TP-RL-1

November 1, 2018

0.1 MVA, Reinforcement Learning TP1

0.2 Dynamic Programming and Reinforcement Learning

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0.3 1. Dynamic Programming

Q1: Implement the discrete MDP model. In this simple MDP it is simple to guess the optimal policy . What is it?
The optimal policy maximizes a fraction γ^t of the reward in expectation. Therefore, we want to reach s_2 and remaining there doing in loop the action a_2. The shortest stochastic path to s_2 is to do a_1 in s_0 and s_1. This means that the optimal policy is 0 = [a1, a1, a2].

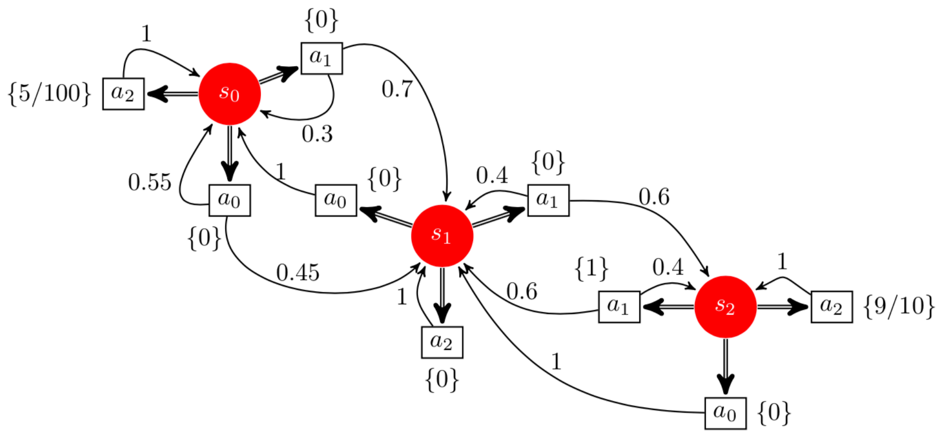
```
In [3]: import numpy as np

        transitions = np.array([[[0.55, 0.3, 1],
                                 [0.45, 0.7, 0],
                                 [0, 0, 0]],
                                [[1, 0, 0],
                                 [0, 0.4, 1],
                                 [0, 0.6, 0]],
                                [[0, 0, 0],
                                 [1, 0.6, 0],
                                 [0, 0.4, 1]]
                              ], dtype=float)
        gamma = 0.95
        rewards = np.array([[0, 0, 0.05],
                            [0, 0, 0],
                            [0, 1, 0.9]], dtype=float)
```

Q2: Implement and run value iteration (VI) in order to identify a 0.01–optimal policy. Recall that the stopping criterion (theory slides) is to stop when

$$v^{k+1}v^k <$$

which implies that [Puterman, 1994, Th. 6.3.1]

$$\overset{+}{v^{k+1}}v < \frac{2}{1}$$

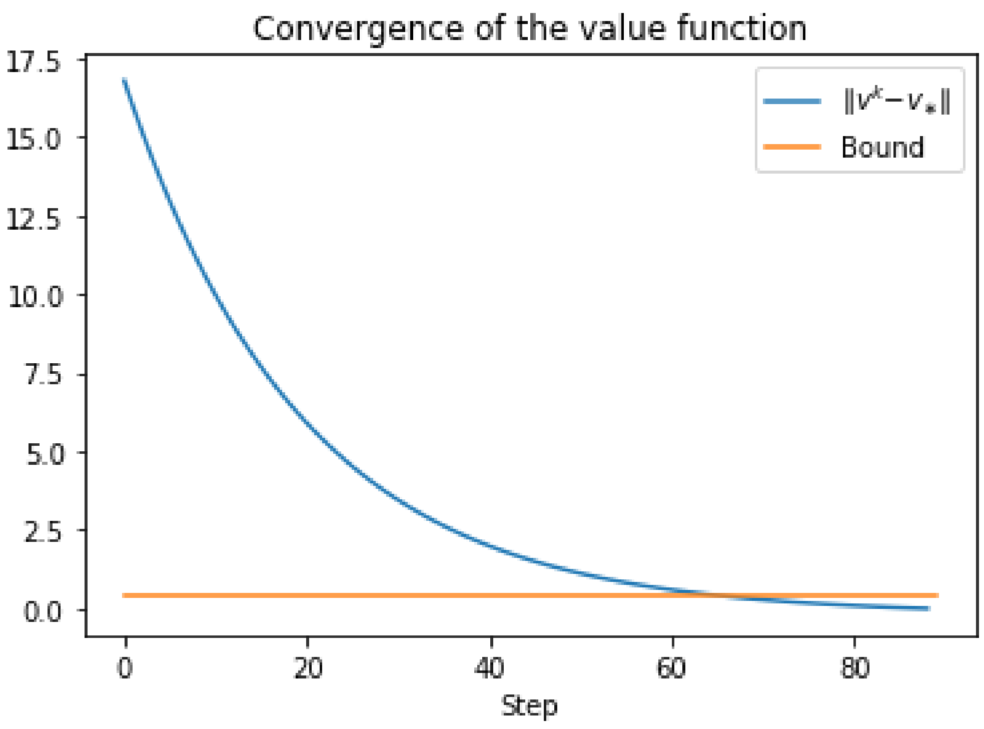


title

Plot $v_k v$ as a function of iteration k. Implement policy evaluation to compute v, i.e. the value function of the optimal policy.

```
In [18]: import matplotlib.pyplot as plt

         def value_iteration(transitions, rewards, eps=1e-2, max_iter=1000, gamma=0.95):
             n_states, n_actions = rewards.shape
             old_value = np.zeros(n_states)
             value = np.zeros(n_states)
             values = []
             error = eps + 1
             n = 0

             while error >= eps and n < max_iter:
                 for s in range(n_states):
                     all_values = np.zeros(n_actions)
                     for a in range(n_actions):
                         all_values[a] = rewards[s, a] + gamma * transitions[s, :, a].dot(valu
                     value[s] = np.max(all_values)
                 error = np.max(value - old_value)
                 values.append(value.copy())
                 old_value = value.copy()
                 n += 1

             return values

         eps = 1e-2
         gamma = 0.95
         values = np.array(value_iteration(transitions, rewards, eps=eps, gamma=gamma))
         v_opt = values[-1]
         errors = np.max(np.abs(values - v_opt), axis=1)
```

```
bound = 2*eps*gamma/(1-gamma)
plt.plot(errors, label="$v^k v_$")
plt.plot([0,len(errors)], [bound, bound], label="Bound")
plt.legend()
plt.xlabel("Step")
plt.title("Convergence of the value function")
plt.show()
```

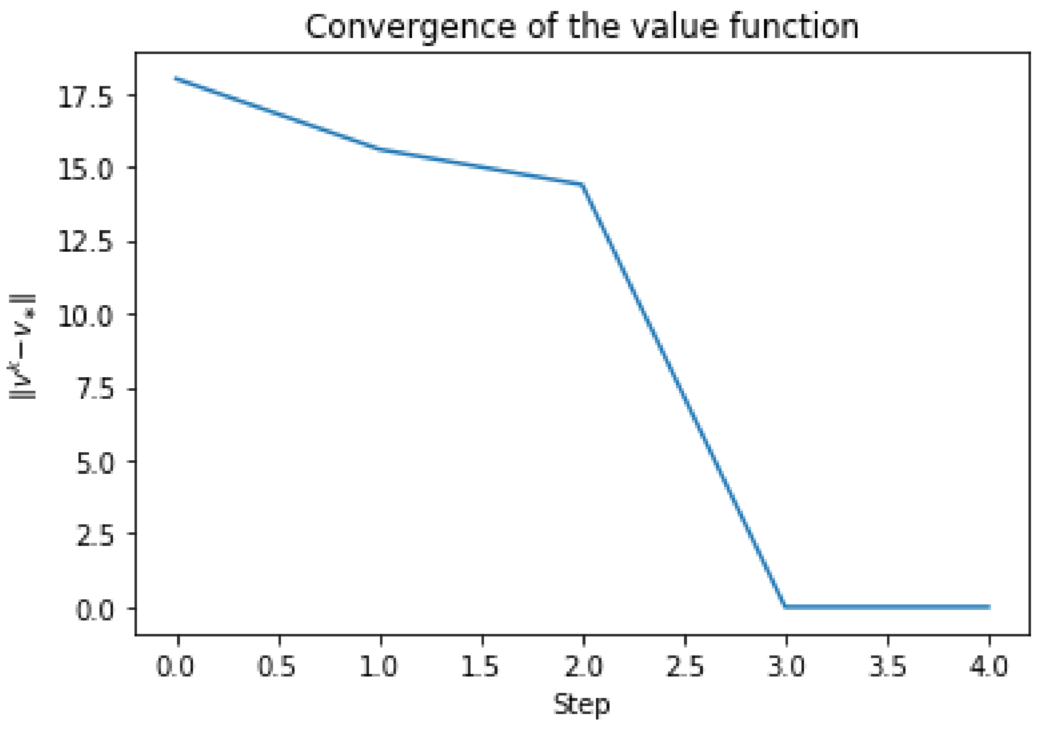


Q3: implement exact policy iteration (PI) with initial policy $_0$ = [a1, a1, a2]. Compare the speed of convergence w.r.t. VI and discuss the relative merits of the two approaches.

The initial policy given is the optimal policy. So we initialize our algorithm with $0 = [a0, a1, a2]$

```
In [21]: from numpy.linalg import inv

         def _policy_evaluation(policy, t, r, gamma):
             n_states, n_actions = r.shape
             r_pi = np.array([r[x, int(policy[x])] for x in range(n_states)])
             p_pi = np.array([[t[x, y, int(policy[x])] for y in range(n_states)]  for x in rang
             v_pi = inv(np.eye(n_states) - gamma*p_pi) @ r_pi
             return v_pi

         def _policy_improvement(v_pi, t, r, gamma):
             n_states, n_actions = r.shape
             greedy = np.zeros(n_actions)
             policy = np.zeros(n_states)
```

```python
    for x in range(n_states):
        for a in range(n_actions):
            greedy[a] = r[x, a] + gamma * t[x, :, a].dot(v_pi)
        policy[x] = np.argmax(greedy)
    return policy

def _has_converged(v, old_v):
    return np.max(np.abs(v - old_v))

def policy_iteration(transitions, rewards, eps=1e-2, max_iter=1000, gamma=0.95):
    n_states, n_actions = rewards.shape
    old_value = np.zeros(n_states)
    v_pi = np.ones(n_states)
    policy = np.zeros(n_states)
    values = []
    error = eps + 1
    n = 0

    while _has_converged(v_pi, old_value) and n < max_iter:
        old_value = v_pi.copy()
        v_pi = _policy_evaluation(policy, transitions, rewards, gamma)
        policy = _policy_improvement(v_pi, transitions, rewards, gamma)
        error = np.max(np.abs(v_pi - old_value))
        values.append(v_pi.copy())
        n += 1

    return values


eps = 1e-2
gamma = 0.95
values = np.array(policy_iteration(transitions, rewards, eps=eps, gamma=gamma))
v_opt = values[-1]
errors = np.max(np.abs(values - v_opt), axis=1)
plt.plot(errors, label="$v^k v_$")
plt.ylabel("$v^k v_$")
plt.xlabel("Step")
plt.title("Convergence of the value function")
plt.show()
```



The convergence of policy iteration is much faster than for value iteration, but each iteration is more computationnally expensive. The convergence of policy iteration is also exact, while it is asymptotic for value iteration.

```
In [23]: print("Time lapse for the policy iteration algorithm")
         %timeit for x in range(100): policy_iteration(transitions, rewards, eps=eps, gamma=gar
         print("Time lapse for the value iteration algorithm")
         %timeit for x in range(100): value_iteration(transitions, rewards, eps=eps, gamma=gamr
```

```
Time lapse for the policy iteration algorithm
61.8 ms ś 5.68 ms per loop (mean ś std. dev. of 7 runs, 10 loops each)
Time lapse for the value iteration algorithm
411 ms ś 2.88 ms per loop (mean ś std. dev. of 7 runs, 1 loop each)
```

0.4 2. Reinforcement Learning

0.4.1 2.1. Policy evaluation.

Consider the deterministic policy that is selecting the action right when available, otherwise up.

Q4: denote with $V_n(x,a)$ the value function estimated using Monte-Carlo, i.e., empirical average:

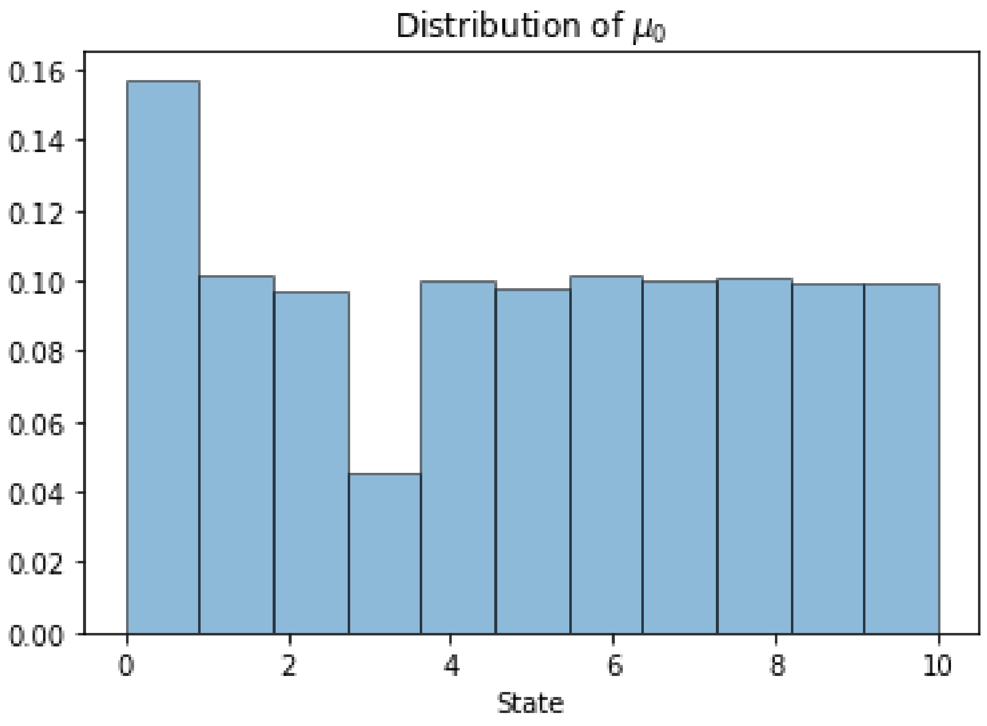
$$V(s) = \frac{1}{N(s)} \sum_{k=1}^{N(s)} \sum_{t=1}^{T_{max}} \gamma^{t-1} r_t^{(k)}$$

with $s_1 = s, a_t(|s_t)$ where $\sum_s N(s) = n$ and $(r_t^{(k)})$ is the sequence of rewards obtained when simulating the k-th trajectory st (using the simulator). Build such estimator and plot $J_n J$ as a function of n, where

$$V = [0.877, 0.928, 0.988, 0, 0.671, 0.994, 0, 0.828, 0.877, 0.934, 0.994]T$$

First, we need to evaluate the distribution of μ by sampling it many times.

```
In [24]: from gridworld import GridWorld1
         env = GridWorld1

         def evaluate_dist_mu(env, n_iter=50000):
             mu0 = [env.reset() for i in range(n_iter)]
             dist_mu, _ = np.histogram(mu0, bins=env.n_states)
             return mu0, dist_mu/n_iter

         mu0, dist_mu = evaluate_dist_mu(env)
         plt.figure()
         plt.title("Distribution of $\mu_0$")
         plt.xlabel("State")
         plt.hist(mu0, bins=env.n_states, density=True, edgecolor='k', alpha=0.5)
         plt.show()
```

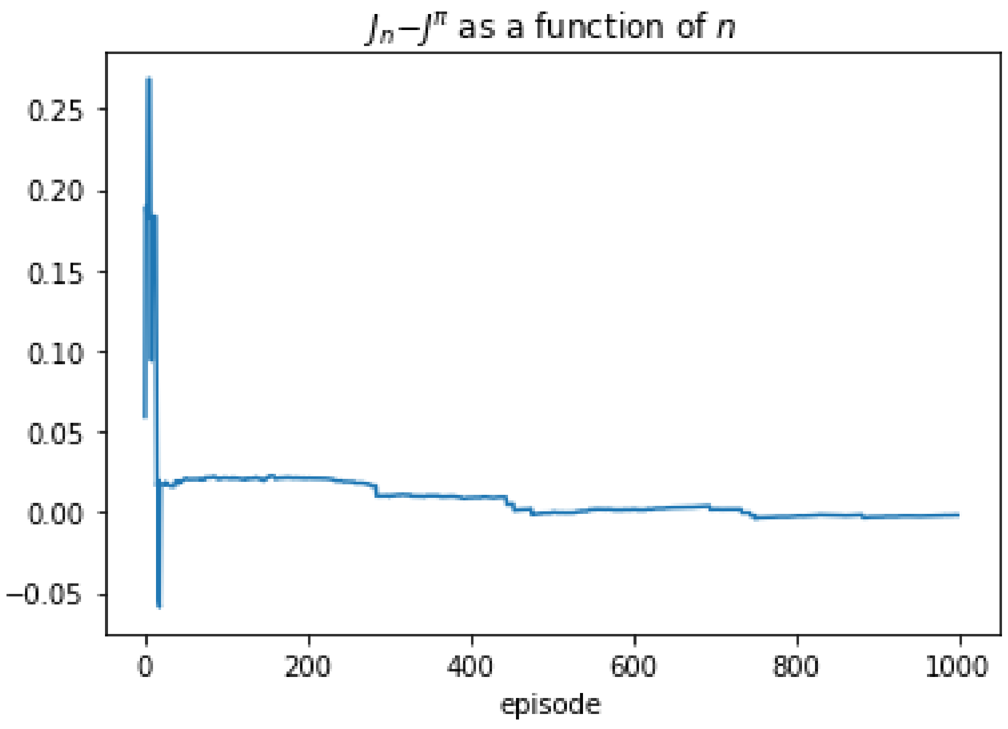


Now, we can do the evaluation of the value function for a given deterministic policy

```
In [126]: def _mc_simulations(env, policy, t_max=100, **kwargs):
              """ Do a Monte Carlo simulations with N episodes. """

              s = env.reset()
              t = 0
              term = False
              rewards = []
              states = [s]
              actions = []
              while t < t_max and not term:
                  a = policy(env.state_actions[s], s, **kwargs)
                  s, r, term = env.step(s, a)
                  rewards.append(r)
                  actions.append(a)
                  states.append(s)
                  t += 1
              return rewards, states, actions

          def _mc_evaluation(rewards, states, actions, n_states, gamma=0.95):
              """
              Evaluate the value function given the Monte-Carlo simulations.
              """
              value = np.zeros(n_states)
              for s in range(n_states):
                  trajectories = _find_trajectories(s, states)
                  Ns = len(trajectories)
                  if Ns == 0:
                      continue
                  acc = 0.0
                  for k in range(Ns):
                      acc += _discounted_reward(rewards[trajectories[k]], gamma)
                  value[s] = acc/Ns
              return value

          def _find_trajectories(s, all_states):
              """
              Return a list of the indices of trajectories that start from s
              """
              trajectories = []
              for k, states in enumerate(all_states):
          #         print(states)
                  if states[0] == s:
                      trajectories.append(k)
              return trajectories

          def _discounted_reward(rewards, gamma=0.95):
              """
              >>> _discounted_reward([0, 0, -1])
```

```
    -0.9025
    """
    acc = 0.0
    for t in range(1, len(rewards)+1):
        acc += gamma**(t-1)*rewards[t-1]
    return acc

def mc_value(env, policy, N=500, gamma=0.95, t_max=100, **kwargs):
    """
    Estimate a value function from Monte-Carlo simulations for a given policy.
    The simulation is run for N episodes.
    Policy parameter is a callback function whose parameter is the set of actions
    """
    values = np.zeros((N, env.n_states))
    rewards, states, actions = [], [], []

    for n in range(N): # episodes
        episodes = _mc_simulations(env, policy, t_max=t_max, **kwargs)
        rewards.append(episodes[0])
        states.append(episodes[1])
        actions.append(episodes[2])
        values[n, :] = _mc_evaluation(rewards, states, actions, env.n_states, gamma=
    return values

def ponderate_value(values, dist_mu):
    return values.dot(dist_mu)

def policy(actions, state):
    if 0 in actions: # turn right
        return 0
    return 3 # go up

# _, dist_mu = evaluate_dist_mu(env)
n_values = 1000
V = mc_value(env, policy, N=n_values)
J = np.zeros(n_values)
for n in range(n_values):
    J[n] = ponderate_value(V[n], dist_mu)

Vpi = np.array([0.877,0.928,0.988,0,0.671,-0.994,0,-0.828,-0.877,-0.934,-0.994])
Jpi = ponderate_value(Vpi, dist_mu)


plt.plot(J-Jpi)
plt.xlabel("episode")
plt.title("$J_n  J^$ as a function of $n$")
```

```
Out[126]: Text(0.5,1,'$J_n  J^$ as a function of $n$')
```



0.4.2 2.2. Policy optimization: the Q-learning algorithm

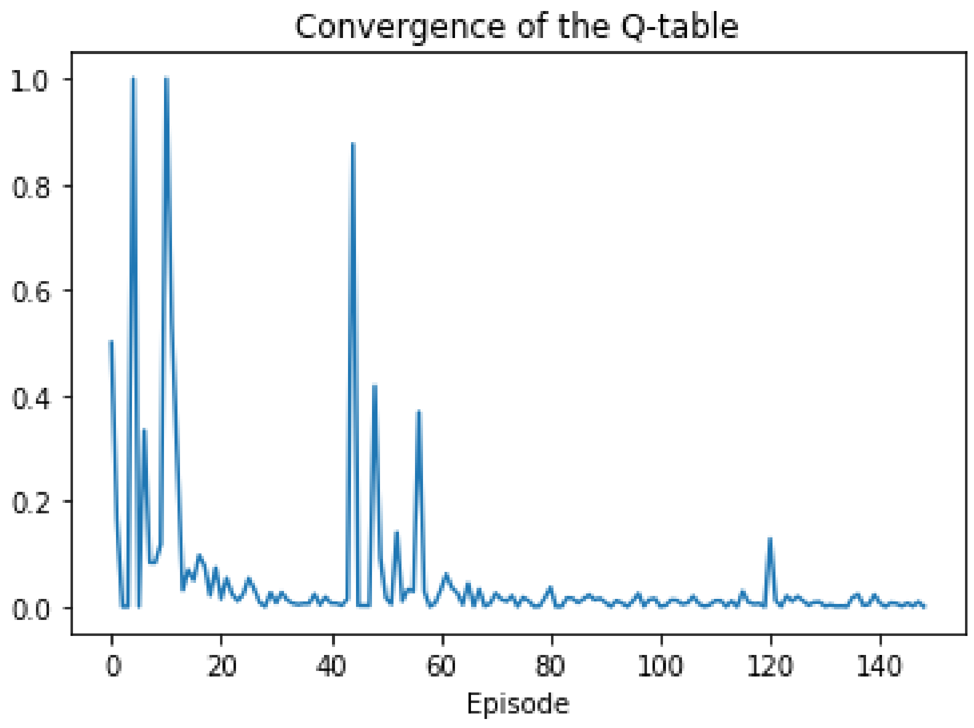
Q5: Describe the (parameters of the) exploration policy and learning rate chosen, and illustrate the convergence of Q-Learning using the following performance metrics:
- Performance over all the other state $||vv^n||$, where $_n$ is the greedy policy w.r.t. Qn at the end of the n-th episode
- Reward cumulated over the episode.
Note that: $v = [0.877, 0.928, 0.988, 0, 0.824, 0.928, 0, 0.778, 0.824, 0.877, 0.828]$

The exploration policy means that most of the time $(1 - \epsilon)$, we chosen the best path exploration policy based on our experiences, while we explore sometimes (ϵ) new path. This means that ϵ fixes the trade-off between exploration and exploitation.

Using an online formulation, the learning rate must follow the Robbins-Monro conditions to provide convergence properties. Here, it is chosen to $\frac{1}{N(s,a)}$ as $\sum_{s,a} \frac{1}{N(s,a)} = +\infty$, while $\sum_{s,a} \frac{1}{N(s,a)} < +\infty$. The Robbins-Monro conditions mean that the learning rate should converge, while being slow enough to estimate the stochastic approximation.

```
In [181]: env = GridWorld1


          def _next_action(Q, state, actions, eps=0.1):
              """ Compute the policy and return the action provided by the policy the given st
              The action is thus defined as $argmax_a Q(x_t, a)$ with an $\epsilon$-greedy exp

              action = actions[np.argmax(Q[state, actions])]
```

```
        # exploration algorithm
        if np.random.random() < eps and  actions != [action]:
            action = np.random.choice(list(set(actions) - set([action])))

        return action


    def _learning_rate(N, state, action):
        return 1/N[state, action]

    def _q_episode(Q, N, env, eps=0.1, gamma=0.95, t_max=1000):
        state = env.reset()
        Q = Q.copy()
        t = 0
        term = False
        old_state = state
        n_actions = len(env.action_names)
        while t < t_max and not term:
            action = _next_action(Q, state, env.state_actions[state], eps=eps)
            N[state, action] += 1
            alpha = _learning_rate(N, state, action)
            nexts, reward, term = env.step(state, action)
            Q[state, action] = (1-alpha)*Q[state, action] + alpha* \
                                    (reward + gamma*np.max(Q[nexts]))
            t += 1
            state = nexts
        return Q, N



    def q_learning(env, gamma=0.95, eps=0.3, max_iter=300, max_err=0):
        """
        Applies to Q Learning algorithm in ordeer to get the best policy.
        The function returns all Q computed at each step.
        """

        n = 0
        n_actions = len(env.action_names)
        Q = np.zeros((env.n_states, n_actions))
        N = np.zeros((env.n_states, n_actions))
        Qs = []
        err = max_err + 1
        while n < max_iter: # and err > max_err: #TODO add convergence criteria
            new_q, N = _q_episode(Q, N, env, eps=eps, t_max=1000, gamma=gamma)
            n += 1
            Qs.append(new_q)
            err = np.max(Q - new_q)
            Q = new_q
        return Qs


    qs = q_learning(env, max_iter=150)
    q = qs[-1]

    # Plotting the convergence of the Q-table
    diff_q = []
    for i in range(1, len(qs)):
        diff_q.append( np.max( np.abs(qs[i] - qs[i-1])) )
    plt.plot(diff_q)
    plt.title("Convergence of the Q-table")
    plt.xlabel("Episode")

    plt.show()
```



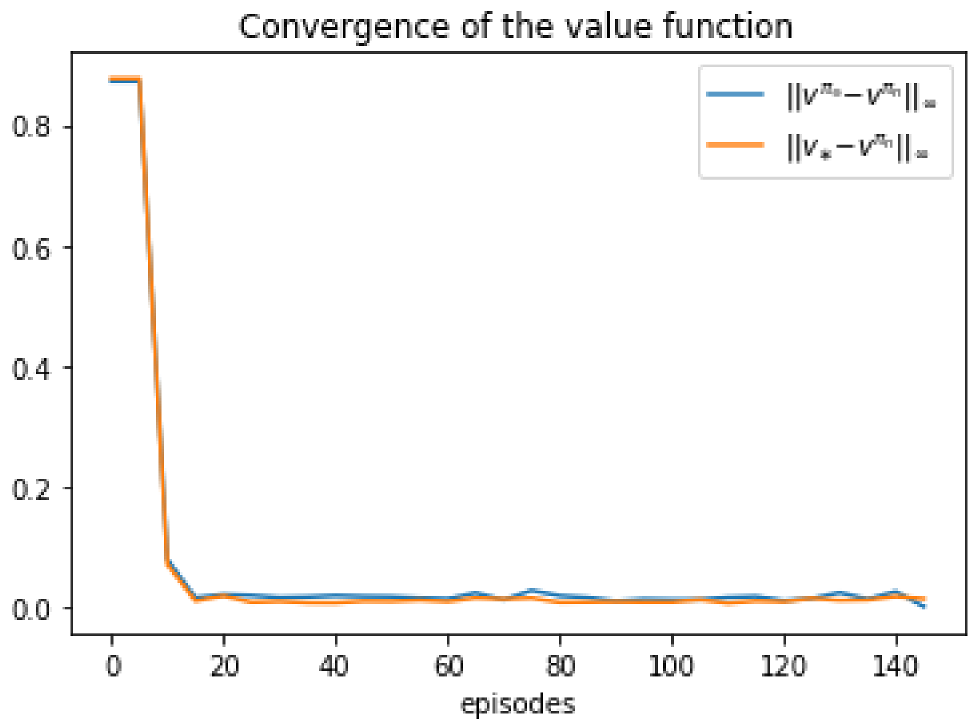
The latter plot shows the convergence of Q. But we should check it has converged to a correct value by rendering the grid.

```
In [160]: import gridrender
          def q_export(q, actions):
              ret = []
              for i in range(q.shape[0]):
                  ret.append(q[i, actions[i]])
```

```
            return ret

        q_exp = q_export(q, env.state_actions)
        gridrender.render_q(env, q_exp)
```

Great! Now, we can deduce from each Q_n its corresponding policy π_n and from π_n, we obtain V_n applying the policy evaluation from Question 4.

```
In [140]: def _q_to_policy(Q, actions):
              """ Return the optimal policy from Q"""
              n_states = Q.shape[0]
              policy = np.zeros(n_states)
              for i in range(n_states):
                  policy[i] = actions[i][np.argmax(Q[i, actions[i]])]
              return policy


          def _q_to_value(Q, env, N=1000):
              """ Return the estimated value function from Q """

              def policy(actions, state, pi=None):
                  return pi[state]

              pi = _q_to_policy(Q, env.state_actions)
              v = mc_value(env, policy, N=N, pi=pi)
              return v[-1]

          def q_to_values(Qs, env, N=1000, step_size=5):
              n_q = len(Qs)
              values = []
              x = range(0, n_q, step_size)

              for i in x:
                  v = _q_to_value(Qs[i], env, N=1000)
                  values.append(v)
              return values


          values = q_to_values(qs, env)

          cv_v = []
          cv_gt = []
          Vstar=[0.877,0.928,0.988,0,0.824,0.928,0,0.778,0.824,0.877,0.828]
          for i in range(len(values)):
              cv_v.append( np.max( np.abs(values[i] - values[-1])) )
              cv_gt.append( np.max( np.abs(values[i] - Vstar)) )
          plt.plot(x, cv_v, label="$||v^{_{\infty}}  v^{_n} ||_$")
          plt.plot(x, cv_gt, label="$||v_  v^{_n} ||_$")
          plt.xlabel("episodes")
          plt.legend()
          plt.title("Convergence of the value function")
          plt.show()
```

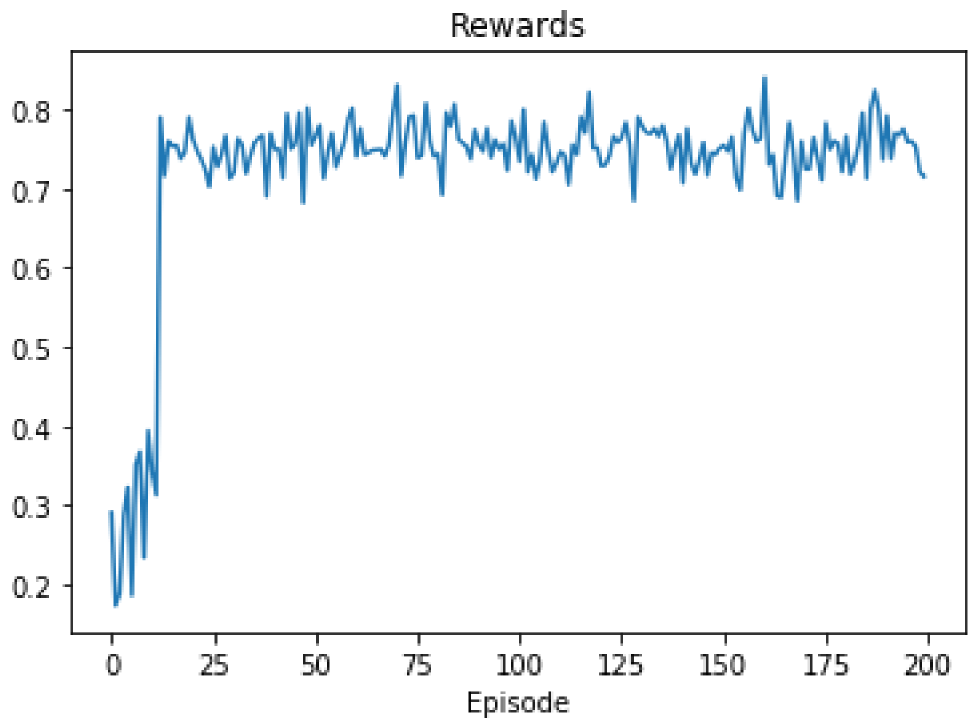


We can also observe the convergence of R_n, the reward obtained when following Q_n without any exploration. As the initial state is a random variable, we do a Monte-Carlo simulation over the trajectories.

```
In [ ]: def reward_evaluation(Q, env, N=100, gamma=0.95, t_max=1000):

            cumulated_rewards = []
            for i in range(N):
                t = 0
                term = False
                rewards = []
                state = env.reset()
                while t < t_max and not term:
                    possible_actions = env.state_actions[state]
                    action = possible_actions[np.argmax(Q[state, possible_actions])]
                    state, reward, term = env.step(state, action)
                    t += 1
                    rewards.append(reward)
                cumulated_rewards.append( _discounted_reward(rewards, gamma) )
            return np.mean(cumulated_rewards)
```

```
        reward_evaluation(qs[i], env)
        Nq = len(qs)
        rewards = np.zeros(Nq)
        for i in range(Nq):
            rewards[i] = reward_evaluation(qs[i], env)
```

```
In [183]: plt.plot(rewards[:200])
          plt.title("Rewards")
          plt.xlabel("Episode")
          plt.show()
```



Q6: Is the optimal policy of an MDP affected by the change of the initial distribution $_0$?
It is not supposed to be related to this parameter. To show it, we can modify the function `env.reset()`:

```
In [196]: from gridworld import GridWorld, grid1

          class GridWorld2(GridWorld):

              def reset(self):
                  n_states = self.n_states
                  u = 0.9
                  a = np.zeros((n_states,))
                  a[1] = 0.5
```

```
        a[4] = -0.8
        u = np.power(np.ones((n_states,)) + a, u)
        p = np.exp(u) / np.sum(np.exp(u))
        x_0 = np.random.choice(np.arange(n_states), p=p)
        return x_0

env2 = GridWorld2(gamma=0.95, grid=grid1)
qs2 = q_learning(env2, max_iter=150)
q2 = qs2[-1]
p = _q_to_policy(q, env2.state_actions)
p2 = _q_to_policy(q2, env2.state_actions)

assert p2.all() == p.all()
```

Great! We get the same policy!!