one outcome where we get a total sum of 6:

Assignment 3, Question 2 (Solution Continued)

Part B:

Now, let's use the tree diagram we constructed again to answer the second part. The question was: What is the probability that the sum of all the cards that you draw is 6?

There is only one outcome where we get a total sum of 6: $C_1 = 2, C_2 = 2, C_3 = 2$.

Assignment 3, Question 2 (Solution Continued)

Part B:

Now, let's use the tree diagram we constructed again to answer the second part. The question was: What is the probability that the sum of all the cards that you draw is 6?

There is only one outcome where we get a total sum of 6: $C_1 = 2, C_2 = 2, C_3 = 2$.

Therefore, we can calculate the probability of this occurring as follows:

$$
\begin{aligned}
P(S = 6) &= P((C_1 = 2) \cap (C_2 = 2) \cap (C_3 = 2)) \\
&= P(C_1 = 1) \cdot P(C_2 = 2 | C_1 = 1) \cdot P(C_3 = 2 | (C_1 = 2) \cap (C_2 = 2)) \\
&= \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{14}
\end{aligned}
$$

Assignment 3, Question 4

Police plan to enforce speed limits by using radar traps at four different locations within the city limits (L_1, L_2, L_3, L_4).

- The radar traps will be operated 40%, 30%, 20%, and 30% of the time respectively.
- A person speeding to work has probabilities of 0.2, 0.1, 0.5, and 0.2 of passing through these locations.

What is the probability that she will receive a speeding ticket?

Assignment 3, Question 4 (Data Summary)

Let L_i be the location of radar trap i. Let T be the event that the person receives a speeding ticket. We can organize all of the data we're given as follows:

Location (L_i)	**Prob. of Passing** $P(L_i)$	**Prob. of Trap** $P(T\|L_i)$
L_1	0.2	0.4
L_2	0.1	0.3
L_3	0.5	0.2
L_4	0.2	0.3

Assignment 3, Question 4 (Solution)

Since the locations (L_1, L_2, L_3, L_4) are mutually exclusive (she can only pass through one at a time) and exhaustive (we assume she must pass through one), we can use the Law of Total Probability to find the probability that she receives a ticket:

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$$
\begin{aligned}
P(T) &= \sum_{i=1}^{4} P(T|L_i) \cdot P(L_i) \\
&= 0.4 \cdot 0.2 + 0.3 \cdot 0.1 + 0.2 \cdot 0.5 + 0.3 \cdot 0.2 \\
&= 0.08 + 0.03 + 0.10 + 0.06 \\
&= 0.27
\end{aligned}
$$

Question 1

Urn A contains 3 red balls and 2 green balls. Urn B contains 2 red balls and 4 green balls. One ball is drawn at random from Urn A and transferred to Urn B (without looking at it). Then, a ball is drawn from Urn B. What is the probability that the ball drawn from Urn B is red?

Question 1

Urn A contains 3 red balls and 2 green balls. Urn B contains 2 red balls and 4 green balls. One ball is drawn at random from Urn A and transferred to Urn B (without looking at it). Then, a ball is drawn from Urn B. What is the probability that the ball drawn from Urn B is red?

Answer:

Let R_1 be the event the transferred ball is red, and G_1 be green. Let R_2 be the event the second ball (drawn from Urn B) is red. To find $P(R_2)$, use the Law of Total Probability:

$$P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|G_1)P(G_1)$$

- $P(R_1) = 3/5$, $P(G_1) = 2/5$
- If R_1 occurs, Urn B now has 3 red, 4 green (7 total). So $P(R_2|R_1) = 3/7$.
- If G_1 occurs, Urn B now has 2 red, 5 green (7 total). So $P(R_2|G_1) = 2/7$.

$$P(R_2) = \frac{3}{7} \cdot \frac{3}{5} + \frac{2}{7} \cdot \frac{2}{5} = \frac{9}{35} + \frac{4}{35} = \frac{13}{35}$$

Question 2

A cloud computing service has three server clusters: A, B, and C. Cluster A handles 50% of the traffic, Cluster B handles 30%, and Cluster C handles 20%. The probability of a server crash is different for each: 1% for A, 2% for B, and 5% for C. A system admin receives a report that a crash has occurred. What is the probability that Cluster C is responsible for the crash?

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Answer:

Let F be the event of a failure (crash). We want to find $P(C|F)$. We know $P(A) = 0.5$, $P(B) = 0.3$, $P(C) = 0.2$, $P(F|A) = 0.01$, $P(F|B) = 0.02$, and $P(F|C) = 0.05$. Applying Bayes' Rule, we get:

$$\begin{aligned} P(C|F) &= \frac{P(F|C)P(C)}{P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C)} \\ &= \frac{0.05 \cdot 0.2}{0.01 \cdot 0.5 + 0.02 \cdot 0.3 + 0.05 \cdot 0.2} = \frac{0.010}{0.021} = \frac{10}{21} \approx 47.6\% \end{aligned}$$