Mathematical Induction and Binomial theorem

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I. E - Subjective Problems

1) Given that (1979)

$$C_1 + 2C_2x + 3C_3x^2 + ...+ 2nC_{2n}x^{2n-1} = 2n(1 + x)^{2n-1}$$

where $C_r = \frac{(2n)!}{r!(2n-r)!}$ $r = 0, 1, 2, ..., 2n$
Prove that

$$C_1^2 - 2C_2^2 + 3C_3^2 - .... - 2nC_{2n}^2 = (-1)^n nC_n.$$

2) Prove that $7^{2n} + \left(2^{3n-2}\right)\left(3^{n-1}\right)$ is divisible by 25 for any natural number n (1982-5M)
3) If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then show that the sum of products of C_i's taken two at a time, represented by $\sum\sum_{0\le i<j\le n} C_iC_j$ is equal to 2^{2n-1}-$\frac{(2n)!}{2(n!)^2}$ (1983-3M)
4) Use mathematical Induction to prove : If n is any odd positive integer , then $n\left(n^2 - 1\right)$ is divisible by 24. (1983-2M)
5) If p be a natural number then prove that $p^{n+1} + (p + 1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n. (1984-4M)
6) Given $s_n = 1 + q + q^2 + ... + q^n$; $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n, q \neq 1$. Prove that

$${}^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 + ... + {}^{n+1}C_ns_n = 2^nS_n$$

(1984-4M)
7) Use method of mathematical Induction $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n > 0$ (1985-5M)
8) Prove by mathematical induction that - $\frac{(2n)!}{2^{2n}(n!)^2} \le \frac{1}{(3n+1)^{\frac{1}{2}}}$ for all postive Integers n.(1987-3M)
9) Let $R = \left(5\sqrt{5} + 11\right)^{2n}$ and $f = R - [R]$, where [] denotes the greatest integer function.Prove that $Rf = 4^{2n+4}$ (1988-5M)
10) Using mathematical induction,prove that

$${}^mC_0{}^nC_k + {}^mC_1{}^nC_{k-1} + ... + {}^mC_k{}^nC_0 = {}^{m+k}C_k$$

(1989-3M)
11) Prove that (1989-5M)

$$C_0 - 2^2C_1 + 3^2C_2 - .... + (-1)^n(n + 1)^2C_n = 0$$

$,n > 2$, where ${}^nC_r$
12) Prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every positive integer n. (1990-2M)
13) Using induction or otherwise , prove that for any non-negative integers m, n, r and k,

$$\sum_{r=0}^{k}(n - m)\frac{(r + m)!}{m!} = \frac{(r + k + 1)!}{k!}[\frac{n}{r + 1} - \frac{k}{r + 2}]$$

(1991-4M)
14) If $\sum_{r=0}^{2n} a_r(x - 2)^r = \sum_{r=0}^{2n} b_r(x - 3)^r$ and $a_k = 1$ for all $k \ge n$ then show that $b_n = {}^{2n+1}C_{n+1}$ (1992-6M)
15) Let $p \le 3$ be an integer and α, β be the roots of $x^2 - (p + 1)x + 1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$
(i) is an integer and (ii) is not divisible by p (1992-6M)