AI24BTECH11010 - Golla Shriram

Question:

If the area of the region bounded by the curve $y^2 = 4ax$ and line x = 4a is $\frac{256}{3}$ sq.units, then using integration, find the value of a, where a > 0. **Solution:**

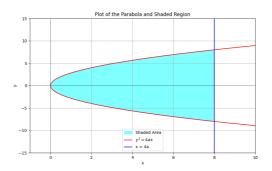


Fig. 0.1: Area bounded by $y^2 = 4ax$ and x = 4a

The given parabola can be expressed as conics with parameters

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix}, f = 0 \tag{0.1}$$

For line x = 4a the parameters are

$$L: \mathbf{x} = \mathbf{h} + \kappa \mathbf{m}, \kappa \in R \tag{0.2}$$

$$\mathbf{h} = \begin{pmatrix} 4a \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.3}$$

To points of intersection of line with conic section

$$\mathbf{x_i} = \mathbf{h} + \kappa_i \mathbf{m} \tag{0.4}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^{\top} \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^{\top} (\mathbf{V} \mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^{\top} (\mathbf{V} \mathbf{h} + \mathbf{u})]^2 - g(\mathbf{h})(\mathbf{m}^{\top} \mathbf{V} \mathbf{m})} \right)$$
(0.5)

Substituting from the above, we get

$$\kappa_i = 4a, -4a \tag{0.6}$$

yeilding the points of intersections

$$\mathbf{x_1} = \begin{pmatrix} 4a \\ 4a \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 4a \\ -4a \end{pmatrix} \tag{0.7}$$

From figure the area bounded by the curve $y^2 = 4ax$ and line x = 4a is given by

$$4\sqrt{a} \int_0^{4a} \sqrt{x} \, dx = \frac{256}{3} \tag{0.8}$$

by solving above equation the value of a = 2