- Research Internship -
The Power of Two Choices in Bike-sharing Systems

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Abstract The use of bike-sharing systems in urban centers is increasing over the years and studies about these systems are very important in order to improve their availability. The operating principle involves four basic steps: customers arrive at stations; pick a bicycle; use it for a while and return it to another or the same station according to their choice. The main problem of these systems are the spatial imbalance of the bike inventory over time, i.e., the appearance of empty and full stations. In the former, the customer cannot pick up a bike and in the latter, the worst case, the user cannot return the bike to the chosen station and he has to search for an available station nearby. This work focuses on evaluating the impact in the system, when users start to return the bike in the less loaded station between two stations near their destination. The analytic results are achieved by using the homogeneous system of bike-sharing. It is important to notice that the entire analysis is made considering steady-state. Otherwise the problem would not be analytically tractable. This study includes the following cases: the users choose the less loaded station between the closest two stations, regarding their destination; the users choose the less loaded stations between two stations belonging to groups of two stations and the effect of the delay in the choice of stations. After comparing these cases, a real bike-sharing system is used to perform a simulation about the impact of the choice in the real system.

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Contents

1 Introduction

The use of bike-sharing systems in urban centers is increasing over the years (see [9]) and the study of such systems is gaining more and more importance. The operating principle involves the arrival of customers at stations, where they pick a bicycle, use it for a while and then, return it to another or the same station accordinf to their choice. The main problem of these systems are the spatial imbalance of the bike inventory over time, i.e., the appearance of empty and full stations (see [6]). In the former, the customer cannot pick up a bike and in the latter, which is more serious, the user cannot return the bike to the chosen station and then, he must search for an available station nearby. This work focuses in evaluating the impact in the system, if users start to return the bike in the less loaded station between two stations near their destination. The analytic results are obtained by using the homogeneous system of bike-sharing, as it is used in [6]. It is important to note that the whole analysis is made in the steady-state regime and for a large number of stations. Otherwise the problem would be analytically untractable.

First of all, it is important to understand that the study of a bike-sharing system and the study of the corresponding queuing system differ by a fixed-point equation, which is briefly explained in chapter 6 or detailed in [6]. That is why we start the study with the corresponding queuing systems, before the bike-sharing systems.

We begin the work by studying a technique called *Pair Approximation*, which can be found in [8] and Mean-field approximation as the system is large. Then, we study the influence of the *local choice*[1] in a *Deposition Model*. We analyze it by *Pair Approximation*, *Mean-field approximation* and compare them with numerical simulations. Then, we study the model of N queues with finite capacity and a general choice policy. Through a perturbation approach for low traffic, we develop an algorithm to find the coefficients of the Taylor expansion of the invariant measure. In the particular case of *local choice*, we were able to deduce the first non-zero term of the expansion for the marginal representing the probability of a queue having a certain amount of units. However, since it is an expansion, it holds only for low traffic. In the attempt to have results for heavy traffic, more precisely traffic rate close to one, since it gives results near to the optimum point (see [6]), we propose a variant of the *local choice* model, which consists in dividing the stations in groups of two and then, making the local choice among these two stations. We guess that this model has a behavior between the case where the users do not choose and the *local choice* case. But we investigate it, because it still holds a reasonable representation of the phenomenon of users choosing. Then, we can apply *Mean-field* techniques to model the bike-sharing system with this choice policy and we obtain several analytic results, which allow us to determine a range of values of the fleet size achieving a *good* performance for the system. Comparing it with the already known result for the case where the users do not choose, which can be found in [6], we conclude that near the optimal condition, the metric of unsatisfied users is roughly divided by two, when users choose.

Then, we analyze the impact of the delay in the choice of each user, i.e., if the user chooses too soon the station where he is going to return the bike, it is possible that this station is no longer the less loaded one when he returns it, which would result in an inefficiency for the system. We obtain its *Mean-field approximation* as we make the system large and we analyze the system through numerical simulations. It confirms the simulations presented in [6]. The conclusion is that if the user chooses when he takes his bike, in the range of realistic values of incoming clients rate and trip time, this effect has little impact on the *performance* of the system.

Data analysis is becoming more important in research. For the bike-sharing system Vélib' in Paris, two kinds of data are available: trips and stations state. We have these data for two years, thanks to JC Decaux (see [1]). Data were used to better understand usage. Via clustering techniques, a few groups of stations with the same behavior were highlight: stations near train stations, in office areas, in living areas, etc (see

[1] Local choice means that the users choose the less loaded station between two neighbor stations.

Côme [3] and Chabchoub [2]). Our aim is totally different. Little has been done to confront models and datas in bike-sharing systems. For us, exploring data is a way to investigate heterogeneity (in time and space) untractable in our models and to test our algorithms on a real bike-sharing system. The question was if, with real trips of Vélib' in Paris, we can *replay* them with users choosing *better* stations close to their destinations and see if the system is more balanced. The conclusion is that choosing the station with the great number of free bikes in the neighborhood when returning the bikes does not make Vélib' behaving better, because of the difference of the station capacities. Vélib' would be better balanced if users choose the most *unbalanced* (will be defined) station in the neighborhood. There are plenty of variants to test: just choosing with an alternative station (more close to a real user behavior), choosing the destination station when taking the bike, and even before, also choosing a *better* source station, etc. Note that for our *replay* system, we keep all the trips, even the truck regulation trips. It allows to compare two systems with the same load. In future works, testing natural regulation made by users themselves, without any trip regulation would be also interesting.

2 The pair approximation equations for queues with local choice

This section was written with the purpose of studying the *pair approximation* technique, as it is found in [8]. Our aim was to show that *pair approximation* is a *mean-field* approximation on a more complex framework. We write the infinitesimal generator and we arrive at the same equations in the paper. In the end of this section, we show that the problem does not have an analytic solution that can be written as the product of two independent quantities.

2.1 Model description

As it is explained in [8], our system is composed of n identical servers that are connected by an undirected graph (V, E), where the set of vertexes is the set of servers $V = 1, ..., n$. Each server serves jobs at rate μ and uses a first-come first-serve discipline. Jobs arrive in the system at rate $n\lambda$. let $Y_{i,j}$ be the proportion of connected pairs that have (i, j) customers and $X_i = \sum_{j\geq 0} Y_{i,j}$ be the proportion of queues that have i customers. Looking at the evolution of $Y_{i,j}$, if an edge (i, j) has a departure on i, which occurs at rate μ, it becomes $(i-1, j)$, if $i > 0$. It becomes $(i + 1, j)$ when there is an arrival on i. This can be caused by two events: : (a) arrival on the edge is chosen at rate $2\lambda/d$, where d is the number of edges, and the packet is allocated to the first server with probability $a(i, j) = 1$ if $i < j$, $a(i, i) = 1/2$ and $a(i, j) = 0$ if $i > j$ or (b) arrival on another neighbor of the first server, each other neighbor of i that has state l induce an arrival on i at rate $2\lambda a(i, l)/d$. Let $Z_{l,i,j}(t)$ be the proportion portion of connected triplets of stations having state (l, i, j). The arrivals on the first server of a pair (i, j) from one of the $d - 1$ other neighbors occur at rate $2\lambda(d - 1)R_{i,j}(t)/d$, where $R_{i,j}(t) = (Z_{i,i,j}(t)/2 + \sum_{l\geq i+1} Z_{l,i,j}(t))/Y_{i,j}(t)$. This shows that, as $X(t)$, the process $Y(t)$ is not a density dependent process because the rates of its transitions involve quantities that depend on triplets. In what follows, we consider a density dependent population process that is an approximation of the original process and has the same transitions but with different rates: in all the rates that involve a quantity $R_{i,j}(t)$, we replace this quantity by $Q_i(t) = (Y_{i,i}/2 + \sum_{j\geq 0} Y i, j)/X_i(t)$. This approximation is called the *pair-approximation* and leads to the following infinitesimal generator. We will not detail the state space of the problem, since this set is too complex and laborious to define and it is outside the scope of this study. For us, finding the infinitesimal generator is enough.

$$
\begin{aligned}
Q\left(y, y + \frac{1}{|E|}\sum_{j=1}^{d}(e_{i-1,k_j} - e_{i,k_j} + e_{k_j,i-1} - e_{k_j,i})\right) &= \frac{\mu}{d}|E|\mathbb{1}_{\{i>0\}}\frac{1}{x_i^{d-1}}\prod_{j=1}^{d} y_{i,k_j} \\
Q\left(y, y + \frac{1}{|E|}\sum_{j=1}^{d}(e_{i+1,k_j} - e_{i,k_j} + e_{k_j,i+1} - e_{k_j,i})\right) &= \frac{2\lambda}{d}|E|\frac{1}{x_i^{d-1}}\left(\prod_{j=1}^{d} y_{i,k_j}\right)\frac{1}{d}\sum_{j=1}^{d} a(i, k_j)
\end{aligned} \tag{1}
$$

The first transition corresponds to the end of service in a queue with i clients. The second transition correspond to the arrival of a client at a station with i clients.

2.2 Model analysis

The dynamical system $(y(t))$ limit of the Markov process $(Y^N(t))$ as $|E|$ tends to infinity is the solution of the following ODE:

$$
\begin{aligned}
\dot{y}_{i,j} =& \mu(y_{i+1,j} - y_{i,j}\mathbb{1}_{\{i>0\}} + y_{j+1,i} - y_{i,j}\mathbb{1}_{\{j>0\}}) \\
&+ \frac{2\lambda}{d}\left(a(i-1, j)y_{i-1,j} + a(j-1, i)y_{j-1,i} - y_{i,j}\right) \\
&+ \frac{2\lambda(d-1)}{d}\left(q_{i-1}\mathbb{1}_{\{i>0\}}y_{i-1,j} + q_{j-1}\mathbb{1}_{\{j>0\}}y_{j-1,i} - (q_i + q_j)y_{i,j}\right)
\end{aligned} \tag{2}
$$

Where $q_i := \frac{1}{x_i}\left(\frac{y_{i,i}}{2} + \sum_{k>i} y_{i,k}\right)$. Summing the equations for all $j \geq 0$, we find

$$\dot{x}_i = \mu(x_{i+1} - x_i \mathbb{1}_{\{i>0\}}) + 2\lambda(q_{i-1}x_{i-1}\mathbb{1}_{\{i>0\}} - q_i x_i) \tag{3}$$

Then, the equilibrium point is given by

$$x_{i+1} = \frac{2\lambda}{\mu} q_i x_i \tag{4}$$

At equilibrium, the rate of arriving clients $N\lambda \cdot 1$ must be equal to the rate of exiting clients $N\mu \cdot (1 - x_0)$. Therefore,

$$x_0 = 1 - \lambda/\mu$$

Using (4) and summing (2) for $i \geq k$ and $j \geq l$, we have, in the equilibrium

$$\begin{aligned} u_{k,l}\mathbb{1}_{\{k>0\}} + u_{l,k}\mathbb{1}_{\{l>0\}} =& \frac{\lambda}{d\mu}\left(2u_{k-1,l}\mathbb{1}_{\{k>0\}} + (u_{l-1,l-1} + u_{l-1,l})\mathbb{1}_{\{l>0\}}\right) \\ &+ \frac{d-1}{d}\left(\frac{u_{k,0}}{u_{k-1,0}} u_{k-1,l}\mathbb{1}_{\{k>0\}} + \frac{u_{l,0}}{u_{l-1,0}} u_{l-1,k}\mathbb{1}_{\{l>0\}}\right), \text{ for } k < l \\ u_{k,k} =& \left(\frac{2\lambda}{d\mu} + \frac{d-1}{d}\frac{u_{k,0}}{u_{k-1,0}}\right) u_{k-1,k} \end{aligned} \tag{5}$$

If we assume that $u_{k,l} = v_k w_l$, with $v_k, w_l \in \mathbb{R}$ and use the second equation of (5), we can find for $d < +\infty$ that $v_k = 2\lambda/\mu$. However, v_k must converge to 0 as k tends to infinity. Therefore, we can't write $u_{k,l}$ as proposed. This illustrates the difficulty of obtaining analytic results through this model. In the attempt of obtaining such results, we propose to study the model of the next section.

3 Deposition Model

We study choice between two neighbors in large queuing systems. For that we associate the corresponding deposition model (without departures). It is a Gates-Wescott model with deposition function values in arithmetic progression. In the Ezanno's these [4, p.66-68], the author achieves analytic results and find a product form for the invariant measure for a model very similar to *local choice*. Nevertheless, for the model, it was proved, in [7], that such a product measure is not the invariant measure. In the same work, however, the product measure corresponding to choice between two at random is derived. In this section, we show the idea of the proof and compare the found expression, which is an approximation, with numerical simulations. Interestingly, the approximation is *good*. In the next subsection, we apply the *pair approximation* technique to model the problem.

3.1 Model description

Consider a set of N sites on a straight line with independent arrival Poisson processes with rate λ on each pair of neighboring sites, where each particle deposes on the site with the smaller number of particles, ties being solved at random. We consider the general framework where the particle chooses with probability $\alpha > 0$, and does not choose with probability $1 - \alpha$. The length and profile processes are defined with a deposition function given by

$$\omega(a,b) = \beta_{\frac{1}{2}(1_{a=0}+1_{b=0})+1_{a>0}+1_{b>0}}. \tag{6}$$

The five deposition values are

$$\beta_0 = \lambda(1-\alpha),\ \beta_{\frac{1}{2}} = \lambda(1-\frac{\alpha}{2}),\ \beta_1 = \lambda,\ \beta_{\frac{3}{2}} = \lambda(1+\frac{\alpha}{2}),\ \beta_2 = \lambda(1+\alpha)$$

with the β_i's in arithmetic progression with common difference $\lambda\alpha/2$.

The lentgh process. For $x = (x(j))_{1\leq j\leq N}$, define $h = (\Delta_1 x, \ldots, \Delta_{N-1}x)$ where $\Delta_j x = x(j) - x(j+1)$ for $1 \leq j \leq N-1$. Then the state process $(X_t^N)_{t\geq 0} = (X_t^N(j))_t$ where $X_t^N(j)$ is the number of particules at site j at time t is a Markov process on the state space $\mathbb{N}^N$ with infinitesimal generator

$$\Omega f(x) = \sum_{i=1}^{N} [f(x+e_i) - f(x)]\omega(\Delta_{i-1}x, -\Delta_i x)$$

where ω is given by equation (6).

The profile (or Gates-Wescott) process. But, for each j, $X_t^N(j)$ tends almost surely to $+\infty$ with time, so let us define the profile process $(H_t^N)_{t\geq 0}$ where $H_t^N = (\Delta_1 X_t^N, \ldots, \Delta_{N-1}X_t^N)$. This process is a Markov process with infinitesimal generator

$$Lg(x) = \sum_{i=1}^{N} [g(h+e_i') - g(h)]\omega(h(i-1), -h(i)) \tag{7}$$

where ω is given by equation (6) and

$$e_i' = e_i - e_{i-1} \text{ if } 1 \leq i \leq N$$

(*periodic case*). The boundary conditions are not so important because we will work on the process on the whole line.

Proposition 1. *If the choice is at random among two sites, then the invariant measure of the infinite site profile process is of product form $m^{\otimes\mathbb{Z}}$ with m measure on $\mathbb{Z}$ given by*

$$\begin{aligned} m(0) &= \frac{1-r}{2}, \\ m(i) &= \frac{1-r^2}{4} r^{|i|-1} \text{ if } |i| \geq 1 \end{aligned} \tag{8}$$

where $r = \beta_{1/2}/\beta_{3/2}$.

Proposition 2. *The profile process with choice among neighbors has no product form measure.*

In other terms, the previous proposition means that, in stationary regime, when the system gets large, the station states are not independent.

Proof. The proof consists of the following. The stationary measure μ verifies that

$$\int Lg d\mu = 0 \tag{9}$$

for all g of the form $g(h) = 1_{h(i)=a_i,\ldots,h(j)=a_j}$ where $i < j$, $a_i, \ldots, a_j \in \mathbb{Z}$ and L is defined by equation (7). We start with $g(h) = 1_{h(i)=a_i}$ where $a_i \in \mathbb{Z}$. If $\mu = m^{\otimes\mathbb{Z}}$ then m is given by equation (8). If $g(h) = 1_{h(i)=a_i,h(i+1)=a_{i+1}}$ where $a_i = a_{i+1} = 0$, equation (9) does not hold. It ends the proof. □

3.2 Simulation and product measure approach

Despite Proposition 2, let us compare the numerical simulation of the deposition process with the expressions given by Proposition 1.

Marginal distribution of profiles as a function of k, with $r = 1/3$ and $N = 1000$.

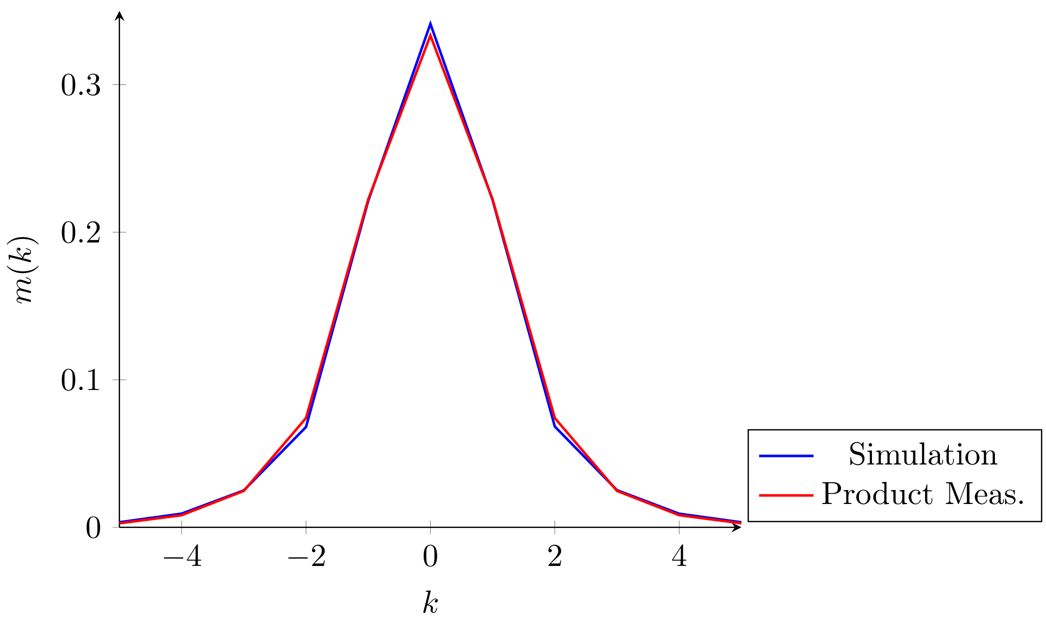

Figure 1: The approximation is relatively close to the simulation $\sqrt{\sum_k |m_1(k) - m_2(k)|^2} \approx 0.012$
.

Marginal distribution of profiles as a function of k, with $r = 0.6$ and $N = 1000$.

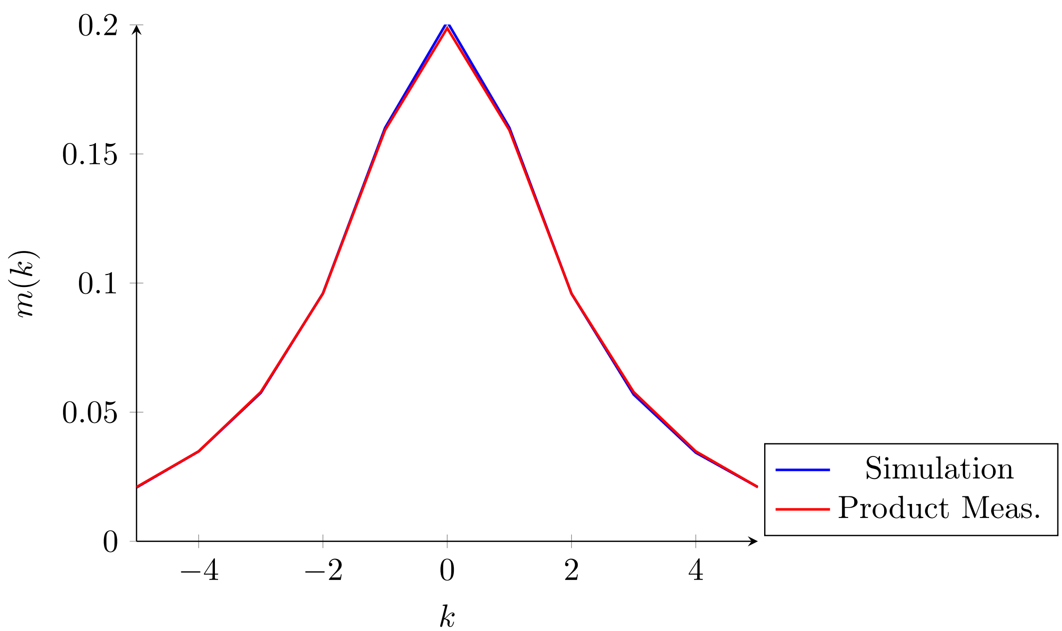

Figure 2: The approximation is even closer to the simulation $\sqrt{\sum_k |m_1(k) - m_2(k)|^2} \approx 0.003$
.

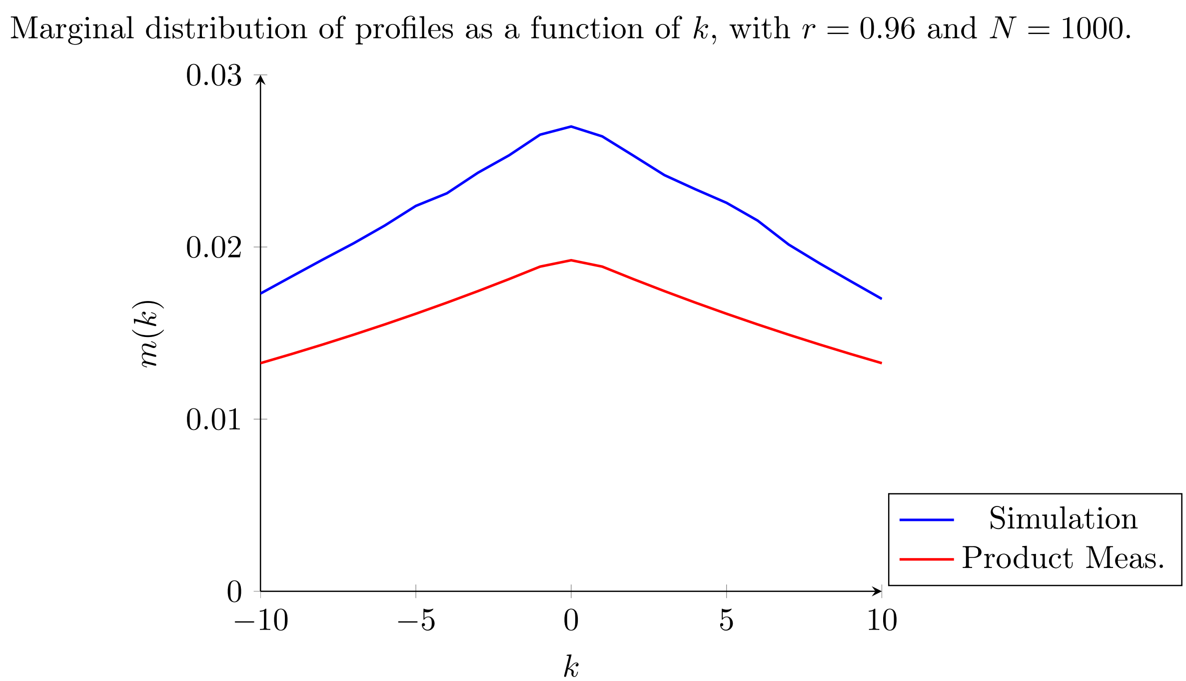

Figure 3: When r gets close to 1, the approximation gets worse again $\sqrt{\sum_k |m_1(k) - m_2(k)|^2} \approx 0.029$
.

The deposition process of *local choice* is very close to the product measure approximation, as we can verify in [7], where simulations for the *local choice* between two queues is considerably different compared to choice between two queues at random.

It is clear that when r approaches 1, the approximation by product measure gets worse. In the attempt to achieve a better approximation, we apply, in the next section, the *pair approximation* technique, presented in Section 2.

3.3 Pair approximation approach

Using the pair approximation in the case $\beta_0 = 0$ ($r = 1/3$) as we used in Section 1.4, we have for all $i, j, k \in \mathbb{Z}$ and the appropriate state space, the following infinitesimal generator,

$$\begin{aligned} Q\left(y, y + \frac{1}{N}(e_{i+1,j} - e_{i,j} + e_{k-1,i+1} - e_{k,i} + e_{l,k-1} - e_{l,k})\right) &= N\lambda(a(i,0) + a(0,k))\frac{y_{l,k}y_{k,i}y_{i,j}}{x_k x_i} \\ Q\left(y, y + \frac{1}{N}(e_{i-1,j+1} - e_{i,j} + e_{k,i-1} - e_{k,i} + e_{j+1,l} - e_{j,l})\right) &= N\lambda(a(0,i) + a(j,0))\frac{yk, iy_{i,j}y_{j,l}}{x_i x_j} \end{aligned} \tag{10}$$

Where $x_i = \sum_k y_{i,k}$ and $a(i,j) = \mathbb{1}_{\{i<j\}} + \frac{1}{2}\mathbb{1}_{\{i=j\}}$. In this case, we do not have the symmetrical property, i.e., $y_{i,j}$ is not necessarily equal to $y_{j,i}$. Instead, we do have $y_{i,j} = y_{-j,-i}$.

As N tends to infinity the system converges to an ODE, for all $i, j \in \mathbb{Z}$

$$\begin{aligned} \dot{y}_{i,j} =& 2\lambda(y_{i-1,j}(a(i-1,0) + q_{i-1}) + y_{i+1,j-1}(a(0,i+1) + a(j-1,0)) \\ &+ y_{i,j+1}(a(0,j+1) + q_{-j-1}) - y_{i,j}(2 + q_i + q_{-j})) \end{aligned} \tag{11}$$

where $q_i = \dfrac{1}{x_i}\left(\dfrac{y_{0,i}}{2} + \sum_{k>0} y_{k,i}\right)$.

The differential equation converges to the invariant measure as $t \to \infty$ which give us a solution closer to the simulation than the product measure approach. Let $m_{sim}(k)$ be the invariant measure by numerical simulation, $m_{pair}(k) = x_k$ the invariant measure by pair approximation and $m_{prod}(k)$ the invariant measure from Proposition 1, for $r = 1/3$, for all $k \in \mathbb{Z}$. Then, $\sqrt{\sum_k |m_{sim}(k) - m_{pair}(k)|^2} \approx 0.004$ is smaller than the previous $\sqrt{\sum_k |m_{sim}(k) - m_{prod}(k)|^2} \approx 0.012$. We achieve, for this case, a better approximation than the product measure approach.

Summing the ODE over $j \in \mathbb{Z}$ we obtain for all $i \in \mathbb{Z}$

$$\dot{x}_i = 2\lambda \left(x_{i+1}(a(0,i+1) + q_{-i-1}) - x_i(1 + q_i + q_{-i}) + x_{i-1}(a(i-1,0) + q_{i-1})\right) \tag{12}$$

At equilibrium, $\dot{x}_i = 0$ and we can solve the recurrence as a function of q_i

$$\begin{aligned} x_1 &= \frac{1}{2}\frac{1+2q_0}{1+q_{-1}} x_0 & & \\ x_i &= \frac{q_{i-1}}{1+q_{-i}} x_{i-1}, & &\text{for } i > 1 \\ x_i &= x_{-i}, & &\text{for } i \in \mathbb{Z} \end{aligned} \tag{13}$$

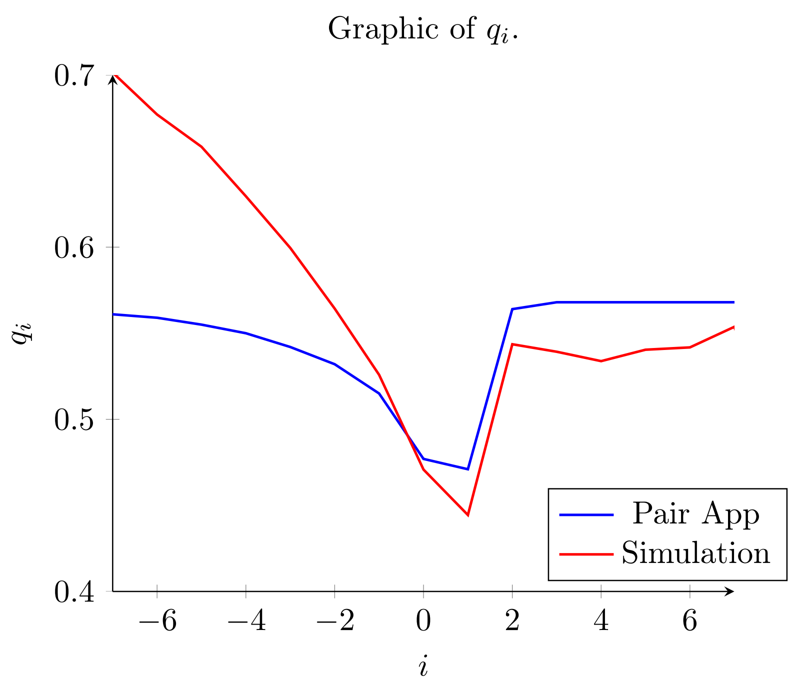

Figure 4: For the pair approximation, q_i seems to be constant and approximately equal to 0.57 for $i \geq 3$. While, for $i \leq -3$, $q_i \approx 0.57 - 0.11^{-i}$.

We can notice that q_i is close to 1/2. If we make $q_i = 1/2$ in (13), for all $i \in \mathbb{Z}$, we find the expression for $m_{prod}(i)$ (Proposition 1). After all, we can convince ourselves that we have good approximations for the model. Nevertheless, as before, we face a problem that does not allow us to achieve relevant analytic results. This led us to retake the queue models, but this time we resort to a perturbation approach, which showed itself much more fertile.

4 Queues with general choice function - Perturbation approach

For a system of N queues, let $n = (n_1, n_2, ..., n_N) \in \mathbb{N}^N$ be the state of the system, where n_i is the number of clients in the queue i. Let $c : \mathbb{N} \times \mathbb{N}^N \longrightarrow \mathbb{R}_+$, be the choice function, so that $\lambda c_i(n)$ is the arrival rate of clients in the queue i if the system is at state n and let μ be the exit rate of a non-empty queue.

Let $(L^N(t)) = ((L_i^N(t)))_{1\leq i\leq N}$, where $L_i^N(t)$ is the number of customers at queue i at time t. $(L^N(t))_{t\geq 0}$ is a Markov process in $\mathbb{N}^N$, irreducible, with jump matrix Q given by $\forall n \in \mathbb{N}^N$, $\forall i \in \{1, 2, ..., N\}$,

$$\begin{aligned} Q(n, n + e_i) &= \lambda c_i(n) \\ Q(n, n - e_i) &= \mu \mathbb{1}_{\{n_i>0\}}, \end{aligned} \tag{14}$$

where e_i is the i-th unity vector of $\mathbb{N}^N$.

From now on, we assume that the choice function c is such that the Markov process is ergodic for a range of values of $\rho := \lambda/\mu$.

Let $y_n(\rho)$ be the invariant measure of $(L^N(t))$, then the equilibrium equations are given by

$$\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}} + \rho\sum_{i=1}^N c_i(n)\right) y_n(\rho) = \sum_{i=1}^N y_{n+e_i}(\rho) + \rho\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}c_i(n-e_i)y_{n-e_i}(\rho), \quad \forall n \in \mathbb{N}^N. \tag{15}$$

Suppose that exists a $\varepsilon > 0$, such that we can write $y_n(\rho)$ as a series expansion of the form

$$y_n(\rho) = \sum_{k\geq 0} \alpha_k(n)\rho^k, \qquad \forall \rho \in [0, \varepsilon[, \forall n \in \mathbb{N}^N, \tag{16}$$

then we know that $\alpha_k(n) = \dfrac{y_n^{(k)}(0)}{k!}$.

Taking the derivative of (15) with respect to ρ, k times, and evaluating it on $\rho = 0$, it holds that, for any $n \in \mathbb{N}^N$ and $k \in \mathbb{N}^*$,

$$\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right) \alpha_k(n) = \sum_{i=1}^N \alpha_k(n+e_i) + \sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}c_i(n-e_i)\alpha_{k-1}(n-e_i) - \left(\sum_{i=1}^N c_i(n)\right)\alpha_{k-1}(n). \tag{17}$$

Lemma 1. *If a function $\alpha : \mathbb{N}^N \longrightarrow \mathbb{R}$ has the following properties for some $k_0 \in \mathbb{N}^*$,*

$$\alpha(n) \geq 0, \quad \forall n \in \mathcal{U}_{k_0} := \left\{n \in \mathbb{N}^N \mid |n| > k_0\right\}, \ |n| := \sum_{i=1}^N n_i$$

$$\sum_{n\in\mathcal{U}_{k_0}} \alpha(n) < \infty$$

and α satisfies the following recurrence equation,

$$\left(\sum_{i=1}^N \mathbb{1}_{\{n_i>0\}}\right) \alpha(n) = \sum_{i=1}^N \alpha(n+e_i), \quad \forall n \in \mathcal{U}_{k_0}, \tag{18}$$

then $\alpha(n) = 0$, for all $n \in \mathcal{U}_{k_0}$.

Proof. Let $k_0 \in \mathbb{N}^*$, $\mathcal{N}_k := \left\{ n \in \mathbb{N}^N \mid \sum\limits_{i=1}^{N} n_i = k \right\}$, $\forall k \in \mathbb{N}$ and $\mathcal{U}_{k_0} := \bigcup\limits_{k > k_0} \mathcal{N}_k$.

First, let us show that for any $k > k_0$,

$$\sum_{n \in \mathcal{N}_k} \sum_{i=1}^{N} \alpha(n + e_i) = \sum_{n \in \mathcal{N}_{k+1}} \left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}} \right) \alpha(n). \tag{19}$$

Each $\alpha(n), n \in \mathcal{N}_{k+1}$ can be written as $\alpha(\hat{n} + e_i)$, for some of the $\hat{n} \in \mathcal{N}_k$. The number of elements in $\mathcal{N}_k$ that can generate n when we sum it with e_i is exactly equal to the number of non-zero n_i, where n_i is the i-th coordinate of the vector n. For each non-zero n_i, we can find an unique $\hat{n}$ such that $\hat{n} + e_i = n$. Therefore, the above equation is true, since $\sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}}$ is the number of non-zero n_i.

Using the recurrence equation, we substitute $\sum_{i=1}^{N} \alpha(n + e_i)$ in the left-hand side of (19) and we have for any $k > k_0$,

$$\sum_{n \in \mathcal{N}_k} \left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}} \right) \alpha(n) = \sum_{n \in \mathcal{N}_{k+1}} \left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}} \right) \alpha(n),$$

which allow us to deduce that for any $k > k_0$,

$$\sum_{n \in \mathcal{N}_k} \left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}} \right) \alpha(n) = C, \tag{20}$$

where $C \in \mathbb{R}_+$ is a positive constant, since $\alpha(n) \geq 0$.

As $\sum_{i=1}^{N} \mathbb{1}_{\{n_i > 0\}} \leq N$, then

$$\sum_{n \in \mathcal{N}_k} N\alpha(n) \geq C$$

If $C > 0$, $\sum\limits_{n \in \mathcal{U}_{k_0}} \alpha(n) = \sum\limits_{k > k_0} \sum\limits_{n \in \mathcal{N}_k} \alpha(n)$ diverges, since $\sum\limits_{n \in \mathcal{N}_k} \alpha(n) \geq \dfrac{C}{N} > 0$.

But, this contradicts the constraint $\sum\limits_{n \in \mathcal{U}_{k_0}} \alpha(n) < \infty$, then $C = 0$. Using the fact that $\alpha(n) \geq 0$ and equation (20), then $\alpha(n) = 0$, for all $n \in \mathcal{U}_{k_0}$. ☐

Lemma 2. *Let $k \in \mathbb{N}$. $\alpha_k(n) = 0$, for all $n \in \mathcal{U}_k$.*

Proof. If we suppose that Lemma 2 is true for $k = k'$, then Lemma 1 guarantees, for $k_0 = k' + 1$, that Lemma 2 is true for $k = k' + 1$. We can apply Lemma 1, because the terms of the equation (17) that has $\alpha_{(k'+1)-1}(n)$ and $\alpha_{(k'+1)-1}(n - e_i)$ vanish for $n \in \mathcal{U}_{k'+1}$, since $n, n - e_i \in \mathcal{U}_{k'}$ and we supposed $\alpha_{k'}(n) = 0$, $\forall n \in \mathcal{U}_{k'}$. Therefore we have an equation like (18). The reason why we can state $\alpha(n) \geq 0$ is because it represents the first possible non-zero coefficient for $y_n(\rho)$. It must be greater or equal to zero, because otherwise it would exist a ρ such that $y_n(\rho) < 0$, which is absurd. For $k' = 0$, it is easy to verify from (15) that the statement is true. Then, by the principle of finite induction the statement is true for all $k \in \mathbb{N}$. ☐

Intuitively speaking, Lemma 2 says that the maximum number of customers in the system at order k is at maximum k.

To organize the order to find the coefficients, let us define $\mathcal{N}_m := \left\{ n \in \mathbb{N}^N \mid |n| = m \right\}, \forall m \in \mathbb{N}$. The **algorithm** to calculate all the coefficients consists in start calculating for $k = 1$, then $k = 2$ and so on. In order to find the coefficients of order k, we calculate all coefficients $\alpha_k(n)$ for $n \in \mathcal{N}_k$, because by Lemma 2, we have all the right-hand side terms of equation (17). The next step is to calculate for $n \in \mathcal{N}_{k-1}$. Since $n + e_i \in \mathcal{N}_k$, we still have all the right-hand side terms of the recurrence equation. After, we calculate the coefficients for $n \in \mathcal{N}_{k-2}$, $n \in \mathcal{N}_{k-3}$ and so on, until $n \in \mathcal{N}_1$. To find $\alpha_k(0_N)$ we use that $\sum\limits_{n\in\mathbb{N}^N} \alpha_k(n) = 0$, if $k > 0$.

Our objective is to find the expansion of $\pi_{i,m}(\rho)$, which represents the probability that the queue i has $m \in \mathbb{N}$ clients. Which is given by

$$\pi_{i,m}(\rho) = \sum_{n\in\mathbb{N}^N | n_i = m} y_n(\rho), \tag{21}$$

As usual, let us assume that $\pi_{i,m}(\rho)$ has an expansion and can be written as

$$\pi_{i,m}(\rho) = \sum_{k\geq 0} \phi_k(i,m)\rho^k, \qquad \forall \rho \in [0,1[, \forall m \in \mathbb{N}, \tag{22}$$

then we know that $\phi_k(i,m) = \dfrac{\pi_{i,m}^{(k)}(0)}{k!}$. We can deduce from (16) that

$$\phi_{i,k}(m) = \sum_{n\in\mathbb{N}^N | n_i = m} \alpha_k(n),$$

and using Lemma 2, we have that

$$\phi_{i,k}(m) = \mathbb{1}_{\{m\leq k\}} \sum_{\substack{n_1+n_2+\ldots+n_N\leq k \\ n_i = m}} \alpha_k(n). \tag{23}$$

Then, a priori, we can find as many coefficients as we want.

4.1 Third order expansion

From now on, let us assume some additional properties for the choice function c,
(H_1) $\sum_{i=1}^N c_i(n) = N$, $\forall n \in \mathbb{N}^N$,
(H_2) $c_i(n)$ is invariant by circular permutation over n and by reflection over n, $\forall n \in \mathbb{N}^N$, $\forall i \in \{1, 2, ..., N\}$.

These assumptions are true for a system with identical queues and that has a fixed total rate of incoming customers.

For $\rho = 0$, the solution is $y_n(0) = \mathbb{1}_{\{n=0_N\}}$, which give us the coefficients for $k = 0$

$$\alpha_0(n) = \mathbb{1}_{\{n=0_N\}}.$$

Using the method described previously, (H_1) and (H_2), we can show that

$$\begin{aligned}
&\alpha_1(0_N) = -N && \\
&\alpha_1(e_i) = 1, && \forall i \in \{1, 2, ..., N\} \\
&\alpha_1(n) = 0, && \text{otherwise}
\end{aligned} \tag{24}$$

It is interesting to notice that these results do not depend on the choice function c. Which means that for ρ sufficiently small, the politic of choice does not influence the system.

For $k = 2$, we use the same arguments to show that for all $i, j \in \{1, 2, ..., N\}$

$$
\begin{aligned}
&\alpha_2(0_N) = \frac{1}{2}(N^2 - Nc_1(e_1)) \\
&\alpha_2(e_i) = -N \\
&\alpha_2(e_i + e_j) = c_i(e_j) \\
&\alpha_2(n) = 0, && \text{otherwise}
\end{aligned}
\tag{25}
$$

For $k = 3$, we show that for all $i, j, l \in \{1, 2, ..., N\}$

$$
\begin{aligned}
&\alpha_3(0_N) = -\sum_{n \neq 0_N} \alpha_3(n) \\
&\alpha_3(e_i) = \frac{1}{2}(N^2 - Nc_1(e_1)) \\
&\alpha_3(e_i + e_j) = \frac{1}{2}\left(\sum_{v=1}^{N} \alpha_3(e_i + e_j + e_v) - 3Nc_i(e_j)\right), && i \neq j \\
&\alpha_3(2e_i) = \frac{1}{2}\left(\sum_{v=1}^{N} c_i(e_v)c_i(e_i + e_v) - 3Nc_i(e_i)\right) \\
&\alpha_3(e_i + e_j + e_l) = \frac{1}{3}(c_i(e_j)c_l(e_i + e_j) + c_j(e_l)c_i(e_j + e_l) + c_l(e_i)c_j(e_l + e_i)), && i \neq j, j \neq l, l \neq i \\
&\alpha_3(2e_i + e_j) = \frac{1}{2}(c_i(e_j)c_i(e_i + e_j) + c_i(e_i)c_j(2e_i)), && i \neq j \\
&\alpha_3(3e_i) = c_1(e_1)c_1(2e_1) \\
&\alpha_3(n) = 0, && \text{otherwise}
\end{aligned}
\tag{26}
$$

We are not going to continue, because the expressions become too complex.

As before, our objective is to calculate $\pi_{i,m}(\rho)$. However, as our system is invariant by circular permutation, we no longer needs to differ which queue we refer, since all have the same law, by (H_2), so using $n_1 = m$, without loss of generality, we can write the probability for a queue to have $m \in \mathbb{N}$ clients as

$$\pi_m(\rho) = \sum_{n \in \mathbb{N}^N | n_1 = m} y_n(\rho), \tag{27}$$

As usual, we assume it has an expansion, that can be written as

$$\pi_m(\rho) = \sum_{k \geq 0} \phi_k(m)\rho^k, \qquad \forall \rho \in [0, 1[, \forall m \in \mathbb{N}, \tag{28}$$

then we know that $\phi_k(m) = \dfrac{\pi_m^{(k)}(0)}{k!}$. From (23), we can deduce that

$$\phi_k(m) = \mathbb{1}_{\{m \leq k\}} \sum_{n_2 + n_3 + ... + n_N \leq k - m} \alpha_k(m, n_2, n_3, ..., n_N). \tag{29}$$

$$\tag{30}$$

For $m = 0$, we can find the closed form, since at equilibrium, the rate of incoming clients and the rate of exiting clients must be the same, i.e., $N\lambda = N(1 - \pi_0)\mu$, then $\pi_0(\rho) = 1 - \rho$.

For $k = 1$, we have

$$\begin{aligned}
&\phi_1(0) = -1\\
&\phi_1(1) = 1\\
&\phi_1(i) = 0, \qquad \forall i > 1
\end{aligned}$$

For $k = 2$,

$$\begin{aligned}
&\phi_2(0) = 0\\
&\phi_2(1) = -c_1(e_1)\\
&\phi_2(2) = c_1(e_1)\\
&\phi_2(i) = 0, \qquad \forall i > 2
\end{aligned}$$

For $k = 3$,

$$\begin{aligned}
&\phi_3(0) = 0\\
&\phi_3(1) = Nc_1(e_1) - \sum_{j=1}^{N} c_1(e_1 + e_j)c_1(e_j)\\
&\phi_3(2) = -Nc_1(e_1) + \sum_{j=2}^{N} c_1(e_1 + e_j)c_1(e_j)\\
&\phi_3(3) = c_1(e_1)c_1(2e_1)\\
&\phi_3(i) = 0, \qquad \forall i > 3
\end{aligned}$$

Additionally, we can show that

$$\phi_k(k) = \prod_{j=1}^{k-1} c_1(je_1), \qquad \forall k > 1$$

Proof. Taking $n = ke_1$ in the equation (17) we have

$$\alpha_k(ke_1) = \sum_{i=1}^{N} \alpha_k(ke_1 + e_i) + c_1((k-1)e_1)\alpha_{k-1}((k-1)e_1) - N\alpha_{k-1}(ke_1)$$

We know that $\alpha_k(ke_1 + e_i) = 0, \forall i$ and $\alpha_{k-1}(ke_1) = 0$, by Lemma 2. Then, using equation (29), we find that $\phi_k(k) = \alpha_k(ke_1)$. Which give us a recurrence equation in $\phi_k(k)$,

$$\phi_k(k) = c_1((k-1)e_1)\phi_{k-1}(k-1), \forall k \in \mathbb{N}^*$$

which leads us to the desired result, since $\phi_1(1) = 1$. □

Note that $\phi_k(k)$ is the first coefficient of the expansion of $\pi_k(\rho)$. This follows directly from Lemma 2 and (29).

The previous results can be summarized with (28) by the following proposition:

Proposition 3. *If the choice function c satisfies $H(1)$ and $H(2)$, then*

$$\begin{aligned}
\pi_0(\rho) &= 1-\rho \\
\pi_1(\rho) &= \rho - c_1(e_1)\rho^2 + \left(N c_1(e_1) - \sum_{j=1}^{N} c_1(e_1+e_j)c_1(e_j) \right)\rho^3 + \mathcal{O}(\rho^4) \\
\pi_2(\rho) &= c_1(e_1)\rho^2 - \left(N c_1(e_1) - \sum_{j=2}^{N} c_1(e_1+e_j)c_1(e_j) \right)\rho^3 + \mathcal{O}(\rho^4) \\
\pi_k(\rho) &= \left(\prod_{j=1}^{k-1} c_1(je_1) \right)\rho^k + \mathcal{O}(\rho^{k+1}), \qquad \forall k \geq 3,
\end{aligned} \tag{31}$$

for ρ that tends to zero.

4.2 No choice: $c_i(n) = 1$

For the trivial case, where each queue receives clients at rate ρ independently and the exit rate is 1, we have $c_i(n) = 1$, $\forall n \in \mathbb{N}^N, \forall i \in \mathbb{N}$. We can easily verify that

$$\alpha_k(n) = (-1)^{k-|n|} \binom{N}{k-|n|} \mathbb{1}_{|n|\leq k}$$

satisfies equation (17), where $|n| = \sum_{i=1}^{N} n_i$. Using equation (29), we have for any $r \in \mathbb{N}$, $0 \leq r \leq k$,

$$\begin{aligned}
\phi_k(k-r) &= (-1)^r \sum_{i=0}^{r} (-1)^i \binom{N-2+i}{i} \binom{N}{r-i} \\
&= (-1)^r \mathbb{1}_{\{r\leq 1\}}.
\end{aligned}$$

The term $\binom{N-2+i}{i}$ comes from the fact that we need to distribute the remaining i customers in the remaining $N-1$ queues. Then, we have the expected result,

$$\pi_m(\rho) = \rho^m - \rho^{m+1}, \qquad \forall m \in \mathbb{N}.$$

4.3 Local choice: $c_i(n) = a(n_i, n_{i-1}) + a(n_i, n_{i+1})$

Taking $c_i(n) = a(n_i, n_{i-1}) + a(n_i, n_{i+1})$, where $a(k,l) := \frac{1}{2}\mathbb{1}_{\{k=l\}} + \mathbb{1}_{\{k<l\}}, \forall k, l \in \mathbb{N}$ with the convention $n_{-1} = n_N$, $n_{N+1} = n_1$ and using Proposition 3, we can write a third order approximation for $\pi_m(\rho)$

$$\begin{aligned}
\pi_0(\rho) &= 1-\rho \\
\pi_1(\rho) &= \rho - \frac{3}{2}\rho^3 + \mathcal{O}(\rho^4) \\
\pi_2(\rho) &= \frac{3}{2}\rho^3 + \mathcal{O}(\rho^4) \\
\pi_m(\rho) &= \mathcal{O}(\rho^{m+1}), \qquad \forall m > 2
\end{aligned}$$

As the amount of cases to analyze grows exponentially with k, it is rather difficult to do it without a program. We will use the following property, which remains to be proved.

$$\forall k \in \mathbb{N}, \exists N_0(k) \text{ such that } \forall N > N_0(k),\ \pi_i^{(k)} \text{ does not depend on } N,\ \forall i \in \mathbb{N}. \tag{32}$$

From this, we can calculate the iteration (15) for some N sufficiently large with a program, leading to the following result

$$
\begin{aligned}
\pi_0(\rho) &= 1 - \rho \\
\pi_1(\rho) &= \rho - \frac{3}{2}\rho^3 + \frac{11}{8}\rho^4 - \frac{7}{3}\rho^5 + \frac{10727}{2880}\rho^6 + \mathcal{O}(\rho^7) \\
\pi_2(\rho) &= \frac{3}{2}\rho^3 - \frac{11}{8}\rho^4 + \frac{47}{24}\rho^5 - \frac{1583}{320}\rho^6 + \mathcal{O}(\rho^7) \\
\pi_3(\rho) &= \frac{3}{8}\rho^5 + \frac{11}{9}\rho^6 + \mathcal{O}(\rho^7) \\
\pi_i(\rho) &= \mathcal{O}(\rho^7), \qquad \forall i > 3.
\end{aligned}
$$

We did not go further than this, because the memory required grows as $\mathcal{O}\left(k^k\right)$.

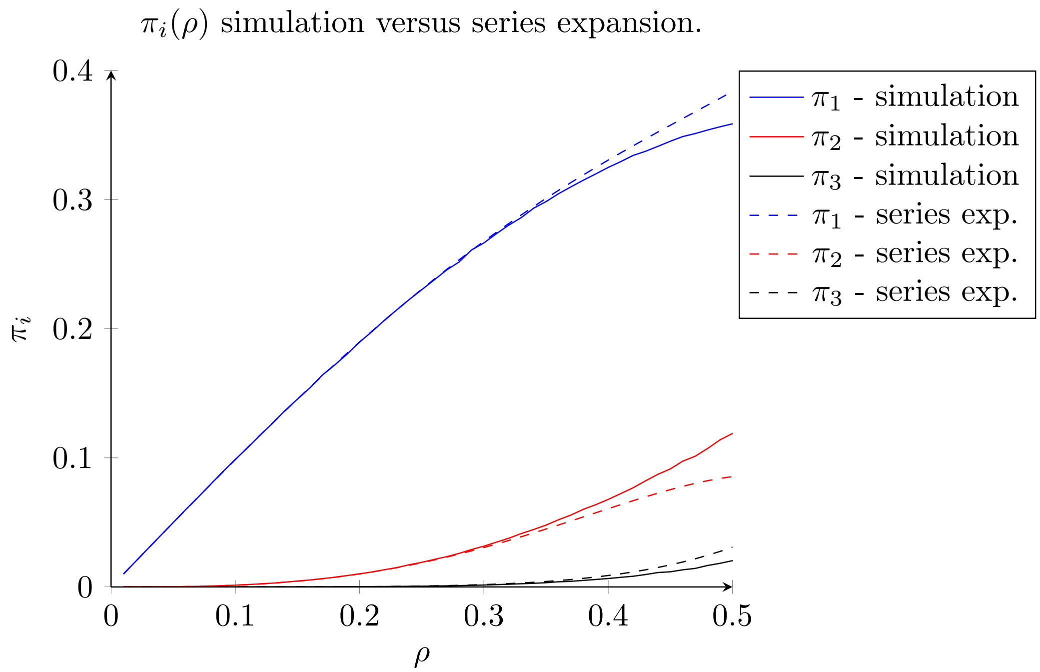

Figure 5: As we can see in the graphic, we have a reasonable approximation for π_i for $\rho < 0.4$, where it starts to diverge.

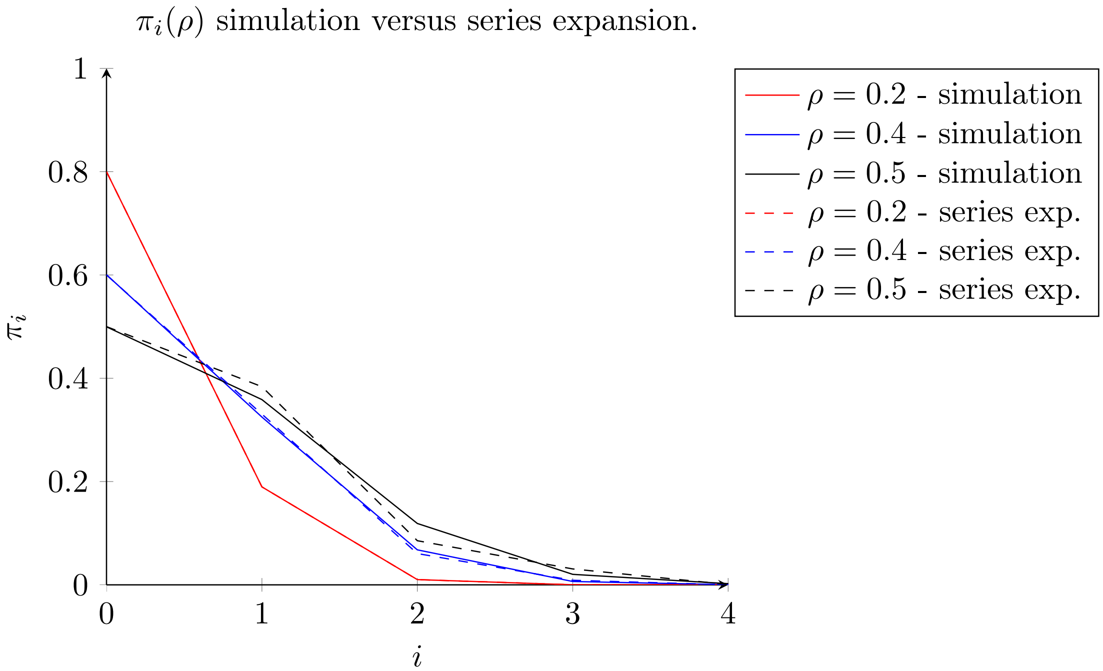

Figure 6: We can only see a difference for $\rho = 0.5$.

Looking how the system evolves with equation (15), i.e., $c_i(n) = 0$ if n_i is greater than $n_{i\pm1}$, we might imagine that $\pi_m(\rho) = \mathcal{O}(\rho^{2m-1})$. In fact, this is true, but to prove that we will need to take a different approach over the process of finding coefficients. First, remember that the sum of all states of order $2m-1$ which have the first queue with m clients (without loss of generality), will give us the $\phi_{2m-1}(m)$. So, saying that the first non-zero coefficient of the expansion of $\pi_m(\rho)$ is $\phi_{2m-1}(m)$ might mean (in this case it means) that from all states of order $2m-1$, the ones who has the greatest number of clients in a queue is the ones who contributes for this coefficient, which would mean that this greatest number is m.
Beyond that, it makes sense to think that the greatest number of clients in a queue in the order $2m-1$ is m, because Lemma 2 guarantees us that the total number of clients in the system at order $2m-1$ cannot be greater than $2m-1$, and as a limitation of the choice function $c_i(n)$, we cannot produce the greatest queue with i clients, if the neighbor queue does not have $i-1$. Then, the maximum possible value that i can assume is given by $i + (i-1) = 2m-1$, which give us $i = m$.

Proposition 4. *Let $c_i(n) = a(n_i, n_{i-1}) + a(n_i, n_{i+1})$ and $m \geq 2$.*

$$\pi_m(\rho) = 12\left(\frac{\rho}{2}\right)^{2m-1} + \mathcal{O}(\rho^{2m}),$$

for ρ that tends to zero.

Proof. First, we have to understand how equation (17) works for $c_i(n) = a(n_i, n_{i-1}) + a(n_i, n_{i+1})$ (remember that $a(k,l) = \frac{1}{2}\mathbb{1}_{\{k=l\}} + \mathbb{1}_{\{k<l\}}$, for $k, l \in \mathbb{N}$), when we are trying to find the states at order k where we have a queue with the greatest number of customers. In other words, we are only interested in the states n at order k, where we have the greatest n_1 or a state that contributes to generate a state that has the greatest n_1 (without loss of generality, since it is invariant by circular permutation) and that, of course, $|\alpha_k(n)| > 0$.

The idea of the proof is given by the fact, that new states can only be primarily generated by the term $\sum_{i=1}^{N} \mathbb{1}_{\{n_i>0\}} c_i(n-e_i)\alpha_{k-1}(n-e_i)$ in (17), which comes from the previous order $k-1$. We said primarily, because this new state can generate new states, because of the term $\sum_{i=1}^{N} \alpha_k(n+e_i)$. But, these new states have less customers, since they are generated by decrementing as much times as possible and in any combination the n_i of the first new state. Notice too, that if these states are indeed new, then it is really easy to calculate them.

We are searching the states that has one queue with the greatest possible number of clients for the order k, because it will be the first term of the expansion. For that, look at equation (29). As the states we are searching have an unique characteristic, it is logical to expect that these states are going to be the new states of its order.

To better visualize the process of finding these states let us make some interactions beginning from the order $k = 1$. Use Figure 7 to help seeing what is going on. When we increment the order, the new primarily states of this order can only differ at maximum one customer from a state of the previous order. Beyond that, we are not interested in states that will not create, eventually, a state where we have the greatest possible queue. That is the case of the state that is dashed in red. This state will never collaborate to create states we are interested in, because in order to do that, it will have to build a state like the one in its left and for that it will spent one increment, which means, it will never contribute to the greatest possible queue.

So, following the rules that we can increment one queue each time we pass to the next order and we can decrement queues resting in the same order, we build Figure 7. Note that we are ignoring the states that are not going to contribute to states we are interested in. The states we are interested in appear in order k odd, since the new greatest possible queue appear only in order odd.

As now, we know the behavior of the system, we can start the proof. Let us begin analyzing the states $n \in \mathcal{N}_k := \left\{ n \in \mathbb{N}^N \mid \sum_{i=1}^{N} n_i = k \right\}$. By Lemma 2, $\alpha_k(n+e_i) = \alpha_{k-1}(n) = 0$, then we can rewrite equation (17) as

$$\left(\sum_{i=1}^{N} \mathbb{1}_{\{n_i>0\}} \right) \alpha_k(n) = \sum_{i=1}^{N} \mathbb{1}_{\{n_i>0\}} c_i(n-e_i) \alpha_{k-1}(n-e_i), \qquad \forall n \in \mathcal{N}_k.$$

Note that, if n_1 is greater than $k/2$ for k even and greater than $(k+1)/2$ for k odd, then the neighbor queues cannot have the same number of customers, since $n \in \mathcal{N}_k$, which imply that $c_1(n-e_1) = 0$. In this case, $|\alpha_k(n)| > 0$ only if there exists a state n, such that $|\alpha(n-e_i)| > 0$, for some $i \neq 1$. However, if that is the case, we can repeat the process for $n - e_i \in \mathcal{N}_{k-1}$ and so on until we have only the n_1 to decrement, which will give us zero. Therefore, in this case $\alpha_k(n) = 0$.

We can use the same arguments for $n \in \mathcal{N}_{k-1}$ and n_1 greater than $(k+1)/2$ for k odd and $k/2$ for k even, since $\alpha_k(n+e_i) = \alpha_{k-1}(n) = 0$, by the above result. Repeating this k times, we show that $\alpha_k(n) = 0$, if any of the queues has more than $(k+1)/2$ customers for k odd and $k/2$ customers for k even.

Now, let us verify the case where $n_1 = (k+1)/2$, k is odd and $n \in \mathcal{N}_k$. If the neighbor queue does not have $(k-1)/2$ customers, we can use the same arguments as before and show that $\alpha_k(n) = 0$. So, without loss of generality, $n_2 = (k-1)/2$ and we have the following recurrence equation, for $k = 2m-1$,

$$2\alpha_{2m-1}(m, m-1, 0, ..., 0) = \frac{1}{2} \alpha_{2(m-1)}(m-1, m-1, 0, ..., 0), \qquad m \geq 2. \tag{33}$$

For k even, the maximum number of customers in a queue is $k/2$. So, for the same number of customers in a queue, the k even always comes after the $k-1$ odd, which means that if it is not zero, the odd coefficient is the first non-zero coefficient in the expansion. Keeping this in mind, we will calculate only the k even that we need to find the $k+1$ odd. Let $k = 2m$, and $n = (m, m, 0, ..., 0)$, then we have

$$2\alpha_{2m}(m, m, 0, ..., 0) = 2\alpha_{2m-1}(m, m-1, 0, ..., 0), \qquad m \geq 2 \tag{34}$$

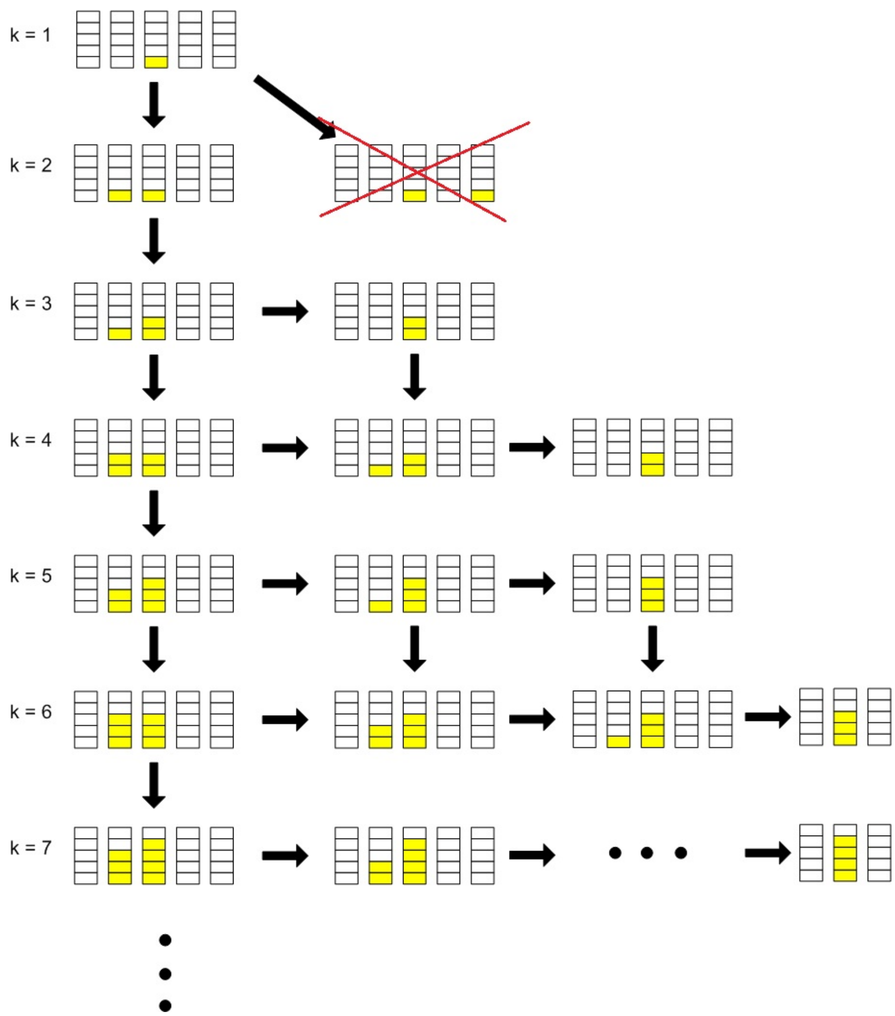

Figure 7: Non-zero states that have the greatest possible queue.

Combining (33) and (34),

$$\alpha_{2m-1}(m, m-1, 0, ..., 0) = \frac{1}{2^2}\alpha_{2m-3}(m-1, m-2, 0, ..., 0), \qquad m \geq 2$$

and then,

$$\begin{aligned}\alpha_{2m-1}(m, m-1, 0, ..., 0) &= \frac{1}{2^{2(m-2)}}\alpha_3(2, 1, 0, ..., 0), && m \geq 3\\ &= \frac{3}{2^{2m-1}}, && \text{by (26)}\end{aligned}$$

Now, the only way to create new states with one queue with m customers at order $2m-1$ is by decrementing the state $n = (m, m-1, 0, ..., 0)$, since the order $2m-2$ does not have states with $n_1 = m$. Then, for

$n = (m, i, 0, ..., 0)$, for $0 < i < m - 1$, we have

$$\begin{aligned} 2\alpha_{2m-1}(m, i, 0, ..., 0) &= \alpha_{2m-1}(m, i+1, 0, ..., 0) \\ \Rightarrow\ \alpha_{2m-1}(m, i, 0, ..., 0) &= \frac{1}{2^{m-1-i}}\alpha_{2m-1}(m, m-1, 0, ..., 0) \\ &= \frac{3}{2^{2m-1}}\frac{1}{2^{m-1-i}}. \end{aligned}$$

For $i = 0$,

$$\begin{aligned} \alpha_{2m-1}(m, 0, 0, ..., 0) &= \alpha_{2m-1}(m, 1, 0, ..., 0) + \alpha_{2m-1}(m, 0, ..., 0, 1) \\ &= \frac{6}{2^{2m-1}}\frac{1}{2^{m-2}}. \end{aligned}$$

Then, from (29), we know that

$$\begin{aligned} \phi_{2m-1}(m) &= \sum_{i=1}^{m-1}\alpha_{2m-1}(m, i, 0, ...0) + \sum_{i=1}^{m-1}\alpha_{2m-1}(m, 0, ..., 0, i) + \alpha_{2m-1}(m, 0, ..., 0) \\ &= 2\sum_{i=1}^{m-1}\alpha_{2m-1}(m, i, 0, ...0) + \alpha_{2m-1}(m, 0, 0, ...0) \\ &= 2\frac{3}{2^{2m-1}} + 2\sum_{i=1}^{m-2}\frac{3}{2^{2m-1}}\frac{1}{2^{m-1-i}} + \frac{6}{2^{2m-1}}\frac{1}{2^{m-2}} \\ &= \frac{6}{2^{2m-1}}\left(\sum_{i=1}^{m-1}\frac{1}{2^{m-1-i}} + \frac{1}{2^{m-2}}\right) \\ &= \frac{12}{2^{2m-1}} \end{aligned}$$

Which give us the desired result, if used in (28). □

Proposition 4 guarantees that for ρ sufficiently small, the probability of having m clients in the queue follows a geometric decay of parameter $\rho^2/4$ as m grows. The following table illustrates where the choice policy of local choice situates.

Choice policy	π_m
No-choice	$\sim \rho^m$
Local choice	$\sim (\rho/2)^{2m-1}$
Random choice	$\sim \rho^{2^m-1}$

In this table the equivalence symbol means that, if $f \sim g$, then $\dfrac{f}{g} \longrightarrow C$, as $\rho \to 0$, where C is a constant. Random choice is the choice policy where the user chooses between two random queues the least loaded one for $N \to \infty$. The result can be found in [10] and [12]. As expected, the performance of *local choice* policy is between the other two policies. However, for low traffic, its behavior[2] is closer to no-choice, than random choice.

[2]behavior in the sense that both are exponential, whilst the third one is double exponential.

5 JSQ study in a 2-queue system

The interest of the study of a system of 2 queues with the choice policy is to model bike-sharing as groups of $N = 2$ stations and the customers choose between these 2 stations the one with more available slots. Our aim is to calculate tight bounds for the mean stationary number of customers $\bar{n}$ per queue. As a matter of fact, the analytic solution of this problem can be found in [5]. However the analytic solution is expressed as an infinite sum and as our next step is to limit the capacity, this result is not sufficient.

5.1 Queues with infinite capacity

Let $(V(t)) = (V_1(t), V_2(t))$, where $V_1(t)$ is the number of customers at the queue with the great number of customers, and $V_2(t)$ is the number of customers at the queue with the least number of customers, at time t. $(V(t))_{t\geq 0}$ is a Markov process on the state space $\mathcal{S}_\infty := \{(i,j) \in \mathbb{N}^2 \mid 0 \leq j \leq i\}$, irreducible, with jump matrix Q given by, for $v \in \mathcal{S}_\infty$,

$$\begin{aligned}
Q(v, v+e_1) &= 2\rho\mathbb{1}_{\{v_1=v_2\}} \\
Q(v, v+e_2) &= 2\rho\mathbb{1}_{\{v_2<v_1\}} \\
Q(v, v-e_1) &= \mathbb{1}_{\{v_1>0\}}(1-\mathbb{1}_{\{v_1=v_2\}}) \\
Q(v, v-e_2) &= \mathbb{1}_{\{v_2>0\}}(1+\mathbb{1}_{\{v_2=v_1\}}),
\end{aligned}$$

where e_k is the k-th unit vector of $\mathbb{N}^2$.

Let us define $(\pi(i,j))_{0\leq j\leq i}$ the invariant measure of the process. Then, the marginal probability π_m for a queue to have m customers is given by, for any $m \in \mathbb{N}$

$$\pi_m = \frac{1}{2}\left(\sum_{j=0}^{m}\pi(m,j) + \sum_{i\geq m}\pi(i,m)\right). \tag{35}$$

The equilibrium equations are given by, $\forall i,j \in \mathbb{N}$ and $i \geq j$,

$$\begin{aligned}
\left(\mathbb{1}_{\{i>0\}} + \mathbb{1}_{\{j>0\}} + 2\rho\right)\pi(i,j) = \pi(i+1,j) &+ \left(\mathbb{1}_{\{i=j+1\}} + 1\right)\pi(i,j+1) \\
&+ 2\rho\left(\mathbb{1}_{\{i=j+1\}}\pi(j,j) + \mathbb{1}_{\{j>0\}}\pi(i,j-1)\right).
\end{aligned} \tag{36}$$

Let the generating function be denoted by $G(x,y) = \sum_{i\geq j\geq 0}\pi(i,j)x^iy^j$, for $(x,y) \in \mathrm{ROC}$[3]. Then, using (36), the generating function is given by

$$\begin{aligned}
\left(2 - x^{-1} - y^{-1} + 2\rho(1-y)\right)G(x,y) = (1-y^{-1})&\left(\pi(0,0) + \sum_{i\geq 0}\pi(i,0)x^i\right) \\
&+ \left(y^{-1} - x^{-1} + 2\rho(x-y)\right)\sum_{i\geq 0}\pi(i,i)x^iy^i.
\end{aligned} \tag{37}$$

It means that the boundary conditions for our recurrence equation is $\pi(i,0)$ and $\pi(i,i)$, for all i greater than 0. However, we do not have them. We need to discover them, using the properties we know about the invariant measure $(\pi(i,j))_{0\leq j\leq i}$. For example, $\pi(i,j) \geq 0$, $\sum_{i,j}\pi(i,j) = 1$, $\lim_{i\to\infty}\pi(i,j) = 0$.

[3]ROC stands for region of convergence. As our objective is to extend the results to the case with finite capacity, where the ROC is the whole complex plane, we will not prove and analyze the ROC for this case. However, this matter is treated carefully in [5].

We can rewrite equation (37) as

$$G(x,y) = \frac{x(y-1)A(x) + (1+2\rho xy)(x-y)B(xy)}{2xy - y - x + 2\rho xy(1-y)} \tag{38}$$

where $A(x) = \pi(0,0) + \sum_{i\geq 0} \pi(i,0)x^i$ and $B(z) = \sum_{i\geq 0} \pi(i,i)z^i$ are the boundary conditions.

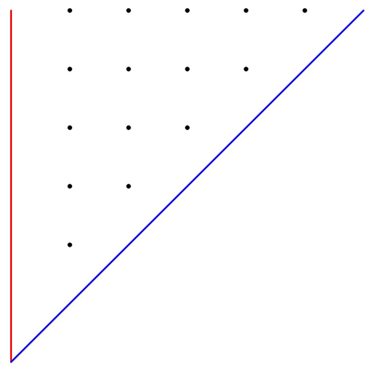

Figure 8: The red line is the boundary condition in A and the blue line is the boundary condition in B.

Let us start by using the fact that

$$\sum_{i\geq j\geq 0} \pi(i,j) = 1$$

since π is a probability measure. Then, let us state that the following limit over the generating function exists and is equal to 1.

$$\lim_{(x,y)\to(1,1)} G(x,y) = 1$$

There is a hidden assumption here that $|\pi(i,j)|$ decreases faster than an exponential of exponent $-(1+\varepsilon)$, for some $\varepsilon > 0$ as $|i+j| \to \infty$. Which is true, since this holds for the known case of no choice between the queues. The invariant measure for the greatest queue must be smaller than two times the invariant measure of the queue with no choice.

As the limit exists, we can approach the point $(1,1)$ in any way and it must give the same result. Then, if we make $y \to 1$, we find

$$G(x,1) = \frac{(1+2\rho x)(x-1)B(x)}{x-1} = (1+2\rho x)B(x)$$

Then, we make $x \to 1$, and as $G(1,1) = 1$,

$$B(1) = \frac{1}{1+2\rho}. \tag{39}$$

This means that the sum of all diagonal elements (the blue line) is equal to $1/(1+2\rho)$. If we approach the point $(1,1)$ by the diagonal, we may find a relation for $A(1)$ as well. If $y = x$, then,

$$G(x,x) = \frac{x(x-1)A(x)}{2x(x-1)(1-\rho x)} = \frac{A(x)}{2(1-\rho x)}$$

And as $x \to 1$, we find

$$A(1) = 2(1-\rho) \tag{40}$$

which is equivalent to the already known relation $\pi_0 = 1-\rho$. Now, if we approach the point $(1,1)$ in any other linear direction with angular coefficient $\alpha \neq \frac{1}{2\rho-1}$ it will give 1. Indeed, for $\alpha \neq \frac{1}{2\rho-1}$,

$$\lim_{x\to 1}\lim_{y\to\alpha(x-1)+1} G(x,y) = \frac{(\alpha-1)(1+2\rho)B(1)-\alpha A(1)}{(2\rho-1)\alpha-1} = \frac{\alpha-1-2\alpha(1-\rho)}{(2\rho-1)\alpha-1} = 1.$$

The reason why for $\alpha = 1/(2\rho-1)$ give us a problem is because this is the slope of the curve where the denominator is 0. For $x = -\frac{y}{2\rho y^2-2(\rho+1)y+1}$ the denominator is 0 and $-\frac{y}{2\rho y^2-2(\rho+1)y+1} = 1 + (2\rho-1)(y-1)+\mathcal{O}(y)$, for y tending to 1, giving us the desired angular coefficient. As we know the generating function does not go to infinity in the convergence region, when the denominator is 0, the numerator must be 0 too. This allow us to use the L'Hôpital's rule when x tends to $x^*(y) = -\frac{y}{2\rho y^2-2(\rho+1)y+1}$.

$$\begin{aligned}
\lim_{x\to x^*(y)} G(x,y) &= \lim_{x\to x^*(y)} \frac{x(y-1)A(x)+(1+2\rho xy)(x-y)B(xy)}{2xy-y-x+2\rho xy(1-y)}\\
&\overset{\mathrm{H}}{=} \lim_{x\to x^*(y)} \frac{(y-1)(A(x)+xA'(x))+(1+2\rho xy)(B(xy)+y(x-y)B'(xy))+2\rho y(x-y)B(xy)}{2y-1+2\rho y(1-y)}\\
&= \frac{(y-1)(A(x^*)+x^*A'(x^*))+(1+2\rho x^*y)(B(x^*y)+y(x^*-y)B'(x^*y))+2\rho y(x^*-y)B(xy)}{2y-1+2\rho y(1-y)}\\
&\underset{y\to 1}{\longrightarrow} (1+2\rho)B(1) = 1.
\end{aligned}$$

As expected, we find 1. We can go beyond that, since we know that the numerator must be 0 for $(x^*(y),y)$ in the region of convergence.

$$x^*(y)(y-1)A(x^*(y)) + (1+2\rho y x^*(y))(x^*(y)-y)B(x^*(y)y) = 0,$$

which can be rewritten as

$$A(x^*(y)) - 2(1+2\rho y x^*(y))(1-\rho y)B(yx^*(y)) = 0. \tag{41}$$

Let us use some notable pairs $(x^*(y),y)$ in equation (41).
For $y = \frac{1}{1+\rho}$, we have $x^*(y) = \frac{1+\rho}{1+\rho^2}$, and then,

$$A\left(\frac{1+\rho}{1+\rho^2}\right) = 2\frac{1+\rho}{1+\rho^2}B\left(\frac{1}{1+\rho^2}\right). \tag{42}$$

For $y = \frac{1}{2\rho}$, we have $x^*(y) = 1$, and then, using (40),

$$\begin{aligned}
B\left(\frac{1}{2\rho}\right) &= \frac{1}{2}A(1)\\
&= 1-\rho.
\end{aligned} \tag{43}$$

For $y = \frac{1}{2}$, we have $x^*(y) = \frac{1}{\rho}$, and then, using also (43),

$$\begin{aligned}
A\left(\frac{1}{\rho}\right) &= 2(2-\rho)B\left(\frac{1}{2\rho}\right)\\
&= 2(2-\rho)(1-\rho).
\end{aligned} \tag{44}$$

For $y = \dfrac{1}{1+2\rho}$, we have $x^*(y) = 1 + 2\rho$, and then, using (39), we can find that

$$\begin{aligned} A(1+2\rho) &= 2(1+\rho)B(1) \\ &= 2\frac{1+\rho}{1+2\rho} \end{aligned} \tag{45}$$

It is possible to find another interesting results if we continue to find pairs $(x^*(y), y)$ in the region of convergence, but for now, these results are enough to prove what we want. The next step is to do the same for the derivative of (41) in respect to y. Taking the derivative we find

$$\begin{aligned} y(1-2\rho y^2)A'(x^*(y)) + 4yx^*(y)(\rho y + y - 1)(1-\rho y)(2\rho y + 2y - 1)B'(yx^*(y)) \\ -2\rho y(4\rho^2 y^2 + 2\rho y^2 - 4\rho y + 1)B(yx^*(y)) = 0 \end{aligned} \tag{46}$$

For $y = \dfrac{1}{1+\rho}$, we have $x^*(y) = \dfrac{1+\rho}{1+\rho^2}$ as before, and then it gives that

$$A'\left(\frac{1+\rho}{1+\rho^2}\right) = 2\rho B\left(\frac{1}{1+\rho^2}\right) \tag{47}$$

Let us calculate the derivative of the generating function. In order to do this, let $y = x \neq 0$,

$$G(x,x) = \frac{A(x)}{2(1-\rho x)},$$

then, taking the derivative with respect to x

$$G_x(x,x) + G_y(x,x) = \frac{A'(x)}{2(1-\rho x)} + \frac{\rho A(x)}{2(1-\rho x)^2},$$

and then, as x tends to 1,

$$G_x(1,1) + G_y(1,1) = \frac{A'(1)}{2(1-\rho)} + \frac{\rho}{1-\rho}. \tag{48}$$

Finally, let us calculate the average number of customers in one queue, which is given by

$$\begin{aligned} \bar{n} &= \sum_{m\geq 0} m\pi_m \\ &= \sum_{m\geq 0} \frac{m}{2}\left(\sum_{j=0}^{m} \pi(m,j) + \sum_{i\geq m} \pi(i,m)\right) \\ &= \frac{1}{2}\sum_{k\geq 0}\sum_{m\geq 0} (m\pi(m,k)\mathbb{1}_{\{k\leq m\}} + m\pi(k,m)\mathbb{1}_{\{k\geq m\}}) \\ &= \frac{1}{2}\left(\sum_{m\geq k\geq 0} m\pi(m,k) + \sum_{k\geq m\geq 0} m\pi(k,m)\right) \\ &= \frac{1}{2}\left(G_x(1,1) + G_y(1,1)\right) \\ &= \frac{A'(1)}{4(1-\rho)} + \frac{\rho}{2(1-\rho)}, \end{aligned} \tag{49}$$

using (48).

Using equations (47), (42), (44) and the fact that $1 \leq \frac{1+\rho}{1+\rho^2} \leq \frac{1}{\rho}$, for $\rho < 1$, we can state that

$$\begin{aligned}
A'(1) &\leq A'\left(\frac{1+\rho}{1+\rho^2}\right) \\
&= 2\rho B\left(\frac{1}{1+\rho^2}\right) \\
&= \rho\frac{1+\rho^2}{1+\rho}A\left(\frac{1+\rho}{1+\rho^2}\right) \\
&\leq \rho\frac{1+\rho^2}{1+\rho}A\left(\frac{1}{\rho}\right) \\
&= \frac{2\rho(1+\rho^2)(2-\rho)(1-\rho)}{1+\rho}.
\end{aligned}$$

From the equilibrium equation (36), we know that $\pi(0,0) = \frac{1}{2\rho}\pi(1,0)$. Then, by definition of A, $\left(1+\frac{1}{\rho}\right)A'(1) \geq A(1)$, and from (40), we have that $A'(1) \geq 2\rho\frac{1-\rho}{1+\rho}$.
Using the two last results and equation (49), we find the following inequalities.

$$\frac{\rho}{1-\rho^2} \leq \bar{n} \leq \frac{\rho}{2(1-\rho)} + \frac{\rho(1+\rho^2)(2-\rho)}{2(1+\rho)}.$$

We can improve the upper bound for $\rho < \frac{1}{2}$, by using that $\frac{1+\rho}{1+\rho^2} \leq 1+2\rho$ and equation (45). It gives that $A\left(\frac{1+\rho}{1+\rho^2}\right) \leq A(1+2\rho) = 2\frac{1+\rho}{1+2\rho}$. This leads to $A'(1) \leq \frac{2\rho(1+\rho^2)}{1+2\rho}$, which, plugged in equation (49), gives the following proposition.

Proposition 5. *Let $\bar{n}$ be the expected stationary number of customers in a queue of a JSQ 2-queue system. Then,*

$$\frac{\rho}{1-\rho^2} \leq \bar{n} \leq \frac{\rho}{2(1-\rho)} + \mathbb{1}_{\{\rho\leq 1/2\}}\frac{\rho(1+\rho^2)}{2(1-\rho)(1+2\rho)} + \mathbb{1}_{\{\rho>1/2\}}\frac{\rho(1+\rho^2)(2-\rho)}{2(1+\rho)}$$

We can show that the difference between the two bounds is always less than 7/48.

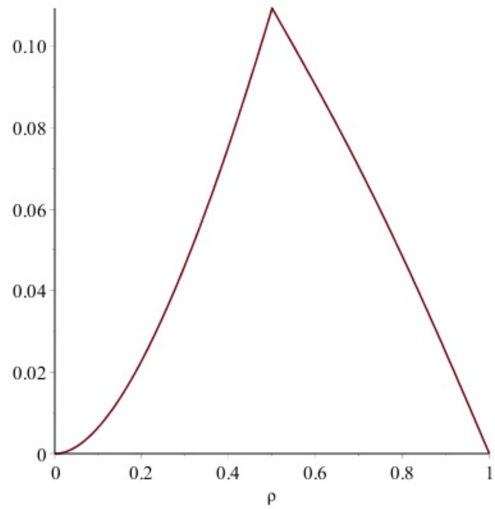

Figure 9: Maximum relative error of $\bar{n}$, if we take the midpoint of the interval in Proposition 5.

$$\sum_{j=0}^{m} \pi(m,j) = \pi(m,m) + \mathbb{1}_{\{m>0\}} 2\rho\pi(m-1,m-1), \qquad \text{proved by using } G(x,1)$$

$$\sum_{i \geq m} \pi(i,m) = 2(1-\rho)(2\rho)^m - \pi(m,m) - \mathbb{1}_{\{m>0\}} 2 \sum_{t=0}^{m-1} \pi(t,t)(2\rho)^{m-t}, \qquad \text{proved by using } G(1,y)$$

$$\pi_m = (1-\rho)(2\rho)^m + \mathbb{1}_{\{m>0\}} \left(\rho\pi(m-1,m-1) - \sum_{t=0}^{m-1} \pi(t,t)(2\rho)^{m-t} \right), \qquad \text{summing the above relations}$$

$$\sum_{k=0}^{\lfloor t/2 \rfloor} \pi(t-k,k) = \frac{\rho^t}{2} \sum_{r=0}^{t} \pi(r,0)\rho^{-r}, \qquad \text{proved by using } G(x,x)$$

$$B\left(\frac{1}{2\rho}\right) = 1-\rho, \qquad \text{proved by using } x^*(1/(2\rho))$$

$$A\left(\frac{1}{\rho}\right) = (1+2\rho)(2-\rho)(1-\rho), \qquad \text{proved by using } x^*(1/2)$$

$$A\left(\frac{1+\rho}{1+\rho^2}\right) = 2\frac{1+2\rho}{1+\rho} B\left(\frac{1}{1+\rho^2}\right), \qquad \text{proved by using } x^*(1/(1+\rho))$$

$$A(1+2\rho) = 2\frac{1+\rho}{1+2\rho} \qquad \text{proved by using } x^*(1/(1+2\rho))$$

$$A\left(\frac{1+\rho}{1+\rho^2}\right) = \frac{2(\rho^2+\rho+1)(1-\rho)}{1+\rho^2} B\left(\frac{(1+\rho)^2}{2\rho(1+\rho^2)}\right)$$

5.2 Queues with finite capacity

Analogous to the previous subsection, let $(V(t)) = (V_1(t), V_2(t))$, where $V_1(t)$ is the number of customers at the queue with the great number of customers, and $V_2(t)$ is the number of customers at the queue with the least number of customers, at time t. $(V(t))_{t\geq 0}$ is a Markov process on $\mathcal{S}_K := \{(i,j) \in \mathbb{N}^2 \mid 0 \leq j \leq i \leq K\}$, irreducible, with jump matrix Q given by, for all $v \in \mathcal{S}_K$,

$$\begin{aligned}
Q(v, v+e_1) &= 2\rho \mathbb{1}_{\{v_1=v_2\}} \mathbb{1}_{\{v_1<K\}} \\
Q(v, v+e_2) &= 2\rho \mathbb{1}_{\{v_2<v_1\}} \\
Q(v, v-e_1) &= \mathbb{1}_{\{v_1>0\}}(1 - \mathbb{1}_{\{v_1=v_2\}}) \\
Q(v, v-e_2) &= \mathbb{1}_{\{v_2>0\}}(1 + \mathbb{1}_{\{v_2=v_1\}}),
\end{aligned}$$

where e_k is the k-th unit vector of $\mathbb{N}^2$ and $K \in \mathbb{N}^*$. Note the only change between this process and the previous one is the first transition, where we have the capacity K of the queue in the characteristic function.

The equilibrium equations are given by, for $(i,j) \in \mathcal{S}_K$,

$$\begin{aligned}
\left(\mathbb{1}_{\{i>0\}} + \mathbb{1}_{\{j>0\}} + 2\rho \mathbb{1}_{\{(i,j)\neq(K,K)\}}\right) \pi(i,j) = \pi(i+1,j)\mathbb{1}_{\{i<K\}} + \left(\mathbb{1}_{\{i=j+1\}} + 1\right) \pi(i,j+1)\mathbb{1}_{\{i>j\}} \\
+ 2\rho\left(\mathbb{1}_{\{i=j+1\}}\pi(j,j) + \mathbb{1}_{\{j>0\}}\pi(i,j-1)\right).
\end{aligned} \tag{50}$$

As before, let us denote the generating funtions as $G(x,y) = \sum_{K\geq i\geq j\geq 0} \pi(i,j)x^i y^j$, for $(x,y) \in \mathbb{C}^2$. In this case, where $K < \infty$, we know that the region of convergence of the generating function is the whole complex plane, since we have a finite number of terms. Then, using (50), we can find that, for $(x,y) \in \mathbb{C}^2$,

$$\begin{aligned}
\left(2 - x^{-1} - y^{-1} + 2\rho(1-y)\right) G(x,y) = 2\rho x^K y^K (1-x)\pi(K,K) + (1-y^{-1})\left(\pi(0,0) + \sum_{i=0}^{K} \pi(i,0)x^i\right) \\
+ \left(y^{-1} - x^{-1} + 2\rho(x-y)\right) \sum_{i=0}^{K} \pi(i,i)x^i y^i,
\end{aligned}$$

which can be rewritten as

$$G(x,y) = \frac{x(y-1)A(x) + (1+2\rho xy)(x-y)B(xy) + 2\rho(xy)^{K+1}(1-x)\pi(K,K)}{2xy - y - x + 2\rho xy(1-y)} \tag{51}$$

where $A(x) = \pi(0,0) + \sum_{i=0}^{K} \pi(i,0)x^i$, $B(z) = \sum_{i=0}^{K} \pi(i,i)z^i$ and $\pi(K,K)$ are the boundary conditions.

As in the previous subsection, let us find $A(1)$ and $B(1)$, using the fact that $G(1,1) = 1$. On one hand, for $x \neq 1$,

$$\begin{aligned}
\lim_{y\to 1} G(x,y) &= \frac{(x-1)(1+2\rho x)B(x) + 2\rho x^{K+1}(1-x)\pi(K,K)}{x-1} \\
&= (1+2\rho x)B(x) - 2\rho x^{K+1}\pi(K,K)
\end{aligned}$$

thus,

$$\lim_{x\to 1} G(x,1) = (1+2\rho)B(1) - 2\rho\pi(K,K)$$

and then,

$$B(1) = \frac{1 + 2\rho\pi(K,K)}{1+2\rho}. \tag{52}$$

On the other hand, still using (51), for $y = x$ and $x \neq 0$, $x \neq 1$,

$$\begin{aligned}
G(x,x) &= \frac{x(x-1)A(x) + 2\rho x^{2K+2}(1-x)\pi(K,K)}{2x(1-x)(\rho x - 1)} \\
&= \frac{A(x) - 2\rho x^{2K+1}\pi(K,K)}{2(1-\rho x)},
\end{aligned} \tag{53}$$

and therefore,

$$\lim_{x\to 1} G(x,x) = \frac{A(1) - 2\rho\pi(K,K)}{2(1-\rho)},$$

then,

$$A(1) = 2(1 - \rho(1 - \pi(K,K))). \tag{54}$$

Now, let us use the fact that, as $G(x,y)$ does not diverge for any $(x,y) \in \mathbb{C}^2$, we can state that when the denominator vanishes in (51), the numerator must vanish too. Then, rearranging the numerator, we find that for $x = x^*(y) = -\dfrac{y}{2\rho y^2 - 2(\rho+1)y + 1}$ that is root of the denominator, using (51),

$$x^*(y)(y-1)A(x^*(y)) + (1 + 2\rho y x^*(y))(x^*(y) - y)B(yx^*(y)) + 2\rho(yx^*(y))^{K+1}(1 - x^*(y))\pi(K,K) = 0, \tag{55}$$

which can be rewritten as,

$$A(x^*(y)) - 2(1-\rho y)(1 + 2\rho y x^*(y))B(yx^*(y)) + 2\rho\frac{1-2\rho y}{y}(yx^*(y))^{K+1}\pi(K,K) = 0, \qquad y \in \mathbb{C}^*. \tag{56}$$

This expression is used to find several useful relations.

Lemma 3. *The invariant measure of both queues having the maximum number of clients is given by*

$$\pi(K,K) = \begin{cases} \dfrac{(1-\rho)(2-\rho)}{(1-\rho)(2\rho)^{-K} + \rho^{-2K} - \rho(2-\rho)}, & \text{for } \rho \in \mathbb{R}_+ \setminus \{0,1,2\} \\ \dfrac{1}{2K + 2^{-K}}, & \text{for } \rho = 1 \\ \dfrac{1}{2 - (K+2)2^{-2K-1}}, & \text{for } \rho = 2. \end{cases}$$

Proof. Let us evaluate equation (56) at some points. For $y = \dfrac{1}{2\rho}$, $x^*(y) = 1$ and using (54),

$$B\left(\frac{1}{2\rho}\right) = 1 - \rho(1 - \pi(K,K)).$$

For $y = \dfrac{1}{2}$, $x^*(y) = \dfrac{1}{\rho}$ and using the previous relation,

$$A\left(\frac{1}{\rho}\right) = 2(2-\rho)(1 - \rho(1 - \pi(K,K))) - 4\rho(1-\rho)(2\rho)^{-K-1}\pi(K,K). \tag{57}$$

For $y = \dfrac{1}{\rho}$, $x^*(y) = \dfrac{1}{\rho}$ and using (55),

$$A\left(\frac{1}{\rho}\right) = 2\rho^{-2K}\pi(K,K). \tag{58}$$

Then, we use equations (57) and (58) to find that

$$\pi(K,K) = \frac{(1-\rho)(2-\rho)}{(1-\rho)(2\rho)^{-K} + \rho^{-2K} - \rho(2-\rho)}. \tag{59}$$

Notice that this function is not defined in $\rho = 1$ or $\rho = 2$, but as we know this quantity varies continuously with ρ, thus, extending it by continuity ends the proof. $\square$

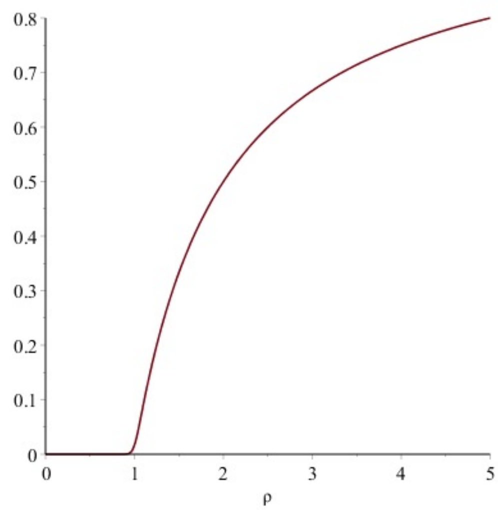

Figure 10: Plot of the function $\pi(K, K)$ for $K = 30$. Note that the function starts to increase rapidly in a region near to $\rho = 1$ and the curve, for $\rho > 1$, is close to the asymptotic curve $\rho \longmapsto 1 - \dfrac{1}{\rho}$, because $\pi(K, K) = 1 - \dfrac{1}{\rho} + \mathcal{O}(\rho^{-K-1})$ as ρ tends to infinity.

Let us calculate the derivative of the generating function. Taking the derivative with respect to x in equation (53) and making x tends to 1, we find, using equation (54),

$$G_x(1,1) + G_y(1,1) = \frac{A'(1)}{2(1-\rho)} + \frac{\rho}{1-\rho}(1 - (2K+1)\pi(K,K)).$$

As in the previous subsection, we can show that the average stationary number of customers in a queue is given by

$$\bar{n} = \frac{1}{2}(G_x(1,1) + G_y(1,1)) = \frac{A'(1)}{4(1-\rho)} + \frac{\rho}{2(1-\rho)}(1 - (2K+1)\pi(K,K)), \tag{60}$$

if we use the previous relation.

Let us use some notable pairs $(x^*(y), y)$ in equation (56). For $y = \dfrac{1}{1+\rho}$, we have $x^*(y) = \dfrac{1+\rho}{1+\rho^2}$ and

$$B\left(\frac{1}{1+\rho^2}\right) = \frac{1+\rho^2}{2(1+\rho)} A\left(\frac{1+\rho}{1+\rho^2}\right) + \frac{\rho(1-\rho)}{1+\rho}\left(\frac{1}{1+\rho^2}\right)^K \pi(K,K). \tag{61}$$

For $y = \dfrac{1}{1+2\rho}$, $x^*(y) = 1 + 2\rho$ and

$$A(1+2\rho) = \frac{2(1+\rho+\rho\pi(K,K))}{1+2\rho}. \tag{62}$$

If we take the derivative of (56) in respect to y and make $y = \dfrac{1}{1+\rho}$, we have $x^*(y) = \dfrac{1+\rho}{1+\rho^2}$ and the following relation

$$A'\left(\frac{1+\rho}{1+\rho^2}\right) = 2\rho B\left(\frac{1}{1+\rho^2}\right) - 2\rho\left(\frac{1}{1+\rho^2}\right)^K \pi(K,K). \tag{63}$$

Now let us construct an upper bound for $\bar{n}$, for $\rho \le 1$. As $1 \le \dfrac{1+\rho}{1+\rho^2}$ for $\rho \le 1$, then

$$A'(1) \le A'\left(\frac{1+\rho}{1+\rho^2}\right).$$

Then, we use equations (61) and (63), to find that

$$A'(1) \le \frac{\rho(1+\rho^2)}{1+\rho}\left(A\left(\frac{1+\rho}{1+\rho^2}\right) - 2\left(\frac{1}{1+\rho^2}\right)^K \pi(K,K)\right). \tag{64}$$

The next step is to use equation (58) and the fact that $\dfrac{1+\rho}{1+\rho^2} \le \dfrac{1}{\rho}$ for $\rho \le 1$, to show that, for $\rho \le 1$,

$$A'(1) \le \frac{2\rho(1+\rho^2)(\rho^{-2K} - (1+\rho^2)^{-K})}{1+\rho}\pi(K,K). \tag{65}$$

Note that we can do the same reasoning to find a lower bound, if $\rho \ge 1$,

$$A'(1) \ge \frac{2\rho(1+\rho^2)(\rho^{-2K} - (1+\rho^2)^{-K})}{1+\rho}\pi(K,K). \tag{66}$$

It yields that for $\rho = 1$, we have the exact value of $A'(1)$. Using Lemma 3, we find that

$$A'(1) = \frac{1-2^{-K}}{K+2^{-K-1}}. \tag{67}$$

Now, let us find a lower bound for $A'(1)$, for all $\rho < 1$. Using the equilibrium equation (50), we know that $\pi(0,0) = \dfrac{1}{2\rho}\pi(1,0)$. Then, by the definition of A, $\left(1+\dfrac{1}{\rho}\right)A'(1) \ge A(1)$, and from (54), we have that, for $\rho < 1$,

$$A'(1) \ge \frac{2\rho}{1+\rho}(1-\rho(1-\pi(K,K))). \tag{68}$$

However, notice that in equation (60), the term on the right-hand side is negative for $\rho > 1$. Then, an upper bound for $A'(1)$ is a lower bound for $\bar{n}$, for $\rho > 1$.

In order to obtain a lower bound for $\bar{n}$ and $\rho > 1$, let us use equations (62), (64) and the fact that $1+2\rho \ge \dfrac{1+\rho}{1+\rho^2}$, for all $\rho \in \mathbb{R}_+$ to find that

$$A'(1) \le \frac{2\rho(1+\rho^2)}{1+2\rho} + \frac{2\rho(1+\rho^2)}{1+\rho}\left(\frac{\rho}{1+2\rho} - (1+\rho^2)^{-K}\right)\pi(K,K) \tag{69}$$

Plugging the previous inequalities in equation (60), we obtain the following proposition.

Proposition 6. *The average number of customers in a queue $\bar{n}$ can be bounded as in the following inequalities.*

$$\bar{n} \le \frac{\rho}{2(1-\rho)}\left(1 - \left((2K+1) - \frac{(1+\rho^2)(\rho^{-2K} - (1+\rho^2)^{-K})}{1+\rho}\right)\pi(K,K)\right), \qquad \textit{for } \rho \in \mathbb{R}_+.$$

$$\bar{n} \ge \frac{\rho}{1-\rho^2}\left(1 - \frac{1}{2}(2K(1+\rho)+1)\pi(K,K)\right), \qquad \textit{for } \rho < 1.$$

$$\bar{n} \ge \frac{\rho}{2(\rho-1)}\left(2K+1 - \frac{\rho(1+\rho^2)}{(1+\rho)(1+2\rho)}\right)\pi(K,K) - \frac{\rho(\rho^2+2\rho+2)}{2(\rho-1)(1+2\rho)}, \qquad \textit{for } \rho > 1,$$

where $\pi(K,K)$ is given by Lemma 3.

Proof. To obtain the first inequality, we use (60), (65) and (66). To obtain the second relation we use (60) and (68). Finally, to obtain the last inequality we use (60) and (69). ☐

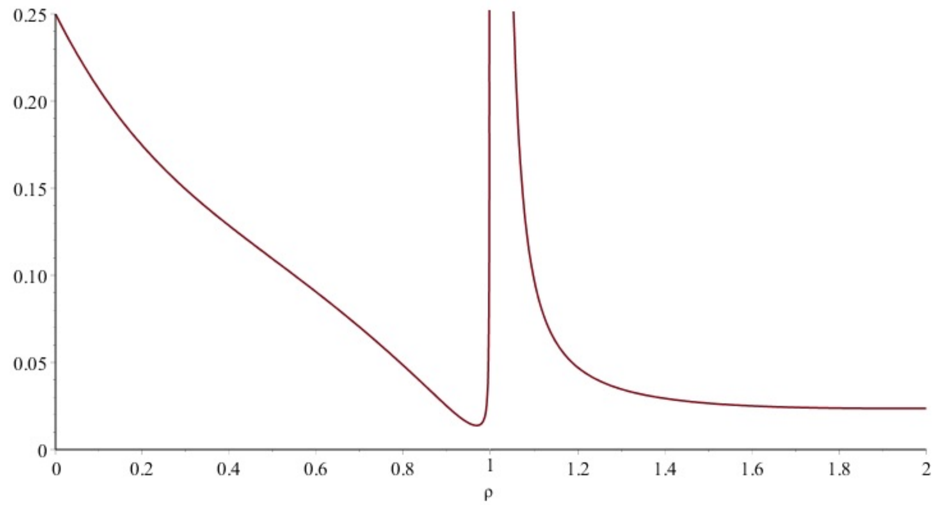

Figure 11: Maximum relative error of $\bar{n}$ for $K = 30$, if we take the midpoint of the interval in Proposition 6. Note that the error goes to infinity as ρ tends to 1. That happens because the lower bound tends to $-\infty$.

As the tighter bound is the upper bound, let us compare it with the simulation.

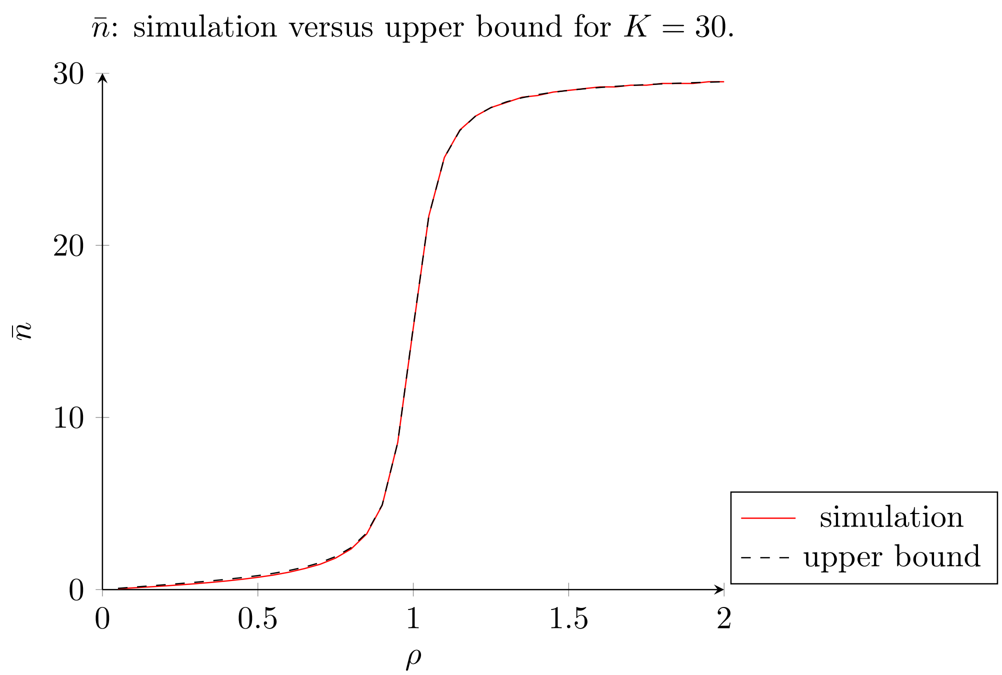

Figure 12: We can only see a difference for $\rho < 0.5$. In fact, it is not only a good upper bound, but it is also a good approximation, since $\sqrt{\int_0^2 (\bar{n}_{upp} - \bar{n}_{sim})^2 d\rho} \approx 0.077$.

In order to better know the behavior of an individual queue, let us calculate the marginals π_0 and π_K, which represent the probability that a given queue is empty or full, respectively.

Proposition 7. *The probability that a given queue is empty π_0 and the probability that a given queue is full π_K are given by*

$$\begin{aligned}
\pi_0 &= 1 - \rho(1 - \pi(K,K)),\\
\pi_K &= \left(1 + \frac{1}{2\rho}\frac{\xi_+^K - \xi_-^K}{(\xi_+ - 1)\xi_+^{K-1} + (1-\xi_-)\xi_-^{K-1}}\right)\pi(K,K),
\end{aligned}$$

where $\xi_\pm = 1 + \rho \pm \sqrt{1+\rho^2}$ and $\pi(K,K)$ is given by Lemma 3.

Proof. The first relation follows immediately from the definition of A and equation (54).
The second relation, we must start by solving the following recurrence, which can be obtained with (50).

$$\begin{aligned}
\pi(K,t+1) - 2(1+\rho)\pi(K,t) + 2\rho\pi(K,t-1) &= 0, \qquad 0 < t < K-1.\\
\pi(K,K) - \rho\pi(K,K-1) &= 0.\\
\pi(K,1) - (1+2\rho)\pi(K,0) &= 0.
\end{aligned}$$

The characteristic equation is given by $\xi^2 - 2(1+\rho)\xi + 2\rho = 0$ and the two solutions are $\xi_\pm = 1 + \rho \pm \sqrt{1+\rho^2}$. Thus, $\pi(K,t)$ is given by

$$\pi(K,t) = C_+\xi_+^t + C_-\xi_-^t, \qquad 0 \le t \le K-1,$$

where C_+ and C_- are constants to be determined with the last equations. Solving the following system of equations

$$\begin{cases} C_+\xi_+^{K-1} + C_-\xi_-^{K-1} = \dfrac{\pi(K,K)}{\rho} \\ C_+(1+2\rho-\xi_+) + C_-(1+2\rho-\xi_-) = 0, \end{cases},$$

with straightforward algebra, we find that $C_\pm = \dfrac{\pm(\xi_\pm - 1)}{(\xi_+ - 1)\xi_+^{K-1} + (1-\xi_-)\xi_-^{K-1}}\dfrac{\pi(K,K)}{\rho}$.

As $\pi_K = \pi(K,K) + \frac{1}{2}\sum_{t=0}^{K-1}\pi(K,t)$, then we can find the desired result. $\square$

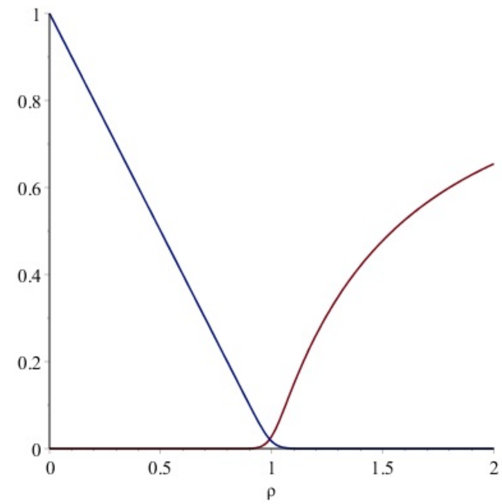

Figure 13: Graphic of π_0 in blue and graphic of π_K in red, for $K = 30$. As expected, π_0 converge to 0 and π_K converge to 1 as ρ tends to infinity.

6 Bike-sharing system with local choice in groups of two stations

The approach of dealing with isolated groups of two stations is somehow coherent with the bike-sharing network of Paris, since the average number of surrounding stations that are less than the reasonable distance of 200 meters is 0.79 (this information is obtained with the data extracted from [1]). Even though, the best approach is still the local choice one (see Subsection 4.3 for details), since the final destination of the users is not a station, their final destination is located between stations, which is not take into account in this model. But, this model is tractable by mean-field techniques, and the typical object is a 2-queue-system with JSQ (join the shortest queue) policy, which has been studied in Section 5.

6.1 Mean-field equations

Consider a homogeneous bike-sharing system with N stations with capacity K and M bikes, arrival rate of users at each station λ and trip times between stations with mean $1/\mu$, all the inter arrival times and trip times are exponentially distributed. Let us denote $s = M/N$ and assume that $\mu = 1$.

Let us define

$$Y^N_{i,j}(t) = \frac{1}{N_p} \sum_{k=1}^{N_p} \mathbb{1}_{\{H_k(t)=i, L_k(t)=j\}}$$

where $H_k(t)$ is the number of bikes in the most loaded station of group k at time t and $L_k(t)$ is the number of bikes at the least loaded station of group k at time t and N_p is the number of groups of two stations. The process $Y^N(t) = (Y^N_{i,j}(t))_{0\le j\le i\le K}$ is Markov on the state space

$$\mathcal{Y}^N = \left\{ y = (y_{i,j})_{0\le j\le i\le K}, y_{i,j} \in \frac{\mathbb{N}}{N_p}, \sum_{0\le j\le i\le K} y_{i,j} = 1, \sum_{0\le j\le i\le K} (i+j)y_{i,j} \le s \right\},$$

irreducible, with jump matrix Q given by, for $y \in \mathcal{Y}^N$,

$$\begin{aligned}
Q\left(y, y + \frac{1}{N_p}(e_{i+1,j} - e_{i,j})\right) &= 2\rho N_p y_{i,j} \mathbb{1}_{\{j=i<K\}} \\
Q\left(y, y + \frac{1}{N_p}(e_{i,j+1} - e_{i,j})\right) &= 2\rho N_p y_{i,j} \mathbb{1}_{\{j<i\}} \\
Q\left(y, y + \frac{1}{N_p}(e_{i-1,j} - e_{i,j})\right) &= N_p y_{i,j} \mathbb{1}_{\{i>0\}}(1 - \mathbb{1}_{\{i=j\}}) \\
Q\left(y, y + \frac{1}{N_p}(e_{i,j-1} - e_{i,j})\right) &= N_p y_{i,j} \mathbb{1}_{\{i>0\}}(1 + \mathbb{1}_{\{i=j\}}),
\end{aligned} \tag{70}$$

where $\rho := \dfrac{\mu}{\lambda}(s - \bar{n})$ and $\bar{n} = \dfrac{1}{2}\sum_{i,j}(i+j)y_{i,j}$.

The dynamical system $(y(t))_{t\ge0}$ limit of the Markov process $(Y^N(t))_{t\ge0}$ as N tends to infinity is the unique solution of the following ODE

$$\begin{aligned}
\dot{y}_{i,j}(t) = {} & y_{i+1,j}(t) + (1 + \mathbb{1}_{\{i=j+1\}})y_{i,j+1} + 2\rho(y_{i,j-1}(t) + \mathbb{1}_{\{i=j+1\}} y_{i-1,j}(t)) \\
& - (2\rho\mathbb{1}_{\{(i,j)\ne(K,K)\}} + \mathbb{1}_{\{i>0\}} + \mathbb{1}_{\{j>0\}})y_{i,j}(t)
\end{aligned} \tag{71}$$

At equilibrium $\dot{y}_{i,j} = 0$, for all $0 \le j \le i \le K$, which give us the same equations as in (50). It means that we can use all the results from the previous section and our $y_{i,j}$ is equal to the $\pi(i,j)$ from the Subsection 5.2. Let us denote π_m, which has the same value as in the previous section, the probability of a station having m bikes.

6.2 Performance metric

Our objective is to optimize the bike-sharing system by choosing the best ratio s of bikes per station. However, we need to decide which metric we want to optimize. In [6], the metric used is the proportion of problematic stations, which is given by $\pi_0 + \pi_K$. However, this does not take into account that, when a client is rejected to return the bike at a given station, he may be rejected again at another station. So there is a set of customers in the pool who have being trying to return the bike, without success. Indeed, this only has effect when we have a great number of bikes with respect to the quantity of customers, but this happens, for example, at night. In order to take this into account, the proposed parameter is to count how many times the customer got unsatisfied multiplied by the probability of the event happening. It is easy to show that our metric to minimize $L \in \mathbb{R}_+$, is given by

$$\begin{aligned} L &= p_0 + (1-p_0)(1-p_K)\sum_{m\geq 1} m p_K^m \\ &= p_0 + \frac{p_K(1-p_0)}{1-p_K}, \end{aligned}$$

where p_0 is the probability for a user to find an empty station and p_K is the probability for the user to not be able to join a queue. For the bike-sharing systems we are studying, the following equality holds $1 - p_0 = \rho(1-p_K)$, where ρ is the ratio between the rate of arriving bikes at a station and the rate of arriving customers at a station. This allow us to deduce that

$$\begin{aligned} L &= p_0 + \rho p_K \\ &= 1 - \rho(1-2p_K). \end{aligned} \tag{72}$$

Note that, $p_0 + \rho p_K$ is a quantity proportional to the rate of unsatisfied customers. So, minimizing L is equivalent to minimize the rate of unsatisfied customers.

6.3 Performance of the 2-choice system

For the JSQ 2-queue system, we can show that $p_0 = \pi_0$ and $p_K = \pi(K,K)$. Then, we find

$$L = 1 - \rho + 2\rho\pi(K,K) \tag{73}$$

As $\rho = \frac{\mu}{\lambda}(s-\bar{n})$, and $\bar{n}$ depends on ρ, we have a fixed-point equation to solve. The parameter to optimize is s and the metric is L, then let us plot L as a function of s. In order to do this, it is necessary to obtain $\bar{n}$, but as we do not have the analytic value of $\bar{n}$, we will use the upper bound $\bar{n}_{upper} \approx \bar{n}$, as we can see in Figure 12, which is given by Proposition 6 and the analytic value of $\pi(K,K)$ is given by Lemma 3. Let us plot also, the curve where users do not choose between the queues. The expressions for L as a function of s, in this case, can be obtained from [6]. Additionally, we plot the local choice, which is defined in Subsection 4.3. As we do not have explicit formulas for local choice, we will simulate the system for $N = 100$.

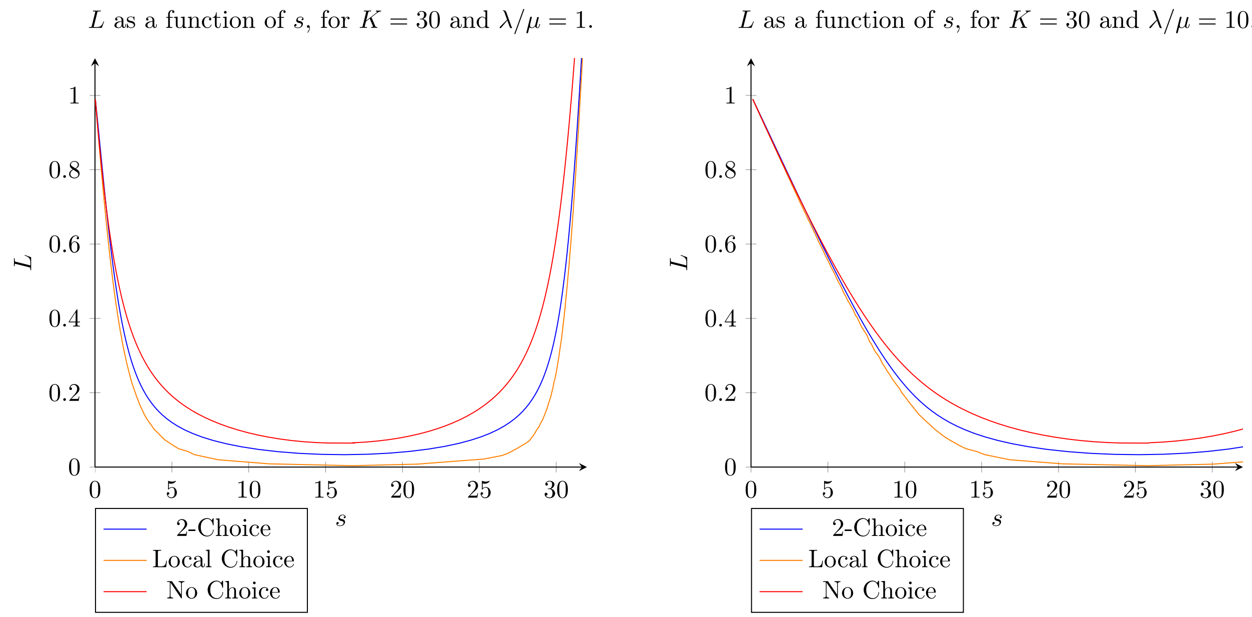

Figure 14: As expected, the worst case is when the users do not choose, the best result is when people adopt the local choice policy and in-between we have the choice in groups of two stations.

As the expressions for L and s, given by equation (73), Proposition 6 and Lemma 3, in the case we are studying are too complex, let us assume that K is sufficiently large to allow us to expand these terms in a power series around $K \to \infty$. However, we still have the problem that the minimum is not at $\rho = 1$, where we have the analytic solutions. In order to solve this problem and at the same time use the fact that K is large, let us make $\rho = 1 + \psi/K$, for any $\psi \in \mathbb{R}$ and then, expand in a power series around $K \to \infty$, leading us to the following proposition.

Proposition 8. *Let* $\psi \in \mathbb{R}$ *and* $\rho = 1 + \dfrac{\psi}{K}$, *then*
if $\psi \neq 0$,

$$\begin{aligned} L &= \frac{\psi}{K}\coth(\psi) + \mathcal{O}\left(\frac{1}{K^2}\right) \\ s &= \frac{K}{2}(1 + \coth(\psi) - \psi^{-1}) + \frac{\psi^2}{4}\operatorname{csch}^2(\psi) + \frac{\mu}{\lambda} + \mathcal{O}\left(\frac{1}{K}\right), \end{aligned}$$

if $\psi = 0$,

$$\begin{aligned} L &= \frac{1}{K + 2^{-K-1}} = \frac{1}{K} + \mathcal{O}\left(\frac{1}{K^2 2^K}\right) \\ s &= \frac{4K^2 + 2K + 1 - 2^{-K}}{8K + 2^{-K+2}} + \frac{\mu}{\lambda} = \frac{K}{2} + \frac{\mu}{\lambda} + \frac{1}{4} + \frac{1}{8K} + \mathcal{O}\left(\frac{1}{2^K}\right), \end{aligned}$$

for K *that tends to infinity.*

Proof. For $\psi = 0$ the proof is trivial from proposition 6. Then, for $\psi \neq 0$, after using the substitution $\rho = 1 + \psi/K$ and Lemma 3, the method to obtain the expression for L from (73) is standard. Nevertheless, the expression for s are not as trivial, since we must first show that the upper bound $\bar{n}_{upper} = \bar{n} + \mathcal{O}(1/K)$. Let us start by showing that we can rewrite equation (64) as an equality plus a $\mathcal{O}(1/K)$. To prove that, it is enough to show that when $\rho = 1 + \psi/K$, then

$$A'\left(\frac{1+\rho}{1+\rho^2}\right) = A'(1) + \mathcal{O}(K^{-1}),$$
$$A\left(\frac{1}{\rho}\right) = A\left(\frac{1+\rho}{1+\rho^2}\right) + \mathcal{O}(K^{-1}).$$

Let us start by using the definition of A, and the fact that $A(z)$, $A'(z)$ and $A''(z)$ converge for all $z \in \mathbb{C}$.

$$\begin{aligned} A'\left(\frac{1+\rho}{1+\rho^2}\right) &= A'\left(1 - \frac{\psi}{2K} + \mathcal{O}(K^{-2})\right) \\ &= \sum_{t=1}^{K} t\pi(t,0)\left(1 - \frac{\psi}{2K} + \mathcal{O}(K^{-2})\right)^{t-1} \\ &= \sum_{t=1}^{K} t\pi(t,0) - \frac{\psi}{2K}\sum_{t=1}^{K} t^2\pi(t,0) + \mathcal{O}(K^{-1}) \\ &= A'(1) - \frac{\psi}{2K}A''(1) + \mathcal{O}(K^{-1}). \end{aligned}$$

We can use the same arguments for $A(1/\rho)$. Then, we use equation (60) and Lemma 3 to prove the proposition for s. □

Note: In fact, the previous proof is not complete. We are still working in justifying the steps to conclude that $A'\left(\frac{1+\rho}{1+\rho^2}\right) = A'(1) + \mathcal{O}(K^{-1})$.

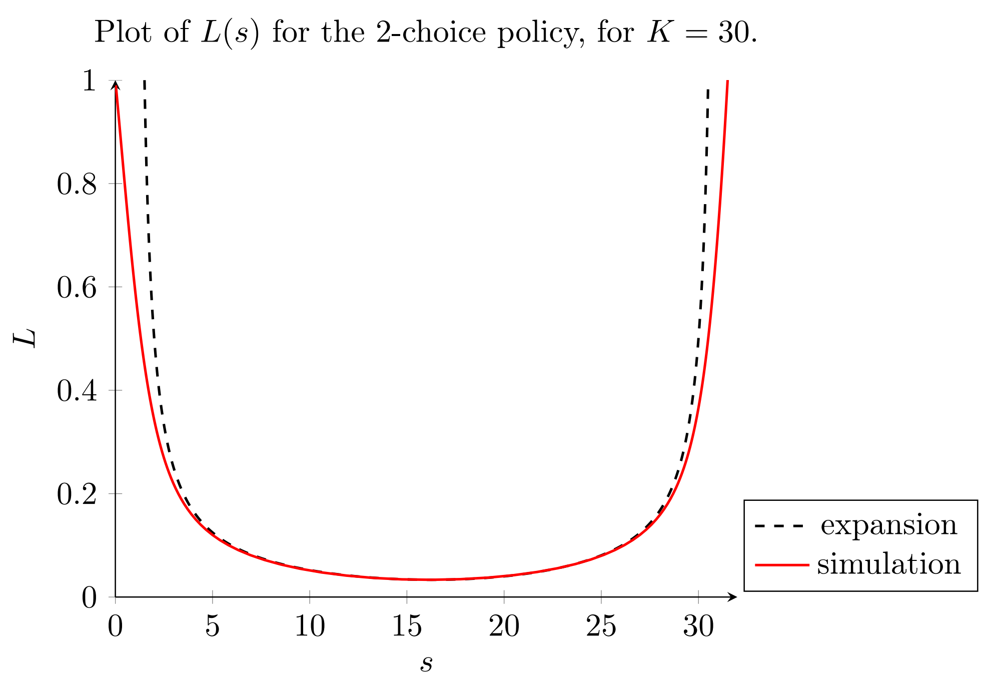

Figure 15: In this graphic, ψ assumed values between $-K$ and K. As we can see, the expansion from Proposition 8 is reliable exactly in the region of interest, which is the optimum region.

Figure 15 shows visually that the proposed expansion, when $\rho = 1 + \psi/K$ around $K \to \infty$ is, indeed, an excellent approach to analyze the optimum region for the fleet size parameter s.

If $|\psi| \leq \ln(3)$, in proposition 8, then,

$$\frac{K}{2}\left(\frac{1}{\ln(3)} - \frac{1}{4}\right) + \frac{9}{64}\ln^2(3) + \mathcal{O}(K^{-1}) \leq s - \frac{\mu}{\lambda} \leq \frac{K}{2}\left(\frac{9}{4} - \frac{1}{\ln(3)}\right) + \frac{9}{64}\ln^2(3) + \mathcal{O}(K^{-1}),$$
$$\frac{1}{K} + \mathcal{O}(K^{-2}) \leq L \leq \frac{5}{4}\frac{\ln(3)}{K} + \mathcal{O}(K^{-2}).$$

Then, using the previous result and for K sufficiently large, we can state that,

$$\frac{1}{3}K + \frac{1}{5} \leq s - \frac{\mu}{\lambda} \leq \frac{2}{3}K + \frac{1}{6},$$
$$\frac{1}{K} \leq L \leq \frac{7}{5K}.$$

Which give us a good idea of the optimum region in which s must belong to. We can easily verify that, on the figures 14 or 15.

For the optimum s point in the system with no-choice policy, we find that $L \approx 2/K$, while in this system we saw that in the optimum point $L \approx 1/K$, which means that we have, at least, half of unsatisfied users, if they start to choose between stations. We say at least half, because, when users start to choose, the best representation of the system is the *local choice*, which is clearly better than the 2-choice model, as we can see in the Figure 14.

7 Bike-sharing systems: Power of two choices with delay

7.1 Model description

Consider a homogeneous bike-sharing system with N stations with capacity C and M bikes, arrival rate of users at each station λ and trip times between stations with mean $1/\mu$, all the inter arrival times and trip times are exponentially distributed. Let us denote $s = M/N$.

In this model, the user chooses, at the time he takes the bike, the least loaded one among two stations at random to return the bike. When he arrives to this station, there are two cases: The station is not full and he returns his bike. The station is full and he chooses again among two stations the least loaded one, and it takes another trip time exponentially distributed with parameter μ, until he succeeds to return.

7.2 The mean-field limit

Let us define

$$Y^N_{k,l}(t) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\tilde{V}_i(t)=k, V_i(t)=l\}}$$

where $\tilde{V}_i(t)$ is the number of bikes coming to station i at time t and $V_i(t)$ is the number of bikes at station i at time t. This model takes into account that the station, where the users want to go is made at the start of the trip.

Let us denote $p_l = \sum_{k=0}^{+\infty}(y_{k,l}(t) + 2\sum_{j=l+1}^{C} y_{k,j}(t)) = y_{\cdot,l}(t) + 2\sum_{j=l+1}^{C} y_{\cdot,j}(t)$. The process $Y^N(t) = (Y^N_{k,l}(t))_{k\in\mathbb{N},0\leq l\leq C}$ is Markov on the state space $\mathcal{L} = \{(k,l) \in \mathbb{N}^2 \mid k \geq 0, 0 \leq l \leq C\}$. Its transitions are given by the infinitesimal generator

$$Q(y, y + \frac{1}{N}(e_{k,l-1} - e_{k,l} + e_{k'+1,l'} - e_{k',l'})) = N\lambda p_{l'} y_{k',l'} y_{k,l} \mathbb{1}_{\{l>0\}}$$
$$Q(y, y + \frac{1}{N}(e_{k-1,l+1} - e_{k,l})) = N\mu k y_{k,l} \mathbb{1}_{\{l<C\}}$$
$$Q(y, y + \frac{1}{N}(e_{k-1,C} - e_{k,C} + e_{k'+1,l'} - e_{k',l'})) = N\mu k p_{l'} y_{k,C} y_{k',l'} \mathbb{1}_{\{l'<C\}}$$

The first transition corresponds to the arrival of a user at a station in state (k, l) who chooses a station with state (k', l'). The second transition corresponds to the return of a bike at a station with state (k, l) and $l < C$. The third transition corresponds to the case a user want to return in a station with (k, C) and has to return to a random station among the non saturated one, in state (k', l').

The dynamical system $(y(t))_{t\geq 0}$ limit of the Markov process $(Y^N(t))_{t\geq 0}$ as N tends to infinity is the unique solution of the ODE

$$\begin{aligned}\dot{y}(t) = \sum_{(k,l)} y_{k,l}(t) \Big(&\mu k \mathbb{1}_{\{l<C\}}(\mathbb{1}_{\{l>0\}} e_{k-1,l+1} - e_{k,l}) + \mu k \mathbb{1}_{\{l=C\}}(1_{k>0} e_{k-1,C} - e_{k,C}) + \mu(e_{k+1,l} - e_{k,l}) p_l \tilde{V}_C \\ &+\lambda \mathbb{1}_{\{l>0\}}(e_{k,l-1} - e_{k,l}) + \lambda(e_{k+1,l} - e_{k,l}) p_l (1 - y_{\cdot,0}(t))\Big), \end{aligned} \tag{74}$$

where $\tilde{V}_C = \sum_m m y_{m,C}(t)$ is the mean number bikes riding to a saturated station. This ODE (74) can be rewritten, for $k \in \mathbb{N}$ and $l < C$,

$$\begin{aligned}\dot{y}_{k,l}(t) &= \mu\left((k+1) y_{k+1,l-1}(t) \mathbb{1}_{\{l>0\}} - k y_{k,l}(t)\right) + (\lambda(1 - y_{\cdot,0}(t)) + \mu \tilde{V}_C) p_l \left(\mathbb{1}_{\{k>0\}} y_{k-1,l}(t) - y_{k,l}(t)\right) \\ &\quad + \lambda\left(y_{k,l+1}(t) - y_{k,l}(t)\mathbb{1}_{\{l>0\}}\right) \\ \dot{y}_{k,C}(t) &= \mu(k+1)(y_{k+1,C-1}(t) + y_{k+1,C}(t)) - \mu k y_{k,C}(t) + \mu \tilde{V}_C(\mathbb{1}_{\{k>0\}} y_{k-1,C}(t) - y_{k,C}(t)) y_{\cdot,C}(t) \\ &\quad - \lambda y_{k,C}(t) + \lambda\left(\mathbb{1}_{\{k>0\}} y_{k-1,C}(t) - y_{k,C}(t)\right) y_{\cdot,C}(t)(1 - y_{\cdot,0}(t)). \end{aligned} \tag{75}$$

For these two systems of ODEs, the equilibrium points must be founded. The existence of the equilibrium points is given by Brower's theorem. The uniqueness is more difficult. We just check it by numerical simulations, which allow us to obtain its numerical value.

7.3 Validation of the approximation

The approximation is validated by simulations of the model. The simulation is made by a program developed in C, whose working principle is to generate random exponential distributed times for stations (with constant λ) and for each client with a bike (with constant μ). Each client has his destination station and trip time determined at the time he leaves a station.

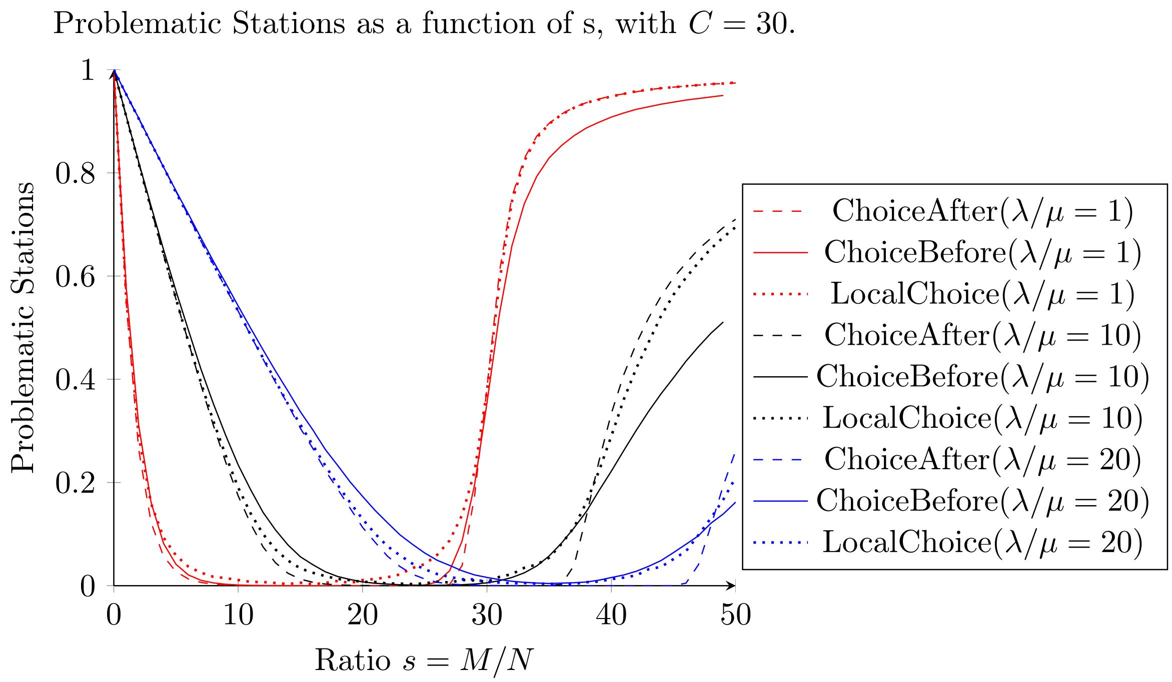

Figure 16: Proportion of problematic stations with a fleet size s.

Figure 16 illustrates the relation between the policies of choice. We opted to plot the proportion of problematic stations $y_0 + y_C$, instead of the metric discussed before, in order to be consistent with [6]. Basically, *choice before* is with *delay*, *choice after* is without *delay* and *local choice* is defined in Subsection 4.3. By the graphics, we see that *choice after* is always better, except for s large, but this is discussed in Subsection 6.2. Regarding *choice before* and *local choice*, we can see that for small values of λ/μ, *local choice* is more significant, however as this ratio increases the *choice before* becomes worse and worse until it surmount the *local choice*.

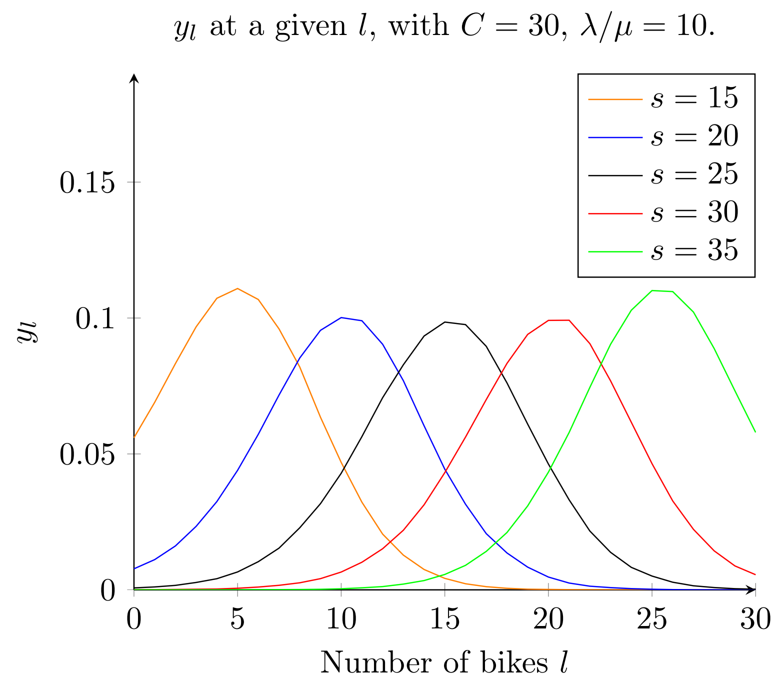

Figure 17: Proportion of stations with l bikes in it.

The graphic of Figure 17 illustrates well why the fleet size around $C/2$ is the best choice. If the fleet size is near C or near 0, we see that the tail of the curve did not decreased enough, so the proportion of problematic stations is high, in these cases.

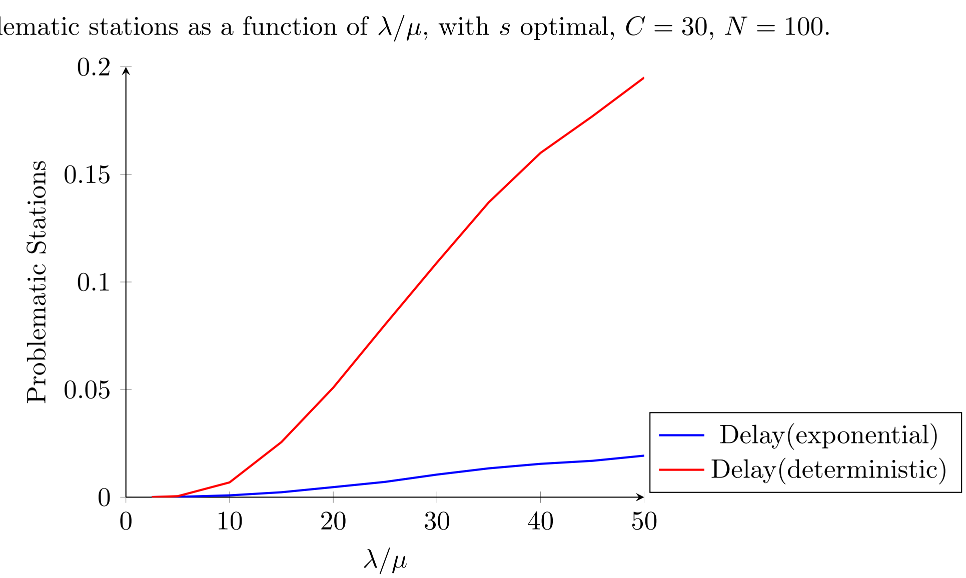

Figure 18: We did not plot when the user choose at the end of the trip, because for s optimum this value is very close to 0 for all λ/μ.

For curiosity, we made the simulation for a deterministic *delay*, instead of an exponential. Surprisingly, the deterministic *delay* is much worse, as we can see in the graphic and, unfortunately, the real system is closer to a deterministic *delay* than from an exponential distributed *delay*. Nevertheless, we do not have a choice regarding this, if we want to deal with this problem introducing the effect of randomness.

8 Implementing local choice in the Vélib' data

In this section we deal with the data from the bike-sharing system of Paris (called Vélib') for the period of one month. The objective is to *replay* the trips such that the customers choose the best station (for the system) to return the bike, keeping the constraint that this station is near their destination. In other words, we are testing the *local choice* (see Subsection 4.3) in a real bike-sharing system.

The data is in public domain and can be found in [1]. This data-set correspond to the trips the users make and it has the following format:

$$"reason; out\_date; out\_station; out\_bikestand; in\_date; in\_station; in\_bikestand",$$

where *reason* is the reason of the trip (a client, maintenance or regulation), *out_ date* and *in_ date* correspond to the date the bike exited and entered in a station, respectively. *Out_ station* and *in_ station* correspond to the identification number of the station where the bike exited and returned, respectively. *Out_ bikestand* and *in_ bikestand* correspond to the bikestand number in the station, where the bike exited and returned, respectively. Unfortunately, this data-set has some flaws. For example, some bike trips does not appear in the data-set. As this error appeared to be around 8%, we can still work with it. We could verify the order of this error, because we had access to another set of data that we call data-set states. This file contains the *state* of all stations each 10 minutes, in average. Basically, the *state* of the station stands for the number of bikes and the number of free slots available in each station. The problem with this data-set is that it is updated irregularly, i.e., the variance of the time between updates is too large and each station is updated independently from the others, they are not updated in a synchronized way. Finally, we use a small data-set containing the geographical position and capacity of each station, in order to list which stations are near a given station.

8.1 Performance metric for the Vélib' system

The metric suggested in Subsection 6.2 cannot be implemented, since we do not know the quantity ρ for each station and therefore, we cannot calculate the rate of unsatisfied users. Even to estimate π_0 and π_K becomes an issue, since we have an error over the measure, i.e., several stations does not reach full capacity or empty state, because of the errors mentioned previously.

Then, we propose a new metric, which is based in the distance (using euclidean norm) from the ideal situation, which is the state where we distribute the parked bikes in a proportional way of the capacity of each station. Let n_s be the total number of bikes that are parked, b_i the number of bikes in station i and C_i the capacity of station i. Then, our performance metric $P(b)$ is given by,

$$P(b) = \sqrt{\sum_{i=1}^{N} \left(\frac{b_i - C_i^*}{C_i - C_i^*} \right)^2}, \tag{76}$$

where $C_i^* = \dfrac{n_s C_i}{\sum_k C_k}$. We divide by $C_i - C_i^*$ to normalize the metric.

8.2 Policies of choice

We decided to test two kind of policies of choice. Let us call the first one, policy A. The policy of choice A consists of changing the *in_station* in the data-set of trips for the station with the greatest number of free slots in a radius of 300 meters. The policy of choice B consists of changing the *in_station* in the data-set of trips for the most available station in a radius of 300 meters. The most available station in the sense of having the greatest number of free slots with respect to the total capacity of the station. It is noteworthy that we apply the policy for users, maintenance and regulation.

As a matter of fact, the policy A is the most intuitive for the users, since each user wants to know the number of free slots in each station. Nevertheless, we will show that the best approach is the policy B.

Regarding the problems of changing the data-set of trips, the most serious one is the fact that, as we are changing the trips, perhaps some trips may not occur, since we may have full/empty stations that were not in the original case. However, we observed, that this event is relatively rare, since applying the policies the system has a better performance. Yet this event occurs and to deal with it, we allow stations to have negative number of bikes or negative number of free slots. Of course, we count them as problematic stations.

The following graphic shows the results of the simulations for a typical day.

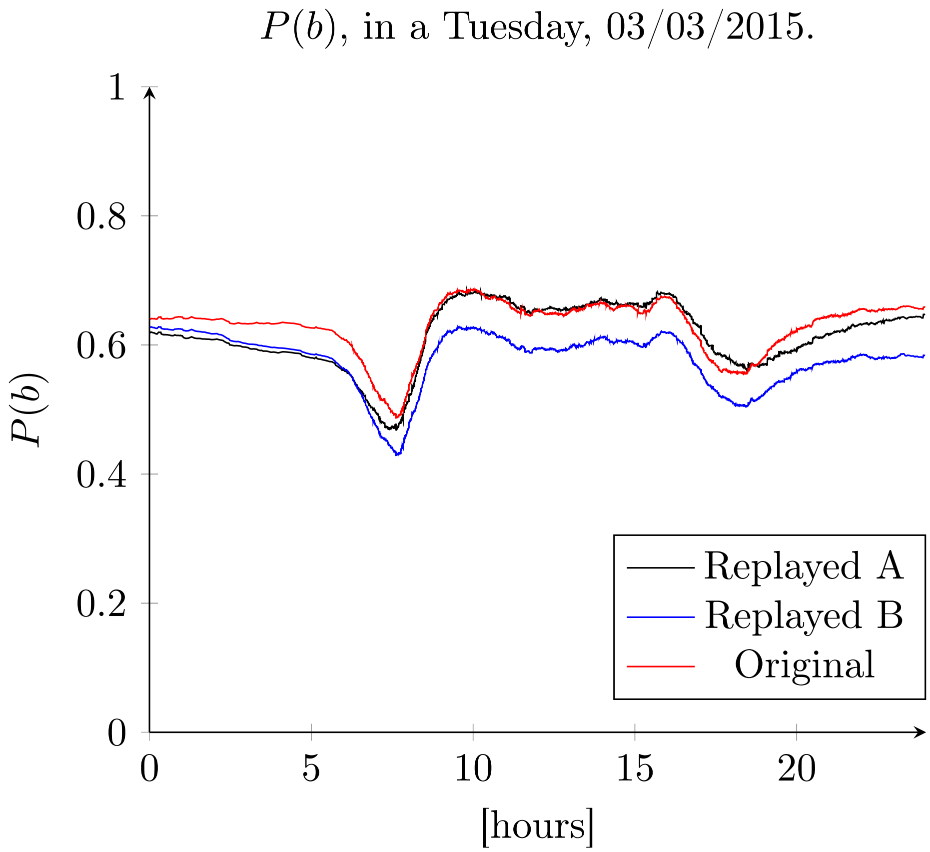

Figure 19: Plot of the performance measure $P(b)$ as a function of time, starting at 00:00 of a Tuesday, when we apply policy A, B and no policy (the original).

As we mentioned, our metric points that the policy B is the best one. Beyond that, we see that the policy A, except for the first hours, almost does not help the system.

9 Conclusion and future work

In this work, we have investigated the influence of the *local choice* on the performance of bike-sharing systems. We used several methods and different models, including *pair approximation*, *mean-field* limit to study the steady-state behavior of the network and compare it with former homogeneous models without *local choice*. Despite the difficulty imposed by the geometry in the problem when we use *local choice*, we still manage to find theoretical results regarding low traffic and a *two-choice* model (see Section 5), which is the closest model to *local choice* we obtained with very relevant results.

While studying this fertile problem, we have find some interesting results, of which the most relevant were the algorithm to find the expansion of a system of queues with general choice function, which can be found in Section 4 and the closed form for the invariant measure of both queues having the maximum number of clients in a JSQ system with 2 queues and finite capacity, as stated in Lemma 3. As far as we know, the closest result to this last result, is found in Flatto [5] and it gives the analytic expression of the same system, but with infinite capacity and for the whole set of values of the invariant measure. Furthermore, the closed form is an infinite sum.

Regarding results about the bike-sharing system, we showed that when the users choose the least loaded station near their destination, the rate of *unsatisfied users*, which is well defined in Subsection 6.2, diminishes roughly by half, by the *two-choice* model. Beyond that, we delimit an optimum region for the fleet size, in order to achieve a *good performance* for the system.

We also have studied the effect of the *delay*, when users choose the least loaded station between two random stations. The *delay* is modeled using the fact that the user will choose his destination station at the beginning of the trip, and the idea is that when the user arrives, perhaps this station is no longer the least loaded one. As a matter of fact, we noticed that this effect was relevant only when the ratio of incoming clients and exiting clients was greater than 5, which is rarely the case for a bike-sharing system. In the case, we used data from Vélib', the bike-sharing system of Paris, to verify that.

The work with the data from Vélib' extended beyond that. We adopted some choice policies for the users and we *replayed* the trips of the users using such policies. After comparing the policies and the original system, with a well defined metric, we found that choosing the station with the greatest number of available slots is not the best strategy, since stations with small capacities may be neglected in the process. By the other hand, when choosing the station with the greatest number of available slots with respect to its total capacity, then, we have an improvement in the system, as it is showed in Figure 19.

As a continuation, deeper analysis over the Vélib' data and choice policies remain to be done and improved. Due to the complex nature of the *local choice* problem, we still lack methods to approach it and a possible way to deal with it, could be to propose a similar model, as we did proposing the *two-choice* model, as we recognize that it still far from the efficiency the *local choice* provides for the system, since it introduces more random and balance effects.

References

[1] Vélib' Data. `https://developer.jcdecaux.com/#/opendata`, accessed Jun 16, 2015.

[2] Y. Chabchoub, C. Fricker. *Classification Of The Vélib' Stations Using Kmeans, Dynamic Time Wraping and DBA Averaging Method.* IWCIM, 1-2 November 2014, Paris, France.

[3] E. Côme, L. Oukhellou. *Model-based count series clustering for Bike Sharing System usage mining, a case study with the Velib system of Paris.* ACM Transactions on Intelligent Systems and Technology, 2014, 27p.

[4] F. Ezanno. *Systèmes de particules en interaction et modèles de déposition aléatoire*. Theses at Aix-Marseille Université, 2012.

[5] L. Flatto, H. P. McKean. *Two Queues in Parallel*. Communications on pure and applied mathematics, Vol. XXX, p.255-263, 1977.

[6] C. Fricker, N. Gast. *Incentives and Redistribution in Homogeneous Bike-sharing Systems with Stations of Finite Capacity*. European Journal on Transportation and Logistics, 2014.

[7] C. Fricker. *Deposition and queuing model with choice between neighbors*. Working paper.

[8] N. Gast. *The Power of Two Choices on Graphs: the Pair-Approximation is Accurate*. `http://www.sigmetrics.org/mama/abstracts/NGast.pdf`, accessed Jun, 25, 2015.

[9] J. Larsen. *Bicycle Share Fact Sheet*. `http://www.earth-policy.org/press_room/C68/bicycle_share_fact_sheet`, accessed Jun 28, 2015.

[10] M. D. Mitzenmacher. *The Power of Two Choices in Randomized Load Balancing*. PhD thesis, University of California at Berkeley, 1996.

[11] P. Robert. *Réseaux de communication, Algorthmes et Probabilités*. MAP554 - département de Mathématiques Appliquées de l'Ecole Polytechinque, 2014.

[12] N. D. Vvedenskaya *et al. A queueing system with a choice of the shorter of two queues - an asymptotic approach*. Problems Inform. Transmission, p.15-27, 1996.