Mathematical Induction and Binomial theorem

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I. E - Subjective Problems

1) Given that (1979)

$$C_1 + 2C_2x + 3C_3x^2 + ... + 2nC_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$$

where $C_r = \frac{(2n)!}{r!(2n-r)!}$ r=0,1,2,..........,2n
Prove that

$$C_1^2 - 2C_2^2 + 3C_3^2 - .... - 2nC_{2n}^2 = (-1)^n nC_n.$$

2) Prove that $7^{2n} + \left(2^{3n-2}\right)\left(3^{n-1}\right)$ is divisible by 25 for any natural number n (1982-5M)
3) If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then show that the sum of products of C_i's taken two at a time, represented by $\sum\sum_{0\le i<j\le n} C_iC_j$ is equal to 2^{2n-1}-$\frac{(2n)!}{2(n!)^2}$ $(1983-3M)$
4) Use mathematical Induction to prove : If n is any odd positive integer , then $n\left(n^2-1\right)$ is divisible by 24. $(1983-2M)$
5) If p be a natural number then prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by p^2+p+1 for every positive integer n. $(1984-4M)$
6) Given $s_n = 1 + q + q^2 + ... + q^n$;
$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n, q \neq 1$.
Prove that

$$^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 + ... + {}^{n+1}C_ns_n = 2^nS_n$$

$(1984-4M)$

7) Use method of mathematical Induction
$2.7^n + 3.5^n - 5$ is divisible by 24 for all $n > 0$
$(1985-5M)$
8) Prove by mathematical induction that -
$\frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{(3n+1)^{\frac{1}{2}}}$ for all postive Integers n.
$(1987-3M)$
9) Let $R = \left(5\sqrt{5}+11\right)^{2n}$ and $f = R - [R]$, where $[d]$enotes the greatest integer function.Prove that $Rf = 4^{2n+4}$ $(1988-5M)$
10) Using mathematical induction,prove that

$$^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + ... + {}^mC_k{}^nC_0 = {}^{m+k}C_k$$

$(1989-3M)$

11) Prove that $(1989-5M)$

$$C_0 - 2^2C_1 + 3^2C_2 - .... + (-1)^n(n+1)^2C_n = 0$$

$,n > 2$, where $C_r = {}^nC_r$
12) Prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every positive integer n. $(1990-2M)$
13) Using induction or otherwise , prove that for any non-negative integers $m, n, randk$,

$$\sum_{r=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}[\frac{n}{r+1} - \frac{k}{r+2}]$$

$(1991-4M)$

14) If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$ then show that $b_n = {}^{2n+1}C_{n+1}$ $(1992-6M)$
15) Let $p \le 3$ be an integer and α,β be the roots of
$x^2 - (p+1)x + 1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$
(i) is an integer and (ii) is not divisible by p $(1992-6M)$