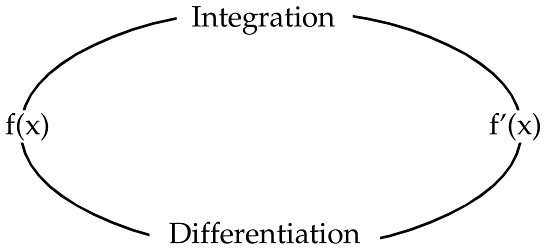
(B) INTEGRAL CALCULUS

8.B.1 INTEGRATION

Integration is the reverse process of differentiation.



We know

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=\frac{(n+1)x^n}{(n+1)}$$

$$\text{or } \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n \qquad \text{.....................(1)}$$

Integration is the inverse operation of differentiation and is denoted by the symbol $\int$.

Hence, from equation (1), it follows that

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}$$

i.e. Integral of x^n with respect to variable x is equal to $\dfrac{x^{n+1}}{n+1}$

Thus if we differentiate $\dfrac{\left(x^{n+1}\right)}{n+1}$ we can get back x^n

Again if we differentiate $\dfrac{\left(x^{n+1}\right)}{n+1}$ + c and c being a constant, we get back the same x^n .

$$\text{i.e. } \frac{d}{dx}\left[\frac{x^{n+1}}{n+1}+c\right]=x^n$$

Hence $\int x^n\,dx = \dfrac{\left(x^{n+1}\right)}{n+1}$ + c and this c is called the constant of integration.

Integral calculus was primarily invented to determine the area bounded by the curves dividing the entire area into infinite number of infinitesimal small areas and taking the sum of all these small areas.

8.B.2 BASIC FORMULAS

i) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$ (If n=-1, $\frac{x^{n+1}}{n+1} = \frac{1}{0}$ which is not defined)

ii) $\int dx = x + c$, since $\int 1dx = \int x^{o}dx = \frac{x1}{1} = x + c$

iii) $\int e^x\,dx = e^x + c$, since $\frac{d}{dx}e^x = e^x$

iv) $\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$, since $\frac{d}{dx}\left(\frac{e^{ax}}{a}\right) = e^{ax}$

v) $\int \frac{dx}{x} = \log x + c$, since $\frac{d}{dx}\log x = \frac{1}{x}$

vi) $\int a^x\,dx = a^x / \log_e a + c$, since $\frac{d}{dx}\left(\frac{a^x}{\log_e^a}\right) = a^x$

Note: *In the answer for all integral sums we add +c (constant of integration) since the differentiation of constant is always zero.*

Elementary Rules:

$\int c\,f(x)\,dx = c\int f(x)\,dx$ where c is constant.

$\int \{ f(x)\,dx \pm g(x)\}\,dx = \int f(x)dx \pm \int g(x)dx$

Examples : Find (a) $\int \sqrt{x}\,dx$, (b) $\int \frac{1}{\sqrt{x}}dx$, (c) $\int e^{-3x}dx$ (d) $\int 3^x dx$ (e) $\int x\sqrt{x}\ dx$.

Solution: (a) $\int \sqrt{x}\ dx = x^{1/2+1} / (1/2+1) = \frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3} + c$

(b) $\int \frac{1}{\sqrt{x}}dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = 2\sqrt{x} + c$ where c is arbitrary constant.

(c) $\int e^{-3x}\ dx = \frac{e^{-3x}}{-3} + c = -\frac{1}{3}e^{-3x} + c$

(d) $\int 3^x\ dx = \frac{3^x}{\log_e 3} + c.$

(e) $\int x\sqrt{x}\ dx. = \int x^{\frac{3}{2}} dx = \dfrac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} dx = \dfrac{2}{5}x^{5/2} + c.$

Examples : Evaluate the following integral:

i) $\int (x + 1/x)^2\, dx = \int x^2\, dx + 2\int dx + \int dx / x^2$

$$= \frac{x^3}{3} + 2x + \frac{x^{-2+1}}{-2+1}$$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

ii) $\int \sqrt{x}\ (x^3+2x-3)\, dx = \int x^{7/2}\, dx + 2\int x^{3/2}\, dx - 3\int x^{1/2}\, dx$

$$= \frac{x^{7/2+1}}{7/2+1} + \frac{2\,x^{3/2+1}}{3/2+1} - \frac{3\,x^{1/2+1}}{1/2+1}$$

$$= \frac{2x^{9/2}}{9} + \frac{4x^{5/2}}{5} \quad -2\,x^{3/2} + c$$

iii) $\int e^{3x} + e^{-4x}\, dx \quad = \int e^{2x}\, dx + \int e^{-4x}\, dx$

$$= \frac{e^{2x}}{2} + \frac{e^{-4x}}{-4} = \frac{e^{2x}}{2} - \frac{1}{4e^{4x}} + c$$

iv) $\int \dfrac{x^2}{x+1} dx = \int \dfrac{x^2-1+1}{x+1} dx$

$$= \int \frac{(x^2-1)}{x+1} dx + \int \frac{dx}{x+1}$$

$$= \int (x-1)dx + \log(x+1) = \frac{x^2}{2} - x + \log(x+1) + c$$

v) $\int \dfrac{x^3 + 5x^2 - 3}{(x+2)} dx$

By simple division $\int \dfrac{x^3 + 5x^2 - 3}{(x+2)} dx$

$$= \int \left\{ x^2 + 3x - 6 + \frac{9}{(x+2)} \right\} dx$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} - 6x + 9\log(x+2) + c$$

8.B.3 METHOD OF SUBSTITUTION (CHANGE OF VARIABLE)

It is sometime possible by a change of independent variable to transform a function into another which can be readily integrated.

We can show the following rules.

To put z = f (x) and also adjust dz = f ′ (x) dx

Example: $\int F\{h(x)\}\ h'(x)\,dx$, take e^z = h(x) and to adjust dz = h′ (x) dx

then integrate $\int F(z)\,dz$ using normal rule.

Example: $\int (2x+3)^7\,dx$

We put (2x + 3) = t $\Rightarrow$ so 2 dx = dt or dx = dt / 2

Therefore $\int (2x+3)^7\,dx = ½\int t^7\,dt = \frac{t^8}{2x8} = \frac{t^8}{16} = \frac{(2x+3)^8}{16} + c$

This method is known as Method of Substitution

Example: $\int \frac{x^3}{(x^2+1)^3}dx$ We put $(x^2 +1) = t$

so 2x dx = dt or x dx = dt / 2

$$= \int \frac{x^2.x}{t^3}dx$$

$$= \frac{1}{2}\int \frac{t-1}{t^3}dt$$

$$= \frac{1}{2}\int \frac{dt}{t^2} - \frac{1}{2}\int \frac{dt}{t^3}$$

$$= \frac{1}{2}\times\frac{t^{-2+1}}{(-2+1)} - \frac{1}{2}\times\frac{t^{-3+1}}{(-3+1)}$$

$$= -\frac{1}{2}\ \frac{1}{t} + \frac{1}{4}\ \frac{1}{t^2}$$

$$= \frac{1}{4}\frac{1}{t^2} - \frac{1}{2}\frac{1}{t}$$

$$= \frac{1}{4} \cdot \frac{1}{\left(x^2+1\right)^2} - \frac{1}{2} \cdot \frac{1}{\left(x^2+1\right)} + c$$

IMPORTANT STANDARD FORMULAE

a) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

b) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$

c) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + c$

d) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right) + c$

e) $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$

f) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right) + c$

g) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right) + c$

h) $\int \frac{f'(x)}{f(x)}\, dx = \log f(x) + c$

Examples: (a) $\int \frac{e^x}{e^{2x} - 4} dx = \int \frac{dz}{z^2 - 2^2}$ where $z = e^x$ $dz = e^x dx$

$$= \frac{1}{4} \log\left(\frac{e^x - 2}{e^x + 2}\right) + c$$

(b) $\int \frac{1}{x + \sqrt{x^2 - 1}} dx = \int \frac{x - \sqrt{x^2 - 1}}{(x + \sqrt{x^2 - 1})\,(x - \sqrt{x^2 - 1})} dx = \int (x - \sqrt{x^2 - 1})\, dx$

$$= \frac{x^2}{2} - \frac{x}{2}\sqrt{x^2 - 1} + \frac{1}{2} \log(x + \sqrt{x^2 - 1}) + c$$

(c) $\int e^x (x^3 + 3x^2) dx = \int e^x \{f(x) + f'(x)\} dx$, where $f(x) = x^3$

[by (e) above)] $= e^x x^3 + c$

8.B.4 INTEGRATION BY PARTS

$$\int uv\,dx = u \int v\,dx - \int \left[\frac{d(u)}{dx} \int v\,dx\right] dx$$

where u and v are two different functions of x

Evaluate:

i) $\int xe^x \, dx$

Integrating by parts we see

$$\int xe^x \, dx = x\int e^x \, dx - \int\left\{\frac{d}{dx}(x)\int e^x \, dx\right\}dx$$

$$= x\,e^x - \int 1.e^x dx = xe^x - e^x + c$$

ii) $\int x \log x \, dx$

Integrating by parts,

$$= \log x \int x \, dx - \int\left\{\frac{d}{dx}(\log x)\int x dx\right\}dx$$

$$= \frac{x^2}{2}\log x - \int\left[\frac{1}{x}.\frac{x^2}{2}\right]dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

iii) $\int x^2 e^{ax} \, dx$

$$= x^2\int e^{ax} \, dx - \int\left\{\frac{d}{dx}(x^2)\int e^{ax} \, dx\right\}dx$$

$$= \frac{x^2}{a}e^{ax} - \int 2x.\frac{e^{ax}}{a} \, dx$$

$$= \frac{x^2}{a}e^{ax} - \frac{2}{a}\int x.e^{ax} dx$$

$$= \frac{x^2}{a}e^{ax} - \frac{2}{a}\times\int e^{ax} \, dx - \int\left[\frac{d}{dx}(x)\int e^{ax} \, dx\right]dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[\frac{xe^{ax}}{a} - \int 1.\frac{e^{ax}}{a}dx\right]$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2}{a^3} e^{ax} + c$$

8.B.5 METHOD OF PARTIAL FRACTION

Type I :

Example: $\int \frac{(3x+2)\,dx}{(x-2)\,(x-3)}$

Solution: let $\frac{(3x+2)}{(x-2)\,(x-3)}$

$$= \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

[Here degree of the numerator must be lower than that of the denominator; the denominator contains non–repeated linear factor]

so $3x + 2 = A\,(x-3) + B\,(x-2)$

We put $x = 2$ and get

$3.2 + 2 = A\,(2–3) + B\,(2–2) \Rightarrow A = –8$

we put $x = 3$ and get

$3.3 + 2 = A\,(3–3) + B\,(3–2) \qquad \Rightarrow B = 11$

$$\int \frac{(3x+2)dx}{(x-2)^2\,(x-3)} \quad -8\int \frac{dx}{x-2} + 11\int \frac{dx}{x-3}$$

$$= -\log(x–2) + 11\log(x-3) + c$$

Type II:

Example: $\int \frac{(3x+2)\,dx}{(x-2)^2\,(x-3)}$

Solution: let $\frac{(3x+2)}{(x-2)^2\,(x-3)} = \frac{A}{(x–2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)}$

or $3x + 2 = A\,(x-2)\,(x-3) + B\,(x-3) + C\,(x-2)^2$

Comparing coefficients of x^2, x and the constant terms of both sides, we find

$A + C = 0$ …………(i)

$–5A + B – 4C = 3$ ……(ii)

$6A – 3B + 4C = 2$ …….(iii)

By (ii) + (iii) A – 2B = 5 ………(iv)

(i) – (iv) 2B + C = –5 …….(v)

From (iv) A = 5 + 2B

From (v) C = –5 – 2B

From (ii) –5 (5 + 2B) + B – 4 (– 5 – 2B) = 3

or – 25 – 10B + B + 20 + 8B = 3

or – B – 5 = 3

or B = – 8, A = 5 – 16 = – 11, from (iv) C = – A = 11

Therefore $\int \frac{(3x+2)\,dx}{(x-2)^2(x-3)}$

$$= -11\int \frac{dx}{(x-2)} - 8\int \frac{dx}{(x-2)^2} + 11\int \frac{dx}{(x-3)}$$

$$= -11 \log (x-2) + \frac{8}{(x-2)} + 11 \log (x-3)$$

$$= 11 \log \frac{(x-3)}{(x-2)} + \frac{8}{(x-2)} + c$$

Type III:

Example: $\int \frac{(3x^2 - 2x + 5)}{(x-1)^2\,(x^2+5)}dx$

Solution: Let $\frac{3x^2-2x+5}{(x-1)^2(x^2+5)} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+5)}$

so $3x^2-2x+5 = A(x^2+5) + (Bx+C)(x-1)$

Equating the coefficients of x^2, x and the constant terms from both sides we get

A + B = 3 …………(i)

C – B = –2 …………(ii)

5A – C = 5 ………….(iii)

by (i) + (ii) A + C = 1 ……… (iv)

by (iii) + (iv) 6A = 6 ……… (v)

or A = 1

therefore B = 3 – 1 = 2 and C = 0

Thus $\int \frac{(3x^2 - 2x + 5)}{(x-1)^2\,(x^2+5)}dx$

$$= \int \frac{dx}{x-1} + \int \frac{2x}{x^2+5} dx$$

$= \log (x–1) + \log (x^2 + 5)$

$= \log (x^2 + 5) (x–1) + c$

Example: $\int \frac{dx}{x(x^3+1)}$

Solution: $\int \frac{dx}{x(x^3+1)}$

$$= \int \frac{x^2 dx}{x^3 (x^3+1)}$$ we put $x^3 = z$, $3x^2\, dx = dz$

$$= \frac{1}{3}\int \frac{dz}{z(z+1)}$$

$$= \frac{1}{3}\int \left(\frac{1}{z} - \frac{1}{z+1}\right) dz$$

$$= \frac{1}{3}\left[\log z - \log (z+1)\right]$$

$$= \frac{1}{3}\log\left(\frac{x^3}{x^3+1}\right)$$

Example : Find the equation of the curve where slope at (x, y) is 9x and which passes through the origin.

Solution: $\frac{dy}{dx} = 9x$

$\therefore \int dy =$ or $y = 9x^2 /2 + c$

Since it passes through the origin, c = 0; thus required eqn . is $9x^2 = 2y$.

8.B.6 DEFINITE INTEGRATION

Suppose F(x) dx = f (x)

As x changes from a to b the value of the integral changes from f (a) to f (b). This is as

$$\int_a^b F(x)\ dx = f(b) - f(a)$$

'b' is called the upper limit and 'a' the lower limit of integration. We shall first deal with indefinite integral and then take up definite integral.

Example: $\int_0^2 x^5\ dx$

Solution: $\int_0^2 x^5\ dx = \frac{x^6}{6}$

$$\int_0^2 x^5\ dx = \left(\frac{x^6}{6}\right)_0^2$$

$$= \frac{1}{6}\ (2^6 - 0) \qquad = 64/6 = 32/3$$

Note: In definite integration the constant (c) **should not be added**

Example: $\int_1^2 (x^2 - 5x + 2)dx$

Solution: $\int_1^2 (x^2 - 5x + 2)dx = \frac{x^3}{3} - \frac{5x^2}{2} + 2x$. Now, $\int_1^2 (x^2 - 5x + 2)dx = \left[\frac{x^3}{3} - \frac{5x^2}{2} + 2x\right]_1^2$

$$= \left[\frac{2^3}{3} - \frac{5x2^2}{2} + 2x2\right] - \left[\frac{1}{3} - \frac{5}{2} + 2\right] = -19/6$$

8.B.7 IMPORTANT PROPERTIES

Important Properties of definite Integral

(I) $\int_a^b f(x)dx = \int_a^b f(t)\,dt$

(II) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(III) $\int_a^b f(x)dx = \int_a^c f(x)\ dx + \int_c^b f(x)dx,\ a < c < b$

(IV) $\int_0^a f(x)\ dx = \int_o^a f(a-x)\ dx$

(V) When f(x) =f (a+x) then $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx$

(VI) $\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx$ if f(–x) = f(x)

= 0 if f(–x) = –f(x)

Example: $\int_0^2 \frac{x^2dx}{x^2+(2-x)^2}$

Solution: Let $I = \int_0^2 \frac{x^2dx}{x^2+(2-x)^2}$

$= \int_0^2 \frac{(2-x)^2\ dx}{(2-x)^2\ +x^2}$ [by prop. IV]

$\therefore\ 21 = \int_0^2 \frac{x^2dx}{x^2+(2-x)^2} + \int_0^2 \frac{(2-x)^2}{(2-x)^2+x^2}\ dx$

$\int_0^2 \frac{x^2+(2-x)^2}{x^2+(2-x)^2} dx$

$= \int_0^2 dx = [x]_0^2 = 2–0 = 2$

or I = 2/2 = 1

Example: Evaluate $\int_{-2}^{2} \frac{x^4\ dx}{a^{10}\ -\ x^{10}}$ $(a > 2)$

Solution: $\frac{x^4\ dx}{a^{10}\ -\ x^{10}} = \frac{x^4\ dx}{(\ a^5\ )^2\ -\ (x^5\ )^2}$

let $x^5 = t$ so that $5x^4\ dx = dt$

Now $\int \frac{x^4dx}{(a^5)^2-(x^5)^2}$

$= \frac{1}{5}\int \frac{5x^4dx}{(a^5)^2-(x^5)^2}$

$= \frac{1}{5}\int \frac{dt}{(a^5)^2-t^2}$

$$= \frac{1}{10a^5} \log \frac{a^5 + x^5}{a^5 - x^5} \qquad \text{(by standard formula b)}$$

Therefore, $\int_{-2}^{2} \frac{x^4 dx}{a^{10} - x^{10}}$

$$= 2\int_0^2 \frac{x^4 dx}{a^{10} - x^{10}} \qquad \text{(by prop. VI)}$$

$$= 2 \times \frac{1}{10a^5} \log\left[\frac{a^5 + x^5}{a^5 - x^5}\right]_0^2$$

$$= \frac{1}{5a^5} \log \frac{a^5 + 32}{a^5 - 32}$$

SUMMARY

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \; n \neq -1$ (If n=-1, $\frac{x^{n+1}}{n+1} = \frac{1}{0}$ which is not defined)
- $\int dx = x$, since $\int 1dx = \int x^{\circ}dx = \frac{x1}{1} = x + c$
- $\int e^x \, dx = e^x + c$, since $\frac{d}{dx} e^x = e^x$
- $\int e^{ax} \, dx = \frac{e^{ax}}{a} + c$, since $\frac{d}{dx}\left(\frac{e^{ax}}{a}\right) = e^{ax}$
- $\int \frac{dx}{x} = \log x + c$, since $\frac{d}{dx} \log x = \frac{1}{x}$
- $\int a^x \, dx = a^x / \log_e a + c$, since $\frac{d}{dx}\left(\frac{a^x}{\log_e^a}\right) = a^x$

  $\int c\, f(x)\, dx = c \int f(x)\, dx$ where c is constant.

  $\int \{ f(x)\, dx \pm g(x)\}\, dx = \int f(x)dx \pm \int g(x)dx$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$

- $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right) + c$
- $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + c$
- $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + c$
- $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + c$
- $\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c$

Integration by parts

Important Properties of definite Integral

- $\int_a^b f(x)dx = \int_a^b f(t)\,dt$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b$
- $\int_0^a f(x)\,dx = \int_o^a f(a-x)\,dx$
- When $f(x) = f(a+x)$ = $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$
- $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

EXERCISE 8(B) [K = CONSTANT]

Choose the most appropriate option (a) (b) (c) or (d).

1. Evaluate $= \int 5x^2\,dx$:

   (a) $5/3x^3 + k$ (b) $\frac{5x^3}{3} + k$ (c) $5x^3$ (d) none of these

2. Integration of $3 - 2x - x^4$ will become

   (a) $-x^2 - x^5/5$ (b) $3x - x^2 - \frac{x^5}{5} + k$ (c) $3x - x^2 + \frac{x^5}{5} + k$ (d) none of these

3. Given $f(x) = 4x^3 + 3x^2 - 2x + 5$ and $\int f(x)\, dx$ is

(a) $x^4 + x^3 - x^2 + 5x$ (b) $x^4 + x^3 - x^2 + 5x + k$

(c) $12x^2 + 6x - 2x^2$ (d) none of these

4. Evaluate $\int (x^2 - 1)\, dx$

(a) $x^5/5 - 2/3\, x^3 + x + k$ (b) $\frac{x^3}{3} - x + k$

(c) $2x$ (d) none of these

5. $\int (1-3x)(1+x)\, dx$ is equal to

(a) $x - x^2 - x^3$ (b) $x^3 - x^2 + x$ (c) $x - x^2 - x^3 + k$ (d) none of these

6. $\int \left[\sqrt{x} - 1/\sqrt{x}\right] dx$ is equal to

(a) $\frac{2}{3} x^{3/2} - 2\, x^{½} + k$ (b) $\frac{2}{3}\sqrt{x} - 2\sqrt{x} + k$ (c) $\frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}} + k$ (d) none of these

7. The integral of $px^3 + qx^2 + rk + w/x$ is equal to

(a) $px^2 + qx + r + k$ (b) $px^3/3 + qx^2/2 + rx$

(c) $3px + 2q - w/x^2$ (d) none of these

8. Use method of substitution to integrate the function $f(x) = (4x + 5)^6$ and the answer is

(a) $1/28\, (4x + 5)^7 + k$ (b) $(4x + 5)^7/7 + k$ (c) $(4x + 5)^7/7$ (d) none of these

9. Use method of substitution to evaluate $\int x(x^2 + 4)^5\, dx$ and the answer is

(a) $(x^2 + 4)^6 + k$ (b) $1/12\, (x^2 + 4)^6 + k$

(c) $(x^2 + 4)^6/ + k$ (d) none of these

10. Integrate $(x + a)^n$ and the result will be

(a) $\frac{(x+a)^{n+1}}{n+1} + k$ (b) $\frac{(x+a)^{n+1}}{n+1}$

(c) $(x + a)^{n+1}$ (d) none of these

11. $\int 8x^2/(x^3 + 2)^3\, dx$ is equal to

(a) $-4/3(x^3 + 2)^2 + k$ (b) $-\frac{4}{3(x^3+2)^2} + k$

(c) $\frac{4}{3(x^3+2)^2}+k$ (d) none of these

12. Using method of partial fraction the integration of f(x) when $f(x) = \frac{1}{x^2 - a^2}$ and the answer is

(a) $\log x - \frac{a}{x+a} + k$ (b) log (x – a) – log (x + a) + k

(c) $\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+k$ (d) none of these

13. Use integration by parts to evaluate $\int x^2 e^{3x} dx$

(a) $x^2 e^{3x}/3 - 2x e^{3x}/9 + 2/27\, e^{3x} + k$ (b) $x^2 e^{3x} - 2x\, e^{3x} + 2e^{3x} + k$

(c) $e^{3x}/3 - x\, e^{3x}/9 + 2e^{3x} + k$ (d) none of these

14. $\int$ logx dx is equal to

(a) x logx + k (b) $x \log x - x^2 + k$ (c) x logx + k (d) none of these

15. $\int xe^x dx$ is

(a) $(x-1)e^x + k$ (b) $(x-1)\, e^x$ (c) $x\, e^x + k$ (d) none of these

16. Evaluate $\int_0^1 (2x^2 - x^3)$ dx and the value is

(a) 4/3 + k (b) 5/12 (c) – 4/3 (d) none of these

17. Evaluate $\int_2^4 (3x-2)^2$ dx and the value is

(a) 104 (b) 100 (c) 10 (d) none of these.

18. Evaluate $\int_0^1 xe^x dx$ dx and the value is

(a) –1 (b) 10 (c) 10/9 (d) +1

19. $\int x^x (1 + \log x)\, dx$ is equal to

(a) $x^x \log x + k$ (b) $e^{x^2} + k$ (c) $\frac{x^2}{2} + k$ (d) $x^x + c$

20. If $f(x) = \sqrt{1 + x^2}$ then $\int f(x)dx$ is

(a) $\frac{2}{3}\, x\,(1 + x^2)^{3/2} + k$
(b) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log(x+\sqrt{x^2+1}) + k$
(c) $\frac{2}{3}\, x\,(1 + x^2)^{3/2} + k$
(d) none of these

21. $\int \frac{\sqrt{2}\left(x^2+1\right)}{\sqrt{x^2+2}}\,dx$ is equal to

(a) $\frac{x}{\sqrt{2}}\left(\sqrt{x^2+2}\right)+k$ (b) $\sqrt{x^2+2} + k$ (c) $1/(x^2 + 2)^{3/2} + k$ (d) none of these

22. $\int (e^x + e^{-x})^2 (e^x - e^{-x})\,dx$ is

(a) $\frac{1}{3}(e^x + e^{-x})^3 + k$
(b) $\frac{1}{2}(e^x - e^{-x})^2 + k$
(c) $e^x + k$
(d) none of these

23. $\int_0^a [\, f(x) + f(-x)\,]\,dx$ is equal to

(a) $\int_0^a 2\, f(x)\,dx$ (b) $\int_{-a}^a f(x)\,dx$ (c) 0 (d) $\int_{-a}^a - f(-x)\,dx$

24. $\int xe^x/(x+1)^2\,dx$ is equal to

(a) $e^x/(x + 1) + k$ (b) $e^x/x + k$ (c) $e^x + k$ (d) none of these

25. $\int (x^4 + 3/x)\,dx$ is equal to

(a) $x^5/5 + 3 \log |x|$
(b) $1/5\, x^5 + 3 \log |x| + k$
(c) $1/5\, x^5 + k$
(d) none of these

26. Evaluate $\int_1^4 (2x + 5)\,dx$ and the value is

(a) 3 (b) 10 (c) 30 (d) none of these

27. $\int_1^2 \frac{2x}{1+x^2}\,dx$ is equal to

(a) $\log_e (5/2)$
(b) $\log_e 5 - \log_e 2$
(c) $\log_e (2/5)$
(d) none of these

28. $\int_0^4 \sqrt{3x+4}\ dx$ is equal to

(a) 9/112 (b) 112/9 (c) 11/9 (d) none of these

29. $\int_0^2 \frac{x+2}{x+1} dx$ is

(a) $2 + \log_e 2$ (b) $2 + \log_e 3$ (c) $\log_e 3$ (d) none of these

e^2

30. The value of $\int_2^3 f(5-x)\,dx - \int_2^3 f(x)\,dx$ is

(a) 1 (b) 0 (c) –1 (d) none of these

31. $\int (x-1)e^x / x^2 dx$ is equal to

(a) $e^x/x + k$ (b) $e^{-x}/x + k$ (c) $-e^x/x + k$ (d) none of these

32. $\int_0^2 3x^2\ dx$ is

(a) 7 (b) –8 (c) 8 (d) none of these

33. Using integration by parts $\int x^3 \log x dx$

(a) $x^4/16 + k$

(b) $x^4/16\ (4 \log x - 1) + k$

(c) $4 \log x - 1 + k$

(d) none of these

34. Evaluate $\int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right) dx$ and the value is

(a) $\log_e |e^x + e^{-x}|$

(b) $\log e\ |e^x + e^{-x}| + k$

(c) $\log_e |e^x - e^{-x}| + k$

(d) none of these

35. If $f'(x) = x - 1$, the equation of a curve $y = f(x)$ passing through the point (1, 0) is given by

(a) $y = x^2 - 2x + 1$

(b) $y = x^2/2 - x + 1$

(c) $y = x^2/2 - x + 1/2$

(d) none of these

ANSWERS

Exercise 8(A)

1. (a)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (c)
9. (a)	**10.** (a)&(b)	**11.** (a)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (c)	**16.** (a)
17. (a)	**18.** (d)	**19.** (c)	**20.** (a)	**21.** (b)	**22.** (c)	**23.** (a)	**24.** (c)
25. (a)	**26.** (a)	**27.** (a)	**28.** (b)	**29.** (a)	**30.** (c)	**31.** (a)	**32.** (b)
33. (a)	**34.** (c)	**35.** (b)					

Exercise 8(B)

1. (b)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (a)	**7.** (d)	**8.** (a)
9. (b)	**10.** (a)	**11.** (b)	**12.** (c)	**13.** (a)	**14.** (d)	**15.** (a)	**16.** (b)
17. (a)	**18.** (d)	**19.** (d)	**20.** (b)	**21.** (a)	**22.** (a)	**23.** (b)	**24.** (a)
25. (b)	**26.** (c)	**27.** (a)	**28.** (b)	**29.** (b)	**30.** (b)	**31.** (a)	**32.** (c)
33. (b)	**34.** (b)	**35.** (c)					

8. C APPLICATION OF INTEGRATION TO COMMERCE AND ECONOMICS

We know that marginal function is obtained by differentiating the total function. Now, when Marginal function is given and initial values are given, then total function can be obtained with the help of integration.

8.C.1 Determination of cost function

If C denotes the total cost and $MC=\frac{dC}{dx}$ is the marginal cost, then we can write $C = C(x) = \int (MC)\,dx + k$, where k is the constant of integration, k, being the constant, is the fixed cost.

Example 8.C.1. The marginal cost function of manufacturing x units of a product is $5 + 16x - 3x^2$. The total cost of producing 5 times is Rs. 500. Find the total cost function.

Solution: Given, $MC = 5 + 16x - 3x^2$

$$\therefore \quad C(x) = \int (5 + 16x - 3x^2)\,dx$$

$$= 5x + 16\frac{x^2}{2} - 3.\frac{x^3}{3} + k$$

$$C(x) = 5x + 8x^2 - x^3 + k$$

When x = 5, C(x) = C(5) = Rs. 500

or, 500 = 25 + 200 – 125 + k

This gives, k= 400

$$\therefore \quad C(x) = 5x + 8x^2 - x^3 + 400$$

Example 8. C.2. The marginal cost (MC) of a product is given to be a constant multiple of number of units (x) produced. Find the total cost function, if fixed cot is Rs. 5000 and the cot of producing 50 units is Rs. 5625

Solution: Here $MC \propto x$ i.e $MC = k_1 x$ (k_1 is a constant)

$$\therefore \quad \frac{dC}{dx} = k_1 x \Rightarrow C = \int k_1 x\,dx + k_2$$

$$\therefore \quad C = k_1\frac{x^2}{2} + k_2$$

Since fixed cost = Rs 5000 $\therefore$ $x = 0 \Rightarrow C = 5000$

$$\therefore \quad 5625 = k_1\frac{2500}{2} + 5000$$

$$\Rightarrow \qquad 625 = 1250\ k_1 \Rightarrow k_1 = \frac{1}{2}$$

Hence $C = \frac{x^2}{4} + 5000$, is the required cost function.

8.C.3.2 Determination of Total Revenue Function

If R (x) denote the total revenue function and MR is the marginal revenue function, then

$$MR = \frac{d}{dx}[R(x)]$$

$\therefore \qquad R(x) = \int (MR)dx + k$ Where k is the constant of integration.

Also, when R (x) is known, the demand function can be found as $p = \frac{Rx}{x}$

Example 8.C.20 The marginal revenue function for a product is given by

$$MR = \frac{6}{(x\text{-}3)^2} - 4.$$

Find the total revenue function and the demand function.

Solution: $MR = \frac{6}{(x\text{-}3)^2} - 4$

$$\therefore \qquad R = \int \left|\frac{6}{(x\text{-}3)^2} - 4\right| dx = \frac{6}{x-3} - 4x + k$$

$X = 0, R = 0 \Rightarrow k = -2$

$R = \frac{6}{x\text{-}3} - 4x - 2$, which is the required revenue function.

Now, $PR = \frac{R}{x} = \frac{6}{x(x\text{-}3)} - 4 - \frac{2}{x}$

$$= -\frac{6}{x(x-3)} - \frac{2}{x} - 4$$

$$= \frac{-6-2x+6}{x(x-3)} - 4$$

$$= \frac{-2}{x-3} - 4 = \frac{2}{3-x} - 4$$

$\therefore$ The demand function is given by $p = \frac{2}{3-x} - 4.$

EXERCISE -8(C)

Choose the most appropriate option (a) (b) (c) or (d)

1. The fixed cost of a new product is Rs. 18000 and the variable cost per unit is Rs 550. If demand function p(x)=4000 -150x, find the break-even values.

   (a) 15, 8 (b) 7, 12 (c) 3, 17 (d) 5, 15

   Using the data (2-4) A company sells its product at Rs.60 per unit. Fixed cost for the company is Rs.18000 and the variable cost is estimated to be 25% of total revenue.

2. Determine: the total revenue function .

   (a) 70x (b) 60x (c) 90x (d) 100x

3. Determine the total cost function

   (a) 19000 + 6x (b) 20000+ 10 x (c) 18000 + 15x (d) 4000 + 5x

4. Determine the breakeven point

   (a) 600 (b) 400 (c) 700 (d) 1000

   Using the data (5-8) The total cost C(x) of a company as $C(x) = 1000 + 25x + 2x^2$ where x is the output.

5. Determine: the average cost

   (a) 1000/x + 25 + 2x (b) 1000/x + 20 + 2x

   (c) 1000/x + 30 + 3x (d) 1000/x + 25 + x

6. Determine the marginal cost.

   (a) 30 + 4x (b) 25 + 4x (c) 50 + 4x (d) 50 + 5x

7. Find the marginal cost when 15 units are produced,

   (a) 60 (b) 90 (c) 80 (d) 85

8. Find the actual cost of producing 15th unit.

   (a) 80 (b) 70 (c) 83 (d) 90

   Using the data (9-12).The total cost function of a firm is given as

   $C(x) = 0.002x^3 - 0.04x^2 + 5x + 1500$, *where x is the output*.

9. Determine: the average cost

   (a) $0.002x^2 – 0.04x + 5 + 1500/x$ (b) $0.002x^2 – 0.05x + 5 + 1500/x$

   (c) $0.002x^2 – 0.05x + 5 + 1000/x$ (d) $0.002x^2 – 0.05x + 5 + 500/x$

10. Determine the marginal average cost (MAC)

(a) $0.004x - 0.08 - 1500/x^2$ (b) $0.004x - 0.04 - 1500/x^2$

(c) $0.004x - 0.04 - 1000/x^2$ (d) $0.001x - 0.04 - 1500/x^2$

(11) Find the marginal cost.

(a) $0.06x^2 - 0.10x + 5$ (b) $0.06x^2 - 0.16x + 5$

(c) $0.06x^2 - 0.08x + 5$ (d) $0.05x^2 - 0.08x + 5$

(12) Find the rate of change of MC with respect to x.

(a) $0.012x - 0.10$ (b) $0.010x - 0.08$

(c) $0.012x + 0.08$ (d) $0.012x - 0.08$

13. The total cost function for a company is given by $C(x) = \frac{3}{4}x^2 - 7x + 27$. Find the level of output for which MC = AC

(a) 8 (b) 6 (c) 9 (d) 10

Using the data (14-17) The demand function for a monopolist is given by $x = 100 - 4p$, where x is the number of units of product produced and old and p is the price per unit.

14. Find total revenue function

(a) $25x - x^2/4$ (b) $25x + x^2/4$

(c) $25x - x^2/2$ (d) $5x - x^2/4$

15. Find average revenue function

(a) 25-x/6 (b) 25-x/4

(c) 5-x/4 (d) 25 + x/4

16. Find marginal revenue function

(a) 25-x/3 (b) 25-x/4 (c) 5-x/2 (d) 25-x/2

17. Find price and quantity at which MR= 0.

(a) 50, 12.5 (b) 70, 12.5 (c) 100, 12.5 (d) 70, 10

Using the data (18-21) A firm knows that the demand function for one of its products is linear. It also knows that it can sell 1000 units when the price is Rs. 4 per unit and it can sell 1500 units when the price is Rs. 2 per unit.

18. Find the demand function.

(a) 2000-250p (b) 2000-5p (c) 2000+5p (d) 2000-25p

19. Find the total revenue function

(a) $8 - x^2/250$ (b) $8x - x^2/50$ (c) $8x - x^2/250$ (d) $8x - x^2/25$

20. Find the average revenue function.

(a) 8-x/50 (b) 8-x/25 (c) 8+x/250 (d) 8-x/250

21. Find the marginal revenue function.

(a) $8 - x/12$ (b) $8 - x/25$ (c) $8 - x/125$ (d) $8 + x/125$

22. A company charge Rs. 15000 for a refrigerator on orders of 20 or less refrigerator. The charge is reduced on every set by Rs. 100 per piece for each piece ordered in excess of 20. Find the largest size order the company should allow so as to receive a maximum revenue.

(a) 85 (b) 80 (c) 100 (d) 70

23. A firm has the following demand and the average cost-functions:

$x = 480 - 20p$ and $AC = 10 + \frac{x}{15}$. Determine the profit maximizing output and price of the monopolist.

(a) 70, 25 (b) 60, 30 (c) 60, 25 (d) 70, 30

Using the data (24-25) The marginal cost of production is $MC = 20 - 0.04x + 0.003x^2$ where x is the number of units produced. The fixed cost is Rs. 7000.

24. Find the total cost function.

(a) $C = 20x - 0.02x^2 + 0.001x^3 + 7000$ (b) $C = -20x - 0.04x^2 + 0.001x^3 + 7000$

(c) $C = 20x + 0.02x^2 + 0.001x^3 + 7000$ (d) $C = 20x - 0.02x^2 + 0.001x^3 - 7000$

25. Find the average cost function.

(a) $AC = 20 - 0.02x + 0.001x^2 + \frac{7000}{x}$ (b) $AC = 20 - 0.02x + 0.001x^2 + \frac{7000}{x}$

(c) $AC = 20 - 0.02x + 0.001x^2 - \frac{7000}{x}$ (d) $AC = 20 + 0.02x + 0.001x^2 + \frac{7000}{x}$

Using the data (26-27) The marginal cost function of manufacturing x units of a product is given by $MC = 3x^2 - 10x + 3$. The total cost of producing one unit of the product is Rs. 7.

26. Find the total cost function

(a) $C = x^3 + 5x^2 + 3x + 7$ (b) $C = x^3 - 5x^2 + 3x + 7$

(c) $C = x^3 + 5x^2 - 3x + 7$ (d) $C = x^3 - 5x^2 - 3x - 7$

27. Find the average cost function.

(a) $AC = x^2 - 5x + 3 + \frac{7}{x}$ (b) $AC = x^2 - 5x + 3 - \frac{7}{x}$

(c) $AC = x^2 - 5x - 3 + \frac{7}{x}$ (d) $AC = x^2 + 5x - 3 + \frac{7}{x}$

Using the data (28-29). The marginal cost function of a commodity is given by $MC = \frac{14000}{\sqrt{7x+4}}$ and the fixed cost is Rs. 18000.

28. Find the total cost function.

(a) $C = 4000\sqrt{7x+4} + 10000$ (b) $C = 4000\sqrt{7x+4} - 10000$

(c) $C = 400\sqrt{7x+4} + 10000$ (d) $C = 4000\sqrt{7x^2+4} + 1000$

29. Find average cost of producing 3 units of the products.

(a) $AC = \frac{4000}{x}\sqrt{7x+2}\ \frac{10000}{x}$ (b) $AC = \frac{4000}{x}\sqrt{7x+4} + \frac{10000}{x}$

(c) $AC = \frac{4000}{x}\sqrt{7x+4} + \frac{10000}{x^2}$ (d) $AC = \frac{4000}{x}\sqrt{7x+4} + \frac{1000}{x}$

30. The marginal revenue of a function $MR = 7\text{-}4x\text{-}x^2$. Find the total Revenue.

(a) $R = 7x - \frac{4x^2}{2} - \frac{x^3}{3}$ (b) $R = 7x + \frac{4x^2}{2} - \frac{x^3}{3}$

(c) $R = 7x - \frac{4x^2}{2} + \frac{x^3}{3}$ (d) $R = 7x + \frac{4x^2}{2} + \frac{x^3}{3}$

ANSWERS

Set C

1 (a)	2 (b)	3 (c)	4 (b)	5 (a)	6 (b)
7 (d)	8 (c)	9 (a)	10 (b)	11 (c)	12 (d)
13 (b)	14 (a)	15 (b)	16 (d)	17 (a)	18 (a)
19 (c)	20 (d)	21 (c)	22 (a)	23 (c)	24 (a)
25 (a)	26 (b)	27 (a)	28 (a)	29 (b)	30 (a)

NOTES

NOTES