Contents

1. Categories

2. Rings

2.1 Ideals

Definition 2.1.1. Let $(R, +, \cdot)$ be a ring (not necessarily commutative nor with unit).

(i) A subset $I \subset R$ is a **left ideal** of R if

  (i.a) $(I, +)$ is an abelian group

  (i.b) For every $a \in R$ and $r \in R$, $ra \in I$.

(ii) A subset $I \subset R$ is a **right ideal** of R if

  (ii.a) $(I, +)$ is an abelian group

  (ii.b) For every $a \in R$ and $r \in R$, $ar \in I$.

(iii) A subset $I \subset R$ that is both a left ideal and a right ideal, is called an **ideal** of R.

If R is a commutative ring, then left and right ideals coincide and are simply called ideals. Moreover, if R is a commutative ring with unit and an ideal $I \subset R$ contains the unit, $1_R \in I$, then $I = R$.

Definition 2.1.2. Let R be a commutative ring with unit. A family $\{f_\lambda\}_{\lambda \in \Lambda}$ of elements of R is a **system of generators of an ideal** $I \subset A$ if every element $f \in I$ can be expressed as a finite linear combination

$$f = a_1 f_{\lambda_1} + \cdots + a_n f_{\lambda_r}, \quad a_1, \ldots, a_r \in R$$

In this case we write $I = (f_\lambda \mid \lambda \in \Lambda)$ to denote that I is generated by the f_λ. If the family $\{f_1, \ldots, f_r\}$ is finite, then $I = (f_1, \ldots, f_r)$ and we say that I is **finitely generated**. If I is generated by a single element, that is, $I = (f) = \{rf \mid r \in R\}$, we say that I is a **principal ideal**.

Hereinafter R will denote a commutative ring with unit. In the coming sections we shall study some manners to construct new ideals from the given ones.

2.1.1 Intersection of ideals

Proposition 2.1.3. Let R be a ring.

(1) If $I, J \subset R$ are ideals, then the intersection $I \cap J$ is an ideal of R.

(2) Given any family $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals of R, the intersection $\bigcap_{\lambda \in \Lambda} I_\lambda \subset R$ is an ideal of R.

2.1.2 Sum of ideals

Definition 2.1.4. Let R be a ring and $\{I_\lambda\}_{\lambda\in\Lambda}$ an arbitary family of ideals of R. We define the **sum ideal** by

$$\sum_{\lambda\in\Lambda} I_\lambda = \left\{\sum\nolimits_{\lambda\in\Lambda} x_\lambda \,|x_\lambda \in I_\lambda,\ x_\lambda = 0 \text{ except for finitely many } \lambda\in\Lambda\right\}$$

If the family $\{I_1,\dots,I_r\}$ of ideals of R is finite, the sum ideal is simply

$$\sum_{\lambda=1}^{r} I_k = \left\{\sum\nolimits_{\lambda=1}^{r} x_\lambda \,|x_\lambda \in I_\lambda\right\}$$

Proposition 2.1.5. Let R be a ring. Then the sum ideal of an arbitrary family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of R is again an ideal of R.

2.1.3 Product of ideals

Definition 2.1.6. Let R be a ring and let I, J be ideals of R. We define the **product ideal** by

$$IJ = \left\{\sum_{\lambda=1}^{n} x_\lambda y_\lambda \mid x_\lambda \in I,\ y_\lambda \in J,\ n>0\right\}$$

If $\{I_1,\dots,I_r\}$ is a finite family of ideals of R, the product ideal is

$$I_1\cdots I_r = \left\{\sum_{\lambda=1}^{n} x_{\lambda,1}\cdots x_{\lambda,r} \mid x_{\lambda,i} \in I_i,\ n>0\right\}$$

Proposition 2.1.7. Let R be a ring and let $\{I_1,\dots,I_r\}$ be a finite family of ideals of R. Then their product ideal $I_1\cdots I_r$ is an ideal of R. Moreover, it is the ideal generated by the set

$$S = \{x_1\cdots x_r \mid x_i \in I_i\}$$

2.1.4 Ideal generated by a subset

In general, an arbitrary subset $S\subset R$ will not be an ideal. Nonetheless, in some cases we shall need the smallest ideal, in the sense of inclusions, that contains S.

Definition 2.1.8. Let R be a commutative ring with unit and $S\subset R$ a subset. The **ideal generated by** S is

$$I = \bigcap_{\substack{J\subset A \text{ ideal}\\ S\subset J}} J$$

Proposition 2.1.9. Let R be a commutative ring with unit and $S\subset R$ a subset. Then the ideal generated by S is an ideal of R.

2.1.5 Radical ideal

Definition 2.1.10. Let R be a commutative ring with unit and $I\subset R$ an ideal. We define the **radical of** I by

$$\mathrm{rad}(I) = \sqrt{I} = \{a\in R \mid a^n \in I \text{ for some } n>0\}$$

We say that I is a **radical ideal** when $I = \mathrm{rad}(I)$.

Proposition 2.1.11. Let R be a commutative ring with unit and $I \subset R$ an ideal.

(1) The radical $\operatorname{rad}(I)$ is an ideal of R.

(2) $I \subset \operatorname{rad}(I) = \operatorname{rad}(\operatorname{rad}(I))$

Definition 2.1.12. Let R be a commutative ring with unit. We define the **nilradical** of R as the radical of the zero ideal, that is,

$$\mathfrak{N}_R = \operatorname{rad}(0) = \{a \in R \mid a^n = 0 \text{ for some } n > 0\}$$

We say that R is **reduced** whenever it has no nilpotent elements different from zero, that is, when $\mathfrak{N}_R = 0$.

Proposition 2.1.13. Let R be a commutative ring with unit. The reduction of R is

$$R_{\text{red}} = {}^{R}/_{\mathfrak{N}_R}$$

which is a reduced ring. If R is already a reduced ring, then $R \simeq R_{\text{red}}$.

2.1.6 Colon and saturation ideals

Definition 2.1.14. Let R be a commutative ring with unit and $I, J \subset R$ ideals of R.

(i) The **colon ideal** of J with respect to I is $(I : J) = \{a \in R \mid aJ \subset I\}$.

(ii) The **annihilator of** J is $(0 : J) = \operatorname{Ann}_R(J)$.

(iii) The **saturation** of J with respect to I is $(I : J^\infty) = \{a \in R \mid aJ^n \subset I \text{ for some } n > 0\}$.

Proposition 2.1.15. Let R be a commutative ring with unit and $I, J \subset R$ ideals of R. Then $(I : J)$, $\operatorname{Ann}_R(J)$ and $(I : J^\infty)$ are ideals of R.

2.1.7 Extension and contraction of ideals

Definition 2.1.16. Let R, S be rings and $f\colon R \to S$ a ring homomorphism. Let $I \subset R$ and $J \subset S$ be ideals. We define:

(i) The **extension** of I by $I^e = \{b_1 f(x_1) + \cdots + b_n f(x_n) \mid x_i \in I,\, b_i \in S,\, n > 0\}$.

(ii) The **contraction** of J by $J^c = f^{-1}(J) = \{a \in R \mid f(a) \in J\}$.

Proposition 2.1.17. Let R, S be rings and $f\colon R \to S$ a ring homomorphism. Let $I \subset R$ and $J \subset S$ be ideals. Then,

(1) The **extension** $I^e \subset S$ is an ideal of S. Moreover, it is the ideal generated by $f(I) \subset S$.

(2) The **contraction** $J^c \subset R$ is an ideal of R.

Proposition 2.1.18. Let R, S be rings and $f\colon R \to S$ a ring homomorphism. Let $I \subset R$ and $J \subset S$ be ideals. Then,

(1) $I \subset I^{ec}$

(2) $I^c = I^{cec}$

(3) $J \supset J^{ce}$

(4) $J^e = J^{ece}$

2.2 Prime and maximal ideals

Definition 2.2.1. Let R be a ring and $I \subset R$ a proper ideal.

(i) We say that the ideal I is **prime** whenever $ab \in I$ implies $a \in I$ or $b \in I$.

(ii) We say that the ideal I is **maximal** if it is not contained in any other proper ideal, that is to say, if $J \subset R$ is a proper ideal and $I \subset J$, then $J = I$.

Example 2.2.2. Consider the ring $\mathbb{Z}$ with the usual addition and multiplication. Then for every prime $p \in \mathbb{Z}$, the ideal $(p) = p\mathbb{Z}$ is maximal. Indeed, assume that $(p) \subset I$ for some proper ideal $I \subset \mathbb{Z}$. Then $I = (a)$ for some $a \in \mathbb{Z}$ because $\mathbb{Z}$ is a principal ideal domain. Therefore $a \mid p$, but since p is prime, there are two possibilities. The first one is $a = \pm 1$, that is to say, $(a) = \mathbb{Z}$ which is not a proper ideal. The second one is $a = \pm p$, thus $(a) = (p)$. Hence (p) is a maximal ideal.

Proposition 2.2.3. Let R be a ring and $I \subset R$ an ideal. If I is a maximal ideal, then it is also a prime ideal.

Proof. Let $a, b \in R$ such that $ab \in I$ and consider the ideal $J = (a) + I$. Since $I \subset J$ and I is a maximal ideal, then either $J = I$ or $J = R$. In the former case we have that $a \in I$ and we are done. In the latter case, there exist $\lambda \in R$ and $c \in I$ satisfying $1 = \lambda a + c$, consequently $b = b \cdot 1 = \lambda ab + bc$. Since $\lambda ab \in I$ and $bc \in I$, we have that $b \in I$. In both cases we deduce that I is a prime ideal. □

The following

Proposition 2.2.4. Let R be a ring and $I \subset R$ an ideal. Then the quotient ring ${}^{R}/_{I}$ is an integral domain if, and only if, I is a prime ideal.

Proof. Assume that ${}^{R}/_{I}$ is an integral domain. Given $a, b \in R$ satisfying $ab = 0$, we have that $\overline{ab} = \overline{a}\,\overline{b} = \overline{0}$, thus either $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, that is to say, $a \in I$ or $b \in I$, so I is a prime ideal.

Conversely, assume that I is a prime ideal and let $\overline{a}, \overline{b} \in {}^{R}/_{I}$ such that $\overline{ab} = \overline{a}\overline{b} = \overline{0}$. Then $ab \in I$, therefore either $a \in I$ or $b \in I$, which implies either $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, that is to say, ${}^{R}/_{I}$ is an integral domain. □

Proposition 2.2.5. Let R be a ring and $I \subset R$ an ideal. Then the quotient ring ${}^{R}/_{I}$ is an field if, and only if, I is a maximal ideal.

Proof. □

The previous propositions give an alternative way to prove that every maximal ideal is prime. If $\mathfrak{m}$ is a maximal ideal, then ${}^{R}/_{\mathfrak{m}}$ is a field and, in particular, an integral domain, so $\mathfrak{m}$ must be a prime ideal.

Definition 2.2.6. Let R be a ring.

(i) The **spectrum of prime ideals** of R is the set of prime ideals of R,

$$\operatorname{Spec} R = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal}\}$$

(ii) The **spectrum of maximal ideals** of R is the set of maximal ideals of R,

$$\operatorname{Spm} R = \{\mathfrak{m} \subset R \mid \mathfrak{m} \text{ is a maximal ideal}\}$$

Definition 2.2.7. We say that a ring R is **local** if it has only one maximal ideal. It is often denoted by $(R, \mathfrak{m})$, where $\mathfrak{m} \subset R$ is the only maximal ideal. The ring R is said to be **semilocal** if it only has finitely many maximal ideals.

Example 2.2.8. The ring $\mathbb{Z}$ with the usual addition and multiplication is not local, since the ideal $p\mathbb{Z}$ is maximal for every prime $p \in \mathbb{Z}$.

The following proposition gives some useful characterisations for maximal ideals.

Proposition 2.2.9. Let A be a ring and $I \subset A$ an ideal. If $A \setminus I \subset A^*$, then A is a local ring and I is its maximal ideal.

Proof. Let $\pi\colon A \to A/I$ be the projection on the quotient ring. Denote the equivalence class of an element $a \in A$ by $\overline{a} = \pi(a)$. Take $\overline{x} \in A/I$ such that $\overline{x} \neq \overline{0}$. Then $x \notin I$ thus x must be a unit of A since $x \in A \setminus A \subset A^*$, so there exists $y \in A$ such that $xy = 1$. By projecting on the quotient we have $\pi(xy) = \pi(x)\pi(y) = \overline{xy} = \overline{1}$. As $x \in A/I$ is an arbitrary element different from zero, we deduce that A/I is a field, so I must be a maximal ideal.

Now let $J \subset A$ be another ideal. The projection $\pi\colon A \to A/I$ is a surjective ring homomorphism, thus the set $\pi(J) \subset A/I$ is an ideal. However, the only ideals of the field A/I are the zero ideal and the total ideal. In the first case $\pi(J) = 0$, thus $J \subset I$. In the second case $\pi(J) = A/I$, which implies $J = R$. Thus every ideal $J \subset A$ is either contained in I or is the total ideal, that is to say, I is the only maximal ideal. □

2.2.1 Extension and contraction of prime ideals

Let $f\colon A \to B$ be a ring homomorphism. It is natural to wonder whether prime and maximal ideals of A are preserved under ideal extensions, and whether prime and maximal ideals of B are preserved under ideal contraction.

Proposition 2.2.10. Let A and B be rings and $f\colon A \to B$ a ring homomorphism. If $J \in \operatorname{Spec} B$, then $J^c \in \operatorname{Spec} A$.

Proof. Let $a, b \in A$ such that $ab \in J^c$. Then $f(ab) = f(a)f(b) \in f(J^c) = J$ and, since $J \subset B$ is a prime ideal, either $f(a) \in J$ or $f(b) \in J$, which implies either $a \in J^c$ or $b \in J^c$, that is to say, $J^c \subset A$ is a prime ideal. □

This need not be the case with maximal ideals, that is to say, if $J \subset B$ is a maximal ideal, then $J^c \subset A$ need not be a maximal ideal.

The same happens to prime ideals: if $I \subset A$ is a prime ideal then $I^e \subset B$ need not be prime.

2.2.2 Existence of maximal ideals

First of all, we should recall Zorn's lemma:

Theorem 2.2.11 (Zorn's lemma)**.** Let S be a non-empty partially ordered set. If every non-empty totally ordered subset of S has an upper bound, then S has a maximal element.

Theorem 2.2.12 (Existence of maximal ideals)**.** Let R be a ring. Then R contains a maximal ideal.

Proof. Let S be the set of ideals of R. It is non-empty as it contains both the zero ideal 0 and the unit ideal R, and a partially ordered set under the order relation of inclusion ($\subset$).

Let $T \subset S$ be a non-empty totally ordered set of ideals of R, that is to say, given two different ideals $I, J \in T$, then either $I \subset J$ or $J \subset I$. We may see the elements of T as an ascending chain $I_1 \subset \cdots \subset I_n \subset \cdots$ of ideals of R. In order to prove the existence of an upper bound of T in S, consider the set $J = \bigcup_{I \in T} I$. We must show that J is an ideal of R and an upper bound for T.

We begin by proving that J is an ideal. Given $a, b \in J$, there exists a "minimal" ideal $I \in T$ such that $a, b \in I$ and either $a \notin I$ or $b \notin I$ for every $I' \subsetneq I$. Since $a - b \in I \subset J$, J is a subgroup of the additive group of R. Now if $a \in I$, for all $\lambda \in R$ it is true that $\lambda a \in I \subset J$, so J is an ideal. This constitutes an upper bound for T in S. Indeed, for otherwise there would exist an ideal $I \in T$ such that $I \subsetneq J$, but this contradicts the construction of J.

Since every non-empty totally ordered subset of S has an upperbound, by Zorn's lemma S has a maximal element, that is, a maximal ideal. $\square$

The existence of maximal ideals theorem yields two immediate corollaries.

Theorem 2.2.13. Let R be a ring and $I \subset R$ an ideal. Then there exists a maximal ideal $\mathfrak{m} \subset R$ that contains I.

Proof. If I is already a maximal ideal, then $\mathfrak{m} = I$. Therefore assume that I is not a maximal ideal. Then the quotient ring ${}^{R}/_{I}$ has a maximal ideal $\tilde{\mathfrak{m}}$ whose preimage $\mathfrak{m} = \pi^{-1}(\tilde{\mathfrak{m}}) \subset R$ is a maximal ideal containing I. $\square$

Corollary 2.2.14. Let R be a ring and let $a \in R \setminus R^*$. Then there exists a maximal ideal $\mathfrak{m}$ that contains a.

Proof. By applying the previous corollary to the principal ideal $I = (a)$, we deduce the existence of a maximal ideal $\mathfrak{m}$ containing I, thus containing a. $\square$

2.3 Localisation

Definition 2.3.1. Let R be a ring. A subset $S \subset R$ is a **multiplicatively closed set** if $1 \in S$ and $st \in S$ whenever $s, t \in S$.

Let R be a ring and $S \subset R$ a multiplicatively closed set. In the cartesian product $R \times S$ consider the relation

$$(a, s) \sim (a', s') \iff (as' - a's)t = 0 \text{ for some } t \in S$$

which is an equivalence relation. It is reflexive since $(a, s) \sim (a, s)$ because $(as - as)1 = 0$. If $(a, s) \sim (a', s')$ then $(as' - a's)t = (a's - as')(-t) = 0$ thus it is reflexive. Finally assume that $(a, s) \sim (a', s')$ and $(a', s') \sim (a'', s'')$, which is equivalent to

$$\begin{aligned} (a, s) \sim (a', s') &\iff (as' - a's)u = 0 &&\iff as'u = a'su \text{ for some } u \in S \\ (a', s') \sim (a'', s'') &\iff (a's'' - a''s')v = 0 &&\iff a's''v = a''s'v \text{ for some } u \in S \end{aligned}$$

In order to prove that $(a, s) \sim (a'', s'')$ we have the following:

$$as''(ss'uv) = as'u(ss''v) = a'su(ss''v) = a's''v(ssu) = a''s'v(ssu) = a''s(ss'uv)$$

By defining $w = ss'uv \in S$, we have shown that $(as'' - a''s)w = 0$ thus the relation is transitive. As we shall immediately see, it is far more natural to write the elements $(a, s) \in R \times S$ as fractions $\frac{a}{s}$. With this notation, the equivalence relation is written as

$$\frac{a}{s} \sim \frac{a'}{s'} \iff (as' - a's)t = 0 \text{ for some } t \in S$$

With this in mind, we define the following set.

Definition 2.3.2. Let R be a ring and $S \subset R$ a multiplicatively closed set. We define the **locatisation of R at S** as

$$S^{-1}R = R \times S \,/\!\sim\, = \left\{ \frac{a}{s} \mid a \in R,\, s \in S \right\} /\!\sim$$

Proposition 2.3.3. Let R be a ring $S \subset R$ a multiplicatively closed set. Then the localisation $S^{-1}R$ is a commutative ring with unit where sum and multiplication are defined as follows:

$$\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'} \qquad \frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$$

Proof. First of all, note that if $u \in S$, then $\frac{a}{s} = \frac{au}{su}$ since $(asu - asu)1 = 0$, so we may simplify the numerator and denominator of fractions as though we were working with school fractions as long as what we are simplyfing is an element of S.

We begin by checking that $(S^{-1}R, +)$ is an abelian group. The sum is an internal operation because $as' + a's \in R$ and $ss' \in S$, so $\frac{as' + a's}{ss'} \in R \times S$. It is associative

$$\left(\frac{a}{s} + \frac{a'}{s'}\right) + \frac{a''}{s''} = \frac{as' + a's}{ss'} + \frac{a''}{s''} = \frac{as's'' + a'ss'' + a''ss'}{ss's''} = \frac{a}{s} + \frac{a's'' + a''s'}{s's''} = \frac{a}{s} + \left(\frac{a'}{s'} + \frac{a''}{s''}\right)$$

The neutral element is $\frac{0}{1}$ since

$$\frac{a}{s} + \frac{0}{1} = \frac{a \cdot 1 + 0 \cdot s}{s \cdot 1} = \frac{a \cdot 1}{s \cdot 1} = \frac{a}{s}$$

The inverse of $\frac{a}{s}$ with respect to the sum is $\frac{-a}{s}$,

$$\frac{a}{s} + \frac{-a}{s} = \frac{as - as}{ss} = \frac{0}{s} = \frac{0 \cdot s}{1 \cdot s} = \frac{0}{1}$$

It is obvious that the sum is commutative since the sum in the numerator is performed in $(R, +)$, which is an abelian group. Note that an element $\frac{a}{s} \in S^{-1}A$ is actually an equivalence class of elements. Thus in order for the sum in $S^{-1}R$ to be well defined, it must not depend on the choice of representant, that is to say, if $\frac{a}{s} \sim \frac{a'}{s'}$ then $\frac{a}{s} + \frac{b}{t} \sim \frac{a'}{s'} + \frac{b}{t}$. To prove this, let $u \in S$ such that $(as' - a's)u = 0$, then

$$\begin{aligned} \frac{a}{s} + \frac{b}{t} \sim \frac{a'}{s'} + \frac{b}{t} \iff \frac{at + bs}{st} \sim \frac{a't + bs'}{s't} &\iff \left[(at + bs)s't - (a't + bs')st\right] w = 0 \text{ for some } w \in S \\ &\iff (as' - a's)ttw = 0 \text{ for some } w \in S \end{aligned}$$

By making $w = u$ we get $utt \in S$ and $(as' - a's)utt = 0$. Hence the sum does not depend on the choice of representant and is well defined.

Now we prove that $(S^{-1}R, \cdot)$ is a commutative semigroup with unit. Given $\frac{a}{s}, \frac{a'}{s'} \in S^{-1}R$ we have $\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \in S^{-1}R$ because $aa' \in R$ and $ss' \in S$, so multiplication is an internal operation. It is also associative

$$\left(\frac{a}{s} \cdot \frac{a'}{s'}\right) \cdot \frac{a''}{s''} = \frac{aa'}{ss'} \cdot \frac{a''}{s''} = \frac{aa'a''}{ss's''} = \frac{a}{s} \cdot \frac{a'a''}{s's''} = \frac{a}{s} \cdot \left(\frac{a'}{s'} \cdot \frac{a''}{s''}\right)$$

and obviously commutative because the products in the numerator and denominator are computed in R,

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} = \frac{a'a}{s's} = \frac{a'}{s'} \cdot \frac{a}{s}$$

The neutral element of $S^{-1}R$ with respect to multiplication is $\frac{1}{1}$,

$$\frac{a}{s} \cdot \frac{1}{1} = \frac{a \cdot 1}{s \cdot 1} = \frac{a}{s}$$

As before, we have to check that multiplication does not depend on the choice of representant, that is to say, if $\frac{a}{s} \sim \frac{a'}{s'}$ then $\frac{a}{s} \cdot \frac{b}{t} \sim \frac{a'}{s'} \cdot \frac{b}{t}$. Let $u \in S$ such that $(as' - a's)u = 0$, then

$$\frac{a}{s} \cdot \frac{b}{t} \sim \frac{a'}{s'} \cdot \frac{b}{t} \iff \frac{ab}{st} \sim \frac{a'b}{s't} \iff (abs't - a'bst)w = 0 \text{ for some } w \in S$$

Finish proof □

Once we have a localisation $S^{-1}R$, we have a ring homomorphism $\varphi \colon R \to S^{-1}R$ sending $a \mapsto \frac{a}{1}$.

Proposition 2.3.4. The ring homomorphism $\varphi \colon R \to S^{-1}R$ is injective if, and only if,

Theorem 2.3.5 (Universal property of localisation)**.** Let A, B be rings and $S \subset A$ a multiplicatively closed set. Let $f \colon A \to B$ a ring homomorphism such that $f(s) \in B$ is a unit for every $s \in A$. Then there exists a unique ring homomorphism $g \colon S^{-1}A \to B$ that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "f"'] & S^{-1}A \arrow[dl, "g"] \\ B & \end{tikzcd}$$

Proof. □

2.4 Chain conditions

Let A be a ring. In this section we focus our attention on ascending chains of ideals of A, that is to say, chains of the form $I_1 \subset I_2 \subset \cdots I_n \subset \cdot$; as well as on descending chains $I_1 \subseteq I_2 \subseteq \cdots I_n \subseteq \cdots$

2.4.1 Noetherian rings

Definition 2.4.1. A commutative ring R is said to be **Noetherian** or to satisfy the **ascending chain condition on ideals** (ACC on ideals) if every increasing chain of ideals in R eventually stabilises, that is, whenever $I_1 \subset \cdots \subset I_n \subset \cdots$ is an increasing chain of ideals of R, then there is $m \geq 1$ such that $I_k = I_m$ for all $k \geq m$.

Example 2.4.2. The ring of integers $\mathbb{Z}$ with the usual sum and multiplication is Noetherian. Since $\mathbb{Z}$ is a principal ideal domain, every ascending chain of ideals can be written as $(a_1) \subset \cdots \subset (a_n) \subset \cdots$ for $a_1, \ldots, a_n \in \mathbb{Z}$. Furthermore, we may assume that all the a_i are positive sine $(a_i) = (-a_i)$. Since a_i divides a_j for every $1 \leq j \leq i$, the ascending chain of ideals is equivalent to the descending chain $a_1 \geq \cdots \geq a_n \geq \cdots$ which is contained in $\mathbb{Z}_{\geq 1}$. Such a chain cannot last indefinitely without becoming stationary at some point, hence the ascending chain of ideals eventually stabilises.

Example 2.4.3. Every field k is Noetherian, for the only ideals it has are the zero ideal and the unit ideal.

The following proposition gives alternative characterisations of Noetherian rings:

Proposition 2.4.4. Let R be a ring. The following are equivalent:

(1) R is a Noetherian ring.

(2) *Finite generation*: every ideal $I \subset R$ is finitely generated.

(3) *Maximality*: every non-empty set S of ideals of R contains a maximal element under inclusion, that is, there exists an ideal $I \in S$ such that if $J \in S$ is another ideal satisfying $I \subset J$, then $I = J$.

Proof. (1) $\Rightarrow$ (2). Assume that there exists an ideal $I \subset R$ that is not finitely generated. Take an element $a_1 \in I$ and let $I_1 = (a_1)$. Since I is not finitely generated, I_1 is properly contained in I, $I_1 \subsetneq I$, thus the set $I \setminus I_1$ is non-empty. Take $a_2 \in I \setminus I_1$ and let $I_2 = (a_1, a_2)$. Again we have $I_1 \subsetneq I_2 \subsetneq I$. More generally, construct the ideal I_{n+1} for $n \geq 1$ as follows: given $I_n = (a_1, \ldots, a_n) \subsetneq I$, take $a_{n+1} \in I \setminus I_n$ and define $I_{n+1} = (a_1, \ldots, a_n, a_{n+1})$, which again satisfies $I_{n+1} \subsetneq I$. This process never ends, since the set $I \setminus I_n$ is non-empty for every $n \geq 1$. Consequently we have an ascending chain $I_1 \subsetneq \cdots \subsetneq I_n \subsetneq \cdots$ of ideals of R that never stabilises, reaching a contradiction.

(2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let $I_1 \subset \cdots \subset I_n \subset \cdots$ be an ascending chain of ideals of R and let $S = \{I_i \mid i \geq 1\}$ be the set of ideals in the chain. By the maximality hypothesis, S contains a maximal element, thus there exists $j \geq 1$ such that if $I_j \subset I_i$ for any $i \geq 1$, then $I_i = I_j$. Consequently the chain stabilises as of j, so the ring is Noetherian.

□

Proposition 2.4.5. Let R be a Noetherian ring and $I \subset R$ an ideal. Then the quotient ring $R/_I$ is Noetherian.

Proposition 2.4.6. Let R be a Noetherian ring and $S \subset R$ a multiplicatively closed set. Then the localisation at S, $S^{-1}R$, is a Noetherian ring.

2.4.2 Hilbert's basis theorem

Theorem 2.4.7 (Hilbert's basis theorem)**.** If R is a Noetherian ring, then the polynomial ring $R[x]$ is also Noetherian.

Corollary 2.4.8. If R is a Noetherian ring, then the polynomial ring $R[x_1, \ldots, x_n]$ is also Noetherian.

2.4.3 Artinian rings

Definition 2.4.9. A commutative ring R is said to be **Artinian** or to satisfy the **descending chain condition on ideals** (DCC on ideals) if every decreasing chain of ideals in R eventually stabilises, that is, whenever $I_1 \supset \cdots \supset I_n \supset \cdots$ is a descending chain of ideals of R, then there is $m \geq 1$ such that $I_k = I_m$ for all $k \geq 1$.

Proposition 2.4.10. Let R be a ring. The following are equivalent:

(1) R is an Artinian ring.

(2) *Minimality*: every non-empty set of ideals of A has a minimal element under inclusion, that is, there exists an ideal $I \in S$ such that if $J \in S$ is another ideal satisfying $J \subset I$, then $J = I$.

Proof. (2) $\Rightarrow$ (1). Let $I_1 \supset \cdots \supset I_n \supset \cdots$ be a descending chain of ideals of R and let $S = \{I_i \mid i \geq 1\}$. By the minimality assumption S has a minimal element, so there exists an ideal $I_j \in S$, where $j \geq 1$, such that if $I_i \subset I_j$ then $I_i = I_j$. Consequently, for every $i \geq j$ we have $I_i = I_j$, so the chain stabilises as of j.

(1) $\Rightarrow$ (2).

□

2.4.4 Dimension????