

# Gas Dynamics and Heat and Mass Transfer

## Fractional Step Method

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# Chapter 1

## Theoretical background

**Theorem 1.0.1** (Helmholtz–Hodge FSM). Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with smooth boundary. Let  $\boldsymbol{\omega}: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a vector field. Then there exist a smooth function  $\varphi: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  and a divergence-less smooth vector field  $\mathbf{a}: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that

$$\boldsymbol{\omega} = \mathbf{a} + \nabla \varphi$$

In addition,

$$\mathbf{a} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega$$

where  $\boldsymbol{\nu}$  denotes the outer normal vector to  $\partial\Omega$ .

**Theorem 1.0.2** (Helmholtz–Hodge). Let  $D \subset \mathbb{R}^3$  be a bounded domain. Then every smooth vector field  $\mathbf{F}: D \rightarrow \mathbb{R}^3$  can be decomposed into a sum  $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ , where  $\mathbf{F}_1$  is an irrotational field and  $\mathbf{F}_2$  is a solenoidal field.

**Theorem 1.0.3** (Helmholtz–Hodge). Let  $\Omega \subset \mathbb{R}^3$  be a contractible bounded open set with smooth boundary. Let  $F: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a  $\mathcal{C}^\infty(\Omega)$  vector field. Then there exist a function  $f \in \mathcal{C}^\infty(\Omega, \mathbb{R})$  and a vector field  $H \in \mathcal{C}^\infty(\Omega, \mathbb{R}^3)$  such that

$$F = \nabla f + \nabla \times H$$

**Theorem 1.0.4.** Let  $\Omega \subset \mathbb{R}^3$  be a contractible bounded open set with smooth boundary. Let  $G: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a divergence-free  $\mathcal{C}^\infty(\Omega)$  vector field, that is to say,  $\nabla \cdot G = 0$ . Then there exists a  $\mathcal{C}^\infty(\Omega)$  vector field  $g: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $\nabla \times g = G$ .

Notice that the previous theorem says nothing about the uniqueness of the field  $g$ .

Assume  $F$  is the Schwartz space

Then see Griffith's of Helmholtz's theorem

## Chapter 2

# Fractional step method (FSM)

In this Chapter

Hereinafter, it is assumed that the studied problems occur in a contractible bounded open set  $\Omega \subset \mathbb{R}^3$  with smooth boundary. Besides, these problems last for finite time, that is to say, the time interval is  $I = [t_0, t_f] \subset \mathbb{R}$ .

### 2.1 First approach to the FSM

#### 2.1.1 Time integration of the Navier–Stokes equations

Recall that the Navier–Stokes equations for incompressible and constant viscosity flows are

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \rho \frac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \mu \Delta \mathbf{v} \end{cases} \quad (2.1.1)$$

where  $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ . By defining the operator

$$\mathbf{R}(\mathbf{v}) = \mu \Delta \mathbf{v} - (\rho \mathbf{v} \cdot \nabla) \mathbf{v} \quad (2.1.2)$$

(2.1.1) may be rewritten as follows:

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{R}(\mathbf{v}) - \nabla p \end{cases} \quad (2.1.3)$$

why integrate the equations?

Let  $[t^n, t^{n+1}] \subset [t_0, t_f]$  be a non-degenerate time interval with  $\Delta t = t^{n+1} - t^n$ . An implicit integration scheme ( $\beta = 1$ ) is used to integrate the continuity equation with respect to time:

$$\int_{t^n}^{t^{n+1}} \nabla \cdot \mathbf{v} \, dt = \left( \beta \nabla \cdot \mathbf{v}^{n+1} + (1 - \beta) \nabla \cdot \mathbf{v}^n \right) \Delta t = \nabla \cdot \mathbf{v}^{n+1} \Delta t = 0$$

Since  $\Delta t > 0$ , it follows that

$$\nabla \cdot \mathbf{v}^{n+1} = 0 \quad (2.1.4)$$

which is the time-integrated continuity equation. As for the momentum equation,

$$\int_{t^n}^{t^{n+1}} \rho \frac{\partial \mathbf{v}}{\partial t} dt = \int_{t^n}^{t^{n+1}} \mathbf{R}(\mathbf{v}) dt - \int_{t^n}^{t^{n+1}} \nabla p dt$$

The left-hand side computation is straightforward as the density is constant and the fundamental theorem of calculus is applied:

$$\int_{t^n}^{t^{n+1}} \rho \frac{\partial \mathbf{v}}{\partial t} dt = \rho \int_{t^n}^{t^{n+1}} \frac{\partial \mathbf{v}}{\partial t} dt = \rho(\mathbf{v}^{n+1} - \mathbf{v}^n)$$

In order to integrate  $\mathbf{R}(\mathbf{v})$ , define

$$\mathfrak{R}(\mathbf{v}, t) = \int_{t_0}^t \mathbf{R}(\mathbf{v}) ds$$

so that

$$\mathfrak{R}(\mathbf{v}, t^{n+1}) - \mathfrak{R}(\mathbf{v}, t^n) = \int_{t^n}^{t^{n+1}} \mathbf{R}(\mathbf{v}) dt \quad (2.1.5)$$

By applying the two-step Adams-Bashforth method, (2.1.5) results in

$$\int_{t^n}^{t^{n+1}} \mathbf{R}(\mathbf{v}) dt = \left( \frac{3}{2} \mathbf{R}(\mathbf{v}^{n+1}) - \frac{1}{2} \mathbf{R}(\mathbf{v}^n) \right) \Delta t$$

Again the implicit integration scheme ( $\beta = 1$ ) is used to compute the third term:

$$\int_{t^n}^{t^{n+1}} \nabla p dt = \left( \beta \nabla p^{n+1} + (1 - \beta) \nabla p^n \right) \Delta t = \nabla p^{n+1} \Delta t$$

Rearranging terms yields the time-integrated momentum equation:

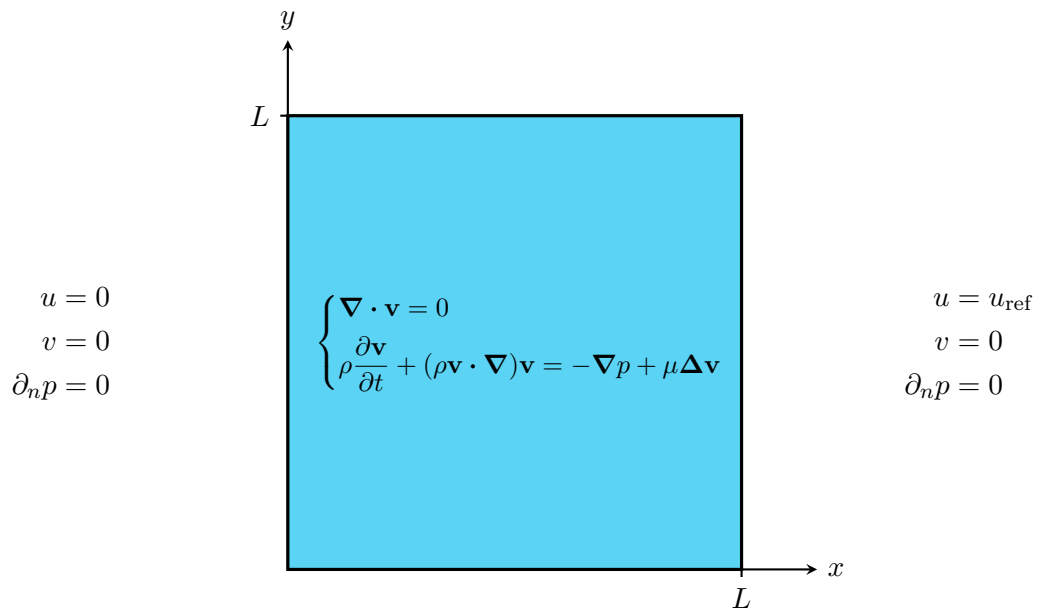
$$\rho \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = \frac{3}{2} \mathbf{R}(\mathbf{v}^{n+1}) - \frac{1}{2} \mathbf{R}(\mathbf{v}^n) - \nabla p^{n+1} \quad (2.1.6)$$

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## Chapter 3

# Lid–Driven Cavity

### 3.1 Statement



**Figure 3.1.1**

$$\int_M d\omega = \int_{\partial M} \omega$$