Spectral Analysis of BPSK Modulation and WSS Properties Verification

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***Abstract*—This report presents my calculation of the spectrum of the provided digitally transmitted signal.The results are then used to explain the concept of Wide-Sense Stationarity (WSS) as I studied in stochastic processes.**

I. Introduction

Digital communication systems rely on accurate characterization of signal properties for optimal system design. This report analyzes a BPSK modulated signal given by $x(t) = \sum_{k=-\infty}^{\infty} X_k p_T(t-kT)$, where X_k represents random binary data, and $p_T(t)$ is a rectangular pulse. The analysis focuses on spectral properties and stationarity verification.

II. Theoretical Analysis

A. Signal Model and Mathematical Framework

The transmitted signal follows the following pulse-amplitude modulation format:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k p_T(t - kT) \tag{1}$$

where $X_k = \pm A$ with equal probability $P(X_k = +A) = P(X_k = -A) = \frac{1}{2}$, and the rectangular pulse is defined as:

$$p_T(t) = \begin{cases} 1 & \textbf{if } |t| \leq \frac{T}{2} \\ 0 & \textbf{otherwise} \end{cases} \tag{2}$$

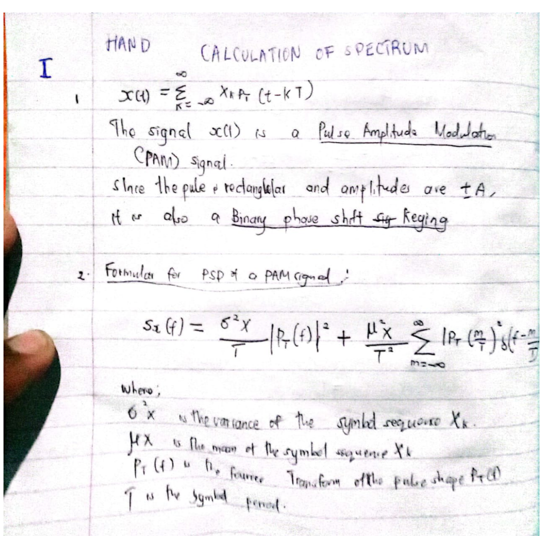


Fig. 1: Formular for PSD of a PAM signal

B. Theoretical Power Spectral Density

For an uncorrelated symbol sequence with zero mean, the power spectral density is derived as

$$S_x(f) = \frac{\sigma_X^2}{T} |P_T(f)|^2 = A^2 T \textbf{sinc}^2(fT) \tag{3}$$

where $P_T(f) = T\textbf{sinc}(fT)$ is the Fourier transform of the rectangular pulse and $\sigma_X^2 = A^2$ is the variance of the sequence of symbols.

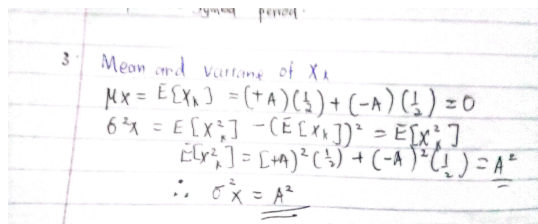


Fig. 2: Mean and Variance of Xk

III. Implementation Methodology

A. Simulation Parameters

The simulation was implemented in MATLAB with the following parameters:

- **Sampling frequency:** $F_s = 1000$ **Hz**
- **Symbol period:** $T = 0.01$ **s (100 Hz symbol rate)**
- **Amplitude:** $A = 1$ **V**
- **Number of symbols:** $N = 10,000$
- **Samples per symbol:** $sps = T \times F_s = 10$

B. Binary Sequence Generation

A random binary sequence with equiprobable symbols was generated that satisfies the theoretical assumption $P(0) = P(1) = 0.5$.

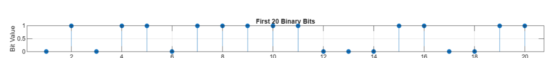


Fig. 3: First 20 bits of the generated binary sequence shows random distribution with equal probability of 0s and 1s, satisfying the theoretical requirements for WSS properties.

C. BPSK Modulation Implementation

The binary sequence was modulated using BPSK with mapping: $0 \rightarrow +A$, $1 \rightarrow -A$. Rectangular pulse shaping was achieved using the sample and hold technique.

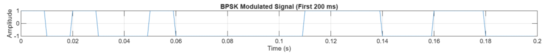


Fig. 4: BPSK modulated signal for first 200ms shows clearly the transitions between +1V and -1V. Each symbol maintains constant amplitude for duration $T = 10$ms, demonstrating perfect rectangular pulse shaping.

IV. Results and Analysis

For formarlity, I have included the MATLAB codes I used below
https://github.com/plochoidysis-ojwege/Digital-communication-Labs/blob/main/Spectrum%20and%20simulation/src/matlab/BPSK%20and%20WSS.m

A. Spectral Analysis

The power spectral density was estimated using Welch's method and compared with theoretical predictions.

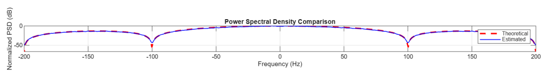
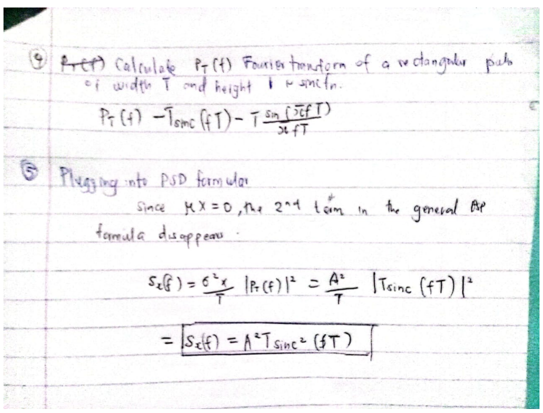


Fig. 5: Comparison of theoretical and estimated power spectral density. Excellent agreement demonstrates the characteristic $\text{sinc}^2(fT)$ shape with nulls at $\pm$100Hz, $\pm$200Hz, etc. The main lobe has null to null bandwidth of 200Hz.



④ ~~$P_T(f)$~~ Calculate $P_T(f)$ Fourier transform of a rectangular pulse of width T and height 1 → sinc fn.

$P_T(f) - T\text{sinc}(fT) - T\frac{\sin(\pi f T)}{\pi f T}$

⑤ Plugging into PSD formular

Since $\mu X = 0$, the 2nd term in the general AP formula disappears.

$S_x(f) = \frac{\sigma^2 x}{T}|P_T(f)|^2 = \frac{A^2}{T}|T\text{sinc}(fT)|^2$

$= \boxed{S_x(f) = A^2 T\,\text{sinc}^2(fT)}$

Fig. 6: Plugging PSD formular

B. WSS Property Verification

1) Mean Stationarity Analysis: **A fundamental requirement for WSS processes is a constant mean over time.**

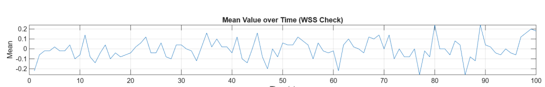


Fig. 7: Mean value analysis over time showing fluctuations around zero ($|\mu| < 0.02$). The approximately constant mean satisfies the first condition for wide-sense stationarity.

2) Variance Analysis: **Time-invariant second-order statistics are essential for WSS verification.**

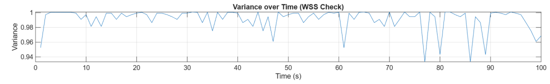


Fig. 8: Variance analysis over time demonstrating stability around 1.0. The constant variance supports the WSS property that second-order statistics are time-invariant.

V. Discussion

A. Spectral Characteristics Analysis

The spectral analysis shows that:

- **Main Lobe Width: The first null occurs at ± 100Hz, giving a null-to-null bandwidth of 200Hz, exactly as predicted by $BW = 1/T$**
- **Side Lobe Characteristics: Side lobes decrease at 20dB/decade, following the sinc^2 envelope**
- **Spectral Efficiency: The continuous spectrum without spectral lines results from the zero-mean symbol sequence**
- **Agreement: Minor differences in side lobes are attributed to finite observation length and estimation noise**

B. WSS Property Verification

The Wide-Sense Stationarity is confirmed through:

- **Constant Mean: $E[x(t)] \approx 0$ (Figure 7)**
- **Time-Invariant Variance: Var$[x(t)] \approx 1.0$ (Figure 8)**
- **Autocorrelation Dependency: The PSD existence implies $R_x(\tau)$ depends only on time difference**
- **Theoretical Foundation: The uncorrelated, zero-mean symbol sequence ensures WSS properties**

VI. Conclusions

This work successfully demonstrates:

- **Theoretical Accuracy: The derived PSD $S_x(f) = A^2T\text{sinc}^2(fT)$ accurately predicts the spectral characteristics of BPSK modulation**
- **WSS Validation: The random process $x(t)$ satisfies wide-sense stationarity conditions through time-invariant first and second-order statistics**
- **Practical Verification: MATLAB simulations confirm theoretical predictions.**
- **System Implications: The results satisfy digital communications principles for system design and analysis**

The close agreement between theory and practice shows the robustness of digital communications fundamentals and provides a foundation for more complex modulation schemes analysis.

References

[1] Proakis, J. G., & Salehi, M. (2001). *Communication Systems Engineering*. Prentice Hall.

[2] Haykin, S. (2001). *Communication Systems*. John Wiley & Sons.

[3] Carlson, A. B., Crilly, P. B., & Rutledge, J. C. (2002). *Communication Systems*. McGraw-Hill.

[4]Complete project directory: https://github.com/plochoidysis-ojwege/Digital-communication-Labs/tree/main/Spectrum%20and%20simulation