Analysis Section
of Spring-mass-damper projects.

General ODE:

$$\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$

Case 1: Overdamped

- $F(t) = 0$
- $\zeta = 1.1$
- $\omega_n = 1.0 \frac{rad}{s}$

$x(0) = 0 m$
$\dot{x}(0) = 1.0 m/s$

$$\dot{x} + 2.1\dot{x} + x = 0$$

$x = e^{rt}$
$\dot{x} = re^{rt}$
$\ddot{x} = r^2 e^{rt}$

$$(r^2 + 2.1 + 1)e^{rt} = 0$$

$$r = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$r_1 = -.647$
$r_2 = -1.558$

$$x(t) = C_1 e^{-.6417t} + C_2 e^{-1.5588t}$$

$x(0) = 0\ m$

$\dot{x}(0) = 1\ m/s$

$0 = C_1 + C_2$

$$\dot{x}(t) = -.6417 e^{-.6417t} C_1 - 1.558 C_2 e^{-1.558t}$$

$C_1 + C_2 = 0$

$-.6417 C_1 - 1.558 C_2 = 1$

$C_1 = 1.09$

$C_2 = -1.09$

$$\boxed{\begin{aligned} x(t) &= 1.09 e^{-.6417t} - 1.09 e^{-1.558t} \\ \dot{x}(t) &= -.699 e^{-.6417t} + 1.7 e^{-1.558t} \end{aligned}}$$

Case 2 : Under damped case

- $\zeta = 0.1$, $\omega_n = 5$ rad/s, $F(t) = 0$
- $X(0) = 0$ m $\quad V(0) = 1$ m/s

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{M}$$

$$\ddot{x} + \dot{x} + 25x = 0$$

$$x = e^{rt}$$
$$\dot{x} = re^{rt}$$
$$\ddot{x} = r^2e^{rt}$$

$$(r^2 + r + 25)\,e^{rt} = 0$$

$$r = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$r_1 = -0.5 \pm 4.975\,i$$

$$x(t) = e^{(-0.5+4.975i)t}$$

$$x(t) = e^{-t/2}\left(\cos(4.975t) + i\sin(4.975t) \right)$$

$$x(t) = e^{-t/2}\left(c_1\cos(4.975t) + c_2\sin(4.975t) \right)$$

$$c_1 = 0$$

$$x(t) = c_1 e^{-t/2}\cos(4.975t) + c_2 e^{-t/2}\sin(4.975t)$$

$$\dot{x}(t) = c_1\left[-\frac{1}{2}e^{-t/2}\cos(4.975t) - 4.975\,e^{-t/2}\sin(4.975t)\right]$$

$$+\ c_2\left[-\frac{1}{2}e^{-t/2}\sin(4.975t) + 4.975\,e^{-t/2}\cos(4.975t)\right]$$

$$4.975\,c_2 = 1 \ ; \quad c_2 = .201$$

$$\boxed{x(t) = .201\,e^{-t/2}\sin(4.975t)}$$

$$\dot{x}(t) = .201\left(-\frac{1}{2}e^{-t/2}\sin(4.975t) + 4.975\,e^{-t/2}\cos(4.975t\right.$$

Case 3: Critically damped

- $\zeta = 1.00$, $\omega_n = 0.5$ rad/s, $F(t)/m = 0$
- $X(0) = 0$ m, $V(0) = 1$ m/s

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$

$$\ddot{x} + \dot{x} + .25x = 0$$

$$r = \frac{-1 \pm \sqrt{1-1}}{2}$$

$$0.5c_1 + c_2 = 2$$

$$r = \frac{-1}{2}$$

$$X(t) = C_1 e^{-1/2t} + C_2 t e^{-1/2t}$$

$$\dot{X}(t) = \frac{-C_1}{2} e^{-1/2t} + C_2\left[e^{-1/2t} - \frac{t}{2} e^{-1/2t}\right]$$

$$C_1 + C_2 = 0 \qquad C_1 = 0$$

$$-0.5C_1 + C_2 = 1 \qquad C_2 = 1$$

$$\boxed{X(t) = t e^{-1/2t}}$$

$$\dot{X}(t) = e^{-1/2t} - \frac{t}{2} e^{-1/2t}$$

Case 4: $\frac{F(t)}{m} = 5$

- $\frac{F(t)}{m} = 5$, $\zeta = 1.1$, $\omega_n = 1.0$

$X(0) = 1\,m$ $\quad \dot{x}(0) = 0.1\,m/s$

$$* \left(X_h(t) = c_1 e^{-.6417t} + c_2 e^{-1.558t} \right) *$$

$$\ddot{x} + 2.1\dot{x} + x = 5$$

$$X_p = 5$$

$$C_1 + C_2 = 1$$

$$-.6417C_1 - 1.558C_2 = 0.1$$

$C_1 = 1.809$
$C_2 = -0.809$

$$\boxed{\begin{aligned} X(t) &= 1.809e^{-.6417t} - 0.809e^{-1.558t} + 5 \\ v(t) &= -1.16e^{-.6417t} + 1.26e^{-1.558t} \end{aligned}}$$

Case 5: $\frac{F}{m} = 5t$

- $\frac{F(t)}{m} = 5t$
  - $\zeta = 0.1$, $\omega_n = 5$ rad/s
  - $x(0) = 5m$, $v(0) = 1m/s$
- $x_h(t) = C_1 e^{-t/2}\left(C_1 \cos(4.975\,t) + C_2 \sin(4.975t)\right)$
- $\ddot{x} + \dot{x} + 25x = 5t$

$$x_p(t) = At + B$$

$$x_p'(t) = A$$

$$x_p''(t) = 0$$

$$A + 25(At + B) = 5t$$

$$(A+B) + 25At = 5t$$

$$A = \frac{1}{5}$$

$$B = -\frac{1}{5}$$

$$x_p(t) = \frac{t}{5} - \frac{1}{5}$$

$$X_h(t) = e^{-t/2}\left(C_1 \cos\left(\frac{4.975t}{}\right) + C_2 \sin\left(4.975t\right)\right)$$

$$C_1 = 5$$

$$X_h(t) = C_1 e^{-t/2} \cos(4.975t) + C_2 e^{-t/2} \sin(4.975t)$$

$$\dot{X}_h(t) = C_1\left(-\frac{1}{2} e^{-t/2} \cos(4.975t) + 4.975 e^{-t/2} \sin(4.975t)\right)$$

$$+ C_2\left(-\frac{1}{2} e^{-t/2} \sin(4.975t) + 4.975 e^{-t/2} \cos(4.975t)\right)$$

$$-\frac{C_1}{2} + 4.975 C_2 = 1$$

$$C_2 = .703517 \qquad C_1 = 5, \ C_2 = .70352$$

$$\boxed{X(t) = 5e^{-t/2} \cos(4.975t) + .703517 \sin(4.975t) e^{-t/2} + \frac{t}{5} - \frac{1}{5}}$$

Case 6: $\frac{F(t)}{m} = 5t^2 + 5t$

- $\zeta = 1.0$, $\omega_n = 0.5$ rad/s
- $x(0) = 10$ m, $v(0) = 2$ m/s
- $x_h(t) = c_1 e^{-1/2 t} + c_2 t e^{-1/2 t}$

$\ddot{x} + \dot{x} + .25x = 5t^2 + 5t$

$x_p(t) = At^2 + Bt + C$

$\dot{x}_p(t) = 2At + B$

$\ddot{x}_p(t) = 2A$

$2A + 2At + B + .25(At^2 + Bt + C) = 5t^2 + 5t$

$2A + B + .25C = 0$

$2At + .25Bt = 5t$

$.25At^2 = 5t^2$

$A = 20$
$B = -140$
$C = 400$

$x_p(t) = 20t^2 - 140t + 400$

$c_1 = 10$, $c_2 = 7$

$-\frac{c_1}{2} + c_2 = 2$

$$\boxed{x(t) = 10e^{-1/2t} + 7te^{-1/2t} + 20t^2 - 140t + 460}$$

$7te^{-1/2t}$

$7\left(e^{-1/2t} - \frac{t}{2}e^{-1/2t}\right)$