Chapter 3. Laplace Transforms

3.0 Introduction.

Pierre-Simon Laplace (*1749-1827*), "The Newton of France".

Laplace transforms convert differentiation into multiplication and integration into division by mapping a function $f(t)$ to a new function $F(s)$.

Definition 0.0. The *Laplace transform* of a function f is $F = \mathcal{L}\{f\}$, where

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\, dt$$

when this integral exists. The variable $s\epsilon\mathbb{C}$ is called the *transform* variable.

Customarily, $\mathcal{L}\{f\}(s)$ is written $F(s)$ using the capital, so $\mathcal{L}\{y\}(s) = Y(s)$.[1]

Since $\mathcal{L}f$ is an integral, it inherits the property of linearity from $\int f\, dt$.[2]

(0.0) $$\mathcal{L}\{c_1 f + c_2 g\} = c_1\mathcal{L}f + c_2\mathcal{L}g$$

Observe $\mathcal{L}e^{at} = \int_0^\infty e^{at}e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a}$. This integral converges when $s - a > 0$, so $s > a$ is an added condition on $\mathcal{L}e^{at} = \frac{1}{s-a}$.

(0.1) $$\mathcal{L}e^{at} = \frac{1}{s-a},\quad s > a\, .$$

The most important feature of the transform comes from just Integrating by parts.[3]

(0.2) $$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\, dt = s\mathcal{L}f - f(0)$$

According to Formula 0.2, the derivative in t is, up to a constant, converted to multiplication by the frequency variable s. This feature of $\mathcal{L}$ allows IVPs to be transformed into algebraic equations.

Example 0.0. Observe what the Laplace transform does when we transform both sides the IVP,

$$y' = -2y + e^{-3t},\quad y(0) = 1\, .$$

Using formulas 0.0, 0.1, 0.2,

$$\mathcal{L}y' = \mathcal{L}\{-2y + e^{-3t}\} \;\Rightarrow\; s\mathcal{L}y - y(0) = -2\mathcal{L}y + \frac{1}{s+3}$$

The IVP becomes an algebraic equation for $\mathcal{L}y$, so we solve it:

$$(s+2)\mathcal{L}y = 1 - \frac{1}{s+3} \;\Rightarrow\; \mathcal{L}y = \frac{1}{s+2} + \frac{1}{(s+2)(s+3)}\, .$$

We only need "un-transform" $\mathcal{L}y$ using the inverse $\mathcal{L}^{-1}$ to find the solution

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2} + \frac{1}{(s+2)(s+3)}\right)\, .$$ □

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[1] We will often suppress the braces as well as the dependence of $\mathcal{L}$ on s and write instead, $\mathcal{L}f = F(s)$ or even $\mathcal{L}y = Y$ for brevity. This custom should be familiar from Calculus, where the antiderivative of $f(t)$ is often written $F(t) + c = \int f dt$ and analogously $\mathcal{L}f$ is another kind of integral of f.

[2] Constants factor out of integrals and integral of the sum is the sum of the integrals.

[3] Verify this in Exercise 4.

Because $\mathcal{L}$ is linear (0.0), it has an inverse that is also linear.

(0.0') $$\mathcal{L}^{-1}\{c_1F(s)+c_2G(s)\}=c_1f(t)+c_2g(t)$$

The Bromwich integral formula[4] for $\mathcal{L}^{-1}$ lies beyond the scope of our course, so we resort to the standard practice of recognizing transforms of known elementary functions in $F(s)$ and inverting their transform formulas on an ad hoc basis.

Each transform formula gives a corresponding *inverse transform* formula.

$$\mathcal{L}\{f(t)\}=F(s) \quad \Leftrightarrow \quad \mathcal{L}^{-1}\{F(s)\}=f(t)$$

Formula 0.1 has the inverse formula:

(0.1') $$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=e^{at}.$$

We can complete the solution of the IVP in Example 0.0 using the inverse formula (0.1').

Example 0.0'. The IVP, $y'=-2y+e^{-3t},\ \ y(0)=1$, of Example 0.0 becomes

$$\mathcal{L}y=\frac{1}{s+2}+\frac{1}{(s+2)(s+3)}.$$

We can't use formula 0.1' on the second term because[5] $\mathcal{L}^{-1}\frac{1}{(s+2)(s+3)}\neq\mathcal{L}^{-1}\frac{1}{s+2}\mathcal{L}^{-1}\frac{1}{s+3}$.

Instead, we use the *partial fractions decomposition* to rewrite

$$\frac{1}{(s+2)(s+3)}=\frac{A}{s+2}+\frac{B}{s+3}.$$

The coefficients $A=1, B=-1$ are chosen so this holds for every value of s.[6]

Using formula (0.1'),

$$y=\mathcal{L}^{-1}\left(\frac{1}{s+2}+\frac{-1}{s+3}+\frac{1}{s+2}\right)=2e^{-2t}-e^{-3t}. \qquad \square$$

Our main interest in the Laplace transform comes in its utility for solving 2nd order linear IVPs associated with simple circuits.[7] Sinusoidal force applied to such a circuit models alternating current (like the electricity powering your home). To this end we need the Laplace transforms of cosine and sine.

By Euler's formula, $e^{i\omega t}=\cos\omega t+i\sin\omega t$, so the Laplace transforms of both can be found as the real and imaginary parts of

$$\mathcal{L}e^{i\omega t}=\frac{1}{s-i\omega}=\frac{1}{s-i\omega}\frac{s+i\omega}{s+i\omega}=\frac{s+i\omega}{s^2+\omega^2}=\frac{s}{s^2+\omega^2}+i\frac{\omega}{s^2+\omega^2}=\mathcal{L}\cos\omega t+i\,\mathcal{L}\sin\omega t$$

where we used formula (0.1) with $a=i\omega$.[8] Though they are simple enough to derive, these formulas are worth memorizing.

(0.3) $$\mathcal{L}\cos\omega t=\frac{s}{s^2+\omega^2}, \quad \mathcal{L}\sin\omega t=\frac{\omega}{s^2+\omega^2}$$

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[4] Thomas Bromwich (1875–1929). When $\Re e\, s\geq a \Rightarrow F(s)$ analytic at s, $\mathcal{L}^{-1}F(s)=\frac{1}{2\pi i}\lim_{b\to\infty}\int_{a-ib}^{a+ib}F(s)e^{st}\,ds$.

[5] Since $\mathcal{L}^{-1}$ is an integral, $\mathcal{L}^{-1}\{fg\}\neq\ \mathcal{L}^{-1}\{f\}\,\mathcal{L}^{-1}\{g\}$. The integral of a product is **not** the product of the integrals.

[6] Partial fractions is explained in any Calculus text. Multiplying through by $(s+3)(s+2)$ shows $1=A(s+2)+B(s+3)$. Equating coefficients in like powers of s shows $1=2A+3B$ and $A+B=0 \Rightarrow A=-1,\ B=1$.

[7] The RLC circuit with resistance R, inductance L, and capacitance C (see 3.1).

[8] The line integral $\mathcal{L}e^{i\omega t}=\int_0^\infty e^{(i\omega-s)t}\,dt$ along the real t-axis is a completely routine integration.

Two other useful results

(0.4) $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

(0.5) $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$

are easy to obtain by integrations. The first, called a *shift formula*, is the result of a direct integration by substitution. The second relies on passing the derivative operator under the integral sign. Both verifications are left for the careful reader.

Example 0.1. Let's find $\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\}$. Comparing (0.3) this will be a cosine with s replaced by $s+3$. Using (0.4) $\mathcal{L}\{e^{-3t}\cos 2t\} = \frac{s+3}{(s+3)^2+4}$ and $\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} = e^{-3t}\cos 2t$. □

Example 0.2. Let's find $\mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2+4}\right\}$. This is like a cosine with s replaced by $s+3$ but lacks $s+3$ in the numerator. To make this work, add and subtract the +3 to rewrite it as

$$\frac{s}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+4} - \frac{3}{(s+3)^2+4}.$$

The first term is the shifted cosine of Example 0.1. The second is like the sine in (0.3) but it has the wrong numerator. We need a 2 not 3 there. Factor out 3 then multiply and divide by 2 to give:

$$\frac{s}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+4} - \left(\frac{3}{2}\right)\frac{2}{(s+3)^2+4}.$$

Now, using linearity and (0.4),

$$\begin{aligned}\mathcal{L}^{-1}\frac{s}{(s+3)^2+4} &= \mathcal{L}^{-1}\frac{s+3}{(s+3)^2+4} - \left(\frac{3}{2}\right)\mathcal{L}^{-1}\frac{2}{(s+3)^2+4}\\ &= e^{-3t}\cos 2t - \frac{3}{2}e^{-3t}\sin 2t. \ \square\end{aligned}$$

Here are three more basic transform formulas.

(0.6) $\mathcal{L}\{1\} = \frac{1}{s}, \quad s>0$

(0.7) $\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0$

(0.8) $\mathcal{L}\{y''\} = s^2\mathcal{L}y - sy(0) - y'(0)$

The first is a simple integration, the second follows applying (0.5) to (0.6) and the last is left to Exercise 25. We can use our formulas to solve 2[nd] order linear IVPs.

Example 0.3. Let's solve $y'' + 4y = e^{2t}$, $y(0) = 0$, $y'(0) = 0$ using the Laplace transform. Formulas 0.8 and 0.1 give

$$\mathcal{L}y'' + 4\mathcal{L}y = \mathcal{L}e^{2t} \Rightarrow s^2\mathcal{L}y + 4\mathcal{L}y = \frac{1}{s-2} \Rightarrow (s^2+4)\mathcal{L}y = \frac{1}{s-2}.$$

Next, the partial fractions decomposition:

$$\frac{1}{(s^2+4)(s-2)} = \frac{As+B}{(s^2+4)} + \frac{C}{s-2} \Rightarrow y = A\cos 2t + \frac{B\sin 2t}{2} + Ce^{2t}.$$

To find $A, B, \& C$, clear fractions and equate coefficients on like powers of s in the result:

$$1 = (As + B)(s - 2) + C(s^2 + 4)$$
$$1 = (A + C)s^2 + (B - 2A)s - 2B + 4C$$
$$\Rightarrow \quad A + C = 0, \; B - 2A = 0, \; -2B + 4C = 1 \; \Rightarrow A = -\frac{1}{8}, B = -\frac{1}{4}, C = \frac{1}{8}$$
$$y = -\frac{\cos 2t}{8} - \frac{\sin 2t}{8} + \frac{e^{2t}}{8}. \qquad \square$$

This example illustrates that Laplace transform, although effective, is not as efficient a method for solving such IVPs as undetermined coefficients. However, there are situations where undetermined coefficients can't be used. We will look at those in the next section.

Exercises 3.0

1. Find $\mathcal{L}\{3e^{-2t} + 2e^{3t}\}$.
2. Find $\mathcal{L}\{\cosh 2t\}$ and $\mathcal{L}\{\sinh 2t\}$.
3. Find $\mathcal{L}\{\cosh 2t - \sinh 2t\}$ and $\mathcal{L}\{\cosh 2t + \sinh 2t\}$.
4. Verify the formula 0.2 by carrying out the integration by parts for $\mathcal{L}\{y'\} = \int_0^\infty y'(t)e^{-st}\,dt$.
5. Find $\mathcal{L}\{3\cos \pi t + 2\sin \pi t\}$.
6. Use Definition 0.0 to verify $\mathcal{L}e^{i\omega t} = \frac{1}{s-i\omega}$ by a routine integration.
7. Find $\mathcal{L}\{e^{2t}\cos 3t + e^{-2t}\sin 3t\}$.
8. Find $\mathcal{L}\{e^{2t}\cos\sqrt{2}t + e^{-2t}\sin\sqrt{2}t\}$.
9. Find $\mathcal{L}^{-1}\left\{\frac{2}{s+4}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{4}{s-4}\right\}$
10. Find $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\}$ [use Formula 0.3].
11. Find $\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{2s+6}{(s+3)^2+4}\right\}$ [Combine formulas (0.3) and (0.4), see Example 0.1].
12. Find $\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^2+4}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{2s-6}{(s-3)^2+4}\right\}$ [Combine formulas (0.3) and (0.4), see Example 0.1].
13. Find $\mathcal{L}^{-1}\left\{\frac{2s}{(s+3)^2+4}\right\}$ [see Example 0.2].
14. Find $\mathcal{L}^{-1}\left\{\frac{2s}{(s-3)^2+16}\right\}$ [see Example 0.2].

    In exercises 15-24 apply formula (0.5) to formula (0.1), (0.3), (0.6), or (0.7) as appropriate.
15. Find $\mathcal{L}\{t^2e^{-3t}\}$.
16. Find $\mathcal{L}\{t^2e^{-5t}\}$.
17. Find $\mathcal{L}\{te^{-2t} + 3t^2e^{-2t}\}$.
18. Find $\mathcal{L}\{te^{-4t} + 2t^2e^{-4t}\}$.
19. Find $\mathcal{L}\{t\cos 2t + t\sin 2t\}$.
20. Find formulas for $\mathcal{L}\{t\cos\omega t\}$ and $\mathcal{L}\{t\sin\omega t\}$.
21. Find formulas for $\mathcal{L}\{t^2\}$ and $\mathcal{L}\{t^3\}$.
22. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$, $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$, $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$.
23. Find a formula for $\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\}$.
24. Verify formula (0.8) by applying formula 0.2 to y''.
25. Use the Gamma function of Definition 9.5 in Chapter 2 to write a formula for $\mathcal{L}\{t^r\}$ for any real number r.
26. Consult a list of Laplace transform formulas to verify your solutions in 11-23.

3.1 Discontinuous Forcing: Heaviside's Step Function

Oliver Heaviside (1875-1925)

The Laplace transform[9] is a powerful tool for solving linear ODEs with discontinuous forcing, such as those common in modeling switched electrical circuits.

Definition 1.0. The *Heaviside unit step function.*

$$u(t) = \begin{cases} 1 & 0 \le t \\ 0 & otherwise \end{cases}$$

The Heaviside step models a switch that is off until it turns on at $t = 0$, delivering a constant unit force. The switch can be turned on and then off, resulting in an *impulse* delivered over a finite duration.

Example 1.0. A constant force f_0, initiated at $t_1 = 2$ then turned off after 4 sec, supplies an *impulse*. It is modeled by $f(t) = (\, u(t-2) - u(t-6)\,) f_0$, using a Heaviside unit step "up" at $t_1 = 2$ followed by a Heaviside step "down" at $t_2 = 6$. The magnitude of the impulse is the integral of the force:

$$\int_{-\infty}^{\infty} f(t)dt = f_0 \int_{-\infty}^{\infty} u(t-2) - u(t-6)\, dt = f_0 \int_2^6 dt = 4f_0$$

Heaviside steps chop off the integral at finite endpoints. The impulse magnitude is just duration times amplitude. □

The Laplace transform of a Heaviside unit step up at $t = a$ is[10]

(1.0) $$\mathcal{L}u(t-a) = \frac{e^{-as}}{s}.$$

When $a = 0$ formula (1.0) describes $\mathcal{L}u(t) = \mathcal{L}\{1\} = \frac{1}{s}$ just a special case of formula (0.1).

Suppose we want to "turn on" a function $f(t)$ at time t_1. This is a delayed effect is modeled as $u(t-t_1)f(t-t_1)$ whose Laplace transform is

$$\mathcal{L}\{u(t-t_1)f(t-t_1)\} = \int_0^{\infty} u(t-t_1)f(t-t_1)e^{-st}\, dt = \int_1^{\infty} e^{-st} f(t-t_1)\, dt\ .$$

The substitution $\tau = t - t_1$ converts the integral into

$$\int_0^{\infty} e^{-s(\tau+t_1)} f(\tau)d\tau = e^{-st_1} \int_0^{\infty} e^{-s(\tau)} f(\tau)d\tau = e^{-st_1}\mathcal{L}f$$

proving the *delay formula:*

(1.1) $$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

When $a = 0$ there is no delay, so $\mathcal{L}\{u(t)f(t)\} = \mathcal{L}\{f(t)\}$.

The inverse formula for (1.1) is

(1.1′) $$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)\ .$$

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[9] Pierre-Simon Laplace (1729-1827) was first to transform a differential equation using an integral $\int_0^{\infty} f(t)x^t\, dt$. Later, with the substitution $x = e^{\ln x}$, so $x^t = e^{t\ln x}$, the integral became $\int_0^{\infty} f(t)e^{t\ln x}\, dt$. This integral converges when $\ln x < 0$. Writing $-s = \ln x < 0$, the transform became $\int_0^{\infty} f(t)e^{-st}\, dt$ in the hands of Oliver Heaviside (1875-1925), the Englishman who developed an "operational calculus" using complex variable methods key to modeling electrical circuits and rewrote Maxwell's equations in modern vector form.

[10] $\mathcal{L}u(t-a) = \int_0^{\infty} u(t-a)e^{-st}\, dt = \int_a^{\infty} e^{-st}\, dt = \lim_{t\to\infty} \frac{e^{-st}}{-s} - \frac{e^{-sa}}{-s} = \frac{e^{-as}}{s}$

Example 1.1. Let's illustrate the use of formulas (1.1) and (1.1').

$\mathcal{L}\left\{u(t-1)e^{-3(t-1)}\right\} = \frac{e^{-s}}{s+3}$ is a simple application of (1.1) because the force $e^{-3(t-1)}$ is shifted so as to begin at $t = 1$ with amplitude $e^{-3(1-1)} = 1$ that it would have had at $t = 0$ without the shift. Compare the transform when the force is e^{-3t} delayed to $t = 1$:[11]

$$\mathcal{L}\{u(t-1)e^{-3t}\} = \mathcal{L}\{u(t-1)\}|_{s+3} = \left.\frac{e^{-s}}{s}\right|_{s+3} = \frac{e^{-(s+3)}}{s+3}.$$

The third example compares another case where the function t is delayed but not shifted. The trick is to rewrite t as $t-a+a$ so it becomes a shift of $t-a$

$$\mathcal{L}\{u(t-a)t\} = \mathcal{L}\{u(t-a)(t-a+a)\} = \mathcal{L}\{u(t-a)(t-a) + au(t-a)\}$$
$$= e^{-as}(\mathcal{L}\{t\} + a\mathcal{L}\{1\}) = e^{-s}\left(\frac{1}{s^2} + \frac{a}{s}\right).$$
$$\mathcal{L}\{u(t-3)t^2\} = \mathcal{L}\{u(t-3)((t-3)^2 - 6(t-3) - (9+6)\} = e^{-3s}\mathcal{L}\{t^2 - 6t - 15\}$$
$$= e^{-3s}\left(\frac{2}{s^3} - \frac{6}{s^2} - \frac{15}{s}\right).$$

The fourth example shows a basic delayed cosine: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-2s}\right\} = u(t-2)\cos 3(t-2)$. □

A Simple Switched RLC Circuit

In electro-magnetism, a conductor is a path along which charges (electrons) travel, such as a wire, and an insulator is everything else. A circuit is a conducting loop. Charges move along a conductor in response to an electromotive force vector $\boldsymbol{E} = -\nabla\varphi$ where φ is a scalar electrical potential. For points a , b along a conductor, the potential difference is $V_{a,b} = \varphi(a) - \varphi(b)$, in *volts* and $\boldsymbol{E}$ points from high to low voltage.

When we think of the flow of electricity in a conductor, we are thinking about the *current* $I = dQ/dt$ where Q is the *charge* at a point. Suppose points a , b are connected by a conductor and satisfy $\varphi(a) > \varphi(b)$. The $\boldsymbol{E}$ vector forces charge down the potential φ toward b, raising its charge, lowering the charge at a, charge flows from a to b. Because the protons can't move much, the negatively charged electrons move from b toward a, lowering the charge at a. This all happens pretty fast in a conductor, so any small region of it will have about the same charge all the time. It is more interesting to consider what happens when a and b are separated by an *insulator*, a gap in the conductor (Figure 3-0.A) that prevents electrons from moving across to equalize the potentials at a and b.

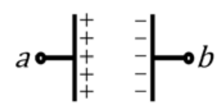

Figure 3-0. A. Capacitor.

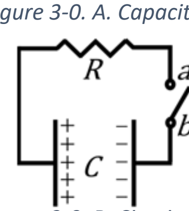

Figure 3-0. B. Circuit with capacitor and.resistance R.

Because opposite charges collect on either side of the gap, a potential difference $V_{a,b}$ is "stored" on the ends of the conductor, right at the gap. Such a device is called a *capacitor*. The capacitor has positive charges on the a side and negative on the b side as shown in Figure 3-0.A. The potential difference $V_{a,b}$ that can be sustained by a capacitor[12] increases with the separation of the plates d, divided by their area A. The *capacity* $C = Q/V_{a,b}$ where Q is the *charge* on the capacitor. A capacitor with large capacity C is a battery, it holds large charge that can provide a current when connected to a circuit as shown in Figure 3-0.B.

When the circuit is interrupted by the open switch, $\varphi(a) > \varphi(b)$ the step down in potential $V_{a,b} > 0$ is the sum of the steps down $V_C = Q/C$ the capacitor and V_{res} at the resistor (representing resistance in the conductor, or a device like a light). When the switch is closed, charge flows around the circuit clockwise,

[11] We use formula 0.4 here to frequency shift the transform of $u(t-1)$ obtained in formula 1.0.

[12] $\epsilon V_{a,b} = Qd/A$ where ϵ is the dielectric permittivity of the insulator between the plates.

because electrons flow counter-clockwise. This flow of charge is the current I that flows through the resistor,[13] perhaps making it glow. Resistance, current, and potential difference across the resistor are related by Ohm's Law: $V_{res} = IR$, so more resistance implies less current or greater potential step down V_{res} across the resistor. Combining the capacitor and resistor in Figure 3-0.B, the step down $V_{a,b} = \frac{Q}{C} + IR$ when the switch is open and $V_{a,b} = \frac{Q}{C} + IR = 0$ when the switch is closed.

For the closed switch, $IR = -Q/C$ and, since $I = dQ/dt$, the equation governing the circuit would be

(1.2) $$dQ/dt = -Q/(RC)$$

whose solution is $Q(t) = Q_0 e^{-t/(RC)}$, exponential decay of the charge on the capacitor. The circuit is relaxing toward the equilibrium state $Q(t) = I(t) = 0$, greater resistance R or capacity C slows the rate of discharge of the capacitor (or battery) through the resistor.

So far we have looked at capacity and resistance but there is one more feature of the RLC circuit, its *self-induced electromotive force* resulting from variation in the magnetic field of the circuit caused when the current in the loop changes. For a constant current, by Ampère's Law, the magnetic field of the circuit has magnitude $B = cI$ for a constant c depending on the configuration and material of the circuit. The magnetic field vector is $\boldsymbol{B} = B\boldsymbol{n}$ with $\boldsymbol{n}$ pointing perpendicular to the current loop.

The normal components (of the curls) in Faraday's Law: $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ integrated over the surface bounded by the circuit's loop, will, by Stokes' theorem give[14]

$$V_{self} = -\frac{dB}{dt} S = -cS\frac{dI}{dt} = -L\frac{dI}{dt}$$

with the middle equality following from Ampère's Law. Since cS depends only on the configuration of the circuit, the *self-inductance of the circuit* is denoted $L = cS > 0$. Usually, this inductance is incorporated in the circuit diagram as an inductor, as shown in Figure 3-0.C.

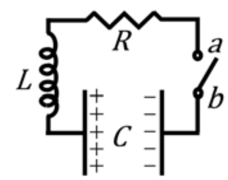

Figure 3-0.C. RLC circuit.

Because the potential step due to V_{self} only occurs with the ciruit loop complete, when the switch is closed, V_{self} imposes an "external" electric field that drives the circuit against V_{res} and V_C . With the switch closed $V_{self} = V_{res} + V_C \Rightarrow -L\frac{dI}{dt} = IR + \frac{Q}{C}$ or, perhaps better: $0 = L\frac{dI}{dt} + IR + \frac{Q}{C}$. Since $I = dQ/dt$,

(1.3) $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

is the unforced, damped harmonic oscillator equation. For small enough resistance, the response $Q(t)$ is an underdamped oscillator. An externally applied potential difference $V(t)$ gives the RLC circuit equation:

(1.4) $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V(t)\ .$$

Note that if $V(t) = L\frac{d^2Q}{dt^2}$, opposing the self-induced electromotive force, $-L\frac{dI}{dt}$, then $IR = -\frac{Q}{C}$ as in equation (1.2) and the circuit relaxes toward equilibrium $Q = I = 0$ by decay of the charge on the capacitor according to the solution $Q(t) = Q_0 e^{-t/RC}$. Because the self-induced potential difference V_{self} is directed opposite to dI/dt , it opposes change in the current in *both* directions. Since current $I = dQ/dt$ is the "velocity" of the charge Q, its derivative dI/dt is the acceleration of

[13] The flow of charge is an effect of electrons drifting in the counterclockwise direction, opposite to the current I.

[14] $\iint \nabla \times \boldsymbol{E} \cdot \boldsymbol{n}\, dS = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{n}\, dS$ where S is the surface bounded by the loop. By Stokes' theorem, the left integral is $\int \boldsymbol{E} \cdot ds = V_{a,b}$ and the right integral is $-\frac{\partial \boldsymbol{B}}{\partial t} S$ because $\boldsymbol{B}$ is perpendicular to S so $\frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{n} = \frac{dB}{dt}$ on S.

charge and L acts on charge in the same way that mass affects motion, imparting inertia, the resistance to acceleration. Inductance is then the "inertia" of the charge flowing around the circuit.

Example 1.2. Suppose a switch interrupts an RLC circuit so that, when the switch is closed the power is "on" and the circuit connects the terminals of a battery to a light. If the battery has enough charge Q, the light will glow when the switch is closed. This situation might be modeled with the equation $Q'' + 2Q' + 4Q = 0$, $Q(0) = 0$, $Q'(0) = 0$, when the power is "off". When the power is "on", charges (electrons) flow from the negative battery terminal, through the circuit to the positive terminal, supplying an electromotive force $b(t)$ while the battery slowly discharges. Suppose the switch is programmed to turn on at $t_1 = 2$ and off at $t_2 = 6$. Let's find the response.

Model the switch with $u(t-2) - u(t-6)$. The IVP is

$$Q'' + 2Q' + 4Q = \big(u(t-2) - u(t-6)\big)b_0\,, \qquad Q(0) = 0, \qquad Q'(0) = 0$$

Transforming, using the delay formula (1.1):

$$(s^2 + 2s + 4)\mathcal{L}Q = e^{-2s}\mathcal{L}b - e^{-6s}\mathcal{L}b \;\Rightarrow\; \mathcal{L}Q = \frac{e^{-2s}}{s^2+2s+4}\mathcal{L}b - \frac{e^{-6s}}{s^2+2s+4}\mathcal{L}b.$$

Suppose $b(t) = b_0$ a constant. Here, $\mathcal{L}b = \mathcal{L}b_0 = \frac{b_0}{s}$, so

$$\mathcal{L}Q = \frac{b_0 e^{-2s}}{s\,(s^2 + 2s + 4)} - \frac{b_0 e^{-6s}}{s\,(s^2 + 2s + 4)}$$

To get back to Q, we need

$$\mathcal{L}^{-1}\frac{e^{-as}}{s\,(s^2 + 2s + 4)}. \tag{1.5}$$

By partial fractions,[15] $\frac{1}{s\,(s^2+2s+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s+2}{(s+1)^2+3}\right)$. Rewriting this as in Examples 0.1 & 0.2:[16]

$$\mathcal{L}^{-1}\frac{1}{s\,(s^2 + 2s + 4)} = \frac{1}{4}\left(1 - e^{-t}\cos\sqrt{3}t - \frac{e^{-t}\sin\sqrt{3}t}{\sqrt{3}}\right). \tag{1.6}$$

Introducing into (1.6) the delay $u(t-a)$ caused by the factor e^{-as} in (1.5) and applying a shift that replaces each occurance of t in (1.6) with $t - a$, according to the inverse delay formula (1.1'):

$$\mathcal{L}^{-1}\frac{e^{-as}}{s\,(s^2 + 2s + 4)} = \frac{u(t-a)}{4}\left(1 - e^{-(t-a)}\cos\sqrt{3}(t-a) - \frac{e^{-(t-a)}\sin\sqrt{3}(t-a)}{\sqrt{3}}\right).$$

The response is:

$$Q(t) = \frac{b_0}{4}u(t-2)\left(1 - e^{-(t-2)}\cos\sqrt{3}(t-2) - \frac{e^{-(t-2)}\sin\sqrt{3}(t-2)}{\sqrt{3}}\right)\ldots$$
$$-\frac{b_0}{4}u(t-6)\left(1 - e^{-(t-6)}\cos\sqrt{3}(t-6) - \frac{e^{-(t-6)}\sin\sqrt{3}(t-6)}{\sqrt{3}}\right). \qquad \square$$

[15] Because $s^2 + 2s + 4 = (s+1)^2 + 3$ has no real roots, $\frac{1}{s\,(s^2+2s+4)} = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+3} \Rightarrow$

$$1 = (A+B)s^2 + (2A+C)s + 4A \Rightarrow A = \tfrac{1}{4},\; B = -\tfrac{1}{4}, \quad C = -\tfrac{1}{2}.$$

[16] $\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+1)^2+3}\right\} = 1 - \mathcal{L}^{-1}\frac{s+2}{(s+1)^2+3} = 1 - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3} + \frac{1}{(s+1)^2+3}\right\}$

The under-damped oscillator of Example 1.2 driven by a rectangular impulse of amplitude $b_0 = 4$ during the interval $2 \le t < 6$ obeys:

$Q'' + 2Q' + 4Q = 4\big(u(t-2) - u(t-6)\big)$.

The steady-state response $Q(t)$ is shown in Figure 3-1. The MatLab function m-file for the response plotted is shown below. [17]

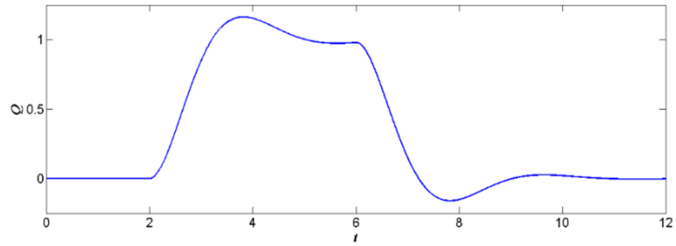

Figure 3-1. Response $Q(t)$ for Example 1.2 with constant $b_0 = 4$.

```
function y = Q(t)
% Response to impulse in Example 1.2 with amplitude b0=4
y=zeros(size(t));
for k=1:size(t,2)
    if t(k)<2
    y(k)=0;
    elseif (t(k)>=2 && t(k)<6)
    y(k)=1-exp(-(t(k)-2)).*(cos(sqrt(3)*(t(k)-2))+sin(sqrt(3)*(t(k)-2))/sqrt(3));
    else
    y(k)=1-exp(-(t(k)-2)).*(cos(sqrt(3)*(t(k)-2))+sin(sqrt(3)*(t(k)-2))/sqrt(3))...
        -(1-exp(-(t(k)-6)).*(cos(sqrt(3)*(t(k)-6))+sin(sqrt(3)*(t(k)-6))/sqrt(3)));
    end
end
```

This function m-file is breaks the response into three intervals: $t < 2$, $2 \le t < 6$, & $6 \le t < \infty$. The response is not only a continuous function but it is differentiable, even though the equation has discontinuous forcing (this works because the equation is second order and the response $Q(t) \in C^1(-\infty < t < \infty)$ but Q is not C^2 at $t = 2, 6$.

Exercises 3.1

1. Express the function $a(t) = \begin{cases} 0 & t < 1 \quad \& \; 2 \le t \\ t-1 & 1 \le t < 2 \end{cases}$ using Heaviside unit steps and make a plot of $a(t)$ for $0 \le t < 4$.
2. Express the function $b(t) = \begin{cases} 0 & t < 3 \quad \& \; 5 \le t \\ t^2 - 9 & 3 \le t < 5 \end{cases}$ using Heaviside unit steps and make a plot of $b(t)$ for $0 \le t < 8$.
3. Write a piecewise constant function $f(t)$ as a linear combination of unit step delays such that $f(t) = 1$ for $0 \le t < \pi/2$, steps down to -1 at $t = \pi/2$, remaining there until $t = \pi$ when it steps back up to 1, remaining there until $t = 3\pi/2$ when it steps down to -1 remaining there until $t = 2\pi$ when it steps down to 0. Make a plot of $f(t)$ for $0 \le t < 3\pi$.
4. Write a function $g(t)$ such that $g(t) = \cos t$ for $0 \le t < \pi/2$ and $g(t) = 0$ for $t \ge \pi/2$.
5. Write a function $h(t)$ such that $h(t) = \sin t$ for $0 \le t < \pi/2$ and $h(t) = 0$ for $t \ge \pi/2$.
6. Give the Laplace transform of the function $a(t)$ in Problem 1.
7. Give the Laplace transform of the function $b(t)$ in Problem 2.
8. Give the Laplace transform of the function $f(t)$ in Problem 3.
9. Give the Laplace transform of the functions $g(t)$ in Problem 4 and $h(t)$ in Problem 5.
10. Find the solution to the IVP: $y'' + 2y' = 4u(t-3) - 4u(t-4)$, $y(0) = 0 = y'(0)$.
11. Find the solution to the IVP: $y'' + 2y' = (t-1)\big(u(t-1) - u(t-2)\big)$, $y(0) = 0 = y'(0)$.
12. Make a plot of your solution in problem 10.
13. Make a plot of your solution in problem 11.

[17] Some versions of MatLab contain the *heaviside* function, (see >>doc heaviside) simplifying impulse response function m-files, like that shown here, eliminating the need for the *if,else,elseif,end* structure.

3.2 The Dirac Delta Function

P.A.M Dirac. Circa 1933.

P.A.M. Dirac (1902-1984) [18] was one of the most influential physicists of the 20th century. In his work,[19] he employed a device δ that he described as an "improper function" to represent a particle as a point mass. Within a few years, mathematicians developed the foundations of a theory of generalized functions, now known as *distributions*,[20] that precisely describes the properties of Dirac's δ. For our limited needs, we will follow Dirac's original discussion.

Definition 2.0. The Dirac δ distribution satisfies:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \text{ and } x \neq 0 \Rightarrow \delta(x) = 0\,.$$

Its most important property is that for $f \epsilon C^0(\mathbb{R})$, integration against δ "samples" the value of f at the point where δ is undefined:

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0).$$

Further, if $f \epsilon C^0(I)$, for I any interval containing $x = a$, then $\int_I \delta(x-a)f(x)dx = f(a)$.

Dirac noted that his δ-function could also be defined as the derivative of the unit step, $u'(x) = \delta(x)$, and so "appears whenever one differentiates a discontinuous function".

The δ-function is also used to model instantaneous impulses, like forces that transfer momentum in elastic collisions as well as forces that act over very short time intervals.

Observe that $\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} \delta(t-a)e^{-st}dx = e^{-as}$, from which we obtain the transform formula

(2.0) $$a \geq 0 \Rightarrow \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

and the inverse formula with $a = 0$ is $\delta(t) = \mathcal{L}^{-1}\{1\}$ (giving another way to define the δ-function).

The δ-function is a model for an instantaneous impulse.

Example 2.0. An undamped harmonic oscillator with natural frequency ω is at rest until, at time $t = 2$, it receives a unit impulse, instantaneously. This is modeled by:

$$y'' + \omega^2 y = \delta(t-2),\;\; y(0) = 0, \qquad y'(0) = 0$$

---

[18] See: Paul A.M. Dirac – Biographical. NobelPrize.org. Nobel Media AB 2018. (Accessed 7-Nov 2018) https://www.nobelprize.org/prizes/physics/1933/dirac/biographical/ . Image by Cambridge University, Cavendish Laboratory [1] - http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Dirac.html, Public Domain, https://commons.wikimedia.org/w/index.php?curid=624388. (Accessed 7-Nov 2018)

[19] *The Principles of Quantum Mechanics* (4th ed.), Oxford, Clarendon Press (1958), §15 The δ Function. First edition published 1930, Oxford University Press.

[20] Generalized functions originated with Sergei Sobolev (1909-1988 Russian) in his 1936 paper "Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales", *Matematicheskiĭ Sbornik*, 1(43) (1): 39–72, but were popularized in the West by Laurent Schwartz's 1951, *Théorie des distributions,* V.*1–2, Hermann.* (Paris, France) describing work for which Schwartz won the Fields medal in Mathematics in 1950.

Transforming,

$$(s^2+\omega^2)\mathcal{L}y = e^{-2s} \Rightarrow \ y = \mathcal{L}^{-1}\frac{e^{-2s}}{s^2+\omega^2} = u(t-2)\frac{\sin\omega(t-2)}{\omega}.$$

The response is a sine wave of amplitude $1/\omega$ that begins oscillating upon impact at $t=2$. □

Example 2.1. Let's solve for the response to an impulse of magnitude 16 delivered instantaneously, using a δ-function force at $t=4$, for the circuit of Example 1.2. The equation is

$$Q''+2Q'+4Q=16\delta(t-4)$$

with steady state ICs: $Q(0)=0,\ Q'(0)=0$. Solving,

$$(s^2+2s+4)\mathcal{L}Q = 16e^{-4s}$$

$$\Rightarrow \ Q = 16\mathcal{L}^{-1}\frac{e^{-4s}}{(s+1)^2+3} = 16\,u(t-4)\left(\frac{e^{-(t-4)}\sin\sqrt{3}(t-4)}{\sqrt{3}}\right).$$ □

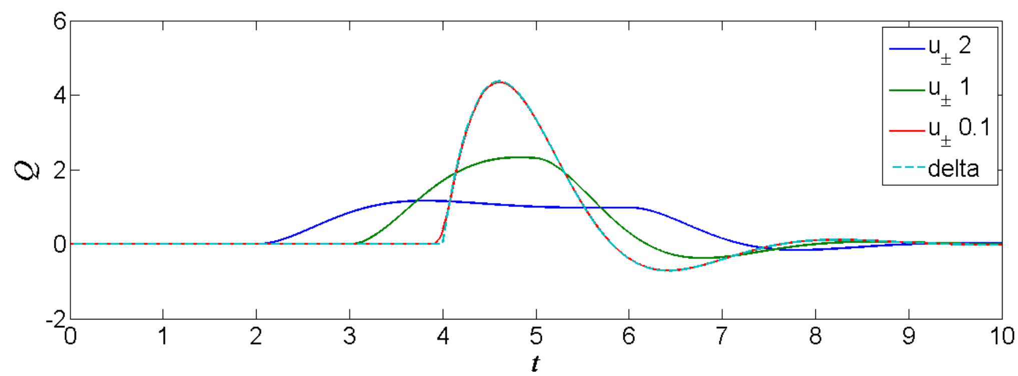

Figure3-2.Comparison of under-damped RCL circuit Q''+2Q'+4Q response to a magnitude 16 impulse delivered with constant amplitudes 4, 8, 80 over intervals 4±2, 4±1, 4±0.1, respectively, all having magnitude 16, versus the instantaneous impulse of magnitude 16, modeled by 16δ(t-4).

In Figure 3-2, the total magnitudes of the four impulses are identical (16 units) but they are delivered to the oscillator over different durations. The slowest impulse is $u_{\pm}$ 2, delivered by a Heaviside step of height 4 during the 4 sec interval $2 \le t < 6$; the response is the broad low amplitude hump. The intermediate duration impulse $u_{\pm}$ 1 is delivered by a Heaviside step of height 8 during the 2 sec interval $3 \le t < 5$. The response to the shortest duration impulse $u_{\pm}$ 0.1, delivered by a Heaviside step of height 80 during the 0.2 sec interval $3.9 \le t < 4.1$, is a very close match to that for the instantaneous magnitude 16 impulse delivered at $t=4$, denoted "delta" examined in Example 2.1. The figure shows that the response to an instantaneous impulse delivered by $f_0\delta(t-a)$ is the limit of the response to constant amplitude impulses of the same magnitude f_0 but decreasing duration, during intervals centered at $t=a$.

Exercises 3.2

1. Solve the IVP: $y''+4y'+20y=50u(t-2)-50u(t-4)$, $y(0)=0=y'(0)$ and plot your solution for $0 \le t \le 10$.
2. Solve the IVP: $y''+4y'+20y=200u(t-2.75)-100u(t-3.25)$, $y(0)=0=y'(0)$ and plot your solution for $0 \le t \le 10$.
3. Solve the IVP: $y''+4y'+20y=100\delta(t-3)$, $y(0)=0=y'(0)$ and plot your solution.
4. Make a plot like Figure 3-2 comparing your solutions to Problems 1, 2, and 3.

5. Solve the IVP: $y'' + y' + 20y = 20u(t-4) - 20u(t-6)$, $y(0) = 0 = y'(0)$ and plot your solution for $0 \le t \le 12$.
6. Solve the IVP: $y'' + y' + 20y = 200u(t-4.9) - 200u(t-5.1)$, $y(0) = 0 = y'(0)$ and plot your solution for $0 \le t \le 12$.
7. Solve the IVP: $y'' + y' + 20y = 40\delta(t-3)$, $y(0) = 0 = y'(0)$ and plot your solution.
8. Make a plot like Figure 3-2 comparing your solutions to Problems 5, 6, and 7.
9. Solve the IVP: $y'' + y = \delta(t) - \delta\left(t - \frac{\pi}{2}\right) + \delta(t-\pi) - \delta\left(t - \frac{3\pi}{2}\right) + \delta(t - 2\pi)$, $y(0) = 0 = y'(0)$ and plot your solution for $0 \le t \le 8$. What appears to be happening? Describe what you see in the amplitude of response. Predict what would happen if the impulses continued in the same pattern and period. Check your prediction by adding another period of impulses to the forcing.
10. Solve the IVP: $y'' + 4y = \delta(t) - \delta\left(t - \frac{\pi}{2}\right) + \delta(t-\pi) - \delta\left(t - \frac{3\pi}{2}\right) + \delta(t - 2\pi)$, $y(0) = 0 = y'(0)$

    and plot your solution for $0 \le t \le 8$. What appears to be happening? Describe what you see in the amplitude of response. Predict what would happen if the impulses continued in the same pattern and period. Check your prediction by adding another period of impulses to the forcing.
11. Solve the IVP: $y'' + 2y = \delta(t) - \delta\left(t - \frac{\pi}{2}\right) + \delta(t-\pi) - \delta\left(t - \frac{3\pi}{2}\right) + \delta(t - 2\pi) - \delta\left(t - \frac{5\pi}{2}\right)$, $y(0) = 0 = y'(0)$ and plot your solution for $0 \le t \le 10$. What appears to be happening? Describe what you see in the amplitude of response. Predict what would happen if the impulses continued in the same pattern and period. Check your prediction by adding another period of impulses to the forcing.
12. Solve the IVP: $y'' + 2y = \delta(t) + \delta\left(t - \frac{\pi}{2}\right) - \delta(t-\pi) - \delta\left(t - \frac{3\pi}{2}\right) + \delta(t - 2\pi) + \delta\left(t - \frac{5\pi}{2}\right)$, $y(0) = 0 = y'(0)$ and plot your solution for $0 \le t \le 10$. What appears to be happening? Describe what you see in the amplitude of response. Predict what would happen if the impulses continued in the same pattern and period. Check your prediction by adding another period of impulses to the forcing.