Inner Product

DT signal : $x[n] = \cos\left(\frac{2\pi n}{10}\right)$

$y[n] = \sin\left[\frac{2\pi n}{10}\right]$

$x[n] = \cos\left(\frac{2\pi n}{10}\right), \quad y[n] = \sin\left(\frac{2\pi n}{10}\right)$

Descrete-Time inner product;

$$\langle x, y\rangle = \sum_{n=n_{start}}^{n_{end}} x[n]\, y^*[n].$$

where $y^*[n] = y[n]$ for real signal

$N = 10$, $n = 0$ to 9:

$$\langle x, y\rangle = \sum_{n=0}^{9} \cos\left(\frac{2\pi n}{10}\right) \sin\left(\frac{2\pi n}{10}\right)$$

since $\cos(\theta)\sin(\theta)$ is orthogonal pair over 1 period

so $0 \leq n \leq 9$

$$\sum_{n=10}^{9} \cos\left(\frac{2\pi n}{10}\right) \sin\left(\frac{2\pi n}{10}\right) = 0$$

$\langle x, y\rangle = 0$ for these DT signals over full period.

Product of two - CT signals

$x(t) = \sin(2\pi t)$, $y(t) = \cos(2\pi t)$

By defination

$$\langle x, y \rangle = \int_{t_{start}}^{t_{end}} x(t)\, y^*(t)\, dt$$

where $y^*(t) = y(t)$ for real signal

$t = \in [0, 1]$ full period

since both $x(t)$ and $y(t)$ have same ~~frequency~~ period $T = 1$

integrate 0 to 1

$$\langle x, y \rangle = \int_0^1 \sin(2\pi t)\, \cos(2\pi t)\, dt.$$

$$\int_0^1 \sin(2\pi t)\, \cos(2\pi t) = 0$$

$= \langle x, y \rangle = 0$ for those CT signals over full period $T = 1$)

Energy and Power from inner product

i) D.T signal

$$E_x = \sum_n |x[n]|^2 = \sum_n x[n]\, x^*[n] = \langle x, x\rangle$$

$x(t) = \sin(2\pi t)$ over $T=1$

ii) C-T signal

$$E_x = \int |x(t)|^2 dt = \int x(t)\, x^*(t) dt = \langle x, x\rangle$$

Let's take energy of
$x(t) = \sin(2\pi t)$ over $T=1$

$$E_x = \int_0^1 \sin^2(2\pi t)\, dt$$

using the identity : $\sin^2(\theta) = \frac{1}{2}[1 - \cos 2\theta]$.

$$E_x = \int_0^1 \frac{1}{2}[1 - \cos(4\pi t)]dt = \frac{1}{2}\int_0^1 1 dt - \frac{1}{2}\int_0^1 \cos(4\pi t$$

1st integral $\int_0^1 1 dt = 1$

2nd integral $\int_0^1 \cos(4\pi t)dt = \frac{1}{4\pi} \sin(4\pi t)\Big|_0^1 = 0$

$$E_x = \frac{1}{2}(1-0) = \frac{1}{2}$$

Power over $[0,1]$

$$P_x = \frac{E_x}{T} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

DT signal

$$E_x = \sum_n |x[n]|^2 = \sum_{n=0}^{9} \cos^2\left(\frac{2\pi n}{10}\right)$$

using the identity

$$\sum_{n=0}^{N-1} \cos^2(A) = \frac{N}{2}, \text{ then } E_x = \frac{10}{5} = \underline{\underline{5}}$$

Power calculations of DT

$$\cancel{P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt}$$

~~for periodic~~

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{21} \sum_{n=0}^{10} \cos^2\left(\frac{2\pi n}{10}\right) =$$

QUESTION 8: Cauchy - Schwartz Inequality

The inequality states

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

since we found $\langle x, y\rangle = 0$, then the inequality holds.

$$0 \leq \|x\| \cdot \|y\|$$

which is true

Excercise of System classification

Test: linearity and time-invariance of the system

$$y(t) = x(t) + x(t-1)$$

A system is stable if it statistics
- ✓ Additivity
- ✓ Homogenuity

Additivity

$$y(t) = y_1(t) + y_2(t)$$

$$\therefore y(t) = [x(t) + x_2(t)] + x_1(t-1) + x_2(t-1)]$$

Scaling

$$y(t) = a x(t) + a x(t-1) = a(x(t) + x(t-1)]$$

Therefore the system is <u>linear</u>

A system is time-invariant if shifting the input by T shifts the output by T

$$y(t-T) = x(t-T) + x(t-1-T)$$

Therefore the system is <u>time-invariant</u>

ii) There fore the system

$$y[n] = x[n] + x[n-1]$$

is <u>lenear and time invariant</u>

Excercise of System classification

Test: linearity and time-invariance of the system

$$y(t) = x(t) + x(t-1)$$

A system is stable if it statisfies
✓ Additivity
✓ Homogeneity

Additivity

$$y(t) = y_1(t) + y_2(t)$$

$$\therefore y(t) = [x(t) + x_2(t)] + x_1(t-1) + x_2(t-1)]$$

Scaling

$$y(t) = ax(t) + ax(t-1) = a(x(t) + x(t-1))]$$

Therefore the system is linear

A system is time-invariant if shifting the input b T shifts the output by T

$$y(t-T) = x(t-T) + x(t-1-T)$$

Therefore the system is time-invariant

ii) There fore the system

$$y[n] = x[n] + x[n-1]$$

is leniear and time invariant

Causality and stability

i) Analyze the causality and stability of the system

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

A system is causal if the output at t depends only on present and past inputs

Since y(t) depends on all inputs value $x(\tau)$ for $\tau \leq t$, it does NOT requir future values.

Therfore the system is <u>causal</u>

A system is stable if bounded input gives a bounded output

if $|x(t)| \leq M$, then
$$|y(t)| = \left| \int_{0}^{t} x(\tau)\, d\tau \right| \leq \int_{-\infty}^{t} |x(\tau)|\, d\tau$$

If $x(t)$ is bounded let entails idefnitoly, $y(t)$ may become unbounded.

Therefore the system is <u>not stable</u>

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ii) Analyze causality and stability of the system

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

<u>Causality check</u>

Since $y[n]$ depends on all past valves $x[k]$ for $k \leq n$
it does not require future values

Therefore the <u>system is causal</u>

stability check

$$|y[n]| = \left|\sum_{k=-\infty}^{n} x[k]\right| \leq \sum_{k=-\infty}^{n} x[k]|$$

If $x[n]$ is bounded but extends indefinitely, $y[n]$ may become unbounded

Therefore the system is not stable.