Proposal 1

Parameter Learning in a Local Volatility Model Using a Particle Filter

In this project we use underlying prices and observed option data to make *online* inference about a local volatility model. We use a parametric model for the local volatility function, and apply a particle filter to construct iterative approximations of the posterior distribution of the parameters on arrival of new underlying and option data. This work is perhaps closest in spirit to Hamida & Cont (2005), but it differs in the sense that they solve a pure calibration problem rather than approximate the unknown parameter's posterior.

Problem Outline

Part 1: Background

Not Included.

Part 2: Implementation Details

To make inference about the parameters θ in $\sigma_{\text{LV}}(S, t; \theta)$, we apply a particle filter to iteratively construct an approximation for the posterior distribution of the parameters on arrival of new underlying and options data.

Define $c_t^M(K, T; \theta)$ to be a model generated call price at time t with strike K and expiry T given a particular set of local volatility parameters θ. These model prices can be computed by solving either the Dupire Equation, or the Black-Scholes Equation, and are comparable to market observed prices $c_t(K, T)$ also at time t. To construct a dynamic system define $s_{1:t} = \{s_1, s_2, \ldots, s_t\}$ to be a set of observed underlying prices which are assumed to have been generated from

$$dS/S = rSdt + \sigma_{\text{LV}}(S, t; \theta)\, dB \tag{1}$$

.

This explicit dependance of s_t on θ is hereafter ommitted for clarity. At each t we also have a set of N observed call prices $\{c_t(K_i, T_i)\}_{i=1}^N$, where K_i and T_i are the strike and expiry of each call in the market (ie $N = K \times T$ where we have K strikes and T expiries).

It is conveniant to package the observed quantities into the set $x_{1:t}$, where at each time t

$$x_t = [s_t, c_t(K_1, T_1; \theta), \ldots, c_t(K_N, T_N; \theta)] \tag{2}$$

The objective of the project is to compute the posterior $p(\theta|x_{1:t})$. To do so we construct a particle approximation for $p(\theta|x_{1:t+1})$ where the weight $\pi_{t+1}^j = p(\theta^j|x_{1:t+1})$ is updated according to the standard rule

$$\pi_{t+1}^j = \frac{p(x_{t+1}|x_t, \theta^j)\pi_t^j}{\sum_j p(x_{t+1}|x_t, \theta^j)\pi_t^j}. \tag{3}$$

and $j = 1, 2, \ldots, J$ denotes a suitably large number of particles in the approximation. The design of the filter can follow Gellert & Schlogl 2018.

To compute $p(x_{t+1}|x_t, \theta^j)$ we follow Hurn Et Al 2012 and note that the likelihood $p(x_{t+1}|x_t, \theta)$ can be written

$$p(x_{t+1}|x_t, \theta) = f(s_{t+1}|s_t, \theta) \prod_{i=1}^N g\left(c_{t+1}(K_i, T_i)|c_{t+1}^M(K_i, T_i), \theta\right) \tag{4}$$

as x_t is fully observed and each option privides independant piece of information. The distribution $g(.)$ is discussed below.

Note that $s_{t+1}|s_t$ is also conditional on θ via the local volatility function $\sigma_{\text{LV}}(S, t; \theta)$ and is distributed according to Equation (1). The transition pdf required to compute the likelihood $f(s_{t+1}|s_t; \theta)$ satisfies a Fokker-Planck Equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial s}\left[\frac{1}{2}\frac{\partial}{\partial s}\left(\sigma^2_{\text{LV}}(s, t; \theta) f\right) - rsf\right] \tag{5}$$

with initial condition $f(s, t_0; \theta) = \delta(s - s_0)$ and relevant boundary conditions. In general this PDE will not have an analytic solution but can be approximated to high accuracy (insert econometric references here).

The distribution $g\left(c_{t+1}(K, T)|c^M_{t+1}(K, T), \theta\right)$ is problematic. Since all option prices are known when conditioned on the underlying and local volatility parameters θ, the problem is overspecified. This motivates the use of a *pricing error* model, which might be either multiplicative or additive (insert references here). For example we might take

$$g\left(c_{t+1}(K, T)|c^M_{t+1}(K, T), \theta\right) \sim \mathbf{N}(c^M_{t+1}(K, T), \sigma^2_M) \tag{6}$$

where σ_M is the spread for each option.

This fully specifies a procedure for computing $p(\theta|x_{1:t})$ via a particle filter, where $x_{1:t}$ includes both underlying and option prices.

Part 3: Reformulation Using Arbitrage Free Implied Volatility Surfaces

A slight reformulation of the problem allows us to use volatility surfaces that are guaranteed to be free from static arbitrage by construction. Previously, we have discussed making inference about the parameters θ in a local volatility function $\sigma_{\text{LV}}(S, t; \theta)$. Instead we can make inference about the implied volatility surface while restricting attention to surfaces free from static arbitrage. From such a surface, we can compute $\sigma_{\text{LV}}(K, T)$ via Dupire's Formula (in the form for implied volatility rather than prices).

Define $\Sigma_t(K_i, T_i)$ to be the Black-Scholes implied volatility of a European call option with price $c_t(K_i, T_i)$ at time t. The collection of $\Sigma_t(K, T)$ for all options in the market defines the implied volatility surface. An equivalent formula to Dupire's Equation is available so that local volatilities can be computed from implied volatilities.

Previously we assumed that $\sigma_{\text{LV}}(S, t; \theta)$ was parameterised by a set θ, which we made inference about using a particle filter. Here, instead we parameterise the implied volatility surface with a set α in $\Sigma(K, T; \alpha)$. We will restrict attention to particular surfaces $\Sigma(K, T; \alpha)$ guaranteed to be free from static arbitrage by construction, such as those introduced in (insert Gatheral references). This is an easier way to enforce no-arbitrage requirements than more procedural methods (eg Fengler 2009). This formulation has a secondary benefit of exploiting rapid closed form solutions for model prices using the standard Black-Scholes formula.

Here I will outline the new procedure more clearly. Define $c^M_t(K, T; \alpha)$ to be the Black-Scholes price at time t of a European call with strike K and expiry T, whose Black-Scholes volatility σ is given by $\sigma = \Sigma(K, T; \alpha)$. These call prices are free of static arbitrage, since $\Sigma(K, T; \alpha)$ is free of static arbitrage by construction. In turn, inference about α defines $\Sigma(K, T; \alpha)$, from which $\sigma_{\text{LV}}(K, T)$ can be calculated easily via Dupire's formula. As above, we construct the filter

$$\pi_{t+1}^{j} = \frac{p(x_{t+1}|x_t, \alpha^j)\pi_t^j}{\sum_j p(x_{t+1}|x_t, \alpha^j)\pi_t^j} \tag{7}$$

following Gellert & Schlogl (2018). A secondary advantage is that by definition $c_t^M(K, T; \alpha)$ is available in closed form via

$$c_t^M(K, T; \alpha) = \text{Call}_{\text{BS}}\left(K, T, \sigma = \Sigma(K, T; \alpha)\right) \tag{8}$$

when computing the option observation term in the likelihood

$$p(x_{t+1}|x_t, \alpha) = f(s_{t+1}|s_t, \alpha) \prod_{i=1}^{N} g\left(c_{t+1}(K_i, T_i)|c_{t+1}^M(K_i, T_i), \alpha\right) \tag{9}$$

Some care is required when computing $f(s_{t+1}|s_t; \alpha^j)$ for the j^{th} particle. Remember that the α refers to parameters defining the implied volatility surface, which does not factor directly in the model for the underlying process

$$dS/S = rdt + \sigma_{\text{LV}}(S, t)\, dB \tag{10}$$

The way to handle this will depend on the approximation techique used, but I don't foresee a problem here. For example, if we exploit the Euler approximation

$$S_{t+1} = S_t + rS_t\Delta + \sigma_{\text{LV}}(S_t, t)\Delta\varepsilon_t \tag{11}$$

we only need a single value for $\sigma_{\text{LV}}(S_t, t)$ and this value can be computed using Dupire's equation applied at the point $\Sigma(K = S_t, T = t; \alpha^j)$.

References

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