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2. Rings

2.1 Prime and maximal ideals

Definition 2.1.1. Let R be a ring and $I \subset R$ a proper ideal.

(i) We say that the ideal I is **prime** whenever $ab \in I$ implies $a \in I$ or $b \in I$.

(ii) We say that the ideal I is **maximal** if it is not contained in any other proper ideal, that is to say, if $J \subset R$ is a proper ideal and $I \subset J$, then $J = I$.

Example 2.1.2. Consider the ring $\mathbb{Z}$ with the usual addition and multiplication. Then for every prime $p \in \mathbb{Z}$, the ideal $(p) = p\mathbb{Z}$ is maximal. Indeed, assume that $(p) \subset I$ for some proper ideal $I \subset \mathbb{Z}$. Then $I = (a)$ for some $a \in \mathbb{Z}$ because $\mathbb{Z}$ is a principal ideal domain. Therefore $a \mid p$, but since p is prime, there are two possibilities. The first one is $a = \pm 1$, that is to say, $(a) = \mathbb{Z}$ which is not a proper ideal. The second one is $a = \pm p$, thus $(a) = (p)$. Hence (p) is a maximal ideal.

Proposition 2.1.3. Let R be a ring and $I \subset R$ an ideal. If I is a maximal ideal, then it is also a prime ideal.

Proof. Let $a, b \in R$ such that $ab \in I$ and consider the ideal $J = (a) + I$. Since $I \subset J$ and I is a maximal ideal, then either $J = I$ or $J = R$. In the former case we have that $a \in I$ and we are done. In the latter case, there exist $\lambda \in R$ and $c \in I$ satisfying $1 = \lambda a + c$, consequently $b = b \cdot 1 = \lambda ab + bc$. Since $\lambda ab \in I$ and $bc \in I$, we have that $b \in I$. In both cases we deduce that I is a prime ideal. □

The following

Proposition 2.1.4. Let R be a ring and $I \subset R$ an ideal. Then the quotient ring ${}^{R}/_{I}$ is an integral domain if, and only if, I is a prime ideal.

Proof. Assume that ${}^{R}/_{I}$ is an integral domain. Given $a, b \in R$ satisfying $ab = 0$, we have that $\overline{ab} = \overline{a}\,\overline{b} = \overline{0}$, thus either $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, that is to say, $a \in I$ or $b \in I$, so I is a prime ideal.

Conversely, assume that I is a prime ideal and let $\overline{a}, \overline{b} \in {}^{R}/_{I}$ such that $\overline{ab} = \overline{a}\overline{b} = \overline{0}$. Then $ab \in I$, therefore either $a \in I$ or $b \in I$, which implies either $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, that is to say, ${}^{R}/_{I}$ is an integral domain. □

Proposition 2.1.5. Let R be a ring and $I \subset R$ an ideal. Then the quotient ring ${}^{R}/_{I}$ is an field if, and only if, I is a maximal ideal.

Proof. □

The previous propositions give an alternative way to prove that every maximal ideal is prime.

If $\mathfrak{m}$ is a maximal ideal, then ${}^{R}/_{\mathfrak{m}}$ is a field and, in particular, an integral domain, so $\mathfrak{m}$ must be a prime ideal.

Definition 2.1.6. Let R be a ring.

(i) The **spectrum of prime ideals** of R is the set of prime ideals of R,

$$\operatorname{Spec} R = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal}\}$$

(ii) The **spectrum of maximal ideals** of R is the set of maximal ideals of R,

$$\operatorname{Spm} R = \{\mathfrak{m} \subset R \mid \mathfrak{m} \text{ is a maximal ideal}\}$$

Definition 2.1.7. We say that a ring R is **local** if it has only one maximal ideal. It is often denoted by $(R, \mathfrak{m})$, where $\mathfrak{m} \subset R$ is the only maximal ideal. The ring R is said to be **semilocal** if it only has finitely many maximal ideals.

Example 2.1.8. The ring $\mathbb{Z}$ with the usual addition and multiplication is not local, since the ideal $p\mathbb{Z}$ is maximal for every prime $p \in \mathbb{Z}$.

The following proposition gives some useful characterisations for maximal ideals.

Proposition 2.1.9. Let A be a ring and $I \subset A$ an ideal. If $A \setminus I \subset A^*$, then A is a local ring and I is its maximal ideal.

Proof. Let $\pi\colon A \to {}^{A}/_{I}$ be the projection on the quotient ring. Denote the equivalence class of an element $a \in A$ by $\overline{a} = \pi(a)$. Take $\overline{x} \in {}^{A}/_{I}$ such that $\overline{x} \neq \overline{0}$. Then $x \notin I$ thus x must be a unit of A since $x \in A \setminus A \subset A^*$, so there exists $y \in A$ such that $xy = 1$. By projecting on the quotient we have $\pi(xy) = \pi(x)\pi(y) = \overline{xy} = \overline{1}$. As $x \in {}^{A}/_{I}$ is an arbitrary element different from zero, we deduce that ${}^{A}/_{I}$ is a field, so I must be a maximal ideal.

Now let $J \subset A$ be another ideal. The projection $\pi\colon A \to {}^{A}/_{I}$ is a surjective ring homomorphism, thus the set $\pi(J) \subset {}^{A}/_{I}$ is an ideal. However, the only ideals of the field ${}^{A}/_{I}$ are the zero ideal and the total ideal. In the first case $\pi(J) = 0$, thus $J \subset I$. In the second case $\pi(J) = {}^{A}/_{I}$, which implies $J = R$. Thus every ideal $J \subset A$ is either contained in I or is the total ideal, that is to say, I is the only maximal ideal. $\square$

2.1.1 Extension and contraction of prime ideals

Let $f\colon A \to B$ be a ring homomorphism. It is natural to wonder whether prime and maximal ideals of A are preserved under ideal extensions, and whether prime and maximal ideals of B are preserved under ideal contraction.

Proposition 2.1.10. Let A and B be rings and $f\colon A \to B$ a ring homomorphism. If $J \in \operatorname{Spec} B$, then $J^c \in \operatorname{Spec} A$.

Proof. Let $a, b \in A$ such that $ab \in J^c$. Then $f(ab) = f(a)f(b) \in f(J^c) = J$ and, since $J \subset B$ is a prime ideal, either $f(a) \in J$ or $f(b) \in J$, which implies either $a \in J^c$ or $b \in J^c$, that is to say, $J^c \subset A$ is a prime ideal. $\square$

This need not be the case with maximal ideals, that is to say, if $J \subset B$ is a maximal ideal, then $J^c \subset A$ need not be a maximal ideal.

The same happens to prime ideals: if $I \subset A$ is a prime ideal then $I^e \subset B$ need not be prime.

2.1.2 Existence of maximal ideals

Firs

Theorem 2.1.11. Let A be a ring and $I \subset A$ an ideal. The there exists a maximal ideal $\mathfrak{m} \subset A$ such that $I \subset \mathfrak{m}$.

2.2 Localisation

Definition 2.2.1. Let R be a ring. A subset $S \subset R$ is a **multiplicatively closed set** if $1 \in S$ and $st \in S$ whenever $s, t \in S$.

Let R be a ring and $S \subset R$ a multiplicatively closed set. In the cartesian product $R \times S$ consider the relation

$$(a, s) \sim (a', s') \iff (as' - a's)t = 0 \text{ for some } t \in S$$

which is an equivalence relation. It is reflexive since $(a, s) \sim (a, s)$ because $(as - as)1 = 0$. If $(a, s) \sim (a', s')$ then $(as' - a's)t = (a's - as')(-t) = 0$ thus it is reflexive. Finally assume that $(a, s) \sim (a', s')$ and $(a', s') \sim (a'', s'')$, which is equivalent to

$$\begin{aligned} (a, s) \sim (a', s') &\iff (as' - a's)u = 0 &&\iff as'u = a'su \text{ for some } u \in S \\ (a', s') \sim (a'', s'') &\iff (a's'' - a''s')v = 0 &&\iff a's''v = a''s'v \text{ for some } u \in S \end{aligned}$$

In order to prove that $(a, s) \sim (a'', s'')$ we have the following:

$$as''(ss'uv) = as'u(ss''v) = a'su(ss''v) = a's''v(ssu) = a''s'v(ssu) = a''s(ss'uv)$$

By defining $w = ss'uv \in S$, we have shown that $(as'' - a''s)w = 0$ thus the relation is transitive. As we shall immediately see, it is far more natural to write the elements $(a, s) \in R \times S$ as fractions $\dfrac{a}{s}$. With this notation, the equivalence relation is written as

$$\frac{a}{s} \sim \frac{a'}{s'} \iff (as' - a's)t = 0 \text{ for some } t \in S$$

With this in mind, we define the following set.

Definition 2.2.2. Let R be a ring and $S \subset R$ a multiplicatively closed set. We define the **locatisation of R at S** as

$$S^{-1}R = R \times S /_{\sim} = \left\{ \frac{a}{s} \mid a \in R,\, s \in S \right\} /_{\sim}$$

Proposition 2.2.3. Let R be a ring $S \subset R$ a multiplicatively closed set. Then the localisation $S^{-1}R$ is a commutative ring with unit where sum and multiplication are defined as follows:

$$\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'} \qquad \frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$$

Proof. First of all, note that if $u \in S$, then $\dfrac{a}{s} = \dfrac{au}{su}$ since $(asu - asu)1 = 0$, so we may simplify the numerator and denominator of fractions as though we were working with school fractions as long as what we are simplyfing is an element of S.

We begin by checking that $(S^{-1}R, +)$ is an abelian group. The sum is an internal operation because $as' + a's \in R$ and $ss' \in S$, so $\dfrac{as' + a's}{ss'} \in R \times S$. It is associative

$$\left(\frac{a}{s} + \frac{a'}{s'}\right) + \frac{a''}{s''} = \frac{as' + a's}{ss'} + \frac{a''}{s''} = \frac{as's'' + a'ss'' + a''ss'}{ss's''} = \frac{a}{s} + \frac{a's'' + a''s'}{s's''} = \frac{a}{s} + \left(\frac{a'}{s'} + \frac{a''}{s''}\right)$$

The neutral element is $\dfrac{0}{1}$ since

$$\frac{a}{s} + \frac{0}{1} = \frac{a \cdot 1 + 0 \cdot s}{s \cdot 1} = \frac{a \cdot 1}{s \cdot 1} = \frac{a}{s}$$

The inverse of $\dfrac{a}{s}$ with respect to the sum is $\dfrac{-a}{s}$,

$$\frac{a}{s} + \frac{-a}{s} = \frac{as - as}{ss} = \frac{0}{s} = \frac{0 \cdot s}{1 \cdot s} = \frac{0}{1}$$

It is obvious that the sum is commutative since the sum in the numerator is performed in $(R, +)$, which is an abelian group. Note that an element $\dfrac{a}{s} \in S^{-1}A$ is actually an equivalence class of elements. Thus in order for the sum in $S^{-1}R$ to be well defined, it must not depend on the choice of representant, that is to say, if $\dfrac{a}{s} \sim \dfrac{a'}{s'}$ then $\dfrac{a}{s} + \dfrac{b}{t} \sim \dfrac{a'}{s'} + \dfrac{b}{t}$. To prove this, let $u \in S$ such that $(as' - a's)u = 0$, then

$$\begin{aligned}\frac{a}{s} + \frac{b}{t} \sim \frac{a'}{s'} + \frac{b}{t} &\iff \frac{at + bs}{st} \sim \frac{a't + bs'}{s't} \iff \left[(at + bs)s't - (a't + bs')st\right] w = 0 \text{ for some } w \in S \\ &\iff (as' - a's)ttw = 0 \text{ for some } w \in S\end{aligned}$$

By making $w = u$ we get $utt \in S$ and $(as' - a's)utt = 0$. Hence the sum does not depend on the choice of representant and is well defined.

Now we prove that $(S^{-1}R, \cdot)$ is a commutative semigroup with unit. Given $\dfrac{a}{s}, \dfrac{a'}{s'} \in S^{-1}R$ we have $\dfrac{a}{s} \cdot \dfrac{a'}{s'} = \dfrac{aa'}{ss'} \in S^{-1}R$ because $aa' \in R$ and $ss' \in S$, so multiplication is an internal operation. It is also associative

$$\left(\frac{a}{s} \cdot \frac{a'}{s'}\right) \cdot \frac{a''}{s''} = \frac{aa'}{ss'} \cdot \frac{a''}{s''} = \frac{aa'a''}{ss's''} = \frac{a}{s} \cdot \frac{a'a''}{s's''} = \frac{a}{s} \cdot \left(\frac{a'}{s'} \cdot \frac{a''}{s''}\right)$$

and obviously commutative because the products in the numerator and denominator are computed in R,

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} = \frac{a'a}{s's} = \frac{a'}{s'} \cdot \frac{a}{s}$$

The neutral element of $S^{-1}R$ with respect to multiplication is $\dfrac{1}{1}$,

$$\frac{a}{s} \cdot \frac{1}{1} = \frac{a \cdot 1}{s \cdot 1} = \frac{a}{s}$$

As before, we have to check that multiplication does not depend on the choice of representant, that is to say, if $\frac{a}{s} \sim \frac{a'}{s'}$ then $\frac{a}{s} \cdot \frac{b}{t} \sim \frac{a'}{s'} \cdot \frac{b}{t}$. Let $u \in S$ such that $(as' - a's)u = 0$, then

$$\frac{a}{s} \cdot \frac{b}{t} \sim \frac{a'}{s'} \cdot \frac{b}{t} \iff \frac{ab}{st} \sim \frac{a'b}{s't} \iff (abs't - a'bst)w = 0 \text{ for some } w \in S$$

Finish proof $\square$

Once we have a localisation $S^{-1}R$, we have a ring homomorphism $\varphi\colon R \to S^{-1}R$ sending $a \mapsto \frac{a}{1}$.

Proposition 2.2.4. The ring homomorphism $\varphi\colon R \to S^{-1}R$ is injective if, and only if,

Theorem 2.2.5 (Universal property of localisation)**.** Let A, B be rings and $S \subset A$ a multiplicatively closed set. Let $f\colon A \to B$ a ring homomorphism such that $f(s) \in B$ is a unit for every $s \in A$. Then there exists a unique ring homomorphism $g\colon S^{-1}A \to B$ that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "f"'] & S^{-1}A \arrow[dl, "g"] \\ B & \end{tikzcd}$$

Proof. □

2.3 Chain conditions

Let A be a ring. In this section we focus our attention on ascending chains of ideals of A, that is to say, chains of the form $I_1 \subset I_2 \subset \cdots I_n \subset \cdot$; as well as on descending chains $I_1 \subseteq I_2 \subseteq \cdots I_n \subseteq \cdots$

Proposition 2.3.1. Let A be a ring. The following conditions are equivalent:

(1) *Finite generation*: every ideal $I \subset A$ is finitely generated.

(2) *Ascending chain condition*: every ascending chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of ideals of A stabilises, that is, there exists $m \geq 1$ such that $I_m = I_n$ for all $n \geq m$.

(3) *Maximality*: every non-empty set of ideals of A has a maximal element.

Proof. contenidos... □