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Volatility

Implied Volatility

Consider a European call option with strike K and expiry T on an underlying whose risk neutral dynamics are described by

$$dS/S = rdt + \sigma dB. \tag{1}$$

The Black-Scholes price of this option c_t(K,T) at time t is given by

$$c_t(K,T) = exp(-r\tau)\mathbf{E}\left[(S_T - K)^+\right] \tag{2}$$

where $\tau = T - t$. All data needed to compute this price is agreed on and publically available, with the exception of σ, the volatility of the underlying asset. As such all options are parameterised by σ - the only piece of the Black-Scholes pricing setup upon which traders may disagree. It can be shown that price c_t is a monotonic (increasing) function of σ so that Black-Scholes options can be quoted interchangeably either as a price or as a volatility. Given an observed price $\widehat{c}_t(K,T)$ in the market, there is a unique volatility $\sigma = \Sigma$ which produces the observed price when inserted in the above Black-Scholes pricing formula. This volaility Σ is the *implied volatility*.

Generally we have a grid of prices for each set of strikes and expiries. By the above logic all option these options are parameterised by the same underlying volatility σ. But why then do we observe different implied volatilities across the grid? Typically we interpret this as a mix of supply/demand issues (eg large fund managers who are net long equities may be willing to pay a premium - a higher implied volatility than expected - to buy some insurance against large equity price falls), or psychological factors like weighting the impact of losses more than gains. However, there may be something more that can be said here.

Local Volatility

Assume that in reality the underlying evolves according to a more complex diffusion process

$$dS/S = rdt + \sigma(S,t)dB \tag{3}$$

so that variance is a *deterministic* function of S and t over the life of the option. Such an asset will experience many different volatility regimes during the evolution $s_t \to s_T$ to expiry. These volatilities are referred to as *local volatility*, as they are different across local regions in S, t space.

Given a current value s_t of the underlying, paths $s_t \to s_T$ that end in the money (and therefore contribute to the expectation determining the Black-Scholes price) are different for options with different strike K and expiry T. Therefore, they will experience different values of $\sigma(s,t)$. The *implied volatility* of this option is the Black-Scholes volatility which reflects this local volatility experience. This is another sense in which it maybe be intuitive for different options $c_t(K,T)$ to have different *implied volatility* - traders are trying to account for possible local volatility regimes price paths that end in the money are likely to experience. It can be shown that the implied volatility $\Sigma(K,T)$ of the option with strike K and expiry T is *approximately* the average local volatility experienced by price paths between s_t and K, where the underlying expires in the money.

Local Volatility from Implied Volatility

It is well known to traders that butterflies

$$\text{Butterfly}(K_2) = c(K_1) - 2c(K_2) + c(K_3) \tag{4}$$

where $K_1 < K_2 < K_3$ are three evenly spaced strikes all at a particular expiry T, give insight into the risk neutral distribution of the underlying S_T at expiry. The Breeden-Litzenberger formula formalises this idea

$$p(s_T = K, T) = exp(-r\tau)\frac{\partial^2 c_t(K,T)}{\partial K^2} \tag{5}$$

. Critically, the left hand side $p(s_T = K, T)$ is the risk neutral probability that the underlying S_t expires at $S_T = K$. This motivates using derivatives of call prices in strike space K to provide information about the underlying dynamics of S_t. This is the insight in Dupire (1994) and the well known PDE he derives.

I find the derivation in Derman & Miller (2016) the most intuitive, and following it is insightful. This is especially true for our look at stochastic volatility models. That said, I will just include the tldr here for brevity. They specify a very flexible local volatility process

$$\begin{aligned} dS/S &= rdt + \sigma(s,t,\ldots)dB \\ &= rdt + \sigma(s,t,\varepsilon)dB \end{aligned}$$

where ε contains other variables in addition to S and t which affect local volatility, possibly including other stochastic processes. Starting at the usual pricing equation

$$c_t(K,T) = exp(-r\tau)\mathbf{E}\left[(S_T - K)H(S_T - K)\right] \tag{6}$$

where $\mathbf{E}$ is a risk neutral expectation over S_T *and* any other stochastic variables that may be in ε. The objective here is to take the *total derivative* of $c_t(K,T)$ with respect to T while keeping strike K constant. This induces a dT as well as a stochastic change dS_T. At a later date I will write notes on the derivation, but that this point it is enough to state that we arrive at the Dupire PDE

$$\frac{\partial C}{\partial T}|_K = -rK\frac{\partial C}{\partial K}|_T + \frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}|_T\mathbf{E}\left[\sigma(S,T,\varepsilon)\right] \tag{7}$$

This PDE should be solved starting at expiry, so has *initial condition* $C = \max(S_T - K, 0)$. The expectation is the diffusion coefficient which must be positive for the PDE to be meaningful.

How is the PDE useful? Typically we rearrange

$$\frac{1}{2}\mathbf{E}\left[\sigma(S,T,\varepsilon)\right] = \frac{\frac{\partial C}{\partial T}|_K + rK\frac{\partial C}{\partial K}|_T}{K^2\frac{\partial^2 C}{\partial K^2}|_T} \tag{8}$$

which is an incredibly useful representation, given that we can (essentially) read off all terms on the right hand side using observed market prices (we have to deal with the problem that prices are only available at discrete points in K, T). We know that prices and implied vols are interchangeable, so it is an exercise in chain rule to derive a similar equation using observed implied volatility.