Chapter 4. Linear Systems

4.0 Linear Algebra Preliminaries

We have already seen linear systems of ODEs in Chapter 2, where we converted second order equations into systems of two equations in order to solve them numerically. In Chapter 2, section 2.6, the second order linear equation

$$y'' + p(t)y' + q(t)y = f(t)$$

was converted into a system by making the substitution: $y_1 = y,\ y_2 = y'$ giving a system:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= f(t) - q(t)y_1 - p(t)y_2 \end{aligned}$$

In this chapter, we will learn how to treat linear systems of ODEs that do not necessarily arise from a single 2[nd] order equation. Such equations appear in the form:

$$\frac{d}{dt}\vec{y} = \vec{F}(t, \vec{y})$$

Where $\vec{y}$ and $\vec{F}$ are vectors of functions. Before embarking on our study of 1[st] order vector ODEs, we begin with four subsections laying out the requisite Linear Algebra preliminaries.

4.0.1 Vectors and Matrices

The solution to a system of n ODEs in the functions $y_1, y_2, \dots y_n$ is a vector, in matrix notation $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ from Calculus are represented in matrix notation as: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Generally, a matrix is an array of entries. Each entry in the array has a row index and a column index.

Definition 0.0. The entries of an $m \times n$ matrix ***A*** are a_{ij} where the row index $i = 1 \dots m$ and column index $j = 1 \dots n$ give the row and column in which a_{ij} is found. The numbers m, n are the *dimensions* of ***A***. We can write $\boldsymbol{A} = (a_{ij})$ when we understand the dimensions are known. Row index is always first.

Example 0.1.0. $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 5 \\ 4 & 1 & 3 \end{bmatrix}$ is a 2×3 matrix. $a_{11} = 2,\ a_{12} = 0,\ a_{13} = 5,\ a_{21} = 4 \dots$. □

A one-column matrix is called a *column vector* or often just a *vector*, we symbolize it with a harpoon, $\vec{v}$, and just use the row index to designate entries.[1] We write $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ for a vector of dimension 2

A one-row matrix is called a *row vector*. We'll use non-bold capitals for rows and bold capitals for matrices in general. If a matrix has two indices it is neither a vector, nor a row vector.

Example 0.1.1. $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix} = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ where $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $A_1 = [1 \quad 0 \quad 3] \dots$. □

The vectors $\vec{a}_k$ in Example 0.1.2 are the *column vectors* or just *columns* of . The row vectors A_k are the *rows* of $\boldsymbol{A}$.

[1] In MatLab, the entries of a vector are also designated using a single index.

The entries of a matrix can be *scaled* by multiplying the matrix by a number. In the algebra of matrices, numbers are called *scalars*. For real matrices a scalar can be any real number and for complex matrices they can be complex.

Example 0.1.2. $\boldsymbol{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times the scalar 3 is $3\boldsymbol{A}=\begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$. □

Definition 0.1. The sum of two $m \times n$ matrices is $\boldsymbol{A}+\boldsymbol{B}=\left(a_{ij}+b_{ij}\right)$.

Matrices add "entry-by-entry", so only matrices with the *same dimensions* can be added.

Definition 0.2. The transpose of an $m \times n$ matrix $\boldsymbol{A}=\left(a_{ij}\right)$ is the $n \times m$ matrix $\boldsymbol{A}^{\mathbf{T}}=\left(a_{ji}\right)$.

The transpose of a row vector $A=[a_1 \quad \cdots \quad a_k]$ is the column vector $A^{\mathrm{T}}=\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$. Transpose swaps rows and columns so $\left(A^{\mathrm{T}}\right)^{\mathrm{T}}=A$.

Multiplication of matrices (including vectors as a basic case) can be entirely defined in terms of the transpose and the "dot" product familiar from Calculus.

Example 0.1.3. In Calculus, the dot product $\left(2\vec{\imath}+3\vec{\jmath}-4\vec{k}\right)\cdot\left(3\vec{\imath}-\vec{\jmath}+2\vec{k}\right)=6-3-8=-5$. In matrix algebra, this can be expressed as either a dot product of vectors, or as a "matrix" product of a row vector with a column vector:

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}=[2 \quad 3 \quad -4]\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}=6-3-8=-5.$$

Literally, the dot in the dot product can be replaced by the "super T" of transpose[2] to make it a matrix product that is found by just adding the products of the entries. □

Definition 0.3. The matrix product of a row A and a column $\vec{b}$ is $[a_1 \quad \cdots \quad a_k]\begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}=a_1b_1+\cdots+a_kb_k$

equivalently, the "dot product" of column vectors $\vec{a}\cdot\vec{b}=\vec{a}^{\mathrm{T}}\vec{b}=a_1b_1+\cdots+a_kb_k$.

Remark 0.0. It is important to notice the **row factor is always on the left in a matrix product**. The product in the reverse order: **(*column*)(*row*) is undefined**. [3]

Definition 0.4. The matrix product $\vec{b}=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}=\begin{bmatrix} a_{11}b_1+\cdots+a_{1n}b_n \\ \vdots \\ a_{m1}b_1+\cdots+a_{mn}b_n \end{bmatrix}=\begin{bmatrix} A_1\vec{b} \\ \vdots \\ A_m\vec{b} \end{bmatrix}$.

Example 0.1.4. $\boldsymbol{A}\vec{b}=\begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix}=\begin{bmatrix} -1(5)+4(6) \\ 2(5)+3(6) \end{bmatrix}=\begin{bmatrix} 19 \\ 26 \end{bmatrix}$ □

[2] In Matlab, the "super T" of transpose is denoted a single-quote.

[3] The dot product in reverse order, $\vec{b}\cdot\vec{a}=\vec{b}^{\mathrm{T}}a$, so the row is again on the left. The results are the same, $\vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}$.

Observe in Example 0.1.4 that $\boldsymbol{A}\vec{b} = 5\begin{bmatrix}-1\\2\end{bmatrix} + 6\begin{bmatrix}4\\3\end{bmatrix}$ so there is another way to think about a matrix times a vector.

Definition 0.4'. The matrix product $\boldsymbol{A}\vec{b} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b_1\vec{a}_1 + \cdots + b_n\vec{a}_1.$

The definitions 0.4 and 0.4' give the same result. You recognize $b_1\vec{a}_1 + \cdots + b_n\vec{a}_n$ as a linear combination of the *vectors* $\vec{a}_1 \cdots \vec{a}_n$. In ODE systems, solutions *are* vectors, so linear combinations of solutions go over into matrix algebra very naturally.

Instead of a matrix times a vector, we can multiply a matrix $\boldsymbol{A}$ times a matrix $\boldsymbol{B}$ by multiplying $\boldsymbol{A}$ by all the column vectors of $\boldsymbol{B}$ and putting the results into a new matrix $\boldsymbol{AB}$ as its columns.

Definition 0.5. For $m \times n$ matrix $\boldsymbol{A}$ and $n \times k$ matrix $= \begin{bmatrix}\vec{b}_1 & \cdots & \vec{b}_k\end{bmatrix}$, their product is

$$\boldsymbol{AB} = \begin{bmatrix}\boldsymbol{A}\vec{b}_1 & \cdots & \boldsymbol{A}\vec{b}_k\end{bmatrix} = \begin{bmatrix} A_1\vec{b}_1 & \cdots & A_1\vec{b}_k \\ \vdots & \ddots & \vdots \\ A_m\vec{b}_1 & \cdots & A_m\vec{b}_k \end{bmatrix}$$

The second expression uses Def 0.4 to calculate the column vectors of $\boldsymbol{AB}$. The rows A_i multiply the columns $\vec{b}_j$ of $\boldsymbol{B}$. These products are defined in Def 0.3; they are equivalent to "dot products".

Remark 0.1. For the products $A_i\vec{b}_j$ to be defined, the rows of $\boldsymbol{A}$ must have the same number of entries as the columns of $\boldsymbol{B}$ so the dimensions of the matrices must match in the inner factors. When multiplying matrices (or vectors) the dimensional product expression:

$$(m \times n)(\, p \times k)\ \underline{\text{must have}}\ n = p.$$

Theorem 0.0. The matrix product $\boldsymbol{AB}$, is *defined* $\Leftrightarrow$ $\boldsymbol{A}$ is $m \times n$ and $\boldsymbol{B}$ is $n \times k$.

Example 0.1.5. If $\boldsymbol{A} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}$ then

$$\boldsymbol{AB} = \begin{bmatrix} [1\ 2]\begin{bmatrix}4\\-3\end{bmatrix} & [1\ 2]\begin{bmatrix}-2\\1\end{bmatrix} \\ [3\ 4]\begin{bmatrix}4\\-3\end{bmatrix} & [3\ 4]\begin{bmatrix}-2\\1\end{bmatrix} \end{bmatrix}$$

has entries equal to products of the rows of $\boldsymbol{A}$ with the columns of $\boldsymbol{B}$, the result is

$$\boldsymbol{AB} = \begin{bmatrix}4-6 & 0\\0 & -6+4\end{bmatrix} = \begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}. \qquad \square$$

Example 0.1.6. If $\boldsymbol{A} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$ then $\boldsymbol{AB}$ is undefined. Also $\boldsymbol{BA}$ is undefined. However,

$$\boldsymbol{AB}^{\mathbf{T}} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\end{bmatrix} = \begin{bmatrix}5 & 11 & 17\\11 & 25 & 39\end{bmatrix}. \qquad \square$$

Recall from Chapter 2, that the inverse of $\boldsymbol{A} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ is $\boldsymbol{A}^{-1} = \frac{1}{det\mathbf{A}}\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}$ and that $\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ the 2×2 *identity* matrix. The matrix $\boldsymbol{B}$ in Example 0.1.7 is almost $\boldsymbol{A}^{-1}$, in fact it's $-2\mathbf{A}^{-1}$.

Square matrices of the same size can always be multiplied together, although changing the order of the factors usually changes the product, unlike our experience with multiplying numbers.

Example 0.1.7. These matrix products, though defined, are unequal:

$$\boldsymbol{AB} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \quad \boldsymbol{BA} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & 1 \end{bmatrix} \qquad \square$$

Basic Matrix Operations Summary

1. Scalar multiples $r\boldsymbol{A} = r(a_{ij}) = (ra_{ij})$ so a multiplying by a number, multiplies all entries.
2. Addition: $(a_{ij}) + (b_{ij}) = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ & \ddots & \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$ so add entry-wise. Subtraction is also entry-wise and the zero matrix is written $\boldsymbol{0}$.
3. $\boldsymbol{A} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \Rightarrow \boldsymbol{A}^{\mathrm{T}} = [a_1 \quad \cdots \quad a_k]$ is "$\boldsymbol{A}$ -transpose" and $[a_1 \quad \cdots \quad a_k]^{\mathrm{T}} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$
4. $[a_1 \quad \cdots \quad a_k]\begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} = a_1b_1 + \cdots + a_kb_k$ a row times a column is a "dot product".
5. $\boldsymbol{AB} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}[\vec{b}_1 \quad \cdots \quad \vec{b}_k] = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}[\vec{b}_1 \quad \cdots \quad \vec{b}_k] = \begin{bmatrix} A_1\vec{b}_1 & \cdots & A_1\vec{b}_k \\ \vdots & \ddots & \vdots \\ A_m\vec{b}_1 & \cdots & A_m\vec{b}_k \end{bmatrix}$

   Because A_i is the i^{th} row of A, the entries in the i^{th} row of AB are the matrix products $A_i\vec{b}_j$.

   The dimensions of the matrices must agree in the middle terms of their dimensional product expression: $(m \times n)\ (n \times k)$ for $\boldsymbol{AB}$ to be defined.

   Usually $\boldsymbol{BA} \neq \boldsymbol{AB}$ even when both $\boldsymbol{BA}$ and $\boldsymbol{AB}$ are defined.
6. A square matrix that has $det\boldsymbol{A} \neq 0$, is invertible and $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$.

Exercise 4.0.1

Find the matrix products or say why they don't exist:

i. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$ *ii.* $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -0.1 & -0.3 \\ -0.4 & -0.2 \end{bmatrix}$ *iii.* $[4 \quad 5 \quad -1]\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$

iv. $\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}[4 \quad 5 \quad -1]$ *v.* $\begin{bmatrix} 1 & 2 & -2 \\ 5 & 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ *vi.* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 2 & -7 \\ 3 & 9 \end{bmatrix}$

4.0.2 Systems of Linear Algebraic Equations in Matrix Form

The system of two linear equations: $\begin{matrix} ax + by = r \\ cx + dy = s \end{matrix}$ represents a pair of lines in the xy-plane. A solution is a point (x, y) that lies on both lines. For a pair of lines in the plane, either

1. *They intersect at one point,*
2. *They are parallel and distinct so they don't intersect, or*
3. *They are parallel and identical so they intersect at all points in the line.*

You should be familiar with solving such a system by substitution. For most of our purposes that will be sufficient. For systems with more equations, it is easier and more reliable to use software. The system can be rewritten as: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$ because $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$.

If we let $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and write the system as $\boldsymbol{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$ to make it look easier, let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} r \\ s \end{bmatrix}$ so it can be written in the standard form:

$$\boldsymbol{A}\vec{x} = \vec{b}\,.$$

If $\boldsymbol{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we can multiply both sides by $\boldsymbol{A}^{-1}$ to get

$$\vec{x} = \mathbf{A}^{-1}\vec{b}$$

and the system is solved

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} dr - bs \\ -cr + as \end{bmatrix}.$$

Evidently, you need $det\mathbf{A} = ad - bc \neq 0$ to solve the system.

Theorem 0.1. For 2×2 linear algebraic systems $\boldsymbol{A}\vec{x} = \vec{b}$,

There is either a unique solution, no solution, or infinitely many solutions.

1. *Unique solution when* $det\boldsymbol{A} \neq 0$
2. *No solution when* $det\boldsymbol{A} = 0$ *for different parallel lines.*
3. *Infinitely many solutions when* $det\boldsymbol{A} = 0$ *for redundant equations.*

If $det\boldsymbol{A} = 0$ then we might have no solution or, sometimes, infinitely many solutions. If we are solving the homogenous system: $\boldsymbol{A}\vec{x} = \vec{0}$, then because $\vec{x} = \vec{0}$ is a solution, it can't have *no* solution, so for $\boldsymbol{A}\vec{x} = \vec{0}$ $det\boldsymbol{A} = 0$ implies infinitely many solutions. This is true more generally.

Theorem 0.1'. $det\boldsymbol{A} = 0 \;\Rightarrow\; \boldsymbol{A}\vec{x} = \vec{0}$ has infinitely many solutions.

Example 0.2.0. $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has **no** solution because $det\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = 0$ **and** the lines are different.

It's a non-homogeneous equation, the right hand side is not 0. But $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has infinitely many solutions $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ -2r \end{bmatrix}$ because the lines are the same. Notice every solution is some scalar r times $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (a solution that is easy to guess). □

Exercise 4.0.2

Show how to obtain the solution or show why no solution is possible.[4]

i. $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ *ii.* $\begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *iii.* $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

4.0.3 Independent Sets of Vectors

The vectors you called $\vec{\imath}, \vec{\jmath}, \vec{k}$ in Calculus are represented by the columns of the matrix $\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We could write the matrix $\boldsymbol{I}_3 = \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \end{bmatrix}$ but in matrix algebra we write $\boldsymbol{I}_3 = [\,\vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3]$ because we can have matrices of any size. The matrix $\boldsymbol{I}_3$ is the "3×3 identity".

Definition 0.6. Matrix $\boldsymbol{I}_n$ is the $n \times n$ identity. Its entries[5] are $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ so it has ones on the main diagonal and zeros elsewhere. The columns of $\boldsymbol{I}_n$ labeled $\vec{e}_1 \cdots \vec{e}_n$ are called the "standard basis" for the n-dimensional real *vector space* $\mathbb{R}^n$**.**

Whenever the product $\boldsymbol{A}\boldsymbol{I}_n$ is defined, $\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}$. Similarly, when $\boldsymbol{I}_n\boldsymbol{A}$ is defined, $\boldsymbol{I}_n\boldsymbol{A} = \boldsymbol{A}$.

Each point in three dimensional space has coordinates (x_1, x_2, x_3) giving the position of the point. The vector from the origin to a point (x_1, x_2, x_3) is the "position vector" $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Example 0.3.0. The position vector for location $(-1, 4)$ is $\vec{r} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ drawn in Figure 4-1. □

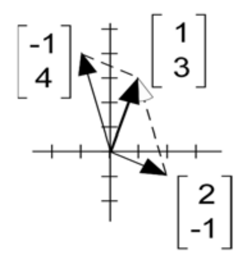

Figure 4-1. Vectors in $\mathbb{R}^2$ with parallelogram for addition.

Example 0.3.1. The vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ all lie in the plane $\mathbb{R}^2$. The sum is the diagonal of the parallelogram with edges along $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ as shown in Figure 4-1. In the figure, an edge of the parallelogram is shown as a "ghost" of $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. □

In matrix notation, position vectors in n-dimensional space are represented as n-vectors. Usually we think of $\mathbb{R}^n$ as the set of all n-dimensional vectors. Because we can add n-vectors and multiply them by scalars, we say $\mathbb{R}^n$ is a *vector space.*

Definition 0.7. For vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbb{R}^n$ and scalars $c_1, \dots, c_k$ the sum $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ is called a *linear combination* of the vectors. The set of all such combinations, Span$(\vec{v}_1, \dots, \vec{v}_k)$ is a vector space.

Example 0.3.2. The vector $3\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Using a matrix product we can write: $\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. □

[4] Solutions: *i.* $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 \\ -14 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$ *ii.* $\begin{bmatrix} x \\ y \end{bmatrix} = r\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *iii.* No solution.

[5] After Leopold Kronecker (1823-1891). The δ here is the discrete Kronecker δ, not to be confused with Dirac's.

If the matrix $\boldsymbol{V}$ has columns $\vec{v}_1, \dots, \vec{v}_k$ then using Def. 0.4', the matrix product $\boldsymbol{V}\vec{c} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$.

Definition 0.8. A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ are *linearly independent* if and only if $\boldsymbol{V}\vec{c} = \vec{0}$ has unique solution $\vec{c} = \vec{0}$.

Independence is a property of a *set* of vectors.

Example 0.3.3. For the vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, it seems pretty clear they are independent because they aren't parallel. The matrix of Example 0.3.2, $\boldsymbol{V} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$ has $det\boldsymbol{V} = 8 - 1 \neq 0$ so $\boldsymbol{V}\vec{c} = \vec{0}$ has **unique** solution, that's $\vec{c} = \vec{0}$, so the columns of $\boldsymbol{V}$ are an *independent set*. □

Two independent vectors in $\mathbb{R}^2$ are not parallel, similarly, three independent vectors in $\mathbb{R}^3$ do not lie in any plane. They point in "independent" directions. Two independent vectors in $\mathbb{R}^3$ determine a plane, an independent triple determines a volume, which in the case of $\mathbb{R}^3$ is the whole space. More generally, if $\vec{v}_1, \dots, \vec{v}_k$ are independent then they determine a k-dimensional space, that is, their linear combinations fill a k-dimensional space. Geometrically, they point in k independent directions and obviously, none of them can be $\vec{0}$, which points in no direction at all.

Example 0.3.4. In $\mathbb{R}^3$ the pair of vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are independent. Their linear combinations $c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ will fill out a plane through the origin. In Calculus, you found their *cross-product* $det\begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \vec{\imath} - \vec{\jmath} - \vec{k}$, which in our notation is $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, determining the equation of that plane to be $x - y - z = 0$. □

In Calculus you learned how to find the determinant of a 3×3 matrix. You can use that to see whether a set of three vectors from $\mathbb{R}^3$ are independent.

Example. 0.3.5. Is the set: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ independent? Clearly, they are pairwise independent because no two are parallel. To check that they are an independent *set* find

$$det\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 1det\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} - 0det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = -1 - 0 + 1 = 0\,.$$

They're not independent as a set, so, they don't point in three independent directions. They lie in a plane determined by the first two, because the third $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is a linear combination of the others. □

Exercise 4.0.3

Which of these sets of vectors are independent?[6]

i. $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$ *ii.* $\begin{bmatrix}2\\1\end{bmatrix}$ *iii.* $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}$ *iv.* $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}$ *v.* $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ *vi.* $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ *vii.* $\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}2\\6\\4\end{bmatrix}$

4.0.4 Eigenvalues and eigenvectors[7]

When a matrix multiplies a vector, it usually changes the vector's direction and length, so $\boldsymbol{A}\vec{v}$ is not the same as $\vec{v}$ (unless $\boldsymbol{A}$ is the identity). For every square matrix $\boldsymbol{A}$ there is at least one direction $\vec{v}$ for which $\boldsymbol{A}\vec{v} = \lambda\vec{v}$ where λ is a scalar. We call that λ an *eigenvalue* for $\boldsymbol{A}$, and every scalar multiple of the vector $\vec{v}$ is an *eigenvector* associated with the eigenvalue λ.

Definition 0.9. For an $n \times n$ matrix $\boldsymbol{A}$, if a vector $\vec{v} \neq \vec{0}$ satisfies $\boldsymbol{A}\vec{v} = \lambda\vec{v}$ for a scalar λ, then λ is an *eigenvalue* and $\vec{v}$ is an *eigenvector* for $\boldsymbol{A}$ associated with λ. We say $(\lambda, \vec{v})$ is an *eigenpair* for $\boldsymbol{A}$.

If $(\lambda, \vec{v})$ is an eigenpair for $\boldsymbol{A}$ then $\boldsymbol{A}\vec{v} - \lambda\,\vec{v} = \vec{0}$ and, since $\vec{v} = \boldsymbol{I}_n\vec{v}$, we can factor out $\vec{v}$:

(0.0) $$(\boldsymbol{A} - \lambda\boldsymbol{I}_n)\vec{v} = \vec{0}.$$

Equation (0.0) is the *eigenvector equation.*[8] By Definition 0.9, $\vec{v} \neq \vec{0}$, so, by Theorem 0.1' the eigenvector equation has infinitely many solutions and $det(\boldsymbol{A} - \lambda\boldsymbol{I}_n) = 0$. You can use that to find the eigenvalues.

Theorem 0.2. The eigenvalues λ of $\mathbf{A}$ are the solutions to the n^{th} degree polynomial equation

(0.1) $$det(\boldsymbol{A} - \lambda\boldsymbol{I}_n) = 0.$$

called the *characteristic equation* for matrix $\boldsymbol{A}$.

Example 0.4.0. Let's find the eigenvalues for $\boldsymbol{A} = \begin{bmatrix}3 & 4\\4 & -3\end{bmatrix}$.

The characteristic equation is

$$det(\boldsymbol{A} - \lambda\boldsymbol{I}) = det\begin{bmatrix}3-\lambda & 4\\4 & -3-\lambda\end{bmatrix} = \lambda^2 - 25 = 0 \;\Rightarrow\; \lambda = \pm 5\,.$$

Remark 0.2. For a 2×2 matrix $\boldsymbol{A}$, the characteristic equation is quadratic. It can have distinct real, complex conjugate, or one (double) real solution. In Chapter 2, we read the characteristic equation directly off the coefficients. We can do nearly the same with 2×2 linear systems. It is easy to see[9] that

$$det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \lambda^2 - tr\boldsymbol{A}\,\lambda + det\boldsymbol{A}\,,\ \text{with}\ \ tr\boldsymbol{A} = a_{11} + a_{22}.$$

When eigenvalues λ are known, eigenvectors are found by solving the eigenvector equation:

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{v} = \vec{0}\,.$$

Example 0.4.0'. The eigenvalues for $\boldsymbol{A} = \begin{bmatrix}3 & 4\\4 & -3\end{bmatrix}$ **are** $\lambda = \pm 5$. The eigenvectors for $\lambda = 5$ satisfy:

$$\left(\begin{bmatrix}3 & 4\\4 & -3\end{bmatrix} - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right)\vec{v} = \begin{bmatrix}-2 & 4\\4 & -8\end{bmatrix}\vec{v} = \vec{0}\,.$$

---

[6] Solutions: *i. Indep. ii. Indep. iii. Dependent iv. Dependent v. Indep vi. Indep. Vii. Dependent.*

[7] "Eigen" is German for "own" so eigenvalues of **A** are "**A**'s own values" also called "**A**'s characteristic" values.

[8] The matrix $\mathbf{A} - \lambda\,\mathbf{I}_n$ is just matrix $\mathbf{A}$ with λ subtracted from the entries on its main diagonal. See Ex. 0.4.0'.

[9] Exercise 4.0.4

Without any effort, it is clear that $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a non-zero solution. Because it is a homogeneous equation, every scalar multiple $r\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a solution. We could choose any *non-zero* r and get an eigenvector for $\lambda = 5$. We just choose $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ because it's easy.

For $\lambda = -5$, we get $\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\vec{v} = \vec{0}$ for the eigenvector equation. This time $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an obvious solution.[10] The eigen-pairs are $\left\{5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ and $\left\{-5, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$. Other choices are possible, but none can have $\vec{v} = \vec{0}$, because eigenvectors can't be zero, they need to point in an eigen-direction. □

Example 0.4.1. Find the eigenvalues and eigenvectors for $\boldsymbol{A} = \begin{bmatrix} 8 & 5 \\ -4 & -1 \end{bmatrix}$.

i. Solve the char. eq'n:[11] $det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \lambda^2 - tr\boldsymbol{A}\lambda + det\boldsymbol{A}$

$$= \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) \;\Rightarrow\; \lambda = 3, 4\,.$$

$$ii.\;\; \lambda = 3 \;\Rightarrow\; (\boldsymbol{A} - \lambda\boldsymbol{I})\vec{v} = \begin{bmatrix} 8-\lambda & 5 \\ -4 & -1-\lambda \end{bmatrix}\vec{v} = \begin{bmatrix} 8-3 & 5 \\ -4 & -1-3 \end{bmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 5 & 5 \\ -4 & -4 \end{bmatrix}\vec{v} = \vec{0} \quad \text{observe } \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ works.}$$

$$iii.\;\; \lambda = 4 \;\Rightarrow\; (\boldsymbol{A} - \lambda\boldsymbol{I})\vec{v} = \begin{bmatrix} 8-4 & 5 \\ -4 & -1-4 \end{bmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 4 & 5 \\ -4 & -5 \end{bmatrix}\vec{v} = \vec{0} \quad \text{observe } \vec{v} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \text{ works.}$$

$$\text{The eigen-pairs are: } (\lambda, \vec{v}) = 3, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } 4, \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

Because for 2×2 systems, we can always solve the quadratic characteristic equation, we will consider the Linear Algebra Preliminaries complete and fill in further details as needed.

Exercise 4.04

Find eigenvalues and eigenvectors for $\boldsymbol{A}$.

i. $\begin{bmatrix} 3 & 5 \\ 5 & 0 \end{bmatrix}$ *ii.* $\begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$ *iii.* $\begin{bmatrix} 0 & -1 \\ 6 & -5 \end{bmatrix}$ *iv.* $\begin{bmatrix} 3 & -3 \\ -5 & 5 \end{bmatrix}$ *v.* $\begin{bmatrix} 2 & 0 \\ 3 & -5 \end{bmatrix}$ *vi.* $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ *vii.* $\begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$ *viii.* $\begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix}$

4.1 Autonomous First Order Linear Systems of ODEs

A single first order ODE looks like $y' = f(t, y)$. A pair of such equations defines an uncoupled system:

$$\begin{aligned} x' &= f_1(t, x) \\ y' &= f_2(t, y) \end{aligned}$$

A solution to such an uncoupled system is a pair of functions $(x(t), y(t))$ giving the path of a point moving in the xy-plane. A triple of such equations describes the motion of a point in 3-dimensional space. Uncoupled systems are solved by the usual methods (separation, variation of parameters) for first order

[10] Because the $8(1) + 4(-2) = 0$ and the equations $8v_1 + 4v_2 = 0$ and $4v_1 + 2v_2 = 0$ give the same line.
[11] Using Remark 0.2.

equations. However, in a coupled system, the solution to one of the equations depends on the solution of the other. A coupled system

$$\begin{aligned} x' &= f_1(t,x,y) \\ y' &= f_2(t,x,y) \end{aligned}$$

requires other methods. We will begin with the simplest such systems.

Autonomous first order systems

An ODE with no explicit time dependence is called *autonomous* and defines a *vector field*. The system:

$$\begin{aligned} x' &= f_1(x,y) \\ y' &= f_2(x,y) \end{aligned}$$

defines a phase velocity vector at every point (x,y) where both $f_1(x,y)$ and $f_2(x,y)$ are defined. The phase velocity vector is $\vec{F}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ and appears on the right side of the vector form for the system:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{F}(x,y)$$

Since $\begin{bmatrix} x \\ y \end{bmatrix}$ is a position in the xy-plane, its time derivative is the velocity. In the autonomous system 1.0 we imagine a thin fluid layer on the xy-plane in which the molecules flow along paths called streamlines that remain fixed in time. The phase velocity vector field $\vec{F}(x,y)$ gives the velocity of the flow at every point in the plane and so the vector $\vec{F}(x,y)$ is tangent to the streamline of the flow at the point (x,y).

Example 1.0. $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ is a simple autonomous first order system. It is *linear* because both coordinates of the vector field, $f_1(x,y) = -y$ and $f_2(x,y) = x$, are linear functions of the variables x, y. We can use MatLab's quiver to plot the vector field near the origin as in Figure 4-2A.

```
>> [x,y]=meshgrid(-2:0.2:2,-2:0.2:2);
>> xdot=-y; ydot= x;
>> quiver(x,y,xdot,ydot)
```

From the vector field, you can guess the solutions flow counter-clockwise around the origin. Solutions may lie on circles or spirals. MatLab can't tell which, because it uses numerical approximations to plot streamlines. An approximate streamline will miss its connection without closing a circle.

```
>> [x,y]=meshgrid(-2:0.2:2,-2:0.2:2);
>> xdot=-y; ydot= x;
>> [ix,iy]=meshgrid(1,0);
>> streamline(x,y,xdot,ydot,ix(:),iy(:));
```

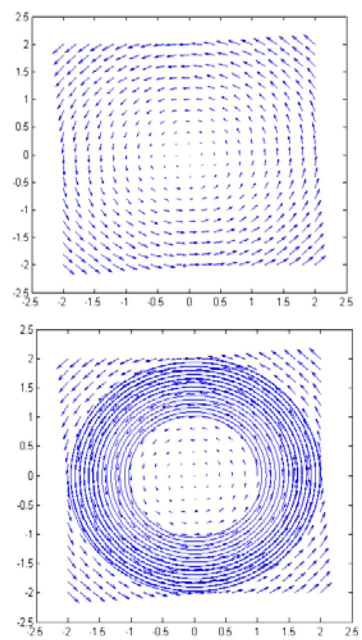

Figure 4-2. A.The vector field plotted by quiver. B. Spiraling approximate streamline.

The streamline plotted in Figure 4-2B appears to be a slowly expanding spiral. The true phase trajectories are circular orbits. Even very small errors will cause a circular streamline to spiral. □

Linear Autonomous First Order Systems

The system of Example 1.0 is linear and, as such, can be written in matrix-vector form:

(1.0) $$\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}.$$

Example 1.0′. The system $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ has matrix $\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We write: $\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$, where the vector $\vec{x}$ takes the place of $\begin{bmatrix} x \\ y \end{bmatrix}$. □

Equation (1.0) has solution $\vec{x} = e^{t\boldsymbol{A}}\vec{x}_0$, where we understand that $e^{t\boldsymbol{A}}$ is a square matrix and $\vec{x}_0 = \vec{x}(0)$. When $t = 0$, the matrix $e^{0\boldsymbol{A}} = e^{\boldsymbol{0}} = \boldsymbol{I}$.

Definition 1.0. For an $n \times n$ matrix $\boldsymbol{A}$, the matrix

$$e^{\boldsymbol{A}t} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} t^k = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2}{2} t^2 + \frac{\boldsymbol{A}^3}{3!} t^3 + \cdots$$

When we take the derivative of this matrix-valued series, understanding that the derivative of a term just takes t^k to kt^{k-1}, so the extra factor of k cancels k out of the factorial in the denominator, we end up with

$$\frac{d}{dt} e^{\boldsymbol{A}t} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^{k+1}}{k!} t^k = \boldsymbol{A}e^{\boldsymbol{A}t} .$$

The matrix $e^{\boldsymbol{A}t}$ is a Fundamental Solution Matrix (FSM) for the system. For an IC $\vec{x}(0) = \vec{x}_0$, the streamline of the IC is traced out by the solution $\vec{x}(t) = e^{\boldsymbol{A}t}\vec{x}_0$ passing through it. For every autonomous linear system, the origin $\vec{0}$ is an equilibrium. The solutions are trajectories of initial conditions in the *phase space* $\mathbb{R}^n$. The trajectory of the origin $\vec{0}$ is just the point $\vec{0}$. The vector field $\boldsymbol{A}\vec{x}$ assigns a *phase velocity vector* to every point $\vec{x}$ in phase space. The phase velocity points in the direction of the *phase flow* and the matrix $e^{\boldsymbol{A}t}$ is a one-parameter family of linear mappings $e^{\boldsymbol{A}t}\colon \mathbb{R}^n \to \mathbb{R}^n$ that maps the entire vector space to the entire vector space as it would appear at time t. We call $e^{\boldsymbol{A}t}$ the "time= t phase flow map". Every point $\vec{x}_0$ in phase space flows along its streamline to reach $e^{\boldsymbol{A}t}\vec{x}_0$ at time= t. We can use MatLab to plot streamlines of the phase flow for 2×2 systems as done in Example 1.0.

In the next three sections, we will see how to find a closed form expression for the flow matrix $e^{\boldsymbol{A}t}$ in the three cases: distinct real, complex conjugate, and equal real eigenvalues of the matrix $\boldsymbol{A}$ using the *eigenvalue method*.

Exercises 4.1

1. Write the system in the matrix-vector form (1.0).

   i. $\begin{aligned} x' &= 3x + 5y \\ y' &= 5x \end{aligned}$ *ii.* $\begin{aligned} x' &= 3x + 5y \\ y' &= -y \end{aligned}$ *iii.* $\begin{aligned} x' &= 3x - 3y \\ y' &= -5x + 5y \end{aligned}$ *iv.* $\begin{aligned} x' &= -2y \\ y' &= 3x \end{aligned}$

2. Use MatLab to plot the vector field and a few streamlines of the flow for the systems in Problem 1.

4.2 Distinct Real Eigenvalues

We solve linear autonomous systems using the *eigenvalue method.* As with linear second order constant coefficient equations, the solutions of the characteristic equation determine the form of the general solution. The idea is simple. A general solution to the $n \times n$ autonomous linear system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$, whose matrix $\boldsymbol{A}$ has real distinct eigenvalues λ_k , $k = 1 \ldots n$ and associated eigenvectors $\vec{v}_k$, is given by

(2.0) $$\vec{x}(t) = \sum_{k=1}^{n} c_k \vec{v}_k e^{\lambda_k t}$$

where the coefficients c_k are arbitrary until determined by initial conditions. In other words, there are n independent solutions $\vec{x}_k(t) = \vec{v}_k e^{\lambda_k t}$ associated with the eigenvalues.[12]

Example 2.0. The system $\frac{d}{dt}\vec{x} = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}\vec{x}$ has the upper triangular matrix of Exercise 4.0.4 ii.

Observe that $\lambda = 3, -1$ easily result.[13] For their eigenvectors:

$$\lambda = 3 \quad \Rightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} 0 & 5 \\ 0 & -4 \end{bmatrix}\vec{v} = \vec{0} \ \Rightarrow \ \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ works.}$$

$$\lambda = -1 \quad \Rightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \quad \Rightarrow \vec{v} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \text{ works.}$$

Eigen-pairs: $\lambda, \vec{v} = 3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $-1, \begin{bmatrix} 5 \\ -4 \end{bmatrix}$.

The solution (2.0) is $$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 5 \\ -4 \end{bmatrix} e^{-t}$$ □

Example 2.1. The system $\frac{d}{dt}\vec{x} = \begin{bmatrix} 8 & 5 \\ -4 & -1 \end{bmatrix}\vec{x} \quad \Rightarrow \quad det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 7\lambda + 12 = 0$

$$\lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) \ \Rightarrow \ \lambda = 3, 4 .$$

$$\lambda = 3 \ \Rightarrow \ (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} 8-\lambda & 5 \\ -4 & -1-\lambda \end{bmatrix}\vec{v} = \begin{bmatrix} 8-3 & 5 \\ -4 & -1-3 \end{bmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 5 & 5 \\ -4 & -4 \end{bmatrix}\vec{v} = \vec{0} \ \Rightarrow \ \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ works.}$$

$$\lambda = 4 \ \Rightarrow \ (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} 8-4 & 5 \\ -4 & -1-4 \end{bmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 4 & 5 \\ -4 & -5 \end{bmatrix}\vec{v} = \vec{0} \ \Rightarrow \ \vec{v} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \text{works.}$$

Eigen-pairs: $\lambda, \vec{v} = 3, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $4, \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Solution: $\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 5 \\ -4 \end{bmatrix} e^{4t}$. □

---

[12] By a standard theorem in Linear Algebra, eigenvectors for distinct real eigenvalues are independent. Consequently, every possible initial position, $x_0 = \sum_{k=1}^{n} c_k \vec{v}_k$ is a linear combination of the eigenvectors, so the solution (2.0) solves the IVP with IC $\vec{x}(0) = \vec{x}_0 \epsilon \mathbb{R}^n$ for a unique choice of constants c_k , $k = 1 \ldots n$.

[13] An upper (or lower) triangular matrix has only zeros below (or above) its main diagonal, in which case the entries on the main diagonal are its eigenvalues, so there's no need to solve the characteristic equation.

Example 2.2. The system $\frac{d}{dt}\vec{x} = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\vec{x} \;\Rightarrow det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 4\lambda = 0 \quad \Rightarrow \lambda = 0, -4\,.$

$$\lambda = 0 \;\Rightarrow\; (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \quad \vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ works}$$

$$\lambda = -4 \;\Rightarrow\; (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} 2+4 & 3 \\ -4 & -6+4 \end{bmatrix}\vec{v} = \begin{bmatrix} 6 & 3 \\ -4 & -2 \end{bmatrix}\vec{v} = \vec{0} \;\Rightarrow\; \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ works.}$$

Eigen-pairs: $\lambda, \vec{v} = 0, \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $-4, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Solution: $\vec{x} = c_1\begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-4t}$. □

Example 2.3. $\frac{d}{dt}\vec{x} = \begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}\vec{x} \;\Rightarrow det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 4\lambda - 8 = 0 \;\Rightarrow\; \lambda = 2 \pm 2\sqrt{3}\,.$

$$\lambda = 2 \pm 2\sqrt{3} \;\Rightarrow\; (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} \mp 2\sqrt{3} & -3 \\ -4 & \mp 2\sqrt{3} \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \quad \vec{v} = \begin{bmatrix} 3 \\ \mp 2\sqrt{3} \end{bmatrix} \text{ works.}$$

$$\text{Eigen-pairs: } \lambda, \vec{v} = 2 \pm 2\sqrt{3}, \begin{bmatrix} 3 \\ \mp 2\sqrt{3} \end{bmatrix}.$$

Solution: $\vec{x} = c_1\begin{bmatrix} 3 \\ -2\sqrt{3} \end{bmatrix}e^{(2+2\sqrt{3})t} + c_2\begin{bmatrix} 3 \\ 2\sqrt{3} \end{bmatrix}e^{(2-2\sqrt{3})t}$. □

Exercises 4.2

1. Give a general solution to the system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for the given matrix $\boldsymbol{A}$.
   i. $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$ *ii.* $\begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix}$ *iii.* $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ *iv.* $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ *v.* $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$
2. Give a general solution to the system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for the given matrix $\boldsymbol{A}$.
   i. $\begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix}$ *ii.* $\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}$ *iii.* $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ *iv.* $\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$ *v.* $\begin{bmatrix} -2 & 5 \\ 4 & 10 \end{bmatrix}$
3. Give a general solution to the system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for the given matrix $\boldsymbol{A}$.
   i. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *ii.* $\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ *iii.* $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ *iv.* $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$
4. Give a general solution to the system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for the given matrix $\boldsymbol{A}$.
   i. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ *ii.* $\begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}$ *iii.* $\begin{bmatrix} 4 & -3 \\ 2 & -2 \end{bmatrix}$ *iv.* $\begin{bmatrix} 5 & 4 \\ 4 & 1 \end{bmatrix}$
5. Use MatLab to plot the vector field and a few streamlines illustrating the flow for the systems in Problem 1.
6. Use MatLab to plot the vector field and a few streamlines illustrating the flow for the systems in Problem 2.
7. Use MatLab to plot the vector field and a few streamlines illustrating the flow for the systems in Problem 3.
8. Use MatLab to plot the vector field and a few streamlines illustrating the flow for the systems in Problem 4.

4.3 Complex Conjugate Eigenvalues

Based on our experience in 4.2, we expect a general solution for a 2×2 system to look like

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

But when the eigenvalues are complex, say, $\lambda = -p \pm i\omega$ the exponentials become $e^{\lambda t} = e^{-pt}(\cos\omega t \pm i\sin\omega t)$. Just as we did when we encountered complex characteristic roots for 2nd order equations, we will again find the two independent real solutions we need are the real and imaginary parts of just one complex solution $\vec{x}_C(t) = \vec{v}e^{-pt}(\cos\omega t + i\sin\omega t)$. The eigenvector $\vec{v}$ is complex too, so we need to multiply the complex $\vec{v}$ with the complex scalar $(\cos\omega t + i\sin\omega t)$ to separate its real and imaginary parts.

Example 3.0. Consider $\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$ where $det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 1 \Rightarrow \lambda = \pm i$ so $\omega = 1$ and the complex solution we need is $\vec{x}_C(t) = \vec{v}(\cos t + i\sin t)$. We discard the other eigenvalue.

The eigenvector $\vec{v}$ is complex: $\lambda = i \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ works.

The complex solution whose real and imaginary parts we need is:

$$\vec{x}_C(t) = \begin{bmatrix} i \\ 1 \end{bmatrix}(\cos t + i\sin t)\,.$$

Multiply the complex scalar into the complex vector then separate real and imaginary parts:

$$\vec{x}_C(t) = \begin{bmatrix} i\cos t - \sin t \\ \cos t + i\sin t \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

The pair of independent real solutions are: $x_1(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$ & $x_2(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$.

The *real* general solution is: $\vec{x}(t) = c_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, which can be written as

$$\vec{x}(t) = \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ □

Example 3.1. Consider $\frac{d}{dt}\vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\vec{x}$ where $det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 2\lambda + 2 \Rightarrow \lambda = 1 \pm i$.

The complex solution we need is $\vec{x}_C(t) = \vec{v}e^{-t}(\cos t + i\sin t)$.

$\lambda = 1 + i \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ works.

The complex solution whose real and imaginary parts we need is:

$$\vec{x}_C(t) = \begin{bmatrix} i \\ 1 \end{bmatrix}e^{t}(\cos t + i\sin t)\,.$$

Multiply the complex scalar into the complex vector then separate real and imaginary parts:

$$\vec{x}_C(t) = \left(\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}\right)e^{t}$$

The independent real solutions are: $x_1(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}e^{t}$, $x_2(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}e^{t}$.

The *real* general solution is:

$$\vec{x}(t) = c_1\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}e^{t} + c_2\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}e^{t}\,.$$ □

In general, cubic and higher degree characteristic polynomials are prohibitively difficult to solve without special software, although some systems with appropriate symmetries can be easily solved explicitly.

Example 3.2. Consider $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ with $\boldsymbol{A} = \begin{bmatrix} -1 & 0 & -4 \\ 0 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix}$

$$det(\boldsymbol{A} - \lambda\boldsymbol{I}) = det\begin{bmatrix} -1-\lambda & 0 & -4 \\ 0 & -1-\lambda & 0 \\ 4 & 0 & -1-\lambda \end{bmatrix} = (-1-\lambda)det\begin{bmatrix} -1-\lambda & -4 \\ 4 & -1-\lambda \end{bmatrix}$$

$$= (-1-\lambda)((\lambda+1)^2+16) \Rightarrow \lambda = -1, -1 \pm 4i.$$

$$\lambda = -1 \Rightarrow \begin{bmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}\vec{v} = 0 \Rightarrow \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ works. } \vec{x}_1(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{-t}$$

$$\lambda = -1+4i \Rightarrow \begin{bmatrix} -4i & 0 & -4 \\ 0 & -4i & 0 \\ 4 & 0 & -4i \end{bmatrix}\vec{v} = 0 \Rightarrow \vec{v} = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} \text{ works.}$$

$$\vec{x}_C(t) = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}e^{-t}(\cos 4t + i\sin 4t) = \begin{bmatrix} i\cos 4t - \sin 4t \\ 0 \\ \cos 4t + i\sin 4t \end{bmatrix}e^{-t}$$

$$\Rightarrow \vec{x}_2(t) = \begin{bmatrix} -\sin 4t \\ 0 \\ \cos 4t \end{bmatrix}e^{-t},\ \vec{x}_3(t) = \begin{bmatrix} \cos 4t \\ 0 \\ \sin 4t \end{bmatrix}e^{-t} \text{ are real solutions.}$$

A general solution is given by $\vec{x}(t) = c_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} -\sin 4t \\ 0 \\ \cos 4t \end{bmatrix}e^{-t} + c_3\begin{bmatrix} \cos 4t \\ 0 \\ \sin 4t \end{bmatrix}e^{-t}$.

Reordering solutions to exhibit the same symmetry as the original matrix:

$$\vec{x}(t) = c_1\begin{bmatrix} \cos 4t \\ 0 \\ \sin 4t \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} -\sin 4t \\ 0 \\ \cos 4t \end{bmatrix}e^{-t}.$$

Writing the solution in more compact form as done in Example 3.0:

$$\vec{x}(t) = \begin{bmatrix} e^{-t}\cos 4t & 0 & -e^{-t}\sin 4t \\ 0 & e^{-t} & 0 \\ e^{-t}\sin 4t & 0 & e^{-t}\cos 4t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Setting $t = 0 \Rightarrow \vec{x}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\Rightarrow \vec{x}(t) = \begin{bmatrix} e^{-t}\cos 4t & 0 & -e^{-t}\sin 4t \\ 0 & e^{-t} & 0 \\ e^{-t}\sin 4t & 0 & e^{-t}\cos 4t \end{bmatrix}\vec{x}(0)$$

$$\Rightarrow \begin{bmatrix} e^{-t}\cos 4t & 0 & -e^{-t}\sin 4t \\ 0 & e^{-t} & 0 \\ e^{-t}\sin 4t & 0 & e^{-t}\cos 4t \end{bmatrix} = e^{t\boldsymbol{A}}, \text{ is the \underline{FSM} for the system with } \boldsymbol{A} = \begin{bmatrix} -1 & 0 & -4 \\ 0 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix}. \quad \square$$

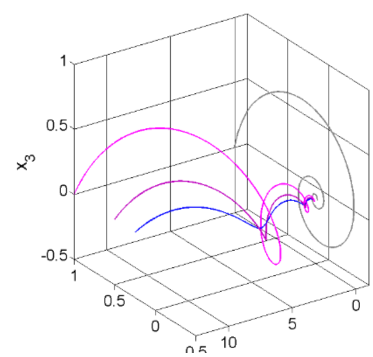

Figure 4-3. Phase trajectories for Example 3.2.

The phase curves pictured in Figure 4-3 have these ICs:

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 4\pi \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 4\pi \\ 0 \end{bmatrix}, \begin{bmatrix} 0.25 \\ 4\pi \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

MatLab commands that produced the plot in Figure 4-3 are shown below.[14]

```
>> plot3(exp(-t).*cos(4*t), 4*pi*exp(-t), exp(-t).*sin(4*t),'LineWidth',2)
>> hold on
>> grid on
>> plot3(0.5*exp(-t).*cos(4*t), 4*pi*exp(-t), 0.5*exp(-t).*sin(4*t),'LineWidth',2)
>> plot3(0.25*exp(-t).*cos(4*t), 4*pi*exp(-t), 0.25*exp(-t).*sin(4*t),'LineWidth',2)
>> plot3(exp(-t).*cos(4*t), 0*exp(-t), exp(-t).*sin(4*t),'LineWidth',2)
```

Exercises 4.3

1. Give a general solution to the system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for the given matrix $\boldsymbol{A}$.

   i. $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ *ii.* $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ *iii.* $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ *iv.* $\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$ *v.* $\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$ *vi.* $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ *vii.* $\begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$
2. Use MatLab to plot the vector field and a few streamlines illustrating the flow for the systems in Problem 1.
3. Give a general solution to the system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for the given matrix $\boldsymbol{A}$.

   i. $\begin{bmatrix} 0 & -4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ *ii.* $\begin{bmatrix} -1 & -4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ *iii.* $\begin{bmatrix} 0 & -8 & 0 \\ 8 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ *iv.* $\begin{bmatrix} -1 & -8 & 0 \\ 8 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ *v.* $\begin{bmatrix} -1 & 0 & -8 \\ 0 & -1 & 0 \\ -8 & 0 & -1 \end{bmatrix}$
4. Use MatLab *plot3* to plot the vector field and a few streamlines illustrating the flow for the systems in Problem 3. Use the *Camera palette* to choose a revealing camera angle for your plot.

4.4 Multiple Eigenvalues and Defect

When a 2×2 system has only one eigenvalue, it is a root of algebraic multiplicity two for the characteristic equation. This means the characteristic equation is $(\lambda - r)^2 = 0$ with $r = \frac{tr\boldsymbol{A}}{2}$. In such a case, sometimes there is only one eigen-direction that works for λ so we get only one independent solution for λ.

Example 4.0. The system $\frac{d}{dt}\vec{x} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\vec{x} \Rightarrow \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$ has algebraic multiplicity 2.

$(\boldsymbol{A} - 3\boldsymbol{I})\vec{v} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ gives one eigenvector but there is <u>not</u> a second independent eigenvector. □

Sometimes there are two independent eigenvectors for a λ of algebraic multiplicity 2.

Example 4.1. The system $\frac{d}{dt}\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\vec{x} \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$ has algebraic multiplicity 2.

$(\boldsymbol{A} - 2\boldsymbol{I})\vec{v} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow$ every $\vec{v} \neq \vec{0}$ is an eigenvector. We can take independent eigenpairs $2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $2, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ giving general solution $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} = \vec{x}(0)e^{2t}$. □

[14] Colors, fonts, labels, and GridLineType were modified in the More Properties Inspector.

Definition 4.0. A real eigenvalue λ that has multiplicity k as a root of the characteristic equation for $\boldsymbol{A}$ is said to have *algebraic multiplicity* k and we write $Algmult(\lambda) = k.$

If h is the number of independent eigen-directions for λ , we say λ has *geometric multiplicity* h and we write $Geomult(\lambda) = h$, in which case there is a set of independent eigenvectors $\vec{v}_1, \dots, \vec{v}_h$ for λ . The vector space $V(\lambda) = Span(\vec{v}_1, \dots, \vec{v}_h)$ is the *eigen-space* for λ and the eigenvectors $\vec{v}_1, \dots, \vec{v}_h$ form a *basis* for $V(\lambda)$, which has dimension h. Every vector in $V(\lambda)$ is an eigenvector for λ.

It is a standard theorem of Linear Algebra that $Geomult(\lambda) \le Algmult(\lambda)$. Example 4.0 is a case where $Geomult(\lambda) < Algmult(\lambda)$ and Example 4.1 has $Geomult(\lambda) = Algmult(\lambda)$. In Example 4.0 the eigenvalue is said to be *defective* because it is missing an eigenvector.

The system in Example 4.1 is uncoupled. This is the only way a 2×2 system with an eigenvalue λ whose $Algmult(\lambda) = 2$ can have $Geomult(\lambda) = Algmult(\lambda)$.

Theorem 4.0. Suppose a 2×2 matrix $\boldsymbol{A}$ has an eigenvalue λ with $Algmult(\lambda) = 2$ then

$$Geomult(\lambda) = 2 \Leftrightarrow \boldsymbol{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

Proof. ($\Leftarrow$) If $\boldsymbol{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ then $(\mathbf{A} - 2\mathbf{I})\vec{v} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow$ every $\vec{v} \neq \vec{0}$ is an eigenvector, in particular, $\vec{e}_1, \vec{e}_2$ are eigenvectors for λ so $Geomult(\lambda) = 2$.

($\Rightarrow$) Suppose $Geomult(\lambda) = 2$ so there are $\vec{v}_1, \vec{v}_2$ independent eigenvectors for λ. Because $\vec{v}_1, \vec{v}_2$ are independent, the eigenspace for $(\lambda) = Span(\vec{v}_1, \vec{v}_2) = \mathbb{R}^2$, so every vector in $\mathbb{R}^2$ is an eigenvector for λ. In particular, $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{e}_1 = \vec{0}$ and $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{e}_2 = \vec{0}$ imply that the first and second columns of $(\boldsymbol{A} - \lambda\boldsymbol{I})$ are both $\vec{0}$, so $(\boldsymbol{A} - \lambda\boldsymbol{I}) = \mathbf{0}$ the 2×2 zero matrix.

Adding $\lambda\boldsymbol{I}$ to both sides gives $\boldsymbol{A} = \lambda\boldsymbol{I} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$. □

Example 4.2. Any matrix $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ with $b \neq 0$, has $\lambda = a, a$, defective by Theorem 4.0 because $Algmult(a) = 2$ but the matrix is not diagonal. This is true for any non-diagonal 2×2 when it has an eigenvalue λ with $Algmult(\lambda) = 2$. □

Definition 4.1. An eigenvalue with $Geomult(\lambda) < Algmult(\lambda)$ is said to be *defective* with

$$Defect(\lambda) = Algmult(\lambda) - Geomult(\lambda) > 0.$$

For an $n \times n$ system, when $Defect(\lambda) > 0$, the eigenvalue method supplies too few independent solutions to satisfy every IC in $\mathbb{R}^n$. For 2×2 systems, a defective eigenvalue gives only one independent solution $\vec{x}_1(t) = \vec{v}e^{\lambda t}$. We need to find a second independent solution using a *generalized eigenvector* associated with the eigenpair $\lambda, \vec{v}$. There are always infinitely many generalized eigenvectors, so they are easy to find.

Definition 4.2. A *rank 2 generalized eigenvector*[15] $\vec{\eta}$ for a defective eigenpair $\lambda, \vec{v}$ is a solution to

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta} = \vec{v}\ .$$

Example 4.3. The system $\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\vec{x} \Rightarrow \lambda = -1\ , -1$, is defective by Theorem 4.0.[16]

[15] A rank 1 generalized eigenvector for the pair $\lambda, \vec{v}$ is just the regular eigenvector $\vec{v}$. See Definition 4.3, below.

[16] This is also apparent by just solving the eigenvector equation.

$\lambda = -1 \;\Rightarrow\; (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \;\Rightarrow\; \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ works but no other independent eigenvector exists. This gives only one solution: $\vec{x}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$. To get another independent solution, we need to find a rank 2 generalized eigenvector $\vec{\eta}$.

Solving $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta} = \vec{v} \;\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow\; \vec{\eta} = \begin{bmatrix} \eta_1 \\ 1 \end{bmatrix}$ works for any η_1. We can take $\eta_1 = 0$, if we want.

It's easy to verify that a second solution is $\vec{x}_2(t) = (\vec{\eta} + t\vec{v}\,)e^{\lambda t} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)e^{-t} = \begin{bmatrix} t \\ 1 \end{bmatrix} e^{-t}$.

A general solution is then: $\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2\begin{bmatrix} t \\ 1 \end{bmatrix} e^{-t}$. □

Theorem 4.1. For an $n \times n$ system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ with an eigenvalue λ of $Defect(\lambda) > 0$, a second independent solution is given by:

$$\vec{x}_2(t) = (\vec{\eta} + t\vec{v}\,)e^{\lambda t},$$

where $\lambda, \vec{v}$ is an eigenpair and $\vec{\eta}$ a rank 2 generalized eigenvector for the pair.

Proof. Substitute $\vec{x}_2(t)$ into the equation. □

For 2×2 systems, any eigenpair $\lambda, \vec{v}$ will yield a generalized eigenvector $\vec{\eta}$ when $Defect(\lambda) = 1$ but in larger systems, when $Geomult(\lambda) > 2$, there may be several independent eigenvectors, say $\vec{v}_1, \vec{v}_2$ for a defective λ . Not every eigenvector for such a λ will have a generalized eigenvector associated with it.

Example 4.4. Consider $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ with $\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \lambda = -1, -1, -1$. $Algmult\,(-1) = 3$.

$(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{v} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \;\Rightarrow\; \vec{v}_1 = \vec{e}_1,\; \vec{v}_2 = \vec{e}_2$ work, giving independent solutions:

$$\vec{x}_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}, \vec{x}_2(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}.$$

Since there is not a third independent eigenvector, $Geomult\,(-1) = 2$, so $Defect(\lambda) = 1 \Rightarrow$ we need a generalized eigenvector to get a third independent solution. Solve $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta} = \vec{v}_1 = \vec{e}_1$.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{\eta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\Rightarrow\; \vec{\eta} = \vec{e}_3 \text{ works.}$$

Observe that the pair $\lambda, \vec{v}_2$ has no generalized eigenvector, because $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta} = \vec{v}_2 = \vec{e}_2$,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{\eta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{has no solutions.}$$

By Theorem 4.1, the third independent solution is $\vec{x}_3(t) = (\vec{\eta} + t\vec{v}_1)e^{\lambda t} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)e^{-t}$

and a general solution is given by

$$\vec{x}(t) = c_1\vec{v}_1 e^{-t} + c_2\vec{v}_2 e^{-t} + c_3(\vec{\eta} + t\vec{v}_1)e^{-t} = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{-t} = \begin{bmatrix} e^{-t} & 0 & te^{-t} \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}\vec{x}(0). \quad □$$

The complete method of generalized eigenvectors, detailed elsewhere,[17] leads to the complete solution of all autonomous linear systems including defects by producing a full set of independent solutions associated with chains of generalized eigenvectors of ranks $r \geq 1$ as needed. We provide only a brief introduction here.

Chains of Generalized Eigenvectors

Definition 4.3. A *rank r generalized eigenvector* $\vec{\eta}$ for a defective eigenpair $\lambda, \vec{v}$ is a solution to

(4.1) $$(\boldsymbol{A} - \lambda\boldsymbol{I})^{r-1}\vec{\eta} = \vec{v}\,.$$

In particular, $\vec{v}$ is a rank 1 generalized eigenvector for its own eigenpair $\lambda, \vec{v}$. We will write:

"$\vec{\eta}$ is an r- gen-e-vector" meaning $\vec{\eta}$ is a rank r generalized eigenvector.

Remark 4.0. The rank 2 case $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta} = \vec{v}$ when multiplied by $(\boldsymbol{A} - \lambda\boldsymbol{I})$ becomes:

$$(\boldsymbol{A} - \lambda\boldsymbol{I})^2\vec{\eta} = \vec{0}$$

Notice this rank 2 equation is also satisfied by the 1-gen-e-vector $\vec{v}$. Similarly, a 2-gen-e-vector $\vec{\eta}$ will also satisfy: $(\boldsymbol{A} - \lambda\boldsymbol{I})^3\vec{\eta} = \vec{0}$ and so on. Generally, an r-gen-e-vector $\vec{\eta}$ is a non-zero solution to

(4.2) $$(\boldsymbol{A} - \lambda\boldsymbol{I})^r\vec{\eta} = \vec{0}$$

that is <u>not</u> a solution to any lower rank equation $(\boldsymbol{A} - \lambda\boldsymbol{I})^p\vec{\eta} = \vec{0}$ where $p < r$. Consequently, when we successively solve the *non-homogeneous* equations

(4.3) $$(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta}_2 = \vec{\eta}_1,\ (\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta}_3 = \vec{\eta}_2,\ \ldots\ (\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta}_p = \vec{\eta}_{p-1}$$

starting from a 1-gen-e-vector $\vec{v} = \vec{\eta}_1$ and proceeding until an equation arises for which $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta}_{p+1} = \vec{\eta}_p$ has no solution, we find a *chain* $\vec{\eta}_1 \ldots \vec{\eta}_p$ of generalized eigenvectors of increasing rank.[18] □

Using Eqns (4.3), we develop a *chain of generalized eigenvectors* that begins from an eigenvector $\vec{v} = \vec{\eta}_1$ at rank 1, and extend the chain as long as generalized eigenvectors can be found. The process terminates when we reach a power p for which $(\boldsymbol{A} - \lambda\boldsymbol{I})^p\vec{\eta} = \vec{0}$ for every $\vec{\eta} \epsilon \mathbb{R}^n$. There is such a p for every eigenpair $\lambda, \vec{v}$. For an eigenvalue λ without defect, the generalized eigenvector chains for each independent eigenvector $\vec{v}$ have length 1, just $\vec{v} = \vec{\eta}_1$. For a λ with $Defect(\lambda) = k > 0$, there are k missing eigenvectors but the sum of the lengths of its chains is exactly $Algmult(\lambda)$; there are k independent higher rank generalized eigenvectors to make up for the missing rank 1 eigenvectors. The higher rank generalized eigenvectors of a chain don't have eigenspaces but the solutions constructed from them lie in $G(\lambda, \vec{\eta}_1) = Span(\vec{\eta}_1 \ldots \vec{\eta}_p)$.

Example 4.5. Consider $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ with $\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \lambda = -1, -1, -1\,.\ Algmult\,(-1) = 3\,.$

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[17] *Differential Equations and Boundary Value Problems: computing and modeling – 3rd ed.* Edwards, C.H. & Penney, D.E. (2004). Chapter 5 §4-5.

[18] Multiplying both sides of the first of Eqns (4.3) by $(\boldsymbol{A} - \lambda\boldsymbol{I})$ gives $(\boldsymbol{A} - \lambda\boldsymbol{I})^2\vec{\eta}_2 = \vec{0}$ since $\vec{\eta}_1$ is an eigenvector. So, $\vec{\eta}_2$ is a solution to (4.2) with $r = 2$ but $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta}_2 = \vec{\eta}_1 \neq \vec{0}$, so $\vec{\eta}_2$ is <u>not</u> solution to (4.2) with $r = 1$. Multiplying the second of Eqns (4.3) by $(\boldsymbol{A} - \lambda\boldsymbol{I})$ gives $(\boldsymbol{A} - \lambda\boldsymbol{I})^2\vec{\eta}_3 = (\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\eta}_2 = \vec{\eta}_1$, which is just (4.1).

$(\boldsymbol{A}-\lambda\boldsymbol{I})\vec{v}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\vec{v}=\vec{0}\ \Rightarrow\ \ \vec{v}_1=\vec{e}_1,$ works, giving independent solution:

$$\vec{x}_1(t)=\begin{bmatrix}1\\0\\0\end{bmatrix}e^{-t}\text{ and }Defect(\lambda)=2.$$

We solve Eqns(4.3) for this system to obtain the remaining independent solutions.
$(\boldsymbol{A}-\lambda\boldsymbol{I})\vec{\eta}_2=\vec{v}_1=\vec{e}_1$.

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\vec{\eta}_2=\begin{bmatrix}1\\0\\0\end{bmatrix}\ \Rightarrow\ \ \vec{\eta}_2=\vec{e}_2\text{ works.}$$

By Theorem 4.1, the second independent solution is $\vec{x}_2(t)=(\vec{\eta}_2+t\vec{v}_1)e^{\lambda t}=\left(\begin{bmatrix}0\\1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\0\end{bmatrix}\right)e^{-t}$

The rank 3 equation in (4.3) is $(\boldsymbol{A}-\lambda\boldsymbol{I})\vec{\eta}_3=\vec{\eta}_2=\vec{e}_2$.

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\vec{\eta}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}\ \Rightarrow\ \ \vec{\eta}_3=\vec{e}_3\text{ works.}$$

It is not difficult to see that

$$\vec{x}_3(t)=\left(\vec{\eta}_3+t\vec{\eta}_2+\frac{t^2}{2}\vec{v}_1\right)e^{\lambda t}=\left(\begin{bmatrix}0\\0\\1\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix}+\frac{t^2}{2}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)e^{-t}$$

and a general solution is given by

$$\vec{x}(t)=c_1\vec{v}_1e^{-t}+c_2(\vec{\eta}_2+t\vec{v}_1)e^{-t}+c_3\left(\vec{\eta}_3+t\vec{\eta}_2+\tfrac{t^2}{2}\vec{v}_1\right)$$

$$=\begin{bmatrix}1&t&\frac{t^2}{2}\\0&1&t\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}e^{-t}=\begin{bmatrix}e^{-t}&te^{-t}&\frac{t^2}{2}e^{-t}\\0&e^{-t}&te^{-t}\\0&0&e^{-t}\end{bmatrix}\vec{x}(0)=e^{t\boldsymbol{A}}\vec{x}(0).\qquad\square$$

The form of $\vec{x}_3(t)$ given in Example 4.5, as well as the form for solutions associated with any r-gen-e-vector $\vec{\eta}_r$, can be obtained by consideration of the matrix exponential of Def. 1.0, for which the solution with $\vec{x}(0)=\vec{\eta}_r$ is given by:

$$e^{t\boldsymbol{A}}\vec{\eta}_r=\left(\boldsymbol{I}+\boldsymbol{A}t+\tfrac{\boldsymbol{A}^2}{2}t^2+\tfrac{\boldsymbol{A}^3}{3!}t^3+\cdots\right)\vec{\eta}_r\,.$$

Rewriting $e^{t\boldsymbol{A}}=e^{t\lambda\boldsymbol{I}}e^{t(\boldsymbol{A}-\lambda\boldsymbol{I})}=e^{\lambda t}e^{t(\boldsymbol{A}-\lambda\boldsymbol{I})}$ gives

$$e^{t\boldsymbol{A}}\vec{\eta}_r=e^{\lambda t}\left(\boldsymbol{I}+t(\boldsymbol{A}-\lambda\boldsymbol{I})+\tfrac{t^2}{2}(\boldsymbol{A}-\lambda\boldsymbol{I})^2+\tfrac{t^3}{3!}(\boldsymbol{A}-\lambda\boldsymbol{I})^3+\cdots\right)\vec{\eta}_r\,.$$

Since $\vec{\eta}_r$ has rank r, by Eqn (4.2), $(\boldsymbol{A}-\lambda\boldsymbol{I})^p\vec{\eta}_r=\vec{0}$ for $p\geq r$ and the exponential series terminates at the term of degree $r-1$. Applying eqns (4.3),

$$\text{(4.4)}\qquad e^{t\boldsymbol{A}}\vec{\eta}_r=e^{\lambda t}\left(\vec{\eta}_r+t(\boldsymbol{A}-\lambda\boldsymbol{I})\vec{\eta}_r+\cdots+\tfrac{t^{r-1}}{(r-1)!}(\boldsymbol{A}-\lambda\boldsymbol{I})^{r-1}\vec{\eta}_r\right)$$

$$=e^{\lambda t}\left(\vec{\eta}_r+t\vec{\eta}_{r-1}+\cdots+\tfrac{t^{r-1}}{(r-1)!}\vec{\eta}_1\right).$$

Because gen-e-vectors in a chain are independent, and gen-evectors from chains that start at independent eigenvectors are also independent, the set of all gen-e-vectors form an independent set of n vectors in $\mathbb{R}^n$, giving an *eigenbasis* in terms of which every IC $\vec{x}(0)$ can be written uniquely as a linear combination.

As a result, there are n independent solutions to the $n \times n$ linear system $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ given by applying equation (4.4) to every gen-e-vector (which will include the ordinary e-vectors as 1-gen-e-vectors). In cases where there are complex conjugate eigenvalues, pairs of solutions result just as in section 4.3. A multiple complex conjugate eigenvalue pair is handled by finding the complex generalized eigenvector chain and taking real and imaginary parts of the complex analogue of Eqn (4.4) for one of the conjugates.

Exercises 4.4

1. Find a general solution to $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for each matrix $\boldsymbol{A}$.

   i. $\begin{bmatrix}-2 & 1\\ 0 & -2\end{bmatrix}$ *ii.* $\begin{bmatrix}4 & -1\\ 1 & 2\end{bmatrix}$ *iii.* $\begin{bmatrix}-3 & -2\\ 2 & -7\end{bmatrix}$ *iv.* $\begin{bmatrix}2 & -1\\ 1 & 0\end{bmatrix}$ *v.* $\begin{bmatrix}4 & -4\\ 9 & -8\end{bmatrix}$

2. Find a general solution to $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for each matrix $\boldsymbol{A}$.

   i. $\begin{bmatrix}2 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}$ *ii.* $\begin{bmatrix}1 & 1 & 2\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}$ *iii.* $\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}$ *iv.* $\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}$

3. Find a general solution to $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for each matrix $\boldsymbol{A}$.

   i. $\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}$ *ii.* $\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}$

4. Find a general solution to $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ for each matrix $\boldsymbol{A}$.

   i. $\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}$ *ii.* $\begin{bmatrix}-1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}$