Model predictive control for bipedal locomotion

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Abstract

Bonsoir à toutes et à tous, nous aimons beaucoup les maths!!!!

1 INTEGRATION RELATIONSHIP

We consider the horizontal motion of the center of mass along one axis, described by its position $x(t)$, velocity $\dot{x}(t)$, acceleration $\ddot{x}(t)$ and jerk $\dddot{x}(t)$ (the time derivative of the acceleration).

Let the sampling period be $T > 0$, and define the discrete-time instants

$$t_k = kT, \quad k \in \mathbb{Z}.$$

We introduce the discrete-time state vector

$$\hat{x}_k = \begin{bmatrix} x_k \\ \dot{x}_k \\ \ddot{x}_k \end{bmatrix} = \begin{bmatrix} x(t_k) \\ \dot{x}(t_k) \\ \ddot{x}(t_k) \end{bmatrix},$$

and the control input (jerk)

$$\dddot{x}_k = \dddot{x}(t_k).$$

We assume that the jerk is *piecewise constant* over each sampling interval:

$$\dddot{x}(t) = \dddot{x}_k, \qquad \forall t \in [t_k, t_{k+1}].$$

Our goal is to derive the discrete-time state update

$$\hat{x}_{k+1} = A\hat{x}_k + B \dddot{x}_k$$

for suitable matrices A and B.

Step 1: Continuous-time relations

By definition,

$$\dot{x}(t) = \frac{d}{dt}x(t), \qquad \ddot{x}(t) = \frac{d}{dt}\dot{x}(t), \qquad \dddot{x}(t) = \frac{d}{dt}\ddot{x}(t).$$

On the interval $[t_k, t_{k+1}]$ we have
$$\dddot{x}(t) = \dddot{x}_k \quad \text{(constant)}.$$

Integrating once from t_k to t gives the acceleration:
$$\ddot{x}(t) = \ddot{x}(t_k) + \int_{t_k}^{t} \dddot{x}(\tau)\, d\tau = \ddot{x}_k + \int_{t_k}^{t} \dddot{x}_k\, d\tau = \ddot{x}_k + (t - t_k)\ \dddot{x}_k.$$

Integrating again gives the velocity:
$$\begin{aligned}
\dot{x}(t) &= \dot{x}(t_k) + \int_{t_k}^{t} \ddot{x}(\sigma)\, d\sigma \\
&= \dot{x}_k + \int_{t_k}^{t} \Big[\ddot{x}_k + (\sigma - t_k)\, \dddot{x}_k\Big] d\sigma \\
&= \dot{x}_k + \ddot{x}_k(t - t_k) + \ \dddot{x}_k \int_{t_k}^{t} (\sigma - t_k)\, d\sigma \\
&= \dot{x}_k + \ddot{x}_k(t - t_k) + \ \dddot{x}_k \cdot \frac{(t - t_k)^2}{2}.
\end{aligned}$$

Integrating a third time gives the position:
$$\begin{aligned}
x(t) &= x(t_k) + \int_{t_k}^{t} \dot{x}(\rho)\, d\rho \\
&= x_k + \int_{t_k}^{t} \Big[\dot{x}_k + \ddot{x}_k(\rho - t_k) + \ \dddot{x}_k \frac{(\rho - t_k)^2}{2}\Big] d\rho \\
&= x_k + \dot{x}_k(t - t_k) + \ddot{x}_k \frac{(t - t_k)^2}{2} + \ \dddot{x}_k \frac{(t - t_k)^3}{6}.
\end{aligned}$$

Step 2: Evaluation at the sampling instant t_{k+1}

We now evaluate these expressions at $t = t_{k+1} = t_k + T$, so that $(t - t_k) = T$.

Acceleration.
$$\ddot{x}_{k+1} = \ddot{x}(t_{k+1}) = \ddot{x}_k + T\ \dddot{x}_k.$$

Velocity.
$$\dot{x}_{k+1} = \dot{x}(t_{k+1}) = \dot{x}_k + T\, \ddot{x}_k + \frac{T^2}{2}\ \dddot{x}_k.$$

Position.
$$x_{k+1} = x(t_{k+1}) = x_k + T\, \dot{x}_k + \frac{T^2}{2}\, \ddot{x}_k + \frac{T^3}{6}\ \dddot{x}_k.$$

We can now assemble these three scalar equations into vector form.

Step 3: Matrix form of the state update

Recall the state vector
$$\hat{x}_k = \begin{bmatrix} x_k \\ \dot{x}_k \\ \ddot{x}_k \end{bmatrix}, \qquad \hat{x}_{k+1} = \begin{bmatrix} x_{k+1} \\ \dot{x}_{k+1} \\ \ddot{x}_{k+1} \end{bmatrix}.$$

The three update equations can be written compactly as

$$\hat{x}_{k+1} = \underbrace{\begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}}_{A} \hat{x}_k + \underbrace{\begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix}}_{B} \dddot{x}_k.$$

Thus, under the assumption of constant jerk on each sampling interval $[t_k, t_{k+1}]$, the discrete-time dynamics are

$$\boxed{\hat{x}_{k+1} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \hat{x}_k + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} \dddot{x}_k}.$$

2 COP & COM RELATIONSHIP

The official relationship between z and x is

$$z = x - \frac{h_{CoM}}{g}\ddot{x}$$

However, in the next relationship, we have

$$z_k = [1 \quad 0 \quad \frac{h_{CoM}}{g}]\hat{x}_k$$

Isn't there a mistake ?????

Let's denote $C = [1 \quad 0 \quad -\frac{h_{CoM}}{g}]^T$.

3 QUADRATIC PROGRAM

We want to choose the jerk (until infinity) that minimizes a certain quantity, which is defined as

$$\sum_{i=k}^{\infty} \frac{1}{2}Q(z_{i+1} - z_{i+1}^{ref})^2 + \frac{1}{2}R\dddot{x}_i^2$$

But wa say that it's enough to solve this program over N steps

$$\sum_{i=k}^{k+N-1} \frac{1}{2}Q(z_{i+1} - z_{i+1}^{ref})^2 + \frac{1}{2}R\dddot{x}_i^2$$

If we want to check the close form for the jerk $\dddot{X}_k = [\dddot{x}_k, ..., \dddot{x}_{k+N-1}]^T$, we have to play with the different equations. Let's rewrite the quantity to minimize, with a more compact form

$$\frac{1}{2}Q\|Z_{k+1} - Z_{k+1}^{ref}\|^2 + \frac{1}{2}R\|\dddot{X}_k\|^2$$

They say in the paper that can write

$$Z_{k+1} = P_x \hat{x}_k + P_u \dddot{X}_k$$

Then, it's basically just the solution of a quadratic problproblem, and can be expressed as

$$\dddot{X}_k = -\left(P_u^\top P_u + \frac{R}{Q} I\right)^{-1} P_u^\top \left(P_x \hat{x}_k - Z_k^{\text{ref}}\right).$$

And we take the first component of that solution, because we want to find the best next jerk. The horizon N is just here to compute it more accurately via planning. Thus, we have the equation

$$\hat{x}_{k+1} = (A - Be^T \left(P_u^\top P_u + \frac{R}{Q} I\right)^{-1} P_u^\top P_x)\hat{x}_k + K Z_k^{ref}$$

Checking the stability of the system would just mean to check that the eigenvalues of the big matrix are always strictly less than 1 in absolute value.

4 Conclusions

Pov c'est la conclusion

References