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Multiattribute decision making based on nonlinear programming methodology and novel score function of interval-valued intuitionistic fuzzy values

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ABSTRACT

In this paper, we propose a novel multiattribute decision making (MADM) method using the nonlinear programming (NLP) methodology and the proposed score function of interval-valued intuitionistic fuzzy values (IVIFVs). Firstly, we propose a new score function of IVIFVs to conquer the drawbacks of the existing score functions of IVIFVs. Then, we construct the converted matrix based on the proposed score function of IVIFVs by calculating the score value of each IVIFV in the decision matrix (DM) offered by the decision maker (DMK). Then, we construct the NLP model via the obtained converted matrix and the interval-valued intuitionistic fuzzy (IVIF) weight of each attribute given by the DMK. Then, we solve the NLP model to obtain the optimal weight for each attribute. Then, based on the obtained converted matrix and the obtained optimal weight of each attribute, we calculate the weighted score of each alternative. Finally, the alternatives are ranked on the basis of the obtained weighted scores of the alternatives. The larger the weighted score of an alternative, the better the preference order of the alternative. The proposed MADM method can overcome the drawbacks of the existing MADM methods. It offers us a very useful approach for MADM in IVIF settings.

1. Introduction

The fuzzy set theory proposed by Zadeh [45] has been applied in many fields [24,25,29,31,32,36,41,44,48]. On the basis of fuzzy sets, Atanassov [4] proposed the theory of intuitionistic fuzzy sets (IFSs). Some multiattribute decision making (MADM) methods [19,20,22,30,35,47,50] have been presented on the basis of IFSs. Athanassov and Gargov [3] extended the theory of IFSs and developed interval-valued intuitionistic fuzzy sets (IVIFSs). In recent years, serval MADM methods [7–17,23,26–28,34,37–40,42,46,49] have been proposed on the basis of IVIFSs in interval-valued intuitionistic fuzzy (IVIF) settings. In [7], Chen and Chiou presented a MADM method on the basis of IVIFSs, particle swarm optimization techniques and the evidential reasoning method. In [8], Chen and Chu presented a MADM method according to the *U*-quadratic distribution of intervals and the transformed matrix in IVIF settings, where it transforms the IVIFVs in the decision matrix (DM) into three intervals to obtain the transformed matrix. Then, it computes the variance and the standard deviation of the transformed matrix and the middle point and the average of the transformed matrix to obtain the *z*-score matrix. After that, it computes the transform weights of the attributes. Finally, it calculates the weighted scores of the alternatives and ranks

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the alternatives on the basis of the weighted scores. In [9], Chen and Fan presented a MADM method according to the largest ranges of evaluating IVIFVs and probability density functions. In [10], Chen and Han presented a MADM method using multiplication operations of IVIFVs and the linear programming (LP) methodology. In [11], Chen and Han presented a MADM method based on the nonlinear programming methodology (NLP) and the ranking method of IVIFVs. In [12], Chen and Huang presented a MADM method based on the LP methodology and IVIFSs. In [13], Chen and Kuo presented a MADM method based on the NLP methodology with hyperbolic function and IVIFVs. In [14], Chen et al. presented a MADM method based on Shannon's information entropy, the NLP methodology and IVIFVs. In [15], Chen and Liao presented a MADM method using the Beta distribution of intervals, the expected values of intervals and their proposed score function of IVIFVs, where it transforms the IVIFVs in the DM into three intervals to obtain the transformed matrix. Then, it constructs the score matrix (SCM) by calculating the score values of the three intervals in the transformed matrix. Then, it transforms the IVIF weights into three intervals and computes the score values of the IVIF weights. After that, it calculates the normalized optimal weights of the attributes. Finally, it calculates the weighted scores of the alternatives and ranks the alternatives on the basis of the weighted scores. In [16], Chen and Tsai presented a MADM method using their proposed score function of IVIFVs and the means and the variances of score matrices, where it constructs the SCM by calculating the score values of the IVIFVs in the DM. Then, on the basis of the SCM, it calculates the mean and the standard deviation of the SCM. After that, it calculates the z-score of the SCM. Then, on the basis of the IVIF weights of the attributes, it calculates the converted weights of the attributes. Finally, it calculates the weighted scores of the alternatives and ranks the alternatives on the basis of the value of the weighted scores. In [17], Chen and Tsai presented a MADM method using their proposed score function of IVIFVs and normalized score matrices, where it constructs the SCM by calculating the score values of the IVIFVs in the DM. Then, on the basis of the SCM, it gets the normalized score matrix. Then, on the basis of the IVIF weights of the attributes, it calculates the optimal weights of the attributes. Then, it gets the weighted normalized DM. Finally, it calculates the weighted scores of the alternatives and ranks the alternatives on the basis of the weighted scores. In [23], Kumar and Chen presented a MADM method based on IVIFVs, the score function of connection numbers and the set pair analysis (SPA) theory. In [26], Li presented a MADM method on the basis of the TOPSIS method, the NLP methodology and IVIFVs. In [27], Liu and Jiang presented a MADM method based on their proposed distance measure of IVIFVs. In [28], Mishra et al. developed a multi-criteria assessment method of programming language using extended MABAC techniques on the basis of divergence measures with IVIFSs. In [34], Rong et al. presented a MADM method using the IVIF weighted copula Bonferroni mean operator of IVIFVs. In [37], Shen et al. presented a MADM method on the basis of binary connection numbers in the set pair analysis under IVIF environments. In [38], Wang and Chen presented a MADM method on the basis of IVIFSs, the LP methodology and the extended TOPSIS method. In [39], Wang and Chen presented an improved MADM method on the basis of their proposed score function of IVIFVs and the LP methodology. In [40], Wang and Chen presented a MADM method on the basis of the LP methodology and their proposed score function and accuracy function of IVIFVs. In [42], Wei et al. presented a MADM method using their proposed information-based score function and the LP methodology. In [46], Zeng et al. presented a MADM method based on IVIFVs, the NLP methodology and the TOPSIS method. In [49], Zhitao and Yingjun proposed a MADM method using the accuracy function of IVIFVs and the IVIF weighted averaging operator in the IVIF settings.

However, the MADM methods presented in [7,10,16,26] and [49] have the following drawbacks:

(1) Chen and Han [11] pointed out that the MADM method presented in [49] has the "the division by zero" problem, which causes it unable to get the preference order (PO) of alternatives in some circumstances.
(2) Chen and Huang [12] pointed out that the MADM method presented in [7] has the drawback that it obtains unreasonable POs of alternatives in some situations.
(3) Chen and Huang [12] pointed out that the MADM method presented in [26] has the "the division by zero" problem, which causes it is unable to get the PO of the alternatives in this circumstance.
(4) Chen and Han [11] pointed out that the MADM method presented in [10] has the infinite loop problem, which causes it unable to get the PO of alternatives in some circumstances.
(5) The MADM method presented in [16] has the shortcoming that the score function presented in [16] has the many to one mapping problem, which transforms different IVIFVs into the same score value, such that it is unable to distinguish the PO of alternatives in some circumstances.

Therefore, with the aim to overcome the shortcoming of the MADM methods presented in [7,10,16,26] and [49], we need to develop a new MADM method to overcome the shortcoming of the MADM methods presented in [7,10,16,26] and [49].

In this paper, we propose a new MADM method on the basis of the NLP methodology and the proposed score function of IVIFVs. The proposed score function of IVIFVs can conquer the shortcoming of existing score functions [5,16,17,43] of IVIFVs. On the basis of the NLP methodology and the proposed score function of IVIFVs, we propose a new MADM method to deal with MADM problems in IVIF settings. Firstly, we propose a new score function of IVIFVs to conquer the shortcomings of the existing score functions of IVIFVs. Then, we construct the converted matrix on the basis of the proposed score function of IVIFVs by calculating the score value of each IVIFV in the decision matrix provided by the decision maker (DMK). Then, we construct the NLP model via the obtained converted matrix and the IVIF weight of each attribute given by the DMK. Then, we solve the NLP model to obtain the optimal weight for each attribute. Then, based on the obtained converted matrix and the obtained optimal weight of each attribute, we calculate the weighted score of each alternative. Finally, the alternatives are ranked on the basis of the obtained weighted scores of the alternatives. The larger the weighted score of an alternative, the better the preference order of the alternative. The proposed MADM method can conquer the shortcomings of the MADM methods presented in [7,10,16,26] and [49]. It offers us a very useful approach for MADM in IVIF settings.

The rest of this paper is presented as follows. In Section 2, we review the definitions of IVIFVs and the score functions of IVIFVs presented in [5,16,17] and [43]. In Section 3, we propose a novel score function of IVIFVs and compare the proposed score function of IVIFVs with the score functions of IVIFVs presented in [5,16,17] and [43] to show that the proposed score function of IVIFVs is better than the score functions of IVIFVs presented in [5,16,17] and [43]. In Section 4, we analyze Chen and Tsai's MADM method [16] and use three counter examples to show the shortcomings of Chen and Tsai's MADM method [16]. In Section 5, we propose a novel MADM method on the basis of the NLP methodology and the proposed score function of IVIFVs. The proposed MADM method can conquer the shortcoming of the MADM methods presented in [7,10,16,26] and [49]. The conclusions are presented in Section 6.

2. Preliminaries

In this section, we review the definitions of IVIFSs, IVIFVs and the score functions of IVIFVs presented in [5,16,17] and [43].

Definition 2.1. [3]. An interval-valued intuitionistic fuzzy set (IVIFS) F $iU = \{u_1, u_2, \cdots, u_n\}$ is represented by $F = \{< u_q, \mu_F(u_q), \upsilon_F(u_q) > | u_q \in U\}$, where $\mu_F(u_q)$ denotes the membership degree and $\upsilon_F(u_q)$ denotes the non-membership degree of element u_q belonging to the IVIFS F, respectively, $\mu_F(u_q) = \left[\rho_q^L, \rho_q^U\right]$, $\upsilon_F(u_q) = \left[\sigma_q^L, \sigma_q^U\right]$, $0 \le \rho_q^L \le \rho_q^U \le 1$, $0 \le \sigma_q^L \le \sigma_q^U \le 1$, $0 \le \rho_q^U + \sigma_q^U \le 1$ and $q = 1, 2, \cdots, n$.

Definition 2.2. [43]. Let $U = \{u_1, u_2, \cdots, u_n\}$ be the universe of discourse. The IVIFV of element u_q in the IVIFS $F = \{< u_q, \mu_F(u_q), \upsilon_F(u_q) > | u_q \in U\}$ *is* represented by $\left(\left[\rho_q^L, \rho_q^U\right], \left[\sigma_q^L, \sigma_q^U\right]\right)$, where $\mu_F(u_q) = \left[\rho_q^L, \rho_q^U\right]$, $\upsilon_F(u_q) = \left[\sigma_q^L, \sigma_q^U\right]$, $0 \le \rho_q^L \le \rho_q^U \le 1$, $0 \le \sigma_q^L \le \sigma_q^U \le 1$, $0 \le \rho_q^U + \sigma_q^U \le 1$ and $q = 1, 2, \cdots, n$.

Definition 2.3. [3,23]. Let $\tilde{f}_1 = ([\rho_1^L, \rho_1^U], [\sigma_1^L, \sigma_1^U])$ and $\tilde{f}_2 = ([\rho_2^L, \rho_2^U], [\sigma_2^L, \sigma_2^U])$ be two IVIFVs. The ranking order (RO) between the *IVIFVs* $\tilde{f}_1$ and $\tilde{f}_2$ is defined as follows:

(1) If $\rho_1^L \ge \rho_2^L$, $\rho_1^U \ge \rho_2^U$, $\sigma_1^L \le \sigma_2^L$ and $\sigma_1^U \le \sigma_2^U$, then $\tilde{f}_1 \ge \tilde{f}_2$.
(2) If $\rho_1^L = \rho_2^L$, $\rho_1^U = \rho_2^U$, $\sigma_1^L = \sigma_2^L$ and $\sigma_1^U = \sigma_2^U$, then $\tilde{f}_1 = \tilde{f}_2$.

Definition 2.4. [16]. The *z*-score z of the corresponding element t in a population is defined as follows:

$$z = \frac{t - \mu}{\delta}, \tag{1}$$

where μ is the population's mean and δ is the population's standard deviation, respectively.

Definition 2.5. [43]. Let $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ be an IVIFV, where $0 \le \rho^L \le \rho^U \le 1$, $0 \le \sigma^L \le \sigma^U \le 1$ and $0 \le \rho^U + \sigma^U \le 1$. Xu's score function α of the *IVIFV* $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ defined as follows:

$$\alpha\left(\tilde{f}\right) = \frac{\rho^L - \sigma^L + \rho^U - \sigma^U}{2}, \tag{2}$$

where $\alpha\left(\tilde{f}\right)$ denotes the score value (SCV) of IVIFV $\tilde{f}$ and $\alpha\left(\tilde{f}\right) \in [-1, 1]$. The larger the SCV $\alpha\left(\tilde{f}\right)$, the better the RO of the IVIFV $\tilde{f}$.

Definition 2.6. [5]. Let $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ be an IVIFV, where $0 \le \rho^L \le \rho^U \le 1$, $0 \le \sigma^L \le \sigma^U \le 1$ *and* $0 \le \rho^U + \sigma^U \le 1$. Bai's score function BSF of the IVIFV $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ is defined as follows:

$$BSF\left(\tilde{f}\right) = \frac{\rho^L + \rho^L(1 - \rho^L - \sigma^L) + \rho^U + \rho^U(1 - \rho^U - \sigma^U)}{2}, \tag{3}$$

where $BSF\left(\tilde{f}\right)$ denotes the SCV of the IVIFV $\tilde{f}$ and $BSF\left(\tilde{f}\right) \in [0,1]$. The larger the SCV $BSF\left(\tilde{f}\right)$, the better the RO of the IVIFV $\tilde{f}$.

Definition 2.7. [16]. Let $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ be an IVIFV, where $0 \le \rho^L \le \rho^U \le 1$, $0 \le \sigma^L \le \sigma^U \le 1$ and $0 \le \rho^U + \sigma^U \le 1$. Chen and Tsai's score function MF of the IVIFV $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ is defined as follows:

$$MF\left(\tilde{f}\right) = \frac{\rho^L + \rho^U - \sigma^L - \sigma^U}{2} + \frac{\rho^L(1-\sigma^L) + \rho^U(1-\sigma^U)}{2 + \sigma^L + \sigma^U} + 1, \tag{4}$$

where $MF\left(\tilde{f}\right)$ denotes the SCV of the IVIFV $\tilde{f}$ and $MF\left(\tilde{f}\right) \in$. The larger the SCV $MF\left(\tilde{f}\right)$, the better the RO of the IVIFV $\tilde{f}$.

Definition 2.8. [17]. Let $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ be an IVIFV, where $0 \le \rho^L \le \rho^U \le 1$, $0 \le \sigma^L \le \sigma^U \le 1$ *and* $0 \le \rho^U + \sigma^U \le 1$. Chen and Tsai's score function TSF of the IVIFV $\tilde{f} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ is defined as follows:

$$TSF\left(\tilde{f}\right) = \frac{\sqrt{\rho^L} + \sqrt{\rho^U} + \sqrt{1-\sigma^L} + \sqrt{1-\sigma^U}}{2}, \tag{5}$$

where $TSF\left(\tilde{f}\right)$ denotes the SCV of the IVIFV $\tilde{f}$ and $TSF\left(\tilde{f}\right) \in [0,2]$. The larger the SCV $TSF\left(\tilde{f}\right)$, the better the RO of the IVIFV $\tilde{f}$.

3. The proposed novel score function of IVIFVs

In this section, we propose a novel score function *HF* of IVIFVs to overcome the drawbacks of Xu's score function α [43], Bai's score function *BSF* [5], Cheng and Tsai's score function *MF* [16] and Cheng and Tsai's score function *TSF* [17] shown in Section 2. The proposed novel score function *HF* of IVIFVs is shown as follows.

Definition 3.1. Let $\tilde{i} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ be an IVIFV, where $0 \le \rho^L \le \rho^U \le 1$, $0 \le \sigma^L \le \sigma^U \le 1$ *and* $0 \le \rho^U + \sigma^U \le 1$. The proposed score function HF of the IVIFV $\tilde{i} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ is defined as follows:

$$HF\left(\tilde{i}\right) = \alpha\left(\tilde{i}\right) + \frac{sin(\rho^L \times \frac{\pi}{2}) + sin(\rho^U \times \frac{\pi}{2}) + sin((1-\sigma^L) \times \frac{\pi}{2}) + sin((1-\sigma^U) \times \frac{\pi}{2})}{2} + 2, \tag{6}$$

where $HF\left(\tilde{i}\right) \in [1,5]$, $\alpha\left(\tilde{i}\right) = \frac{\rho^L - \sigma^L + \rho^U - \sigma^U}{2}$ is the score value of the IVIFV $\tilde{i} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ obtained by Xu's score function α [43] shown in Eq. (2), $\alpha\left(\tilde{i}\right) \in [-1,1]$ and "$\frac{sin(\rho^L \times \frac{\pi}{2}) + sin(\rho^U \times \frac{\pi}{2}) + sin((1-\sigma^L) \times \frac{\pi}{2}) + sin((1-\sigma^U) \times \frac{\pi}{2})}{2}$" $\in [0,2]$. The larger the value of $HF\left(\tilde{i}\right)$, the larger the IVIFV $\tilde{i}$.

In the following, we present some properties of the proposed score function *HF* of IVIFVs, shown as follows.

Property 3.1. *Let* $\tilde{i}_1 = ([\rho_1^L, \rho_1^U], [\sigma_1^L, \sigma_1^U])$ *and* $\tilde{i}_2 = ([\rho_2^L, \rho_2^U], [\sigma_2^L, \sigma_2^U])$ *be two IVIFVs. If* $\tilde{i}_1 > \tilde{i}_2$, *then* $HF\left(\tilde{i}_1\right) > HF\left(\tilde{i}_2\right)$. *If* $\tilde{i}_1 < \tilde{i}_2$, *then* $HF\left(\tilde{i}_1\right) < HF\left(\tilde{i}_2\right)$. *If* $\tilde{i}_1 = \tilde{i}_2$, *then* $HF\left(\tilde{i}_1\right) = HF\left(\tilde{i}_2\right)$.

Proof. Based on Eq. (6), we get.

$$HF\left(\tilde{i}_1\right) = \frac{\rho_1^L - \sigma_1^L + \rho_1^U - \sigma_1^U}{2} + \frac{sin(\rho_1^L \times \frac{\pi}{2}) + sin(\rho_1^U \times \frac{\pi}{2}) + sin((1-\sigma_1^L) \times \frac{\pi}{2}) + sin((1-\sigma_1^U) \times \frac{\pi}{2})}{2} + 2,$$

$$HF\left(\tilde{i}_2\right) = \frac{\rho_2^L - \sigma_2^L + \rho_2^U - \sigma_2^U}{2} + \frac{sin(\rho_2^L \times \frac{\pi}{2}) + sin(\rho_2^U \times \frac{\pi}{2}) + sin((1-\sigma_2^L) \times \frac{\pi}{2}) + sin((1-\sigma_2^U) \times \frac{\pi}{2})}{2} + 2.$$

Then, we get.

$$HF\left(\tilde{i}_1\right) - HF\left(\tilde{i}_2\right) = \frac{\rho_1^L - \sigma_1^L + \rho_1^U - \sigma_1^U}{2} + \frac{sin(\rho_1^L \times \frac{\pi}{2}) + sin(\rho_1^U \times \frac{\pi}{2}) + sin((1-\sigma_1^L) \times \frac{\pi}{2}) + sin((1-\sigma_1^U) \times \frac{\pi}{2})}{2} + 2$$

$$-\frac{\rho_2^L-\sigma_2^L+\rho_2^U-\sigma_2^U}{2}-\frac{sin(\rho_2^L\times\frac{\pi}{2})+sin(\rho_2^U\times\frac{\pi}{2})+sin((1-\sigma_2^L)\times\frac{\pi}{2})+sin((1-\sigma_2^U)\times\frac{\pi}{2})}{2}-2$$

$$=\frac{\rho_1^L-\sigma_1^L+\rho_1^U-\sigma_1^U+sin(\rho_1^L\times\frac{\pi}{2})+sin(\rho_1^U\times\frac{\pi}{2})+sin((1-\sigma_1^L)\times\frac{\pi}{2})+sin((1-\sigma_1^U)\times\frac{\pi}{2})}{2}$$

$$-\left(\frac{\rho_2^L-\sigma_2^L+\rho_2^U-\sigma_2^U+sin(\rho_2^L\times\frac{\pi}{2})+sin(\rho_2^U\times\frac{\pi}{2})+sin((1-\sigma_2^L)\times\frac{\pi}{2})+sin((1-\sigma_2^U)\times\frac{\pi}{2})}{2}\right)$$

$$=\frac{(\rho_1^L-\rho_2^L)+(\sigma_2^L-\sigma_1^L)+(\rho_1^U-\rho_2^U)+(\sigma_2^U-\sigma_1^U)}{2}$$

$$+\frac{(sin(\rho_1^L\times\frac{\pi}{2})-sin(\rho_2^L\times\frac{\pi}{2}))+(sin(\rho_1^U\times\frac{\pi}{2})-sin(\rho_2^U\times\frac{\pi}{2}))}{2}$$

$$+\frac{(sin((1-\sigma_1^L)\times\frac{\pi}{2})-sin((1-\sigma_2^L)\times\frac{\pi}{2}))+(sin((1-\sigma_1^U)\times\frac{\pi}{2})-sin((1-\sigma_2^U)\times\frac{\pi}{2}))}{2}.$$

(i) If $\tilde{i}_1>\tilde{i}_2$, where $\tilde{i}_1=([\rho_1^L,\rho_1^U],[\sigma_1^L,\sigma_1^U])$ and $\tilde{i}_2=([\rho_2^L,\rho_2^U],[\sigma_2^L,\sigma_2^U])$, then according to ***Definition 2.3***, it can be seen that $\rho_1^L>\rho_2^L$, $\rho_1^U>\rho_2^U$, $\sigma_1^L<\sigma_2^L$ and $\sigma_1^U<\sigma_2^U$. From ***Definition 2.2***, we can see that $0\le\rho_1^L\le\rho_1^U\le1$, $0\le\sigma_1^L\le\sigma_1^U\le1$, $0\le\rho_2^L\le\rho_2^U\le1$, $0\le\sigma_2^L\le\sigma_2^U\le1$. Because $\rho_1^L>\rho_2^L$, $\rho_1^U>\rho_2^U$, $\sigma_1^L<\sigma_2^L$ and $\sigma_1^U<\sigma_2^U$, it can be seen that $sin(\rho_1^L\times\frac{\pi}{2})>sin(\rho_2^L\times\frac{\pi}{2})$, $sin(\rho_1^U\times\frac{\pi}{2})>sin(\rho_2^U\times\frac{\pi}{2})$, $sin(\sigma_1^L\times\frac{\pi}{2})<sin(\sigma_2^L\times\frac{\pi}{2})$ and $sin(\sigma_1^U\times\frac{\pi}{2})<sin(\sigma_2^U\times\frac{\pi}{2})$. Because $sin(\sigma_1^L\times\frac{\pi}{2})<sin(\sigma_2^L\times\frac{\pi}{2})$, we get $sin((1-\sigma_1^L)\times\frac{\pi}{2})>sin((1-\sigma_2^L)\times\frac{\pi}{2})$; because $sin(\sigma_1^U\times\frac{\pi}{2})<sin(\sigma_2^U\times\frac{\pi}{2})$, we get $sin((1-\sigma_1^U)\times\frac{\pi}{2})>sin((1-\sigma_2^U)\times\frac{\pi}{2})$. Thus, we get $sin(\rho_1^L\times\frac{\pi}{2})-sin(\rho_2^L\times\frac{\pi}{2})>0$, $sin(\rho_1^U\times\frac{\pi}{2})-sin(\rho_2^U\times\frac{\pi}{2})>0$, $sin((1-\sigma_1^L)\times\frac{\pi}{2})-sin((1-\sigma_2^L)\times\frac{\pi}{2})>0$ and $sin((1-\sigma_1^U)\times\frac{\pi}{2})-sin((1-\sigma_2^U)\times\frac{\pi}{2})>0$. Because.

$$HF\left(\tilde{i}_1\right)-HF\left(\tilde{i}_2\right)=\frac{(\rho_1^L-\rho_2^L)+(\sigma_2^L-\sigma_1^L)+(\rho_1^U-\rho_2^U)+(\sigma_2^U-\sigma_1^U)}{2}$$

$$+\frac{(sin(\rho_1^L\times\frac{\pi}{2})-sin(\rho_2^L\times\frac{\pi}{2}))+(sin(\rho_1^U\times\frac{\pi}{2})-sin(\rho_2^U\times\frac{\pi}{2}))}{2}$$

$$+\frac{(sin((1-\sigma_1^L)\times\frac{\pi}{2})-sin((1-\sigma_2^L)\times\frac{\pi}{2}))+(sin((1-\sigma_1^U)\times\frac{\pi}{2})-sin((1-\sigma_2^U)\times\frac{\pi}{2}))}{2}$$

and because $sin(\rho_1^L\times\frac{\pi}{2})-sin(\rho_2^L\times\frac{\pi}{2})>0$, $sin(\rho_1^U\times\frac{\pi}{2})-sin(\rho_2^U\times\frac{\pi}{2})>0$, $sin((1-\sigma_1^L)\times\frac{\pi}{2})-sin((1-\sigma_2^L)\times\frac{\pi}{2})>0$ and $sin((1-\sigma_1^U)\times\frac{\pi}{2})-sin((1-\sigma_2^U)\times\frac{\pi}{2})>0$, we get $HF\left(\tilde{i}_1\right)-HF\left(\tilde{i}_2\right)>0$. That is, $HF\left(\tilde{i}_1\right)>HF\left(\tilde{i}_2\right)$. Therefore, if the IVIFVs $\tilde{i}_1>\tilde{i}_2$, then $HF\left(\tilde{i}_1\right)>HF\left(\tilde{i}_2\right)$.

(ii) If $\tilde{i}_1<\tilde{i}_2$, where $\tilde{i}_1=([\rho_1^L,\rho_1^U],[\sigma_1^L,\sigma_1^U])$ and $\tilde{i}_2=([\rho_2^L,\rho_2^U],[\sigma_2^L,\sigma_2^U])$, then according to ***Definition 2.3***, it can be seen that $\rho_1^L<\rho_2^L$, $\rho_1^U<\rho_2^U$, $\sigma_1^L>\sigma_2^L$ and $\sigma_1^U>\sigma_2^U$. From ***Definition 2.2***, we can see that $0\le\rho_1^L\le\rho_1^U\le1$, $0\le\sigma_1^L\le\sigma_1^U\le1$, $0\le\rho_2^L\le\rho_2^U\le1$, $0\le\sigma_2^L\le\sigma_2^U\le1$. Because $\rho_1^L<\rho_2^L$, $\rho_1^U<\rho_2^U$, $\sigma_1^L>\sigma_2^L$ and $\sigma_1^U>\sigma_2^U$, it can be seen that $sin(\rho_1^L\times\frac{\pi}{2})<sin(\rho_2^L\times\frac{\pi}{2})$, $sin(\rho_1^U\times\frac{\pi}{2})<sin(\rho_2^U\times\frac{\pi}{2})$, $sin(\sigma_1^L\times\frac{\pi}{2})>sin(\sigma_2^L\times\frac{\pi}{2})$ and $sin(\sigma_1^U\times\frac{\pi}{2})>sin(\sigma_2^U\times\frac{\pi}{2})$. Because $sin(\sigma_1^L\times\frac{\pi}{2})>sin(\sigma_2^L\times\frac{\pi}{2})$, we get $sin((1-\sigma_1^L)\times\frac{\pi}{2})<sin((1-\sigma_2^L)\times\frac{\pi}{2})$; because $sin(\sigma_1^U\times\frac{\pi}{2})>sin(\sigma_2^U\times\frac{\pi}{2})$, we get $sin((1-\sigma_1^U)\times\frac{\pi}{2})<sin((1-\sigma_2^U)\times\frac{\pi}{2})$. Because $sin(\rho_1^L\times\frac{\pi}{2})<sin(\rho_2^L\times\frac{\pi}{2})$, $sin(\rho_1^U\times\frac{\pi}{2})<sin(\rho_2^U\times\frac{\pi}{2})$, $sin((1-\sigma_1^L)\times\frac{\pi}{2})<sin((1-\sigma_2^L)\times\frac{\pi}{2})$ and $sin((1-\sigma_1^U)\times\frac{\pi}{2})<sin((1-\sigma_2^U)\times\frac{\pi}{2})$, we get $sin(\rho_1^L\times\frac{\pi}{2})-sin(\rho_2^L\times\frac{\pi}{2})<0$, $sin(\rho_1^U\times\frac{\pi}{2})-sin(\rho_2^U\times\frac{\pi}{2})<0$, $sin((1-\sigma_1^L)\times\frac{\pi}{2})-sin((1-\sigma_2^L)\times\frac{\pi}{2})<0$ and $sin((1-\sigma_1^U)\times\frac{\pi}{2})-sin((1-\sigma_2^U)\times\frac{\pi}{2})<0$. Because.

$$HF\left(\tilde{i}_1\right)-HF\left(\tilde{i}_2\right)=\frac{(\rho_1^L-\rho_2^L)+(\sigma_2^L-\sigma_1^L)+(\rho_1^U-\rho_2^U)+(\sigma_2^U-\sigma_1^U)}{2}$$

$$+\frac{(sin(\rho_1^L\times\frac{\pi}{2})-sin(\rho_2^L\times\frac{\pi}{2}))+(sin(\rho_1^U\times\frac{\pi}{2})-sin(\rho_2^U\times\frac{\pi}{2}))}{2}$$

$$+\frac{(sin((1-\sigma_1^L)\times\frac{\pi}{2})-sin((1-\sigma_2^L)\times\frac{\pi}{2}))+(sin((1-\sigma_1^U)\times\frac{\pi}{2})-sin((1-\sigma_2^U)\times\frac{\pi}{2}))}{2}$$

and because $sin(\rho_1^L \times \frac{\pi}{2}) - sin(\rho_2^L \times \frac{\pi}{2}) < 0$, $sin(\rho_1^U \times \frac{\pi}{2}) - sin(\rho_2^U \times \frac{\pi}{2}) < 0$, $sin((1-\sigma_1^L) \times \frac{\pi}{2}) - sin((1-\sigma_2^L) \times \frac{\pi}{2}) < 0$ and $sin((1-\sigma_1^U) \times \frac{\pi}{2}) - sin((1-\sigma_2^U) \times \frac{\pi}{2}) < 0$, we get $HF\left(\tilde{i}_1\right) - HF\left(\tilde{i}_2\right) < 0$. That is, $HF\left(\tilde{i}_1\right) < HF\left(\tilde{i}_2\right)$. Therefore, if the IVIFVs $\tilde{i}_1 < \tilde{i}_2$, then $HF\left(\tilde{i}_1\right) < HF\left(\tilde{i}_2\right)$.

(iii) If $\tilde{i}_1 = \tilde{i}_2$, where $\tilde{i}_1 = ([\rho_1^L, \rho_1^U], [\sigma_1^L, \sigma_1^U])$ and $\tilde{i}_2 = ([\rho_2^L, \rho_2^U], [\sigma_2^L, \sigma_2^U])$, then according to ***Definition 2.3***, it can be seen that $\rho_1^L = \rho_2^L$, $\rho_1^U = \rho_2^U$, $\sigma_1^L = \sigma_2^L$ and $\sigma_1^U = \sigma_2^U$. From ***Definition 2.2***, we can see that $0 \le \rho_1^L \le \rho_1^U \le 1$, $0 \le \sigma_1^L \le \sigma_1^U \le 1$, $0 \le \rho_2^L \le \rho_2^U \le 1$, $0 \le \sigma_2^L \le \sigma_2^U \le 1$. Because $\rho_1^L = \rho_2^L$, $\rho_1^U = \rho_2^U$, $\sigma_1^L = \sigma_2^L$ and $\sigma_1^U = \sigma_2^U$, it can be seen that $sin(\rho_1^L \times \frac{\pi}{2}) = sin(\rho_2^L \times \frac{\pi}{2})$, $sin(\rho_1^U \times \frac{\pi}{2}) = sin(\rho_2^U \times \frac{\pi}{2})$, $sin(\sigma_1^L \times \frac{\pi}{2}) = sin(\sigma_2^L \times \frac{\pi}{2})$ and $sin(\sigma_1^U \times \frac{\pi}{2}) = sin(\sigma_2^U \times \frac{\pi}{2})$. Because $sin(\sigma_1^L \times \frac{\pi}{2}) = sin(\sigma_2^L \times \frac{\pi}{2})$, we get $sin((1-\sigma_1^L) \times \frac{\pi}{2}) = sin((1-\sigma_2^L) \times \frac{\pi}{2})$; because $sin(\sigma_1^U \times \frac{\pi}{2}) = sin(\sigma_2^U \times \frac{\pi}{2})$, we get $sin((1-\sigma_1^U) \times \frac{\pi}{2}) = sin((1-\sigma_2^U) \times \frac{\pi}{2})$. Because $sin(\rho_1^L \times \frac{\pi}{2}) = sin(\rho_2^L \times \frac{\pi}{2})$, $sin(\rho_1^U \times \frac{\pi}{2}) = sin(\rho_2^U \times \frac{\pi}{2})$, $sin((1-\sigma_1^L) \times \frac{\pi}{2}) = sin((1-\sigma_2^L) \times \frac{\pi}{2})$, $sin((1-\sigma_1^U) \times \frac{\pi}{2}) = sin((1-\sigma_2^U) \times \frac{\pi}{2})$, we get $sin(\rho_1^L \times \frac{\pi}{2}) - sin(\rho_2^L \times \frac{\pi}{2}) = 0$, $sin(\rho_1^U \times \frac{\pi}{2}) - sin(\rho_2^U \times \frac{\pi}{2}) = 0$, $sin((1-\sigma_1^L) \times \frac{\pi}{2}) - sin((1-\sigma_2^L) \times \frac{\pi}{2}) = 0$ and $sin((1-\sigma_1^U) \times \frac{\pi}{2}) - sin((1-\sigma_2^U) \times \frac{\pi}{2}) = 0$. Because.

$$HF\left(\tilde{i}_1\right) - HF\left(\tilde{i}_2\right) = \frac{(\rho_1^L - \rho_2^L) + (\sigma_2^L - \sigma_1^L) + (\rho_1^U - \rho_2^U) + (\sigma_2^U - \sigma_1^U)}{2}$$

$$+\frac{(sin(\rho_1^L \times \frac{\pi}{2}) - sin(\rho_2^L \times \frac{\pi}{2})) + (sin(\rho_1^U \times \frac{\pi}{2}) - sin(\rho_2^U \times \frac{\pi}{2}))}{2}$$

$$+\frac{(sin((1-\sigma_1^L) \times \frac{\pi}{2}) - sin((1-\sigma_2^L) \times \frac{\pi}{2})) + (sin((1-\sigma_1^U) \times \frac{\pi}{2}) - sin((1-\sigma_2^U) \times \frac{\pi}{2}))}{2}$$

and because $sin(\rho_1^L \times \frac{\pi}{2}) - sin(\rho_2^L \times \frac{\pi}{2}) = 0$, $sin(\rho_1^U \times \frac{\pi}{2}) - sin(\rho_2^U \times \frac{\pi}{2}) = 0$, $sin((1-\sigma_1^L) \times \frac{\pi}{2}) - sin((1-\sigma_2^L) \times \frac{\pi}{2}) = 0$ and $sin((1-\sigma_1^U) \times \frac{\pi}{2}) - sin((1-\sigma_2^U) \times \frac{\pi}{2}) = 0$, we get $HF\left(\tilde{i}_1\right) - HF\left(\tilde{i}_2\right) = 0$. That is, $HF\left(\tilde{i}_1\right) = HF\left(\tilde{i}_2\right)$. Therefore, if the IVIFVs $\tilde{i}_1 = \tilde{i}_2$, then $HF\left(\tilde{i}_1\right) = HF\left(\tilde{i}_2\right)$. **Q. E. D.**

Property 3.2. *Let* $\tilde{i}$ *be an IVIFV, where* $\tilde{i} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$, $0 \le \rho^L \le \rho^U \le 1$, $0 \le \sigma^L \le \sigma^U \le 1$ *and* $0 \le \rho^U + \sigma^U \le 1$. *Then,* $HF\left(\tilde{i}\right) \in$.

Proof. Because $0 \le \rho^L \le \rho^U \le 1$ and $0 \le \sigma^L \le \sigma^U \le 1$, we get $0 \le sin(\rho^L \times \frac{\pi}{2}) \le sin(\rho^U \times \frac{\pi}{2}) \le 1$ and $0 \le sin((1-\sigma^U) \times \frac{\pi}{2}) \le sin((1-\sigma^L) \times \frac{\pi}{2}) \le 1$, where $sin(\rho^L \times \frac{\pi}{2}) \in [0,1]$, $sin(\rho^U \times \frac{\pi}{2}) \in [0,1]$, $sin((1-\sigma^L) \times \frac{\pi}{2}) \in [0,1]$ and $sin((1-\sigma^U) \times \frac{\pi}{2}) \in [0,1]$. Because the IVIFV $\tilde{i} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$, from Eq. (6), it can be seen that $HF\left(\tilde{i}\right) = \alpha\left(\tilde{i}\right) + \frac{sin(\rho^L \times \frac{\pi}{2}) + sin(\rho^U \times \frac{\pi}{2}) + sin((1-\sigma^L) \times \frac{\pi}{2}) + sin((1-\sigma^U) \times \frac{\pi}{2})}{2} + 2$, where $\alpha\left(\tilde{i}\right) = \frac{\rho^L - \sigma^L + \rho^U - \sigma^U}{2}$ is the score value of the IVIFV $\tilde{i} = ([\rho^L, \rho^U], [\sigma^L, \sigma^U])$ obtained by Xu's score function α [43] shown in Eq. (2), $\alpha\left(\tilde{i}\right) \in [-1,1]$, $sin(\rho^L \times \frac{\pi}{2}) \in [0,1]$, $sin(\rho^U \times \frac{\pi}{2}) \in [0,1]$, $sin((1-\sigma^L) \times \frac{\pi}{2}) \in [0,1]$ and $sin((1-\sigma^U) \times \frac{\pi}{2}) \in [0,1]$, we get $HF\left(\tilde{i}\right) \in [1,5]$. ***Q. E. D.***

Property 3.3. *If the IVIFV* $\tilde{i} = ([0,0],[1,1])$, *then* $HF\left(\tilde{i}\right) = 1$.

Proof. Because the *IVIFV* $\tilde{i} = ([0,0],[1,1])$, $\alpha\left(\tilde{i}\right) = \frac{0-1+0-1}{2} = -1$ and $sin(0 \times \frac{\pi}{2}) = 0$, based on Eq. (6), we get $HF\left(\tilde{i}\right) = \alpha\left(\tilde{i}\right) + \frac{sin(0 \times \frac{\pi}{2}) + sin(0 \times \frac{\pi}{2}) + sin(0 \times \frac{\pi}{2}) + sin(0 \times \frac{\pi}{2})}{2} + 2 = -1 + \frac{0}{2} + 2 = 1$ Q. E. D.

Property 3.4. *If the IVIFV* $\tilde{i} = ([1,\ 1],[0,\ 0])$, *then* $HF\left(\tilde{i}\right) = 5$.

Proof. Because the *IVIFV* $\tilde{i} = ([1, 1], [0, 0])$, $\alpha(\tilde{i}) = \frac{1-0+1-0}{2} = 1$ and $sin(1 \times \frac{\pi}{2}) = 1$, based on Eq. (6), we get. $HF(\tilde{i}) = \alpha(\tilde{i}) + \frac{sin(1\times\frac{\pi}{2})+sin(1\times\frac{\pi}{2})+sin(1\times\frac{\pi}{2})+sin(1\times\frac{\pi}{2})}{2} + 2 = 1 + \frac{4}{2} + 2 = 5$ Q. E. D.

In what follows, we use some examples to show that the proposed score function *HF* can overcome the drawbacks of Xu's score function α [43], Bai's score function *BSF* [5], Cheng and Tsai's score function *MF* [16] and Cheng and Tsai's score function *TSF* [17], where they have the drawbacks that they cannot distinguish the ranking order (RO) of IVIFVs in some circumstances.

Example 3.1. Let $\tilde{i}_1 = ([0.15, 0.15], [0.15, 0.15])$ and $\tilde{i}_2 = ([0.25, 0.25], [0.25, 0.25])$ be two IVIFVs.

(1) According to Xu's score function α [43] shown in Eq. (2), Xu's score function α gets $\alpha(\tilde{i}_1) = 0$ and $\alpha(\tilde{i}_2) = 0$. Because $\alpha(\tilde{i}_1) = \alpha(\tilde{i}_2)$, where $\alpha(\tilde{i}_1) = 0$ and $\alpha(\tilde{i}_2) = 0$, the RO of $\tilde{i}_1 = ([0.15, 0.15], [0.15, 0.15])$ and $\tilde{i}_2 = ([0.25, 0.25], [0.25, 0.25])$ obtained by Xu's score function [43] is "$\tilde{i}_1 = \tilde{i}_2$". Thus, Xu's score function α [43] has the shortcoming that it is not able to distinguish the RO between the IVIFVs $\tilde{i}_1$ and $\tilde{i}_2$ in this circumstance.

(2) According to Bai's score function *BSF* [5] shown in Eq. (3), Bai's score function *BSF* gets $BSF(\tilde{i}_1) = 0.255$ and $BSF(\tilde{i}_2) = 0.375$. Because $BSF(\tilde{i}_1) < BSF(\tilde{i}_2)$, the RO between the IVIFVs $\tilde{i}_1 = ([0.15, 0.15], [0.15, 0.15])$ and $\tilde{i}_2 = ([0.25, 0.25], [0.25, 0.25])$ obtained by Bai's score function *BSF* [5] is "$\tilde{i}_1 < \tilde{i}_2$".

(3) According to Chen and Tsai's score function *MF* [16] shown in Eq. (4), Chen and Tsai's score function *MF* gets $MF(\tilde{i}_1) = 1.111$ and $MF(\tilde{i}_2) = 1.150$. Because $MF(\tilde{i}_1) < MF(\tilde{i}_2)$, the RO between the IVIFVs $\tilde{i}_1 = ([0.15, 0.15], [0.15, 0.15])$ and $\tilde{i}_2 = ([0.25, 0.25], [0.25, 0.25])$ obtained by Chen and Tsai's score function *MF* [16] is "$\tilde{i}_1 < \tilde{i}_2$".

(4) According to Chen and Tsai's score function *TSF* [17] shown in Eq. (5), Chen and Tsai's score function *TSF* gets $TSF(\tilde{i}_1) = 1.309$ and $TSF(\tilde{i}_2) = 1.366$. Because $TSF(\tilde{i}_1) < TSF(\tilde{i}_2)$, the RO between the IVIFVs $\tilde{i}_1 = ([0.15, 0.15], [0.15, 0.15])$ and $\tilde{i}_2 = ([0.25, 0.25], [0.25, 0.25])$ obtained by Chen and Tsai's score function *TSF* [17] is "$\tilde{i}_1 < \tilde{i}_2$".

(5) According to the proposed score function *HF* shown in Eq. (6), the proposed score function *HF* gets $HF(\tilde{i}_1) = 3.206$ and $HF(\tilde{i}_2) = 3.307$. Because $HF(\tilde{i}_1) < HF(\tilde{i}_2)$, the RO between the IVIFVs $\tilde{i}_1 = ([0.15, 0.15], [0.15, 0.15])$ and $\tilde{i}_2 = ([0.25, 0.25], [0.25, 0.25])$ obtained by the proposed score function *HF* is "$\tilde{i}_1 < \tilde{i}_2$".

Example 3.2. Let $\tilde{i}_3 = ([0.50, 0.50], [0.35, 0.35])$ and $\tilde{i}_4 = ([0.40, 0.40], [0.25, 0.25])$ be two IVIFVs.

(1) According to Xu's score function α [43] shown in Eq. (2), Xu's score function α gets $\alpha(\tilde{i}_3) = 0.150$ and $\alpha(\tilde{i}_4) = 0.150$. Because $\alpha(\tilde{i}_3) = \alpha(\tilde{i}_4)$, where $\alpha(\tilde{i}_3) = 0.150$ and $\alpha(\tilde{i}_4) = 0.150$, the RO between the IVIFVs $\tilde{i}_3 = ([0.50, 0.50], [0.35, 0.35])$ and $\tilde{i}_4 = ([0.40, 0.40], [0.25, 0.25])$ obtained by Xu's score function [43] is "$\tilde{i}_3 = \tilde{i}_4$". Thus, Xu's score function α [43] has the shortcoming that it is not able to distinguish the RO between the IVIFVs $\tilde{i}_3$ and $\tilde{i}_4$ in this circumstance.

(2) According to Bai's score function BSF [5] shown in Eq. (3), Bai's score function BSF gets $BSF\left(\tilde{i}_3\right) = 0.575$ and $BSF\left(\tilde{i}_4\right) = 0.540$. Because $BSF\left(\tilde{i}_3\right) > BSF\left(\tilde{i}_4\right)$, the RO between the IVIFVs $\tilde{i}_3 = ([0.50, 0.50], [0.35, 0.35])$ and $\tilde{i}_4 = ([0.40, 0.40], [0.25, 0.25])$ obtained by Bai's score function BSF [5] is "$\tilde{i}_3 > \tilde{i}_4$".

(3) According to Chen and Tsai's score function MF [16] shown in Eq. (4), Chen and Tsai's score function MF gets $MF\left(\tilde{i}_3\right) = 1.391$ and $MF\left(\tilde{i}_4\right) = 1.390$. Because $MF\left(\tilde{i}_3\right) > MF\left(\tilde{i}_4\right)$, the RO between the IVIFVs $\tilde{i}_3 = ([0.50, 0.50], [0.35, 0.35])$ and $\tilde{i}_4 = ([0.40, 0.40], [0.25, 0.25])$ obtained by Chen and Tsai's score function MF [16] is "$\tilde{i}_3 > \tilde{i}_4$".

(4) According to Chen and Tsai's score function TSF [17] shown in Eq. (5), Chen and Tsai's score function TSF gets $TSF\left(\tilde{i}_3\right) = 1.513$ and $TSF\left(\tilde{i}_4\right) = 1.498$. Because $TSF\left(\tilde{i}_3\right) > TSF\left(\tilde{i}_4\right)$, the RO between the IVIFVs $\tilde{i}_3 = ([0.50, 0.50], [0.35, 0.35])$ and $\tilde{i}_4 = ([0.40, 0.40], [0.25, 0.25])$ obtained by Chen and Tsai's score function TSF [17] is "$\tilde{i}_3 > \tilde{i}_4$".

(5) According to the proposed score function HF shown in Eq. (6), the proposed score function HF gets $HF\left(\tilde{i}_3\right) = 3.710$ and $HF\left(\tilde{i}_4\right) = 3.662$. Because $HF\left(\tilde{i}_3\right) > HF\left(\tilde{i}_4\right)$, the RO between the IVIFVs $\tilde{i}_3 = ([0.50, 0.50], [0.35, 0.35])$ and $\tilde{i}_4 = ([0.40, 0.40], [0.25, 0.25])$ obtained by the proposed score function HF is "$\tilde{i}_3 > \tilde{i}_4$".

Example 3.3. Let $\tilde{i}_5 = ([0.35, 0.40], [0.35, 0.40])$ and $\tilde{i}_6 = ([0.25, 0.40], [0.25, 0.40])$ be two IVIFVs.

(1) According to Xu's score function α [43] shown in Eq. (2), Xu's score function α gets $\alpha\left(\tilde{i}_5\right) = 0$ and $\alpha\left(\tilde{i}_6\right) = 0$. Because $\alpha\left(\tilde{i}_5\right) = \alpha\left(\tilde{i}_6\right)$, where $\alpha\left(\tilde{i}_5\right) = 0$ and $\alpha\left(\tilde{i}_6\right) = 0$, the RO between the IVIFVs $\tilde{i}_5 = ([0.35, 0.40], [0.35, 0.40])$ and $\tilde{i}_6 = ([0.25, 0.40], [0.25, 0.40])$ obtained by Xu's score function α [43] is "$\tilde{i}_5 = \tilde{i}_6$". Thus, Xu's score function α [43] has the shortcoming that it is not able to distinguish the RO between $\tilde{i}_5$ and $\tilde{i}_6$ in this circumstance.

(2) According to Bai's score function BSF [5] shown in Eq. (3), Bai's score function BSF gets $BSF\left(\tilde{i}_5\right) = 0.468$ and $BSF\left(\tilde{i}_6\right) = 0.428$. Because $BSF\left(\tilde{i}_5\right) > BSF\left(\tilde{i}_6\right)$, the RO between the IVIFVs $\tilde{i}_5 = ([0.35, 0.40], [0.35, 0.40])$ and $\tilde{i}_6 = ([0.25, 0.40], [0.25, 0.40])$ obtained by Bai's score function BSF [5] is "$\tilde{i}_5 > \tilde{i}_6$".

(3) According to Chen and Tsai's score function MF [16] shown in Eq. (4), Chen and Tsai's score function MF gets $MF\left(\tilde{i}_5\right) = 1.170$ and $MF\left(\tilde{i}_6\right) = 1.161$. Because $MF\left(\tilde{i}_5\right) > MF\left(\tilde{i}_6\right)$, the RO between the IVIFVs $\tilde{i}_5 = ([0.35, 0.40], [0.35, 0.40])$ and $\tilde{i}_6 = ([0.25, 0.40], [0.25, 0.40])$ obtained by Chen and Tsai's score function MF [16] is "$\tilde{i}_5 > \tilde{i}_6$".

(4) According to Chen and Tsai's score function TSF [17] shown in Eq. (5), Chen and Tsai's score function TSF gets $TSF\left(\tilde{i}_5\right) = 1.402$ and $TSF\left(\tilde{i}_6\right) = 1.387$. Because $TSF\left(\tilde{i}_5\right) > TSF\left(\tilde{i}_6\right)$, the RO between the IVIFVs $\tilde{i}_5 = ([0.35, 0.40], [0.35, 0.40])$ and $\tilde{i}_6 = ([0.25, 0.40], [0.25, 0.40])$ obtained by Chen and Tsai's score function TSF [17] is "$\tilde{i}_5 > \tilde{i}_6$".

(5) According to the proposed score function HF shown in Eq. (6), the proposed score function HF gets $HF\left(\tilde{i}_5\right) = 3.386$ and $HF\left(\tilde{i}_6\right) = 3.352$. Because $HF\left(\tilde{i}_5\right) > HF\left(\tilde{i}_6\right)$, the RO between the IVIFVs $\tilde{i}_5 = ([0.35, 0.40], [0.35, 0.40])$ and $\tilde{i}_6 = ([0.25, 0.40], [0.25, 0.40])$ obtained by the proposed score function HF is "$\tilde{i}_5 > \tilde{i}_6$".

Example 3.4. Let $\tilde{i}_7 = ([0.15, 0.40], [0.30, 0.40])$ and $\tilde{i}_8 = ([0.25, 0.40], [0.40, 0.40])$ be two IVIFVs.

(1) According to Xu's score function α [43] shown in Eq. (2), Xu's score function α gets $\alpha\left(\tilde{i}_7\right) = -0.075$ and $\alpha\left(\tilde{i}_8\right) = -0.075$. Because $\alpha\left(\tilde{i}_7\right) = \alpha\left(\tilde{i}_8\right)$, where $\alpha\left(\tilde{i}_7\right) = -0.075$ and $\alpha\left(\tilde{i}_8\right) = -0.075$, the RO between the IVIFVs $\tilde{i}_7 = ([0.15, 0.40], [0.30, 0.40])$ and $\tilde{i}_8 = ([0.25, 0.40], [0.40, 0.40])$ obtained by Xu's score function [43] is "$\tilde{i}_7 = \tilde{i}_8$". Thus, Xu's score function [43] has the shortcoming that it is not able to distinguish the RO between the IVIFVs $\tilde{i}_7$ and $\tilde{i}_8$ in this circumstance.

(2) According to Bai's score function *BSF* [5] shown in Eq. (3), Bai's score function *BSF* gets $BSF\left(\tilde{i}_7\right) = 0.356$ and $BSF\left(\tilde{i}_8\right) = 0.409$. Because $BSF\left(\tilde{i}_7\right) < BSF\left(\tilde{i}_8\right)$, the RO between the IVIFVs $\tilde{i}_7 = ([0.15, 0.40], [0.30, 0.40])$ and $\tilde{i}_8 = ([0.25, 0.40], [0.40, 0.40])$ obtained by Bai's score function *BSF* [5] is "$\tilde{i}_7 < \tilde{i}_8$".

(3) According to Chen and Tsai's score function *MF* [16] shown in Eq. (4), Chen and Tsai's score function *MF* gets $MF\left(\tilde{i}_7\right) = 1.053$ and $MF\left(\tilde{i}_8\right) = 1.064$. Because $MF\left(\tilde{i}_7\right) < MF\left(\tilde{i}_8\right)$, the RO between the IVIFVs $\tilde{i}_7 = ([0.15, 0.40], [0.30, 0.40])$ and $\tilde{i}_8 = ([0.25, 0.40], [0.40, 0.40])$ obtained by Chen and Tsai's score function *MF* [16] is "$\tilde{i}_7 < \tilde{i}_8$".

(4) According to Chen and Tsai's score function *TSF* [17] shown in Eq. (5), Chen and Tsai's score function gets $TSF\left(\tilde{i}_7\right) = 1.316$ and $TSF\left(\tilde{i}_8\right) = 1.341$. Because $TSF\left(\tilde{i}_7\right) < TSF\left(\tilde{i}_8\right)$, the RO between the IVIFVs $\tilde{i}_7 = ([0.15, 0.40], [0.30, 0.40])$ and $\tilde{i}_8 = ([0.25, 0.40], [0.40, 0.40])$ obtained by Chen and Tsai's score function *TSF* [17] is "$\tilde{i}_7 < \tilde{i}_8$".

(5) According to the proposed score function *HF* shown in Eq. (6), the proposed score function *HF* gets $HF\left(\tilde{i}_7\right) = 3.186$ and $HF\left(\tilde{i}_8\right) = 3.219$. Because $HF\left(\tilde{i}_7\right) < HF\left(\tilde{i}_8\right)$, the RO between the IVIFVs $\tilde{i}_7 = ([0.15, 0.40], [0.30, 0.40])$ and $\tilde{i}_8 = ([0.25, 0.40], [0.40, 0.40])$ obtained by the proposed score function *HF* is "$\tilde{i}_7 < \tilde{i}_8$".

Example 3.5. Let $\tilde{i}_9 = ([0, 0], [0.15, 0.15])$ and $\tilde{i}_{10} = ([0, 0], [0.23, 0.23])$ be two IVIFVs.

(1) According to Xu's score function α [43] shown in Eq. (2), Xu's score function α gets $\alpha\left(\tilde{i}_9\right) = -0.150$ and $\alpha\left(\tilde{i}_{10}\right) = -0.230$. Because $\alpha\left(\tilde{i}_9\right) > \alpha\left(\tilde{i}_{10}\right)$, the RO between the IVIFVs $\tilde{i}_9 = ([0, 0], [0.15, 0.15])$ and $\tilde{i}_{10} = ([0, 0], [0.23, 0.23])$ obtained by Xu's score function [43] is "$\tilde{i}_9 > \tilde{i}_{10}$".

(2) According to Bai's score function *BSF* [5] shown in Eq. (3), Bai's score function *BSF* gets $BSF\left(\tilde{i}_9\right) = 0$ and $BSF\left(\tilde{i}_{10}\right) = 0$. Because $BSF\left(\tilde{i}_9\right) = BSF\left(\tilde{i}_{10}\right)$, where $BSF\left(\tilde{i}_9\right) = 0$ and $BSF\left(\tilde{i}_{10}\right) = 0$, the RO between the IVIFVs $\tilde{i}_9 = ([0, 0], [0.15, 0.15])$ and $\tilde{i}_{10} = ([0, 0], [0.23, 0.23])$ obtained by Bai's score function *BSF* [5] is "$\tilde{i}_9 = \tilde{i}_{10}$". Thus, Bai's score function *BSF* [5] has the shortcoming that it is not able to distinguish the RO between the IVIFVs $\tilde{i}_9$ and $\tilde{i}_{10}$ in this circumstance.

(3) According to Chen and Tsai's score function *MF* [16] shown in Eq. (4), Chen and Tsai's score function *MF* gets $MF\left(\tilde{i}_9\right) = 0.850$ and $MF\left(\tilde{i}_{10}\right) = 0.770$. Because $MF\left(\tilde{i}_9\right) > MF\left(\tilde{i}_{10}\right)$, the RO between the IVIFVs $\tilde{i}_9 = ([0, 0], [0.15, 0.15])$ and $\tilde{i}_{10} = ([0, 0], [0.23, 0.23])$ obtained by Chen and Tsai's score function *MF* [16] is "$\tilde{i}_9 > \tilde{i}_{10}$".

(4) According to Chen and Tsai's score function TSF [17] shown in Eq. (5), Chen and Tsai's score function TSF gets $TSF\left(\tilde{i}_9\right) = 0.922$ and $TSF\left(\tilde{i}_{10}\right) = 0.877$. Because $TSF\left(\tilde{i}_9\right) > TSF\left(\tilde{i}_{10}\right)$, the RO between the IVIFVs $\tilde{i}_9 = ([0,0],[0.15,0.15])$ and $\tilde{i}_{10} = ([0,0],[0.23,0.23])$ obtained by Chen and Tsai's score function TSF [17] is "$\tilde{i}_9 > \tilde{i}_{10}$".

(5) According to the proposed score function HF shown in Eq. (6), the proposed score function HF gets $HF\left(\tilde{i}_9\right) = 2.822$ and $HF\left(\tilde{i}_{10}\right) = 2.705$. Because $HF\left(\tilde{i}_9\right) > HF\left(\tilde{i}_{10}\right)$, the RO between the IVIFVs $\tilde{i}_9 = ([0,0],[0.15,0.15])$ and $\tilde{i}_{10} = ([0,0],[0.23,0.23])$ obtained by the proposed score function HF is "$\tilde{i}_9 > \tilde{i}_{10}$".

Example 3.6. Let $\tilde{i}_{11} = ([0.30,0.30],[0.20,0.20])$ and $\tilde{i}_{12} = ([0.20,0.40],[0.20,0.20])$ be two IVIFVs.

(1) According to Xu's score function α [43] shown in Eq. (2), Xu's score function α gets $\alpha\left(\tilde{i}_{11}\right) = 0.100$ and $\alpha\left(\tilde{i}_{12}\right) = 0.100$. Because $\alpha\left(\tilde{i}_{11}\right) = \alpha\left(\tilde{i}_{12}\right)$, where $\alpha\left(\tilde{i}_{11}\right) = 0.100$ and $\alpha\left(\tilde{i}_{12}\right) = 0.100$, the RO between the IVIFVs $\tilde{i}_{11} = ([0.30,0.30],[0.20,0.20])$ and $\tilde{i}_{12} = ([0.20,0.40],[0.20,0.20])$ obtained by Xu's score function [43] is "$\tilde{i}_{11} = \tilde{i}_{12}$". Thus, Xu's score function [43] has the shortcoming that it is not able to distinguish the RO of $\tilde{i}_{11}$ and $\tilde{i}_{12}$ under this circumstance.

(2) According to Bai's score function BSF [5] shown in Eq. (3), Bai's score function BSF gets $BSF\left(\tilde{i}_{11}\right) = 0.450$ and $BSF\left(\tilde{i}_{12}\right) = 0.440$. Because $BSF\left(\tilde{i}_{11}\right) > BSF\left(\tilde{i}_{12}\right)$, the RO between the IVIFVs $\tilde{i}_{11} = ([0.30,0.30],[0.20,0.20])$ and $\tilde{i}_{12} = ([0.20,0.40],[0.20,0.20])$ obtained by Bai's score function BSF [5] is "$\tilde{i}_{11} > \tilde{i}_{12}$".

(3) According to Chen and Tsai's score function MF [16] shown in Eq. (4), Chen and Tsai's score function MF gets $MF\left(\tilde{i}_{11}\right) = 1.300$ and $MF\left(\tilde{i}_{12}\right) = 1.300$. Because $MF\left(\tilde{i}_{11}\right) = MF\left(\tilde{i}_{12}\right)$, where $MF\left(\tilde{i}_{11}\right) = 1.300$ and $MF\left(\tilde{i}_{12}\right) = 1.300$, the RO between the IVIFVs $\tilde{i}_{11} = ([0.30,0.30],[0.20,0.20])$ and $\tilde{i}_{12} = ([0.20,0.40],[0.20,0.20])$ obtained by Chen and Tsai's score function MF [16] is "$\tilde{i}_{11} = \tilde{i}_{12}$". Thus, Chen and Tsai's score function MF [16] has the shortcoming that it is not able to distinguish the RO between the IVIFVs $\tilde{i}_{11} = ([0.30,0.30],[0.20,0.20])$ and $\tilde{i}_{12} = ([0.20,0.40],[0.20,0.20])$ in this circumstance.

(4) According to Chen and Tsai's score function TSF [17] shown in Eq. (5), Chen and Tsai's score function TSF gets $TSF\left(\tilde{i}_{11}\right) = 1.442$ and $TSF\left(\tilde{i}_{12}\right) = 1.434$. Because $TSF\left(\tilde{i}_{11}\right) > TSF\left(\tilde{i}_{12}\right)$, the RO between the IVIFVs $\tilde{i}_{11} = ([0.30,0.30],[0.20,0.20])$ and $\tilde{i}_{12} = ([0.20,0.40],[0.20,0.20])$ obtained by Chen and Tsai's score function TSF [17] is "$\tilde{i}_{11} > \tilde{i}_{12}$".

(5) According to the proposed score function HF shown in Eq. (6), the proposed score function HF gets $HF\left(\tilde{i}_{11}\right) = 3.505$ and $HF\left(\tilde{i}_{12}\right) = 3.499$. Because $HF\left(\tilde{i}_{11}\right) > HF\left(\tilde{i}_{12}\right)$, the RO between the IVIFVs $\tilde{i}_{11} = ([0.30,0.30],[0.20,0.20])$ and $\tilde{i}_{12} = ([0.20,0.40],[0.20,0.20])$ obtained by the proposed score function HF is "$\tilde{i}_{11} > \tilde{i}_{12}$".

Example 3.7. Let $\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$ and $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$ be two IVIFVs.

(1) According to Xu's score function $\propto$ [43] shown in Eq. (2), Xu's score function $\propto$ gets $\propto\left(\tilde{i}_{13}\right) = 0.010$ and $\propto\left(\tilde{i}_{14}\right) = 0$. Because $\propto\left(\tilde{i}_{13}\right) > \propto\left(\tilde{i}_{14}\right)$, the RO between the IVIFVs $\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$ and $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$ obtained by Xu's score function [43] is "$\tilde{i}_{13} > \tilde{i}_{14}$".

(2) According to Bai's score function *BSF* [5] shown in Eq. (3), Bai's score function *BSF* gets $BSF\left(\tilde{i}_{13}\right) = 0.039$ and $BSF\left(\tilde{i}_{14}\right) = 0.020$. Because $BSF\left(\tilde{i}_{13}\right) > BSF\left(\tilde{i}_{14}\right)$, the RO between the IVIFVs $\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$ and $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$ obtained by Bai's score function *BSF* [5] is "$\tilde{i}_{13} > \tilde{i}_{14}$".

(3) According to Chen and Tsai's score function *MF* [16] shown in Eq. (4), Chen and Tsai's score function *MF* gets $MF\left(\tilde{i}_{13}\right) = 1.030$ and $MF\left(\tilde{i}_{14}\right) = 1.010$. Because $MF\left(\tilde{i}_{13}\right) > MF\left(\tilde{i}_{14}\right)$, the RO between the IVIFVs $\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$ and $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$ obtained by Chen and Tsai's score function *MF* [16] is "$\tilde{i}_{13} > \tilde{i}_{14}$".

(4) According to Chen and Tsai's score function *TSF* [17] shown in Eq. (5), Chen and Tsai's score function *TSF* gets $TSF\left(\tilde{i}_{13}\right) = 1.095$ and $TSF\left(\tilde{i}_{14}\right) = 1.095$. Because $TSF\left(\tilde{i}_{13}\right) > TSF\left(\tilde{i}_{14}\right)$, where $TSF\left(\tilde{i}_{13}\right) = 1.095$ and $TSF\left(\tilde{i}_{14}\right) = 1.095$, the RO between the IVIFVs $\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$ and $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$ obtained by Chen and Tsai's score function *TSF* [17] is "$\tilde{i}_{13} = \tilde{i}_{14}$". Thus, Chen and Tsai's score function *TSF* [17] has the shortcoming that it is not able to distinguish the RO between the IVIFVs $\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$ and $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$ in this circumstance.

(5) According to the proposed score function *HF* shown in Eq. (6), the proposed score function *HF* gets $HF\left(\tilde{i}_{13}\right) = 3.041$ and $HF\left(\tilde{i}_{14}\right) = 3.016$. Because $HF\left(\tilde{i}_{13}\right) > HF\left(\tilde{i}_{14}\right)$, the RO between the IVIFVs $\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$ and $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$ obtained by the proposed score function *HF* is "$\tilde{i}_{13} > \tilde{i}_{14}$".

In summary, Table 1 shows a comparison of the ROs of IVIFVs obtained by different score functions. From Table 1, it can be seen that the proposed score function *HF* can overcome the shortcomings of Xu' score function $\propto$ [43], Bai's score function *BSF* [5], Chen and Tsai's score function *MF* [16] and Chen and Tsai's score function *TSF* [17].

4. Analyze the drawbacks of Chen and Tsai's MADM method

In this section, we analyze the shortcomings of Chen and Tsai's MADM method [16]. Let $T_1, T_2, \cdots$, and T_m be alternatives, let $S_1, S_2, \cdots$, and S_n be attributes, and let $\tilde{s}_q = \left(\left[k_q^L, k_q^U\right], \left[c_q^L, c_q^U\right]\right)$ be the IVIF weight of attribute S_q, where $0 \le k_q^L \le k_q^U \le 1$,

Table 1
A comparison of the ROs of IVIFVs obtained by different score functions.

IVIFVs	Xu' score function $\propto$ [43]	Bai's score function *BSF* [5]	Chen and Tsai's score function *MF* [16]	Chen and Tsai's score function *TSF* [17]	The proposed score function *HF*
$\tilde{i}_1 = ([0.15, 0.15], [0.15, 0.15])$, $\tilde{i}_2 = ([0.25, 0.25], [0.25, 0.25])$	$\tilde{i}_1 = \tilde{i}_2$	$\tilde{i}_1 < \tilde{i}_2$	$\tilde{i}_1 < \tilde{i}_2$	$\tilde{i}_1 < \tilde{i}_2$	$\tilde{i}_1 < \tilde{i}_2$
$\tilde{i}_3 = ([0.50, 0.50], [0.35, 0.35])$, $\tilde{i}_4 = ([0.40, 0.40], [0.25, 0.25])$	$\tilde{i}_3 = \tilde{i}_4$	$\tilde{i}_3 > \tilde{i}_4$	$\tilde{i}_3 > \tilde{i}_4$	$\tilde{i}_3 > \tilde{i}_4$	$\tilde{i}_3 > \tilde{i}_4$
$\tilde{i}_5 = ([0.35, 0.40], [0.35, 0.40])$, $\tilde{i}_6 = ([0.25, 0.40], [0.25, 0.40])$	$\tilde{i}_5 = \tilde{i}_6$	$\tilde{i}_5 > \tilde{i}_6$	$\tilde{i}_5 > \tilde{i}_6$	$\tilde{i}_5 > \tilde{i}_6$	$\tilde{i}_5 > \tilde{i}_6$
$\tilde{i}_7 = ([0.15, 0.40], [0.30, 0.40])$, $\tilde{i}_8 = ([0.25, 0.40], [0.40, 0.40])$	$\tilde{i}_7 = \tilde{i}_8$	$\tilde{i}_7 < \tilde{i}_8$	$\tilde{i}_7 < \tilde{i}_8$	$\tilde{i}_7 < \tilde{i}_8$	$\tilde{i}_7 < \tilde{i}_8$
$\tilde{i}_9 = ([0, 0], [0.15, 0.15])$, $\tilde{i}_{10} = ([0, 0], [0.23, 0.23])$	$\tilde{i}_9 > \tilde{i}_{10}$	$\tilde{i}_9 = \tilde{i}_{10}$	$\tilde{i}_9 > \tilde{i}_{10}$	$\tilde{i}_9 > \tilde{i}_{10}$	$\tilde{i}_9 > \tilde{i}_{10}$
$\tilde{i}_{11} = ([0.30, 0.30], [0.20, 0.20])$, $\tilde{i}_{12} = ([0.20, 0.40], [0.20, 0.20])$	$\tilde{i}_{11} = \tilde{i}_{12}$	$\tilde{i}_{11} > \tilde{i}_{12}$	$\tilde{i}_{11} = \tilde{i}_{12}$	$\tilde{i}_{11} > \tilde{i}_{12}$	$\tilde{i}_{11} > \tilde{i}_{12}$
$\tilde{i}_{13} = ([0, 0.04], [0.01, 0.01])$, $\tilde{i}_{14} = ([0.01, 0.01], [0.01, 0.01])$	$\tilde{i}_{13} > \tilde{i}_{14}$	$\tilde{i}_{13} > \tilde{i}_{14}$	$\tilde{i}_{13} > \tilde{i}_{14}$	$\tilde{i}_{13} = \tilde{i}_{14}$	$\tilde{i}_{13} > \tilde{i}_{14}$

$0 \le c_q^L \le c_q^U \le 1$, $0 \le k_q^U + c_q^U \le 1$ and $q = 1, 2, \cdots, n$. Assume that the DM $\widetilde{D} = (\widetilde{f}_{pq})_{m\times n}$ given by the decision maker (DMK) is as follows:

$$\widetilde{D} = \begin{array}{c} \\ T_1 \\ T_2 \\ \vdots \\ T_m \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & \cdots & S_n \end{array} \\ \begin{pmatrix} \widetilde{f}_{11} & \widetilde{f}_{12} & \cdots & \widetilde{f}_{1n} \\ \widetilde{f}_{21} & \widetilde{f}_{22} & \cdots & \widetilde{f}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{f}_{m1} & \widetilde{f}_{m2} & \cdots & \widetilde{f}_{mn} \end{pmatrix} \end{array},$$

where $\widetilde{f}_{pq} = \left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)$ is an IVIFV, $0 \le \rho_{pq}^L \le \rho_{pq}^U \le 1$, $0 \le \sigma_{pq}^L \le \sigma_{pq}^U \le 1$, $0 \le \rho_{pq}^U + \sigma_{pq}^U \le 1$, $p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$. Chen and Tsai's MADM method [16] is reviewed as follows:

Step 1: On the basis of Eq. (4) and the DM $\widetilde{D} = \left(\widetilde{f}_{pq}\right)_{m\times n} = \left(\left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)\right)_{m\times n}$, construct the score matrix (SCM) $D = (f_{pq})_{m\times n}$, where

$$f_{pq} = \begin{cases} MF\left(\widetilde{f}_{pq}\right), \textit{if } S_q \textit{ is a benifit type attribute} \\ 3 - MF\left(\widetilde{f}_{pq}\right), \textit{if } S_q \textit{ is a cost type attribute} \end{cases}, \tag{7}$$

$MF(\widetilde{f}_{pq}) = \frac{\rho_{pq}^L+\rho_{pq}^U-\sigma_{pq}^L-\sigma_{pq}^U}{2} + \frac{\rho_{pq}^L\left(1-\sigma_{pq}^L\right)+\rho_{pq}^U\left(1-\sigma_{pq}^U\right)}{2+\sigma_{pq}^L+\sigma_{pq}^U} + 1$, $MF(\widetilde{f}_{pq}) \in [0, 3]$, $p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$.

Step 2: On the basis of the obtained SCM $D = (f_{pq})_{m\times n}$, compute the mean μ of the SCM $D = (f_{pq})_{m\times n}$, where $\mu = \frac{\sum_{p=1}^{m}\sum_{q=1}^{n} f_{pq}}{m\times n}$, $p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$ and compute the standard deviation (STD) δ of the SCM $D = (f_{pq})_{m\times n}$, where $\delta = \sqrt{\frac{1}{m\times n}\sum_{p=1}^{m}\sum_{q=1}^{n}\left(f_{pq}-\mu\right)^2}$, $p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$.

Step 3: On the basis of Eq. (1), compute the z-score z_{pq} of each element f_{pq} in the obtained SCM $D = (f_{pq})_{m\times n}$ to construct the standard score matrix (SSCM) $Z = (z_{pq})_{m\times n}$, where.

$$z_{pq} = \begin{cases} \frac{f_{pq}-\mu}{\delta}, & \text{if } \sigma \neq 0 \\ 0, & \text{if } \sigma = 0 \end{cases} \tag{8}$$

where $p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$.

Step 4: On the basis of Eq. (4) and the IVIF weight $\widetilde{s}_q$ of attribute S_q, where $\widetilde{s}_q = \left(\left[k_q^L, k_q^U\right], \left[c_q^L, c_q^U\right]\right)$, calculate the converted weight s_q of attribute S_q, where $s_q = \frac{k_q^L+k_q^U-c_q^L-c_q^U}{2} + \frac{k_q^L\left(1-c_q^L\right)+k_q^U\left(1-c_q^U\right)}{2+c_q^L+c_q^U} + 1$, $s_q \in [0, 3]$ and $q = 1, 2, \cdots, n$.

Step 5: On the basis of the obtained converted weights $s_1, s_2, \ldots$, and s_q of the attributes $S_1, S_2, \ldots$, and S_q, respectively, and the obtained SSCM Z, calculate the weighted score ws_p of alternative T_p, shown as follows:

$$ws_p = \sum_{q=1}^{n} s_q \times z_{pq}, \tag{9}$$

where $1 \le p \le m$. According to the weighted scores $ws_1, ws_2, \ldots$, and ws_m of the alternatives $T_1, T_2, \ldots$, and T_m, respectively, rank the alternatives $T_1, T_2, \ldots$, and T_m. The larger the value of ws_p, the better the preference order (PO) of alternative T_p, where $p = 1, 2, \cdots, m$.

However, Chen and Tsai's MADM method [16] has the shortcoming that it is not able to distinguish the POs of alternatives in some circumstances. In what follows, we use three counter examples to illustrate the shortcomings of Chen and Tsai's MADM method [16].

Example 4.1. Let T_1 and T_2 be two alternatives, let S_1 and S_2 be two benefit type attributes and let $\widetilde{s}_1$ and $\widetilde{s}_2$ be the IVIF weights of the attributes S_1 and S_2 given by the DMK, respectively, shown as follows:

$$\widetilde{s}_1 = ([0.15, 0.25], [0.30, 0.30]),$$

$$\widetilde{s}_2 = ([0.25, 0.50], [0.35, 0.35]).$$

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{2\times 2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{2\times 2}$ given by the DMK is shown as follows:

$$\tilde{D} = \left(\tilde{f}_{pq}\right)_{2\times 2} = \begin{array}{c} \\ T_1 \\ T_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} ([0.30, 0.60], [0, 0.10]) & ([0.20, 0.30], [0, 0]) \\ ([0.50, 0.60], [0, 0.30]) & ([0.10, 0.40], [0, 0]) \end{pmatrix} \end{array}.$$

where $p = 1, 2$ and $q = 1, 2$.

[Step 1] Because S_1 and S_2 are benefit type attributes, on the basis of Eq. (7) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{2\times 2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{2\times 2}$, Chen and Tsai's MADM method [16] constructs the SCM $D = (f_{pq})_{2\times 2}$, where $f_{pq} = MF(\tilde{f}_{pq}) = \frac{\rho^L_{pq}+\rho^U_{pq}-\sigma^L_{pq}-\sigma^U_{pq}}{2} + \frac{\rho^L_{pq}\left(1-\sigma^L_{pq}\right)+\rho^U_{pq}\left(1-\sigma^U_{pq}\right)}{2+\sigma^L_{pq}+\sigma^U_{pq}} + 1$, $p = 1, 2$, $q = 1, 2$, $MF\left(\tilde{f}_{11}\right) = 1.800$, $MF\left(\tilde{f}_{12}\right) = 1.500$, $MF\left(\tilde{f}_{21}\right) = 1.800$, $MF\left(\tilde{f}_{22}\right) = 1.500$, and

$$D = \left(f_{pq}\right)_{2\times 2} = \begin{array}{c} \\ T_1 \\ T_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} 1.8 & 1.5 \\ 1.8 & 1.5 \end{pmatrix} \end{array}.$$

[Step 2] On the basis of the obtained SCM $D = (f_{pq})_{2\times 2}$, Chen and Tsai's MADM method [16] computes the mean μ and the STD δ of the SCM $D = (f_{pq})_{2\times 2}$, where $\mu = \frac{\sum_{p=1}^{2}\sum_{q=1}^{2} f_{pq}}{4} = 1.65$ and

$$\delta = \sqrt{\frac{1}{4}\sum_{p=1}^{2}\sum_{q=1}^{2}\left(f_{pq} - \mu\right)^2} = 0.150.$$

[Step 3] On the basis of Eq. (8), the obtained mean μ of SCM $D = (f_{pq})_{2\times 2}$, where $\mu = 1.65$, and the obtained STD δ of the SCM $D = (f_{pq})_{2\times 2}$, where $\delta = 0.150$, Chen and Tsai's MADM method [16] computes the z-score z_{pq} of each element f_{pq} in the obtained SCM $D = (f_{pq})_{2\times 2}$ to construct the SSCM $Z = (z_{pq})_{2\times 2}$, where $p = 1, 2$, $q = 1, 2$, $z_{11} = \frac{f_{11}-\mu}{\delta} = 1$, $z_{12} = \frac{f_{12}-\mu}{\delta} = -1$, $z_{21} = \frac{f_{21}-\mu}{\delta} = 1$, $z_{22} = \frac{f_{22}-\mu}{\delta} = -1$, and

$$Z = \left(z_{pq}\right)_{2\times 2} = \begin{array}{c} \\ T_1 \\ T_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \end{array}.$$

[Step 4] On the basis of Eq. (4) and the IVIF weights $\tilde{s}_1$ and $\tilde{s}_2$ of the attributes S_1 and S_2, respectively, where $\tilde{s}_1 = ([0.15, 0.25], [0.30, 0.30])$ and $\tilde{s}_2 = ([0.25, 0.50], [0.35, 0.35])$, Chen and Tsai's MADM method [16] computes the converted weights s_1 and s_2 of the attributes S_1 and S_2, respectively, where $s_1 = MF\left(\tilde{s}_1\right) = 1.008$ and $s_2 = MF\left(\tilde{s}_2\right) = 1.206$.

[Step 5] On the basis of Eq. (9), the obtained converted weights s_1 and s_2 of the attributes S_1 and S_2, respectively, where $s_1 = 1.008$ and $s_2 = 1.206$, and the obtained SSCM $Z = (z_{pq})_{2\times 2}$, Chen and Tsai's MADM method [16] computes the weighted score ws_p of alternative T_p, where $ws_p = \sum_{q=1}^{2} s_q \times z_{pq}$, $p = 1, 2$, $ws_1 = -0.198$ and $ws_2 = -0.198$. Because the weighted scores $ws_1 = ws_2$, where $ws_1 = -0.198$ and $ws_2 = -0.198$, the PO of the alternatives T_1 and T_2 obtained by Chen and Tsai's MADM method [16] is: "$T_1 = T_2$". However, from the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{2\times 2}$ provided by the DMK, we can observe that the evaluating IVIFVs of the attributes S_1 and S_2 with respect to the alternatives T_1 and T_2, respectively, are not the same. Thus, the PO of the alternatives T_1 and T_2 should not be the same. Hence, Chen and Tsai's MADM method [16] has the shortcoming that it is not able to distinguish the PO of the alternatives T_1 and T_2 in this circumstance.

Example 4.2. Let T_1, T_2 and T_3 be three alternatives, let S_1 and S_2 be two benefit type attributes and let $\tilde{s}_1$ and $\tilde{s}_2$ be the IVIF weights of the attributes S_1 and S_2 given by the DMK, respectively, shown as follows:

$\tilde{s}_1 = ([0.15, 0.35], [0.10, 0.30])$,

$\tilde{s}_2 = ([0.35, 0.50], [0.25, 0.50])$.

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times2}$ given by the DMK is shown as follows:

$$\tilde{D} = (\tilde{f}_{pq})_{3\times2} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cc} \quad S_1 \quad\quad\quad\quad\quad\quad\quad & \quad\quad\quad\quad S_2 \end{array} \\ \begin{pmatrix} ([0.10, 0.50], [0.20, 0.20]) & ([0.50, 0.60], [0.10, 0.10]) \\ ([0.30, 0.30], [0.10, 0.30]) & ([0.30, 0.60], [0, 0]) \\ ([0.20, 0.40], [0.20, 0.20]) & ([0.40, 0.70], [0.10, 0.10]) \end{pmatrix} \end{array},$$

where $p = 1, 2, 3$ and $q = 1, 2$.

[Step 1] Because S_1 and S_2 are benefit type attributes, on the basis of Eq. (7) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times2}$, Chen and Tsai's MADM method [16] constructs the SCM $D = (f_{pq})_{3\times2}$, where f_{pq}=$MF(\tilde{f}_{pq}) = \frac{\rho^L_{pq}+\rho^U_{pq}-\sigma^L_{pq}-\sigma^U_{pq}}{2} + \frac{\rho^L_{pq}\left(1-\sigma^L_{pq}\right)+\rho^U_{pq}\left(1-\sigma^U_{pq}\right)}{2+\sigma^L_{pq}+\sigma^U_{pq}} + 1$, $p = 1, 2, 3$, $q = 1, 2$, $MF(\tilde{f}_{11}) = 1.300$, $MF\left(\tilde{f}_{12}\right) = 1.900$, $MF\left(\tilde{f}_{21}\right) = 1.300$, $MF\left(\tilde{f}_{22}\right) = 1.900$, $MF\left(\tilde{f}_{31}\right) = 1.300$, $MF\left(\tilde{f}_{32}\right) = 1.900$, and

$$D = (f_{pq})_{3\times2} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} 1.3 & 1.9 \\ 1.3 & 1.9 \\ 1.3 & 1.9 \end{pmatrix} \end{array}.$$

[Step 2] On the basis of the obtained SCM $D = (f_{pq})_{3\times2}$, Chen and Tsai's MADM method [16] computes the mean μ and the STD δ of the SCM $D = (f_{pq})_{3\times2}$, where $\mu = \frac{\sum_{p=1}^{3}\sum_{q=1}^{2} f_{pq}}{6} = 1.6$ and

$$\delta = \sqrt{\frac{1}{6}\sum_{p=1}^{3}\sum_{q=1}^{2}\left(f_{pq}-\mu\right)^2} = 0.3.$$

[Step 3] On the basis of Eq. (8), the obtained mean μ of SCM $D = (f_{pq})_{3\times2}$, where $\mu = 1.6$, and the obtained STD δ of the SCM $D = (f_{pq})_{3\times2}$, where $\delta = 0.3$, Chen and Tsai's MADM method [16] computes the z-score z_{pq} of each element f_{pq} in the obtained SCM $D = (f_{pq})_{3\times2}$ to construct the SSCM $Z = (z_{pq})_{3\times2}$, where $p = 1, 2, 3$, $q = 1, 2$, $z_{11} = \frac{f_{11}-\mu}{\delta} = -1$, $z_{12} = \frac{f_{12}-\mu}{\delta} = 1$, $z_{21} = \frac{f_{21}-\mu}{\delta} = -1$, $z_{22} = \frac{f_{22}-\mu}{\delta} = 1$, $z_{31} = \frac{f_{31}-\mu}{\delta} = -1$, $z_{32} = \frac{f_{32}-\mu}{\delta} = 1$, and

$$Z = (z_{pq})_{3\times2} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix} \end{array}.$$

[Step 4] On the basis of Eq. (4) and the IVIF weights $\tilde{s}_1$ and $\tilde{s}_2$ of the attributes S_1 and S_2, respectively, where $\tilde{s}_1 = ([0.15, 0.35], [0.10, 0.30])$ and $\tilde{s}_2 = ([0.35, 0.50], [0.25, 0.50])$, Chen and Tsai's MADM method [16] computes the converted weights s_1 and s_2 of the attributes S_1 and S_2, respectively, where $s_1 = MF\left(\tilde{s}_1\right) = 1.208$ and $s_2 = MF\left(\tilde{s}_2\right) = 1.236$.

[Step 5] On the basis of Eq. (9), the obtained converted weights s_1 and s_2 of the attributes S_1 and S_2, respectively, where $s_1 = 1.208$ and $s_2 = 1.236$, and the obtained SSCM $Z = (z_{pq})_{3\times2}$, Chen and Tsai's MADM method [16] computes the weighted score ws_p of alternative T_p, where $ws_p = \sum_{q=1}^{2} s_q \times z_{pq}$, $p = 1, 2, 3$, $ws_1 = 0.028$, $ws_2 = 0.028$ and.

$ws_3 = 0.028$. Because the weighted scores $ws_1 = ws_2 = ws_3$, where $ws_1 = 0.028$, $ws_2 = 0.028$ and $ws_3 = 0.028$, the PO of the alternatives T_1, T_2 and T_3 obtained by Chen and Tsai's MADM method [16] is: "$T_1 = T_2 = T_3$". However, from the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times2}$ provided by the DMK, we can observe that the evaluating IVIFVs of the attributes S_1 and S_2 with respect the alternatives T_1, T_2 and T_3 are not the same. Thus, the PO of the alternatives T_1, T_2 and T_3 should not be the same. Hence, Chen and Tsai's MADM method [16] has the shortcoming that it is not able to distinguish the PO of the alternatives T_1, T_2 and T_3 in this circumstance.

Example 4.3. Let T_1, T_2 and T_3 be three alternatives, let S_1, S_2 and S_3 be three benefit type attributes and let $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ be the IVIF weights of the attributes S_1, S_2 and S_3 provided by the DMK, respectively, shown as follows:

$$\tilde{s}_1 = ([0.10, 0.30], [0.20, 0.20]),$$

$$\tilde{s}_2 = ([0.30, 0.50], [0.40, 0.40]),$$

$$\tilde{s}_3 = ([0.20, 0.50].[0.30, 0.30]).$$

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times 3} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times 3}$ provided by the DMK is shown as follows:

$$\tilde{D} = (\tilde{f}_{pq})_{3\times 3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} ([0.30, 0.30], [0.50, 0.50]) & ([0.30, 0.30], [0, 0]) & ([0.50, 0.50], [0, 0]) \\ ([0.30, 0.30], [0.40, 0.60]) & ([0, 0.70], [0, 0.10]) & ([0.40, 0.60], [0, 0]) \\ ([0.20, 0.40], [0.50, 0.50]) & ([0.10, 0.50], [0, 0]) & ([0.30, 0.70], [0, 0]) \end{pmatrix} \end{array},$$

where $p = 1, 2, 3$ and $q = 1, 2, 3$.

[Step 1] Because S_1, S_2 and S_3 are benefit type attributes, on the basis of Eq. (7) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times 3} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times 3}$, Chen and Tsai's MADM method [16] constructs the SCM $D = (f_{pq})_{3\times 3}$, where f_{pq}=$MF(\tilde{f}_{pq}) = \frac{\rho^L_{pq}+\rho^U_{pq}-\sigma^L_{pq}-\sigma^U_{pq}}{2} + \frac{\rho^L_{pq}\left(1-\sigma^L_{pq}\right)+\rho^U_{pq}\left(1-\sigma^U_{pq}\right)}{2+\sigma^L_{pq}+\sigma^U_{pq}} + 1$, $p = 1, 2, 3$, $q = 1, 2, 3$, $MF\left(\tilde{f}_{11}\right) = 0.900$, $MF\left(\tilde{f}_{12}\right) = 1.600$, $MF\left(\tilde{f}_{13}\right) = 2$, $MF\left(\tilde{f}_{21}\right) = 0.900$, $MF\left(\tilde{f}_{22}\right) = 1.600$, $MF\left(\tilde{f}_{23}\right) = 2$, $MF\left(\tilde{f}_{31}\right) = 0.900$, $MF\left(\tilde{f}_{32}\right) = 1.600$, $MF\left(\tilde{f}_{33}\right) = 2$, and

$$D = (f_{pq})_{3\times 3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} 0.9 & 1.6 & 2 \\ 0.9 & 1.6 & 2 \\ 0.9 & 1.6 & 2 \end{pmatrix} \end{array}.$$

[Step 2] On the basis of the obtained SCM $D = (f_{pq})_{3\times 3}$, Chen and Tsai's MADM method [16] computes the mean μ and the STD δ of the SCM $D = (f_{pq})_{3\times 3}$, where $\mu = \frac{\sum_{p=1}^{3}\sum_{q=1}^{3} f_{pq}}{9} = 1.5$ and $\delta = \sqrt{\frac{1}{9}\sum_{p=1}^{3}\sum_{q=1}^{3}\left(f_{pq} - \mu\right)^2} = 0.455$.

[Step 3] On the basis of Eq. (8), the obtained mean μ of the SCM $D = (f_{pq})_{3\times 3}$, where $\mu = 1.5$, and the obtained STD δ of the SCM $D = (f_{pq})_{3\times 3}$, where $\delta = 0.455$, Chen and Tsai's MADM method [16] computes the z-score z_{pq} of each element f_{pq} in the obtained SCM $D = (f_{pq})_{3\times 3}$ to construct the SSCM $Z = (z_{pq})_{3\times 3}$, where $p = 1, 2, 3$, $q = 1, 2, 3$, $z_{11} = \frac{f_{11}-\mu}{\delta} = -1.319$, $z_{12} = \frac{f_{12}-\mu}{\delta} = 0.22$, $z_{13} = \frac{f_{13}-\mu}{\delta} = 1.099$, $z_{21} = \frac{f_{21}-\mu}{\delta} = -1.319$, $z_{22} = \frac{f_{22}-\mu}{\delta} = 0.22$, $z_{23} = \frac{f_{23}-\mu}{\delta} = 1.099$, $z_{31} = \frac{f_{31}-\mu}{\delta} = -1.319$, $z_{32} = \frac{f_{32}-\mu}{\delta} = 0.22$, $z_{33} = \frac{f_{33}-\mu}{\delta} = 1.099$, and

$$Z = (z_{pq})_{3\times 3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} -1.319 & 0.22 & 1.099 \\ -1.319 & 0.22 & 1.099 \\ -1.319 & 0.22 & 1.099 \end{pmatrix} \end{array}.$$

[Step 4] On the basis of Eq. (4) and the IVIF weights $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ of the attributes S_1, S_2 and S_3, respectively, where $\tilde{s}_1 = ([0.10, 0.30], [0.20, 0.20])$, $\tilde{s}_2 = ([0.30, 0.50], [0.40, 0.40])$ and $\tilde{s}_3 = ([0.20, 0.50], [0.30, 0.30])$, Chen and Tsai's MADM method [16] computes the converted weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, where $s_1 = MF\left(\tilde{s}_1\right) = 1.133$, $s_2 = MF\left(\tilde{s}_2\right) = 1.171$ and $s_3 = MF\left(\tilde{s}_3\right) = 1.238$.

[Step 5] On the basis of Eq. (9), the obtained converted weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, where $s_1 = 1.133$, $s_2 = 1.171$ and $s_3 = 1.238$, and the obtained SSCM $Z = (z_{pq})_{3\times3}$, Chen and Tsai's MADM method [16] computes the weighted score ws_p of alternative T_p, where $ws_p = \sum_{q=1}^{3} s_q \times z_{pq}$, $p = 1, 2, 3$, $ws_1 = 0.124$, $ws_2 = 0.124$ and $ws_3 = 0.124$. Because the weighted scores $ws_1 = ws_2 = ws_3$, where $ws_1 = 0.124$, $ws_2 = 0.124$ and $ws_3 = 0.124$, the PO of the alternatives T_1, T_2 and T_3 obtained by Chen and Tsai's MADM method [16] is: "$T_1 = T_2 = T_3$". However, from the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3}$ provided by the DMK, we can observe that the evaluating IVIFVs of the attributes S_1, S_2 and S_3 with respect to the alternatives T_1, T_2 and T_3 are not the same. Thus, the PO of the alternatives T_1, T_2 and T_3 should be not be the same. Hence, Chen and Tsai's MADM method [16] has the shortcoming that it is not able to distinguish the PO of the alternatives T_1, T_2 and T_3 under this circumstance.

In order to overcome the shortcomings of Chen and Tsai's MADM method [16], we will propose a new MADM method on the basis of the proposed score function HF shown in Eq. (6) and the NLP methodology in the next section.

5. A new MADM method based on the NLP methodology and the proposed score function of IVIFVs

In this section, we propose a new MADM method on the basis of the NLP methodology and the proposed score function HF of IVIFVs. Assume that $T_1, T_2, \cdots$, and T_m are the alternatives, assume that $S_1, S_2, \cdots$, and S_n are the attributes, and assume that $\tilde{D} = \left(\tilde{f}_{pq}\right)_{m\times n} = \left(\left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)\right)_{m\times n}$ is the DM given by the decision maker (DMK), where $\tilde{f}_{pq} = \left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)$ is an IVIFV, where $0 \le \rho_{pq}^L \le \rho_{pq}^U \le 1$, $0 \le \sigma_{pq}^L \le \sigma_{pq}^U \le 1$, $0 \le \rho_{pq}^U + \sigma_{pq}^U \le 1$, $p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$. Assume that the IVIF weight of attribute S_q given by the DMK is $\tilde{s}_q$, where $\tilde{s}_q = \left(\left[k_q^L, k_q^U\right], \left[c_q^L, c_q^U\right]\right)$ is an IVIFV, where $0 \le k_q^L \le k_q^U \le 1$, $0 \le c_q^L \le c_q^U \le 1$, $0 \le k_q^U + c_q^U \le 1$ and $q = 1, 2, \cdots, n$. The proposed method for MADM is shown as follows:

Step 1: On the basis of the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{m\times n} = \left(\left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)\right)_{m\times n}$ given by the DMK and the proposed score function HF of IVIFVs shown in Eq. (6), construct the converted matrix $D = (f_{pq})_{m\times n} = \left(HF\left(\tilde{f}_{pq}\right)\right)_{m\times n}$, where

$$f_{pq} = \begin{cases} HF\left(\tilde{f}_{pq}\right), \textit{if } S_q \textit{ is a benifit type attribute} \\ 6 - HF\left(\tilde{f}_{pq}\right), \textit{if } S_q \textit{ is a cost type attribute} \end{cases} \tag{10}$$

$$HF\left(\tilde{f}_{pq}\right) = \frac{\rho_{pq}^L - \sigma_{pq}^L + \rho_{pq}^U - \sigma_{pq}^U}{2} + \frac{sin\left(\rho_{pq}^L \times \frac{\pi}{2}\right) + sin\left(\rho_{pq}^U \times \frac{\pi}{2}\right) + sin\left(\left(1 - \sigma_{pq}^L\right) \times \frac{\pi}{2}\right) + sin\left(\left(1 - \sigma_{pq}^U\right) \times \frac{\pi}{2}\right)}{2} + 2,$$

$p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$.

Step 2: On the basis of the deviation variables [18], the obtained converted matrix $D = (f_{pq})_{m\times n}$ and the IVIF weight $\tilde{s}_q = \left(\left[k_q^L, k_q^U\right], \left[c_q^L, c_q^U\right]\right)$ of attribute S_q given by the DMK, where $q = 1, 2, \cdots, n$, build the following NLP model:

$$Maximize \sum_{q=1}^{n} tanh\left(\sum_{p=1}^{m}\sum_{x=1}^{m}\left|f_{pq}{}^{s_q} - f_{xq}{}^{s_q}\right|\right) + \sum_{p=1}^{m}\sum_{q=1}^{n} tanh\left(f_{pq}{}^{s_q}\right) - \sum_{q=1}^{n}\left(o_q^- + o_q^+\right)$$

$$s.t.\begin{cases} k_q^L - o_q^- \le s_q \le 1 - c_q^L + o_q^+ \\ \sum_{q=1}^{n} s_q = 1 \\ 0 \le s_q \le 1 \\ 0 \le o_q^- \le 1 \\ 0 \le o_q^+ \le 1 \\ q = 1, 2, \cdots, n \end{cases} \tag{11}$$

Solve the NLP model to obtain the optimal left shift value o_q^-, the optimal right shift value o_q^+ and the optimal weights s_1, $s_2, \cdots$, and s_n of the attributes $S_1, S_2, \cdots$, and S_n, respectively, where $q = 1, 2, \cdots, n$.

Step 3: On the basis of the converted matrix $D = (f_{pq})_{m\times n}$ obtained by the proposed score function HF of IVIFVs in **Step 1**, the optimal weight s_q of attribute S_q obtained by the NLP model in **Step 2**, compute the weighted score ws_p of alternative T_p, where

$$ws_p = \sum_{q=1}^{n} \left(f_{pq}\right)^{s_q}, \tag{12}$$

$p = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, n$.

Step 4: Rank the alternatives $T_1, T_2, \cdots$, and T_m according to the obtained weighted scores $ws_1, ws_2, \cdots$, and ws_m. The larger the weighted score ws_p, the better the PO of alternative T_p, where $p = 1, 2, \cdots, m$.

In order to show that the proposed MADM method can conquer the shortcomings of Chen and Tsai's MADM method [16], we apply the proposed MADM method to deal with the three examples shown in **Section 4** to show that that the proposed MADM method can conquer the shortcomings of Chen and Tsai's MADM method [16].

Example 5.1. Let T_1 and T_2 be two alternatives, let S_1 and S_2 be two benefit type attributes and let $\tilde{s}_1$ and $\tilde{s}_2$ be the IVIF weights of the attributes S_1 and S_2 provided by the DMK, respectively, shown as follows:

$$\tilde{s}_1 = ([0.15, 0.25], [0.30, 0.30]),$$

$$\tilde{s}_2 = ([0.25, 0.50], [0.35, 0.35]).$$

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{2\times 2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{2\times 2}$ provided by the DMK is as follows:

$$\tilde{D} = \left(\tilde{f}_{pq}\right)_{2\times 2} = \begin{array}{c c} & \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{array}{c} T_1 \\ T_2 \end{array} & \begin{pmatrix} ([0.30, 0.60], [0, 0.10]) & ([0.20, 0.30], [0, 0]) \\ ([0.50, 0.60], [0, 0.30]) & ([0.10, 0.40], [0, 0]) \end{pmatrix} \end{array},$$

where $p = 1, 2$ and $q = 1, 2$.

[Step 1] Because the attributes S_1 and S_2 are benefit type attributes, on the basis of Eq. (10) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{2\times 2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{2\times 2}$, we get the converted matrix $D = \left(f_{pq}\right)_{2\times 2} = \left(HF\left(\tilde{f}_{pq}\right)\right)_{2\times 2}$, where $HF\left(\tilde{f}_{pq}\right) = \frac{\rho^L_{pq} - \sigma^L_{pq} + \rho^U_{pq} - \sigma^U_{pq}}{2} + \frac{\sin\left(\rho^L_{pq}\times\frac{\pi}{2}\right) + \sin\left(\rho^U_{pq}\times\frac{\pi}{2}\right) + \sin\left(\left(1-\sigma^L_{pq}\right)\times\frac{\pi}{2}\right) + \sin\left(\left(1-\sigma^U_{pq}\right)\times\frac{\pi}{2}\right)}{2} + 2$, $p = 1, 2$, $q = 1, 2$, $f_{11} = 4.025$, $f_{12} = 3.632$, $f_{21} = 4.104$ and $f_{22} = 3.622$. Therefore, we get the converted matrix $D = \left(f_{pq}\right)_{2\times 2}$, shown as follows:

$$D = \left(f_{pq}\right)_{2\times 2} = \begin{array}{c c} & \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{array}{c} T_1 \\ T_2 \end{array} & \begin{pmatrix} 4.025 & 3.632 \\ 4.104 & 3.622 \end{pmatrix} \end{array}.$$

[Step 2] On the basis of Eq. (11), the converted matrix $D = \left(f_{pq}\right)_{2\times 2}$ obtained in **Step 1** and the IVIF weights $\tilde{s}_1$ and $\tilde{s}_2$ of the attributes S_1 and S_2 given by the DMK, respectively, we construct the following NLP model:

$$Maximize \sum_{q=1}^{2} tanh\left(\sum_{p=1}^{2}\sum_{x=1}^{2}\left|{f_{pq}}^{s_q} - {f_{xq}}^{s_q}\right|\right) + \sum_{p=1}^{2}\sum_{q=1}^{2} tanh\left({f_{pq}}^{s_q}\right) - \sum_{q=1}^{2}\left(o^-_q + o^+_q\right)$$

$$s.t.\begin{cases} k^L_q - o^-_q \le s_q \le 1 - c^L_q + o^+_q \\ \sum_{q=1}^{2} s_q = 1 \\ 0 \le s_q \le 1 \\ 0 \le o^-_q \le 1 \\ 0 \le o^+_q \le 1 \\ q = 1, 2 \end{cases}$$

After solving the NLP model, we get the optimal left shift values o^-_1 and o^-_2, the optimal right shift values o^+_1 and o^+_2 and the optimal weights s_1 and s_2 of the attributes S_1 and S_2, respectively, where $o^-_1 = 0$, $o^-_2 = 0$, $o^+_1 = 0$, $o^+_2 = 0$, $s_1 = 0.533$ and $s_2 = 0.467$.

[Step 3] On the basis of Eq. (12), the converted matrix $D = \left(f_{pq}\right)_{2\times 2}$ obtained in **Step 1** and the optimal weights s_1 and s_2 of the attributes S_1 and S_2, respectively, obtained in **Step 2**, we calculate the weighted score ws_p of alternative T_p, where $p = 1, 2$,

$$ws_1 = {f_{11}}^{s_1} + {f_{12}}^{s_2} = 4.025^{0.533} + 3.632^{0.467} = 3.927,$$

$$ws_2 = {f_{21}}^{s_1} + {f_{22}}^{s_2} = 4.104^{0.533} + 3.622^{0.467} = 3.946.$$

[Step 4] Because $ws_2 > ws_1$, where $ws_1 = 3.927$ and $ws_2 = 3.946$, the PO of the alternatives T_1 and T_2 is: $T_2 \succ T_1$. Thus, the proposed MADM method can overcome the drawback of Chen and Tsai's MADM method [16], which it is not able to distinguish the PO of the alternatives T_1 and T_2, as shown in ***Example 4.1***.

Example 5.2. Let T_1, T_2 and T_3 be three alternatives, let S_1 and S_2 be two benefit type attributes and let $\tilde{s}_1$ and $\tilde{s}_2$ be the IVIF weights of the attributes S_1 and S_2 provided by the DMK, respectively, shown as follows:

$$\tilde{s}_1 = ([0.15, 0.35], [0.10, 0.30]),$$

$$\tilde{s}_2 = ([0.35, 0.50], [0.25, 0.50]).$$

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times 2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times 2}$ provided by the DMK is as follows:

$$\tilde{D} = (\tilde{f}_{pq})_{3\times 2} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cc} \qquad\qquad S_1 \qquad\qquad & \qquad\qquad S_2 \qquad\qquad \end{array} \\ \begin{pmatrix} ([0.10, 0.50], [0.20, 0.20]) & ([0.50, 0.60], [0.10, 0.10]) \\ ([0.30, 0.30], [0.10, 0.30]) & ([0.30, 0.60], [0, 0]) \\ ([0.20, 0.40], [0.20, 0.20]) & ([0.40, 0.70], [0.10, 0.10]) \end{pmatrix} \end{array},$$

where $p = 1, 2, 3$ and $q = 1, 2$.

[Step 1] Because the attributes S_1 and S_2 are benefit type attributes, on the basis of Eq. (10) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times 2} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times 2}$, we get the converted matrix $D = (f_{pq})_{3\times 2} = \left(HF\left(\tilde{f}_{pq}\right)\right)_{3\times 2}$, where $HF\left(\tilde{f}_{pq}\right) = \frac{\rho^L_{pq} - \sigma^L_{pq} + \rho^U_{pq} - \sigma^U_{pq}}{2} + \frac{\sin\left(\rho^L_{pq} \times \frac{\pi}{2}\right) + \sin\left(\rho^U_{pq} \times \frac{\pi}{2}\right) + \sin\left(\left(1 - \sigma^L_{pq}\right) \times \frac{\pi}{2}\right) + \sin\left(\left(1 - \sigma^U_{pq}\right) \times \frac{\pi}{2}\right)}{2} + 2$, $p = 1, 2, 3$, $q = 1, 2$, $f_{11} = 3.483$, $f_{12} = 4.196$, $f_{21} = 3.493$, $f_{22} = 4.082$, $f_{31} = 3.499$ and $f_{32} = 4.177$. Therefore, we get the converted matrix $D = (f_{pq})_{3\times 2}$, shown as follows:

$$D = (f_{pq})_{3\times 2} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & \quad S_2 \end{array} \\ \begin{pmatrix} 3.483 & 4.196 \\ 3.493 & 4.082 \\ 3.499 & 4.177 \end{pmatrix} \end{array}.$$

[Step 2] On the basis of Eq. (11), the converted matrix $D = (f_{pq})_{3\times 2}$ obtained in **Step 1** and the IVIF weights $\tilde{s}_1$ and $\tilde{s}_2$ of the attributes S_1 and S_2 given by the DMK, respectively, we construct the following NLP model:

$$Maximize \sum_{q=1}^{2} tanh\left(\sum_{p=1}^{3}\sum_{x=1}^{3} \left|{f_{pq}}^{s_q} - {f_{xq}}^{s_q}\right|\right) + \sum_{p=1}^{3}\sum_{q=1}^{2} tanh({f_{pq}}^{s_q}) - \sum_{q=1}^{2}\left(o_q^- + o_q^+\right)$$

$$s.t.\begin{cases} k_q^L - o_q^- \le s_q \le 1 - c_q^L + o_q^+ \\ \sum_{q=1}^{2} s_q = 1 \\ 0 \le s_q \le 1 \\ 0 \le o_q^- \le 1 \\ 0 \le o_q^+ \le 1 \\ q = 1, 2 \end{cases}$$

After solving the NLP model, we get the optimal left shift values o_1^- and o_2^-, the optimal right shift values o_1^+ and o_2^+ and the optimal weights s_1 and s_2 of the attributes S_1 and S_2, respectively, where $o_1^- = 0$, $o_2^- = 0$, $o_1^+ = 0$, $o_2^+ = 0$, $s_1 = 0.423$ and $s_2 = 0.577$.

[Step 3] On the basis of Eq. (12), the converted matrix $D = (f_{pq})_{3\times 2}$ obtained in **Step 1** and the optimal weights s_1 and s_2 of the attributes S_1 and S_2, respectively, obtained in **Step 2**, calculate the weighted score ws_p of alternative T_p, where $p = 1, 2, 3$,

$$ws_1 = {f_{11}}^{s_1} + {f_{12}}^{s_2} = 3.483^{0.423} + 4.196^{0.577} = 3.983,$$

$$ws_2 = {f_{21}}^{s_1} + {f_{22}}^{s_2} = 3.493^{0.423} + 4.082^{0.577} = 3.949,$$

$$ws_3 = {f_{31}}^{s_1} + {f_{32}}^{s_2} = 3.499^{0.423} + 4.177^{0.577} = 3.980.$$

[Step 4] Because $ws_1 > ws_3 > ws_2$, where $ws_1 = 3.983$, $ws_2 = 3.949$ and $ws_3 = 3.980$, the PO of the alternatives T_1, T_2 and T_3 is: $T_1 > T_3 > T_2$. Thus, the proposed MADM method can overcome the drawback of Chen and Tsai's MADM method [16], which it is not able to distinguish the PO of the alternatives T_1, T_2 and T_3 in this situation, as shown in ***Example 4.2***.

Example 5.3. Let T_1, T_2 and T_3 be three alternatives, let S_1, S_2 and S_3 be three benefit type attributes and let $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ be the IVIF weights of the attributes S_1, S_2 and S_3 provided by the DMK, respectively, shown as follows:

$$\tilde{s}_1 = ([0.10, 0.30], [0.20, 0.20]),$$

$$\tilde{s}_2 = ([0.30, 0.50], [0.40, 0.40]),$$

$$\tilde{s}_3 = ([0.20, 0.50], [0.30, 0.30]).$$

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times3}$ provided by the DMK is as follows:

$$\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} ([0.30, 0.30], [0.50, 0.50]) & ([0.30, 0.30], [0, 0]) & ([0.50, 0.50], [0, 0]) \\ ([0.30, 0.30], [0.40, 0.60]) & ([0, 0.70], [0, 0.10]) & ([0.40, 0.60], [0, 0]) \\ ([0.20, 0.40], [0.50, 0.50]) & ([0.10, 0.50], [0, 0]) & ([0.30, 0.70], [0, 0]) \end{pmatrix} \end{array},$$

where $p = 1, 2, 3$ and $q = 1, 2, 3$.

[Step 1] Because the attributes S_1, S_2 and S_3 are benefit type attributes, on the basis of Eq. (10) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3} = \left(\left(\left[\rho^L_{pq}, \rho^U_{pq}\right], \left[\sigma^L_{pq}, \sigma^U_{pq}\right]\right)\right)_{3\times3}$, we get the converted matrix $D = (f_{pq})_{3\times3} = \left(HF\left(\tilde{f}_{pq}\right)\right)_{3\times3}$, where $HF\left(\tilde{f}_{pq}\right) = \frac{\rho^L_{pq} - \sigma^L_{pq} + \rho^U_{pq} - \sigma^U_{pq}}{2} + \frac{sin\left(\rho^L_{pq}\times\frac{\pi}{2}\right) + sin\left(\rho^U_{pq}\times\frac{\pi}{2}\right) + sin\left(\left(1-\sigma^L_{pq}\right)\times\frac{\pi}{2}\right) + sin\left(\left(1-\sigma^U_{pq}\right)\times\frac{\pi}{2}\right)}{2} + 2$, $p = 1, 2, 3$, $q = 1, 2, 3$, $f_{11} = 2.961$, $f_{12} = 3.754$, $f_{13} = 4.207$, $f_{21} = 2.952$, $f_{22} = 3.739$, $f_{23} = 4.198$, $f_{31} = 2.956$, $f_{32} = 3.732$ and $f_{33} = 4.172$. Therefore, we get the converted matrix $D = (f_{pq})_{3\times3}$, shown as follows:

$$D = (f_{pq})_{3\times3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} 2.961 & 3.754 & 4.207 \\ 2.952 & 3.739 & 4.198 \\ 2.956 & 3.732 & 4.172 \end{pmatrix} \end{array}.$$

[Step 2] On the basis of Eq. (11), the converted matrix $D = (f_{pq})_{3\times3}$ obtained in **Step 1** and the IVIF weights $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ of attributes S_1, S_2 and S_3 given by the DMK, construct the following NLP model:

$$Maximize \sum_{q=1}^{3} tanh\left(\sum_{p=1}^{3}\sum_{x=1}^{3}\left|{f_{pq}}^{s_q} - {f_{xq}}^{s_q}\right|\right) + \sum_{p=1}^{3}\sum_{q=1}^{3} tanh\left({f_{pq}}^{s_q}\right) - \sum_{q=1}^{3}\left(o^-_q + o^+_q\right)$$

$$s.t.\begin{cases} k^L_q - o^-_q \le s_q \le 1 - c^L_q + o^+_q \\ \sum_{q=1}^{3} s_q = 1 \\ 0 \le s_q \le 1 \\ 0 \le o^-_q \le 1 \\ 0 \le o^+_q \le 1 \\ q = 1, 2, 3 \end{cases}$$

After solving the NLP model, we get the optimal left shift values o_1^-, o_2^- and o_3^-, the optimal right shift values o_1^+, o_2^+ and o_3^+ and the optimal weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, where $o_1^- = 0$, $o_2^- = 0$, $o_3^- = 0$, $o_1^+ = 0$, $o_2^+ = 0$, $o_3^+ = 0$, $s_1 = 0.295$, $s_2 = 0.347$ and $s_3 = 0.359$.

[Step 3] On the basis of Eq. (12), the converted matrix $D = (f_{pq})_{3\times3}$ obtained in **Step 1** and the optimal weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, obtained in **Step 2,** calculate the weighted score ws_p of alternative T_p, where $p = 1, 2, 3$,

$$ws_1 = {f_{11}}^{s_1} + {f_{12}}^{s_2} + {f_{13}}^{s_3} = 2.961^{0.295} + 3.754^{0.347} + 4.207^{0.359} = 4.635,$$

$$ws_2 = {f_{21}}^{s_1} + {f_{22}}^{s_2} + {f_{23}}^{s_3} = 2.952^{0.295} + 3.739^{0.347} + 4.198^{0.359} = 4.63,$$

$$ws_3 = {f_{31}}^{s_1} + {f_{32}}^{s_2} + {f_{33}}^{s_3} = 2.956^{0.295} + 3.732^{0.347} + 4.172^{0.359} = 4.626.$$

[Step 4] Because $ws_1 > ws_2 > ws_3$, where $ws_1 = 4.635$, $ws_2 = 4.63$ and $ws_3 = 4.626$, the PO of the alternatives T_1, T_2 and T_3 is: $T_1 > T_2 > T_3$. Thus, the proposed MADM method can overcome the drawback of Chen and Tsai's MADM method [16], which it is not able to distinguish the PO of the alternatives T_1, T_2 and T_3 in this situation, as shown in ***Example 4.3***.

In what follows, we use two examples to show that the proposed MADM method can also conquer the drawbacks of the MADM methods presented in [7,10,26] and [49].

Example 5.4. [8,9,11,12,15,16,23]**.** Let T_1, T_2 and T_3 be three alternatives, let S_1, S_2 and S_3 be three benefit type attributes and let $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ be the IVIF weights of the attributes S_1, S_2 and S_3 provided by the DMK, respectively, shown as follows:

$$\tilde{s}_1 = ([0.25, 0.25], [0.25, 0.25]),$$

$$\tilde{s}_2 = ([0.35, 0.35], [0.40, 0.40]),$$

$$\tilde{s}_3 = ([0.30, 0.30], [0.65, 0.65]).$$

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3} = \left(\left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)\right)_{3\times3}$ provided by the DMK is as follows:

$$\tilde{D} = (\tilde{f}_{pq})_{3\times3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} ([0.30, 0.30], [0.10, 0.10]) & ([0.60, 0.60], [0.25, 0.25]) & ([0.80, 0.80], [0.20, 0.20]) \\ ([0.20, 0.20], [0.15, 0.15]) & ([0.68, 0.68], [0.20, 0.20]) & ([0.45, 0.45], [0.50, 0.50]) \\ ([0.20, 0.20], [0.45, 0.45]) & ([0.70, 0.70], [0.05, 0.05]) & ([0.60, 0.60], [0.30, 0.30]) \end{pmatrix} \end{array},$$

where $p = 1, 2, 3$ and $q = 1, 2, 3$.

[Step 1] Because the attributes S_1, S_2 and S_3 are benefit type attributes, on the basis of Eq. (10) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3} = \left(\left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)\right)_{3\times3}$, we get the converted matrix $D = (f_{pq})_{3\times3} = \left(HF\left(\tilde{f}_{pq}\right)\right)_{3\times3}$, where $HF\left(\tilde{f}_{pq}\right) = \frac{\rho_{pq}^L - \sigma_{pq}^L + \rho_{pq}^U - \sigma_{pq}^U}{2} + \frac{\sin\left(\rho_{pq}^L \times \frac{\pi}{2}\right) + \sin\left(\rho_{pq}^U \times \frac{\pi}{2}\right) + \sin\left(\left(1 - \sigma_{pq}^L\right) \times \frac{\pi}{2}\right) + \sin\left(\left(1 - \sigma_{pq}^U\right) \times \frac{\pi}{2}\right)}{2} + 2$, $p = 1, 2, 3$, $q = 1, 2, 3$, $f_{11} = 3.642$, $f_{12} = 4.083$, $f_{13} = 4.502$, $f_{21} = 3.331$, $f_{22} = 4.307$, $f_{23} = 3.307$, $f_{31} = 2.819$, $f_{32} = 4.538$ and $f_{33} = 4$. Therefore, we get the converted matrix $D = (f_{pq})_{3\times3}$, shown as follows:

$$D = (f_{pq})_{3\times3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array} \begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} 3.642 & 4.083 & 4.502 \\ 3.331 & 4.307 & 3.307 \\ 2.819 & 4.538 & 4 \end{pmatrix} \end{array}.$$

[Step 2] On the basis of Eq. (11), the converted matrix $D = (f_{pq})_{3\times3}$ obtained in **Step 1** and the IVIF weights $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ of the attributes S_1, S_2 and S_3 given by the DMK, respectively, we construct the following NLP model:

$$Maximize \sum_{q=1}^{3} tanh\left(\sum_{p=1}^{3}\sum_{x=1}^{3}\left|{f_{pq}}^{s_q} - {f_{xq}}^{s_q}\right|\right) + \sum_{p=1}^{3}\sum_{q=1}^{3} tanh\left({f_{pq}}^{s_q}\right) - \sum_{q=1}^{3}\left(o_q^- + o_q^+\right)$$

$$
s.t.\begin{cases} k_q^L - o_q^- \le s_q \le 1 - c_q^L + o_q^+ \\ \sum_{q=1}^{3} s_q = 1 \\ 0 \le s_q \le 1 \\ 0 \le o_q^- \le 1 \\ 0 \le o_q^+ \le 1 \\ q = 1, 2, 3 \end{cases}
$$

After solving the NLP model, we get the optimal left shift values o_1^-, o_2^- and o_3^-, the optimal right shift values o_1^+, o_2^+ and o_3^+ and the optimal weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, where $o_1^- = 0$, $o_2^- = 0$, $o_3^- = 0$, $o_1^+ = 0$, $o_2^+ = 0$, $o_3^+ = 0$, $s_1 = 0.300$, $s_2 = 0.350$ and $s_3 = 0.350$.

[Step 3] On the basis of Eq. (12), the converted matrix $D = (f_{pq})_{3\times3}$ obtained in **Step 1** and the optimal weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, obtained in **Step 2**, calculate the weighted score ws_p of alternative T_p, where $p = 1, 2, 3$, $ws_1 = {f_{11}}^{s_1} + {f_{12}}^{s_2} + {f_{13}}^{s_3} = 4.803$, $ws_2 = {f_{21}}^{s_1} + {f_{22}}^{s_2} + {f_{23}}^{s_3} = 4.622$ and $ws_3 = {f_{31}}^{s_1} + {f_{32}}^{s_2} + {f_{33}}^{s_3} = 4.687$.

[Step 4] Because $ws_1 > ws_3 > ws_2$, where $ws_1 = 4.803$, $ws_2 = 4.622$ and $ws_3 = 4.687$, the PO of the alternatives T_1, T_2 and T_3 is: $T_1 \succ T_3 \succ T_2$.

In the following, we compare the POs of the alternatives obtained by different MADM methods for ***Example 5.4***. The proposed MADM method and the MADM methods presented in [8,9,11,12,15,16,23] obtain the same PO "$T_1 \succ T_3 \succ T_2$" of the alternatives. In [12], Chen and Huang pointed out that the MADM method presented in [26] has the "the division by zero" problem, which causes it unable to get the PO of the alternatives in this circumstance. In [12], Chen and Huang Chan and Huang also pointed out that the MADM method presented in [7] has the shortcoming that it obtains an unreasonable PO of the alternatives in this circumstance. Therefore, the proposed MADM method can conquer the shortcomings of the MADM methods presented in [7] and [26].

Example 5.5. [8,9,11,12,15,16,23]. Let T_1, T_2 and T_3 be three alternatives, let S_1, S_2 and S_3 be three benefit type attributes and let $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ be the IVIF weights of the attributes S_1, S_2 and S_3 provided by the DMK, respectively, shown as follows:

$$\tilde{s}_1 = ([0, 0], [0, 0]),$$

$$\tilde{s}_2 = ([0, 0], [0, 0]),$$

$$\tilde{s}_3 = ([0, 0], [0, 0]).$$

Assume that the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3} = \left(\left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)\right)_{3\times3}$ provided by the DMK is as follows:

$$
\tilde{D} = (\tilde{f}_{pq})_{3\times3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array}\begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} ([0.40, 0.40], [0.50, 0.50]) & ([0.40, 0.40], [0.50, 0.50]) & ([0.40, 0.40], [0.50, 0.50]) \\ ([0.40, 0.40], [0.50, 0.50]) & ([0.40, 0.40], [0.50, 0.50]) & ([0.40, 0.40], [0.50, 0.50]) \\ ([0.40, 0.40], [0.50, 0.50]) & ([0.40, 0.40], [0.50, 0.50]) & ([0.40, 0.40], [0.50, 0.50]) \end{pmatrix} \end{array},
$$

where $p = 1, 2, 3$ and $q = 1, 2, 3$.

[Step 1] Because the attributes S_1, S_2 and S_3 are benefit type attributes, on the basis of Eq. (10) and the DM $\tilde{D} = \left(\tilde{f}_{pq}\right)_{3\times3} = \left(\left(\left[\rho_{pq}^L, \rho_{pq}^U\right], \left[\sigma_{pq}^L, \sigma_{pq}^U\right]\right)\right)_{3\times3}$, we get the converted matrix $D = (f_{pq})_{3\times3} = \left(HF\left(\tilde{f}_{pq}\right)\right)_{3\times3}$, where $HF\left(\tilde{f}_{pq}\right) = \frac{\rho_{pq}^L - \sigma_{pq}^L + \rho_{pq}^U - \sigma_{pq}^U}{2} + \frac{\sin\left(\rho_{pq}^L \times \frac{\pi}{2}\right) + \sin\left(\rho_{pq}^U \times \frac{\pi}{2}\right) + \sin\left(\left(1-\sigma_{pq}^L\right) \times \frac{\pi}{2}\right) + \sin\left(\left(1-\sigma_{pq}^U\right) \times \frac{\pi}{2}\right)}{2} + 2$, $p = 1, 2, 3$, $q = 1, 2, 3$, $f_{11} = 3.195$, $f_{12} = 3.195$, $f_{13} = 3.195$, $f_{21} = 3.195$, $f_{22} = 3.195$, $f_{23} = 3.195$, $f_{31} = 3.195$, $f_{32} = 3.195$ and $f_{33} = 3.195$, Therefore, we get the converted matrix $D = (f_{pq})_{3\times3}$, shown as follows:

$$
D = (f_{pq})_{3\times3} = \begin{array}{c} \\ T_1 \\ T_2 \\ T_3 \end{array}\begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} 3.195 & 3.195 & 3.195 \\ 3.195 & 3.195 & 3.195 \\ 3.195 & 3.195 & 3.195 \end{pmatrix} \end{array}.
$$

[Step 2] Based on Eq. (11), the converted matrix $D = (f_{pq})_{3\times3}$ obtained in **Step 1** and the IVIF weights $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$ of the attributes S_1, S_2 and S_3 given by the DMK, respectively, we construct the following NLP model:

$$Maximize \sum_{q=1}^{3} tanh\left(\sum_{p=1}^{3}\sum_{x=1}^{3}\left|f_{pq}{}^{s_q} - f_{xq}{}^{s_q}\right|\right) + \sum_{p=1}^{3}\sum_{q=1}^{3} tanh(f_{pq}{}^{s_q}) - \sum_{q=1}^{3}\left(o_q^- + o_q^+\right)$$

$$s.t.\begin{cases} k_q^L - o_q^- \le s_q \le 1 - c_q^L + o_q^+ \\ \sum_{q=1}^{3} s_q = 1 \\ 0 \le s_q \le 1 \\ 0 \le o_q^- \le 1 \\ 0 \le o_q^+ \le 1 \\ q = 1, 2, 3 \end{cases}$$

After solving the NLP model, we get the optimal left shift values o_1^-, o_2^- and o_3^-, the optimal right shift values o_1^+, o_2^+ and o_3^+ and the optimal weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, where $o_1^- = 0$, $o_2^- = 0$, $o_3^- = 0$, $o_1^+ = 0$, $o_2^+ = 0$, $o_3^+ = 0$, $s_1 = 0.333$, $s_2 = 0.333$ and $s_3 = 0.333$.

[Step 3] On the basis of Eq. (12), the converted matrix $D = (f_{pq})_{3\times3}$ obtained in **Step 1** and the obtain optimal weights s_1, s_2 and s_3 of the attributes S_1, S_2 and S_3, respectively, obtained in **Step 2**, calculate the weighted score ws_p of alternative T_p, where $p = 1, 2, 3$, $ws_1 = f_{11}{}^{s_1} + f_{12}{}^{s_2} + f_{13}{}^{s_3} = 4.417$, $ws_2 = f_{21}{}^{s_1} + f_{22}{}^{s_2} + f_{23}{}^{s_3} = 4.417$ and $ws_3 = f_{31}{}^{s_1} + f_{32}{}^{s_2} + f_{33}{}^{s_3} = 4.417$.

[Step 4] Because $ws_1 = ws_2 = ws_3$, where $ws_1 = 4.417$, $ws_2 = 4.417$ and $ws_3 = 4.417$, the PO of the alternatives T_1, T_2 and T_3 is: $T_1 = T_2 = T_3$.

In the following, we compare the POs of the alternatives obtained by different MADM methods. The proposed MADM method and the MADM methods given in [8,9,11,12,15,16,23] obtains the same PO "$T_1 = T_2 = T_3$" of the alternatives. In [11], Chen and Han pointed out that the MADM methods presented in [26] and [49] has the "the division by zero" problem, which causes them unable to get the PO of alternatives in this circumstance. In [11], Chen and Han also pointed out that the MADM method presented in [10] has the "infinite loop" problem, which causes it unable to get the PO of alternatives in this circumstance. Therefore, the proposed MADM method can conquer the shortcoming of the MADM methods presented in [10,26] and [49].

6. Conclusions

In this paper, we have proposed a new multiattribute decision making (MADM) method based on the nonlinear programming methodology and the proposed novel score function of IVIFVs. From Table 1, it can be seen that the proposed score function of IVIFVs can conquer the shortcoming of the score functions of IVIFVs presented in [5,16,17] and [43]. From the examples shown in Section 5, it can be seen that the proposed MADM method can conquer the shortcoming of the MADM methods presented in [7,10,16,26] and [49]. The proposed MADM method offers a very useful approach for MADM in interval-value intuitionistic fuzzy settings. In [1], Akram and Shahzadi presented a MADM method on the basis of q-rung orthopair fuzzy Yager aggregation operators. In [2], Akram et al. presented extension of Einstein geometric operators for MADM in q-rung orthopair fuzzy environments. In [6], Biswas and Deb presented a MADM method on the basis of Pythagorean fuzzy Schweizer and Sklar power aggregation operators. In [21], Giri et al. presented a grey relational analysis method for MADM on the basis of single-valued trapezoidal neutrosophic numbers. In [33], Qin et al. presented a MADM method on the basis of picture fuzzy Archimedean power Maclaurin symmetric mean operators. It is worth of future research to develop new MADM methods based on [1,2,6,21,33].

Data availability

Data will be made available on request.

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References

[1] M. Akram, G. Shahzadi, A hybrid decision-making model under q-rung orthopair fuzzy Yager aggregation operators, Granular Comput. 6 (4) (2021) 763–777.
[2] M. Akram, G. Shahzadi, X. Peng, Extension of Einstein geometric operators to multiattribute decision-making under q-rung orthopair fuzzy information, Granular Comput. 6 (4) (2021) 779–795.
[3] K. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets Syst. 31 (3) (1989) 343–349.

[4] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1) (1986) 87–96.
[5] Z.Y. Bai, An interval-valued intuitionistic fuzzy TOPSIS method based on an improved score function, The Scientific World Journal 2013 (2013) Article ID 879089, 6 pages (http://dx.doi.org/10.1155/2013/879089).
[6] A. Biswas, N. Deb, Pythagorean fuzzy Schweizer and Sklar power aggregation operators for solving multi-attribute decision-making problems, Granular Comput. 6 (4) (2021) 991–1007.
[7] S.M. Chen, C.H. Chiou, Multiattribute decision making based on interval-valued intuitionistic fuzzy sets, PSO techniques, and evidential reasoning methodology, IEEE Trans. Fuzzy Syst. 23 (6) (2015) 1905–1916.
[8] S.M. Chen, Y.C. Chu, Multiattribute decision making based on *U*-quadratic distribution of intervals and the transformed matrix in interval-valued intuitionistic fuzzy environments, Inf. Sci. 537 (2020) 30–45.
[9] S.M. Chen, K.Y. Fan, Multiattribute decision making based on probability density functions and the variances and standard deviations of largest ranges of evaluating interval-valued intuitionistic fuzzy values, Inf. Sci. 490 (2019) 329–343.
[10] S.M. Chen, W.H. Han, A new multiattribute decision making method based on multiplication operations of interval-valued intuitionistic fuzzy values and linear programming methodology, Inf. Sci. 429 (2018) 421–432.
[11] S.M. Chen, W.H. Han, An improved MADM method using interval-valued intuitionistic fuzzy values, Inf. Sci. 467 (2018) 489–505.
[12] S.M. Chen, Z.C. Huang, Multiattribute decision making based on interval-valued intuitionistic fuzzy values and linear programming methodology, Inf. Sci. 381 (2017) 341–351.
[13] S.M. Chen, L.W. Kuo, Multiattribute decision making based on non-linear programming methodology with hyperbolic function and interval-valued intuitionistic fuzzy values, Inf. Sci. 453 (2018) 379–388.
[14] S.M. Chen, L.W. Kuo, X.Y. Zou, Multiattribute decision making based on Shannon's information entropy, non-linear programming methodology, and interval-valued intuitionistic fuzzy values, Inf. Sci. 465 (2018) 404–424.
[15] S.M. Chen, W.T. Liao, Multiple attribute decision making using Beta distribution of intervals, expected values of intervals, and new score function of interval-valued intuitionistic fuzzy values, Inf. Sci. 579 (2021) 863–887.
[16] S.M. Chen, C.A. Tsai, Multiattribute decision making using novel score function of interval-valued intuitionistic fuzzy values and the means and the variances of score matrices, Inf. Sci. 577 (2021) 748–768.
[17] S.M. Chen, K.Y. Tsai, Multiattribute decision making based on new score function of interval-valued intuitionistic fuzzy values and normalized score matrices, Inf. Sci. 575 (2021) 714–731.
[18] T.Y. Chen, A comparative analysis of score functions for multiple criteria decision making in intuitionistic fuzzy settings, Inf. Sci. 181 (17) (2011) 3652–3676.
[19] Y. Cheng, Y. Li, J. Yang, Multi-attribute decision-making method based on a novel distance measure of linguistic intuitionistic fuzzy sets, J. Intell. Fuzzy Syst. 40 (1) (2021) 1147–1160.
[20] P. Dutta, Multi-criteria decision making under uncertainty via the operations of generalized intuitionistic fuzzy numbers, Granular Comput. 6 (2) (2021) 321–337.
[21] B.C. Giri, M.U. Molla, P. Biswas, Grey relational analysis method for SVTrNN based multi-attribute decision making with partially known or completely unknown weight information, Granular Comput. 5 (4) (2020) 561–570.
[22] G. Jingni, X. Junxiang, H. Zhenggang, Study on location selection of client-supplied goods and materials support center for the Sichuan-Tibet railway based on dynamic intuitionistic fuzzy multi-attribute decision-making, J. Intell. Fuzzy Syst. 40 (3) (2021) 5669–5679.
[23] K. Kumar, S.M. Chen, Multiattribute decision making based on interval-valued intuitionistic fuzzy values, score function of connection numbers, and the set pair analysis theory, Inf. Sci. 551 (2021) 100–112.
[24] N. Kumar, S. Susan, Particle swarm optimization of partitions and fuzzy order for fuzzy time series forecasting of COVID-19, Appl. Soft Comput. 110 (2021) 107611.
[25] M.L. Lagunes, O. Castillo, J. Soria, F. Valdez, Optimization of a fuzzy controller for autonomous robot navigation using a new competitive multi-metaheuristic model, Soft. Comput. 25 (17) (2021) 11653–11672.
[26] D.F. Li, TOPSIS-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets, IEEE Trans. Fuzzy Syst. 18 (2) (2010) 299–311.
[27] Y. Liu, W. Jiang, A new distance measure of interval-valued intuitionistic fuzzy sets and its application in decision making, Soft. Comput. 24 (9) (2020) 6987–7003.
[28] A.R. Mishra, A. Chandel, D. Motwani, Extended MABAC method based on divergence measures for multi-criteria assessment of programming language with interval-valued intuitionistic fuzzy sets, Granular Comput. 5 (1) (2020) 97–117.
[29] Z.M. Ma, Z.S. Xu, Computation of generalized linguistic term sets based on fuzzy logical algebras for multi-attribute decision making, Granular Comput. 5 (1) (2020) 17–28.
[30] J. Mahanta, S. Panda, A novel distance measure for intuitionistic fuzzy sets with diverse applications, Int. J. Intell. Syst. 36 (2) (2021) 615–627.
[31] M.K. Muni, D.R. Parhi, P.B. Kumar, S. Kumar, Motion control of multiple humanoids using a hybridized prim's algorithm-fuzzy controller, Soft. Comput. 25 (2) (2021) 1159–1180.
[32] P. Phochanikorn, C. Tan, W. Chen, Barriers analysis for reverse logistics in Thailand's palm oil industry using fuzzy multi-criteria decision-making method for prioritizing the solutions, Granular Comput. 5 (4) (2020) 419–436.
[33] Y. Qin, X. Cui, M. Huang, Y. Zhong, Z. Tang, P. Shi, Multiple-attribute decision-making based on picture fuzzy Archimedean power Maclaurin symmetric mean operators, Granular Comput. 6 (3) (2021) 737–761.
[34] Y. Rong, Y. Liu, Z. Pei, Interval-valued intuitionistic fuzzy generalised Bonferroni mean operators for multi-attribute decision making, Int. J. Fuzzy Syst. 23 (6) (2021) 1728–1754.
[35] M.R. Seikh, U. Mandal, Intuitionistic fuzzy Dombi aggregation operators and their application to multiple attribute decision-making, Granular Comput. 6 (3) (2021) 473–488.
[36] S. Sen, K. Patra, S.K. Mondal, A new approach to similarity measure for generalized trapezoidal fuzzy numbers and its application to fuzzy risk analysis, Granular Comput. 6 (3) (2021) 705–718.
[37] Q. Shen, X. Huang, Y. Liu, Y. Jiang, K. Zhao, Multiattribute decision making based on the binary connection number in set pair analysis under an interval-valued intuitionistic fuzzy set environment, Soft. Comput. 24 (10) (2020) 7801–7809.
[38] C.Y. Wang, S.M. Chen, Multiple attribute decision making based on interval-valued intuitionistic fuzzy sets, linear programming methodology, and the extended TOPSIS method, Inf. Sci. 397 (2017) 155–167.
[39] C.Y. Wang, S.M. Chen, An improved multiattribute decision making method based on new score function of interval-valued intuitionistic fuzzy values and linear programming methodology, Inf. Sci. 411 (2017) 176–184.
[40] C.Y. Wang, S.M. Chen, A new multiple attribute decision making method based on linear programming methodology and novel score function and novel accuracy function of interval-valued intuitionistic fuzzy values, Inf. Sci. 438 (2018) 145–155.
[41] H.Y. Wang, S.M. Chen, Evaluating students' answerscripts using fuzzy numbers associated with degrees of confidence, IEEE Trans. Fuzzy Syst. 16 (2) (2008) 403–415.
[42] A.P. Wei, D.F. Li, P.P. Lin, B.Q. Jiang, An information-based score function of interval-valued intuitionistic fuzzy sets and its application in multiattribute decision making, Soft. Comput. 25 (3) (2021) 1913–1923.
[43] Z.S. Xu, Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making, Control Decision 22 (2) (2007) 215–219.
[44] Y. Yoshida, Dynamic risk-sensitive fuzzy asset management with coherent risk measures derived from decision maker's utility, Granular Comput. 6 (1) (2021) 19–35.

[45] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (3) (1965) 338–353.
[46] S. Zeng, S.M. Chen, K.Y. Fan, Interval-valued intuitionistic fuzzy multiple attribute decision making based on nonlinear programming methodology and TOPSIS method, Inf. Sci. 506 (2020) 424–442.
[47] S. Zeng, S.M. Chen, L.W. Kuo, Multiattribute decision making based on novel score function of intuitionistic fuzzy values and modified VIKOR method, Inf. Sci. 488 (2019) 76–92.
[48] J. Zhang, W.H. Chen, X. Lu, J. Wen, Fuzzy controller synthesis for nonlinear neutral state-delayed systems with impulsive effects, Inf. Sci. 555 (2021) 293–313.
[49] Z. Zhitao, Z. Yingjun, Multiple attribute decision making method in the frame of interval-valued intuitionistic fuzzy sets, in: Proceedings of the 2011 Eighth International Conference on Fuzzy Systems and Knowledge Discovery, Shanghai, China (2011) 192-196.
[50] X.Y. Zou, S.M. Chen, K.Y. Fan, Multiple attribute decision making using improved intuitionistic fuzzy weighted geometric operators of intuitionistic fuzzy values, Inf. Sci. 535 (2020) 242–253.