Chapter 2. Second Order Linear Equations

2.0 Introduction.

Definition 0.0. An ODE in variables t, y that includes the n^{th} derivative of y but no higher derivatives is called an n^{th} *order ODE*.

Example 0.0. $\frac{d^2y}{dt^2} = -9.8$ is a 2nd order linear ODE for the position $y(t)$ (in meters) of an object of any mass dropped from some height above the Earth, neglecting air resistance, with time t measured in seconds. This equation can be solved by performing two integrations.

$$\int \frac{d^2y}{dt^2}\,dt = -9.8\int dt \;\;\Rightarrow\;\; \frac{dy}{dt} = -9.8t + c_1$$

$$\int \frac{dy}{dt}\,dt = \int -9.8t + c_1\,dt \;\;\Rightarrow\;\; y = -4.9t^2 + c_1 t + c_2$$

Each integration results in an arbitrary constant, so the general solution to this second order linear equation depends on the two arbitrary constants, $c_1 = y'(0)$ and $c_2 = y(0)$. □

As you might guess, an n^{th} order equation requires *n* integrations and so there will be *n* arbitrary constants in the general solution. As it turns out, the *n* arbitrary constants can be determined by specifying *n* conditions, namely, *n* algebraic equations in the *n* unknown constants. These equations derive from initial conditions, of which there must then be *n* ICs. Each IC specifies the value of the solution y or one of its first *n*-1 derivatives at the initial time.

Theorem 0.0. A complete set of initial conditions for an n^{th} order linear equation is a list:

$$y(t_0) = y_0\,, \qquad y'(t_0) = y'_0\,, \qquad \dots \quad, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

where $y_0', \dots y_0^{(n-1)}$ are constants giving initial values for the derivatives of the desired solution.

Example 0.1. The equation $\frac{d^2y}{dt^2} = -9.8$ requires initial conditions for both $y(t_0)$ and $y'(t_0)$ to specify a solution to an IVP. We might give, $y(0) = 100,\; y'(0) = 0$ for an object dropped from 100 meters at $t_0 = 0$. □

Fortunately, due to Newton's 2nd law $F = ma$, where $a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$ is acceleration, in many basic applications the equation of interest is of only second order. In this chapter, we will treat only 2nd order *linear* equations, leaving nonlinear equations to Chapter 5.

Definition 0.1. A second order *linear* ODE in t, y is a 2nd order ODE that can be written as

(0.1) $$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t).$$

A *solution* to Eq. 0.1 is a twice differentiable function $y(t)$ that satisfies Eq. 0.1 for all t in a non-empty open interval I.

Definition. The set of functions that are *n*-times continuously differentiable on an interval I is denoted

$$C^n(I).$$

Example 0.2. A function $y\epsilon C^2(I)$ is at least twice differentiable on I. A function $f\epsilon C^0(I)$ is at least continuous on the interval I. Every function in $C^n(I),\; n \geq 1$ is also in $C^{n-1}(I)$ and so in $C^0(I)$. □

Observe that for any t where $a(t) = 0$, Eq. 0.1 momentarily becomes a first order equation. Values of t where $a(t) = 0$ are called *singular points* for the equation. The solution near a singular point is a delicate matter and will not be treated here. Instead, we give the standard EU theorem for linear 2nd order equations. [1]

Theorem 0.1. (Linear EU) The IVP:

(0.2) $$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t)\,,\;\; y(t_0) = y_0\,, y'(t_0) = y_0'$$

has unique solution on any open interval I where p, q, f are continuous and $t_0 \epsilon I$.

Probably the simplest proof of Theorem 0.1 uses the associated *system* of differential equations obtained from Eq. 0.2 by making the substitution: $y = y_1,\; y' = y_2$ giving:

(0.3) $$\begin{aligned} y_1' &= y_2 \\ y_2' &= f(t) - q(t)y_1 - p(t)y_2 \end{aligned}$$

Interpreting Eq. 0.3 as a differential equation in vectors,[2] by letting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ so that

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

Substituting the matrix function $P(t) = \begin{bmatrix} 0 & -1 \\ q(t) & p(t) \end{bmatrix}$ and the vector function $\boldsymbol{F}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$ into the equation gives

(0.4) $$\frac{dy}{dt} + P(t)\boldsymbol{y} = \boldsymbol{F}(t)$$

which is a linear first order system of ODEs[3] but is of the same form as the first order scalar equation $\frac{dy}{dt} + p(t)y = f(t)$ we solved in Chapter 1 by the integrating factor method.

Since the derivative of a vector or matrix valued function is just the vector or matrix of the derivatives and the same goes for the integral, the first order linear system 0.4 can be solved using the integrating matrix factor $R(t) = \exp\int P(t)dt$ when $P(t)$ is continuous and we understand that $R(t)$ is a matrix valued function that satisfies the first order homogeneous matrix system:

$$\frac{dR}{dt} = P(t)R.$$

Evidently, multiplying Eq. 0.4 by the integrating factor $R(t)$ simplifies 0.4, after integration, to

$$R(t)\boldsymbol{y} = \int R(t)\boldsymbol{F}(t)dt + \boldsymbol{c}$$

where $R(t)\boldsymbol{F}(t)$ is integrable because $\boldsymbol{F}(t)$ is continuous and we already know $R(t)$ is differentiable. The solution to 0.4 satisfying the vector of ICs $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$ is then

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[1] The standard EU theorem for higher order equations is the most natural generalization and our proof goes over easily to the general higher order case.

[2] We will denote vector quantities with bold font. A vector quantity is just a list of ordinary functions or numbers. In the vector setting, ordinary numbers are called *scalars* and ordinary functions are called *scalar functions*.

[3] Vector ODEs are called *systems of ODEs,* or just *systems* for short We will treat them in detail in Chapter 4.

(0.5) $$\boldsymbol{y} = \exp\left(-\int_{t_0}^{t} P(s)ds\right)\left(\int_{t_0}^{t} R(s)\boldsymbol{F}(s)ds + \boldsymbol{y}_0\right).$$

The existence part of Theorem 0.1 is now proven because the first coordinate of the vector $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is just $y(t)$ that satisfies the IVP 0.2 of the Theorem. Uniqueness is left as an exercise.

Definition 0.2. Operator notation substitutes D for the derivative operator $\frac{d}{dt}$ so that the second derivative operator is denoted D^2. The null operator D^0 is not typically written, as it has no effect on a function.

In operator notation, the linear second order equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t)$ becomes

$$\left(D^2 + p(t)D + q(t)\right)y = f(t).$$

The algebra of linear operators applies to the parenthetical expression in operator notation so that left-multiplication of the function y by the sum of linear operators distributes y to each of the operators. The second order linear equation of Theorem 0.1 becomes

$$D^2y + p(t)Dy + q(t)y = f(t).$$

Because we will have frequent occasion to refer to this equation throughout the chapter, we observe the convention that substitutes a single symbol for the general non-singular second order linear operator.

(0.6) $$Ly = D^2y + p(t)Dy + q(t)y\,.$$

Example 0.3. The equation $\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y = \cos t$ becomes, in operator notation, $Ly = \cos t$, where we understand from context that the functions $p(t) = 3t$ and $q(t) = 1$. □

Exercises 2.0

1. Solve $\frac{d^2y}{dt^2} + 32 = 0\,,\ y(0) = 100\,,\ y'(0) = 0.$
2. Solve $\frac{d^3y}{dt^3} + 1 = 0\,,\ y(0) = 100\,,\ y'(0) = 0\,,\ y''(0) = -32.$
3. Write the Euler equation[4] $at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + c = 0$ where $a \neq 0$ in the form of Theorem 0.1 and determine the t-intervals for which the theorem guarantees unique solutions for all ICs $y(t_0) = y_0\,, y'(t_0) = y_0'$.
4. Write the Euler equation of Problem 3 in D-operator notation.
5. Substitute $y_r = t^r$ into the Euler equation $Ly = t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = 0$ to find two values of r that will make y_r a solution to $Ly = 0$.
6. Substitute $y_r = t^r$ into the Euler equation $Ly = t^2\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = 0$ to find two values of r that will make y_r a solution to $Ly = 0$.
7. Substitute $y_r = t^r$ into the Euler equation $Ly = t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0$ to find two values of r that will make y_r a solution to $Ly = 0$.
8. Find a formula for the values of r that make $y_r = t^r$ a solution to: $at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0$ where $(a-b)^2 - 4ac > 0$.[5]

2.1 Homogeneous Equations $Ly = 0$: Superposition and Independence

As we saw in the Introduction, the second order linear ODE needs two ICs $y(t_0) = y_0\,, y'(t_0) = y_0'$. Because the ICs are associated somehow with two otherwise arbitrary constants, we can guess that all

[4] These are also known as Cauchy-Euler equations.

[5] This problem is borrowed from *Notes on Diffy Qs* by J. Lebl, 2014, p. 50.

solutions must be constructed from two solutions, one associated with each arbitrary constant. This is indeed the case.

Definition 1.0. The equation $Ly = D^2y + p(t)Dy + q(t)y = 0$ is second order linear *homogeneous*.

Theorem 1.0. (Superposition) If y_1 and y_2 are solutions to a homogeneous second order linear equation, $Ly = 0$, then $L(c_1y_1 + c_2y_2) = 0$ for any constants c_1 and c_2.

Proof. $Lcy = D^2cy + p(t)Dcy + q(t)cy = cLy$ so $Ly_k = 0 \Rightarrow c_kLy_k = 0 = Lc_ky_k$ and so
$Ly_1 = Ly_2 = 0 \Rightarrow L(c_1y_1) = L(c_2y_2) = 0.$ Furthermore,

$$\begin{aligned} L(y+z) &= D^2(y+z) + p(t)D(y+z) + q(t)(y+z) \\ &= D^2y + D^2z + p(t)(Dy + Dz) + q(t)y + q(t)z \\ &= D^2y + p(t)Dy + q(t)y + D^2z + p(t)Dz + q(t)z = Ly + Lz \end{aligned}$$

So $L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2 = 0$ and the theorem is proved. □

Superposition means that any solutions of a homogeneous linear equation can be combined to give more solutions by multiplying them by any pair of constants and adding the results. This kind of a combination of functions is called a *linear combination*.

Definition 1.1. If $y_1, \dots, y_n$ are functions of t and $c_1, \dots, c_n$ are constants, then the function

$$c_1y_1(t) + \dots + c_ny_n(t)$$

is a *linear combination* of $y_1, \dots, y_n$.

Theorem 1.01. (Superposition rephrased) If $y_1, \dots, y_n$ are solutions to a homogeneous linear equation, $Ly = 0$ then any linear combination of $y_1, \dots, y_n$ is also a solution.

Example 1.0. $Ly = y'' - 4y = 0$, has solutions $y_1 = e^{2t}$ and $y_2 = e^{-2t}$ so every linear combination $y(t) = c_1e^{2t} + c_2e^{-2t}$ is also a solution. In fact, this $y(t)$ is a general solution to $Ly = 0$. □

Example 1.1. $Ly = y'' + 4y = 0$, has solutions $y_1 = \cos 2t$ and $y_2 = \sin 2t$ so every linear combination $y(t) = c_1 \cos 2t + c_2 \sin 2t$ is too. This $y(t)$ is a general solution to $Ly = 0$. □

The solutions $y(t) = c_1y_1 + c_2y_2$ are general solutions to the equations in Examples 1.0 &1.1 because every solution to every IVP can be found by specifying the right choices for c_1, c_2. In order to do this, the solutions y_1, y_2 must be linearly independent.

Definition 1.2. Functions $y_1, \dots, y_n$ are *linearly independent* on an interval I if *all* linear combinations satisfy

$$c_1y_1(t) + \dots + c_ny_n(t) = 0 \quad \Rightarrow \quad c_1 = \dots = c_n = 0.$$

Otherwise, they are *linearly dependent*.

When y_1, y_2 are linearly independent, every choice of constants c_1, c_2 gives a different solution because different choices of constants give different values for the ICs. By the Linear EU Theorem 0.1, uniqueness of solution to IVPs (assuming continuity of p, q) implies different ICs determine different solutions.

For pairs y_1, y_2, dependence is easy to spot because there must be a linear combination that is identically zero with at least one $c_k \neq 0$, making y_1, y_2 multiples of each other.

Theorem 1.1. A pair of functions, y_1, y_2 are independent on an interval $I \Leftrightarrow$ they are *not* constant multiples of each other.

The proof is left as an exercise.

Example 1.2. Functions t^n and t^m are independent when $n \neq m$, because, using Def. 1.2, $c_1 t^n + c_2 t^m = 0$ with, say $c_1 \neq 0$, $\Rightarrow c_1 t^{n-m} + c_2 = 0 \Rightarrow$ $t^{n-m} = \frac{c_2}{c_1}$ but t^{n-m} is non-constant. We could equally well use Theorem 1.1 and observe when $n \neq m$, that $t^n = ct^m \Rightarrow$ $t^{n-m} = c$, again impossible when $n \neq m$.

Consequently, if we form different combinations $y(t) = c_1 t^n + c_2 t^m$ for solutions t^n and t^m to a second order $Ly = 0$, we know that every choice of constants gives a different solution. □

Similar reasoning shows t^n and $\cos t$ or e^t would be independent on any interval.

Remark 1.0. The pair of functions $f(t)$, 0 is always dependent because $(0)f(t) = 0$, so 0 is a multiple of $f(t)$ even though $f(t)$ is not a multiple of 0.

It is easy to spot dependent pairs of non-zero functions because they have to be the same up to a constant multiple.

Example 1.3. Let $f(t) = 3t^2 - 2t + 1$ and $g(t) = 6t^2 - 4t + 2$. It should be easy to spot dependence because $g(t) = 2f(t)$. Every coefficient in $g(t)$ is twice the same one in $f(t)$. □

Example 1.4. The functions $f(t) = e^{at}$ and $g(t) = e^{bt}$ are independent whenever $a \neq b$ because $c_1 e^{at} = -c_2 e^{bt} \Rightarrow$ $c_1 e^{(a-b)t} = -c_2 \Rightarrow c_1 = c_2 = 0$ or $e^{(a-b)t}$ is constant, which it is not. □

We are interested in determining whether a pair of solutions y_1, y_2 to $Ly = 0$ are independent because we want to write a general solution to $Ly = 0$ as a linear combination of solutions. The number of solutions we need to combine should be equal to the number of arbitrary constants, which is just the order of the equation, in this case 2. More than two solutions would be redundant because we only need one solution to account for each constant. If our solutions are dependent, then one is a multiple of the other, so we can combine them and use just one arbitrary constant. Our goal for the remainder of this section is to sort this out in detail, finishing with Abel's Theorem[6] that shows we need 2 independent solutions for a general solution to a 2nd order linear equation. For an n^{th}order equation we would need n independent solutions.

Generally, independence can be decided for sets of n solutions to a homogeneous n^{th}order linear ODE by calculating their Wronskian[7] determinant.

Definition 1.3. Functions $y_1, \dots, y_n \epsilon C^n(I)$ determine a function

$$W(y_1, \dots, y_n) = \det \begin{bmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \epsilon C^1(I)$$

called the *Wronskian determinant* of the set functions $\{y_k | \, k = 1 \dots n\}$.

[6] Abel's (Theorem 1.4) is as close to a "Fundamental Theorem of 2nd Order Linear ODEs" as you will get.

[7] So named for polymath Jozéf Hoëné Wronski (1778-1853).

Formulas for determinants of matrices are easily found elsewhere. We give only the one that we will use in this course.

Definition 1.4. The determinant of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = ad - bc$.

Theorem 1.2. A pair[8] of solutions y_1, y_2 to the 2nd order linear ODE $Ly = 0$ are linearly independent on an interval I if and only if their Wronskian determinant $W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \neq 0$ on I.

Example 1.5. Functions $y_1 = \cos \omega t$ and $y_2 = \sin \omega t$ are solutions to $y'' + \omega^2 y = 0$ $(\omega \neq 0)$ on $(-\infty, \infty)$ and they are independent because their Wronskian determinant
$W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t \end{bmatrix} = \omega \cos^2 \omega t + \omega \sin^2 \omega t = \omega \neq 0$.
As mentioned above, the general solution will be $y(t) = c_1 \cos \omega t + c_2 \sin \omega t$. □

Definition 1.5. A pair[9] of solutions y_1, y_2 to the 2nd order linear ODE $Ly = 0$ form a *fundamental set* for $Ly = 0$ on an interval I, if for every IC: $y(t_0) = y_0$, $y'(t_0) = y_0'$ with $t_0 \epsilon I$, there is a linear combination $c_1 y_1 + c_2 y_2$ giving a solution to the IVP.

When y_1, y_2 form a fundamental set, the general linear combination $c_1 y_1 + c_2 y_2$ with arbitrary c_1, c_2 is then called a *general solution* to $Ly = 0$.

Example 1.6. The solutions $y_1 = \cos t$, $y_2 = \sin t$ for $y'' + y = 0$ (from Example 1.5 with $\omega = 1$) are independent. We want to see if they are a fundamental set so that $y(t) = c_1 \cos t + c_2 \sin t$ is a general solution to $y'' + y = 0$. Let's find c_1, c_2 giving the solution to the IVP with ICs $y(t_0) = y_0$, $y'(t_0) = y_0'$.

We need to solve for c_1, c_2 in the system of equations

$$\begin{aligned} y_0 &= c_1 \cos t_0 + c_2 \sin t_0 \\ y_0' &= -c_1 \sin t_0 + c_2 \cos t_0 \end{aligned}$$

Equivalently, in matrix form $\begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} \cos t_0 & \sin t_0 \\ -\sin t_0 & \cos t_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ which has unique solution,

$\begin{bmatrix} \cos t_0 & \sin t_0 \\ -\sin t_0 & \cos t_0 \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ as long as the inverse of the matrix is defined. □

Theorem 1.3. The inverse $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is well-defined by

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

when $\det A \neq 0$. Furthermore, $A^{-1} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the 2×2 *identity matrix*.

[8] This theorem can be extended to sets of *n* solutions to an *n*th order homogeneous linear equation.
[9] This definition can be extended to sets of *n* solutions to an *n*th order homogeneous linear equation.

Proof. Multiply $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the 2×2 identity.

If $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is any vector, then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. □

Because the Wronskian $W(\cos t, \sin t) = \det\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$, the matrix inverse needed to solve $\begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} \cos t_0 & \sin t_0 \\ -\sin t_0 & \cos t_0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is well-defined at any t_0. Multiplying both sides by the inverse from Theorem 1.3:

$$\begin{bmatrix} \cos t_0 & \sin t_0 \\ -\sin t_0 & \cos t_0 \end{bmatrix}^{-1}\begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} \cos t_0 & \sin t_0 \\ -\sin t_0 & \cos t_0 \end{bmatrix}^{-1}\begin{bmatrix} \cos t_0 & \sin t_0 \\ -\sin t_0 & \cos t_0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

gives for the left side:

$$\begin{bmatrix} \cos t_0 & -\sin t_0 \\ \sin t_0 & \cos t_0 \end{bmatrix}\begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} y_0 \cos t_0 - y_0' \sin t_0 \\ y_0 \sin t_0 + y_0' \cos t_0 \end{bmatrix}$$

and on the right just $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Consequently, $\begin{bmatrix} y_0 \cos t_0 - y_0' \sin t_0 \\ y_0 \sin t_0 + y_0' \cos t_0 \end{bmatrix}$ gives the unique values of the constants c_1, c_2 that make $y(t) = c_1 \cos t + c_2 \sin t$ a solution to the IVP of Example 1.6. In particular,

$$y(t) = (y_0 \cos t_0 - y_0' \sin t_0)\cos t + (y_0 \sin t_0 + y_0' \cos t_0)\sin t$$

is the *unique* solution to the IVP $y'' + y = 0\,, y(t_0) = y_0\,, y'(t_0) = y_0'$.

Abel's Theorem[10]

Of the many profound and brilliant theorems Abel produced in his short career, this, perhaps the very least of them, fits within the scope of our course and is in some regard The Fundamental Theorem of 2nd Order Linear ODEs. In order to avoid more extensive use of systems of ODEs, we restrict the presentation to 2nd order equations for clarity, although this theorem can be proved for *n*th order linear equations as well, by passing to their associated first order systems.[11]

Theorem 1.4 (Abel 1827) Solutions y_1, y_2 to the 2nd order linear ODE $Ly = 0$ form a fundamental set on an interval $I \Leftrightarrow \exists\, t_0 \epsilon\, I$ where the Wronskian $W(y_1, y_2)(t_0) \neq 0$.

Proof. *Part 1*. To see that y_1, y_2 is a fundamental set for $Ly = 0$, we write the matrix form (as done in Example 1.5) of the system of equations we need to solve for the values of c_1, c_2 that give a solution $c_1 y_1 + c_2 y_2$ to the IVP $Ly = 0, y(t_0) = y_0\,, y'(t_0) = y_0'$:

$$\begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Using Theorem 1.3, this system has the unique solution

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{W(t_0)}\begin{bmatrix} y_2'(t_0) & -y_2(t_0) \\ -y_1'(t_0) & y_1(t_0) \end{bmatrix}\begin{bmatrix} y_0 \\ y_0' \end{bmatrix} \Leftrightarrow W(t_0) \neq 0\,.$$

So, as long as $W(y_1, y_2)(t_0) \neq 0$, the solutions y_1, y_2 are a fundamental set for ICs at t_0.

[10] Niels Henrik Abel (1802-1829) Norwegian mathematical prodigy first proved this in 1827.

[11] Joseph Liouville (1809-1882) French mathematician first proved this in 1838. For proof see Hartman, P. *Ordinary Differential equations* 2nd ed. Birkhäuser, Boston; Basel; Stuttgart. Reprinted 1982. Chapter IV §8.

Part 2. Abel's Theorem says more. It says that there merely needs to exist *some* $t_0 \epsilon I$ where $W(t_0) \neq 0$ to ensure y_1, y_2 form a fundamental set for *every* $t_0 \epsilon I$. In other words, we need to show that if, at *one* $t_0 \epsilon I$, $W(t_0) \neq 0$ then $W(t) \neq 0$ at *all* $t \epsilon I$.

Because $W = y_1 y_2' - y_2 y_1'$ and since y_1, y_2 are solutions on I to the 2nd order ODE $Ly = 0$, y_1'', y_2'' are continuous there, so $W \epsilon C^1(I)$ and using the product rule, $\frac{d}{dt} W = y_1 y_2'' - y_2 y_1''$. Suppressing t dependence, we multiply $Ly_2 = 0$ by y_1 and similarly $Ly_1 = 0$ by y_2, giving

$$y_1 L y_2 = y_1 y_2'' + y_1 y_2' \, p + y_1 y_2 \, q = 0 \quad \text{and}$$
$$y_2 L y_1 = y_2 y_1'' + y_2 y_1' \, p + y_2 y_1 \, q = 0.$$

Subtracting these gives

$$y_1 y_2'' - y_2 y_1'' + (y_1 y_2' - y_2 y_1') \, p = 0\,.$$

Substituting $\frac{d}{dt} W$ for $y_1 y_2'' - y_2 y_1''$ and W for $y_1 y_2' - y_2 y_1'$ gives $\frac{d}{dt} W = -pW$ so

(1.0) $$W(t) = W(t_0) \exp\left(-\int_{t_0}^{t} p(s) ds\right).$$

This means that if $\exists t_0 \epsilon\, I$ where $W(y_1, y_2)(t_0) \neq 0$, then $W(y_1, y_2)(t) \neq 0$ for all $t \epsilon\, I$. □

Abel observed that, given a solution y_1 of the 2nd order linear $Ly = 0$, another independent solution can be found using Eq.1.0 by solving it for y_2' in terms of y_1, y_1' and y_2 to get a first order ODE for y_2. This 1st order equation can then be solved for y_2 independent from y_1 to obtain a general solution. Formula 1.0 is known as Abel's formula and besides giving the general solution once one non-zero solution is known, gives the Wronskian without even having to solve $Ly = 0$.[12]

Exercises 2.1

1. Verify the claim of Example 1.0: $Ly = y'' - 4y = 0$, has solutions $y_1 = e^{2t}$ and $y_2 = e^{-2t}$, by substituting them into the equation.
2. Verify the claim of Example 1.1: $Ly = y'' + 4y = 0$, has solutions $y_1 = \cos 2t$ and $y_2 = \sin 2t$, by substituting them into the equation.
3. Verify that $Ly = y'' - 2y' + y = 0$, has solutions $y_1 = 2e^t$ and $y_2 = -3e^t$ by substituting them into the equation. Are these solutions independent? Explain your conclusion. If so, write a general solution.
4. Verify that $Ly = y'' + 4y' + 4y = 0$, has solutions $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$ by substituting them into the equation. Are these solutions independent? Explain your conclusion. If so, write a general solution.
5. This concerns solutions of the ODE $Ly = y'' - y = 0$.
   a. Verify by substitution that $y_1 = \cosh t$ and $y_2 = \sinh t$ are solutions to $Ly = 0$ and use the Wronskian to show these are independent..
   b. Verify by substitution that $y_3 = e^t$ and $y_4 = e^{-t}$ are solutions to $Ly = 0$. Use the Wronskian to show these solutions are independent.
   c. Write the formulas for $\cosh t$ and $\sinh t$ in terms of e^t and e^{-t}. Use pairwise Wronskians to examine whether the pair of solutions $\{\cosh t\ , e^t\}$ is independent.and also whether $\{\sinh t\ , e^t\}$ is an independent set.
   d. Use Definition 1.2 to settle the issue of independence for each set of solutions (these are not pairs, so you can't use the Wronskian):
      i. $y_1 = \cosh t, y_2 = \sinh t\,,\ y_3 = e^t$
      ii. $y_1 = \cosh t, y_2 = \sinh t\,,\ y_4 = e^{-t}$
      iii. $y_1 = \cosh t, y_2 = \sinh t\,,\ y_3 = e^t, y_4 = e^{-t}$ independent?

[12] The general n^{th} order form of Eq.1.0 was given by Liouville (1838) and by Jacobi (1845) and is named for either of them by different authors. Hartman, P. ibid.

6. Write four different general solutions for $Ly = y'' - y = 0$ by combining solutions from problem 5.
7. Prove Theorem 1.1.
8. Substitute $y = t^r$ into the Euler equation $t^2y'' + 3ty + y = 0$ to find a solution. Write the equation in the form of Theorem 0.1 and use Abel's formula 1.0 to find a second solution. Write the general solution and find the solution to the IVP with ICs $y(1) = 2\,, y'(1) = 0$.(Moderately difficult)

2.2 Homogeneous Constant Coefficient Linear Equations

The second order constant coefficient homogeneous equation $ay'' + by' + cy = 0$ can always be solved exactly by finding the characteristic roots r_1, r_2 for its characteristic equation. More generally,

Definition 2.0. The n^{th}order constant coefficient homogeneous linear ODE:

$$(*) \qquad a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

has characteristic polynomial: $a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$. The zeros of this polynomial are the characteristic values r for the ODE. These values r are the roots of the *characteristic equation*: $a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$ and so are called *characteristic roots* as well.

Theorem 2.0. If r is a characteristic value for equation $(*)$ then $y_r(t) = e^{rt}$ is a solution to $(*)$.

Proof. Substituting $y_r^{(n)}(t) = r^n e^{rt}$ into $(*)$ $\Rightarrow$

$$0 = a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = e^{rt}(a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0)$$

so the characteristic polynomial is zero only when $(*)$ holds. □

Example 2.0. The equation $y'' + 5y' + 6 = 0$ has characteristic poly, $r^2 + 5r + 6$ so the characteristic values are solutions to $r^2 + 5r + 6 = 0$. Factoring, $(r + 3)(r + 2) = 0$, so characteristic roots are $r = -3, -2$ and we have two solutions $y_{-3}(t) = e^{-3t}\,, y_{-2}(t) = e^{-2t}$. They are independent, so they make a fundamental set and a general solution is

$$y(t) = c_1 e^{-3t} + c_2 e^{-2t}.$$ □

Example 2.1. The equation $y'' + 5y' = 0$ has characteristic poly, $r^2 + 5r = r(r + 5)$ so the characteristic roots are $r = -5, 0$. The solutions: $y_{-5}(t) = e^{-5t}$, $y_0(t) = 1$ are independent, so they make a fundamental set and a general solution is

$$y(t) = c_1 e^{-5t} + c_2.$$ □

By Theorem 2.0, we can solve the ODE $(*)$ by finding the zeros r of the characteristic polynomial. Second order ODEs of type $(*)$ look like $ay'' + by' + cy = 0$ and have quadratic polynomials that can always be solved using the quadratic formula.[13] Depending on the discriminant, $b^2 - 4ac$, the characteristic roots must be of one of these types:

$$b^2 - 4ac > 0 \;\Rightarrow r_\pm \text{ unequal reals,}$$
$$b^2 - 4ac < 0 \;\Rightarrow r_\pm \text{ complex conjugates,}$$
$$b^2 - 4ac = 0 \;\Rightarrow r_\pm \text{ equal reals.}$$

[13] ODEs of type $(*)$ have n^{th} degree characteristic polynomials that can often be factored (by symbolic algebra software) but, in principle, always factor into a product $(r - r_1) \ldots (r - r_n)$ where some of the r_k may be complex. Because the operator is real, the characteristic values group into real solutions and complex conjugate pairs.

The examples 2.0 & 2.1 illustrated the case of unequal real roots.

Complex conjugate characteristic roots

Example 2.2. The equation $y'' + 4y = 0$ has characteristic poly, $r^2 + 4$ so the characteristic values are imaginary $r = \pm 2i$ and the solutions: $y_+(t) = e^{2it}$, $y_-(t) = e^{-2it}$ are independent, so they make a fundamental set and the general *complex* solution is

$$y_C(t) = c_1 e^{2it} + c_2 e^{-2it}.$$

Because we are usually only interested in *real* solutions, we need to find which constants give independent real solutions and which do not. □

Generally, for an equation $Ly = 0$, such as $(*)$ with real coefficients a_k we know from superposition that if $y(t)$ is any solution then $cy(t)$ is also a solution for any constant c. In particular, a general solution in complex-valued functions allows arbitrary constants to be complex $c_k = a_k + ib_k$. The general solution in complex-valued functions includes all the real solutions because every real number $a = a + 0i$ is also in the complex field. The real numbers are a *sub-field* of the field of complex numbers.

Definition 2.1. Let $\mathbb{R}$ represent the field of real numbers. The field of complex numbers is

$$\mathbb{C} = \{a + ib \mid (a, b) \epsilon \mathbb{R}^2,\ i = \sqrt{-1}\}.$$

The real and imaginary parts of a complex number $c = a + ib$ are $\Re e(c) = a,\ \Im m(c) = b$.

The *conjugate* of $c = a + ib$ is $\bar{c} = a - ib$ (so $\Im m(\bar{c}) = -\Im m(c)$) ;

its *absolute value*, or *modulus* is $|a + ib| = \sqrt{c\bar{c}} = \sqrt{a^2 + b^2}$ and $c = a + ib = 0 \Leftrightarrow a = b = 0$.

Because the set $\mathbb{C}$ of all complex numbers is a *field*[14], they can be added and multiplied, and their fractions are well-defined as long as the denominator is not zero.

Example 2.3. The complex numbers $2 + 3i$ and $2 - 3i$ are conjugates. Their product: $(2 + 3i)(2 - 3i) = 4 + 9 = 13$ is real, although their quotient:

$$\frac{2-3i}{2+3i} = \frac{(2-3i)^2}{13} = \frac{4-12i+9i^2}{13} = -\frac{5}{13} - \frac{12}{13}i$$

is not. Notice also $\frac{1}{2+3i} = \frac{2-3i}{13}$. More generally, $\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{\sqrt{a^2+b^2}}$. □

Returning to our complex solutions $y_C(t)$ of a real equation $Ly = 0$, such as $(*)$, we can write

$$y_C = \Re e(y_C) + i\,\Im m(y_C).$$

Theorem 2.1. For the linear constant coefficient *real* differential operator,

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 D^0,$$

the real and imaginary parts $y_1 = \Re e(y_C)$ and $y_2 = \Im m(y_C)$ of a complex-valued solution y_C to $Ly = 0$ are real solutions to $Ly = 0$.

[14] A *field* is an algebraic structure whose full definition can be found in any text on Abstract Algebra. Familiar examples are the fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, denoting the rational, real, and complex numbers respectively.

Proof. Taking the conjugate on both sides of $Ly_C = 0$ gives $L\overline{y_C} = 0$ because L is real[15], so both $y_C, \overline{y_C}$ are solutions to $Ly = 0$. By the superposition Theorem 1.01, the linear combinations:

$$y_1 = \frac{y_C + \overline{y_C}}{2} = \Re e(y_C)\,, \qquad y_2 = \frac{y_C - \overline{y_C}}{2} = \Im m(y_C)$$

are also solutions to $Ly = 0$. □

Returning to the equation $y'' + 4y = 0$ of Example 2.2 with two complex solutions

$$y_+(t) = e^{2it}\,, \qquad y_-(t) = e^{-2it},$$

We now have four solutions,

$$y_1(t) = \Re e\left(e^{2it}\right), \qquad y_2(t) = \Im m\left(e^{2it}\right), \qquad y_3 = \Re e\left(e^{-2it}\right), \qquad y_4(t) = \Im m\left(e^{-2it}\right).$$

That's too many, we only need two.

Theorem 2.2. (Euler's Formula) $e^{i\theta} = \cos\theta + i\sin\theta$.

The proof is an instructive exercise following easily from comparing MacLaurin[16] series.

Applying Euler's formula, $\theta = 2t \;\Rightarrow\; e^{2it} = \cos 2t + i\sin 2t \;\Rightarrow$

$$y_1(t) = \cos 2t,\;\; y_2(t) = \sin 2t.$$

And similarly[17], $e^{-2it} = \cos(-2t) + i\sin(-2t) = \cos 2t - i\sin 2t \;\Rightarrow$

$$y_3(t) = \cos 2t,\;\; y_4(t) = -\sin 2t.$$

Only two of the solutions are independent; we take the easiest ones, $y_1(t) = \cos 2t$, $y_2(t) = \sin 2t$, for a fundamental set.

We are now well positioned to easily solve Eq.$(*)$ when the characteristic values are complex.

Example 2.4. Let's find a general solution to $y'' + 9y = 0$.

Characteristic equation $r^2 + 9 = 0 \;\Rightarrow\; r = \pm 3i \;\Rightarrow\; y_c = e^{3it} \;\Rightarrow$

$$y(t) = c_1\cos(3t) + c_2\sin(3t) \text{ is a general solution.}$$ □

Example 2.5. Consider $y'' + 2y' + 2y = 0$ whose characteristic polynomial is $r^2 + 2r + 2 = 0$.

The characteristic values, $r = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i \;\Rightarrow$ one complex solution (using $-1 \pm i$) is

$$y_C = e^{(-1+i)t} = e^{-t}e^{it} = e^{-t}(\cos t + i\sin t) \;\Rightarrow$$

$$y(t) = c_1e^{-t}\cos t + c_2e^{-t}\sin t \text{ is a general real solution.}$$ □

Generalizing from Example 2.5 implies:

Theorem 2.3. A second order linear constant coefficient $Ly = 0$ with complex characteristic values:

$$r = \alpha \pm i\omega$$

has a general (real) solution: $y(t) = e^{\alpha t}(c_1\cos\omega t + c_2\sin\omega t)$.

Example 2.6. The ODE $y'' + 4y' + 9y = 0$, whose characteristic equation $r^2 + 4r + 9 = 0$

[15] Every real number, function or operator, is equal to its own conjugate, in particular $\bar{L} = L$.
[16] Colin MacLaurin (1698-1746) Scottish mathematical prodigy – entered Glasgow University at age 11.
[17] Because cosine is an even function and sine is an odd function, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

has roots $r = \frac{-4 \pm \sqrt{16-36}}{2} = -2 \pm \sqrt{5}i$, by Theorem 2.3 has

$$y(t) = e^{-2t}\left(c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t\right) \text{ for a general solution.} \qquad \square$$

Equal characteristic roots

Example 2.7. The ODE $y'' + 4y' + 4y = 0$, whose characteristic equation $r^2 + 4r + 4 = 0$ has equal roots $r = -2, -2$, has only one solution $y_1(t) = e^{-2t}$ given by Theorem 2.0. To get another, we guess that maybe $y_2(t) = te^{-2t}$ will work. Substituting it into the ODE proves that it does. The guessed solution $y_2(t) = te^{-2t}$ is also independent of y_1 so

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} = e^{-2t}(c_1 + c_2 t) \text{ is a general solution.} \qquad \square$$

Generally, when a second order $ay'' + by' + cy = 0$ equation of the type $(*)$ has equal roots, the general solution is obtained exactly as done in Example 2.7.

Theorem 2.4. A constant coefficient ODE $ay'' + by' + cy = 0$ with only one characteristic root r, has a general solution $$y(t) = e^{rt}(c_1 + c_2 t) \quad . \qquad \square$$
The proof is left as an exercise.

Example 2.8. The ODE $y'' + 2y' + y = 0$, whose characteristic equation $r^2 + 2r + 1 = 0$ has only one root $r = -1$ giving only one solution $y_1(t) = e^{-2t}$. By Theorem 2.4 a general solution is

$$y(t) = e^{-t}(c_1 + c_2 t). \qquad \square$$

Exercises 2.2

1. Give a general solution to
   i. $y'' + 3y' + 2y = 0$ *ii.* $y'' - 12y = 0$ *iii.* $y'' + 7y' + 12y = 0$
   iv. $y'' + 3y' = 0$ *v.* $y'' - 3y' = 0$ *vi.* $2y'' - 7y' + 4y = 0$
2. Solve the IVPs for Problem 1 with the IC $y(0) = 1, y'(0) = 0$.
3. Give a general solution to
   i. $y'' + 3y = 0$ *ii..* $y'' + 12y = 0$ *iii.* $3y'' - 3y' + 3y = 0$
   iv. $y'' - 7y' + 13y = 0$ *v.* $y'' + 2y' + 4y = 0$ *vi.* $y'' + 4y' + 9y = 0$
4. Solve the IVPs for Problem 3 with the IC $y(0) = 1, y'(0) = 0$.
5. Solve the IVPs for Problem 3 with the IC $y(0) = 0, y'(0) = 1$.
6. Give a general solution to
   i. $y'' + 6y' + 9y = 0$ *ii.* $y'' + 8y' + 16y = 0$ *iii.* $y'' + 10y' + 25y = 0$
7. Solve the IVPs for Problem 6 with the IC $y(0) = 1, y'(0) = 0$.
8. Solve the IVPs for Problem 6 with the IC $y(0) = 0, y'(0) = 1$.
9. Give a general solution to
   i. $y'' + y' + 4y = 0$ *ii.* $y'' + 2y' = 0$ *iii.* $y'' + 16y = 0$
   iv. $y'' + y' + y = 0$ *v.* $y'' + 8y' + 16y = 0$
10. Solve the IVPs for Problem 9 with the IC $y(0) = 1, y'(0) = 0$.

11. This exercise proves the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$ of Theorem 2.2.

i. Find the MacLaurin series for e^x , substitute $i\theta$ for x and write out the first 9 terms and simplify the powers of i appearing in these terms as much as you can.

ii. Find the MacLaurin series for $\cos\theta$ and for $\sin\theta$ and write the first 5 terms of each.

iii. Cleverly combine the series found in parts (a) and in part (b) to prove Euler's formula.

iv. Type up Theorem 2.2 with a proof constructed from the results of parts (a-c). Use your own English words to briefly fill in between equations, so that the proof reads like your own professional explanation. This is meant to be a competition for top score.

12. Substitute the function $y_2(t) = te^{-2t}$ into the equation $y'' + 4y' + 4y = 0$ to show that it indeed, is a solution. Next use the Wronskian determinant to show that the solution given in Example 2.7 is indeed, a general solution to the equation in question.
13. Consider $y'' - 2by' + b^2y = 0$. Find a pair of independent solutions. Use the Wronskian to show they are independent. This exercise effectively proves Theorem 2.4.
14. Write a proof of Theorem 2.4 using your work from Problem 13. Be sure to begin with the statement of the theorem and put explanations between the equations in your proof. This is a competition for high score based on originality and professionalism. It should be typed.

2.3 Mechanical Vibrations 1: Unforced Harmonic Oscillators

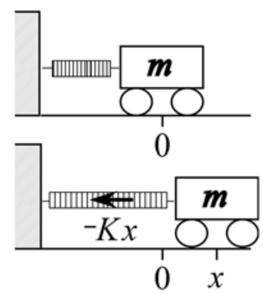

Figure 2-1.Car of mass m tied to a wall by a linear spring. At equilibrium (above) and displaced to $x>0$ (below).

Consider a mechanical system composed of a linear spring anchored to a wall and a car of mass m connected to the other end of the spring as pictured in Figure 2-1. Newton's 2nd Law gives a 2nd order ODE that relates the forces determining the motion of the car. The force applied by the spring is proportional to the amount it is stretched or compressed. Such springs are *linear* springs.

Definition 3.0. *Linear springs obey Hooke's Law*: The spring will be stretched or compressed a distance ΔL by a force $F = K \cdot \Delta L$. The constant $K > 0$ is called the spring constant.

For a linear spring, the greater the constant K, the stiffer the spring. When the car is displaced to position x, the force on the car applied by the spring will be $-Kx$, so that when the spring is stretched, $x > 0$, the spring pulls the car with a "restoring" force $-Kx$ (pointing in the negative direction) as it tries to restore itself to equilibrium length. The equilibrium of the spring+car system is depicted in the upper portion of Figure 2-1. The force $-Kx$ in the lower frame of Figure 2-1 is applied to the car by the spring, when displaced $x > 0$ beyond equilibrium.

Unforced, un-damped, or "simple" harmonic oscillator: $x'' + \omega_0^2\, x = 0$

Let's find the ODE modeling the motion of the car under the assumption that the only force the car feels is the force of the spring. Imagine we pull the car out to some initial $x_0 > 0$ and just let go of it. The spring will pull it back toward equilibrium $x = 0$ but the car's momentum will carry it past equilibrium, compressing the spring to some $x < 0$. Newton's 2nd Law says that at any time t, the force applied by the spring to the car at x will be $F = ma$, which is

$$m\frac{d^2x}{dt^2} = -Kx \;\;\Rightarrow\;\; \frac{d^2x}{dt^2} + \frac{K}{m}x = 0\,.$$

Because $\frac{K}{m} > 0$, Example 1.5 applies with $\omega^2 = \frac{K}{m}$ so that

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

is a general solution. The frequency $\sqrt{\frac{k}{m}}$ is called the *natural frequency* of the oscillator.[18] Our initial conditions are: $x(0) = x_0$, $x'(0) = 0$ because we "just let go of it". The solution of the IVP is $x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right)$. Figure 2-2 depicts the solution a mass-spring system having natural frequency $\omega_0 = \sqrt{\frac{k}{m}} = 2$. The *period* of the oscillation is $\frac{2\pi}{\omega_0}$; the *amplitude of the response* (motion of the car) is $x_0 = 1$ and depends on the ICs. This is an example of a *simple harmonic oscillator*. There are other possibilities for the initial conditions. We may not "just let it go" from some initial x_0. We might give it a little push toward equilibrium or away from it. Those ICs have $x_0' \neq 0$. To understand how different ICs affect c_1, c_2 it is especially convenient to rewrite the solution in *phase-amplitude form*.

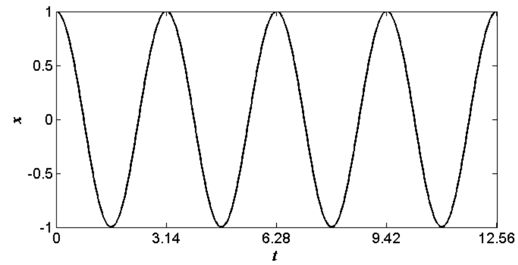

Figure 2-2. Simple harmonic oscillator mass-spring response amplitude 1 , frequency 2.

Definition 3.1. $x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ is re-written in *phase-amplitude form* as

$$x(t) = R\cos(\omega_0 t - \varphi)$$

where the *amplitude* $R = \sqrt{c_1^2 + c_2^2}$ and $\frac{c_1}{R} = \cos\varphi$, $\frac{c_2}{R} = \sin\varphi$ define the *phase angle* φ .

Example 3.1. Let's write $x(t) = 4\cos(\pi t) + 3\ \sin(\pi t)$ in phase-amplitude form.

The natural frequency is $\omega_0 = \pi$ and $R = \sqrt{16+9} = 5$ so $\frac{4}{5} = 0.8 = \cos\varphi$, $\frac{3}{5} = 0.6 = \sin\varphi$. The point $(0.8 , 0.6)$ lies in the quadrant QI so the phase angle $\varphi = \arctan\left(\frac{0.6}{0.8}\right) = \arctan\frac{3}{4}$, the same as $\varphi = \arccos(0.8) = \arcsin(0.6) \approx 0.6435$. The phase-amplitude form is

$$x(t) = 4\cos(\pi t - \arcsin 0.6) \qquad \square$$

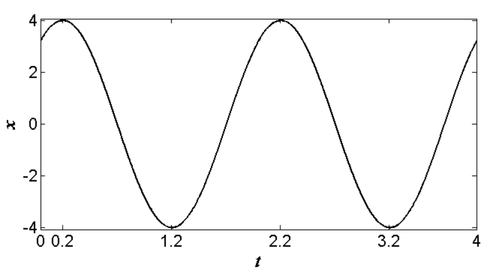

Figure 2-3. Oscillator of Example 3.1 with $R = 4$, $\omega = \pi$, $\varphi \approx 0.205$.

The function $x(t)$ in Example 3.1 is pictured in Figure 2-3. The cosine curve first reaches maximum amplitude $x = 4$ at

$$\hat{t} = \frac{\varphi}{\omega_0} = \frac{\arcsin 0.6}{\pi} \approx 0.205 .$$

The angle $\hat{t} = \frac{\varphi}{\omega_0}$ is the *actual phase shift*. In terms of the actual phase shift, the amplitude-phase form is:

$$x(t) = R\cos\big(\omega_0(t - \hat{t})\big) .$$

Example 3.2. Let's write $x(t) = -6\cos(2t) - 8\sin(2t)$ in phase-amplitude form.

$R = \sqrt{36+64} = 10$ and $\frac{-6}{10} = -0.6 = \cos\varphi$, $\frac{-8}{10} = -0.8 = \sin\varphi$. You may have learned to use $\varphi = \arctan\left(\frac{0.8}{0.6}\right) = \arctan\frac{4}{3}$ but the angle φ satisfying $(-0.6, -0.8) = (\cos\varphi\ , \sin\varphi)$ lies in quadrant

[18] Because it depends on the mass and spring characteristics m, K and has nothing to do with the initial condition, the natural frequency is a property intrinsic to the oscillator; it is usually denoted ω_0 .

QIII, outside the range of the arctangent.[19] So, $\arctan\frac{4}{3} \approx 0.9273$ gives the wrong phase angle, as do $\arccos(-0.6)$ and $\arcsin(-0.8)$ in QII and QIV respectively. It's best to always make a quick diagram like Figure 2-4 to find the phase angle's quadrant, determine the reference angle and adjust it to bring φ into the correct quadrant.

$$x(t) = 10\cos\left(2t - \left(\pi + \arctan\tfrac{4}{3}\right)\right). \quad \square$$

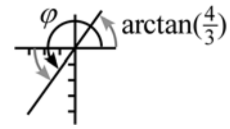

Figure2-4. Phase angle φ for Example 3.2 lies in QIII. Arctan(4/3) gives the reference angle in QI.

Example 3.3. Let's solve the equation $\frac{d^2x}{dt^2} + \frac{K}{m}x = 0$ when $K = 3$ and $m = 4$ with ICs $x(0) = 4$, $x'(0) = 2$. The natural frequency is $\omega_0 = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$. Write

$$x(t) = 4\cos(\omega_0 t) + \frac{2}{\omega_0}\sin(\omega_0 t) = 4\cos(\omega_0 t) + \frac{4}{\sqrt{3}}\sin(\omega_0 t).$$

Get the amplitude: $R = \sqrt{c_1^2 + c_2^2} = \sqrt{16 + \frac{16}{3}} = 4\sqrt{\frac{4}{3}} = \frac{8}{\sqrt{3}}$, and the phase angle satisfies $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = (\cos\varphi, \sin\varphi)$, which puts φ in QI, in the range of arctan, arccos, and arcsin, so we could use any of these to find the phase angle $\varphi = \arcsin\frac{1}{2} = \pi/6$ giving

$$x(t) = \frac{8}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right). \quad \square$$

The unforced, damped harmonic oscillator, $mx'' + cx' + Kx = 0$

More generally, the oscillator may be subject to linear drag[20] proportional to the car's velocity, and directed in opposition to its progress. This sort of drag is called *damping* and it applies a force to the car in addition to the force due to the spring. Including damping, the equation for the car's motion becomes:

$$mx'' = -Kx - cx' \;\Rightarrow\; x'' + \frac{c}{m}x' + \frac{K}{m}x = 0.$$

This is a constant coefficient homogeneous 2nd order linear equation. We can always solve it. In 2.2 we learned it has three possibilities for the characteristic roots, depending on the discriminant,

$$\left(\frac{c}{m}\right)^2 - 4\frac{K}{m} > 0 \;\Rightarrow\; \textit{real unequal roots}$$

$$\left(\frac{c}{m}\right)^2 - 4\frac{K}{m} < 0 \;\Rightarrow\; \textit{complex conjugate roots}$$

$$\left(\frac{c}{m}\right)^2 - 4\frac{K}{m} = 0 \;\Rightarrow\; \textit{only one real root.}$$

For the harmonic oscillator, because c, m, K are all positive, the possible responses are somewhat limited. In either case of real roots, they are negative, the complex roots both have $\Re e(r) < 0$, so in every case there is a factor of $e^{-\lambda t}$, $\lambda > 0$ present in the solution. All solutions then will decay exponentially toward zero, as $t \to \infty$.

[19] Arctangent only returns angles in QI & QII, we need an angle in QIII.

[20] Most authors describe it as "friction" although mechanical friction is not well modelled linearly.

Recalling that the natural frequency of the un-damped oscillator, is $\omega_0 = \sqrt{\frac{k}{m}}$, and letting $2p = \frac{c}{m}$, the unforced, damped, harmonic oscillator equation becomes

(3.0) $$x'' + 2px' + \omega_0^2 x = 0.$$

The roots take the particularly simple form $r = -p \pm \sqrt{p^2 - \omega_0^2}$ and the three cases of characteristic roots become

$$p^2 > \omega_0^2 \Rightarrow real\ unequal\ roots, \quad overdamped$$

$$p^2 < \omega_0^2 \Rightarrow complex\ conjugate\ roots, \quad underdamped$$

$$p^2 = \omega_0^2 \Rightarrow one\ real\ root, \quad critically\ damped.$$

Example 3.4. $x'' + 6x' + 5x = 0$ has $p = 3 \Rightarrow p^2 = 9 > 5 = \omega_0^2 \Rightarrow$ two real roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -3 \pm 2 = -1, -5\,.$$

General solution is $y(t) = c_1 e^{-t} + c_2 e^{-5t}$. These solutions are called *overdamped* because the damping is sufficient to overcome oscillation. Solutions with $c_1 c_2 > 0$ do not pass through the equilibrium position at $x = 0$, when $c_1 c_2 < 0$, the solution crosses $x = 0$ exactly once. □

Example 3.5. $x'' + 6x' + 10x = 0$ has $p^2 = 9 < 10 = \omega_0^2 \Rightarrow$ complex conjugate roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -3 \pm \sqrt{-1} = -3 \pm i$$

and general solution: $y(t) = c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t$. These solutions are called *underdamped* because the damping is insufficient to overcome oscillation. All non-equilibrium solutions pass through the equilibrium $x = 0$ infinitely often with reduced frequency $\sqrt{\omega_0^2 - p^2} < \omega_0$ and longer period $\frac{2\pi}{\sqrt{\omega_0^2 - p^2}} > \frac{2\pi}{\omega_0}$ than the undamped oscillator. □

Example 3.6. $x'' + 6x' + 9x = 0$ has $p^2 = 9 = \omega_0^2 \Rightarrow$ one real root

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -3$$

and general solution $y(t) = c_1 e^{-3t} + c_2\, t\, e^{-3t}$. □

Critical damping (Example 3.6) lies on the boundary between under- and over-damped responses. In the critical case, the damping overcomes oscillation but even the tiniest reduction in damping will bring it back.[21] Solutions of the critical case appear to be overdamped when $c_2 = 0$ but otherwise they pass through equilibrium just once, when $c_1 + c_2 t = 0$, so they initially appear to oscillate but then decay exponentially to equilibrium, much as an overdamped solution does. Near critical damping is good for a shock absorber in a mechanical device. There is only a single exact value of the damping coefficient that gives the critical response. In designing real mechanical systems, it is impossible to achieve such precision. Regardless, gradual wear on the damper eventually results in an underdamped response, as seen when

[21] This is because decreasing p by, say $\varepsilon > 0$, makes $(p-\varepsilon)^2 < \omega_0^2$ so the radical $\sqrt{p^2 - \omega_0^2}$ becomes $\sqrt{(p-\varepsilon)^2 - \omega_0^2} < 0 \Rightarrow r = -p + \varepsilon \pm \sqrt{\omega_0^2 - (p-\varepsilon)^2}$, having complex roots.

automotive shock absorbers, damping the suspension's springs, wear out leading to a "soft" ride with an oscillating "springy" response to bumps in the road.

Exercises 2.3

1. Write the equation of a car of mass m attached to a wall by a linear spring with constant K. Identify the natural frequency and period.
   a. $m = 2\,kg\,,\ K = 6\frac{kg}{s^2}$ b. $m = 2\,kg\,,\ K = 1.6\frac{kg}{s^2}$
   c. $m = 2\,kg\,,\ K = 0.8\frac{kg}{s^2}$ d. $m = 2\,kg\,,\ K = 3.2\frac{kg}{s^2}$
2. Solve the equations of Problem 1. Write the general solution in phase-amplitude form.
3. For each general solution obtained in Problem 2, introduce the initial conditions given here and plot your solution for 3 periods of the oscillation.
   a. $x(0) = 1,\ x'(0) = 0$ b. $x(0) = 0,\ x'(0) = 1$ c. $x(0) = 1,\ x'(0) = -1$
4. Write the equation of a car of mass $10\,kg$ attached to a wall by a linear spring with constant K, experiencing linear damping with coefficient c. Identify the natural frequency ω_0 and find the associated period. Find the characteristic values and write the general solution.
   a. $K = 6.4\frac{kg}{s^2}\,,\ c = 0$ b. $K = 6.4\frac{kg}{s^2}\,,\ c = 8\frac{kg}{s}$
   c. $K = 6.4\frac{kg}{s^2}\,,\ c = 5\frac{kg}{s}$ d. $K = 6.4\frac{kg}{s^2}\,,\ c = 2\frac{kg}{s}$
5. For each equation of Problem 4, find the reduced frequency of response, the frequency reduction from the undamped oscillator due to the damping, and the period for the damped oscillator.
6. Write the equation of a car of mass $2\,kg$ attached to a wall by a linear spring with constant K, experiencing linear damping with coefficient c . Write the general solution.
   a. $K = 6\frac{kg}{s^2}\,,\ c = 0$ b. $K = 6.4\frac{kg}{s^2}\,,\ c = 5\frac{kg}{s}$ c. $K = 8\frac{kg}{s^2}\,,\ c = 2\frac{kg}{s}$
7. For equations of Problem 6, write the general solution with oscillatory part in phase-amplitude form.
8. For equations of Problem 6, write the solution to the IVP with ICs given below and the oscillatory part in phase-amplitude form. Find the actual phase shift $\hat{t}$ and write the solution again in terms of $\hat{t}$. Plot the solutions for 3 periods of the oscillation from each of these initial conditions. Show results for each equation in Problem 6 on a single plot.
   a. $x(0) = 1,\ x'(0) = 0$ b. $x(0) = 0,\ x'(0) = 1$ c. $x(0) = 1,\ x'(0) = -1$
9. Write the equation for a car of mass $1\,kg$ attached to a wall by a linear spring with constant $K = 4\,kg/s^2$, experiencing linear damping with coefficient c given below. Write the general solution.
   a. $c = 6\frac{kg}{s}$ b. $c = 4\frac{kg}{s}$ c. $c = 4.1\frac{kg}{s}$ d. $c = 3.9\frac{kg}{s}$
10. Find and plot solutions to the equations for (a) and (b) in Problem 9 for the ICs:
    $x(0) = 5,\ x'(0) = 2$ showing the solutions for 9a & b on a single plot.[22]
11. Find and plot the solutions to the equations (b), (c), and (d) in Problem 9 for the ICs:
    $x(0) = 5,\ x'(0) = 2$ showing the solutions for 9c & d on a single plot.[23]
12. Explain in under 100 words what your plots in Problem 10 show you.
13. Explain in under 100 words what your plots in Problem 11 show you.

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[22] Make sure your plot shows enough time for the solution to (9b) to reach zero. (NEW ICs 1,-4)

[23] Make sure your plot shows enough time for the solution to (9b) to reach zero.(NEW ICs 1,-4)

2.4 Second Order Non-Homogeneous Linear Equations

Recall Theorem 0.1 which we repeat here for convenience.

Theorem 4.0. (Linear 2nd order EU) The IVP:

(4.0) $$y'' + p(t)y' + q(t)y = f(t)\,, \quad y(t_0) = y_0\,, y'(t_0) = y_0'$$

has unique solution on any open interval I where p, q, f are continuous and $t_0 \epsilon I$.

Definition 4.0. The non-homogeneous equation

(4.1) $$y'' + p(t)y' + q(t)y = f(t)$$

has *complementary* or *associated homogeneous* equation: $y'' + p(t)y' + q(t)y = 0$

Theorem 4.1. Let $y_p(t)$ be any solution to (4.1) on an interval I where $p, q, f \epsilon C^0(I)$ and let

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

be a general solution to the associated homogeneous equation, then a general solution to (4.1) is given by

$$y(t) = y_p(t) + y_h(t).$$

Example 4.0. Let's find a general solution to $y'' + 5y' + 6y = 9t$.

We know the general solution to the associated homogeneous $y'' + 5y' + 6y = 0$ is

$$y_h = c_1 e^{-2t} + c_2 e^{-3t}$$

The right side is polynomial so we try a polynomial as a candidate for $y_p = At + B$ where A, B are constants. Substituting into the non-homogeneous equation gives:

$$y_p'' + 5y_p' + 6y_p = 0 + 5A + 6(At + B) = 9t \Rightarrow$$

$$6At + 5A + 6B = 9t \;\Rightarrow\; 6A = 9 \;\&\; 5A + 6B = 0 \;\Rightarrow\; y_p = \tfrac{9}{6}t - \tfrac{5}{4} = 1.5t - 1.25 \text{ works.}$$

A general solution is then:

$$y(t) = y_p(t) + y_h(t) = 1.5t - 1.25 + c_1 e^{-2t} + c_2 e^{-3t}. \qquad \square$$

There must be more to this, and there are, in fact, at least five general methods that are guaranteed to produce solutions for every constant coefficient 2nd order linear non-homogeneous equation like 4.1. In fact, we already showed in the Introduction, while proving Theorem 0.1, Eqn(0.5) gives the unique solution to every 2nd order linear non-homogeneous IVP on an interval of continuity of p, q, f. Before briefly introducing two classical methods that apply to constant coefficient equations, we observe a classical theorem related to forced linear equations of all orders.

Theorem 4.2. (Superposition of forced responses). Suppose $Ly = f_1(t) + f_2(t)$ is a linear ODE, and $Ly_1 = f_1(t)$ and $Ly_2 = f_2(t)$, then $L(y_1 + y_2) = f_1(t) + f_2(t)$.

The proof is left an exercise. Let's compare an example with this theorem to fix ideas.

Example 4.1. Let's find a general solution to $y'' + 5y' + 6y = 9t + e^{-t}$.

We know the general solution to the associated homogeneous $y'' + 5y' + 6y = 0$ is

$$y_h = c_1 e^{-2t} + c_2 e^{-3t}$$

We already solved $Ly = 9t$ in Example 4.0. In Th 4.2, we have solved $Ly = f_1(t) = 9t$, giving $y_{p_1} = 1.5t - 1.25$, now we only need to solve $Ly = f_2(t) = e^{-t}$. Guessing a solution $y_{p_2} = Ae^{-t}$ Substituting into the non-homogeneous equation gives:

$$Ly_{p_2} = y_{p_2}'' + 5y_{p_2}' + 6y_{p_2} = Ae^{-t} - 5Ae^{-t} + 6Ae^{-t} = Ae^{-t}(1 - 5 + 6) = 2Ae^{-t}$$

And $Ly_{p_2} = f_2(t) = e^{-t} \Rightarrow 2Ae^{-t} = e^{-t} \Rightarrow \mathrm{A} = \frac{1}{2}$.

A general solution $y(t) = y_{p_1}(t) + y_{p_2}(t) + y_h(t) = 1.5t - 1.25 + \frac{e^{-t}}{2} + c_1e^{-2t} + c_2e^{-3t}$ □

You can see in Example 4.1 that when the functions f_1, f_2 on the right hand side are of different types, say exponentials with polynomials, and maybe some trig functions, the equations you would need to solve to find all of the coefficients would be more of a challenge but each part can done separately using the superposition Theorem 4.2.

Method of Undetermined Coefficients

The method used in Examples 4.0 & 4.1 to find a solution y_p relies on the ability to guess the form of y_p. For a polynomial right side, and a constant coefficient operator L, the highest degree term on the left will be no higher than that on the right, so a polynomial guess like that in our Examples will almost always yield a y_p despite the tedious work involved in finding all of the coefficients. There are some exceptions and we'll need a rule to handle them.

Here is an exceptional case: $y'' = 9t + 1$. The general solution to the associated homogeneous equation is $y_h(t) = c_1t + c_2$ so if we put $y_p = At + B$ into the non-homogeneous equation we get $0 = 9t + 1$ and the coefficients A, B that determine y_p are, *undetermined*! In fact, no choice of A, B will make $At + B$ a solution to the non-homogeneous equation, because $At + B$ is a solution to the *homogeneous* equation. Here's a little but important theorem.

Theorem 4.3. If $y_1(t)$ is a solution to the homogeneous equation $Ly = 0$, and $Ly = f(t)$ is an associated non-homogeneous equation, then $y_1(t)$ is **not** a solution to $Ly = f(t)$.

Proof. Because $Ly_1 = 0$ surely, $Ly_1 \neq f(t)$ when $f(t) \neq 0$ as it is in the non-homogeneous equation. □

Remark 4.0. Never use a y_p that is a solution to the associated $Ly = 0$ because **it will not work**.

Bearing Remark 4.0 in mind, our first step in solving a non-homogeneous equation should be to solve the associated $Ly = 0$ to rule out any impossible y_p. We still need to use a y_p that "looks like" the right hand side, just make sure it isn't a homogeneous solution.

Exponential forcing functions

Example 4.2. Let's find a general solution to $Ly = y'' + 5y' + 6y = e^{-t}$.

We know the general solution to $Ly = 0$ is

$$y_h = c_1e^{-2t} + c_2e^{-3t}$$

So, $y_p = Ae^{-t}$ looks like the right side of $Ly = e^{-t}$ and is *possible*, because it is not a solution to the $Ly = 0$. Substituting this y_p into $Ly = e^{-t}$:

$$y_p'' + 5y_p' + 6y_p = Ae^{-t} - 5Ae^{-t} + 6Ae^{-t} = e^{-t}(A - 5A + 6A) = 2Ae^{-t} = e^{-t}$$

$$\Rightarrow \quad A = \frac{1}{2}.$$

A general solution is $y(t) = y_p + y_h = \frac{e^{-t}}{2} + c_1e^{-2t} + c_2e^{-3t}$. □

Compare the next example.

Example 4.3. Let's find a general solution to $Ly = y'' + 5y' + 6y = e^{-2t}$.

We know the general solution to $Ly = 0$ is

$$y_h = c_1 e^{-2t} + c_2 e^{-3t}$$

So, $y_p = Ae^{-2t}$ looks like the right side of $Ly = e^{-2t}$ but is **not** possible, because it is a solution to $Ly = 0$. Recalling the trick from the single real root case, we multiply Ae^{-2t} by t to get a better candidate, $y_p = Ate^{-2t}$. Substituting y_p into $Ly = e^{-2t}$ determines A:

$$y_p' = (A - 2At)e^{-2t}$$

$$y_p'' = -2Ae^{-2t} - 2(A - 2At)e^{-2t} = (-4A + 4At)e^{-2t}$$

$$Ly_p = y_p'' + 5y_p' + 6y_p = (-4A + 4At)e^{-2t} + 5(A - 2At)e^{-2t} + 6Ate^{-2t}$$

$$= (A + 4At - 10At + 6At)e^{-2t} = Ae^{-2t}$$

$$Ly_p = e^{-2t} \Rightarrow \; Ae^{-2t} = e^{-2t} \quad \Rightarrow \quad A = 1\,.$$

A general solution is $\quad y(t) = y_p + y_h = te^{-2t} + c_1 e^{-2t} + c_2 e^{-3t}$. □

The "trick" used in Example 4.3 illustrates a more general rule.

Rule 4.0. The 2nd order constant coefficient linear ODE $Ly = f_0 e^{rt}$ has a solution

$$y_p = At^k e^{rt}$$

where k is the the multiplicity of r as a zero of the characteristic polynomial for L. In particular, when r is not a root, $k = 0$, for r one of two distinct real roots $k = 1$, and for r the only real root $k = 2$.[24]

Example 4.4. Let's find a general solution to $Ly = y'' + 2y' + y = 4e^{-t}$.

First, solve $Ly = 0$, which is the equal roots case, where the root $r = -1$ has multiplicity $k = 2$,

$$y_h = (c_1 + c_2 t)e^{-t}.$$

Rule 4.0 implies $k = 2$ and $y_p = At^2 e^{-t}$ will work in $Ly = 4e^{-t}$. We need multiple applications of the product rule to calculate Ly_p, so it's best to be organized.

$$y_p' = 2Ate^{-t} - At^2 e^{-t} \;=\; A(2t - t^2)e^{-t}$$

$$y_p'' = A(2 - 2t)e^{-t} - A(2t - t^2)e^{-t} \;=\; A(2 - 4t + t^2)e^{-t}$$

Now, $\quad L(At^2 e^{-t}) = A(2 - 4t + t^2)e^{-t} + 2A(2t - t^2)e^{-t} + At^2 e^{-t}$

(combining like powers of t) $\quad = e^{-t}(A - 2A + A)t^2 + e^{-t}(-4A + 4A)t + 2Ae^{-t} = 2Ae^{-t}$

$$= 4e^{-t} \Rightarrow A = 2\,.$$

A general solution is $\quad y(t) = y_p + y_h = 2t^2 e^{-t} + c_1 e^{-t} + c_2 te^{-t}$. □

How would we find a general solution to $Ly = y'' + 2y' + y = 3te^{-t}$? Our rule doesn't apply to this $f(t)$ but in the spirit of the rule, we might think $y_p = At^2 e^{-t}$ because it's not a solution to $Ly = 0$. However, we saw in Example 4.4 that $L(At^2 e^{-t}) = 2Ae^{-t}$, which can't equal $3te^{-t}$ for any value A. Using $At^2 e^{-t} + Bte^{-t}$ won't help, because the operator is linear, so $L(At^2 e^{-t} + Bte^{-t}) = L(At^2 e^{-t}) + L(Bte^{-t}) = L(At^2 e^{-t}) + 0 = 2Ae^{-t}$, again, it won't work. Maybe $y_p = At^3 e^{-t}$ would work. Let's see what happens.

$$y_p' = 3At^2 e^{-t} - At^3 e^{-t} \;=\; A(3t^2 - t^3)e^{-t}$$

$$y_p'' = A(6t - 3t^2)e^{-t} - A(3t^2 - t^3)e^{-t} \;=\; A(6t - 6t^2 + t^3)e^{-t}$$

[24] This Rule can be extended to higher order equations where k is the the multiplicity of r as a zero of the characteristic polynomial, and multiplicities higher than 2 are possible.

Now, $L(At^3e^{-t}) = A(6t - 6t^2 + t^3)e^{-t} + 2A(3t^2 - t^3)e^{-t} + At^3e^{-t}$
$= t^3(A - 2A + A)e^{-t} + t^2(-6A + 6A)e^{-t} + 6Ate^{-t} = 6Ate^{-t} \Rightarrow A = \frac{1}{2}$.

It works because of the fortuitous cancellations in the t^3 and t^2 coefficients. It is easy to see $y_p = At^3e^{rt}$ works for any $L = (D - r)^2$, finding[25] $A = f_0/6$ when $f(t) = f_0te^{rt}$. A similar calculation[26] shows $y_p = \frac{f_0t^{n+2}e^{rt}}{(n+2)(n+1)}$ will be a solution when $f(t) = f_0t^ne^{rt}$, proving the following rule.

Rule 4.1. The constant coefficient linear ODE $Ly = (D - r)^2y = f_0t^ne^{rt}$ has a solution

$$y_p = At^{n+2}e^{rt}.$$

In particular, the coefficient $A = \frac{f_0}{(n+2)(n+1)}$.

Remark 4.1. This rule covers every equation $Ly = (D - r)^2y = f_0t^ne^{rt}$ and by Theorem 4.2 (Superposition of forced responses) it covers every constant coefficient equation

$$Ly = (D - r)^2y = F(t)e^{rt}$$

where $F(t) = f_nt^n + \cdots + f_1t + f_0$ is a polynomial, because the solution

$$y_p = A_nt^{n+2}e^{rt} + \cdots A_1t^3e^{rt} + A_0t^2e^{rt}$$

can be constructed by superposition from the Rule 4.1 solutions for

$$Ly = (D - r)^2y = f_kt^ke^{rt}, \quad k = 1 \dots n. \qquad \square$$

Distinct Real Roots

We now turn to the distinct real roots case, $Ly = (D - r_1)(D - r_2)y = f_0t^ne^{rt}$. First observe that when $Le^{rt} \neq 0$ ($r_1 \neq r \neq r_2$), we might expect $y_p = At^ne^{rt}$. However, the fortuitous cancellation we observed above does not occur, so we need to use $y_p = A(t)e^{rt}$, where $A(t) = A_nt^n + \cdots + A_1t + A_0$ is a polynomial and we need to find all $n + 1$ coefficients.

Example 4.5. Let's find a general solution to $Ly = y'' + 5y' + 6y = 3te^{-t}$.

We know the general solution to the associated homogeneous $y'' + 5y' + 6y = 0$ is

$$y_h = c_1e^{-2t} + c_2e^{-3t}$$

If we try $y_p = Ate^{-t}$ we get $y_p' = (A - At)e^{-t}$ and $y_p'' = -Ae^{-t} - (A - At)e^{-t} = -Ae^{-t}$ so $Ly_p = -Ae^{-t} + 5(A - At)e^{-t} + 6Ate^{-t} = 4Ae^{-t} + Ate^{-t}$, which needs to be $3te^{-t}$ but isn't.

The only option is to use $y_p = (A_1t + A_2)\, e^{-t}$ to pick up the extra term in Ly_p. $\square$

Because we need a complete general polynomial $A(t) = A_nt^n + \cdots + A_1t + A_0$ for a one-term forcing function $f_0t^ne^{rt}$, there is no point to using superposition of forced responses to treat different

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[25] Put $y_p = At^3e^{rt}$ into $Ly = (D - r)^2y = y'' - 2ry' + r^2y$.

$$y_p' = 3At^2e^{rt} + rAt^3e^{rt} = A(3t^2 + rt^3)e^{rt}$$
$$y_p'' = A(6t + 3rt^2)e^{rt} + rA(3t^2 + rt^3)e^{rt} = A(6t + 6rt^2 + r^2t^3)e^{rt}$$
$$Ly_p = A(6t + 6rt^2 + r^2t^3)e^{rt} - 2rA(3t^2 + rt^3)e^{rt} + r^2At^3e^{rt}$$
$$= Ae^{rt}(6t + 6rt^2 - 6rt^2 + r^2t^3 - 2r^2t^3 + r^2t^3) = 6Ate^{rt}.$$

[26] Put $y_p = At^{n+2}e^{rt}$ into $Ly = (D - r)^2y = y'' - 2ry' + r^2y$. Leaving a few details to the reader,

$$y_p' = A((n+2)t^{n+1} + rt^{n+2})e^{rt}$$
$$y_p'' = A\big((n+2)(n+1)t^n + 2r(n+2)t^{n+1} + r^2t^{n+2}\big)e^{rt}$$
$$Ly_p = A\big((n+2)(n+1)t^n + 2r(n+2)t^{n+1} + r^2t^{n+2}\big)e^{rt} - 2r\,A\big((n+2)t^{n+1} + rt^{n+2}\big)e^{rt} + r^2At^{n+2}e^{rt}$$
$$= Ae^{rt}\big((n+2)(n+1)t^n + 2r(n+2)t^{n+1} - 2r(n+2)t^{n+1}\big) + Ae^{rt}(r^2t^{n+2} - 2r^2t^{n+2} + r^2t^{n+2})$$
$$= A(n+2)(n+1)t^ne^{rt}.$$

terms in a multi-term forcing function $F(t)e^{rt}$ when $F(t)$ is a polynomial. It turns out to work the same way for the case

$$Ly = (D - r_1)(D - r_2)y = f_0 t^n e^{rt}$$

in which the exponential factor in the forcing function satisfies $Le^{rt} = 0$. The only difference is that, when the exponential in the forcing function satisfies $Le^{rt} = 0$, the polynomial in y_p is $t^k A(t)$, k the multiplicity of r as a root of the characteristic polynomial for L .

Rule 4.2. The 2nd order constant coefficient linear ODE $Ly = (D - r_1)(D - r_2)y = f_0 t^n e^{rt}$, has

$$y_p = A(t)e^{rt}, \quad A(t) = A_n t^n + \cdots + A_1 t + A_0 \quad \Leftrightarrow \quad Le^{rt} \neq 0$$
$$y_p = t^k A(t)e^{rt}, \quad A(t) = A_n t^n + \cdots + A_1 t + A_0 \quad \Leftrightarrow \quad Le^{rt} = 0$$

as a solution, where k is the multiplicity of r as a root of the characteristic polynomial for L.[27]

Periodic forcing

There are rules, similar to Rule 4.0 and Rule 4.1 that apply to periodic forcing. First, we look at a few classic examples.

Example 4.6. Let's find a general solution for $Ly = y'' + 5y' + 6y = \cos t$.

The general solution to $Ly = 0$ is $y_h = c_1 e^{-2t} + c_2 e^{-3t}$, so the forcing function is not a solution to $Ly = 0$. We *want* to use $y_p = A\cos t$ but **it won't work**, because the y' term would be the only term with $A\sin t$.[28] We need

$$y_p = A\cos t + B\sin t$$
$$y_p' = -A\sin t + B\cos t$$
$$y_p'' = -A\cos t - B\sin t$$
$$Ly = -A\cos t - B\sin t + 5(-A\sin t + B\cos t) + 6(A\cos t + B\sin t)$$
$$= (-A + 5B + 6A)\cos t + (-B - 5A + 6B)\sin t = \cos t$$
$$\Rightarrow \quad 5(A + B) = 1 \quad \& \quad 5(B - A) = 0 \quad \Rightarrow \quad A = B = 0.1.$$

A general solution $y(t) = y_p + y_h = 0.1\cos t + 0.1\sin t + c_1 e^{-2t} + c_2 e^{-3t}$ □

The first example shows the wisdom of using $y_p = A\cos\omega t + B\sin\omega t$ even when $f(t) = \cos\omega t$ or similarly, when $f(t) = \sin\omega t$, because terms involving the co-function arise in y_p'. The next example shows when it is safe to use $y_p = A\cos\omega t$.

Example 4.7. Let's find a general solution for $Ly = y'' + y = \cos 2t$.

The general solution to $Ly = 0$ is $y_h = c_1\cos t + c_2\sin t$, so the forcing function is not a solution to $Ly = 0$. Generally, we would use $y_p = A\cos 2t + B\sin 2t$, but this equation has no y' term or sine functions so we should feel safe using $y_p = A\cos 2t$.

$$y_p' = -2A\sin 2t \quad \& \quad y_p'' = -4A\cos 2t \quad \Rightarrow \quad Ly_p = -4A\cos 2t + A\cos 2t = -3A\cos 2t$$
$$Ly_p = \cos 2t \quad \Rightarrow \quad A = -1/3$$

A general solution is $y(t) = y_p + y_h = -\frac{\cos 2t}{3} + c_1\cos t + c_2\sin t$. □

To every rule there seems to be an exception. Here, expect the exception when $f(t)$ is a solution to the associated homogeneous equation $Ly = 0$. The next example illustrates what happens.

[27] By now it should be clear that the coefficient f_0 in a forcing function has no bearing on our choice of the *form* for y_p, since it's only effect is to multiply all coefficients in y_p by f_0.

[28] Because $(A\cos t)'' = -A\cos t$, the other terms $y'' + 6y = -A\cos t + 6A\cos t = 5A\cos t$ and so $L(A\cos t) = 5A\cos t - 5A\sin t = \cos t \Rightarrow A = 1/5 \;\; \& \;\; A = 0$, which is impossible.

Example 4.8. Let's find a general solution for $Ly = y'' + y = \cos t$.

Again $y_h = c_1 \cos t + c_2 \sin t$ but now the forcing function is a solution to $Ly = 0$, so we know to try something different. Using $y_p = At \cos t$ will **not** work because the product rule will produce sine terms. We need $y_p = At \cos t + Bt \sin t$.

$$\begin{aligned} y_p' &= A(\cos t - t \sin t) + B(\sin t + t \cos t) \\ y_p'' &= A(-\sin t - \sin t - t\cos t) + B(\cos t + \cos t - t \sin t) \\ Ly_p &= A(-\sin t - \sin t - t\cos t) + B(\cos t + \cos t - t \sin t) \\ &\quad + At\cos t + Bt \sin t \\ &= 2B \cos t - 2A \sin t \\ Ly_p &= \cos t \;\; \Rightarrow \;\; A = 0, \;\; B = 1/2. \end{aligned}$$

A general solution is $y(t) = y_p + y_h = \frac{t \sin t}{2} + c_1 \cos t + c_2 \sin t$. □

Example 4.9. Let's find a general solution for $Ly = y'' + 2y' + 2y = \cos t$.

Here, $y_h = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$. Some people get fooled by the appearance of $f(t) = \cos t$ in y_h and fail to recognize that $Lf(t) = L \cos t \neq 0$ since $\cos t$ lacks the factor e^{-t} it needs to be a homogeneous solution y_h. So, [29]

$$\begin{aligned} y_p &= A\cos t + B \sin t \;\; \Rightarrow \\ y_p' &= -A \sin t + B \cos t \\ y_p'' &= -A \cos t - B \sin t \, . \end{aligned}$$

$$\begin{aligned} Ly_p &= -A\cos t - B \sin t + 2(-A \sin t + B \cos t) + 2(A \cos t + B \sin t) \\ &= (A + 2B)\cos t + (-2A + B)\sin t \;=\; \cos t \end{aligned}$$

$$\Rightarrow \;\; A + 2B = 1 \quad \& \; -2A + B = 0 \quad \Rightarrow \;\; B = 2A \;\; \& \;\; 5A = 1 \Rightarrow A = 0.2, \;\; B = 0.4$$

$$y_p = \;\; 0.2 \cos t + 0.4 \sin t$$

A general solution is $y(t) = y_p + y_h = \;\; 0.2 \cos t + 0.4 \sin t + c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$. □

Rule 4.3. For the 2nd order constant coefficient linear ODE $Ly = f_0 \cos \omega t$, or the ODE $Ly = f_0 \sin \omega t$

$L \cos \omega t \neq 0 \;\; \Leftrightarrow \;\; y_p = A \cos \omega t + B \sin \omega t$ is a solution;

$L \cos \omega t = 0 \;\; \Leftrightarrow \;\; y_p = t(A \cos \omega t + B \sin \omega t)$ is a solution.

More extensive treatments[30] of the Method of Undetermined Coefficients cover wider classes of forcing functions. For our purposes, we will need no more than our four rules.

Exercises 2.4

1. Find a general solution to each equation.
   i. $y'' - 4y' - 5y = 2t + 3$ *ii.* $y'' - 3y' + 2y = 4t + 1$ *iii.* $y'' + 6y' + 8y = t^2 - 2t$
2. Find a general solution to each equation.
   i. $y'' - 4y' - 5y = 2e^t$ *ii.* $y'' - 3y' + 2y = 2e^{-t}$ *iii.* $y'' + 6y' + 8y = e^{4t}$
3. Find a general solution to each equation.
   i. $y'' - 4y' - 5y = 2e^{-t}$ *ii.* $y'' - 3y' + 2y = 2e^t$ *iii.* $y'' + 6y' + 8y = e^{-4t}$
4. Find a general solution to each equation.
   i. $y'' - 2y' + y = e^t$ *ii.* $y'' + 4y' + 4y = 2e^{-2t}$ *iii.* $y'' + 6y' + 9y = e^{-3t}$
5. Find a general solution to each equation.
   i. $y'' - 4y' - 5y = te^{-t}$ *ii.* $y'' - 3y' + 2y = 2te^t$ *iii.* $y'' + 6y' + 8y = t^2 e^{-4t}$

---

[29] Note the distribution of signs among y_p, y_p' and y_p'' as written in this stack of equations. The lower triangle is negative, upper triangle positive, a helpful pattern to recall.

[30] Trench, W. *Elementary Differential Equations and Boundary Value Problems,* among many others.

6. Find a general solution to each equation.
   i. $y'' - 2y' + y = te^t$ *ii.* $y'' + 4y' + 4y = te^{-2t}$ *iii.* $y'' + 6y' + 9y = t^2e^{-3t}$
7. Find a general solution to each equation.
   i. $y'' + 11y' + 28y = 1 + te^{-4t}$ *ii.* $y'' + 4y' + 4y = (t^2 + t)e^{-2t}$ *iii.* $y'' + 3y' + 2y = \sinh t$
8. Find a general solution to each equation.
   i. $y'' + 2y = e^{-t}$ *ii.* $y'' + 9y = e^{-3t}$ *iii.* $y'' + 9y = t^2e^{-t}$
9. Find a general solution to each equation.
   i. $y'' + 2y = \cos 2t$ *ii.* $y'' + 9y = 4\cos t$ *iii.* $y'' + 16y = \cos 2t$
10. Make plots of your solutions in Problem 9 using the "steady-state" ICs $y(0) = 0,\ y'(0) = 0$.
11. Find a general solution to each equation.
    i. $y'' + 2y = \cos\sqrt{2}t$ *ii.* $y'' + 9y = 4\cos 3t$ *iii.* $y'' + 16y = \cos 4t$
12. Make plots of your solutions in Problem 11 using the"steady-state" ICs $y(0) = 0,\ y'(0) = 0$.
13. Find a general solution to each equation.
    i. $y'' + 2y' + 2y = \cos\sqrt{2}t$ *ii.* $y'' + 2y' + 9y = 4\sin 3t$ *iii.* $y'' + 8y' + 20y = \cos 2\sqrt{5}t$
14. Make plots of your solutions in Problem 13 using the"steady-state" ICs $y(0) = 0,\ y'(0) = 0$.
15. Find a general solution to each equation.
    i. $y'' + 2y' + 2y = \cos t$ *ii.* $y'' + 2y' + 9y = 4\sin 2\sqrt{2}t$ *iii.* $y'' + 8y' + 20y = \cos 4t$
16. Make plots of your solutions in Problem 15 using the"steady-state" ICs $y(0) = 0,\ y'(0) = 0$.

2.5 Variation of Parameters

The method of undetermined coefficients in 2.6 requires new rules for each different type of forcing function. While it is helpful for the few cases we presented in Rules 4.0-4.3, the proliferation of rules makes it unwieldy for many purposes. In this section, we look at a method that will handle any ODE:

$$R(t)y'' + P(t)y' + Q(t)y = F(t)$$

where $R(t), P(t), Q(t), F(t)\epsilon C^0(I)$ for some open interval I, where $R(t) \neq 0$ so the EU Theorem 0.1 applies. Since $(t) \neq 0$, write this equation in the form of Theorem 0.1:

(5.0) $$Ly = y'' + p(t)y' + q(t)y = f(t)$$

To solve Eqn(5.0) we need a particular solution y_p. Assuming we were able to find a solution y_1 for associated homogeneous equation $Ly = 0$, Abel's formula will produce a second, independent solution , y_2. A general form for y_p is

$$y_p = u_1(t)y_1 + u_2(t)y_2.$$

We intend to substitute it into $Ly = F(t)$ to determine the functions u_1 and u_2.

$$y_p' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2$$

Before continuing, we make the simplifying assumption that:

(5.1) $$u_1'y_1 + u_2'y_2 = 0$$

so that $$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'.$$

Collecting terms in multiples of the u_k, leaving the terms $u_1'y_1' + u_2'y_2'$ for last,

$$Ly_p = u_1y_1'' + p(t)u_1y_1' + q(t)u_1y_1 + u_2y_2'' + p(t)u_2y_2' + q(t)u_2y_2 + u_1'y_1' + u_2'y_2'$$

Factoring out the u_k:

$$Ly_p = u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + u_1'y_1' + u_2'y_2'$$

$= u_1' y_1' + u_2' y_2'$

The last equality follows because $y_k'' + p y_k' + q y_k = 0$, as y_k are solutions to $Ly = 0$.
Since, also $Ly_p = f(t)$,

(5.2) $$u_1' y_1' + u_2' y_2' = f(t).$$

The problem is reduced to a system of two equations, (5.1 & 5.2) in the two unknowns, u_1', u_2'. We can solve the system[31] for

(5.3) $$u_1' = -y_2 \frac{f}{W} \quad \& \quad u_2' = y_1 \frac{f}{W}$$

which reduces the problem to a pair of quadratures

(5.4) $$u_1 = -\int y_2 \frac{f}{W}\, dt \;\; \& \;\; u_2 = \int y_1 \frac{f}{W}\, dt$$

determined up to a constant of integration, which can be taken to be zero.

Theorem 5.0. Formulas (5.4) give a particular solution $y_p = -y_1 \int y_2 \frac{f}{W}\, dt + y_2 \int y_1 \frac{f}{W}\, dt$ to Eqn (5.0) where y_1 , y_2 are solutions to $Ly = 0$.

When the indefinite integrals can be found, the quadratures are accomplished symbolically using zero constants of integration. If the integrations are done numerically[32], a starting point t_0 needs to be given for the integrals, initial conditions determined at t_0 for the known solutions y_1 , y_2 and inherited by the Wronskian $W(y_1, y_2)(t)$.

Example 5.1. Let's solve the Euler equation $t^2 y'' + 3ty' - 3y = F(t)$ which has a pair of independent solutions $y_1 = t,\ y_2 = t^{-3}$ with $W(t, t^{-3}) = -4t^{-3}$.

Write the ODE in the form of Eqn.(5.0): $Ly = y'' + \frac{3y'}{t} - \frac{3y}{t^2} = f(t) = \frac{F(t)}{t^2}$, with $p(t) = \frac{3}{t}$ and $q(t) = -\frac{3}{t^2}$.

$$u_1 = -\int t^{-3} \frac{f(t)}{-4t^{-3}}\, dt = \frac{1}{4}\int f(t)\, dt \quad \& \quad u_2 = \int t \frac{f(t)}{-4t^{-3}} dt \;\Rightarrow$$

$$y_p = \frac{t}{4}\int f(t) dt - \frac{t^{-3}}{4}\int t^4 f(t) dt$$ □

---

[31] Multiplying (5.1) by y_2' and (5.2) by y_2 gives:

$$u_1' y_1 y_2' + u_2' y_2 y_2' = 0$$
$$u_1' y_1' y_2 + u_2' y_2' y_2 = y_2 f.$$

Subtracting the second from the first, the $u_2' y_2 y_2'$ terms are eliminated leaving

$$u_1' y_1 y_2' - u_1' y_1' y_2 = -y_2 f$$

Factoring out u_1' and recalling the Wronskian $W(y_1, y_2) = y_1' y_2 - y_1' y_2$, we can write this as:

$$u_1' = -y_2 \frac{f}{W}.$$

Similarly, multiplying (5.1) by y_1' and (5.2) by y_1 and subtracting the results gives:

$$u_2' = y_1 \frac{f}{W}.$$

[32] For numerical integrations, Abel's formula (1.0) gives $W = W(t_0) \exp\left(-\int_{t_0}^{t} p(t) ds\right)$ and formulas (5.4) become

$$u_1 = -\frac{1}{W_0}\int_{t_0}^{t} y_2(\tau) f(\tau) e^{\int_{t_0}^{\tau} p(s) ds}\, d\tau \quad \& \quad u_2 = \frac{1}{W_0}\int_{t_0}^{t} y_1(\tau) f(\tau) e^{\int_{t_0}^{\tau} p(s) ds}\, d\tau .$$

Exercises 2.5

1. Consider the Euler equation $Ly = t^2\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = F(t)$. Find two values of r that will make y_r a solution to $Ly = 0$. Find a general solution to $Ly = F(t)$ expressed in terms of quadratures.
2. Consider the Euler operator $Ly = t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y$. Find two values of r that will make y_r a solution to $Ly = 0$. Find a general solution to the non-homogeneous $Ly = t^3$.
3. Solve $Ly = t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = F(t)$ for each of the following forcing functions:
   i. $F(t) = t^4$ *ii.* $F(t) = t\cos 2t$ *iii.* $F(t) = t^3e^{-t}$ *iv.* $F(t) = t^2\cosh t$

2.6 Numerical Solution of 2nd order $Ly = f(t)$

The proof of the existence and uniqueness Theorem 0.1 in the Introduction uses the associated *system* of differential equations obtained from the IVP

$$y'' + p(t)y' + q(t)y = f(t)\,,\ \ y(t_0) = y_0\,, y'(t_0) = y_0'$$

by making the substitution: $y = y_1,\ y' = y_2$ giving a system of equations:

(6.0)
$$\begin{aligned} y_1' &= y_2 \\ y_2' &= f(t) - q(t)y_1 - p(t)y_2 \end{aligned}$$

System 6.0 is the standard form into which a 2nd order linear equation is usually put to perform a numerical integration. The ICs are then $y_1(t_0) = y_0\,,\ \ y_2(t_0) = y_0'$

Example 6.0. Let's use ode45 to numerically solve $y'' + 6y' + \pi^2y = \cos 3t\ ,\ \ y(0) = 1\,, y'(0) = 0.$

Write the system:
$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \cos 3t - \pi^2y_1 - 6y_2 \end{aligned}$$

Create a function M-File that constructs a *vector* ***dydt*** that can be passed to a solver. Set the vector of ICs and the time span for the integrator, run it and plot the solution (Fig.2-5). □

```
function dydt=d2f(t,y)
dydt=[y(2);cos(3*t)-pi^2*y(1)-6*y(2)];
end
```

```
>> y0=[1;0]; tspan=[0 8];
>> [t,y]=ode45(@d2f,tspan,y0);
>> h=plot(t,y(:,1),'-',t,y(:,2),'--','LineWidth',2);
```

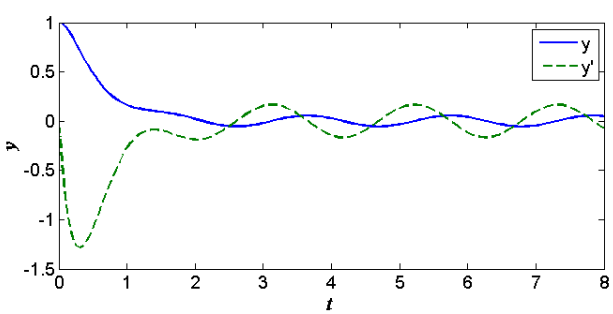

Figure 2-5. Solution $y=y_1$ and derivative $y'=y_2$ for the IVP of Example 6.0.

Exercises 2.6

1. Write the system form of each equation.
i. $y'' + 9y = \cos 3t$
ii. $y'' + 4y' + 9y = \cos 3t$
iii. $y'' + 6y' + 9y = \cos 3t$
iv. $y'' + 8y' + 9y = \cos 3t$

2. For the equations in Problem 1, write a function M-file for dydt, use ode45 to solve the system as an IVP with ICs below. Plot your solution with axis labels, a legend, and a title that shows the equation and ICs plotted.
i. $y(0) = 1\,, y'(0) = 0$
ii. $y(0) = 0\,, y'(0) = 0$
iii. $y(0) = 1\,, y'(0) = 1$

3. Write the system form of each equation.
   i. $y'' + 9y + y^3 = \cos 3t$
   ii. $y'' + 4y' + 9y + y^3 = 4\cos 3t$
   iii. $y'' + 6y' + 9y + y^3 = 6\cos 3t$
   iv. $y'' + 8y' + 9y + y^3 = 8\cos 3t$
4. For the equations in Problem 3, write a function M-file for dydt, use ode45 to solve the system as an IVP with ICs below. Plot your solution with axis labels, a legend, and a title that shows the equation and ICs plotted.
   i. $y(0) = 1, y'(0) = 0$ *ii.* $y(0) = 1, y'(0) = 1$ *iii.* $y(0) = 0, y'(0) = 0$

2.7 Mechanical Vibrations 2: Periodically Forced Harmonic Oscillators

Un-damped Periodically Forced Harmonic Oscillators: Beats & Resonance

Un-damped periodically forced oscillators exhibit several interesting behaviors that have important practical applications, from acoustic phenomena to artificial hearts, from radio signals and sonar sensing to bridge and spacecraft design.

Definition 7.0. The *periodically forced, un-damped harmonic oscillator equation*:

(7.0) $$y'' + \omega_0^2 y = f_0 \cos \omega t$$

has *steady state* initial conditions, $y(0) = 0 = y'(0)$.

We suppose that the frequencies ω, ω_0 are always positive, to simplify the exposition and because they are counts of the number of cycles per 2π.

Part 1. Beats

In cases when the frequencies in equation (7.0) are unequal, $\omega_0 \neq \omega \Rightarrow L\cos\omega t \neq 0$, so Rule 4.3 says, $y_p = A\cos\omega t + B\sin\omega t$ is a solution. Furthermore, $y_p'' + \omega_0^2 x_p = -\omega^2 x_p + \omega_0^2 x_p = f_0 \cos\omega t \Rightarrow$

$$y_p = \frac{f_0}{\omega_0^2 - \omega^2}\cos\omega t\,.$$

The general solution is

$$y(t) = \frac{f_0}{\omega_0^2-\omega^2}\cos\omega t + c_1\cos\omega_0 t + c_2 \sin\omega_0 t\,.$$

The steady-state solution satisfies $y(0) = 0 = \frac{f_0}{\omega_0^2-\omega^2}\cos\omega t + c_1$ and $y'(0) = 0 \Rightarrow c_2 = 0$, so the steady-state solution is $\bar{y}(t) = \frac{f_0}{\omega_0^2-\omega^2}(\cos\omega t - \cos\omega_0 t)\,.$

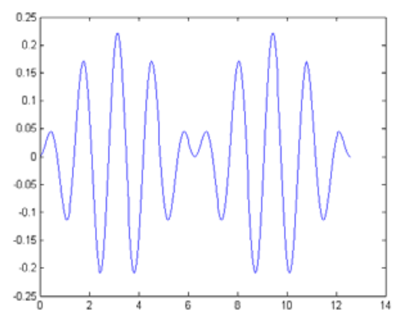

Figure 2-6. Two beats of the response for the solution in Example 7.0. The wave completes 4.5 cycles in each beat.

Example 7.0. $y'' + 16y = \cos 5t$ has steady state solution:

$$\bar{y}(t) = \frac{1}{16-25}(\cos 4t - \cos 5t) = -\frac{1}{9}(\cos 4t - \cos 5t)$$

plotted in Figure 2-6. The response wave is like a periodic drum beat with apparent maxima near $\pi, 3\pi$ and minima (quiet between the beats) near $0, 2\pi, 4\pi$. □

Rewriting $\bar{y}(t)$ as a product of sines,[33] the steady state solution is

(7.1) $$\bar{y}(t) = \frac{2f_0}{\omega_0^2-\omega^2}\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$$

[33] Substitutions: $A - B = \omega t$, $A + B = \omega_0 t$ in the identity $\cos(A-B) - \cos(A+B) = 2\sin A \sin B \Rightarrow$ the steady-state solution $\bar{y}(t) = \frac{2f_0}{\omega_0^2-\omega^2}\sin\frac{\omega_0 t+\omega t}{2}\sin\frac{\omega_0 t - \omega t}{2}$. Factoring t out of the numerators gives (7.1)

Notice the sine wave frequencies, $\frac{\omega_0+\omega}{2} > \frac{\omega_0-\omega}{2}$, the sine function $\sin\left(\frac{\omega_0+\omega}{2}t\right)$ cycles faster than the other. For $\left|\frac{\omega_0+\omega}{\omega_0-\omega}\right| > n$, the faster sine wave, completes more than n cycles during a single cycle of the slower wave. In Example 7.0, $\left|\frac{\omega_0+\omega}{\omega_0-\omega}\right| = 9$, the fast wave has frequency 4.5 and the slow wave has frequency 0.5 producing the beats pictured in Figure 2-6, where $n = 9$. Beats result from a slow periodic variation in the amplitude of a higher frequency sine wave. If we combine the amplitude & slower wave,

$$A(t) = \frac{2f_0}{\omega_0^2-\omega^2}\sin\left(\frac{\omega_0-\omega}{2}t\right),$$

is the slowly varying amplitude, the steady state solution describes an amplitude-modulated sine wave:

$$\bar{y}(t) = A(t)\sin\left(\frac{\omega_0+\omega}{2}t\right).$$

The ratio $\left|\frac{\omega_0+\omega}{\omega_0-\omega}\right| \approx n$, the number of cycles of the fast wave in one cycle of the slower wave. The slowly varying amplitude wave produces two beats per period: one for its maximum $A(t) = \frac{2f_0}{\omega_0^2-\omega^2}$ the other for its minimum $A(t) = -\frac{2f_0}{\omega_0^2-\omega^2}$. The **beat frequency is** $|\omega - \omega_0|$ and the **beat period is** $\frac{2\pi}{|\omega_0-\omega|}$.

Example 7.1. Let's examine the forced harmonic oscillator $y'' + 4\pi^2 y = \cos 6t$ for beats. The natural frequency is $\omega_0 = 2\pi$; forcing frequency is $\omega = 6$ which is near to $2\pi \approx 6.28$. The steady state solution will be:

$$\bar{y}(t) = \frac{2}{4\pi^2 - 36}\sin((\pi+3)t)\sin((\pi-3)t)$$
$$\approx 0.575\sin(6.14\,t)\sin(0.14\,t)$$

The slow frequency is $\pi - 3 \approx 0.14$ so the beat frequency is ≈ 0.28. The ratio $\left|\frac{\omega_0+\omega}{\omega_0-\omega}\right| = \left|\frac{2\pi+6}{2\pi-6}\right| \approx 43$ is about the number of **cycles** of the fast wave in one **cycle** of the slower wave. The slow wave produces **two beats per cycle** giving ≈ 21.7 fast cycles per beat. Alternatively, since the slow period is $\frac{2\pi}{\pi-3}$ the beat period is half that, $\frac{\pi}{\pi-3} \approx 22.19$. The fast sine's period is $\frac{2\pi}{\pi+3} \approx 1.02$ so the solution wave $\bar{y}(t)$completes $\frac{\pi+3}{2(\pi-3)} \approx 21.7$ cycles per beat. Solutions are pictured in Figure 2-7. □

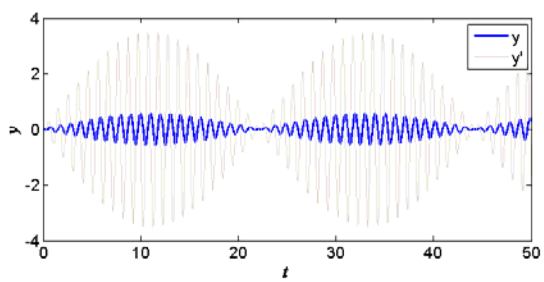

Figure2-7. Steady-state solution $\bar{y}(t)$ (blue) to the forced harmonic oscillator equation of Example 7.1. Beat period $\frac{\pi}{\pi-3} \approx 22.2$ amplitude max $\frac{2f_0}{|\omega_0^2-\omega^2|} \approx 0.575$

In Example 7.1, the forcing frequency is near to the natural frequency of the oscillator, $\omega_0 = 2\pi \approx 6.28 \approx 6 = \omega$. As the two frequencies get closer, the beat period increases and beat amplitude $\frac{2f_0}{|\omega_0^2-\omega^2|}$ grows. Figure 2-8 shows the effect that forcing nearer to the natural frequency has on the oscillator's response. The solution there has $\omega = 6.25$ so that its beat period $\frac{\pi}{\pi-3.125} \approx 189.3$ and amplitude max $\frac{2f_0}{|\omega_0^2-\omega^2|} \approx 4.8$. Both the beat period and amplitude max are dramatically increased.

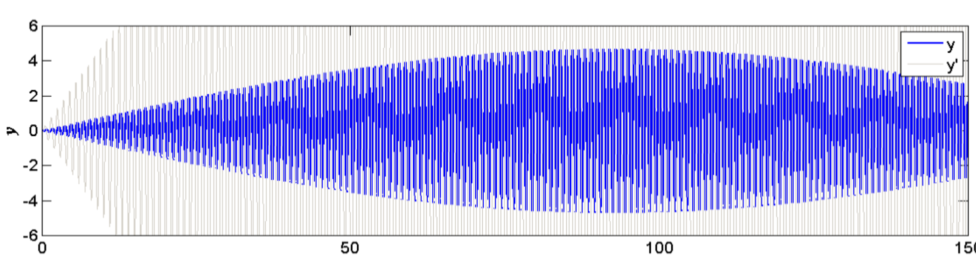

Figure2-8.Oscillator of Example 7.1 forced at ω =6.25. Beat period ≈ 189 and amplitude max≈ 4.8.

Part 2. Resonance

Periodic forcing at exactly the oscillator's natural frequency results in the phenomena called *resonance*. Figures 2-7, 2-8 show beat period and amplitude both increase as the forcing ω approaches natural frequency. When it exactly equals the natural frequency, the period of the beat becomes infinite[34] and the amplitude increases without bound. Applying Rule 4.3 to the resonantly forced oscillator

(7.2) $$y'' + \omega_0^2 y = f_0 \cos \omega_0 t$$

gives, after a brief calculation, a particular solution $y_p = \frac{f_0}{2\omega_0} t \sin \omega_0 t$. The general solution is

(7.3) $$y(t) = \frac{f_0}{2\omega_0} t \sin \omega_0 t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

Steady state ICs $y(0) = 0 \Rightarrow c_1 = 0$ and $\bar{y}'(t) = \frac{f_0}{2\omega_0}(\sin \omega_0 t + t\omega_0 \cos \omega_0 t) + c_2\omega_0 \cos \omega_0 t \Rightarrow$

$$\bar{y}'(0) = 0 = c_2\omega_0 \Rightarrow c_2 = 0.$$

So, the steady-state solution is

(7.4) $$\bar{y}(t) = \frac{f_0}{2\omega_0} t \sin \omega_0 t .$$

Using the slowly varying amplitude idea of Part 1, we can write $A(t) = \frac{f_0}{2\omega_0} t$ for the amplitude in $\bar{y}(t)$ and observe the amplitude grows linearly as a multiple of t.

Example 7.2. Let's re-examine the oscillator of Example 7.1 for resonant response in

$$y'' + 4\pi^2 y = \cos 2\pi t \;\Rightarrow\; \bar{y}(t) = \frac{t \sin 2\pi t}{4\pi}.$$

Figure 2-9 shows the response.

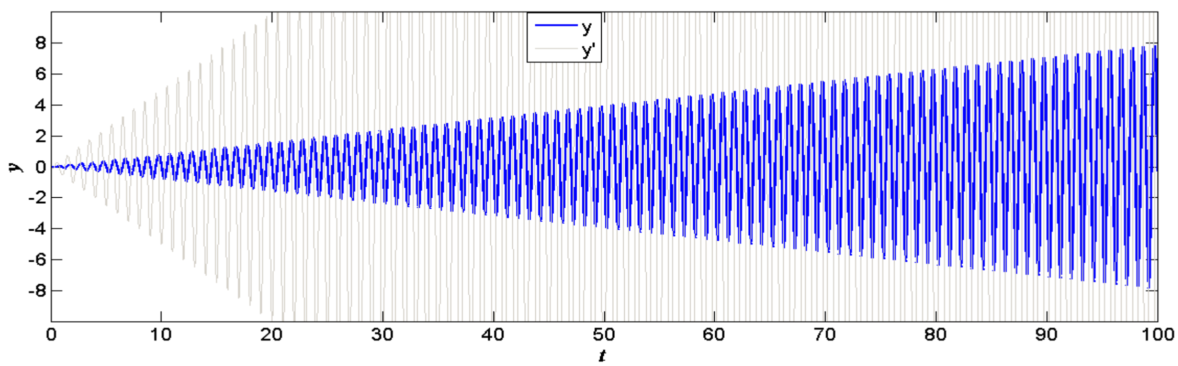

Figure 2-9.Steady state resonant response of Example 7.2.

Exercises 2.7

1. $x'' + 25x = \cos 4t$, $x(0) = 0,\ x'(0) = 0$. Solve the IVP and plot your solution using MatLab "plot". Show enough time to see at least two beats. Find the beat period and frequency. Find the approximate maximum amplitude, and number of fast cycles per beat.
2. $x'' + 25x = \cos 4.5t$, $x(0) = 0,\ x'(0) = 0$. Solve the IVP and plot your solution using MatLab "plot". Show enough time to see at least two beats. Find the beat period and frequency. Find the approximate maximum amplitude, and number of fast cycles per beat.
3. $x'' + 25x = \cos 5t$, $x(0) = 0,\ x'(0) = 0$. Solve the IVP and plot your solution using MatLab "plot". Show enough time to see at least ten zeros of your solution.

[34] Of course, there really is no beat in this case. The amplitude just increases for all time. No mechanical system could sustain such an increase forever, because strain due to the large amplitude oscillations would eventually cause the system to tear itself apart. Recall the excitation of resonant vibration in a wine glass by a sustained musical note at the right frequency will excite the glass until it breaks, ending the experiment.

4. $x'' + 60x = \cos 8t$, $x(0) = 0,\ x'(0) = 0$. Solve the IVP and plot your solution using MatLab "plot". Show enough time to see at least one beat. Find the beat period and frequency. Find the approximate maximum amplitude and number of fast cycles per beat.
5. $x'' + 64x = \cos 8t$, $x(0) = 0,\ x'(0) = 0$. Solve the IVP and plot your solution using MatLab "plot". Show enough time to see at least ten zeros of your solution.
6. $x'' + 81x = \cos 9t$, $x(0) = 0,\ x'(0) = 0$. Solve the IVP and plot your solution using MatLab "plot". Show enough time to see at least ten zeros of your solution.

2.8 Mechanical Vibrations 3: Periodically Forced Damped Oscillator

We are now ready for the most commonly applied harmonic oscillator equation. In all macro-scale mechanical devices, there is some loss of energy through friction, heating of components, or some other form of drag that is conceptually simplest to understand in the second order linear model we will study here. Additionally, this equation arises as the model for response of a simple RCL circuit[35] to a sinusoidal alternating current. For simplicity, we consider the mechanical model first.

Definition 8.0. The *periodically forced, damped harmonic oscillator equation*:

(8.0) $$y'' + 2py' + \omega_0^2 y = f_0 \cos \omega t$$

has *steady state* initial conditions, $y(0) = 0 = y'(0)$.

The homogeneous solutions to Eqn.(8.0) are $y_h = c_1 e^{-pt} \cos\sqrt{\omega_0^2 - p^2}\,t + c_2 e^{-pt} \sin\sqrt{\omega_0^2 - p^2}\,t$ when under-damped, $y_h = e^{-pt}(c_1 + c_2 t)$ when critically damped, and $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ when overdamped with $r_k = -p + (-1)^k\sqrt{p^2 - \omega_0^2}$. The forcing function $f_0 \cos \omega t$ is clearly not in any case a solution to the homogeneous Equation, so by Rule 4.3 $y_p = A\cos\omega t + B \sin \omega t$ is a solution to (8.0).

Equation (8.0) can model a mass-spring system where the mass is driven periodically with force $F_0 \cos \omega t$ while attached by a spring to a fixed support, like the mass in Figure 2-10. This mass-spring system can be given the somewhat realistic feature of linear drag with coefficient c. The equation for the motion of the mass would be $my'' + cy' + Ky = F_0 \cos \omega t$ where $c/m = 2p$, $K/m = \omega_0^2$ and $F_0/m = f_0$ in (8.0). Choosing units in which the mass $m = 1$, so $c = 2p$, $K = \omega_0^2$ & $F_0 = f_0$, (8.0) becomes

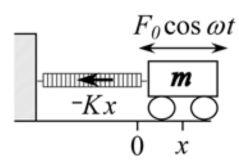

Fig. 2-10. Periodically forced mass-spring system.

(8.1) $$y'' + cy' + Ky = f_0 \cos \omega t\ ,$$

with particular solution $y_p = A\cos\omega t + B\sin\omega t$ and general solution $y(t) = y_p + y_h$.

Because $c > 0 \Rightarrow p > 0$, the homogeneous response $y_h \to 0$ so $y(t)$ decays to the steady state oscillator:

(8.2) $$y_p = A\cos\omega t + B\sin\omega t = R\cos(\omega t - \varphi)$$

for *every* initial condition.

The amplitude $R = \sqrt{A^2 + B^2}$ of the steady-state solution 8.2 depends on $p,\ \omega_0^2,\ f_0,\ \&\ \omega^2$ according to[36]

(8.3) $$R^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + (2p\omega)^2}\ .$$

[35] Resistor-Capacitor-Inductor circuit where R is resistance in the circuit, C capacitance, and L inductance of an inductor (e.g. a current bearing coil) influencing the circuit.
[36] Derivation of this formula is detailed in Exercises 10 & 11.

Practical Resonance

Since p, ω_0^2 depend on the mass-spring system, we assume the forcing frequency ω is tunable. The steady-state response amplitude R will be maximized by tuning ω to minimize the denominator in 8.3. We can recognize this denominator as a quadratic function g of the non-negative variable $u = \omega^2$, where

$$g(u) = (\omega_0^2 - u)^2 + 4p^2u.$$

Since g is a positive quadratic polynomial, its minimum occurs at the value of $u = \omega^2$ where

$$\frac{dg}{du} = 4p^2 - 2(\omega_0^2 - u) = 0.$$

Solving for $u = \omega_0^2 - 2p^2$, substituting ω^2 for u gives, $\omega^2 = \omega_0^2 - 2p^2$, so the value

(8.4) $$\omega_{res} = \sqrt{\omega_0^2 - 2p^2}\,,$$

is the frequency of *practical resonance* for Eqns.(8.0 & 8.1), at which the amplitude of response R will be maximized. In particular, we observe the condition:

(8.5) $$\omega_0^2 > 2p^2$$

must hold for practical resonance to occur in the forced damped linear oscillator 8.0 with $p > 0$. In the damped mass-spring system (8.1), where $c > 0$, the condition for practical resonance is

(8.6) $$2K > c^2\,.$$

Example 8.1. We wish to maximize the amplitude of response in the mass-spring system with equation $y'' + 2y' + 4y = \cos\omega t$ by tuning the forcing frequency ω. First, we check that $2K > c^2$ so practical resonance exists. Here, $2K = 8 > c^2 = 4$, so we can proceed by tuning the forcing frequency to ω_{res} given by (8.4) $\omega_{res} = \sqrt{K - \frac{c^2}{2}} = \sqrt{2}$. Using this forcing frequency, with $p = 1$, $\omega_0^2 = 4$ in Eqn.(8.3), $R^2 = \frac{1}{(4-2)^2 + (2\sqrt{2})^2} = \frac{1}{12}$ and the steady-state solution is[37]

(8.7) $$y_p(t) = \frac{1}{2\sqrt{3}}\cos\left(\sqrt{2}t - \varphi\right).$$ □

Example 8.2. Suppose we wish to tune ω, so as to maximize the amplitude of response in the mass-spring system with equation $y'' + 3y' + 4y = 10\cos\omega t$. We find $2K = 8 < c^2 = 9$, so there is no practical resonance. Equations 8.2 and 8.3 still hold, giving the steady-state solution[38]

(8.8) $$y_p(t) = \frac{10\cos(\omega t - \varphi)}{\sqrt{(4-\omega^2)^2 + (3\omega)^2}} = \frac{10\cos(\omega t - \varphi)}{\sqrt{\omega^4 + \omega^2 + 16}} \le \frac{10}{4} = 2.5\,.$$

The amplitude decreases as ω^2 increases. We would then maximize the response amplitude by taking $\omega^2 \to 0$, to minimize the denominator.

The forcing would then be the constant $f_0 = 10$ and y_p would, of course, **not** be of the form $A\cos\omega t + B\sin\omega t$; it would be just a constant solution $y_p = 2.5$. □

The case of Example 8.2 illustrates what happens in the absence of practical resonance. The value of ω^2 that maximizes R by minimizing the denominator $g(\omega^2)$ in (8.3) turns out to be $\omega^2 = 0$ so the maximum amplitude R_{res} is not achieved. All frequencies $\omega > 0$ lead to the steady-state solution (8.8). As the frequency of the forcing is reduced to zero, the amplitude $R(\omega) \to R(0) = |f_0|/\omega_0^2$ but

[37] Determination of the phase angle $\varphi = \arctan\sqrt{2}$ is left as Exercise 12.
[38] The phase angle is $\varphi = \arctan\frac{c\omega}{\omega_0^2 - \omega^2} = -\arctan 1.2$ (see Exercise 13).

the period of the response, $2\pi/\omega \to \infty$. We can conclude from this that for a damping coefficient $c^2 > 2K$, the absence of practical resonance just means $R(\omega)$ decreases with increasing ω , so the amplitude $R(\omega) < |f_0|/K$ for all ω.

Exercises 2.8

1. Find the general solution to $y'' + 4y' + 9y = \cos\omega t$ for each of the following values of ω:
   i. $\omega = 0.5$ *ii.* $\omega = 1$ *iii.* $\omega = 2$
2. For each of the frequencies in Problem 1, find the solution to the IVP $y'' + 4y' + 9y = \cos\omega t$, $y(0) = y_0$, $y'(0) = 0$, using $y_0 = -2, 0, 2$ and make one plot for each ω showing the three solutions for the three different ICs. Be sure you plot the solutions on a long enough time interval to see at least 4 periods of the steady state solution and include a legend in each plot.
3. Find the general solution to $y'' + 6y' + 20y = \cos\omega t$ for each of the following values of ω:
   i. $\omega = 1$ *ii.* $\omega = 2$ *iii.* $\omega = 3$
4. For each of the frequencies in Problem 3, find the solution to the IVP $y'' + 6y' + 20y = \cos\omega t$, $y(0) = y_0$, $y'(0) = 0$, using $y_0 = -2, 0, 2$ and make one plot for each ω showing the three solutions for the three different ICs. Be sure you plot the solutions on a long enough time interval to see at least 4 periods of the steady state solution and include a legend in each plot.
5. Use MatLab ode45 to solve the IVPs in problems 2 & 4. Making the corresponding plots using your numerical solutions. [Recall you will need to save the [*t*,*y*] value arrays with different names for each solution in each plot.]
6. Find the steady-state solution and find the frequency of practical resonance.
   i. $y'' + 4y' + 12y = \cos\omega t$ *ii.* $y'' + 4y' + 20y = \cos\omega t$ *iii.* $y'' + 4y' + 40y = \cos\omega t$
7. Find the steady-state solution and find the frequency of practical resonance.
   i. $y'' + 2y' + 20y = \cos\omega t$ *ii.* $y'' + 4y' + 20y = \cos\omega t$ *iii.* $y'' + 6y' + 20y = \cos\omega t$
8. Find the steady-state solution, frequency of practical resonance and maximum amplitude of resonant response.
   i. $y'' + 4y' + 12y = \cos\omega t$ *ii.* $y'' + 4y' + 20y = \cos\omega t$ *iii.* $y'' + 4y' + 40y = \cos\omega t$
9. Find the steady-state solution, frequency of practical resonance and maximum amplitude of resonant response.
   i. $y'' + 2y' + 20y = \cos\omega t$ *ii.* $y'' + 4y' + 20y = \cos\omega t$ *iii.* $y'' + 6y' + 20y = \cos\omega t$
10. Substitute $y_p = A\cos\omega t + B\sin\omega t$ into Eqn.(8.0), to obtain a pair of equations satisfied by the coefficients A & B in terms of $2p\omega$, $\omega_0^2 - \omega^2$, & f_0 . Show how to solve this system of equations to arrive at these formulas for A & B:
$$A = \frac{(\omega_0^2 - \omega^2)f_0}{(\omega_0^2 - \omega^2)^2 + (2p\omega)^2}\ , \quad B = \frac{2p\omega f_0}{(\omega_0^2 - \omega^2)^2 + (2p\omega)^2}$$
11. Show how the formulas for A & B in Problem 10 lead to Eqn.(8.3).
12. Use the formulas from Problem 10 to find the phase shift φ in Example 8.1.
13. Use the formulas from Problem 10 to find the phase shift φ in Example 8.2.
14. Use the formulas from Problem 10 to find the steady-state response to practical resonant forcing in phase amplitude form.
    i. $y'' + 4y' + 12y = \cos\omega t$ *ii.* $y'' + 4y' + 20y = \cos\omega t$ *iii.* $y'' + 4y' + 40y = \cos\omega t$
15. Use the formulas from Problem 10 to find the steady-state response to practical resonant forcing in phase amplitude form. Plot your solutions.
    i. $y'' + 2y' + 20y = \cos\omega t$ *ii.* $y'' + 4y' + 20y = \cos\omega t$ *iii.* $y'' + 6y' + 20y = \cos\omega t$
16. Find the steady-state solutions to
    i. $y'' + 3y' + 4y = 6\cos t$ *ii.* $y'' + 3y' + 4y = 6\cos(0.1t)$ *iii.* $y'' + 3y' + 4y = 6$.
    Plot them together for $0 \le t \le 10\pi$. Briefly explain the plot in light of the discussion after Example 8.2.
17. Find the steady-state solutions to
    i. $y'' + 6y' + 8y = 5\cos(0.5t)$ *ii.* $y'' + 6y' + 8y = 5\cos(0.05t)$ *iii.* $y'' + 6y' + 8y = 5$.
    Plot them together for $0 \le t \le 10\pi$. Briefly explain the plot in light of the discussion after Example 8.2.

2.9 Series Solutions of $Ly = 0$

Part 1. Introduction

A number of important examples in applications depend on our ability to obtain explicit expressions for solutions of certain non-constant coefficient homogeneous second order equations. We have already worked with Euler equations $Ly = at^2y'' + bty' + cy = 0$ into which we substituted $y = t^r$ to get a quadratic we could solve for r. Other important examples, while similar in this regard, do not have solutions in terms of elementary functions but their solutions can be obtained as power series. Power series solutions have advantages over numerical solutions, especially because the behavior of the solutions, can sometimes be characterized independent of the choice of initial conditions.

As you may recall, a Taylor series can be developed[39] for $f(t)$ at t_0 when it has infinitely many continuous derivatives on an open interval I about the point t_0. In solving ODEs using series, we look at it from the opposite direction. The solution to $Ly = 0$ is assumed to be given by a power series

$$y(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^k$$

in which the coefficients $a_k,\ k = 0 \dots \infty$ determine the function $y(t)$. By substituting the infinite series into the ODE, we hope to determine the coefficients, much as we did with the method of undetermined coefficients in 2.4. Under certain standard conditions, the coefficients a_k determine a series that converges near the point t_0, so the series gives a solution to the ODE.

Example 9.0. The Airy[40] equation $Ly = y'' - t\,y = 0$ can be solved near $t_0 = 0$ by substituting the power series $y(t) = \sum_{k=0}^{\infty} a_k t^k$ into the ODE. We write:

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \quad \text{so} \quad ty = a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + \cdots$$

$$y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots$$

$$y''(t) = 2a_2 + (2\cdot 3)a_3 t + (3\cdot 4)a_4 t^2 + (4\cdot 5)a_5 t^3 \dots \quad \text{and} \quad Ly = y'' - t\,y \Rightarrow$$

$$Ly = (2a_2 + (2\cdot 3)a_3 t + (3\cdot 4)a_4 t^2 + \cdots) - (a_0 t + a_1 t^2 + a_2 t^3 + a_3 t^4 + \cdots) \Rightarrow$$

$$Ly = 2a_2 + (6a_3 - a_0)t + (12a_4 - a_1)t^2 + (20a_5 - a_2)t^3 + (30a_6 - a_3)t^4 \dots$$

The Airy equation is $Ly = 0$ so every coefficient in Ly must be zero. That gives a list of equations that determines most of the coefficients:

$$2a_2 = 0, \quad 6a_3 = a_0, \quad 12a_4 = a_1, \quad 20a_5 = a_2, \quad 30a_6 = a_3, \quad 42a_7 = a_4 \dots$$

Solving these for the later coefficients in terms of earlier ones:

$$a_2 = 0, \quad a_3 = \frac{a_0}{6}, \quad a_4 = \frac{a_1}{12}, \quad a_5 = a_2 = 0, \quad a_6 = \frac{a_3}{30} = \frac{a_0}{180},$$

$$a_7 = \frac{a_4}{42} = \frac{a_1}{(3\cdot 4)(6\cdot 7)} = \frac{a_1}{504}, \quad a_8 = a_5 = 0 \dots$$

The coefficients a_0 and a_1 are the constant terms in (t), $y'(t)$, so they are determined by the ICs $y(0) = a_0$, $y'(0) = a_1$. Since all other coefficients depend on a_0 and a_1, *all* coefficients are determined by the ICs. We can write the solution to any IVP at $t_0 = 0$ as

$$y(t) = a_0 + a_1 t + 0t^2 + \frac{a_0}{6}t^3 + \frac{a_1}{12}t^4 + 0t^5 + \frac{a_0}{180}t^6 + \frac{a_1}{504}t^7 + \cdots.$$

[39] Coefficients are $f^{(k)}(t_0)/n!$.

[40] G.B. Airy (1801-1892) British astronomer and mathematician, encountered the equation in optics. It is also related to quantum tunneling in the time-independent, one-dimensional Schrödinger equation.

Further, by separating terms in a_0 and a_1, and factoring these out

$$y(t) = a_0\left(1 + \frac{t^3}{6} + \frac{t^6}{180} + \cdots\right) + a_1\left(t + \frac{t^4}{12} + \frac{t^7}{504} + \cdots\right).$$

We can identify a fundamental set of solutions at $t_0 = 0$:

$$y_1(t) = 1 + \frac{t^3}{6} + \frac{t^6}{180} + \frac{t^9}{12960} + \cdots \quad \text{and} \quad y_2(t) = t + \frac{t^4}{12} + \frac{t^7}{504} + \frac{t^{10}}{45360} + \cdots$$

for which the solution with ICs $y(0) = y_0$, $y'(0) = y_0'$ is $y(t) = y_0\, y_1(t) + y_0'\, y_2(t)$. □

Briefly, the only question left is convergence of the series. The series in Example 9.0 surely have radius of convergence $\rho = \infty$. A general theorem on radius of convergence is repeated here for convenience.

Theorem 9.0. (Ratio Test[41]) $\sum_{k=0}^{\infty} a_k(t - t_0)^k$ converges absolutely for $0 < |t - t_0| < \rho$ when

$$\rho = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|.$$

Because $\sum_{k=0}^{\infty} a_k(t - t_0)^k$ may converge only at $t = t_0$, it may not converge on any open interval $0 < |t - t_0| < \rho$, in which case it would be a *divergent* series.[42] We will give the standard EU theorem guaranteeing the series converge on some interval $I = (t_0 - \rho, t_0 + \rho)$.

Definition 9.0. The interval $I = (t_0 - \rho, t_0 + \rho)$ in Theorem 9.0 is the *interval of convergence* for the series and ρ is called the *radius of convergence* of the series in Theorem 9.0.

Theorem 9.1. Every series $\sum_{k=0}^{\infty} a_k(t - t_0)^k$ convergent in I represents a C^∞ function in I.

Proof. We need the derivative series $\frac{d}{dt}\sum_{k=0}^{\infty} a_k(t - t_0)^k = \sum_{k=1}^{\infty} k a_k(t - t_0)^{k-1}$ convergent in I. The ratio test for the derivative series has $\lim\limits_{k\to\infty}\left|\frac{k a_k}{(k+1)a_{k+1}}\right| = \lim\limits_{k\to\infty}\left|\frac{k}{k+1}\frac{a_k}{a_{k+1}}\right| = \lim\limits_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \rho$. So, the derivative series has the same interval of convergence and applying this to the derivative series, its derivative $\frac{d^2}{dt^2}\sum_{k=0}^{\infty} a_k(t - t_0)^k$ also has radius of convergence ρ, and so on. □

Definition 9.1. The set of C^∞ functions representable by powers series $\sum_{k=0}^{\infty} a_k(t - t_0)^k$ convergent in I is called $\mathcal{A}(I)$, the set of *analytic* functions on I.

All polynomials and the familiar functions $\exp t, \cos t, \sin t$ are analytic on $(-\infty, \infty)$. Any ratio of polynomials is analytic wherever the denominator is non-zero. All powers t^r are analytic for $t > 0$. Finite sums and products, as well as many well-defined compositions of analytic functions on an interval I are again analytic on I. We will use all of these facts without proof.

In section 2.5 we showed how to use variation of parameters to produce a solution to

$$R(t)y'' + P(t)y' + Q(t)y = F(t)$$

where $R, P, Q, F \epsilon C^0(I)$ and $R(t) \neq 0$ on some open interval I, given a solution $y_1(t)$ to the homogeneous equation. We wrote this equation in the form of the EU Theorem 0.1:

$$Ly = y'' + p(t)y' + q(t)y = f(t)$$

[41] See any Calculus text.

[42] Furthermore, it may diverge at one or both endpoints of the closed interval $I = [t_0 - \rho, t_0 + \rho]$ but we will avoid such details, since we are only interested in solutions, which must exist on open intervals.

guaranteeing a unique solution for every IC at $t_0 \epsilon I$. Here we will suppose that $R, P, Q, F \epsilon\, \mathcal{A}(I)$, so they can all be represented by convergent power series in I about some initial time $t_0 \epsilon I$. Leaving the non-homogeneous problem to variation of parameters, we can focus on the homogeneous equation.

Definition 9.2. t_0 is an *ordinary point* for

(9.0) $$R(t)y'' + P(t)y' + Q(t)y = 0$$

if there is an open interval with $t_0 \epsilon I$ where $p(t) = \frac{P(t)}{R(t)}, q(t) = \frac{Q(t)}{R(t)}$ are analytic on I. Otherwise, t_0 is a *singular point*.

The simplest examples of analytic functions are polynomials, which are just power series with only finitely many non-zero coefficients. It turns out that a number of important examples in Science and Engineering concern 2nd order linear ODEs with polynomial coefficients. When R, P, Q in Def. 9.2 are polynomials then, supposing they have no common factors, t_0 is an ordinary point if $R(t_0) \neq 0$ and singular if $R(t_0) = 0$. If t_0 is ordinary, $p(t), q(t)$ will be analytic on an open interval about t_0 wherein both of their Taylor series converge. The radius of convergence of both the series for $p(t), q(t)$ is $\rho =$ the distance from t_0 to the nearest zero of R in the complex plane.

Theorem 9.2. For t_0 an ordinary point for $R(t)y'' + P(t)y' + Q(t)y = 0$, with R, P, Q polynomials[43], then every solution can be represented

$$y(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^k$$

as a series solution convergent on an interval $I = (t_0 - \rho, t_0 + \rho)$ where $\rho =$ minimum distance from t_0 to any complex number c for which $R(c) = 0$.

Theorem 9.2 guarantees that for polynomial coefficient functions, we can always develop power series solutions much as was done in Example 9.0 for the Airy equation as long as t_0 is ordinary. The calculation of coefficients is tedious but the results give rise to interesting special functions with important applications in Applied Mathematics, Statistics, Physics, and Engineering.[44]

Example 9.2. Hermite's[45] equation $Ly = y'' - 2t\,y' + 2ny = 0$ can be solved near $t_0 = 0$ by substituting the power series $y(t) = \sum_{k=0}^{\infty} a_k t^k$ into the ODE. We write:

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \qquad y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots$$

$$y'' = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 30a_6 t^4 + 42a_7 t^5 \ldots$$

$$-2ty' = \quad -2a_1 t - 4a_2 t^2 - 6a_3 t^3 - 8a_4 t^4 - 10a_5 t^5 - \cdots$$

$$2ny = 2n\,a_0 + 2na_1 t + 2na_2 t^2 + 2na_3 t^3 + 2na_4 t^4 + 2na_5 t^5 + \cdots$$

$$Ly = 2a_2 + 2n\,a_0 + (6a_3 - 2a_1 + 2na_1)t + (12a_4 - 4a_2 + 2na_2)t^2$$
$$+(20a_5 - 6a_3 + 2na_3)t^3 + (30a_6 - 8a_4 + 2na_4)t^4 + (42a_7 - 10a_5 + 2na_5)t^5 + \cdots$$

$$a_2 = -na_0,\ a_3 = \frac{2(1-n)a_1}{6},\ a_4 = \frac{2(2-n)a_2}{12}, a_5 = \frac{2(3-n)a_3}{20}, a_6 = \frac{2(4-n)a_4}{30}, a_7 = \frac{2(5-n)a_5}{42}, \ldots$$

Choosing ICs so $a_0 = 1, a_1 = 0$ gives a solution

$$y_{1,n} = 1 - nt^2 - \frac{n(2-n)}{6}t^4 - \frac{n(2-n)(4-n)}{90}t^6 - \cdots$$

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[43] We note that R is not identically zero, whenever we write an operator as second order.

[44] A good start on methods can be found in Chapter 7 of either Lebl or Trench (op cit). The interested reader should further consult the literature on Airy, Bessel, Chebyshev, Legendre, and Hermite equations.

[45] Named for Charles Hermite (French 1822-1901) but originated 1810, Pierre-Simon Laplace (French 1749-1827).

Choosing ICs so $a_0 = 0, a_1 = 1$ gives another solution

$$y_{2,n} = t + \frac{1-n}{3}t^3 + \frac{(1-n)(3-n)}{30}t^5 + \cdots$$

Integer values $n = 0,2,4, \ldots$ lead to even polynomials:

$$y_{1,0} = 1,\ \ y_{1,2} = 1 - 2t^2,\ \ y_{1,4} = 1 - 4t^2 + \frac{8}{6}t^4$$

Integer values $n = 1,3,5, \ldots$ lead to odd polynomials:

$$y_{2,1} = t,\ \ y_{2,3} = t - \frac{2}{3}t^3,\ \ y_{2,5} = t - \frac{4}{3}t^3 + \frac{8}{30}t^5$$

Rewritten and rescaled to have positive leading coefficients ascending in powers of two, these polynomial solutions to the family of Hermite ODEs exhibit an orthogonal basis for the space of all polynomials:[46]

$H_0 = 1$
$H_1 = 2t$
$H_2 = 4t^2 - 2$
$H_3 = 8t^3 - 12t$
$H_4 = 16t^4 - 48t^2 + 12$
$H_5 = 32t^5 - 160t^3 + 120t \ldots$ □

Regular Singular Points

Because of the importance of solutions to 2nd order linear equations with singular points in applications, we can't conclude our discussion of power series solutions without briefly entering into the matter.

Definition 9.3. Values t_0 for which $R(t_0) = 0$ but both limits

$$\lim_{t\to t_0}(t - t_0)\frac{P(t)}{R(t)} \quad \& \quad \lim_{t\to t_0}(t - t_0)^2\frac{Q(t)}{R(t)} \tag{9.1}$$

exist and are finite are called *regular singular points* for $R(t)y'' + P(t)y' + Q(t)y = 0$.

Euler equations

Euler equations $t^2y'' + bty' + cy = 0$ are simple examples of equations with a regular singular point at $t_0 = 0$. We can always find a pair of independent solutions to an Euler equation. We observed previously that the substitution of $y = t^r$ yields the quadratic equation[47]

$$r(r-1) + br + c = 0$$

with solutions $r = \frac{1-b}{2} \pm \frac{\sqrt{(b-1)^2-4c}}{2}$ wherein, if we let $2\beta = b - 1$, the exponents

$$r_{\pm} = -\beta \pm \sqrt{\beta^2 - c}$$

will be real distinct for $\beta^2 - c > 0$, complex conjugate for $\beta^2 - c < 0$, and single real $r = -\beta$ when $c = \beta^2$.

In the first case we obtain a pair of independent solutions: $y_1 = t^{r_+}, y_2 = t^{r_-}$. In the complex case, say $r = -\beta + i\omega$, since $y_C = t^{-\beta+i\omega}$ is a complex solution to a real equation, both its real and imaginary parts[48] are real independent solutions,

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[46] Hermite polynomials H_k provide localized wave forms for the probability densities of the steady-state solutions

$$\varphi_k(x) = \left(2^k k!\sqrt{\pi}\right)^{-\frac{1}{2}}\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} H_k\left(\frac{m\omega}{\hbar}x\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

to the Schrodinger equation for the quantum harmonic oscillator.

[47] A special case of the *indicial equation* of Frobenius' Method.

[48] Noting $t^{-\beta+i\omega} = t^{-\beta}e^{\ln t^{i\omega}} = t^{-\beta}e^{i\omega\ln t} = t^{-\beta}(\cos(\omega\ln t) + i\sin(\omega\ln t))$ by Euler's formula.

$$\Re(y_C) = t^{-\beta}\cos(\omega \ln t)\,,\ \Im(y_C) = t^{-\beta}\sin(\omega \ln t).$$

For the single real case, $r = -\beta$ one solution is $y_1 = t^{-\beta}$ and a second independent solution is $y_2 = t^{-\beta}\ln t$. □

This situation occurs in more general settings. Cases of second order linear equations where the polynomial coefficients are at most quadratic find a wide variety of physical applications. One of the most famous is the Bessel[49] equation, whose solutions, *Bessel functions,* are extremely common in the physical literature and, like the Hermite polynomials, are special solutions to a parameterized family of homogeneous linear ODEs. The Bessel functions occur when solving the wave equation in material media.

Bessel Functions

Definition 9.4. Bessel's differential equation of *order*[50] $p \geq 0$ is

$$t^2y'' + ty' + (t^2 - p^2)y = 0.$$

Solutions to Bessel's equation are found by substituting $y(t) = t^r \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} a_k t^{k+r}$ into the equation:

$$y = a_0t^r + a_1t^{r+1} + a_2t^{r+2} + a_3t^{r+3} + \cdots$$
$$y' = a_0rt^{r-1} + (r+1)a_1t^r + (r+2)a_2t^{r+1} + (r+3)a_3t^{r+2} + \cdots$$
$$y'' = a_0r(r-1)t^{r-2} + (r+1)ra_1t^{r-1} + (r+2)(r+1)a_2t^r + (r+3)(r+2)a_3t^{r+1} + \ldots$$

$$t^2y'' = a_0r(r-1)t^r + (r+1)ra_1t^{r+1} + (r+2)(r+1)a_2t^{r+2} + (r+3)(r+2)a_3t^{r+3} + \ldots$$
$$ty' = a_0rt^r + (r+1)a_1t^{r+1} + (r+2)a_2t^{r+2} + (r+3)a_3t^{r+3} + \cdots$$
$$t^2y = a_0t^{r+2} + a_1t^{r+3} + a_2t^{r+4} + a_3t^{r+5} + \cdots$$
$$-p^2y = -p^2(a_0t^r + a_1t^{r+1} + a_2t^{r+2} + a_3t^{r+3} + \cdots)$$
$$Ly = t^r[\,a_0(r^2-p^2) + a_1((r+1)^2-p^2)t + (a_2((r+2)^2-p^2) + a_0)t^2$$
$$+(a_3((r+3)^2-p^2) + a_1)t^3 + \cdots]$$

The indicial equation is $r^2 - p^2 = 0 \;\Rightarrow\; r = \pm p$.

Since the ODE is linear homogenous, we are free to scale y by any constant, so we set $a_0 = 1$.

For p not an integer, one solution is found by substituting $r = p$ into $Ly = 0$. Observe that

$a_1((p+1)^2 - p^2) = a_1(p^2 + 2p + 1 - p^2) = a_1(2p+1) = 0 \Rightarrow p = -\frac{1}{2}\ or\ a_1 = 0.$

If $a_1 = 0$, the coefficient $a_3((p+3)^2 - p^2) + a_1 = a_3((p+3)^2 - p^2) = 0 \Rightarrow p = -\frac{3}{2}\ or\ a_3 = 0$ and so, either all odd coefficients $a_{2n+1} = 0$, or $p = \pm\frac{2n+1}{2}$ for some positive integer n and the odd coefficients beyond a_{2n+1} are nonzero.

Using non-integer $r = -p \;\Rightarrow$

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[49] Named for Wilhelm G. Bessel (Prussian 1784-1846) who generalized them in his 1824 study of perturbations of planetary motions in the three-body problem, although some cases (probably integer) were discovered in connection with oscillations of hanging chain and vibrating membranes by Daniel Bernoulli (1700-1782) and Euler respectively. https://www.britannica.com/science/Bessel-function (accessed 23-Oct, 2018).

[50] "Order" is the commonly used expression for the value of the parameter in a parameterized *family* of ODEs, as here in the family of Bessel equations, collectively referred to as The Bessel equation. Do not confuse it with the order of the ODE, which is, of course, 2, the maximum number of differentiations of y.

$a_1((-p+1)^2-p^2)=a_1(p^2-2p+1-p^2)=a_1(-2p+1)=0 \Rightarrow r=-p=\frac{-1}{2}$ *or* $a_1=0$. And we again find odd coefficients are zero unless $p=\pm\frac{2n+1}{2}$ for some integer n.

Evidently, $p=\pm\frac{2n+1}{2}$ is a special case to deal with later. So, assuming non-integer $p\neq\pm\frac{2n+1}{2}$, all odd coefficients $a_{2k+1}=0$.

Turning to the even ones: $a_2((p+2)^2-p^2)+a_0=a_2(4p+4)+a_0=0 \Rightarrow a_2=\frac{-a_0}{4(p+1)}$,

$$a_{2k}((p+2k)^2-p^2)+a_{2(k-1)}=a_{2k}(4kp+4k^2)+a_{2(k-1)}=0 \Rightarrow$$

(9.2) $$a_{2k}=\frac{-a_{2(k-1)}}{4k(p+k)}$$

is the recursion relation satisfied by all the coefficients.

Expanding Eqn(9.2) backward to $a_2=-\frac{1}{4(p+1)}$ and $a_0=1$ (noting $p+1=p+k-(k-1)$)

$$a_{2k}=\frac{-a_{2(k-1)}}{4k(p+k)}=\frac{-1}{4k(p+k)}\frac{-a_{2(k-2)}}{4(k-1)(p+k-1)}=\frac{-1}{4k(p+k)}\frac{-1}{4(k-1)(p+k-1)}\frac{-a_{2(k-3)}}{4(k-2)(p+k-2)}=\cdots$$

$$=\frac{(-1)^k}{4^k k!(p+k)(p+k-1)\cdots(p+2)(p+1)} \text{ for } p>0 \text{ and } a_{2k}=\frac{(-1)^k}{4^k k!(k-p)(k-1-p)\cdots(1-p)} \text{ for } r=-p<0.$$

For non-integer $p\neq n\pm\frac{1}{2}$, an independent set of solutions is given by the series:

$$y_1(t)=\sum_{k=0}^{\infty}\frac{(-1)^k\,t^{2k+p}}{4^k k!\,(p+k)(p+k-1)\cdots(p+2)(p+1)}$$

$$y_2(t)=\sum_{k=0}^{\infty}\frac{(-1)^k\,t^{2k-p}}{4^k k!\,(k-p)(k-1-p)\cdots(2-p)(1-p)}$$

But these aren't quite yet Bessel functions. They need to be scaled properly. To do this we need one of the most useful of the special functions.

Definition 9.5. The G*amma function* is the continuous interpolation of the factorial given by

$$\Gamma(x+1)=\int_0^{\infty}t^x e^{-t}\,dt\,.$$

Gamma works just like a factorial for non-integer values x but it's one unit shifted, since

$$\Gamma(x+1)=x\Gamma(x)\Rightarrow\Gamma(n+1)=n\Gamma(n)=n!$$

The products in the denominators of y_1 , y_2 would be the factorials of $(p+k)$ and of $(k-p)$ if p were an integer and they continued down to a final factor of 1. We observe that multiplying the "unfinished factorial" in y_1 by the remaining factors in $\Gamma(p+1)$ will "finish" it:

(9.3) $$(p+k)(p+k-1)\cdots(p+2)(p+1)\cdot\Gamma(p+1)=\Gamma(p+k+1)$$

and the other one in y_2 is "finished" by the factors in $\Gamma(1-p)$:

$$(k-p)(k-1-p)\cdots(2-p)(1-p)\cdot\Gamma(1-p)=\Gamma(k-p+1).$$

Finally, other (normalization) concerns require a factor of 2^p in the denominator.

Definition 9.6. The *Bessel functions of the first kind* for non-integer, positive $p\neq n\pm\frac{1}{2}$ are

$$\mathcal{J}_p(t)=\frac{1}{2^p}\sum_{k=0}^{\infty}\frac{(-1)^k\,t^{2k+p}}{4^k k!\Gamma(k+p+1)}$$

$$\mathcal{J}_{-p}(t)=\frac{1}{2^{-p}}\sum_{k=0}^{\infty}\frac{(-1)^k\,t^{2k-p}}{4^k k!\Gamma(k-p+1)}\ .$$

For $= n \geq 0$, an integer,

$$\mathcal{J}_n(t) = \frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k\, t^{2k+n}}{4^k k!\,\Gamma(k+n+1)} = \frac{1}{2^n}\sum_{k=0}^{\infty}\frac{(-1)^k\, t^{2k+n}}{4^k k!\,(k+n)!} = (-1)^n\mathcal{J}_{-n}(t).$$

Because $\mathcal{J}_n = (-1)^n\mathcal{J}_{-n}$ are linearly dependent, there must be a second independent solution to Bessel's n^{th} order equation. This solution is obtained as a limit of linear combinations of non-integer Bessel functions.

Definition 9.7. The *Bessel function of the second kind* for integer $p = n \geq 0$ is

$$Y_n(t) = \lim_{p\to n}\frac{\cos(p\pi)\,\mathcal{J}_p(t) - \mathcal{J}_{-p}(t)}{\sin(p\pi)}.$$

Finally, we are left with the case of half-integral order $p = n \pm \frac{1}{2}$.

Recall:

$$Ly = t^r[\,a_0(r^2-p^2) + a_1((r+1)^2-p^2)t + (a_2((r+2)^2-p^2)+a_0)t^2$$
$$+(a_3((r+3)^2-p^2)+a_1)t^3+\cdots]$$

Rewrite it for $r^2 = p^2$ as

$$Ly = t^p[a_1((p+1)^2-p^2)t + (a_2((p+2)^2-p^2)+a_0)t^2 + (a_3((p+3)^2-p^2)+a_1)t^3+\cdots]$$
$$= t^p[a_1(2p+1)t + (a_2(4p+4)+a_0)t^2 + (a_3(6p+9)+a_1)t^3+\cdots].$$

Let $p = \frac{-1}{2}$.

$$Ly = t^{-1/2}[a_1(-1+1)t + (a_2(-2+4)+a_0)t^2 + (a_3(-3+9)+a_1)\,t^3+\cdots]$$
$$= t^{-1/2}[(2a_2+a_0)t^2 + (6a_3+a_1)\,t^3 + (12a_4+a_2)\,t^4 + (20a_5+a_3)\,t^5+\cdots]$$
$$a_2 = -\frac{a_0}{2}\,,\ a_3 = -\frac{a_1}{6},\ a_4 = -\frac{a_2}{12},\ \ a_5 = -\frac{a_3}{20},\ldots$$

Taking the undetermined $a_0 = 1, a_1 = 0$ gives $y_1 = t^{-1/2}\left[1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{6!} + \cdots\right] = \frac{\cos t}{\sqrt{t}}$ as a solution.

Taking $a_0 = 0, a_1 = 1$ gives $y_2 = t^{-1/2}\left[t - \frac{t^3}{6} + \frac{t^5}{5!} + \cdots\right] = \frac{\sin t}{\sqrt{t}}$ a second independent solution.

A little rescaling leads to the definition of $\mathcal{J}_{\frac{1}{2}}\,, \mathcal{J}_{-\frac{1}{2}}$.

Definition 9.8. The *half-integral Bessel functions* are

$$\mathcal{J}_{\frac{1}{2}} = \sqrt{\frac{2}{\pi t}}\sin t\,,\quad \mathcal{J}_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi t}}\cos t,$$

$$\mathcal{J}_{n+\frac{3}{2}} = -t^{\left(n+\frac{1}{2}\right)}\frac{d}{dx}\left(t^{-\left(n+\frac{1}{2}\right)}\mathcal{J}_{n+\frac{1}{2}}(t)\right)\,,\ \mathcal{J}_{n-\frac{1}{2}} = t^{-\left(n+\frac{1}{2}\right)}\frac{d}{dx}\left(t^{\left(n+\frac{1}{2}\right)}\mathcal{J}_{n+\frac{1}{2}}(t)\right).$$

The half-integral Bessel functions for a few values of n are:

$$\mathcal{J}_{\frac{3}{2}}(t) = \sqrt{\frac{2}{\pi t}}\left(\frac{\sin t}{t} - \cos t\right),\qquad \mathcal{J}_{\frac{5}{2}}(t) = \sqrt{\frac{2}{\pi t}}\left(\left(\frac{3}{t^2}-1\right)\sin t - \frac{3}{t}\cos t\right),$$

$$\mathcal{J}_{-\frac{3}{2}}(t) = \sqrt{\frac{2}{\pi t}}\left(-\frac{\cos t}{t} - \sin t\right),\qquad \mathcal{J}_{-\frac{5}{2}}(t) = \sqrt{\frac{2}{\pi t}}\left(\left(\frac{3}{t^2}-1\right)\cos t + \frac{3}{t}\sin t\right).$$

Concerning the calculation of the Bessel functions for non-integer $p = \frac{1}{2} \pm n$, the formulas:

(9.4) $$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}\ ,\ \Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}$$

are beautiful, if not terribly useful. Certainly, the consequences $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$ are nice. For $p=\frac{1}{k}\pm n$ with integer k, the following are useful:

(9.5) $\Gamma\left(\frac{1}{k}+n\right)=\Gamma\left(\frac{1}{k}\right)\frac{(kn-(k-1))!^{(k)}}{k^n}$ where $!^{(k)}$ denotes the k^{th} multifactorial[51].

Example 9.3. Using calculated values[52] for $\Gamma\left(\frac{1}{3}\right)$, $\Gamma\left(\frac{1}{4}\right)$ Formula 9.5 gives

$\Gamma\left(\frac{1}{3}+4\right)=\Gamma\left(\frac{1}{3}\right)\frac{(3\cdot 4-2)!!!}{3^4}=\Gamma\left(\frac{1}{3}\right)\frac{(10\cdot 7\cdot 5\cdot 2)}{3^4}=\Gamma\left(\frac{1}{3}\right)\frac{700}{81}\approx 267.89385347077476\cdot\frac{7}{81}\approx 23.151320670.$

$\Gamma\left(\frac{1}{4}+5\right)=\Gamma\left(\frac{1}{4}\right)\frac{(4\cdot 5-3)!^{(4)}}{4^5}=\Gamma\left(\frac{1}{4}\right)\frac{(17\cdot 13\cdot 9\cdot 5\cdot 1)}{1024}=\Gamma\left(\frac{1}{4}\right)\frac{9945}{1024}\approx 3.62560990822190831\cdot\frac{9945}{1024}\approx 35.2116118528$ □

Exercises 2.9

1. Find the first five non-zero terms of a solution to $Ly=y''+2ty=0$ near $t_0=0$.
2. Find the first five non-zero terms of pair of independent solutions to $y''+t\,y=0$ at $t_0=0$ satisfying $y_1(0)=1,\ y_1'(0)=0,\ y_2(0)=0,\ y_2'(0)=1$.
3. Find the first five non-zero terms of a pair of independent solutions to $y''-2t\,y=0$ at $t_0=0$ satisfying $y_1(0)=1,\ y_1'(0)=0,\ y_2(0)=0,\ y_2'(0)=1$.
4. Find the first five non-zero terms of a pair of independent solutions to $y''+2t\,y=0$ at $t_0=0$ satisfying $y_1(0)=1,\ y_1'(0)=0,\ y_2(0)=0,\ y_2'(0)=1$.
5. Find the first five non-zero terms of a solution to Hermite's $Ly=y''-2t\,y'=0$ near $t_0=0$.
6. Find the first five non-zero terms of a solution to Hermite's $Ly=y''-2t\,y'+2y=0$ near $t_0=0$.
7. Find the polynomials $y_{1,6}$, $y_{2,7}$ for Hermite's equations and give the Hermite polynomials H_6, H_7.
8. Give the general solution to the Euler equation $t^2y''+3ty'+5y=0$.
9. Give the general solution to the Euler equation $t^2y''+4ty'+3y=0$.
10. Find $\Gamma(3),\Gamma(5),\Gamma(7)$ and use the formula (9.4) to find $\Gamma(3/2)$, $\Gamma(5/2)$, $\Gamma(7/2)$ given $\Gamma(1/2)=\sqrt{\pi}$.
11. Find $\Gamma(1),\Gamma(9)$ and use the formula (9.4) to find $\Gamma(-3/2),\Gamma(-9/2)$ given $\Gamma(1/2)=\sqrt{\pi}$.
12. Find the first four terms of $\mathcal{J}_{1/4}(t)$ in terms of $\Gamma(1/4)$ using formula 9.5.
13. Find the first four terms of $\mathcal{J}_{-1/4}(t)$ in terms of $\Gamma(1/4)$ using formula 9.5.
14. Find the first four terms of $\mathcal{J}_0(t), Y_0(t)$.
15. Plot the functions you found in Problem 14.
16. Plot Matlab's version of $\mathcal{J}_0(t)$, $Y_0(t)$ for enough time to see the difference against the plots in # 15.
17. Find the first four terms of $\mathcal{J}_{-1}(t),\mathcal{J}_1(t)$.
18. Plot the functions you found in Problem 17.
19. Plot Matlab's version of $\mathcal{J}_{-1}(t)$, $\mathcal{J}_1(t)$ for enough time to see the difference against the plots in # 18.
20. Find the first four terms of $\mathcal{J}_{7/2}(t)$.
21. Find the first four terms of $\mathcal{J}_{-7/2}(t)$.
22. Plot the functions you found in Problems 20&21.
23. Plot Matlab's version of $\mathcal{J}_{7/2}(t)$, $\mathcal{J}_{-7/2}(t)$ for enough time to see the difference against the plots in # 22.

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[51] Multifactorial $n!^{(k)}=\begin{cases}1 & 0<n\le k\\ n(n-k)!^{(k)} & \end{cases}$.

[52] The values of $\Gamma\left(\frac{1}{3}\right)$, $\Gamma\left(\frac{1}{4}\right)$ were retrieved from the *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org, 23-Oct 2018.