Course on Numerical Methods in Heat Transfer and Fluid Dynamics

Fractional Step Method

Staggered and Collocated Meshes

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Contents

Objectives

- Solve the NS equations using the FSM
- To understand the key features of the Fractional Step Method
- Study the checkerboard problem and review the different existing solutions
- Implement a CFD code for structured and staggered or collocated meshes
- Verification of the developed code using different benchmark case data

Introduction to Fractional Step Method

The fractional step method (**FSM**) **is a** common technique for solving the incompressible NS equations. The main reasons for this success are basically:

- Better performance than other methods, e.g SIMPLE-like algorithms
- Code simplicity

Main issues to bear in mind:

- **FSM** are also referred to as **projection methods** because it can be interpreted as a projection into a divergence-free velocity space.
- The ***intermediate (or predictor) velocity***, is an approximate solution of the momentum equations, but it cannot satisfy the incompressibility constraint at the next time level.
- The **pressure Poisson equation** determines the minimum perturbation that will make the predictor velocity incompressible.

(*) SIMPLE stands for Semi-Implicit Method for Pressure-Linked Equations

Theoretical background: the Helmholtz-Hodge theorem

Theorem: *A given vector field* $\boldsymbol{\omega}$*, defined in a bounded domain* Ω *with smooth boundary* $\delta\Omega$*, is uniquely decomposed in a pure gradient field and a divergence-free vector parallel to* $\delta\Omega$

$$\boldsymbol{\omega} = \boldsymbol{a} + \nabla\varphi$$

where,

$$\nabla \cdot \boldsymbol{a} = \mathbf{0} \qquad \boldsymbol{a} \in \Omega$$

The theorem also applies for periodic inflow/outflow conditions.

The proof of the theorem can be found in the extra material of the course entitled: "Introduction to the Fractional Step Method".

Application of the HH theorem to NS equations (1/4)

Navier-Stokes (NS) equations for incompressible and constant viscosity flows:

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\rho \frac{\partial v}{\partial t} + (\rho \boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \mu \Delta \boldsymbol{v} \quad \text{or} \quad \rho \frac{\partial v}{\partial t} = \mathbf{R}(\boldsymbol{v}) - \boldsymbol{\nabla} p$$

where $\boldsymbol{v} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$ and $\mathbf{R}(\boldsymbol{v}) = -(\rho \boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \mu \Delta \boldsymbol{v}$

Time integration of NS equations gives:

$$\nabla \cdot \boldsymbol{v}^{n+1} = 0$$

$$\rho \frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^{n}}{\Delta t} = \frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^{n}) - \frac{1}{2}\boldsymbol{R}(\boldsymbol{v}^{n-1}) - \boldsymbol{\nabla} p^{n+1}$$

Momentm equations are integrated at time instant (n+1/2) while continuity equations is implicitly integrated.

Application of the HH theorem to NS equations (2/4)

Now, if we introduce the following unique decomposition (thanks to the HH theorem),

$$\boldsymbol{v}^p = \boldsymbol{v}^{n+1} + \frac{\Delta t}{\rho}\nabla p^{n+1} \qquad \text{(where } \nabla \cdot \boldsymbol{v}^{n+1} = \boldsymbol{0}\text{)}$$

we can transform the original momentum equation to the following velocity projection equation,

$$\rho\frac{\boldsymbol{v}^p - \boldsymbol{v}^n}{\Delta t} = \frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^n) - \frac{1}{2}\boldsymbol{R}(\boldsymbol{v}^{n-1})$$

Application of the HH theorem to NS equations (3/4)

An equation for the pressure can be derived from the velocity decomposition equation if the divergence operator is applied,

$$\nabla \cdot \boldsymbol{v}^{n+1} = \nabla \cdot \boldsymbol{v}^{p} - \nabla \cdot \left(\frac{\Delta t}{\rho} \nabla p^{n+1}\right)$$

Since $\nabla \cdot \boldsymbol{v}^{n+1} = \mathbf{0}$, a final Poisson equation for the pressure is found,

$$\Delta p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \boldsymbol{v}^{p}$$

Application of the HH theorem to NS equations (4/4)

Finally, $\boldsymbol{v}^{n+1}$ results from the original decomposition,

$$\boldsymbol{v}^{n+1} = \boldsymbol{v}^{p} - \frac{\Delta t}{\rho} \nabla p^{n+1}$$

Therefore, at each time step the following equations give a unique $\boldsymbol{v}^{n+1}$ and $\boldsymbol{\nabla} p^{n+1}$. In summary:

$$\left.\begin{array}{ll}
1. & \text{Evaluation of } \boldsymbol{R}(\boldsymbol{v}^n) \\
2. & \boldsymbol{v}^p = \boldsymbol{v}^n + \frac{\Delta t}{\rho}\left[\frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^n) - \frac{1}{2}\boldsymbol{R}(\boldsymbol{v}^{n-1})\right] \\
3. & \Delta p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \boldsymbol{v}^p \\
4. & \boldsymbol{v}^{n+1} = \boldsymbol{v}^p - \frac{\Delta t}{\rho} \nabla p^{n+1}
\end{array}\right\} \text{FSM}$$

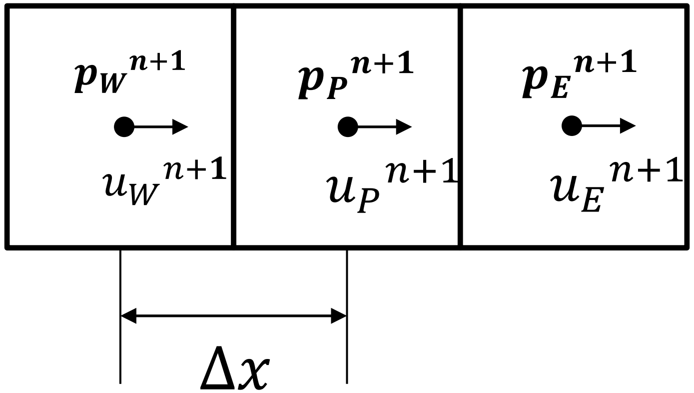
The checkerboard problem (1/3)

If we focus ourselves in the 1D spatial discretization of the step 3 of the previously described FSM, and after applying finite differences at node P:

$$\boldsymbol{v}^{n+1} = \boldsymbol{v}^{p} - \frac{\Delta t}{\rho} \nabla p^{n+1}$$

For the x-component of the velocity ($\boldsymbol{v}=u\boldsymbol{i}+v\boldsymbol{j}+w\boldsymbol{k}$) at node P:

$$u_P{}^{n+1} = u_P{}^{p} - \frac{\Delta t}{\rho}\left(\frac{p_E{}^{n+1} - p_W{}^{n+1}}{2\Delta x}\right)$$

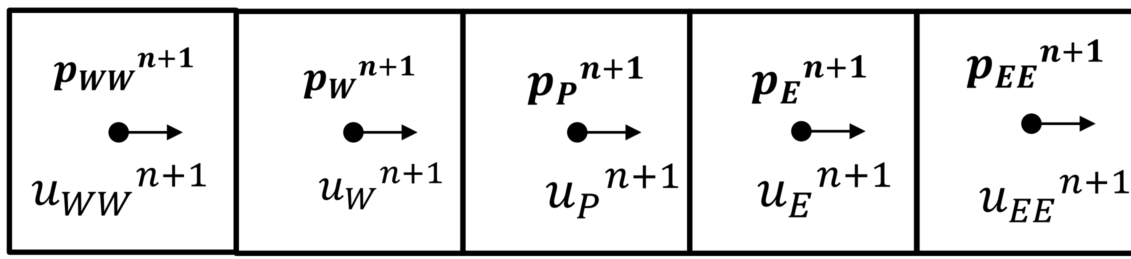


Therefore, the discrete approximation of ∇p^{n+1} at node P is independent of $p_P{}^{n+1}$.

The checkerboard problem (2/3)

We can obtain converged velocity fields for unphysical pressure distributions. For example,

$p_{WW}{}^{n+1} = 100$
$p_{W}{}^{n+1} = 0$
$p_{P}{}^{n+1} = 100$
$p_{E}{}^{n+1} = 0$
$p_{EE}{}^{n+1} = 100$



This final "unphysical" pressure field verifies $\nabla p^{n+1} = 0$!

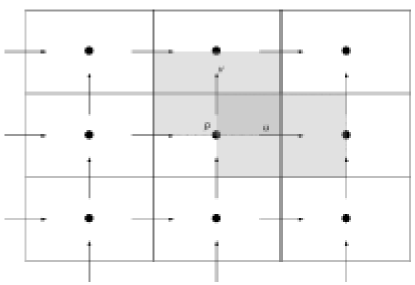
Remember, ∇p^{n+1} at node P is independent of $p_P{}^{n+1}$.

We need a smarter strategy to couple $\boldsymbol{\nabla} p^{n+1}$ with the velocity field v^{n+1}!

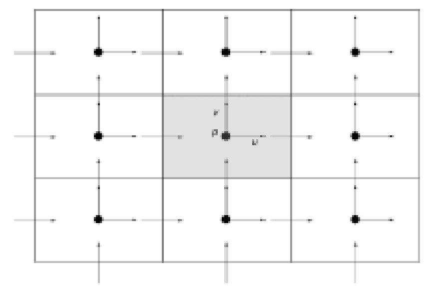
The checkerboard problem (3/3)

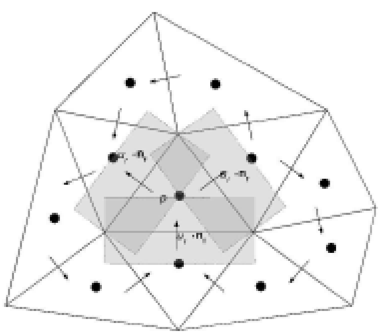
Two possible solutions have been developed to solve the checkerboard problem,

Staggered meshes ***Collocated meshes***

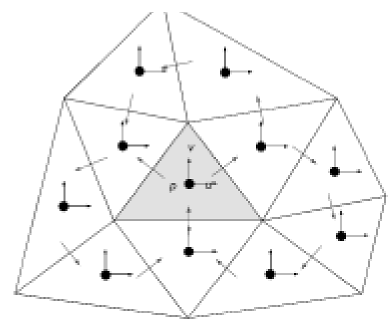


Structured meshes



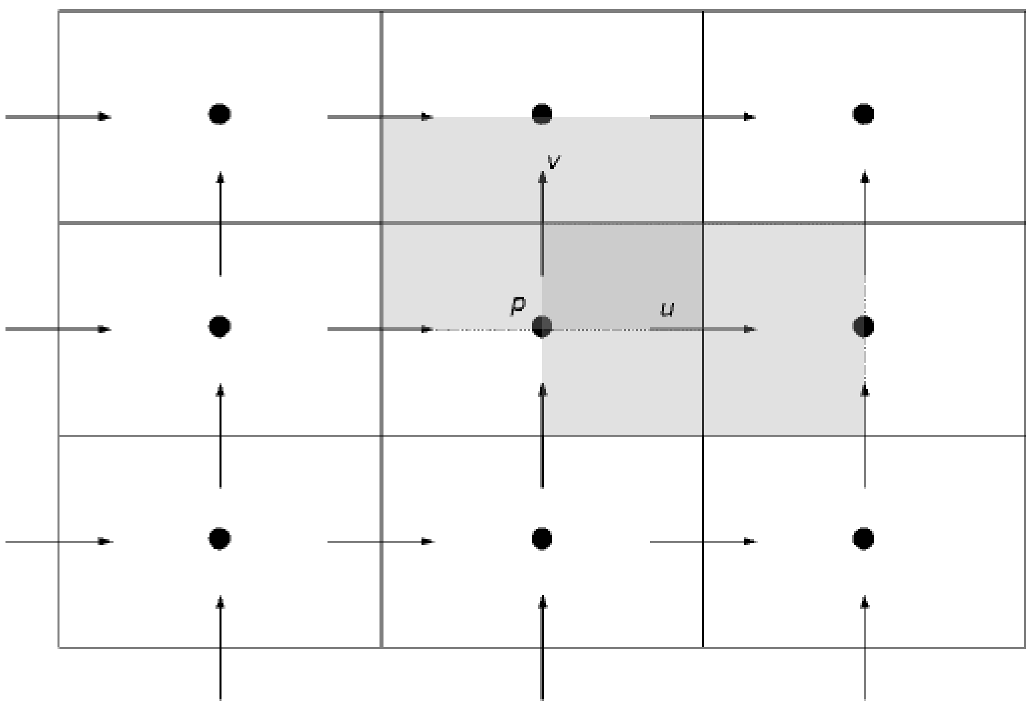


Unstructured meshes



Attention is now focused on staggered meshes. Collocated meshes are explained in the Appendix.

FSM for staggered meshes (1/13)



- Staggered velocity mesh solves the checkerboard problem.
- Easy to implement on structured meshes.
- But on unstructured meshes, it is difficult to implement !!
- Widely used for academic purposes.
- Next lesson will be focused on collocated arrangement (*Unit 4: FSM. Part 2: Collocated Meshes*).

FSM for staggered meshes (2/13)

$t = t_{steady}$

Summary:

1. $\boldsymbol{R}(\boldsymbol{v}^n)$
2. $\boldsymbol{v}^p = \boldsymbol{v}^n + \frac{\Delta t}{\rho}[\frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^n) - \frac{1}{2}R(\boldsymbol{v}^{n-1})]$
3. $\Delta p^{n+1} = \frac{\rho}{\Delta t}\nabla \cdot \boldsymbol{v}^p$
4. $\boldsymbol{v}^{n+1} = \boldsymbol{v}^p - \frac{\Delta t}{\rho}\nabla p^{n+1}$
5. *Choose your new* $\Delta t = min(\Delta t_c, \Delta t_d)$

At each time step

…

… and finish when, e.g., the steady state is reached

$t = 0$

The unsteady resolution advances with adaptive time steps until a specified condition is reached , e.g. steady state.

Next slides show the evaluation details of the different terms.

FSM Step 1: Stagg-x mesh (3/13)

Step 1 FSM (x component of $\boldsymbol{v}$): $u^P = u^n + \frac{\Delta t}{\rho}[\frac{3}{2}R(u^n) - \frac{1}{2}R(u^{n-1})]$

where :

$$R(u) = -(\rho\boldsymbol{v}\cdot\nabla)u + \mu\Delta u$$

If we integrate $\mathrm{R}(u)$ over the staggered-x control volume and then the Gauss theorem is applied:

$$\int_{\Omega_x} R(u)d\Omega_x = -\int_{\Omega_x}(\rho\boldsymbol{v}\cdot\nabla)ud\Omega_x + \int_{\Omega_x}\mu\Delta u d\Omega_x =$$

$$= -\int_{\partial\Omega_x}(\rho\boldsymbol{v})u\cdot\boldsymbol{n}dS + \int_{\partial\Omega_x}\mu\nabla u\cdot\boldsymbol{n}dS$$

FSM Step 1: Stagg-x mesh (4/13)

$$\int_{\Omega_x} R(u)d\Omega_x = -\int_{\partial\Omega_x} (\rho\boldsymbol{v})u \cdot \boldsymbol{n}dS + \int_{\partial\Omega_x} \mu\nabla u \cdot \boldsymbol{n}dS$$

$R(u)\Omega_{xP}$= - $[\dot{m}_e u_e$ - $\dot{m}_w u_w + \dot{m}_n u_n$ - $\dot{m}_s u_s]$+

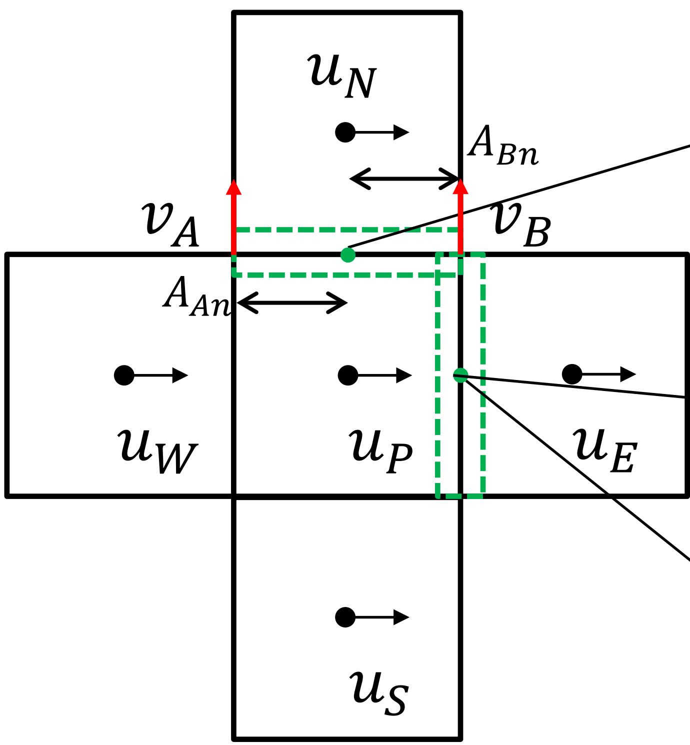
$$[\mu_e \frac{u_E - u_P}{d_{EP}} A_e - \mu_w \frac{u_P - u_W}{d_{WP}} A_w + \mu_n \frac{u_N - u_P}{d_{NP}} A_n - \mu_s \frac{u_P - u_S}{d_{SP}} A_s]$$

where $\dot{m}_e = (\rho u)_e A_e$, $\dot{m}_w = (\rho u)_w A_w$, $\dot{m}_n = (\rho v)_n A_n$, $\dot{m}_s = (\rho v)_s A_s$ (here $\dot{m}$ is positive in the positive coordinate direction).

But, how can we evaluate, the volumetric flow rate and the transport property (i.e. momentum)?:

$(\rho u)_e$, $(\rho v)_n$, $(\rho u)_w$, $(\rho v)_s$??? and u_e, u_n, u_w, u_s ???

FSM Step 1: Stagg-x mesh (5/13)



$$\dot{m}_n = (\rho v)_A A_{An} + (\rho v)_B A_{Bn}$$

u_e= UDS(u_P,u_E)

u_e= CDS(u_P,u_E)

u_e= QUICK(u_P,u_E, u_W, u_{EE}, u_{WW})

u_e=...

See lecture : *Numerical Solution of Convection-Diffussion equation*

$$\dot{m}_e = \frac{(\rho u)_E + (\rho u)_P}{2} A_e$$

FSM Step 1: Stagg-y mesh (6/13)

Step 1 FSM (y component of $\boldsymbol{v}$): $v^P = v^n + \frac{\Delta t}{\rho}[\frac{3}{2}R(v^n)\ \frac{1}{2}R(v^{n-1})]$

where :

$$R(v)\Omega_{yP} \approx - [\dot{m}_e v_e - \dot{m}_w v_w + \dot{m}_n v_n - \dot{m}_s v_s] +$$

$$[\mu_e \frac{v_E - v}{d_{EP}} A_e - \mu_w \frac{v_P - v_W}{d_{WP}} A_w + \mu_n \frac{v_N - v_P}{d_{NP}} A_n - \mu_s \frac{v_P - v_S}{d_{SP}} A_s]$$

where,

- $\dot{m}_e = (\rho u)_e A_e$, $\dot{m}_n = (\rho v)_n A_n$, $\dot{m}_w = (\rho u)_w \mathrm{A}_w$, $\dot{m}_s = (\rho v)_s A_s$; volumetric fluxes are evaluated with mass conserving interpolations
- v_e, v_n, v_w, v_s are evaluated with convective numerical schemes

FSM Step 2: Main mesh (7/13)

$$\Delta p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \boldsymbol{v}^{\boldsymbol{p}}$$

$$\int_{\Omega} \Delta p^{n+1} d\Omega = \frac{\rho}{\Delta t} \int_{\Omega} \nabla \cdot \boldsymbol{v}^{\boldsymbol{p}} d\Omega$$

$$\int_{\partial\Omega} \nabla p^{n+1} \cdot \boldsymbol{n} dS = \frac{\rho}{\Delta t} \int_{\partial\Omega} \boldsymbol{v}^{\boldsymbol{p}} \cdot \boldsymbol{n} dS$$

$$\frac{p_E^{n+1} - p_P^{n+1}}{d_{EP}} A_e - \frac{p_P^{n+1} - p_W^{n+1}}{d_{WP}} A_w + \frac{p_N^{n+1} - p_P^{n+1}}{d_{NP}} A_n - \frac{p_P^{n+1} - p_S^{n+1}}{d_{SP}} A_s =$$

$$\frac{1}{\Delta t}[(\rho u^P)_e A_e - (\rho u^P)_w A_w + (\rho v^P)_n A_n - (\rho v^P)_s A_s]$$

FSM Step 2: Main mesh (8/13)

$$a_P p_P^{n+1} = a_E p_E^{n+1} + a_W p_W^{n+1} + a_N p_N^{n+1} + a_S p_S^{n+1} + b_P$$

$$a_P = a_E + a_W + a_N + a_S$$

$$a_E = \frac{A_e}{d_{EP}} \qquad a_N = \frac{A_n}{d_{NP}}$$

$$a_W = \frac{A_w}{d_{WP}} \qquad a_S = \frac{A_s}{d_{SP}}$$

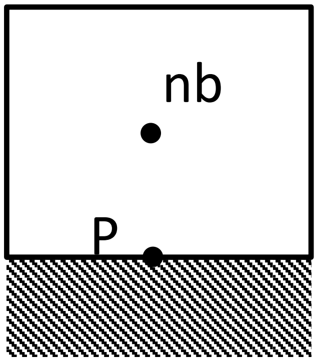
$$b_P = -\frac{1}{\Delta t}\left[(\rho u^P)_e A_e - (\rho u^P)_w A_w + (\rho v^P)_n A_n - (\rho v^P)_s A_s\right]$$

Any of the linear solvers developed for the conduction exercises can be used here (Jacobi, Gauss-Seidel, line-by-line, etc.)

See "Unit 2: Numerical Solution of Conduction" for more information about linear systems solvers.

FSM Step 2: Boundary conditions (9/13)

- **Wall boundary condition:**



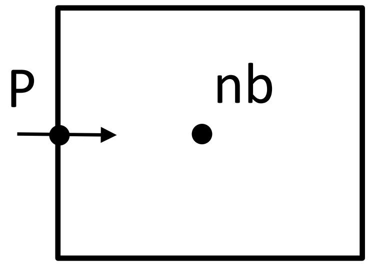
- Since a boundary layer is created at the wall $\frac{\partial p}{\partial n} = 0$

$$a_P = 1$$

$$a_{nb} = 1 \quad a_{i \neq nb} = 0$$

FSM Step 2: Boundary conditions (10/13)

- **Prescribed velocity:**



From,

$$\boldsymbol{v}^{n+1} = \boldsymbol{v}^{P} - \frac{\Delta t}{\rho} \nabla p^{n+1}$$

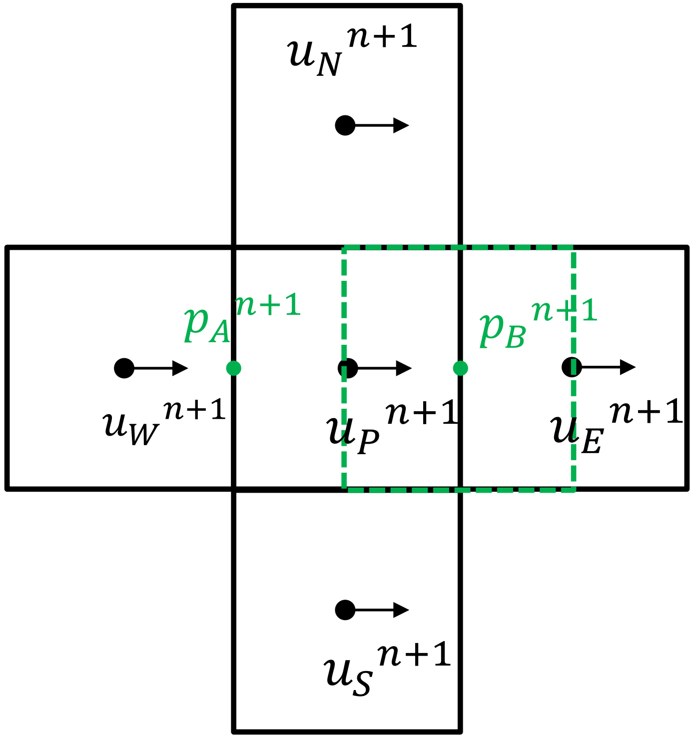
if $\boldsymbol{v}^{n+1}{}_{P}$ is known, we can set $\boldsymbol{v}^{P} = \boldsymbol{v}^{n+1}{}_{P}$. Thus,

$$\frac{\partial p}{\partial n} = 0$$

$$a_P = 1$$

$$a_{nb} = 1 \quad a_{i \neq nb} = 0$$

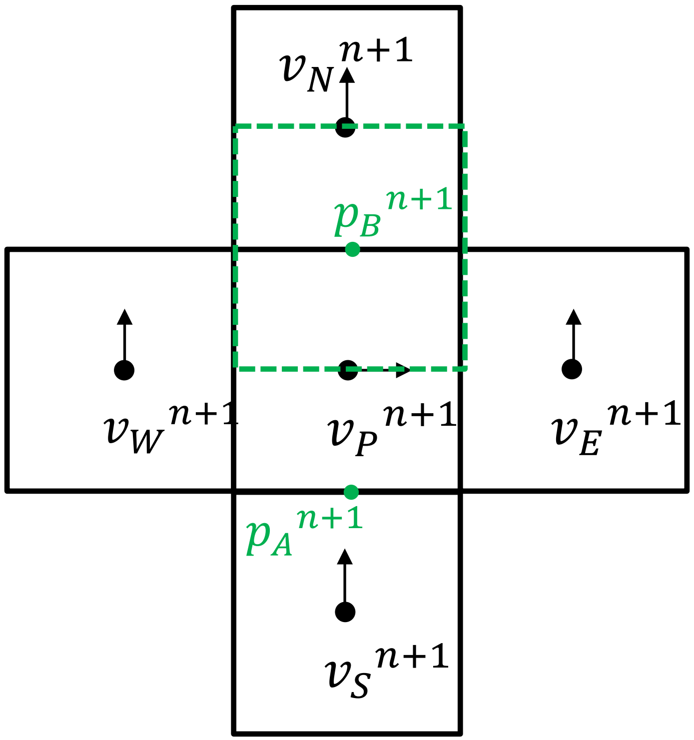
FSM Step 3: Stagg-x mesh (11/13)



$$u_P^{n+1} = u_P{}^P - \frac{\Delta t}{\rho}\left(\frac{\partial p}{\partial x}\right)_P^{n+1}$$

$$u_P^{n+1} = u_P{}^P - \frac{\Delta t}{\rho} \cdot \frac{p_B^{n+1} - p_A^{n+1}}{d_{BA}}$$

FSM Step 3: Stagg-y mesh (12/13)



$$v_P^{n+1} = v_P{}^P - \frac{\Delta t}{\rho}\left(\frac{\partial p}{\partial y}\right)_P^{n+1}$$

$$v_P^{n+1} = v_P{}^P - \frac{\Delta t}{\rho} \cdot \frac{p_B^{n+1} - p_A^{n+1}}{d_{BA}}$$

FSM Step 4: Choice of the time step (13/13)

CFL (Courant-Friedrich-Levy) condition:

$$\Delta t_c = \min\left(0.35\,\frac{\Delta x}{|\boldsymbol{v}|}\right)$$

$$\Delta t_d = \min\left(0.20\,\frac{\rho \Delta x^2}{\mu}\right)$$

$$\Delta t = \min(\Delta t_c, \Delta t_d)$$

More advanced ways to find the optimal Δt can be found in: *"A self-adaptive strategy for the time integration of Navier-Stokes equations",* FX Trias, O Lehmkuhl, Numerical Heat Transfer, Part B: Fundamentals 60 (2), 116-134, 2011.

Non-uniform meshes

- Consider a line segment joining two points $\vec{x}_1$ and $\vec{x}_2$
- On the line, $\vec{x}_i = \vec{x}_1 + s_i(\vec{x}_2 - \vec{x}_1)$ (s_i is a stretching function, from 0 to 1)
- Hyperbolic concentration: $s_i = 1 + \dfrac{tanh\left[k\left(\frac{i}{N}-1\right)\right]}{tank(k)}$

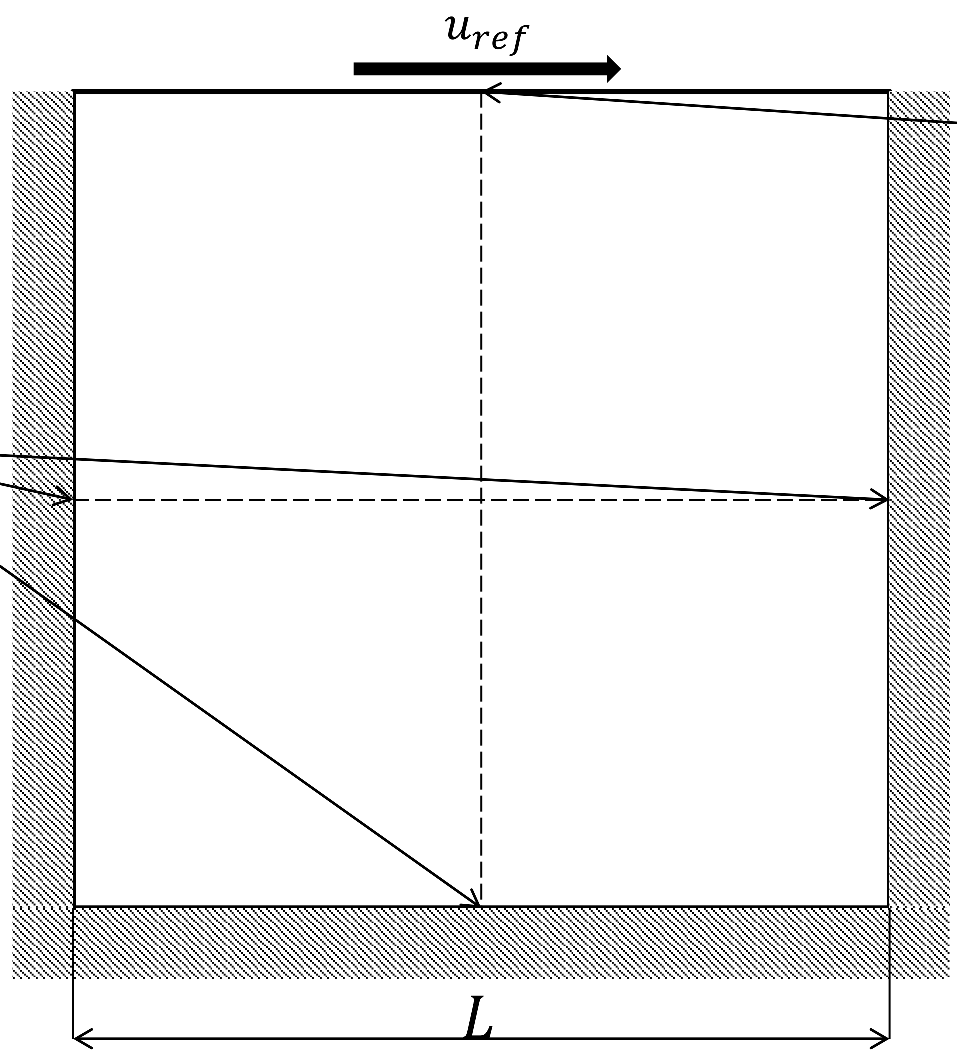
  where $i = 0,1,2, \dots, N$, and k is the stretching factor (e.g. $k = 0.0001$: uniform distribution; $k = 3$: strong concentration towards $\vec{x}_1$)

Exercise: Lid-Driven Cavity (1/4)

$u = u_{ref}$

$v = 0$

$\frac{\partial p}{\partial y} = 0$



$u = 0$

$v = 0$

$\frac{\partial p}{\partial n} = 0$

$$Re = \frac{\rho u_{ref} L}{\mu}$$

Note: pressure level of the fluid can be specified giving the pressure in an internal node ($p_P = p_{ref}$)

Exercise: Driven Cavity, u in the vertical centre line (2/4)

129-grid pt. no.	y	Re 100	400	1000	3200	5000	7500	10,000
129	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
126	0.9766	0.84123	0.75837	0.65928	0.53236	0.48223	0.47244	0.47221
125	0.9688	0.78871	0.68439	0.57492	0.48296	0.46120	0.47048	0.47783
124	0.9609	0.73722	0.61756	0.51117	0.46547	0.45992	0.47323	0.48070
123	0.9531	0.68717	0.55892	0.46604	0.46101	0.46036	0.47167	0.47804
110	0.8516	0.23151	0.29093	0.33304	0.34682	0.33556	0.34228	0.34635
95	0.7344	0.00332	0.16256	0.18719	0.19791	0.20087	0.20591	0.20673
80	0.6172	−0.13641	0.02135	0.05702	0.07156	0.08183	0.08342	0.08344
65	0.5000	−0.20581	−0.11477	−0.06080	−0.04272	−0.03039	−0.03800	0.03111
59	0.4531	−0.21090	−0.17119	−0.10648	−0.86636	−0.07404	−0.07503	−0.07540
37	0.2813	−0.15662	−0.32726	−0.27805	−0.24427	−0.22855	−0.23176	−0.23186
23	0.1719	−0.10150	−0.24299	−0.38289	−0.34323	−0.33050	−0.32393	−0.32709
14	0.1016	−0.06434	−0.14612	−0.29730	−0.41933	−0.40435	−0.38324	−0.38000
10	0.0703	−0.04775	−0.10338	−0.22220	−0.37827	−0.43643	−0.43025	−0.41657
9	0.0625	−0.04192	−0.09266	−0.20196	−0.35344	−0.42901	−0.43590	−0.42537
8	0.0547	−0.03717	−0.08186	−0.18109	−0.32407	−0.41165	−0.43154	−0.42735
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Results extracted from, "High-Re Solutions for Incompressible Flow Using Navier-Stokes Equations and a Multigrid Method", Ghia et al., Journal of Computational Physics 48, 387-411 (1982).

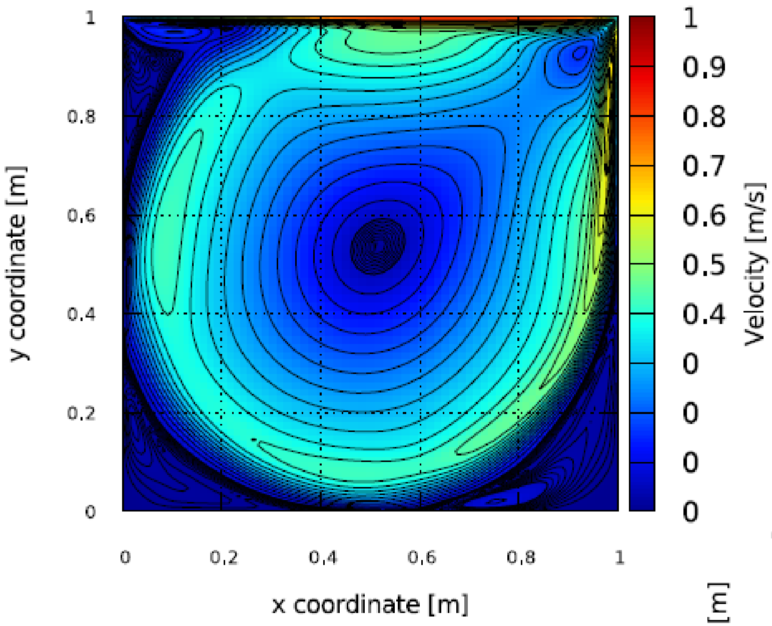
Exercise: Driven Cavity, v in the horizontal centre line (3/4)

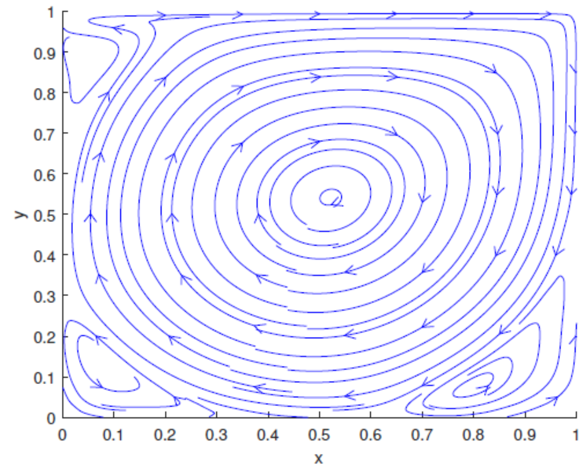
129-grid pt. no.	x	Re 100	400	1000	3200	5000	7500	10,000
129	1.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
125	0.9688	−0.05906	−0.12146	−0.21388	−0.39017	−0.49774	−0.53858	−0.54302
124	0.9609	−0.07391	−0.15663	−0.27669	−0.47425	−0.55069	−0.55216	−0.52987
123	0.9531	−0.08864	−0.19254	−0.33714	−0.52357	−0.55408	−0.52347	−0.49099
122	0.9453	−0.10313	−0.22847	−0.39188	−0.54053	−0.52876	−0.48590	−0.45863
117	0.9063	−0.16914	−0.23827	−0.51550	−0.44307	−0.41442	−0.41050	−0.41496
111	0.8594	−0.22445	−0.44993	−0.42665	−0.37401	−0.36214	−0.36213	−0.36737
104	0.8047	−0.24533	−0.38598	−0.31966	−0.31184	−0.30018	−0.30448	−0.30719
65	0.5000	0.05454	0.05186	0.02526	0.00999	0.00945	0.00824	0.00831
31	0.2344	0.17527	0.30174	0.32235	0.28188	0.27280	0.27348	0.27224
30	0.2266	0.17507	0.30203	0.33075	0.29030	0.28066	0.28117	0.28003
21	0.1563	0.16077	0.28124	0.37095	0.37119	0.35368	0.35060	0.35070
13	0.0938	0.12317	0.22965	0.32627	0.42768	0.42951	0.41824	0.41487
11	0.0781	0.10890	0.20920	0.30353	0.41906	0.43648	0.43564	0.43124
10	0.0703	0.10091	0.19713	0.29012	0.40917	0.43329	0.44030	0.43733
9	0.0625	0.09233	0.18360	0.27485	0.39560	0.42447	0.43979	0.43983
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

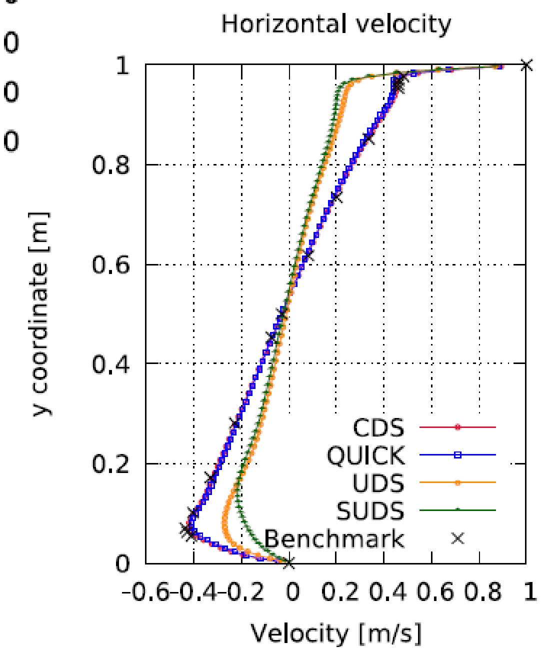
Results extracted from, "High-Re Solutions for Incompressible Flow Using Navier-Stokes Equations and a Multigrid Method", Ghia et al., Journal of Computational Physics 48, 387-411 (1982).

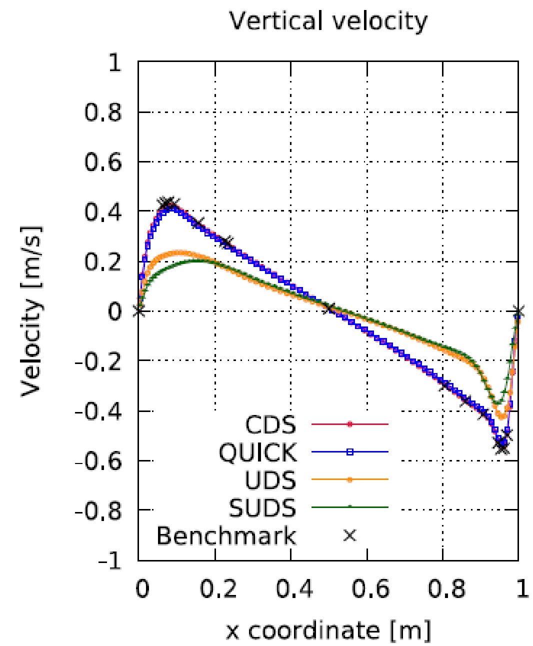
Exercise: Driven Cavity, $Re = 5000$ (4/4)

Velocity contours and stream lines (Re=5000)



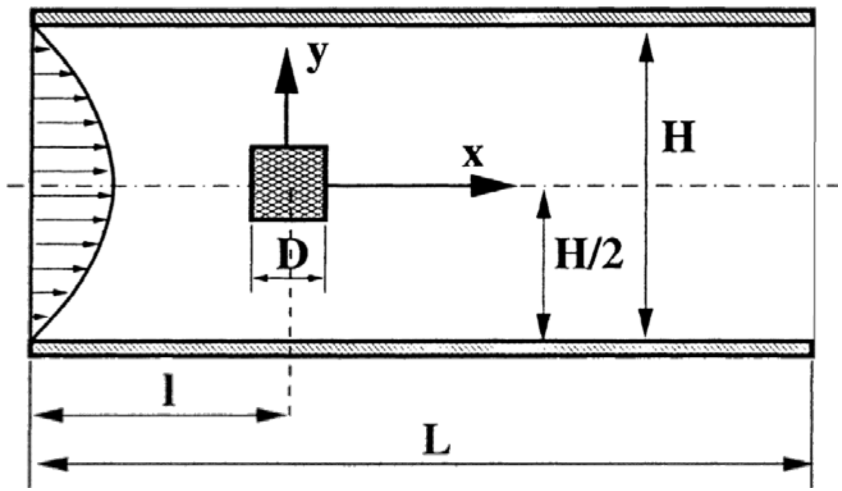


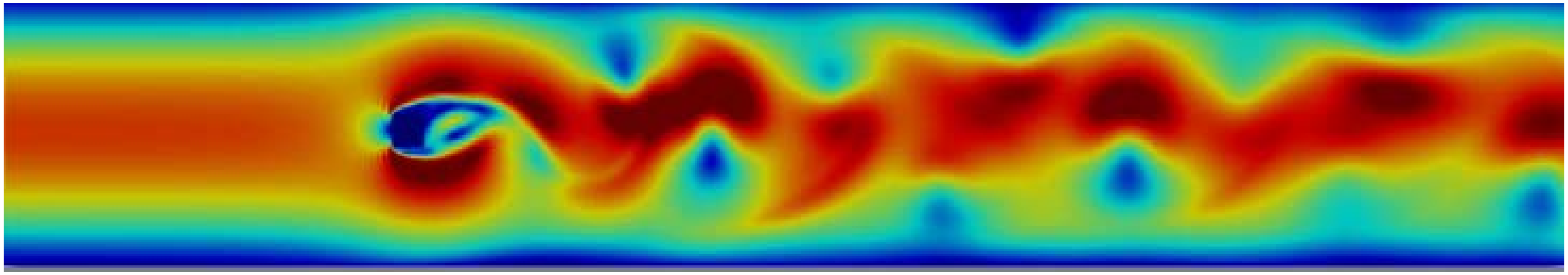




Other interesting exercises (1/3)

- Flow over a square cylinder

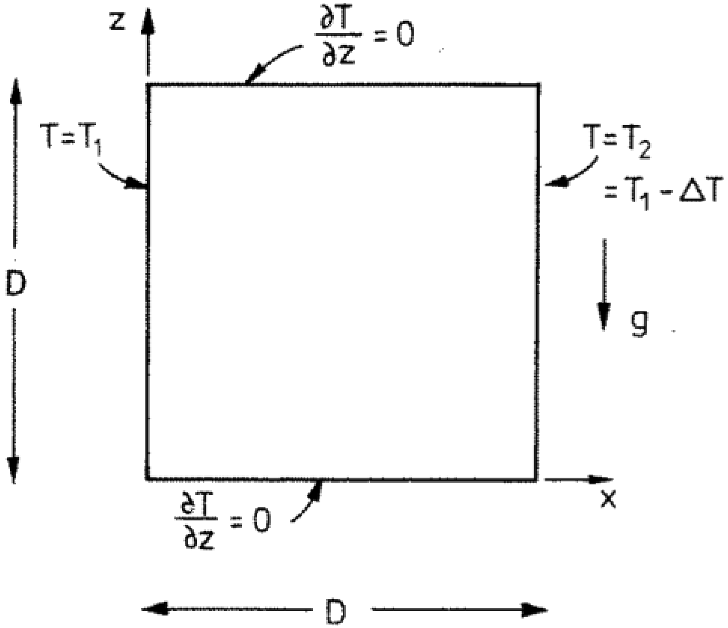


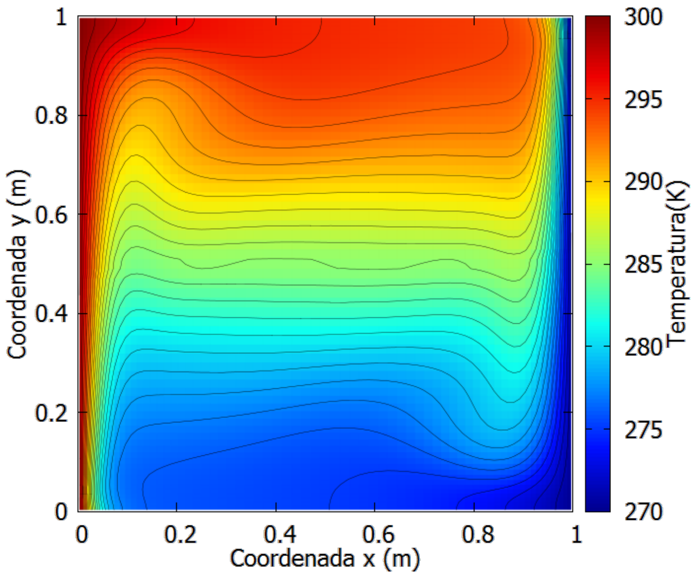


M. Breuer, J. Bernsdorf, T. Zeiser, and F. Durst. Accurate computations of the laminar flow past a square cylinder based on two different methods: lattice-Boltzmann and finite volume. *International Journal of Heat and Fluid Flow*, 21:186–196, 2000.

Other interesting exercises (2/3)

- Differentially heated cavity (energy equation has also been taken into account)





G. De Vahl Davis and I. P. Jones. Natural convection in a square cavity: a comparison exercise. *International Journal for Numerical methods in Fluids*, 3:227–248, 1983.

Other interesting exercises (3/3)

- The differentially heated cavity case also involved the resolution of the discretized energy equation:

$$\rho c_p \frac{T^{n+1} - T^n}{\Delta t} = \frac{3}{2} R_T(T^n) - \frac{1}{2} R_T(T^{n-1})$$

where $R_T(T) = -\rho c_p \boldsymbol{v} \cdot \nabla T + \nabla \cdot (\lambda \nabla T)$.

- The global algorithm follows these steps at each Δt:

  1. $R_T(T^n)$ and $\boldsymbol{R}(\boldsymbol{v}^n)$
  2. $T^{n+1} = T^n + \frac{\Delta t}{\rho c_p}\left[\frac{3}{2} R_T(T^n) - \frac{1}{2} R_T(T^{n-1})\right]$ and $\boldsymbol{v}^p = \boldsymbol{v}^n + \frac{\Delta t}{\rho}\left[\frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^n) - \frac{1}{2} R(\boldsymbol{v}^{n-1})\right]$
  3. $\Delta p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \boldsymbol{v}^p$
  4. $\boldsymbol{v}^{n+1} = \boldsymbol{v}^p - \frac{\Delta t}{\rho} \nabla p^{n+1}$
  5. *New* $\Delta t = min(\Delta t_c, \Delta t_d, \Delta t_t)$, where $\Delta t_t = 0.20 \frac{\Delta x^2}{\lambda/\rho c_p}$

Summary

- The basics concepts for solving NS equations using the FSM have been studied.
- An introduction to the checkerboard problem and its possible solutions have been presented.
- An staggered mesh code for the solution of NS equations should be developed by the student.
- The developed code must be verified through direct comparison with benchmark data of a driven cavity at different Re.

Bibliography

- *A. J. Chorin, "Numerical Solution of the Navier-Stokes Equations", Journal of Computational Physics 22, 745-762 (1968).*
- *N. N. Yanenko, "The Method of Fractional Steps", Springer-Verlag, 1971.*
- *"Introduction to the Fractional-Step Method", CTTC report.*
- *U. Ghia, K.N. Ghia, C.T. Shin, "High-Re Solutions for Incompressible Flow Using Navier-Stokes Equations and a Multigrid Method", Journal of Computational Physics 48, 387-411, 1982.*
- *Suhas V. Patankar, "Numerical Heat Transfer and Fluid Flow", Hemisphere Publishing Corporation, McGraw-Hill Book Company, 1980.*

- *Note: references related to* more advanced ways to find the optimal Δt, *or the three different benchmark cases here proposed, are given in the corresponding slides.*

Annex:
FSM for collocated meshes

Introduction

- This interesting option of using just a single CV (see slide 12) needs to face the checkerboard problem.
- As it was mentioned before, we have to distinguish between the variable to be transported (e.g. $\boldsymbol{v}$ or T) and the mass fluxes ($\dot{m}_f$) at the CV faces.
- Momentum equation is discretized in two steps and at the main CV (x, y or z component of the velocity):

$$\rho\frac{\boldsymbol{v}_P^{n+1}-\boldsymbol{v}_P^{p}}{\Delta t}=-(\nabla p)_P^{n+1} \quad (1a)$$

$$\rho\frac{\boldsymbol{v}_P^{p}-\boldsymbol{v}_P^{n}}{\Delta t}=\frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^{n})_P-\frac{1}{2}\boldsymbol{R}(\boldsymbol{v}^{n-1})_P \quad (1b)$$

where $\boldsymbol{R}(\boldsymbol{v})_P\Omega_P=\mp\sum\dot{m}_f\boldsymbol{v}_f+\sum\mu_f\frac{\boldsymbol{v}_F-\boldsymbol{v}_P}{d_{PF}}A_f \quad (f=e,w,n,s)$

Continuity equation and physical velocities

- Taking the divergence of eq. (1a), considering the continuity equation, $(\nabla \cdot \boldsymbol{v})_P^{n+1} = 0$, and discretizing in the usual way along the CV faces, nodal pressures are obtained:

$$\sum \frac{p_F^{n+1} - p_P^{n+1}}{d_{PF}} A_f = \frac{\rho}{\Delta t} \sum \left(\boldsymbol{v}_f^p \cdot \boldsymbol{n}_f \right) A_f \quad (2)$$

- Physical nodal velocities are finally obtained from eq. (1a):

$$\boldsymbol{v}_P^{n+1} = \boldsymbol{v}_P^p - \frac{\Delta t}{\rho} (\nabla p)_P^{n+1} \quad (3)$$

- The key point is the evaluation of the mass fluxes at the CV faces.

Mass fluxes at the CV faces (1/2)

- Mass flow rate that is exiting the CV face *f*:

$$\dot{m}_f^{n+1} = \rho \boldsymbol{v}_f^{n+1} \cdot \boldsymbol{n}_f A_f \qquad (4a)$$

- Introducing eq. (3):

$$\dot{m}_f^{n+1} = \rho \left(\boldsymbol{v}_f^p \cdot \boldsymbol{n}_f - \frac{\Delta t}{\rho} (\nabla p)_f^{n+1} \cdot \boldsymbol{n}_f \right) A_f \qquad (4b)$$

- After discretizing the pressure gradient at the CV faces:

$$\dot{m}_f^{n+1} = \rho \left(\boldsymbol{v}_f^p \cdot \boldsymbol{n}_f - \frac{\Delta t}{\rho} \frac{p_F^{n+1} - p_P^{n+1}}{d_{PF}} \right) A_f \qquad (4c)$$

- Finally, intermediate velocities at the CV faces are obtained by interpolation of the nodal values, i.e.

$$\boldsymbol{v}_f^p \cdot \boldsymbol{n}_f = \overline{\left(\boldsymbol{v}_P^p\right)} \cdot \boldsymbol{n}_f = \frac{1}{2}\left(\boldsymbol{v}_P^p + \boldsymbol{v}_F^p\right) \cdot \boldsymbol{n}_f.$$

Mass fluxes at the CV faces (2/2)

- Eq. (4c) shows that CV mass fluxes directly depend on pressure difference at the neighbors nodes. This is a relevant issue.
- It is interesting to see the final form of eq. (4c) after the inter-mediate velocities are interpolated, i.e. introducing eq. (3):

$$\dot{m}_f^{n+1} = \rho\left[\overline{(\boldsymbol{v}_P^{n+1})}\cdot\boldsymbol{n}_f + \frac{\Delta t}{\rho}\overline{(\nabla p)_P^{n+1}}\cdot\boldsymbol{n}_f - \frac{\Delta t}{\rho}\frac{p_F^{n+1}-p_P^{n+1}}{d_{PF}}\right]A_f \quad (5a)$$

- This equation can be rewritten in the following form:

$$\dot{m}_f^{n+1} = \rho\overline{(\boldsymbol{v}_P^{n+1})}\cdot\boldsymbol{n}_f A_f + \Delta t\left[\overline{(\nabla p)_P^{n+1}}\cdot\boldsymbol{n}_f - \frac{p_F^{n+1}-p_P^{n+1}}{d_{PF}}\right]A_f \quad (5b)$$

- Therefore, the third term of this equation (in brackets) acts a correction term which stabilized the convergence process.

Global algorithm

- Global algorithm (unlike the previously described method for staggered meshes, only a single CV is now used):

1. $\boldsymbol{R}(\boldsymbol{v}^n)_P \Omega_P =- [\dot{m}_e \boldsymbol{v}_e + \dot{m}_w \boldsymbol{v}_w + \dot{m}_n \boldsymbol{v}_n + \dot{m}_s \boldsymbol{v}_s] + [\mu_e \frac{\boldsymbol{v}_E - \boldsymbol{v}_P}{d_{EP}} A_e + \mu_w \frac{\boldsymbol{v}_W - \boldsymbol{v}_P}{d_{WP}} A_w + \mu_n \frac{\boldsymbol{v}_N - \boldsymbol{v}_P}{d_{NP}} A_n + \mu_s \frac{\boldsymbol{v}_S - \boldsymbol{v}_P}{d_{SP}} A_s]$
2. $\boldsymbol{v}_P^p = \boldsymbol{v}_P^n + \frac{\Delta t}{\rho} \left[\frac{3}{2} \boldsymbol{R}(\boldsymbol{v}^n)_P - \frac{1}{2} \boldsymbol{R}(\boldsymbol{v}^{n-1})_P \right]$
3. $\sum \frac{p_F^{n+1} - p_P^{n+1}}{d_{PF}} A_f = \frac{\rho}{\Delta t} \sum \left(\boldsymbol{v}_f^p \cdot \boldsymbol{n}_f \right) A_f \rightarrow p_P^{n+1}$
4. $\boldsymbol{v}_P^{n+1} = \boldsymbol{v}_P^p - \frac{\Delta t}{\rho} (\nabla p)_P^{n+1}$
5. $\dot{m}_f^{n+1} = \rho \overline{(\boldsymbol{v}_P^{n+1})} \cdot \boldsymbol{n}_f A_f + \Delta t \left[\overline{(\nabla p)_P^{n+1}} \cdot \boldsymbol{n}_f - \frac{p_F^{n+1} - p_P^{n+1}}{d_{PF}} \right] A_f$
6. Go to a new time step with $\Delta t = min(\Delta t_c, \Delta t_d)$