Gas Dynamics and Heat and Mass Transfer

Fractional Step Method

Student: Pedro López Sancha

Professor: Carlos-David Pérez Segarra

Aerospace Technology Engineering
The School of Industrial, Aerospace and Audiovisual Engineering of Terrassa
Technic University of Catalonia

February 20, 2022

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa

Contents

Chapter 1

Theoretical background

Theorem 1.0.1 (Helmholtz–Hodge FSM)**.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let $\boldsymbol{\omega} \colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then there exist a smooth function $\varphi \colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ and a divergence-less smooth vector field $\mathbf{a} \colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\boldsymbol{\omega} = \mathbf{a} + \boldsymbol{\nabla}\varphi$$

In addition,

$$\mathbf{a} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega$$

where $\boldsymbol{\nu}$ denotes the outer normal vector to $\partial\Omega$.

Theorem 1.0.2 (Helmholtz–Hodge)**.** Let $D \subset \mathbb{R}^3$ be a bounded domain. Then every smooth vector field $\mathbf{F} \colon D \to \mathbb{R}^3$ can be decomposed into a sum $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where $\mathbf{F}_1$ is an irrotational field and $\mathbf{F}_2$ is a solenoidal field.

Theorem 1.0.3 (Helmholtz–Hodge)**.** Let $\Omega \subset \mathbb{R}^3$ be a contractible bounded open set with smooth boundary. Let $F \colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^\infty(\Omega)$ vector field. Then there exist a function $f \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ and a vector field $H \in \mathcal{C}^\infty(\Omega, \mathbb{R}^3)$ such that

$$F = \boldsymbol{\nabla} f + \boldsymbol{\nabla} \times H$$

Theorem 1.0.4. Let $\Omega \subset \mathbb{R}^3$ be a contractible bounded open set with smooth boundary. Let $G \colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a divergence–free $\mathcal{C}^\infty(\Omega)$ vector field, that is to say, $\boldsymbol{\nabla} \cdot G = 0$. Then there exists a $\mathcal{C}^\infty(\Omega)$ vector field $g \colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ such that $\boldsymbol{\nabla} \times g = G$.

Notice that the previous theorem says nothing about the uniqueness of the field g.

Assume F is the Schwartz space

Then see Griffith's of Helmholtz's theorem

Chapter 2

Fractional step method (FSM)

In this Chapter

Hereinafter, it is assumed that the studied problems occur in a contractible bounded open set $\Omega \subset \mathbb{R}^3$ with smooth boundary. Besides, these problems last for finite time, that is to say, the time interval is $I = [t_0, t_f] \subset \mathbb{R}$.

2.1 First approach to the FSM

2.1.1 Time integration of the Navier–Stokes equations

Recall that the Navier–Stokes equations for incompressible and constant viscosity flows are

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \rho \dfrac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mu \boldsymbol{\Delta} \mathbf{v} \end{cases} \tag{2.1.1}$$

where $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. By defining the operator

$$\mathbf{R}(\mathbf{v}) = \mu \boldsymbol{\Delta} \mathbf{v} - (\rho \mathbf{v} \cdot \nabla)\mathbf{v} \tag{2.1.2}$$

(2.1.1) may be rewritten as follows:

$$\begin{cases} \nabla \cdot \mathbf{v} = 0 \\ \rho \dfrac{\partial \mathbf{v}}{\partial t} = \mathbf{R}(\mathbf{v}) - \nabla p \end{cases} \tag{2.1.3}$$

why integrate the equations?

Let $[t^n, t^{n+1}] \subset [t_0, t_f]$ be a non–degenerate time interval with $\Delta t = t^{n+1} - t^n$. An implicit integration scheme ($\beta = 1$) is used to integrate the continuity equation with respect to time:

$$\int_{t^n}^{t^{n+1}} \nabla \cdot \mathbf{v} \, \mathrm{d}t = \left(\beta \nabla \cdot \mathbf{v}^{n+1} + (1-\beta) \nabla \cdot \mathbf{v}^n \right) \Delta t = \nabla \cdot \mathbf{v}^{n+1} \Delta t = 0$$

Since $\Delta t > 0$, it follows that

$$\nabla \cdot \mathbf{v}^{n+1} = 0 \tag{2.1.4}$$

which is the time–integrated continuity equation. As for the momentum equation,

$$\int_{t^n}^{t^{n+1}} \rho \frac{\partial \mathbf{v}}{\partial t}\,\mathrm{d}t = \int_{t^n}^{t^{n+1}} \mathbf{R}(\mathbf{v})\,\mathrm{d}t - \int_{t^n}^{t^{n+1}} \boldsymbol{\nabla} p\,\mathrm{d}t$$

The left–hand side computation is straightforward as the density is constant and the fundamental theorem of calculus is applied:

$$\int_{t^n}^{t^{n+1}} \rho \frac{\partial \mathbf{v}}{\partial t}\,\mathrm{d}t = \rho \int_{t^n}^{t^{n+1}} \frac{\partial \mathbf{v}}{\partial t}\,\mathrm{d}t = \rho(\mathbf{v}^{n+1} - \mathbf{v}^n)$$

In order to integrate $\mathbf{R}(\mathbf{v})$, define

$$\mathfrak{R}(\mathbf{v}, t) = \int_{t_0}^{s} \mathbf{R}(\mathbf{v})\,\mathrm{d}s$$

so that

$$\mathfrak{R}(\mathbf{v}, t^{n+1}) - \mathfrak{R}(\mathbf{v}, t^n) = \int_{t^n}^{t^{n+1}} \mathbf{R}(\mathbf{v})\,\mathrm{d}t \tag{2.1.5}$$

By applying the two–step Adams–Bashforth method, (2.1.5) results in

$$\int_{t^n}^{t^{n+1}} \mathbf{R}(\mathbf{v})\,\mathrm{d}t = \left(\frac{3}{2}\mathbf{R}(\mathbf{v}^{n+1}) - \frac{1}{2}\mathbf{R}(\mathbf{v}^n)\right)\Delta t$$

Again the implicit integration scheme ($\beta = 1$) is used to compute the third term:

$$\int_{t^n}^{t^{n+}} \boldsymbol{\nabla} p\,\mathrm{d}t = \left(\beta \boldsymbol{\nabla} p^{n+1} + (1-\beta)\boldsymbol{\nabla} p^n\right)\Delta t = \boldsymbol{\nabla} p^{n+1}\,\Delta t$$

Rearranging terms yields the time–integrated momentum equation:

$$\rho \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = \frac{3}{2}\mathbf{R}(\mathbf{v}^{n+1}) - \frac{1}{2}\mathbf{R}(\mathbf{v}^n) - \boldsymbol{\nabla} p^{n+1} \tag{2.1.6}$$