Computational Physics

Exercise 6 - Form factor of a two-boson bound state

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1. Form factor and partial wave expansion

A (handwritten) derivation of the term in task 1 and 2 is provided in a separated scan.

2. Implementation of form factor

The notebook from the lectures were used and extended at the end of the document.

For the implementation of the form factor we are interested in the arguments of the spatial wave function Ψ_{ll_z} and the spherical harmonics Y_{l,l_z}: For Ψ_{ll_z} we need the absolute value of $\vec{p'}$ and $\vec{p'} - \vec{q}/2$ and for Y_{l,l_z} the polar and azimuthal angle of $\vec{p'}$ and $\vec{p'} - \vec{q}/2$. From the vectors given in the exercise sheet we find

$$|\vec{p'}| = p' \tag{1}$$
$$|\vec{p'} - \vec{q}/2| = \sqrt{p'^2 + q^2/4 - p'x'q} \tag{2}$$
$$\phi = \arcsin{(y/x)} = 0 \qquad \text{for all vectors, since } y = 0 \tag{3}$$
$$\theta = \arccos{(x = \cos{(\theta)})} \qquad \text{for } \vec{p'} \tag{4}$$
$$\theta = \arccos{(z/r)} = \arccos{(z/\sqrt{z^2 + x^2})} \qquad \text{for } \vec{p'} - \vec{q}/2 \tag{5}$$

where for the last equation we have $x = p'\sqrt{1 - x'^2}$, $z = p'x' - q/2$ and in general $x' = \cos\theta$. The wave function for $\Psi_{ll_z}(p')$ was obtained from the TwoBody-class with grid points p', while $\Psi_{ll_z}(|\vec{p'} - \vec{q}/2|)$ was obtained via interpolation with the help of cubic splines. The splines were calculated for each value of x and p', such that we calculate $\Psi_{ll_z}(|\vec{p'} - \vec{q}/2|)$ right before the summation, with the corresponding weights to the integral. The spherical harmonics Y_{l,l_z} where simply evaluated by calculating the angles for different x and p' and plugging them right into the function.

3. Implementing

The implementation of the form factor can be seen in the notebook sent with this homework. It can be found at the end of the *TwoBody* class at the interpolation part with some comments, in order to explain the procedure.

4. Testing

In the following figure1 a plot of the form factor is presented for a q that varies from 0-10 and a value of $\Lambda = 1200$ and the rest of parameters were taken from the data given.

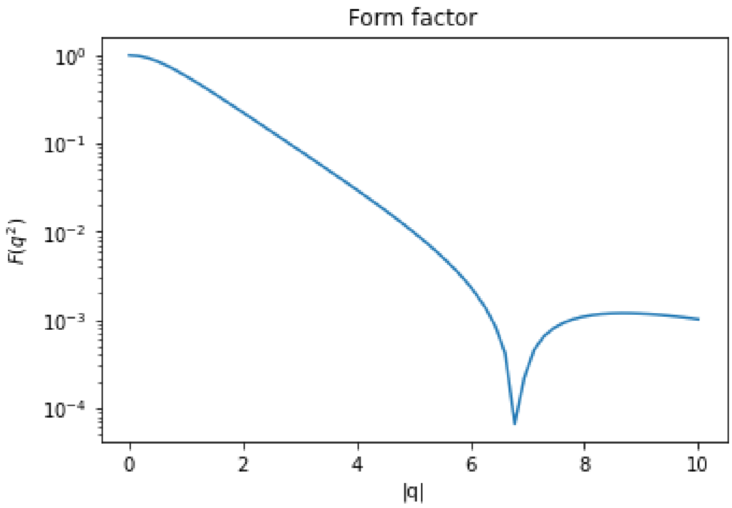

Figure 1: Form factor over q for $\Lambda = 1200$

The plot seems reasonable as it looks similar to the plots we have seen in the class.

5. $F(0) = 1$ and $< r^2 >$

A (handwritten) derivation of the equations is given in a separated scan.

6. Plotting for different Λ

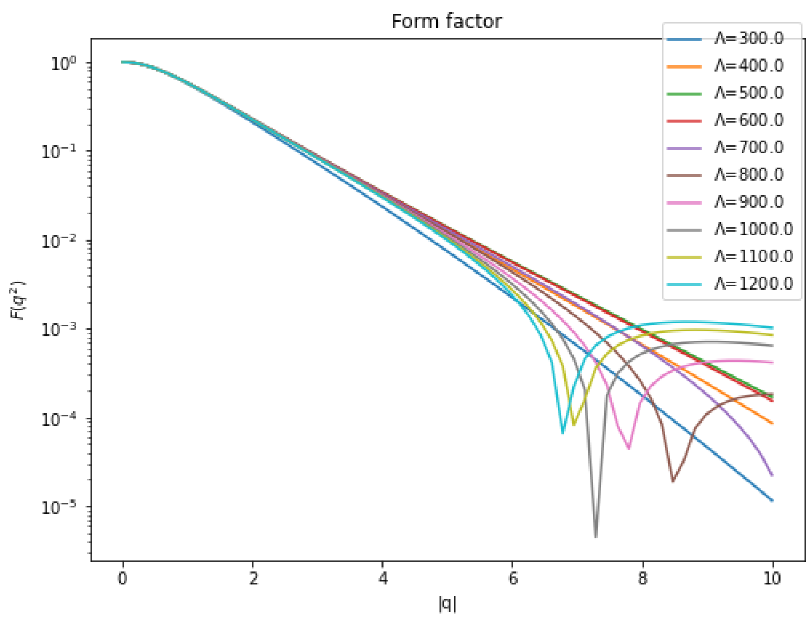

Figure 2: Form factor over q for several values of

We see that for low values of q the form factors behave similarly which is the expecting result, as the behavior or the 2-body system would not vary much when the momentum transfer is low. As we increase the value of the momentum transfer the system becomes more dependant on the cut-off. For low values of Λ(¡600) there is no singularity observed in the range examined. For higher values of Λ the system is more sensible on the momentum transfers at the point where there are specific values that can make the system "break", the singularity points.

A. Derivations

1) Show $\langle \psi \vec P' | \rho(\vec q) | \psi \vec P\rangle = F(\vec q^{\,2})\,\delta(\vec P' - \vec q - \vec P)$

use unity operator of Jacobi momentum eigenstates (see lecture 7, p.3)

$$\int d^3p'\, d^3P'\, |\vec p'\,\vec P'\rangle\langle \vec p'\,\vec P'|$$

• $\langle \psi \vec P'|\rho(\vec q)|\psi \vec P\rangle = \langle \psi\vec P'|\mathbb{1}\,\rho(\vec q)\,\mathbb{1}|\psi \vec P\rangle$

$$= \int d^3p'\,d^3P'\,d^3p\,d^3P\,\langle\psi\vec P'|\vec p'\vec P'\rangle\langle\vec p'\vec P'|\rho(\vec q)|\vec p\vec P\rangle\langle\vec p\vec P|\psi\vec P\rangle$$

$$= \int d^3p'\, \cancel{d^3P'}\,\cancel{d^3p}\,\cancel{d^3P}\,\langle\psi|\vec p'\rangle\underbrace{\langle\vec P'|\vec P'\rangle}_{1}\underbrace{\langle\vec p'|\vec p\rangle}_{1}\langle\vec P'|\rho(\vec q)|\vec P\rangle\langle\vec p|\psi\rangle\underbrace{\langle\vec P|\vec P\rangle}_{1}$$

integrate out

$$= \int d^3p'\,\langle\psi|\vec p'\rangle\underbrace{\langle\vec P'|\rho(\vec q)|\vec P\rangle}\langle\vec p|\psi\rangle$$

$$= \int d^3p'\,\psi^*(\vec p')\,\psi(\vec p)\,\delta(\vec P'-\vec q-\vec P)$$

~~$\vec p \to$ {rotational invariance}~~ $\vec p = +\frac{1}{2}(\vec k_1 - \vec k_2)$ ⊛

$\vec P = \vec k_1 + \vec k_2$
$\vec P' = \vec k_1' + \vec k_2$
$\vec p = 1/2\,(\vec k_1 - \vec k_2)$
$\vec p' = 1/2\,(\vec k_1' - \vec k_2)$
$\vec q = \vec k_1' - \vec k_1$

with $\vec p' = \frac{1}{2}(\vec k_1' - \vec k_2)$ and $-\frac{1}{2}\vec q = (\vec k_1 - \vec k_1')\,1/2$

$$\Rightarrow \left(\vec p' - \tfrac{1}{2}\vec q\right) = \tfrac{1}{2}(\cancel{\vec k_1'} - \vec k_2) + \tfrac{1}{2}(\vec k_1 - \cancel{\vec k_1'}) = \left(\tfrac{1}{2}(\vec k_1 - \vec k_2)\right) \circledast$$

$$\Rightarrow \int d^3p'\,\psi^*(\vec p')\,\psi(\vec p' - \tfrac{1}{2}\vec q)\,\delta(\vec P'-\vec q-\vec P)$$

$$= F(\vec q^{\,2})\,\delta(\vec P'-\vec q-\vec P)$$

2) Partial wave decomposition: $\psi(\vec p) = \sum_{lm}\psi_{lm}(p)\,Y_{lm}(\hat p)$

→ consider only component $l\,l_z$ (one sum)

→ integrate over $x = \cos\theta$

→ $\vec q = q\,e_z$ → l_z conserved

→ $d^3p' = dp'\,p'^2\,d\varphi\,d\theta\,\sin\theta$ $\qquad \left|\frac{dx}{d\theta}\right| = \sin\theta \Leftrightarrow d\theta = \frac{dx}{\sin\theta}$

$\qquad = dp'\,p'^2\,d\varphi\,dx$

Figure 3: First page of derivations

$$\Rightarrow F(\vec{q}^{\,2}) = \int dp'\, p'^2 \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_{-1}^{1} dx\; \psi^*(\vec{p}\,')\, \psi(\vec{p}\,' - \vec{q}/2)$$

$$= 2\pi \int dp'\, p'^2 \int_{-1}^{1} dx\; \psi^*_{l l_z}(|\vec{p}\,'|)\, \psi_{l l_z}(|\vec{p}\,' - \vec{q}/2|)\, Y^*_{l l_z}(\hat{\vec{p}}\,')\, Y_{l l_z}(\widehat{\vec{p}\,' - \vec{q}/2})$$

5) Show $F(0) = 1$ & $\left.\frac{\partial F(\vec{q}^{\,2})}{\partial \vec{q}^{\,2}}\right|_{q^2=0} = -\frac{1}{6}\langle r^2\rangle$

$$F(0) = 2\pi \int dp'\, p'^2 \int_{-1}^{1} dx\; Y^*_{l l_z}(\hat{\vec{p}}\,')\, Y_{l l_z}(\hat{\vec{p}}\,')\, \psi^*_{l l_z}(|\vec{p}\,'|)\, \psi_{l l_z}(|\vec{p}\,'|)$$

$$= \underbrace{\int dp'\, p'^2\, |\psi_{l l_z}(|\vec{p}\,'|)|^2}_{=1} \cdot \underbrace{\int_{-1}^{1} dx\, 2\pi\, |Y_{l l_z}(\hat{\vec{p}}\,')|^2}_{=1} = 1$$

$$F(\vec{q}^{\,2}) = \int d^3r\, \exp(-i\vec{r}\vec{q})\, F(\vec{r}) \qquad \text{(Fourier-Trafo)}$$

$$\vec{r}\cdot\vec{q} = r\cdot q\cdot \cos\theta$$

expand for small q

$$\exp(-i\vec{r}\vec{q}) = 1 - i\vec{r}\vec{q} - \frac{1}{2}(\vec{r}\vec{q})^2 + \dots$$

$$\Rightarrow F(\vec{q}^{\,2}) \approx \int d^3r\, \left(1 - i\vec{r}\vec{q} - \frac{1}{2}(\vec{r}\vec{q})^2 + \dots\right) F(\vec{r})$$

$$= \int d^3r\, F(r) - i\int d^3r\, \vec{r}\cdot\vec{q}\, F(r) - \frac{1}{2}\int d^3r\, r^2 q^2 \cos^2\theta\, F(\vec{r})$$

$$\Rightarrow \left.\frac{\partial F(\vec{q}^{\,2})}{\partial q^2}\right|_{q^2=0} \approx -\frac{1}{2}\int d^3r\, r^2 \cos^2\theta\, F(\vec{r})$$

$$d^3r = r^2 \sin\theta\, d\varphi\, d\theta\, dr$$

$$= -\frac{1}{2}\int dr\, r^4 \underbrace{\int d\varphi}_{2\pi} \underbrace{\int_{1}^{-1} dx\, x^2}_{2/3}\, F(\vec{r})$$

$$x = \cos\theta \qquad dx = -\sin\theta\, d\theta$$

$$d\theta = -\frac{dx}{\sin\theta}$$

$$-r^2\, d\varphi\, dx\, dr$$

$$= -\frac{1}{6}\underbrace{\int dr\, r^4\, 4\pi\, F(\vec{r})}_{=:\langle r^2\rangle} = -\frac{1}{6}\langle r^2\rangle$$

Figure 4: Second page of derivations