Chapter 5. NonLinear Systems

5.0 Autonomous Systems

Autonomous systems give the velocity vector at every point in phase space for a *flow*. As we saw with linear systems, through each initial point $\vec{x}_0$ in phase space there should be a solution curve along a streamline for the flow. The new twist with non-linear systems is that we can almost never find an explicit formula for the solutions. For non-linear autonomous systems we can plot the velocity vector field and from it, deduce qualitative characteristics of solutions. We can calculate numerical solutions to the system to very close approximation on finite time intervals but the asymptotic behavior of solutions is determined through qualitative analysis of the phase portrait.

Autonomous systems take the form

(0.0) $$\frac{d}{dt}\vec{x} = \vec{f}(\vec{x})$$

with, $\vec{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ vector of potentially non-linear functions of n variables $x_1, \dots x_n$ and $\mathbb{R}^n$ the phase space for the system. The vector $\vec{x}(t)$ specifies the position of a solution in phase space. The left hand side of the equation, $\frac{d}{dt}\vec{x}$ is the phase-velocity vector for $\vec{x}(t)$. The right hand side specifies a phase velocity vector $\vec{f}(\vec{x})$ at each position $\vec{x} \epsilon \mathbb{R}^n$ and is called a *vector field*. You already know that a vector field on the plane $\mathbb{R}^2$ is envisioned by plotting a grid of vectors in the plane. You know that the solution curves for an autonomous linear system must be tangent to the vectors of the vector field. The same is true for non-linear systems.

In the case of linear systems, the velocity vector at a position $\vec{x}$ is found by the matrix multiplication by the system's matrix $\boldsymbol{A}$ and the solution through initial condition $\vec{x}_0 = \vec{x}(0)$ is $\vec{x}(t) = e^{t\boldsymbol{A}}\vec{x}_0$. For nonlinear systems, the velocity at position $\vec{x}$ is found by calculating the value of $\vec{f}(\vec{x})$ but there is no general formula for the solution through an IC. In cases where a solution exists, we write $\vec{x}(t) = \varphi_t(\vec{x}_0)$ for the solution with $\vec{x}_0 = \vec{x}(0)$. For non-linear systems, the vector field may be badly behaved, so there may be places where there is no solution or non-uniqueness of solution. Just as with scalar equations, we have the EU theorem that guarantees existence and uniqueness of solution in a neighborhood of $\vec{x}_0$ as long as $\vec{f}$ is differentiable in a neighborhood of $\vec{x}_0$.

Definition 0.0. For $\vec{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ where each coordinate $f_i: \mathbb{R}^n \to \mathbb{R}$, let $\partial_j f_i = \frac{\partial f_i}{\partial x_j}$.

If all of the functions f_i are differentiable in some open region $\mathcal{U}$ in $\mathbb{R}^n$, and all $\partial_j f_i$ are continuous on $\mathcal{U}$, then $\vec{f}$ is differentiable on $\mathcal{U}$ and the $n \times n$ matrix $D\vec{f} = (\partial_j f_i)$ is the derivative of $\vec{f}$ (on $\mathcal{U}$).

The *gradient* of f_i is the row vector $\nabla f_i = [\partial_1 f_i \quad \cdots \quad \partial_n f_i]$ and $D\vec{f} = (\partial_j f_i) = \begin{bmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{bmatrix}$ is the derivative of the *vector field* $\vec{f}$.

Example 0.0. $\frac{d}{dt}\vec{x} = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$ is a non-linear system. Here $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and the left side is $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ clearly interpreted as the velocity of a moving point in the x_1x_2-plane. The vector on the right side is $\vec{f}(\vec{x})$ giving the phase velocity vector field. The gradients of the functions in $\vec{f}$ are

$$\nabla f_1(x_1 \quad x_2) = \nabla(x_2) = \begin{bmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \end{bmatrix} = [0 \quad 1]$$

and

$$\nabla f_2(x_1 \quad x_2) = \nabla(-\sin x_1) = \begin{bmatrix} \frac{\partial(-\sin x_1)}{\partial x_1} & \frac{\partial(-\sin x_1)}{\partial x_2} \end{bmatrix} = [-\cos x_1 \quad 0]$$

The derivative of $\vec{x}$ is the 2×2 matrix $D\,\vec{f}(\vec{x}) = \begin{bmatrix} \nabla x_2 \\ \nabla(-\sin x_1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{bmatrix}$.

Since the entries in D$\vec{f}$ are continuous everywhere, $\vec{f}$ is differentiable on all of $\mathbb{R}^n$. □

Theorem 0.0 (Non-linear Autonomous EU) If $\vec{f}$ is a differentiable vector field on a bounded, open rectangle *B* in $\mathbb{R}^n$, then there is an open rectangle $\mathcal{U}$ within *B,* an $\varepsilon > 0$, and a one-parameter family of differentiable functions $\varphi_t\colon \mathcal{U} \to B$ such that $\forall \vec{x}_0 \in \mathcal{U}$, $\vec{x}(t) = \varphi_t(\vec{x}_0)$ gives the unique solution for $|t| < \varepsilon$ to the IVP

$$\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}); \quad \vec{x}(0) = \vec{x}_0 .$$

Definition 0.1 The functions φ_t given in Theorem 0.0 are called "time = *t flow diffeomorphisms*"[1] for the vector field $\vec{f}$.

Theorem 0.0 looks more complicated than it really is. It says that when $\vec{f}$ is differentiable on a "big" bounded rectangle *B* then there is a "little" rectangle $\mathcal{U}$ inside of the big rectangle where the equation has unique solutions for *all* of the initial conditions in $\mathcal{U}$. Also those unique solutions *all* exist for at least some small interval of time $|t| < \varepsilon$. In fact, the flow diffeomorphisms φ_t deform $\mathcal{U}$ (stretch or contract or bend or rotate) but never stretch it enough to leave the big rectangle *B* while $|t| < \varepsilon$. Because of uniqueness, the rectangle $\mathcal{U}$ is not folded, each $\varphi_t\colon \mathcal{U} \to \varphi_t(\mathcal{U})$ is one-to-one. The flow moves each $\vec{x}_0$ to $\varphi_t(\vec{x}_0)$ so the whole rectangle of initial conditions flows forward in time for *t* > 0 along streamlines of the flow φ.

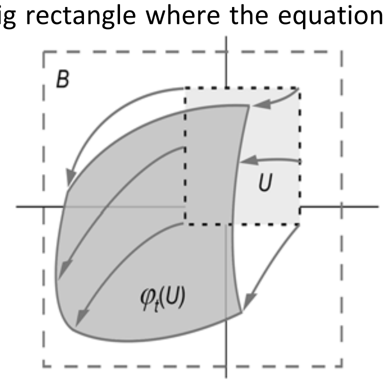

Figure 5-1. Action of the flow φ_t on rectangle $\mathcal{U}$ in B.

You have already seen the flows for the linear systems $\frac{d}{dt}\vec{x} = \boldsymbol{A}\vec{x}$ in phase portraits. Their flow diffeomorphisms just multiply the initial condition vectors by $e^{\boldsymbol{A}t}$. The flows for non-linear systems typically have no explicit formula, so numerical solutions (from Euler's or a similar method) are used to approximate $\varphi_t(\vec{x}_0)$ at some sequence of times $t_1, t_2, \ldots t_k$. For low-dimensional systems ($n \le 3$), phase portraits can be plotted. For higher-dimensional systems ($n \ge 4$), the geometry describing the flows is similar, much as in the case of linear systems, but is

[1] A diffeomorphism is a function $\varphi\colon \mathcal{U} \to \mathcal{V}$ that is one-to-one and differentiable with differentiable inverse. The sets $\mathcal{U}$ and *V* would then be called *diffeomorphic*, approximately meaning "smoothly deformable into each other". The flow diffeomorphisms φ_t smoothly deform regions $\mathcal{U}$ into their futures $\varphi_t(\mathcal{U})$. Imagine a gently flowing fluid.

not visually accessible through phase portraits. Our further discussion will treat techniques that apply to interpretation of phase portraits for autonomous non-linear systems in $\mathbb{R}^2$.

5.1 Equilibrium Point Analysis

Linear systems always have one equilibrium: $\vec{x} = \vec{0}$, which is the only stationary point of the flow unless there is a zero eigen-value whose entire eigen-space is stationary. Non-linear systems can have several equilibria. Typically, each isolated equilibrium has a local flow that is approximated by the flow of an associated linear system. Just as the tangent line approximates the curve of a differentiable function in a neighborhood of the point of tangency, so the flow for the linear system with matrix $D\vec{f}(\vec{x}_0)$ can, under certain conditions, approximate that of $\vec{f}$ near an isolated equilibrium point $\vec{x}_0$ (where $\vec{f}(\vec{x}_0) = \vec{0}$). For this reason, our first act in studying a non-linear system is always to find the equilibria.

Example 1.0. Find the equilibria for the system $\frac{d}{dt}\vec{x} = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$ of Example 0.0. We set the vector-field $\begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and solve for the equilibrium. Clearly, this requires $x_2 = 0$ and $\sin x_1 = 0$ simultaneously. The equilibria are then the points: $\vec{x}_k = \begin{bmatrix} \pm k\pi \\ 0 \end{bmatrix}$, k=0,1,2,…. □

Example 1.1. Find the equilibria for $\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix}$. Set $\begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Again $x_2 = 0$ at every equilibrium. Now

$$-x_1 + x_1^3 = 0 \Rightarrow x_1(x_1^2 - 1) = 0 \Rightarrow x_1 = -1, 0, 1$$

The equilibria are $\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. □

For each equilibrium $\vec{x}_k$, we study the flow locally near $\vec{x}_k$.

Suppose $\vec{x}$ is near $\vec{x}_k$ then because $D\vec{f}$ is the derivative of $\vec{f}$,

$$\lim_{\boldsymbol{x} \to \boldsymbol{x_k}} \vec{f}(\vec{x}) - \vec{f}(\vec{x}_k) - D\vec{f}(\vec{x}_k)(\vec{x} - \vec{x}_k) = \vec{0}$$

so $\vec{f}(\vec{x}) \to \vec{f}(\vec{x}_k) + D\vec{f}(\vec{x}_k)(\vec{x} - \vec{x}_k)$, and because $\vec{x}_k$ is an equilibrium, $\vec{f}(\vec{x}_k) = \vec{0}$ so

$$\vec{f}(\vec{x}) \to D\vec{f}(\vec{x}_k)(\vec{x} - \vec{x}_k) \quad \text{as } \vec{x} \to \vec{x}_k$$

The act of replacing $\vec{f}(\vec{x})$ by $D\vec{f}(\vec{x}_k)(\vec{x} - \vec{x}_k)$ for $\vec{x}$ near $\vec{x}_k$ is called *linearization at* $\vec{x}_k$. It is similar to replacing a curve by its tangent line at a point of tangency.

Localizing our view, we choose local coordinates in which $\vec{x}_k$ becomes the origin and let the local coordinate vector be $\vec{u} = \vec{x} - \vec{x}_k$ we have

$$\frac{d\vec{u}}{dt} = \frac{d(\vec{x} - \vec{x}_k)}{dt} = \frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \to D\vec{f}(\vec{x}_k)(\vec{x} - \vec{x}_k) = D\vec{f}(\vec{x}_k)\vec{u}$$

and so, when $\vec{x}$ is near $\vec{x}_k$ we have $\vec{u} = \vec{x} - \vec{x}_k \approx \vec{0}$ and

(1.0) $$\frac{d\vec{u}}{dt} = D\vec{f}(\vec{x}_k)\vec{u} + o(|\vec{u}|^2)$$

where the last term $o\left(|\vec{u}|^{\mathbf{2}}\right)$ stands for a vector whose length goes to 0 faster than $|\vec{u}|^2$ as $|\vec{u}| \to 0$.

This says that when $\vec{x}$ is near $\vec{x}_k$, the vector field $\vec{f}$ is approximated by the vector field for the linear system

$$(1.1) \qquad \frac{d\vec{u}}{dt} = D\vec{f}(\vec{x}_k)\vec{u}\,.$$

It does not say the solutions are approximately the same. However, the streamlines for the linear system (1.1) give us *qualitative information* about the *asymptotic behavior* of solutions to $\frac{d\vec{x}}{dt} = \vec{f}(x)$ near $\vec{x}_k$.

Definition 1.0. The system (1.1) is *the linearization* of the system $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$ at its equilibrium $\vec{x}_k$.

The eigenvalues and eigenvectors of the matrix $D\vec{f}(\vec{x}_k)$ tell us about the flow near $\vec{x}_k$.

Example 1.2. Let's find the linearization at an equilibroium for $\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix}$ from Example 1.1.

We found equilibria: $\vec{x}_{1,2,3} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let's find the linearization at $\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. First we get the Jacobian of $\vec{f}$:

$$D\vec{f}(\vec{x}) = D\begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix} = \begin{bmatrix} \nabla x_2 \\ \nabla(-x_1 + x_1^3) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 3\,x_1^2 & 0 \end{bmatrix}.$$

At $\vec{x}_1$ this gives: $D\vec{f}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ for the coefficient matrix in the linearization:

$$\frac{d}{dt}\vec{u} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\vec{u}$$

The eigenvalues for $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ are $\lambda = \pm\sqrt{2}$ so the linear system has a saddle at its equilibrium in the $\vec{u}$-plane. This means the NL system has a saddle at its equilibrium $\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Eigen-vectors can be found for these eigenvalues, giving eigenpairs: $\left\{\pm\sqrt{2} \quad \begin{bmatrix} 1 \\ \pm\sqrt{2} \end{bmatrix}\right\}$. The phase portrait for the linearization is a saddle, so $\vec{x}_1$ is a saddle point for the NL system. □

Comparison of the phase portraits in figures 5-2A & 5-2B reveals the qualitative similarity of the flows for the non-linear (NL) system near the equilibrium at $\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and the local linearization near $\vec{u} = \vec{0}$. This similarity between the flows makes the local phase portrait of the non-linear system appear to be the same as that of the linear system but *bent*, as if it were printed on a curved surface.

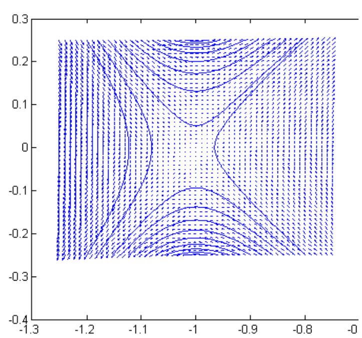

Figure 5-2A. Phase portrait for NL system in Example 1.2 near $\vec{x}_1$.

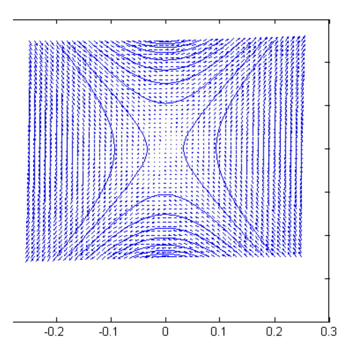

Figure 5-2B. Local linearization near $\vec{x}_1$ as seen in the $\vec{u}$-plane.

The mathematics that describes the relationship between these similar flows, although beyond the scope of our work, is conceptually simple. It is easy to imagine smoothly deforming the portrait for the nonlinear flow so the streamlines of the linear system in a in a small neighborhood $\mathcal{V}$ of $\vec{x}_1$ (shown in Fig. 5-2A) coincide exactly with those of the linear system in a neighborhood $\mathcal{U}$ near $\vec{u} = \vec{0}$ (shown in Fig. 5-2B). By the inverse deformation, the streamlines of the linear system in $\mathcal{U}$ would be made to coincide with those of the non-linear system in $\mathcal{V}$ near $\vec{x}_1$. These deformations correspond to *continuous* functions: $h: \mathcal{V} \to \mathcal{U}$, and $h^{-1}: \mathcal{U} \to \mathcal{V}$. Generally, such (bi-continuous) functions are called *homeomorphisms.*[2] Because $\vec{h}$ carries solutions of one differential equation to solutions of the other, it is also *parameterization preserving*, it respects the flow. In cases where this can be done, we say the non-linear equation is *locally almost linear at* $\vec{x}_k$.

5.2 Locally Almost Linear Systems of NLODEs

After we find the equilibria for a nonlinear system, we linearize at each equilibrium $\vec{x}_k$. We hope that the flow of the non-linear system is a homeomorphic deformation of the flow of the linearization, in a neighborhood $\mathcal{V}$ of $\vec{x}_k$. If so, the non-linear system is *locally almost linear* at $\vec{x}_k$. Based on the eigenvalues for the linearization, we can determine whether or not this is the case. It may help to formalize this discussion with a couple of definitions.

Definition 2.0. Flow diffeomorphisms $\varphi_t: \mathcal{V} \to \varphi_t(\mathcal{V})$ and $\psi_t: \mathcal{U} \to \psi_t(\mathcal{U})$ are *topologically conjugate* if there exists a homeomorphism $h: \mathcal{V} \cap \varphi_t(\mathcal{V}) \to \mathcal{U} \cap \psi_t(\mathcal{U})$ such that $\psi_t \circ h = h \circ \varphi_t$.

Conjugate flows satisfy $h^{-1} \circ \psi_t \circ h = \varphi_t$ on a neighborhood $\mathcal{V}$ of an equilibrium.

Definition 2.1. Suppose $\psi_t(\vec{u})$ is the flow for the linearization at an equilibrium $\vec{x}_k$ of $\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}\,)$ and φ_t is the local nonlinear flow near $\vec{x}_k$. If φ_t and ψ_t are topologically conjugate then

$$\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}\,) \text{ is } \textit{locally almost linear at } \vec{x}_k \,.$$

Theorem 2.0 (Hartman & Grobman 1962) If $\vec{x}_k$ is an equilibrium for $\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}\,)$ where:

i) In some open neighborhood of $\vec{x}_k$ the function $\vec{f}(\vec{x}) - D\vec{f}(\vec{x}_k)(\vec{x} - \vec{x}_k)$ is differentiable, and

ii) No eigenvalue of $D\vec{f}(\vec{x}_k)$ has zero real part,

then there is an open neighborhood $\mathcal{V}$ about $\vec{x}_k$ where $\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}\,)$ is almost linear.

The Hartman-Grobman theorem is key to understanding the dynamics of autonomous non-linear systems near their equilibria. According to this theorem, the flow for the NLODE (0.0) is a continuous deformation of that for its linearization (1.1), as long as all eigenvalues of $D\vec{f}(\vec{x}_k)$ have non-zero real parts. For the planar systems we have studied, this means that a node for the linearization guarantees either a node or focus (spiral point) of the same stability type for the non-linear system. A saddle in the linearization guarantees a saddle in the non-linear system. Furthermore, in the case of the saddle, the stable and unstable eigenspaces for the linearization have continuous deformations called the stable and unstable "manifolds" of the non-linear system's equilibrium. The eigenspaces of the linearization are *tangent* to these manifolds at the equilibrium.

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[2] Topology is the study of sets on which continuous functions can be defined. Two sets U and V between which a homeomorphism exists are called homeomorphic, because it is possible to continuously "map" the points of U one-to-one onto those of V. Continuity requires that sets are not torn or punctured, one-to-one that no points get smashed.

Now let's find the local linearization approximating the system of Example 1.1 at $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$D\vec{f}(\vec{x}\,) = D\begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix} = \begin{bmatrix} \nabla x_2 \\ \nabla(-x_1 + x_1^3) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 3\,x_1^2 & 0 \end{bmatrix}$ which at $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives: $\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x}$ for the linearized system. The eigenvalues for $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\lambda = \pm i$, so the linearization has a center at $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Because the real parts of the eigenvalues are zero, the Hartman-Grobman Theorem does not guarantee that the phase portrait for the non-linear system will be topologically conjugate to a center near $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The phase portrait generated by MatLab for the system of Example 1.2 is shown in Figure 5-3. It illustrates a problem with center equilibria. In the phase portrait, the center at $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ appears to be a spiral point, even though by other methods it can be shown that periodic solutions surround the origin but were broken by numerical errors as we have seen before. To show the existence of the periodic solutions, need to show the system is conservative, that there is no dissipation or gain of "energy" causing solutions to spiral.

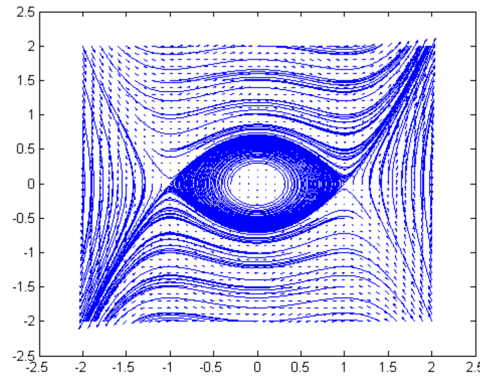

Figure 5-3. Matlab's approximate phase portrait for the NL system of Ex 1.2.

Conservative Systems

Defintion 2.2. A system $\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}\,)$ is *conservative* if there exists a function $H(\vec{x})$ that is constant along its solutions. Such a function is called *an integral* of the system.

Clearly, if $\vec{x}(t)$ is a solution then the integral must satisfy $\frac{d}{dt}H(\vec{x}(t)) = 0$. The chain rule implies:

$$\frac{d}{dt}H(\vec{x}(t)) = \frac{dH}{dx_1}\frac{dx_1}{dt} + \cdots + \frac{dH}{dx_n}\frac{dx_n}{dt} = \nabla H \cdot \frac{d\,\vec{x}}{dt} = 0.$$

And since $\vec{x}(t)$ is a solution to the system, $\nabla H \cdot \vec{f} = 0$. This proves a useful little theorem:

Theorem 2.1. $\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}\,)$ is *conservative* if and only if there exists a function $H(\vec{x})$ with $\nabla H \cdot \vec{f} = 0$.

Example 2.3. Is the system $\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix}$ conservative? We need to try to find an integral $H(x_1\,, x_2\,)$ for which $\nabla H \cdot \vec{f} = [\partial_1 H \quad \partial_2 H] \cdot \begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix} = x_2\partial_1 H + (-x_1 + x_1^3)\partial_2 H = 0$.

Letting $\partial_1 H = -x_1 + x_1^3$ and $\partial_2 H = -x_2$ is an obvious choice. To find out whether such a function exists, we do partial integrations:[3]

$$\int \partial_1 H\,\partial x_1 = \int -x_1 + x_1^3\ dx_1 = -\frac{{x_1}^2}{2} + \frac{x_1^4}{4} + a(x_2)$$

$$\int \partial_2 H\,\partial x_2 = \int -x_2\ dx_2 \qquad = -\frac{{x_2}^2}{2} + b(x_1)$$

[3] Recall exact differentials in Chapter 1 Definition 11.1.

And since both should result in the same function H, we need:

$$H = -\frac{{x_1}^2}{2} + \frac{x_1^4}{4} + a(x_2) = -\frac{{x_2}^2}{2} + b(x_1)\,.$$

So $a(x_2) = -\frac{{x_2}^2}{2}$ and $b(x_1)\ -\frac{{x_1}^2}{2} + \frac{x_1^4}{4}$ will work, making $H = -\frac{{x_1}^2}{2} + \frac{x_1^4}{4} - \frac{{x_2}^2}{2}$ an integral. □

Because the integral H is constant on solutions, the phase portrait for a conservative system in $\mathbb{R}^2$ consists of the level curves of H. Because H is constant on its level curves, a conservative system cannot have spiral foci at which infinitely many different level curves twist arbitrarily tightly around the focus equilibrium. Using Matlab's *contour* function, the level curves of H can be plotted to good approximation, using a sufficiently fine meshgrid. We can also use the properties of level curves together with the vector field, linearizations at equilibria and numerical solutions, to qualitatively represent the phase portrait.

Example 2.4. Using Matlab's *contour*. We plot the phase portrait for the conservative $\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix}$ by plotting contours for the integral $H = -\frac{{x_1}^2}{2} + \frac{x_1^4}{4} - \frac{{x_2}^2}{2}$ which are the same as the contours of

$$4H = x_1^4 - 2{x_1}^2 - 2{x_2}^2\,.$$

In Matlab execute >>doc contour; to open the help file on the contour plotting function. Scroll down to the example where contour is used to plot levels of a function $z = f(x, y)$. You can see the following command lines will produce a contour plot for the integral $4H$.

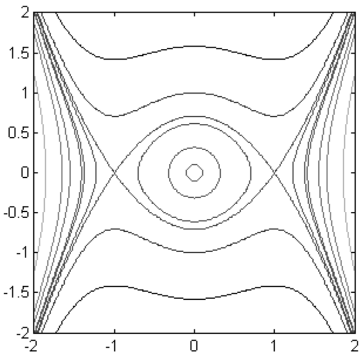

Figure 5-4. Contours for the NL system of Examples 1.2, 2.3 & 2.4.

```
>> [x,y]=meshgrid(-2:0.025:2,-2:0.025:2);
>> z=(x.^4-2*x.^2-2*y.^2);
>> contour(x,y,z,[-5 -2 -1 -.75 -.2 -.02 1 2 5] );
>> colormap Gray
```

The contours in Figure 5-4 show that the equilibrium at the origin is indeed a center. They also show that a pair of solutions connect the saddle points.

To produce a quality phase portrait, we need to indicate the flow directions on the streamlines. In this system, $\frac{d}{dt}x_1 = x_2$ so, the flow moves in the direction of increasing x_1 when x_2 is positive and decreasing x_1 when x_2 is negative. This means the arrows in the upper half-plane point to the right and in the lower half-plane they point left. Along the x_1-axis they are vertical as appropriate to maintain continuous flow. Figure 5-5 shows the arrows on the contours. Points are also inserted at the equilibria and the upper *saddle connection* trajectory is thickened. The region bounded by the two saddle connections is filled with periodic orbits. These are not circles or ellipses, although they become asymptotically circular near the equilibrium at (0,0). □

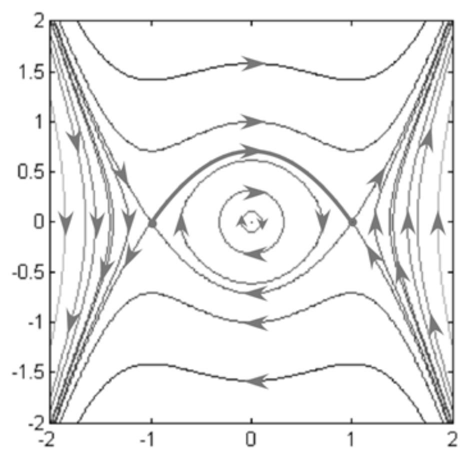

Figure 5-5. Completed phase portrait for the NL system of Examples 1.2, 2.3 & 2.4.

Without the use of a contour plotter we can still use the fact that a system is conservative to give a qualitative picture of the phase portrait based on an approximate numerical phase portrait like that shown in Figure 5-3. Because trajectories for a conservative system are level curves of an integral, isolated equilibria can only be saddles and centers - there can be no spiral points or nodes. Based on the numerical approximation in Figure 5-3, we would conclude that, since the system is conservative, the correct phase portrait must have a center at the origin (0,0). The spirals in Figure 5-3 are numerical errors. Based on equilibrium analysis, we know the saddles are at $(-1,0)$ and $(1,0)$. The integral $4H$ used in Example 2.4 depends on ${x_2}^2$ so it has symmetry about the x_1-axis. It also contains only even powered terms in x_1, so it has symmetry about the x_2 axis as well. These symmetries of the integral's level curves guarantee the saddle connections completing a qualitative picture of the phase portrait that must be similar to that in Figure 5-5, information we can obtain without resorting to a contour plotter.