Section 0. Categories and functors.

0.1 Categories.

Definition. A category $\mathcal{C}$ consists of a collection of objects $ob(\mathcal{C})$ and for each pair of objects $A, B \in ob(\mathcal{C})$ we have a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of morphisms from A to B. For every triple $A, B, C \in ob(\mathcal{C})$ we have a composition law

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,C) \\ (f,g) & \mapsto & g \circ f \end{array}$$

subject to the following conditions:

C1) *Associativity*: Given $A, B, C, D \in ob(\mathcal{C})$ and the composition of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have $h \circ (g \circ f) = (h \circ g) \circ f$.

C2) *Identity*: For every $B \in ob(\mathcal{C})$ there exists an identity morphism $Id_B \in \mathrm{Hom}_{\mathcal{C}}(B, B)$ such that $Id_B \circ f = f$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \circ Id_B = g$ for any $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

C3) The sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{C}}(A', B')$ are disjoint unless $A = A'$ and $B = B'$.

Examples. Some categories that we encountered in previous courses are:

1) **Sets**: Objects are sets and morphisms are functions between sets.
2) $\mathbf{Vect}_{\mathbb{K}}$: Objects are $\mathbb{K}$-vector spaces and morphisms are linear maps.
3) **Top**: Objects are topological spaces and morphisms are continuous maps.
4) $\boldsymbol{\mathcal{G}}$: Objects are groups and we consider homomorphisms of groups.
5) $\boldsymbol{\mathcal{A}}\mathbf{b}$: Objects are abelian groups and we consider homomorphisms of groups.
6) **Diff**: Objects are differentiable manifolds and morphisms are differentiable maps.
7) **Rings**: Objects are rings and we consider homomorphism of rings.
8) $\boxed{\mathbf{Mod(A)}}$: Objects are modules over an associative, commutative ring A with unit and we consider homomorphisms of modules. If the ring is not commutative we have to distinguish *left* modules from *right* modules.

Definition. Given a category $\mathcal{C}$ we define its *opposite* or *dual* as the category $\mathcal{C}^\circ$ having the same objects, i.e. $ob(\mathcal{C}^\circ) = ob(\mathcal{C})$, but the morphisms go in the other direction $\mathrm{Hom}_{\mathcal{C}^\circ}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Definition. We say that $\mathcal{B}$ is a subcategory of $\mathcal{C}$ if

i) $ob(\mathcal{B}) \subseteq ob(\mathcal{C})$

ii) For every $A, B \in ob(\mathcal{B}) \subseteq ob(\mathcal{C})$ we have $\mathrm{Hom}_{\mathcal{B}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$.

iii) The composition laws and the identity are the same.

Whenever $\mathrm{Hom}_{\mathcal{B}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ we say that the subcategory is *full*.

Example. $\mathcal{A}\mathbf{b} \subseteq \mathcal{G}$ is a full subcategory but $\mathcal{G} \subseteq \mathbf{Sets}$ is not.

Definition. A morphism $f : A \to B$ in a category $\mathcal{C}$ is an *isomorphism* if there exists $g : B \to A$ such that $g \circ f = Id_A$ and $f \circ g = Id_B$. In particular we will say that the objects $A, B \in ob(\mathcal{C})$ are isomorphic. We will say that f is a *monomorphism* (resp. epimorphism) if $f \circ g = f \circ h$ implies $g = h$ (resp. $g \circ f = h \circ f$ implies $g = h$).

Remark. Any isomomorphism is necessarily both a monomorphism and an epimorphism, but the converse need not be true. A category is called *balanced* if any morphism which is both a monomorphism and an epimorphism is an isomorphism.

0.2 Functors.

Definition. Let $\mathcal{B}$ and $\mathcal{C}$ be categories. A *covariant functor* $F : \mathcal{B} \to \mathcal{C}$ is a rule that assigns

i) An object $F(A) \in ob(\mathcal{C})$ for every object $A \in ob(\mathcal{B})$.

ii) For every morphism $f : A \to B$ in $\mathcal{B}$ we have a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{C}$ satisfying:

  F1) $F(Id_A) = Id_{F(A)}$ for every $A \in ob(\mathcal{B})$.

  F2) Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{B}$ we have $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* $F : \mathcal{B} \to \mathcal{C}$ satisfies $i)$ and

ii') For every morphism $f : A \to B$ in $\mathcal{B}$ we have a morphism $F(f) : F(B) \to F(A)$ in $\mathcal{C}$ satisfying $F1)$ and

  F2') Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{B}$ we have $F(g \circ f) = F(f) \circ F(g)$.

Remark. A contravariant functor $F : \mathcal{B} \to \mathcal{C}$ is a covariant functor $F : \mathcal{B} \to \mathcal{C}^\circ$.

Examples. 1) *Forgetful functor:* $\boldsymbol{\mathcal{G}} \subseteq \textbf{Sets}$, $\textbf{Rings} \subseteq \boldsymbol{\mathcal{G}}$, ...

2) *Functor of points:* Fix $S \in ob(\mathcal{C})$ and consider $F^S(A) = \mathrm{Hom}_{\mathcal{C}}(A, S)$ for all $A \in ob(\mathcal{C})$. We get a contravariant functor $F^S : \mathcal{C} \to \textbf{Sets}$ with $F^S(A) = \mathrm{Hom}_{\mathcal{C}}(A, S)$ for all $A \in ob(\mathcal{C})$.

Definition. Let $F : \mathcal{B} \to \mathcal{C}$ be a covariant functor (analogously for contravariant).

1) We say that F is *full, faithful, fully faithful* if the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{B}}(A, B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(F(A), F(B)) \\ f & \longmapsto & F(f) \end{array}$$

is surjective, injective, bijective respectively.

2) We say that F is *essentially surjective* or *dense* if for all $C \in ob(\mathcal{C})$ there exists $B \in ob(\mathcal{B})$ such that $C \cong F(B)$.

Definition. Let $F, G : \mathcal{B} \to \mathcal{C}$ be covariant functors (analogously for contravariant). A morphism between F and G is a law that assigns, to each object $B \in ob(\mathcal{B})$, a morphism $\tau_B : F(B) \to G(B)$ such that, for every $f \in \mathrm{Hom}_{\mathcal{B}}(A, B)$ we have a commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\tau_B"'] & G(B) \end{tikzcd}$$

If τ_B is an isomorphism for all $B \in ob(\mathcal{B})$ then we say that τ is a *natural equivalence* of functors. Then there exists τ^{-1} and we denote $F \simeq G$.

Definition. We say that the categories $\mathcal{B}$ and $\mathcal{C}$ are *naturally equivalents* if there exist functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ such that $G \circ F \simeq Id_{\mathcal{B}}$ and $F \circ G \simeq Id_{\mathcal{C}}$.

Theorem. *The categories $\mathcal{B}$ and $\mathcal{C}$ are* naturally equivalents *if and only if there exists a fully faithful and essentially surjective covariant functor $F : \mathcal{B} \to \mathcal{C}$.*

0.3 Products and coproducts.

Definition. Let $\mathcal{C}$ be a category. We say that $A \in ob(\mathcal{C})$ is an *initial object* if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ only has one element for all $B \in ob(\mathcal{C})$. We say that $A \in ob(\mathcal{C})$ is a *final object* if $\mathrm{Hom}_{\mathcal{C}}(B, A)$ only has one element for all $B \in ob(\mathcal{C})$. If an initial object is also a final object we say that it is a *zero object* and we denote it as $\mathbf{0} \in ob(\mathcal{C})$.

Lemma. *Initial (resp. final) objects are unique up to isomorphism.*

0.3.1 Products and coproducts.

Let $\mathscr{C}$ be a category. Consider a family $\{A_i\}_{i\in I}$ where $A_i \in ob(\mathscr{C})$ and I is a set of indices. We construct new categories $\mathscr{P}$ and $\mathscr{Q}$ as follows:

- $ob(\mathscr{P}) = \{(B, f_i)_{i\in I} \mid B \in ob(\mathscr{C}),\ \ f_i \in \mathrm{Hom}_{\mathscr{C}}(B, A_i)\}$.
- $\mathrm{Hom}_{\mathscr{P}}((B, f_i), (C, g_i)) = \{h \in \mathrm{Hom}_{\mathscr{C}}(B, C) \mid g_i \circ h = f_i \ \ \forall i \in I\}$.

$$\begin{tikzcd} B \arrow[r, "f_i"] \arrow[d, "h"'] & A_i \\ C \arrow[ur, "g_i"'] & \end{tikzcd}$$

- $ob(\mathscr{Q}) = \{(B, f_i)_{i\in I} \mid B \in ob(\mathscr{C}),\ \ f_i \in \mathrm{Hom}_{\mathscr{C}}(A_i, B)\}$.
- $\mathrm{Hom}_{\mathscr{Q}}((B, f_i), (C, g_i)) = \{h \in \mathrm{Hom}_{\mathscr{C}}(B, C) \mid h \circ f_i = g_i \ \ \forall i \in I\}$.

$$\begin{tikzcd} B \arrow[d, "h"'] & A_i \arrow[l, "f_i"'] \arrow[dl, "g_i"] \\ C & \end{tikzcd}$$

Definition. A final object of the category $\mathscr{P}$, if it exists, is called *product* of the objects $\{A_i\}_{i\in I}$. An initial object of the category $\mathscr{Q}$, if it exists, is called *coproduct* of $\{A_i\}_{i\in I}$.

Notation. Usually we will denote the product as $\prod_{i\in I} A_i$ and the coproduct either as $\coprod_{i\in I} A_i$ or $\bigoplus_{i\in I} A_i$ depending on the category we are working with. The universal property these objects satisfy can be visualized in the following diagrams:

$$\begin{tikzcd} & & A_i \\ B \arrow[r, dotted] \arrow[urr, bend left, "f_i"] \arrow[drr, bend right, "f_j"'] & \prod_{i\in I} A_i \arrow[ur, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\ & & A_j \end{tikzcd} \qquad \begin{tikzcd} & & A_i \arrow[dl, "\lambda_i"'] \arrow[dll, bend right, "f_i"'] \\ B & \coprod_{i\in I} A_i \arrow[l, dotted] & \\ & & A_j \arrow[ul, "\lambda_j"] \arrow[ull, bend left, "f_j"] \end{tikzcd}$$

0.3.2 Pullbacks and pushouts.

Let $\mathscr{C}$ be a category and fix an object $S \in ob(\mathscr{C})$. We construct new categories $\mathscr{P}_S$ and $\mathscr{Q}_S$ as follows:

- $ob(\mathscr{P}_S) = \{(A, f) \mid A \in ob(\mathscr{C}),\ \ f \in \mathrm{Hom}_{\mathscr{C}}(A, S)\}$.

· $\mathrm{Hom}_{\mathscr{P}_S}((A,f),(B,g)) = \{h \in \mathrm{Hom}_{\mathscr{C}}(A,B) \mid g \circ h = f\}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & S \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

· $ob(\mathscr{Q}_S) = \{(A,f) \mid A \in ob(\mathscr{C}),\ \ f \in \mathrm{Hom}_{\mathscr{C}}(S,A)\}$.

· $\mathrm{Hom}_{\mathscr{Q}_S}((A,f),(B,g)) = \{h \in \mathrm{Hom}_{\mathscr{C}}(A,B) \mid h \circ f = g\ \ \forall i \in I\}$.

$$\begin{tikzcd} A \arrow[d, "h"'] & S \arrow[l, "f"'] \arrow[dl, "g"] \\ B & \end{tikzcd}$$

Definition. The *fiber product* or *pullback* of (A,f) and (B,g) is the product of these two objects in $\mathscr{P}_S$. The *fiber coproduct* or *pushout* of (A,f) and (B,g) is the coproduct of these two objects in $\mathscr{Q}_S$.

Notation. Usually we will denote the pullback as $A \times_S B$ and the pushout as $A \sqcup_S B$ but it will depend on the category we are working with. The universal property these objects satisfy can be visualized in the following diagrams:

$$\begin{tikzcd} C \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddrr, bend left] \arrow[ddr, bend right] & & \\ & A \times_S B \arrow[r, "h"] \arrow[d] \arrow[dr] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & S \end{tikzcd} \qquad \begin{tikzcd} C & & \\ & A \sqcup_S B \arrow[ul, dotted] & A \arrow[l, "h"'] \arrow[ull, bend right] \\ & B \arrow[u] \arrow[uul, bend left] & S \arrow[l, "g"] \arrow[u, "f"'] \arrow[ul] \arrow[uull, bend right] \end{tikzcd}$$

0.4 Limits.

Let $\mathscr{C}$ be a category and let I be a *preordered set*, i.e. we have a relation $\leq$, satisfying the reflexive and transitive properties. Quite often we will consider a *directed set* when the preordered set also satisfies that for all $i, j \in I$ there exists $\ell \in I$ such that $i \leq \ell$ and $j \leq \ell$.

0.4.1 Inverse limits.

Consider a family $\{A_i\}_{i \in I}$ where $A_i \in ob(\mathscr{C})$ satisfying:

- For all $i, j \in I$ with $i \leq j$ we have $f_{j,i} \in \mathrm{Hom}_{\mathscr{C}}(A_j, A_i)$ such that
    i) $f_{i,i} = Id_{A_i}$.
    ii) $f_{j,i} \circ f_{k,j} = f_{k,i}$, for all $i \leq j \leq k$.

Definition. We say that $(\{A_i\}_{i\in I}, \{f_{j,i}\}_{i,j\in I})$ is a *projective* or *inverse system*.

Consider the category $\mathscr{P}$

- $ob(\mathscr{P}) = \{(B, f_i)_{i\in I} \mid B \in ob(\mathscr{C}),\ \ f_i \in \mathrm{Hom}_{\mathscr{C}}(B, A_i),\ \ f_{j,i} \circ f_j = f_i\ \ i \leq j\}$.
- $\mathrm{Hom}_{\mathscr{P}}((B, f_i), (C, g_i)) = \{h \in \mathrm{Hom}_{\mathscr{C}}(B, C) \mid g_i \circ h = f_i\ \ \forall i \in I\}$.

Definition. A final object of the category $\mathscr{P}$ is called the *projective* or *inverse limit* of $\{A_i\}_{i\in I}$. We denote it as $\varprojlim A_i$ and the universal property it satisfies reads as

$$\begin{tikzcd}
 & & A_i \\
B \arrow[urr, bend left, "f_i"] \arrow[drr, bend right, "f_j"'] \arrow[r, dotted] & \varprojlim A_i \arrow[ur, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\
 & & A_j \arrow[uu, "f_{j,i}"']
\end{tikzcd}$$

Remark. If I is trivially ordered, i.e. $i \leq j$ iff $i = j$, then $\varprojlim A_i = \prod_{i\in I} A_i$.

Examples. 1) $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ ring of p-adic numbers.

2) Let $\mathbb{F}|\mathbb{K}$ be a Galois extension. Then $\mathrm{Gal}(\mathbb{F}|\mathbb{K}) = \varprojlim \mathrm{Gal}(\mathbb{L}|\mathbb{K})$ where the field extensions $\mathbb{F}|\mathbb{L}|\mathbb{K}$ are finite Galois.

3) Let R be a commutative ring with unit and $\mathfrak{a} \subseteq R$ an ideal. Then, the $\mathfrak{a}$-adic completion of R is $\widehat{R}^{\mathfrak{a}} = \varprojlim R/\mathfrak{a}^i$.

0.4.2 Direct limits.

Consider a family $\{A_i\}_{i\in I}$ where $A_i \in ob(\mathscr{C})$ satisfying:

- For all $i, j \in I$ with $i \leq j$ we have $f_{i,j} \in \mathrm{Hom}_{\mathscr{C}}(A_i, A_j)$ such that
    i) $f_{i,i} = Id_{A_i}$.
    ii) $f_{j,k} \circ f_{i,j} = f_{i,k}$, for all $i \leq j \leq k$.

Definition. We say that $(\{A_i\}_{i\in I}, \{f_{i,j}\}_{i,j\in I})$ is an *inductive* or *direct system*.

Consider the category $\mathcal{Q}$

- $ob(\mathcal{Q}) = \{(B, f_i)_{i\in I} \mid B \in ob(\mathcal{C}),\ \ f_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B),\ \ f_j \circ f_{i,j} = f_i\ \ i \le j\}$.
- $\mathrm{Hom}_{\mathcal{Q}}((B, f_i),(C, g_i)) = \{h \in \mathrm{Hom}_{\mathcal{C}}(B, C) \mid h \circ f_i = g_i\ \ \forall i \in I\}$.

Definition. An initial object of the category $\mathcal{Q}$ is called the *inductive* or *direct limit* of $\{A_i\}_{i\in I}$. We denote it as $\varinjlim A_i$ and the universal property it satisfies reads as

$$\begin{tikzcd}
 & & A_i \arrow[dll, bend right, "f_i"'] \arrow[dl, "\lambda_i"'] \arrow[dd, "f_{i,j}"] \\
B & \varinjlim A_i \arrow[l, dotted] & \\
 & & A_j \arrow[ull, bend left, "f_j"] \arrow[ul, "\lambda_j"]
\end{tikzcd}$$

Remark. If I is trivially ordered then $\varinjlim A_i = \bigsqcup_{i\in I} A_i$.

Examples. 1) $\mathbb{Q}/\mathbb{Z} = \varinjlim C_i$, where C_i is the cyclic group of order i.

2) Let $\overline{\mathbb{K}}$ be the algebraic closure of a field $\mathbb{K}$. Then $\overline{\mathbb{K}} = \varinjlim \mathbb{L}$, where the field extensions $\mathbb{L}|\mathbb{K}$ are finite.

3) Let R be a commutative ring with unit and M a R-module. Then $M = \varinjlim M_i$ where $M_i \subseteq M$ are finitely generated submodules.

0.5 Abelian categories.

Definition. A category $\mathcal{C}$ is *preadditive* if

i) For all $A, B \in ob(\mathcal{C})$, we have that $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group such that the composition of morphisms is distributive with respect to the group operation.

ii) There exists a zero object $\mathbf{0} \in ob(\mathcal{C})$.

We say that $\mathcal{C}$ is *additive* if it satisfies $i), ii)$ and the equivalent conditions:

iii) $\mathcal{C}$ admits finite products.

iii') $\mathcal{C}$ admits finite coproducts.

Definition. A functor $F : \mathcal{B} \to \mathcal{C}$ between preadditive categories is *additive* if it is a homomorphism of groups.

Definition. Let $\mathcal{C}$ be a preadditive category and let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ be a morphism. We define the *kernel* of f as a morphism $\iota : \ker(f) \to A$, with $\ker(f) \in ob(\mathcal{C})$, such that

i) $f \circ \iota = 0$.

ii) For all $g : C \to A$ such that $f \circ g = 0$ there exists a unique $\overline{g} : C \to \ker(f)$ such that $\iota \circ \overline{g} = g$.

We define the *cokernel* of f as a morphism $\pi : B \to \mathrm{Coker}(f)$, with $\mathrm{Coker}(f) \in ob(\mathcal{C})$, such that

i) $\pi \circ f = 0$.

ii) For all $h : B \to C$ such that $h \circ \pi = 0$ there exists a unique $\overline{h} : \mathrm{Coker}(f) \to C$ such that $\overline{h} \circ \pi = h$.

Proposition. Let $\mathcal{C}$ be a preadditive category with kernels and cokernels. Then we have that $\iota : \ker(f) \to A$ is a monomorphism and $\pi : B \to \mathrm{Coker}(f)$ is an epimorphism.

Definition. Let $\mathcal{C}$ be a preadditive category with kernels and cokernels and let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ be a morphism. We define the *image* of f as the kernel of the cokernel of f. We define the *coimage* of f as the cokernel of the kernel of f.

Notice that we have

$$\begin{tikzcd} \ker(f) \arrow[r, "\iota"] & A \arrow[r, "f"] \arrow[d, "\tau"'] & B \arrow[r, "\pi"] & \mathrm{Coker}(f) \\ & \mathrm{Coim}(f) & \mathrm{Im}(f) \arrow[u, "j"'] & \end{tikzcd}$$

We have $f \circ \iota = 0$ and thus, by property *ii)* for cokernels, there exists $\upsilon : \mathrm{Coim}(f) \dashrightarrow B$ such that $\upsilon \circ \tau = f$. We also have $\pi \circ \upsilon = 0$ because $0 = \pi \circ f = \pi \circ \upsilon \circ \tau$ and τ is an epimorphism. Using property *ii)* for kernels we get $\overline{f} : \mathrm{Coim}(f) \dashrightarrow \mathrm{Im}(f)$ such that $j \circ \overline{f} = \upsilon$. Therefore we have the commutative diagram

$$\begin{tikzcd} \ker(f) \arrow[r, "\iota"] & A \arrow[r, "f"] \arrow[d, "\tau"'] & B \arrow[r, "\pi"] & \mathrm{Coker}(f) \, . \\ & \mathrm{Coim}(f) \arrow[ur, dotted, "\upsilon"] \arrow[r, dotted, "\overline{f}"'] & \mathrm{Im}(f) \arrow[u, "j"'] & \end{tikzcd}$$

In particular, $\overline{f}$ is the unique morphism such that $f = j \circ \overline{f} \circ \tau$.

Definition. An additive category $\mathcal{C}$ is *abelian* if there exist kernels and cokernels and $\overline{f} : \mathrm{Coim}(f) \dashrightarrow \mathrm{Im}(f)$ is an isomorphism for any morphism f in $\mathcal{C}$.

Section 1. Rings and Ideals

1.1 Quick review on rings.

Definition. We say that A is a ring if it is a set with two internal operations
Sum:

$$\begin{array}{rcl} A \times A & \longrightarrow & A \\ (a,b) & \longmapsto & a+b \end{array}$$

Product:

$$\begin{array}{rcl} A \times A & \longrightarrow & A \\ (a,b) & \longmapsto & ab \end{array}$$

satisfying for all $a, b, c \in A$

1) $(A, +)$ is an abelian group.
2) · $a(b+c) = ab + ac$.
   · $(a+b)c = ac + bc$.
   · $(ab)c = a(bc)$.

Unless otherwise stated we will also assume the following properties:

3) **Commutative:** $ab = ba$ for all $a, b \in A$
4) **Unit element:** There exists 1_A s.t. $a1_A = 1_A a = a$

Examples. 1) $\mathbb{N}$ not a ring.

2) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.
3) $\mathbb{K}[x]$, $\mathbb{K}[[x]]$ where $\mathbb{K}$ is a field.
4) $A[x]$, $A[[x]]$ where A is a ring.
5) Euclidean $\Rightarrow$ PID $\Rightarrow$ UFD

Definition. We say that $B \subseteq A$ is a subring if it is closed for the sum, the product and $1_B = 1_A$.

Proposition. $B \subseteq A$ is a subring if and only if for all $a, b \in B$ we have $a - b \in B$, $ab \in B$ and $1_A \in B$.

Definition. Some concepts that will be useful:

- **Zero divisor:** $a \in A$ s.t. $\exists b \neq 0 \;\; ab = 0$.
- **Integral domain:** Ring A with no $(\neq 0)$ zero divisors
- **Nilpotent:** $a \in A$ s.t. $a^n = 0$ for some $n > 0$.
- **Unit:** $a \in A$ s.t. $ab = 1$ for some $b \in A$.

$$A^* := \{a \in A \mid a \text{ unit}\}$$

- **Field:** Ring A s.t. $1_A \neq 0$ and $A^* = A \setminus \{0\}$.

Definition. Let A, B be rings. We say that $f : A \longrightarrow B$ is a ring homomorphism if for all $a, b \in A$:

- $f(a + b) = f(a) + f(b)$
- $f(ab) = f(a)f(b)$
- $f(1_A) = 1_B$

We set: $\mathrm{Hom}(A, B) = \{f : A \longrightarrow B \mid f \text{ ring homomorphism}\}$

Definition. We say:

- f monomorphism if f injective.
- f epimorphism if f surjective.
- f isomorphism if f bijective.

1.2 Ideals.

Definition. Let $(A, +, \cdot)$ be a commutative ring with unit. We say that $I \subseteq A$ is an ideal if:

1) $(I, +)$ abelian group.
2) For all $a \in A$ and all $x \in I$, $ax \in I$

Remark. If $1_A \in I$ then $I = A$.

1.2.1 Generators of an ideal.

- The **principal ideal** generated by $x \in A$ is

$$I = (x) = \{ax \mid a \in A\}.$$

- A family $\{f_\lambda\}_{\lambda \in \Lambda}$ of elements $f_\lambda \in A$ is a system of generators of an ideal I if any element $f \in I$ can be expressed as a finite linear combination

$$f = a_1 f_{\lambda_1} + \cdots + a_r f_{\lambda_r} \qquad a_i \in A$$

We denote $I = (f_\lambda \mid \lambda \in \Lambda)$, and $I = (f_1, \ldots, f_r)$ if $\Lambda = \{1, \ldots, r\}$ is finite.

1.2.2 Operations with ideals.

- **Intersection:**
  - $I \cap J = \{x \in A \mid x \in I, x \in J\}$ is an ideal.
  - $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an ideal for any family $\{I_\lambda\}_{\lambda \in \Lambda}$
- **Sum:**
  - $I + J = \{x + y \in A \mid x \in I, y \in J\}$ is the ideal generated by $I \cup J$.
  - $\sum_{\lambda \in \Lambda} I_\lambda = \{x_1 + \cdots + x_n \in A \mid x_i \in I_{\lambda_i}, n > 0\}$ ideal generated by $\bigcup_{\lambda \in \Lambda} I_\lambda$.
- **Product:**
  - $IJ = \{x_1 y_1 + \cdots + x_n y_n \in A \mid x_i \in I, y_i \in J, n > 0\}$ is the ideal generated by $\{xy \mid x \in I, y \in J\}$.
  - Analogous for finite products $I_1 \cdots I_r$.
- **Radical:**
  - $\mathrm{rad}(I) = \sqrt{I} := \{a \in A \mid a^n \in I, \ \ n >> 0\} \subseteq A$ is an ideal.
  - $I \subseteq \mathrm{rad}(I) = \mathrm{rad}(\mathrm{rad}(I))$.
  - I is a radical ideal if $I = \mathrm{rad}(I)$.
  - $\mathrm{rad}(0) = \{a \in A \mid a^n = 0, \ \ n >> 0\} \subseteq A$ is the nilradical of A.
  - We say that A is reduced if it has no nilpotents, i.e. $\mathrm{rad}(0) = 0$.
  - $A_{\mathrm{red}} = A/\mathrm{rad}(0)$.

- **Colon ideal:**
    - $(I : J) := \{a \in A \mid aJ \subseteq I\} \subseteq A$ is an ideal
    - $(0 : J) = \mathrm{Ann}_A(J)$ annihilator of J.
- **Saturation:**
    - $(I : J^\infty) := \{a \in A \mid \exists n > 0 \text{ s.t. } aJ^n \subseteq I\} \subseteq A$ is an ideal
- Let $f : A \longrightarrow B$ be a ring homomorphism. Let $I \subseteq A$ and $J \subseteq B$ be ideals. Then:
    - **Extension:** $I^e := \{b_1 f(x_1) + \cdots + b_r f(x_r) \in B \mid b_i \in B, x_i \in I\}$ is the ideal generated by $f(I)$.
    - **Contraction:** $J^c = f^{-1}(J) := \{a \in A \mid f(a) \in J\}$ is an ideal.

Proposition. We have:

- $I \subseteq I^{ec}, \quad I^c = I^{cec}$.
- $J \supseteq J^{ce}, \quad J^e = J^{ece}$.

1.3 Quotient ring.

Let $I \subseteq A$ be an ideal. We define:

$$A/I = \{\overline{a} = a + I \mid a \in A\}$$

Remark. $a + I = b + I$ if and only if $a - b \in I$

Proposition. $(A/I, +, \cdot)$ is a ring with the operations:

- **sum:** $\overline{a} + \overline{b} = \overline{a + b}$
- **product:** $\overline{a}\overline{b} = \overline{ab}$

Remark. The quotient morphism

$$\begin{array}{rcl} \pi : A & \longrightarrow & A/I \\ a & \longmapsto & \overline{a} \end{array}$$

is surjective and $\mathrm{Ker}\,\pi = I$.

Proposition. There is a bijection

$$\begin{array}{ccc} \Big\{ \text{Ideals } J \subseteq A \text{ s.t. } I \subseteq J \Big\} & \leftrightarrow & \Big\{ \text{Ideals of } A/I \Big\} \\ J & \longrightarrow & J/I \end{array}$$

1.4 Prime and maximal ideals.

Definition. Let $I \subseteq A$ be a proper ideal.

1) We say that I is **prime** if for all $a, b \in A$

$$ab \in I \Longrightarrow a \in I \text{ or } b \in I$$

2) We say that I is **maximal** if it is maximal w.r.t. inclusion.

Proposition. A maximal ideal is a prime ideal.

Notation. We denote

- $\mathrm{Spec}A = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ prime}\}$
- $\mathrm{Max}A = \{\mathfrak{m} \subseteq A \mid \mathfrak{m} \text{ maximal}\}$

Proposition. We have:

1) $\mathfrak{p} \in \mathrm{Spec}A \Longleftrightarrow A/\mathfrak{p}$ domain.

2) $\mathfrak{m} \in \mathrm{Spec}A \Longleftrightarrow A/\mathfrak{m}$ field.

Definition. We say that $(A, \mathfrak{m})$ is a local ring if $\mathrm{Max}A = \{\mathfrak{m}\}$. We say that A is semilocal if it only has a finite number of maximal ideals.

1.4.1 Extension and contraction of prime ideals.

Let $f : A \longrightarrow B$ be a ring homomorphism. Let $I \subseteq A$ and $J \subseteq B$ be ideals. Then:

- $J \in \mathrm{Spec}B \Rightarrow J^c \in \mathrm{Spec}A$.
- $J \in \mathrm{Max}B \not\Rightarrow J^c \in \mathrm{Max}A$.
- $I \in \mathrm{Spec}A \not\Rightarrow I^e \in \mathrm{Spec}B$.

1.4.2 Existence of maximal ideals.

Zorn's lemma: Every partially ordered set s.t. every chain has an upper bound contains a maximal element.

Theorem. *Let A be a ring and $I \subseteq A$ an ideal. Then there exists a maximal ideal $\mathfrak{m} \subseteq A$ s.t. $I \subseteq \mathfrak{m}$.*

1.5 Ring of fractions.

Definition. Let A be a ring. A set of elements $S \subseteq A$ is a **multiplicatively closed set** if $1 \in S$ and $s, t \in S \Rightarrow st \in S$.

Definition. Let A be a ring and $S \subseteq A$ a multiplicatively closed set. We define

$$S^{-1}A = {A \times S}\big/_{\sim} = \left\{ \frac{a}{s} \mid a \in A, \;\; s \in S \right\}\big/_{\sim}$$

where $\frac{a}{s} \sim \frac{a'}{s'}$ if and only if $\exists t \in S$ such that $t(as' - a's) = 0$. We have that $\sim$ is an equivalence relation so it satisfies

- *Reflexive:* $\frac{a}{s} \sim \frac{a}{s}$.
- *Symmetric:* $\frac{a}{s} \sim \frac{a'}{s'}$ if and only if $\frac{a'}{s'} \sim \frac{a}{s}$.
- *Transitive:* If $\frac{a}{s} \sim \frac{a'}{s'}$ and $\frac{a'}{s'} \sim \frac{a''}{s''}$ then $\frac{a}{s} \sim \frac{a''}{s''}$.

Proposition. $(S^{-1}A/I, +, \cdot)$ is a commutative ring with unit with the operations:

- **sum:** $\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}$
- **product:** $\frac{a}{s}\frac{a'}{s'} = \frac{aa'}{ss'}$

Remark. We have a ring homomorphism $\varphi : A \to S^{-1}A$ sending $a \to \frac{a}{1}$ which is injective if and only if S has no zero divisors.

1.5.1. Universal property

Let A be a ring and $S \subseteq A$ a multiplicatively closed set and $S^{-1}A$ its ring of fractions with the ring homomorphism $\varphi : A \to S^{-1}A$. Then, $S^{-1}A$ is determined, up to a unique isomorphism, by the universal property:

Let $f : A \to B$ be a ring homomorphism s.t. $f(s) \in B$ is a unit for all $s \in S$. Then there exists a unique ring homomorphism $g : S^{-1}A \to B$ s.t. $g \circ \varphi = f$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "f"'] & S^{-1}A \arrow[dl, dotted, "g"] \\ B & \end{tikzcd}$$

Remark. If C is a ring satisfying the universal property then $C \cong S^{-1}A$.

1.5.2. Examples

The main examples we are going to consider:

1) **Total fraction ring:** $\mathrm{Tot}(A) := S^{-1}A$ where $S = \{a \in A \mid a \text{ is not a zero divisor}\}$.

   Remark. $S = A \setminus \{0\}$ when A is a domain.

2) **Localization at an element:** $A_f := S^{-1}A$ where $S = \{f^n \mid n \geq 0\}$ for a given $f \in A$.

3) **Localization at a prime ideal:** $A_\mathfrak{p} := S^{-1}A$ where $S = A \setminus \mathfrak{p}$ for a given prime ideal $\mathfrak{p} \subseteq A$.

   Proposition. $\frac{a}{s} \in A_\mathfrak{p} \Leftrightarrow a \notin \mathfrak{p}$

   Proposition. $A_\mathfrak{p}$ is a local ring.

1.5.3. Ideals in the ring of fractions

Let A be a ring and $S \subseteq A$ a multiplicatively closed set and $S^{-1}A$ its ring of fractions with the ring homomorphism $\varphi : A \to S^{-1}A$. Given an ideal $I \subseteq A$ we may consider:

- $I^e = IS^{-1}A = \{\frac{a_1}{1}\frac{b_1}{s_1} + \cdots + \frac{a_r}{1}\frac{b_r}{s_r} \mid a_i \in \varphi(I), b_i \in A, s_i \in S\}$
- $S^{-1}I = \left\{\frac{a}{s} \mid a \in I,\ s \in S\right\}\big/\sim$

Proposition. We have $IS^{-1}A = S^{-1}I$ and every ideal $J \subseteq S^{-1}A$ is of this form, i.e. $\exists I \subseteq A$ s.t. $J = S^{-1}I$.

Theorem. *We have*

1) $S^{-1}I = S^{-1}A \quad \Leftrightarrow \quad I \cap S \neq \emptyset$.

2) $\mathfrak{p} \in \mathrm{Spec}A$ *s.t.* $\mathfrak{p} \cap S = \emptyset \quad \Rightarrow \quad S^{-1}\mathfrak{p} \in \mathrm{Spec}S^{-1}A$

3) *There is a bijection*

$$\begin{array}{ccc} \left\{\mathfrak{p} \in \mathrm{Spec}A \mid \mathfrak{p} \cap S = \emptyset\right\} & \leftrightarrow & \mathrm{Spec}S^{-1}A \\ \mathfrak{p} & \longrightarrow & S^{-1}\mathfrak{p} \\ \mathfrak{q}^c = \mathfrak{q} \cap A & \longleftarrow & \mathfrak{q} \end{array}$$

Examples. 1) $A_f := S^{-1}A$ where $S = \{f^n \mid n \geq 0\}$ for a given $f \in A$.

$$\mathrm{Spec}A_f = \{\mathfrak{p} \in \mathrm{Spec}A \mid f \notin \mathfrak{p}\} = \mathrm{Spec}A \setminus V(f)$$

2) $A_\mathfrak{p} := S^{-1}A$ where $S = A \setminus \mathfrak{p}$ for a given prime ideal $\mathfrak{p} \subseteq A$.

$$\mathrm{Spec}A_\mathfrak{p} = \{\mathfrak{q} \in \mathrm{Spec}A \mid \mathfrak{q} \subseteq \mathfrak{p}\}$$

1.5.4. Localization and ring homomorphisms

Proposition. Let $f : A \longrightarrow B$ be a ring homomorphism. Let $S \subseteq A$ and $T \subseteq B$ be multiplicatively closed subsets s.t. $f(S) \subseteq T$. Then, $\exists!\ g : S^{-1}A \longrightarrow T^{-1}B$ s.t. the diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi"'] & B \arrow[d, "\psi"] \\ S^{-1}A \arrow[r, "g"'] & T^{-1}B \end{tikzcd}$$

Corollary. *Let $f : A \longrightarrow A/I$ for a given ideal $I \subseteq A$. Let $\overline{S} \subseteq A/I$ be the multiplicatively closed set associated to $S \subseteq A$. Then*

$$\overline{S}^{-1}A/I \cong {}^{S^{-1}A}\big/_{S^{-1}I}$$

Definition. The **residue field** of a ring A w.r.t a prime ideal $\mathfrak{p} \in \mathrm{Spec}A$ is

$$k(\mathfrak{p}) = {}^{A_\mathfrak{p}}\big/_{\mathfrak{p}A_\mathfrak{p}}$$

1.5.5. Fiber of a ring homomorphism

Let $f : A \longrightarrow B$ be a ring homomorphism. Recall that, given $\mathfrak{q} \in \mathrm{Spec}B$ we have $\mathfrak{q}^c \in \mathrm{Spec}A$. Thus we have a not necessarily surjective map

$$\begin{array}{rcl} f^* : \mathrm{Spec}B & \longrightarrow & \mathrm{Spec}A \\ \mathfrak{q} & \longmapsto & \mathfrak{q}^c \end{array}$$

Definition. Given $\mathfrak{p} \in \mathrm{Spec}A$, its fiber is

$$(f^*)^{-1}(\mathfrak{p}) := \{\mathfrak{q} \in \mathrm{Spec}B \mid \mathfrak{q}^c = \mathfrak{p}\}$$

Proposition. Let $f : A \longrightarrow B$ be a ring homomorphism. Given $\mathfrak{p} \in \mathrm{Spec}A$ we have

$$(f^*)^{-1}(\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$$

1.6 Chain conditions.

Proposition. The following are equivalent:

0) *Finite generation:* Every ideal $I \subseteq A$ is finitely generated.

1) *Ascending chain condition:* Every ascending chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq \cdots$ of ideals of A stabilizes. That is, there exists $m \geq 1$ such that $I_m = I_n$ for all $n \geq m$.

2) *Maximality:* Every non-empty set of ideals of A has a maximal element.

Definition. We say that A is Noetherian if it satisfies the conditions of the proposition.

Proposition. The following are equivalent:

1) *Descending chain condition:* Every descending chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_i \supseteq \cdots$ of ideals of A stabilizes. That is, there exists $m \geq 1$ such that $I_m = I_n$ for all $n \geq m$.

2) *Minimality:* Every non-empty set of ideals of A has a minimal element.

Definition. We say that A is Artinian if it satisfies the conditions of the proposition.

Examples. 1) $\mathbb{Z}$, k field are Noetherian.

2) A Noetherian, $I \subseteq A$ ideal $\Rightarrow$ A/I Noetherian.

3) A Noetherian, $S \subseteq A$ multiplicatively closed set $\Rightarrow$ $S^{-1}A$ Noetherian.

Theorem. **(Hilbert basis theorem)** *A Noetherian $\Rightarrow$ $A[x]$ Noetherian.*

Proposition. Let A be an Artinian ring. Then:

1) $\mathfrak{p} \in \mathrm{Spec} A \Rightarrow \mathfrak{p} \in \mathrm{Max} A$

2) $\mathrm{Max} A$ is a finite set.

Definition. Let A be a ring. A chain of prime ideals of length n is

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

with $\mathfrak{p}_i \in \mathrm{Spec}A$.

Definition. Let A be a ring.

- The **Krull dimension** of A is

$$\dim A = \sup\{n \mid \exists \text{ chain of prime ideals of length n } \ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$$

- The **height** of $\mathfrak{p} \in \mathrm{Spec}A$ is

$$\mathrm{ht}\,\mathfrak{p} = \sup\{n \mid \exists \text{ chain of prime ideals of length n } \ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}$$

- The **height** of an ideal $I \subseteq A$ is

$$\mathrm{ht}\, I = \inf\{\mathrm{ht}\,\mathfrak{p} \mid I \subseteq \mathrm{ht}\mathfrak{p}, \ \mathfrak{p} \in \mathrm{Spec}A\}$$

Theorem.

$$A \ \textit{Artinian} \ \Leftrightarrow \ A \ \ \textit{Noetherian and} \ \dim A = 0$$

Remark. $k[x]$ is Noetherian but not Artinian.

1.7 Primary decomposition.

Definition. Let $\mathfrak{q} \in A$ be a proper ideal. We say that $\mathfrak{q}$ is **primary** if for all $a, b \in A$

$$ab \in \mathfrak{q}, \ \ a \notin \mathfrak{q} \ \implies b^n \in \mathfrak{q} \ \text{ for some } \ n >> 0$$

Remark. We have

- $\mathfrak{p} \in \mathrm{Spec}A$ prime ideal $\Rightarrow$ $\mathfrak{p}$ primary ideal
- Let $f : A \longrightarrow B$ be a ring homomorphism. Then,

$$\mathfrak{q} \subseteq B \ \text{ primary ideal} \Rightarrow \ \mathfrak{q}^c \subseteq A \ \text{ primary ideal}$$

- Let $\pi : A \longrightarrow A/I$ for a given ideal $I \subseteq A$. There is a bijection

$$\begin{array}{ccc} \left\{ \text{Primary ideals } \ \mathfrak{q} \subseteq A \ \text{ s.t. } \ I \subseteq \mathfrak{q} \right\} & \leftrightarrow & \left\{ \text{Primary ideals of } A/I \right\} \\ \mathfrak{q} & \longrightarrow & \mathfrak{q}/I \end{array}$$

Proposition. $\mathfrak{q}$ primary ideal $\Rightarrow$ $\operatorname{rad}(\mathfrak{q})$ prime ideal.

Definition. We say that $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\operatorname{rad}(\mathfrak{q}) = \mathfrak{p}$.

Remark. $\operatorname{rad}(\mathfrak{q})$ prime ideal $\not\Rightarrow$ $\mathfrak{q}$ primary ideal.

Proposition. Let $\mathfrak{q} \subseteq A$ be an ideal s.t. $\operatorname{rad}(\mathfrak{q}) = \mathfrak{m}$ is a maximal ideal. Then $\mathfrak{q}$ is primary.

Definition. Let $I \subseteq A$ be an ideal.

- A **primary decomposition** of I is

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

  with $\mathfrak{q}_i$ primary.

- It is **minimal** or reduced if

  i) $\operatorname{rad}(\mathfrak{q}_1) = \mathfrak{p}_1, \cdots, \operatorname{rad}(\mathfrak{q}_n) = \mathfrak{p}_n$ are different.

  ii) $\bigcap_{i \neq j} \mathfrak{q}_i \not\subset \mathfrak{q}_j$ for all $j = 1, \ldots, n$.

Remark. Given a primary decomposition we may always find a minimal primary decomposition.

Definition. Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a minimal primary decomposition of an ideal I with $\operatorname{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$. The **associated primes** of I are

$$\operatorname{Ass}(A/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

1.7.1 Unicity of primary decomposition.

Theorem. *Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a minimal primary decomposition of an ideal I with $\operatorname{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$. Then*

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \{\operatorname{rad}(I : a) \mid a \in A, \;\; \operatorname{rad}(I : a) \text{ prime}\}$$

In particular, they only depend of I.

Lemma. *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and $a \in A$. Then:*

- $a \in \mathfrak{q} \Rightarrow (\mathfrak{q} : a) = A$.
- $a \notin \mathfrak{q} \Rightarrow (\mathfrak{q} : a)$ *$\mathfrak{p}$-primary.*
- $a \notin \mathfrak{p} \Rightarrow (\mathfrak{q} : a) = \mathfrak{q}$.

Theorem. *Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a minimal primary decomposition of an ideal I with $\operatorname{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$. Then, the minimal primary components $\{\mathfrak{q}_i \mid \mathfrak{p}_i \text{ minimal prime}\}$ are uniquely determined by I.*

Proposition. Let $S \subseteq A$ be a multiplicatively closed set and $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal. Then:

- $\mathfrak{p} \cap S \neq \emptyset \Rightarrow S^{-1}\mathfrak{q} = S^{-1}A$.
- $\mathfrak{p} \cap S = \emptyset \Rightarrow S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary and $S^{-1}\mathfrak{q} \cap A = \mathfrak{q}$.

Proposition. Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m \cap \mathfrak{q}_{m+1} \cap \cdots \cap \mathfrak{q}_n$ be a minimal primary decomposition of an ideal I with $\operatorname{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$. Assume

- $\mathfrak{p}_i \cap S = \emptyset$ for $i = 1, \ldots, m$
- $\mathfrak{p}_i \cap S \neq \emptyset$ for $i = m+1, \ldots, n$.

Then:

- $S^{-1}I = S^{-1}\mathfrak{q}_1 \cap \cdots \cap S^{-1}\mathfrak{q}_m$.
- $S^{-1}I \cap A = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$.

1.7.1 Existence of primary decomposition.

Definition. Let $I \in A$ be a proper ideal. We say that I is **irreducible** if for a decomposition $I = J_1 \cap J_2$ we have $I = J_1$ or $I = J_2$.

Proposition. Let A be a Noetherian ring. Then, any ideal $I \subseteq A$ admits a decomposition

$$I = I_1 \cap \cdots \cap I_n$$

with I_i irreducible.

Proposition. Let A be a Noetherian ring and $I \subseteq A$ an ideal. Then

$$I \text{ irreducible} \Rightarrow I \text{ primary}$$

Theorem. **(Emmy Noether)**
Let A be a Noetherian ring. Then, any ideal $I \subseteq A$ admits a minimal primary decomposition.

Section 2. Modules.

2.1 Modules.

Unless otherwise stated we will always assume that A is a commutative ring with unit 1.

Definition. We say that M is an A-module if

1) $(M,+)$ is an abelian group.

2) A acts linearly on M. We have:

$$\begin{array}{ccc} A \times M & \longrightarrow & M \\ (a,m) & \longmapsto & am \end{array}$$

   satisfying for all $a,b \in A$ and all $m,n \in M$

   - $a(m+n) = am + an$.
   - $(a+b)m = am + bm$.
   - $(ab)m = a(bm)$.
   - $1\,m = m$

Remark. If A is not commutative we have to distinguish between *left* modules with the action $(a,m) \longmapsto am$ and *right* modules with the action $(a,m) \longmapsto ma$.

Examples. 1) Let $I \subseteq A$ be an ideal. Then I is an A-module.

2) Let $I \subseteq A$ be an ideal. Then:

   - A/I is an A-module with the action $(a,\overline{b}) \longmapsto \overline{ab}$.
   - A/I is an A/I-module with the action $(\overline{a},\overline{b}) \longmapsto \overline{ab}$.

3) Let $A = \mathbb{K}$ be a field. M is a $\mathbb{K}$-module if and only if M is a $\mathbb{K}$-vector space.

4) Let $A = \mathbb{Z}$. M is a $\mathbb{Z}$-module if and only if M is an abelian group.

5) Let $f : A \to B$ be a ring homomorphism. Then:

   - B is an A-module with the action $(a,b) \longmapsto f(a)\,b$.
   - A B-module M is an A-module with the action $(a,m) \longmapsto f(a)\,m$.

Definition. Let M, N be A-modules. We say that $f : M \to N$ is an homomorphism of A-modules if

- $f(m + m') = f(m) + f(m')$ for all $m, m' \in M$.
- $f(am) = am$ for all $a \in A$ and all $m \in M$.

Definition. We consider the category of A-modules $\mathbf{Mod(A)}$ where the objects are A-modules and for each pair $M, N \in ob(\mathbf{Mod(A)})$ we define

$$\mathrm{Hom}_A(M, N) := \{f : M \to N \mid f \ \ A\text{−module homomorphism}\}.$$

Remark. $\mathrm{Hom}_A(M, N)$ is an A-module if we consider:

- $(f + g)(m) = f(m) + g(m)$ for all $m \in M$.
- $(af)(m) = af(m)$ for all $a \in A$ and all $m \in M$.

Examples. 1) We have an isomorphism $\mathrm{Hom}_A(A, N) \cong N$ sending $f : A \to N$ to $f(1)$.

2) Let $A = \mathbb{K}$ be a field and $M = \mathbb{K}^n$. Then $\mathrm{Hom}_A(M, M) \cong M_n(\mathbb{K})$.

Definition. Let M be an A-module.

1) *Submodule:* $N \subseteq M$ is a submodule if
   - $n, n' \in N \Rightarrow n + n' \in N$.
   - $a \in A, \ \ n \in N \Rightarrow an \in N$.

2) *Quotient:* M/N is an A-module with the action $(a, \overline{m}) \longmapsto \overline{am}$.

Lemma. *Let $f : M \to N$ be an homomorphism of A-modules. Then:*

- *$P \subseteq M$ submodule $\Rightarrow f(P) \subseteq N$ is a submodule.*
- *$Q \subseteq N$ submodule $\Rightarrow f^{-1}(Q) \subseteq M$ is a submodule.*

In particular $\ker f = f^{-1}(0) \subseteq M$ *and* $\mathrm{Im} f = f(M) \subseteq N$ *are submodules.*

Lemma. *Let $f : M \to N$ be an homomorphism of A-modules. Then:*

- *f is a monomorphism if and only if* $\ker f = 0$.
- *f is an epimorphism if and only if* $\mathrm{Im} f = N$.

Definition. We say that $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is a short exact sequence of A-modules if f is a monomorphism, g is an epimorphism and $\ker g = \mathrm{Im} f$.

2.2 Some operations on modules.

Let M be an A-module and $\{M_i\}_{i\in\Lambda}$ a family of submodules of M.

- **Intersection:** $\bigcap_{i\in\Lambda} M_i \subseteq M$ is a submodule.
- **Sum:** $\sum_{i\in\Lambda} M_i := \{m_{i_1} + \cdots + m_{i_n} \mid m_{i_j} \in M_{i_j}\} \subseteq M$ is a submodule.

Proposition. We have

i) $M_1+(M_2\cap M_3) \subseteq (M_1+M_2)\cap(M_1+M_3)$. Equality holds if $M_1 \subseteq M_2$ or $M_1 \subseteq M_3$.

ii) $M_1\cap(M_2+M_3) \supseteq (M_1\cap M_2)+(M_1\cap M_3)$. Equality holds if $M_1 \supseteq M_2$ or $M_1 \supseteq M_3$.

Proposition. i) Let $N_2 \subseteq N_1 \subseteq M$ be submodules. Then

$$\frac{M/N_2}{N_1/N_2} \cong \frac{M}{N_1}.$$

ii) Let $N_1, N_2 \subseteq M$ be submodules. Then

$$\frac{N_1+N_2}{N_2} \cong \frac{N_1}{N_1\cap N_2}.$$

- **Product by an ideal:** Let $I \subseteq A$ be an ideal. We define

$$IM := \{a_1m_1 + \cdots + a_nm_n \mid a_i \in I,\ \ m_i \in M_i\} \subseteq M$$

- **Colon ideal:** Let $N_1, N_2 \subseteq M$ be submodules. Then

$$(N_1 : N_2) := \{a \in A \mid aN_2 \subseteq N_1\} \subseteq A$$

  is an ideal. A particular case is the annihilator of a module

$$\mathrm{Ann}_A(M) = (0 : M) = \{a \in A \mid aM = 0\}.$$

- **Radical:** Let $N \subseteq M$ be a submodule. Then

$$\mathrm{rad}_M(N) := \{a \in A \mid a^nM \subseteq N,\ \ n >> 0\} \subseteq A$$

  is an ideal.

The category of modules has products and coproducts.

- **Direct product:** It is the cartesian product $\prod_{i\in\Lambda} M_i$.
- **Direct sum:** It is the submodule $\bigoplus_{i\in\Lambda} M_i \subseteq \prod_{i\in\Lambda} M_i$ generated by elements $(m_i)_{i\in\Lambda}$ such that at most finitely many components m_i are non-zero.

Remark. If Λ is a finite set of indices, then $\bigoplus_{i\in\Lambda} M_i = \prod_{i\in\Lambda} M_i$.

2.3 Generators of modules.

Let M be an A-module and $S = \{m_i\}_{i\in\Lambda} \subset M$ a subset of elements of M. We define

$$\langle S\rangle = \{a_{i_1}m_{i_1} + \cdots + a_{i_n}m_{i_n} \mid a_{i_j} \in A, \;\; s_{i_j} \in S\}.$$

Definition. Let M be an A-module.

1) We say that S is a system of generators of M if $M = \langle S\rangle$.

2) We say that $\{m_i\}_{i\in\Lambda}$ are linearly independent if for every subset $\{m_1, \dots, m_n\}$ we have that $a_1m_1 + \cdots + a_nm_n = 0_M$ implies $a_1 = \cdots = a_n = 0_A$.

3) A basis of M is a system of generators that is linearly independent.

Remark. Consider the morphism

$$\begin{array}{rcl} \varphi : A^{(\Lambda)} & \longrightarrow & M \\ (a_i)_{i\in\Lambda} & \longmapsto & \sum_{i\in\Lambda} a_im_i \end{array}$$

Then

- φ epimorphism $\Leftrightarrow$ $\{m_i\}_{i\in\Lambda}$ are generators.
- φ monomorphism $\Leftrightarrow$ $\{m_i\}_{i\in\Lambda}$ are linearly independent.

Definition. We say that an A-module M is free if $M \cong \bigoplus_{i\in\Lambda} M_i$ where we have A-module isomorphisms $M_i \cong A$. We define the rank of M as $\mathrm{rank}_A(M) = \#\Lambda$.

Remark. M is a free A-module if and only if there exists a basis $\{m_i\}_{i\in\Lambda}$ of M. Thus

$$M \cong \bigoplus_{i\in\Lambda} Am_i.$$

2.3.1 Finitely generated modules.

Definition. Let M be an A-module.

1) We say that $M = \langle S\rangle$ is finitely generated if $\#S < \infty$.

2) We say that $M = \langle S\rangle$ is cyclic if $\#S = 1$.

Lemma. *If M is a cyclic module then $M \cong A/I$, where $I \subseteq A$ is an ideal.*

Proposition. M is a finitely generated A-module if and only if M is a quotient of A^n for some $n >> 0$.

Remark. Let M be a finitely generated A-module and $N \subseteq M$ a submodule. Then N is not necessarily finitely generated.

2.4 Chain conditions.

Proposition. Let M be an A-module. The following are equivalent:

0) *Finite generation:* Every submodule of M is finitely generated.

1) *Ascending chain condition:* Every ascending chain $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq \cdots$ of submodules of M stabilizes. That is, there exists $m \geq 1$ such that $N_m = N_n$ for all $n \geq m$.

2) *Maximality:* Every non-empty set of submodules of M has a maximal element.

Definition. We say that M is Noetherian if it satisfies the conditions of the proposition.

Proposition. Let M be an A-module. The following are equivalent:

1) *Descending chain condition:* Every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_i \supseteq \cdots$ of submodules of M stabilizes. That is, there exists $m \geq 1$ such that $N_m = N_n$ for all $n \geq m$.

2) *Minimality:* Every non-empty set of submodules of M has a minimal element.

Definition. We say that M is Artinian if it satisfies the conditions of the proposition.

Examples. 1) $\mathbb{Z}$ is a Noetherian $\mathbb{Z}$-module but is not Artinian.

2) A finite dimensional $\mathbb{K}$-vector space is a Noetherian and Artinian $\mathbb{K}$-module.

3) $\mathbb{Q}$ is neither a Noetherian nor an Artinian $\mathbb{Z}$.

Proposition. Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules. Then

1) M is a Noetherian if and only if M', M'' are Noetherian.

2) M is a Artinian if and only if M', M'' are Artinian

In particular, every submodule of a Noetherian (resp. Artinian) module is Noetherian (resp. Artinian) and every quotient of a Noetherian (resp. Artinian) module is Noetherian (resp. Artinian).

Corollary. *Let $M_1, \ldots, M_n$ be a finite set of A-modules. Then $\bigoplus_{i=1}^{n} M_i$ is Noetherian (resp. Artinian) if and only if M_i is Noetherian (resp. Artinian) for all i.*

Corollary. *Let A be a Noetherian (resp. Artinian) and M a finitely generated A-module. Then M is a Noetherian (resp. Artinian) A-module.*

Remark. If M is a Noetherian A-module then it is finitely generated, but the converse is not true.

2.5 Sequences of modules.

Consider a sequence

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

where M_i is an A-module for all $i \in \mathbb{Z}$ and $f_i : M_i \to M_{i+1}$ is an homomorphism of A-modules.

Definition. 1) It is a *complex* of A-modules if $f_{i+1} \circ f_i = 0\ \forall i$. That is $\mathrm{Im} f_i \subseteq \ker f_{i+1}$.

2) It is an *exact sequence* of A-modules if $\mathrm{Im} f_i = \ker f_{i+1}$ for all i.

Remark. Every long exact sequence splits into short exact sequences

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
& \begin{array}{c}\ker f_{i-1}\\ \| \\ \mathrm{Im} f_{i-2}\end{array} \arrow[ur] \arrow[dr] & & & & \begin{array}{c}\ker f_{i+1}\\ \| \\ \mathrm{Im} f_{i}\end{array} \arrow[ur] \arrow[dr] & & & \\
\cdots \arrow[ur] \arrow[rr] & & M_{i-1} \arrow[rr, "f_{i-1}"] \arrow[dr] & & M_i \arrow[rr, "f_i"] \arrow[ur] & & M_{i+1} \arrow[rr, "f_{i+1}"] \arrow[dr] & & \cdots \\
& & & \begin{array}{c}\ker f_{i}\\ \| \\ \mathrm{Im} f_{i-1}\end{array} \arrow[ur] \arrow[dr] & & & & \begin{array}{c}\ker f_{i+2}\\ \| \\ \mathrm{Im} f_{i+1}\end{array} \arrow[ur] \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Example. The *trivial* short exact sequence is $0 \longrightarrow M' \longrightarrow M' \oplus M'' \longrightarrow M'' \longrightarrow 0$ sending $m' \in M'$ to $(m', 0)$ and $(m', m'') \in M' \oplus M''$ to m''.

Remark. Given a short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ we say that M is an extension of M' and M''. In general we have non trivial extensions.

Definition. We say that two short exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M_1 \longrightarrow M'' \longrightarrow 0\,, \quad 0 \longrightarrow M' \longrightarrow M_2 \longrightarrow M'' \longrightarrow 0$$

are equivalent if there exists a homomorphism $\varphi : M_1 \to M_2$ such that we have a commutative diagram

$$\begin{tikzcd} & & M_1 \arrow[dd, "\varphi"] \arrow[dr] & & \\ 0 \arrow[r] & M' \arrow[ur] \arrow[dr] & & M'' \arrow[r] & 0 \\ & & M_2 \arrow[ur] & & \end{tikzcd}$$

Lemma. *If it exists, φ is an isomorphism.*

Definition. Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules. We say that it *splits* if it is equivalent to the trivial sequence.

Proposition. Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules. The following are equivalent:

1) The sequence splits.
2) $\exists\ r : M \to M'$ such that $r \circ f = Id_{M'}$. We say that r is a retraction of f.
3) $\exists\ s : M'' \to M$ such that $g \circ s = Id_{M''}$. We say that s is a section of g.

2.5.1 Exact functors.

Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules.

Definition. Let $F : \mathbf{Mod(A)} \to \mathbf{Mod(A)}$ be a covariant functor. Then

1) F is *left exact* if we have a short exact sequence

$$0 \longrightarrow F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'') \longrightarrow$$

2) F is *right exact* if we have a short exact sequence

$$\longrightarrow F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'') \longrightarrow 0$$

Definition. Let $F : \mathbf{Mod(A)} \to \mathbf{Mod(A)}$ be a contravariant functor. Then

1) F is *left exact* if we have a short exact sequence

$$0 \longrightarrow F(M'') \xrightarrow{F(g)} F(M) \xrightarrow{F(f)} F(M') \longrightarrow$$

2) F is *right exact* if we have a short exact sequence

$$\longrightarrow F(M'') \xrightarrow{F(g)} F(M) \xrightarrow{F(f)} F(M') \longrightarrow 0$$

2.6 The Hom functor.

Given M, N A-modules, we have that $\mathrm{Hom}_A(M,N)$ is an A-module as well. Therefore we can define two different functors $\mathrm{Hom}_A(M,-)$ and $\mathrm{Hom}_A(-,N)$ in the category $\mathbf{Mod(A)}$.

Proposition. In the category $\mathbf{Mod(A)}$ we have:

1) The functor $\mathrm{Hom}_A(M,-)$ is covariant and left exact.

2) The functor $\mathrm{Hom}_A(-,N)$ is contravariant and left exact.

Definition. Let P, E be A-modules

1) We say that P is *projective* if the functor $\mathrm{Hom}_A(P,-)$ is exact.

2) We say that E is *injective* if the functor $\mathrm{Hom}_A(-,E)$ is exact.

Remark. Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules. We have:

1) P is *projective* if and only if $\forall \varphi \in \mathrm{Hom}_A(P, M'')$ there exists $\psi \in \mathrm{Hom}_A(P,M)$ such that $\varphi = g \circ \psi$.

$$\begin{tikzcd} & P \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] & \\ M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

2) E is *injective* if and only if $\forall \varphi \in \mathrm{Hom}_A(M', E)$ there exists $\psi \in \mathrm{Hom}_A(M,E)$ such that $\varphi = \psi \circ f$.

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M' \arrow[u, "\varphi"] \arrow[r, "f"] & M \arrow[ul, dotted, "\psi"'] \end{tikzcd}$$

Proposition. Let M be a free A-module. Then M is projective.

Proposition. The following are equivalent:

1) P is projective.

2) Every short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow P \longrightarrow 0$ splits.

3) P is a direct summand of a free module.

Remark. A projective module is not necessarily a free module.

2.7 The tensor product functor.

Definition. The tensor product of two A-modules M, N is the pair $(M \otimes_A N, g)$ where $M \otimes_A N$ is an A-module and $g : M \times N \to M \otimes_A N$ is an A-bilineal homomorphism satisfying the following universal property:

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[d, "f"'] & M \otimes_A N \arrow[dl, dotted, "\varphi"] \\ P & \end{tikzcd}$$

For all (P, f) with P an A-module and $f : M \times N \to P$ A-bilineal homomorphism there exists a unique homomorphism of A-modules $\varphi : M \otimes_A N \to P$ such that $f = \varphi \circ g$.

Theorem. *Tensor product exists and is unique. That is, if (P, f) satisfies the same universal property as $(M \otimes_A N, g)$ then there exist a unique isomorphism $\psi : M \otimes_A N \to P$ such that*

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[d, "f"'] & M \otimes_A N \arrow[dl, dotted, "\psi"] \\ P & \end{tikzcd}$$

Corollary. *$M \otimes_A N$ is the A-module generated by the elements $\{m \otimes n \mid m \in M, \ n \in N\}$ satisfying the properties*

- $(m + m') \otimes n = m \otimes n + m' \otimes n$.
- $m \otimes (n + n') = m \otimes n + m \otimes n'$.
- $am \otimes n = m \otimes an = a(m \otimes n)$.

for all $m, m' \in M$, $n, n' \in N$ and $a \in A$.

Proposition. Let $\{m_i\}_{i \in \Lambda}$ and $\{n_j\}_{j \in \Delta}$ be sets of generators of M and N respectively. Then $\{m_i \otimes n_j\}_{(i,j) \in \Lambda \times \Delta}$ is a set of generators of $M \otimes_A N$.

Proposition. Let $f : M \to N$ and $g : M' \to N'$ be homomorphism of A-modules. Then there exists an homomorphism of A-modules $f \otimes g : M \otimes_A N \to M' \otimes_A N'$ sending $m \otimes n \to f(m) \otimes g(n)$.

Theorem. *There are natural isomorphisms:*

1) $M \otimes_A N \cong N \otimes_A M$.

2) $M \otimes_A N \otimes_A P \cong (M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$.

3) $M \otimes_A \left(\bigoplus_{i\in\Lambda} N_i\right) \cong \bigoplus_{i\in\Lambda}(M \otimes_A N_i)$.

Theorem. *There are natural isomorphisms:*

1) $A \otimes_A M \cong M$.

2) $A/I \otimes_A M \cong M/IM$, *where* $I \subseteq A$ *is an ideal.*

3) $A/I \otimes_A A/J \cong A/I + J$, *where* $I, J \subseteq A$ *are ideals.*

Proposition. In the category $\mathbf{Mod(A)}$ we have that $- \otimes_A N$ is a covariant right exact functor.

Definition. We say that N is a *flat* A-module if the functor $- \otimes_A N$ is exact.

Proposition. We have that free and projective A-modules are flat.

2.8 The localization functor.

Let $S \subseteq A$ be a multiplicatively closed set. That is $1 \in S$ and $s, t \in S \Rightarrow st \in S$. We define

$$S^{-1}A = \left\{ \frac{a}{s} \mid a \in A,\ \ s \in S \right\}\Big/\sim$$

where $\frac{a}{s} \sim \frac{a'}{s'}$ if and only if $\exists t \in S$ such that $t(as' - a's) = 0$. We have that $\sim$ is an equivalence relation so it satisfies

· *Reflexive:* $\frac{a}{s} \sim \frac{a}{s}$.

· *Symmetric:* $\frac{a}{s} \sim \frac{a'}{s'}$ if and only if $\frac{a'}{s'} \sim \frac{a}{s}$.

· *Transitive:* If $\frac{a}{s} \sim \frac{a'}{s'}$ and $\frac{a'}{s'} \sim \frac{a''}{s''}$ then $\frac{a}{s} \sim \frac{a''}{s''}$.

If we define the sum and the product in $S^{-1}A$ as $\frac{a}{s} + \frac{a'}{s'} = \frac{as'+a's}{ss'}$ and $\frac{a}{s}\frac{a'}{s'} = \frac{aa'}{ss'}$ we have that $S^{-1}A$ is a commutative ring with unit. In particular we have a ring homomorphism $\varphi : A \to S^{-1}A$ sending $a \to \frac{a}{1}$ which is injective if and only if S has no zero divisors.

Examples. The main examples we are going to consider:

1) $A_f := S^{-1}A$ where $S = \{f^n \mid n \geq 0\}$ for a given $f \in A$.

2) $A_{\mathfrak{p}} := S^{-1}A$ where $S = A \setminus \mathfrak{p}$ for a given prime ideal $\mathfrak{p} \subseteq A$.

3) $\mathrm{Tot}(A) := S^{-1}A$ where $S = A \setminus \{0\}$ when A is a domain.

More generally we are going to consider the localization of A-modules

Definition. Let $S \subseteq A$ be a multiplicatively closed set and M an A-module. We define

$$S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, \quad s \in S \right\} \Big/ \sim$$

where $\frac{m}{s} \sim \frac{m'}{s'}$ if and only if $\exists t \in S$ such that $t(ms' - m's) = 0$.

Proposition. We have that $S^{-1}M$ is an A-module but we also have that $S^{-1}M$ is an $S^{-1}A$-module.

Proposition. We have that

$$\begin{array}{rcl} \varphi : S^{-1}A \otimes_A M & \longrightarrow & S^{-1}M \\ \frac{a}{s} \otimes m & \longmapsto & \frac{am}{s} \end{array}$$

is a natural isomorphism of A-modules.

Proposition. In the category $\mathbf{Mod(A)}$ we have that $S^{-1}- = S^{-1}A \otimes_A -$ is a covariant exact functor. In particular $S^{-1}A$ is a flat A-module.

2.9 Algebras.

Definition. Let $f : A \to B$ be a ring homomorphism.

1) *Restriction of scalars:* Let N be a B-module. We have that N is also an A-module with the product $an := f(a)n$.

2) *Extension of scalars:* Let M be an A-module. We have that $B \otimes_A M$ has a structure as B-module with the product $b(b' \otimes m) := bb' \otimes m$.

Definition. Let A be a commutative ring with unit and let B be a ring. We say that B is an A-algebra if there exists a ring homomorphism $f : A \to B$.

Remark. B is an A-algebra if and only if B is a ring that has an A-module structure. Notice that we can define $f : A \to B$ that sends the unit $1 \to b$ for a given $b \in B$ and then $a \to ab$.

Examples. 1) A is an A-algebra.

2) Every ring is a $\mathbb{Z}$-algebra.

3) $M_n(A)$ is a non-commutative A-algebra.

4) A $\mathbb{K}$-algebra is a ring containing a field $\mathbb{K}$.

Definition. We say that $\varphi : B \to C$ is a homomorphism of A-algebras if it is a homomorphism of rings that makes the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & B \arrow[d, "\varphi"] \\ & C \end{tikzcd}$$

Remark. $\varphi : B \to C$ is a homomorphism of A-algebras if it is a homomorphism of rings that is also a homomorphism of A-modules.

Definition. We say that the ring homomorphism $f : A \to B$ is finite and B is a finite A-algebra if B is finitely generated as A-module.

Definition. We say that the ring homomorphism $f : A \to B$ is of finite type and B is a finitely generated A-algebra there exists $x_1, \ldots, x_n \in B$ such that every $b \in B$ is a polynomial in $x_1, \ldots, x_n$ with coefficients in $f(A)$.

Example. $A[x]$ is a finitely generated A-algebra but is not finitely generated as A-module.

We can consider the category A-**Alg** of A-algebras. The coproduct in this category is the tensor product of A-algebras.

Proposition. Let B and C be A-algebras. Then the A-module $B \otimes_A C$ has a structure of A-algebra.

Section 3. Algebraic Varieties.

3.1 Algebraic varieties over a field $\mathbb{K}$.

Definition. Let $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ be a set of polynomials with coefficients over a field $\mathbb{K}$. The (affine) algebraic variety associated to these polynomials is

$$V = V(f_1, \ldots, f_m) := \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid f_i(x_1, \ldots, x_n) = 0 \;\; i = 1, \ldots, m\}.$$

Remark. 1) V only depends on the ideal $J = (f_1, \ldots, f_m)$. That is

$$V = V(J) := \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid f(x_1, \ldots, x_n) = 0 \;\; \forall f \in J\}.$$

2) The polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian by the Hilbert basis theorem. Therefore, any ideal J is finitely generated so there exist $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ such that $J = (f_1, \ldots, f_m)$.

3) We have a map of sets

$$\begin{array}{ccc} \left\{ \text{Ideals of } \mathbb{K}[x_1, \ldots, x_n] \right\} & \longrightarrow & \left\{ \text{(affine) Algebraic Varieties of } \mathbb{K}^n \right\} \\ J & \longrightarrow & V(J) \end{array}$$

Proposition. Algebraic varieties satisfy the following properties:

1) $J_1 \subseteq J_2 \Rightarrow V(J_1) \supseteq V(J_2)$.

2) $V(0) = \mathbb{K}^n$.

3) $V((1)) = \emptyset$.

4) $V(J_1 \cap J_2) = V(J_1 J_2) = V(J_1) \cup V(J_2)$.

5) $V(\sum_{\ell \in \Lambda} J_\ell) = \bigcap_{\ell \in \Lambda} V(J_\ell)$.

6) $V(J) = V(\operatorname{rad}(J))$.

Definition. Let $S \subseteq \mathbb{K}^n$ be a set. The ideal associated to S is

$$I(S) := \{f \in \mathbb{K}[x_1, \ldots, x_n] \mid f(x_1, \ldots, x_n) = 0 \;\; \forall (x_1, \ldots, x_n) \in S\}.$$

Example. Let $S = \{p\}$ be a set consisting of a single point $p = (p_1, \ldots, p_n) \in \mathbb{K}^n$. Then

$$I(S) = (x_1 - p_1, \ldots, x_n - p_n).$$

This is a maximal ideal of $K[x_1, \ldots, x_n]$ that we will denote as $\mathfrak{m}_p = (x_1 - p_1, \ldots, x_n - p_n)$.

Proposition. We have the following properties:

1) $S_1 \subseteq S_2 \Rightarrow I(S_1) \supseteq I(S_2)$.
2) $I(\emptyset) = \mathbb{K}[x_1, \ldots, x_n]$.
3) $I(\mathbb{K}^n) = 0$ if $\mathbb{K}$ is an infinite field.
4) $I(\bigcup_{\ell \in \Lambda} S_\ell) = \bigcap_{\ell \in \Lambda} I(S_\ell)$.
5) $I(S_1 + S_2) \subseteq I(S_1 \cap S_2)$.
6) $I(S)$ is a radical ideal.

Remark. We have maps

$$\begin{array}{ccc} \left\{ \text{ Ideals of } \mathbb{K}[x_1, \ldots, x_n] \right\} & \underset{I}{\overset{V}{\leftrightarrows}} & \left\{ \text{Subsets of } \mathbb{K}^n \right\} \\ J & \longrightarrow & V(J) \\ I(S) & \longleftarrow & S \end{array}$$

Proposition. We have:

1) $S \subseteq V(I(S))$.
2) $J \subseteq I(V(J))$.

In particular, $I(S) = I(V(I(S)))$ and $V(J) = V(I(V(J)))$.

Remark. We want to restrict to radical ideals on the left hand side and to algebraic sets on the right hand side.

$$\begin{array}{ccc} \left\{ \text{ Radical Ideals of } \mathbb{K}[x_1, \ldots, x_n] \right\} & \underset{I}{\overset{V}{\leftrightarrows}} & \left\{ \text{(affine) Algebraic Varieties of } \mathbb{K}^n \right\} \\ I & \longrightarrow & V(I) \\ I(W) & \longleftarrow & W \end{array}$$

We point out that, in general, we do not have a bijection between both sets.

Theorem. *(Weak Nullstellensatz) Let $\mathbb{K}$ be a field and A a finitely generated $\mathbb{K}$-algebra. If A is a field, then $\mathbb{K} \subseteq A$ is a finite (algebraic) extension of fields.*

Corollary. *Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field. Let $\mathfrak{m} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a maximal ideal. Then, there exists a point $p \in \mathbb{K}^n$ such that $\mathfrak{m} = \mathfrak{m}_p$.*

Corollary. *Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field. Let $J \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be a proper ideal. Then $V(J) \neq \emptyset$.*

Theorem. **(Hilbert Nullstellensatz)** *Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field. Let $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. Then*

$$I(V(J)) = \mathrm{rad}(J).$$

Corollary. *Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field. Let $J \subsetneq \mathbb{K}[x_1, \ldots, x_n]$ be a proper ideal. Then*

$$I(V(J)) = \mathrm{rad}(J) = \bigcap_{\mathfrak{m} \supseteq J} \mathfrak{m}.$$

Definition. We define the **Zariski closure** of a subset $S \subseteq \mathbb{K}^n$ as

$$\overline{S} = VI(S)$$

Remark. $\overline{S}$ is the smallest algebraic variety containing S.

Definition. Let $V \subseteq \mathbb{K}^n$ be an algebraic variety. We say that W is irreducible if whenever we have a decomposition $W = W_1 \cup W_2$, with W_1, W_2 algebraic varieties, then either $W = W_1$ or $W = W_2$.

Proposition. Let $W \subseteq \mathbb{K}^n$ be an algebraic variety. Then W is irreducible if and only if $I(W)$ is a prime ideal.

2.2 Finitely generated $\mathbb{K}$-algebras and algebraic varieties.

Definition. Given $f \in \mathbb{K}[x_1, \ldots, x_n]$ we consider the polynomial function

$$\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{f} & \mathbb{K} \\ (x_1, \ldots, x_n) & \longrightarrow & f(x_1, \ldots, x_n) \end{array}$$

Let $W \subseteq \mathbb{K}^n$ be an algebraic variety. A *regular function* on W is the restriction to W of a polynomial function.

$$\begin{array}{ccc} W & \xrightarrow{f_{|W}} & \mathbb{K} \\ (x_1, \ldots, x_n) & \longrightarrow & f(x_1, \ldots, x_n) \end{array}$$

By abuse of notation we will say $f : W \longrightarrow \mathbb{K}$ is a regular function.

Definition. Let $W \subseteq \mathbb{K}^n$ be an algebraic variety. We define

$$A(W) = \{f : W \longrightarrow \mathbb{K} \mid f \text{ regular function}\}.$$

Proposition. $A(W)$ is a reduced finitely generated $\mathbb{K}$-algebra.

Let $\mathbb{K}$-**Alg** be the category of $\mathbb{K}$-algebras. We may consider the full subcategory of reduced finitely generated $\mathbb{K}$-algebras, that we will denote $\mathbb{K}\text{-}\mathbf{Alg}_{\text{red,fg}}$, which has:

· *Objects:* A reduced finitely generated $\mathbb{K}$-algebra.

· *Morphisms:* $f : A \longrightarrow B$ morphism of $\mathbb{K}$-algebras for all $A, B \in ob(\mathbb{K}\text{-}\mathbf{Alg}_{\text{red,fg}})$

On the other hand we want to consider the category $\mathbf{Var}_{\mathbb{K}}$ of algebraic varieties over a field $\mathbb{K}$, which has:

· *Objects:* $W \subseteq \mathbb{K}^n$ algebraic variety (n arbitrary).

· *Morphisms:* Let $W \subseteq \mathbb{K}^n$ and $W' \subseteq \mathbb{K}^m$ be algebraic varieties. A morphism of algebraic varieties $\varphi : W \longrightarrow W'$ is the restriction

$$\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{\varphi} & \mathbb{K}^m \\ \cup & & \cup \\ W & \xrightarrow{\varphi_{|W}} & W' \end{array}$$

of a polynomial function

$$\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{\varphi} & \mathbb{K}^m \\ (x_1, \ldots, x_n) & \longrightarrow & \big(f_1(x_1, \ldots, x_n), \ldots, f_1(x_1, \ldots, x_n)\big) \end{array}$$

Notice that we require $\operatorname{Im}(\varphi_{|W}) \subseteq W'$ and that $f_{i|W} \in A(W)$ are regular functions.

Definition. Let $\varphi : W \longrightarrow W'$ be a morphism of algebraic varieties. We say that φ is an isomorphism if there exists a morphism of algebraic varieties $\psi : W' \longrightarrow W$ such that $\varphi \circ \psi = \mathrm{Id}_W$ and $\psi \circ \varphi = \mathrm{Id}_{W'}$.

Definition. A bijective morphism $\varphi : W \longrightarrow W'$ is not necessarily an isomorphism.

Proposition. There exists a contravariant functor $F : \mathbf{Var}_{\mathbb{K}} \longrightarrow \mathbb{K}\text{-}\mathbf{Alg}_{\text{red,fg}}$ sending W to the algebra of regular functions $A(W)$.

Remark. Taking the opposite category, we get the covariant functor of regular functions $F : \mathbf{Var}_{\mathbb{K}} \longrightarrow \big(\mathbb{K}\text{-}\mathbf{Alg}_{\text{red,fg}}\big)^{\circ}$

Theorem. *Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field. Then, the functor of regular functions*

$$F : \mathbf{Var}_{\mathbb{K}} \longrightarrow \big(\mathbb{K}\text{-}\mathbf{Alg}_{\mathit{red,fg}}\big)^{\circ}$$

establishes an equivalence of categories.

Corollary. *Let $\mathbb{K}$ be any field and $\varphi : W \longrightarrow W'$ a morphism of algebraic varieties. Then $F(\varphi)$ is an isomorphism if and only if $F(\varphi) : A(W') \longrightarrow A(W)$ is an isomorphism.*

Definition. Given $W_1 \subseteq \mathbb{K}^n$ and $W_2 \subseteq \mathbb{K}^m$ algebraic varieties, we have that the cartesian product $W_1 \times W_2$ is an algebraic variety.

Proposition. We have $A(W_1 \times W_2) = A(W_1) \otimes_{\mathbb{K}} A(W_2)$

Corollary. *Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field. The tensor product of reduced finitely generated $\mathbb{K}$-algebras is reduced.*

2.3 Zariski topology.

2.3.1 Zariski topology in $\mathbb{K}^n$.

Definition. $\mathbb{K}^n$ is a topological space with the Zariski topology defined by:

- *Closed Sets:* $\Big\{V(I) \mid I \subseteq \mathbb{K}[x_1, \dots, x_n] \text{ ideal}\Big\}$.
- *Open Sets:* $\Big\{\mathbb{K}^n \setminus V(I) \mid I \subseteq \mathbb{K}[x_1, \dots, x_n] \text{ ideal}\Big\}$.

Remark. $\mathbb{K}^n$ and $\emptyset$ are open and closed sets at the same time.

Proposition. A basis of open sets for the Zariski topology is given by

$$\{D(f) \mid f \in \mathbb{K}[x_1, \dots, x_n]\}$$

where, for a given f, we define

$$D(f) = \mathbb{K}^n \setminus V(f) = \{(a_1, \dots, a_n) \in \mathbb{K}^n \mid f(a_1, \dots, a_n) \neq 0]\}.$$

Proposition. Let $f : \mathbb{K}^n \longrightarrow \mathbb{K}$ be a polynomial function. Then f is continuous with the Zariski topology.

Corollary. *Let $\varphi : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ be a polynomial function. Then φ is continuous with the Zariski topology.*

Proposition. Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field (or just an infinite field). Let U and U' be two non-empty Zariski open sets. Then

$$U \cap U' \neq \emptyset.$$

Remark. The Zariski topology is not Hausdorff (T2). It is Fréchet (T1).

Remark. If $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the Zariski topology is coarser than the Euclidean topology.

2.3.2 Zariski topology in an algebraic variety W.

Let $W \subseteq \mathbb{K}^n$ be an algebraic variety. Let $J \subseteq A(W)$ be an ideal and let $Z \subseteq W$ be an algebraic subvariety. Then we may define

$$V_W(J) := \{(x_1, \dots, x_n) \in \mathbb{K}^n \mid f(x_1, \dots, x_n) = 0 \ \forall f \in J\}.$$

$$I_W(Z) := \{f \in A(W) \mid f(x_1, \dots, x_n) = 0 \ \forall (x_1, \dots, x_n) \in Z\}.$$

We have maps

$$\begin{array}{ccc} \Big\{ \text{Radical Ideals of } A(W) \Big\} & \overset{V_W}{\underset{I_W}{\leftrightarrows}} & \Big\{ \text{Algebraic Subvarieties of } W \Big\} \\ J & \longrightarrow & V_W(J) \\ I_W(Z) & \longleftarrow & Z \end{array}$$

Let $\mathbb{K} = \overline{\mathbb{K}}$ be an algebraically closed field. The Hilbert Nullstellensatz in this setting ensures that $I_W(V_W(J)) = \operatorname{rad}(J)$.

Definition. W is a topological space with the Zariski topology induced by the topology of $\mathbb{K}^n$.

- *Closed Sets:* $\Big\{V_W(J) \mid J \subseteq A(W) \text{ ideal}\Big\}$.
- *Open Sets:* $\Big\{A(W) \setminus V_W(J) \mid J \subseteq A(W) \text{ ideal}\Big\}$.

A basis of open sets is given by

$$D_W(f) = W \setminus V_W(f) = \{(a_1, \dots, a_n) \in W \mid f(a_1, \dots, a_n) \neq 0]\}.$$

Proposition. Let $f : W \longrightarrow \mathbb{K}$ be a regular function. Then f is continuous with the Zariski topology.

Corollary. *Let $\varphi : W \longrightarrow W'$ be a morphism of algebraic varieties. Then φ is continuous with the Zariski topology.*

2.4 Spectrum of a ring.

Definition. Let A be a commutative ring with unit. Given an ideal $J \subseteq A$ we define:

$$V(J) = \{\mathfrak{p} \in \operatorname{Spec} A \mid J \subseteq \mathfrak{p}\}.$$

Remark. Let $(a_1, \dots, a_r)$ be the ideal generated by a set of elements $S = \{a_1, \dots, a_r\} \subseteq A$. Then

$$V(S) := V((a_1, \dots, a_r))$$

Proposition. The following properties are satisfied:

1) $J_1 \subseteq J_2 \Rightarrow V(J_1) \supseteq V(J_2)$.

2) $V(0) = \mathbb{K}^n$.

3) $V((1)) = \emptyset$.

4) $V(J_1 \cap J_2) = V(J_1 J_2) = V(J_1) \cup V(J_2)$.

5) $V(\sum_{\ell \in \Lambda} J_\ell) = \bigcap_{\ell \in \Lambda} V(J_\ell)$.

6) $V(J) = V(\mathrm{rad}(J))$.

Proposition.

$$X = \mathrm{Spec} A = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ prime ideal}\}$$

is a topological space with the **Zariski topology** given by:

· *Closed Sets:* $\Big\{V(J) \mid J \subseteq A \text{ ideal}\Big\}$.

· *Open Sets:* $\Big\{X \setminus V(J) \mid J \subseteq A \text{ ideal}\Big\}$.

· *Basis of Open Sets:* $\Big\{D(f) = X \setminus V(f) \mid f \in A\Big\}$

Remark. The Zariski topology is not Hausdorff (T2). It is Fréchet (T1).

Definition. We define the **Zariski closure** of a subset $W \subseteq \mathrm{Spec} A$ as

$$\overline{W} = V(\bigcap_{\mathfrak{p} \in W} \mathfrak{p})$$

Proposition. $\{\mathfrak{p}\} \subseteq \mathrm{Spec} A$ is closed if and only if $\mathfrak{p}$ is maximal.

Definition. The non-closed points of $\mathrm{Spec} A$ are generic points.

2.4.1 A Hilbert Nullstellensatz type correspondence.

Definition. Let $W \subseteq X = \mathrm{Spec}A$ be a set. The ideal associated to W is

$$I(W) := \bigcap_{\mathfrak{p} \in W} \mathfrak{p}.$$

Proposition. We have the following properties:

1) $W_1 \subseteq W_2 \Rightarrow I(W_1) \supseteq I(W_2)$.
2) $I(\emptyset) = A$.
3) $I(X) = \bigcap_{\mathfrak{p} \in X} \mathfrak{p} = \eta(A)$ nilradical of A.
4) $I(\bigcup_{\ell \in \Lambda} W_\ell) = \bigcap_{\ell \in \Lambda} I(W_\ell)$.
5) $I(W_1 + W_2) \subseteq I(W_1 \cap W_2)$.
6) $I(W)$ is a radical ideal.

Proposition. We have the following properties:

1) $\overline{W} = V(I(W))$.
2) $I(V(J)) = \mathrm{rad}(J)$.
3) $V(J) = V(J') \;\Leftrightarrow\; \mathrm{rad}(J) = \mathrm{rad}(J')$.

Remark. We have a bijection

$$\begin{array}{ccc} \left\{ \text{Radical ideals of } A \right\} & \underset{I}{\overset{V}{\leftrightarrows}} & \left\{ \text{Subsets of } \mathrm{Spec}A \right\} \\ J & \longrightarrow & V(J) \\ I(W) & \longleftarrow & W \end{array}$$

We have a contravariant functor:

$$\begin{array}{ccl} \mathrm{Spec} : \mathbf{Rings} & \longrightarrow & \mathbf{Top} \\ A & \longrightarrow & \mathrm{Spec}A \\ f : A \to B & \longrightarrow & f^* : \mathrm{Spec}B \to \mathrm{Spec}A \\ & & \qquad\qquad \mathfrak{p} \to f^*(\mathfrak{p}) = \mathfrak{p}^c \end{array}$$

Proposition. Let $f : A \to B$ a ring homomorphism and $f^* : Y = \mathrm{Spec}B \to X = \mathrm{Spec}A$. Then:

1) $(f^*)^{-1}(D_X(g)) = D_Y(f(g))$ for any $g \in A$.

2) $(f^*)^{-1}(V(I)) = V(I^e)$ for any ideal $I \subseteq A$

3) $\overline{f^*(V(J))} = V(J^c)$ for any ideal $J \subseteq B$.

Corollary. *f^* is continuous w.r.t. the Zariski topolology.*

Proposition. $\overline{f^*(\mathrm{Spec}B)} = \mathrm{Spec}A$ if and only if Ker $f \subseteq \eta(A)$.

Irreducibility:

Definition. Let X be a topological space. We say that X is irreducible if $X = V_1 \cup V_2 \Rightarrow X = V_1$ or $X = V_2$, where V_1, V_2 are closed sets.

Proposition. Let $X = \mathrm{Spec}A$ and $W \subseteq X$. Then:

1) W irreducible $\Leftrightarrow$ $I(W)$ prime

2) $W = V(J)$ irreducible $\Leftrightarrow$ $\mathrm{rad}(J)$ prime

Corollary. $\mathrm{Spec}A$ *irreducible* $\Leftrightarrow$ $\eta(A)$ *prime*

Noetherianity:

Definition. Let X be a topological space. We say that X is Noetherian if it satisfies ACC for open sets (resp. DCC for closed sets).

Proposition. A Noetherian $\Rightarrow$ $\mathrm{Spec}A$ Noetherian.

Connectedness:

Assume that $A \cong A_1 \times A_2$, then $\mathrm{Spec}A = \mathrm{Spec}A_1 \sqcup \mathrm{Spec}A_2$ not connected.

Proposition. There exist rings A_1, A_2 s.t. $A \cong A_1 \times A_2$ if and only if there exist $e \neq 0, 1$ s.t. $e^2 = e$ (idempotent element).

Section 4. An introduction to Homological Algebra.

4.1 Complexes of modules.

Unless otherwise stated we will always assume that A is a commutative ring with unit 1. We will consider the category $\mathbf{Comp(A)}$ of complexes of A-modules.

Definition. We define:

· **Homological complex:** $M_\bullet$: $\cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$

s.t. $d_n \circ d_{n+1} = 0$

· **Cohomological complex:** $M^\bullet$: $\cdots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} Mn+1 \longrightarrow \cdots$

s.t. $d^n \circ d_{n-1} = 0$

Definition. A morphism $f_\bullet : M_\bullet \longrightarrow N_\bullet$ of (homological) complexes of A-modules is a sequence of morphisms of A-modules $f_n : M_n \longrightarrow N_n$ s.t. the diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & M_n \arrow[r, "d_n"] \arrow[d, "f_n"] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & N_{n+1} \arrow[r, "d'_{n+1}"'] & N_n \arrow[r, "d'_n"'] & N_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Analogously for cohomological complexes.

Remark. $f_\bullet$ isomorphism of complexes $\Leftrightarrow$ f_n isomorphism $\forall n$

Some operations on complexes are:

- **Shift:** $M[\ell]_\bullet$ is the complex with $M[\ell]_n = M_{\ell+n}$.
- **Tensor product:** $M_\bullet \otimes N_\bullet$ is the complex with $(M_\bullet \otimes N_\bullet)_n = \bigoplus_{i+j=n} M_i \otimes_A N_j$
- **Hom:** $\mathrm{Hom}(M_\bullet, N_\bullet)_\bullet$ is the complex with $\mathrm{Hom}(M_\bullet, N_\bullet)_n = \prod_i \mathrm{Hom}(M_i, N_{i+n})$
- **Kernel and image:** Given a morphism of complexes $f_\bullet : M_\bullet \longrightarrow N_\bullet$ we may define $\mathrm{Ker} f_\bullet \subseteq M_\bullet$ and $\mathrm{Im} f_\bullet \subseteq N_\bullet$.

4.1.1 Homology and cohomology.

Definition. We define:

- **Homology:** Let $M_\bullet$: $\quad \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$
  be an homological complex. We define
  $$H_n(M_\bullet) = {\text{Ker } d_n}\big/{\text{Im } d_{n+1}}$$
  - Elements of $M_n \rightsquigarrow n$-chains.
  - $Z_n(M_\bullet) = \text{Ker } d_n \rightsquigarrow n$-cycles.
  - $B_n(M_\bullet) = \text{Im } d_{n+1} \rightsquigarrow n$-boundaries.

- **Cohomology:** Let $M^\bullet$: $\quad \cdots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} Mn+1 \longrightarrow \cdots$
  be a cohomological complex. We define
  $$H^n(M_\bullet) = {\text{Ker } d^n}\big/{\text{Im } d^{n-1}}$$
  - Elements of $M^n \rightsquigarrow n$-cochains.
  - $Z_n(M_\bullet) = \text{Ker } d^n \rightsquigarrow n$-cocycles.
  - $B_n(M_\bullet) = \text{Im } d^{n-1} \rightsquigarrow n$-coboundaries.

Proposition. A morphism of complexes $f_\bullet : M_\bullet \longrightarrow N_\bullet$ induces a morphism
$$f_* = H_n(f_\bullet) : H_n(M_\bullet) \longrightarrow H_n(N_\bullet)$$

Remark. For all n we have a covariant functor $H_n : \mathbf{Comp(A)} \longrightarrow \mathbf{Mod(A)}$.

Let $0 \longrightarrow M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \longrightarrow 0$ be a short exact sequence of complexes. Then we have
$$H_n(M'_\bullet) \xrightarrow{f} H_n(M_\bullet) \xrightarrow{g} H_n(M''_\bullet)$$

Proposition. Let $0 \longrightarrow M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \longrightarrow 0$ be a short exact sequence of complexes. Then, for all n we have a **connecting morphism**
$$\partial_n : H_n(M''_\bullet) \longrightarrow H_{n-1}(M'_\bullet)$$

Theorem. *Let $0 \longrightarrow M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \longrightarrow 0$ be a short exact sequence of complexes. Then we have the* **homology long exact sequence**
$$\cdots \longrightarrow H_n(M'_\bullet) \xrightarrow{(f_*)_n} H_n(M_\bullet) \xrightarrow{(g_*)_n} H_n(M''_\bullet) \xrightarrow{\partial_n} H_{n-1}(M'_\bullet) \longrightarrow \cdots$$

Theorem. *Given a commutative diagram of complexes*

$$\begin{tikzcd} 0 \arrow[r] & M'_\bullet \arrow[r, "f_\bullet"] \arrow[d, "h'_\bullet"] & M_\bullet \arrow[r, "g_\bullet"] \arrow[d, "h_\bullet"] & M''_\bullet \arrow[r] \arrow[d, "h''_\bullet"] & 0 \\ 0 \arrow[r] & N'_\bullet \arrow[r, "f'_\bullet"'] & N_\bullet \arrow[r, "g'_\bullet"'] & N''_\bullet \arrow[r] & 0 \end{tikzcd}$$

then we get

$$\begin{tikzcd} \cdots \arrow[r] & H_n(M'_\bullet) \arrow[r, "(f_*)_n"] \arrow[d, "(h'_*)_n"] & H_n(M_\bullet) \arrow[r, "(g_*)_n"] \arrow[d, "(h_*)_n"] & H_n(M''_\bullet) \arrow[r, "\partial_n"] \arrow[d, "(h''_*)_n"] & H_{n-1}(M'_\bullet) \arrow[r] \arrow[d, "(h'_*)_{n-1}"] & \cdots \\ \cdots \arrow[r] & H_n(N'_\bullet) \arrow[r, "(f'_*)_n"] & H_n(N_\bullet) \arrow[r, "(g'_*)_n"] & H_n(N''_\bullet) \arrow[r, "\partial'_n"] & H_{n-1}(N'_\bullet) \arrow[r] & \cdots \end{tikzcd}$$

Definition. A morphism $f_\bullet : M_\bullet \longrightarrow N_\bullet$ of complexes is **homotopically zero** if there exist morphisms $\{s_n : M_n \longrightarrow N_{n+1}\}$ s.t. $\forall n \quad f_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$

$$\begin{tikzcd} \cdots \arrow[r] & M_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & M_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "s_n"] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"] & \cdots \\ \cdots \arrow[r] & N_{n+1} \arrow[r, "d'_{n+1}"'] & N_n \arrow[r, "d'_n"'] & N_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Definition. We say that the morphisms of complexes $f_\bullet, g_\bullet : M_\bullet \longrightarrow N_\bullet$ are **homotopically equivalent** if $f_\bullet - g_\bullet$ is homotopically zero.

Theorem. *If $f_\bullet, g_\bullet : M_\bullet \longrightarrow N_\bullet$ are homotopically equivalent then, $\forall n$ we have*

$$(f_*)_n = (g_*)_n : H_n(M_\bullet) \longrightarrow H_n(N_\bullet)$$

4.2 Free, projective and injective resolutions.

4.2.1 Free resolutions.

Definition. Let M be an A-module. A **free resolution** of M is an exact sequence

$$\mathbb{F}_\bullet : \qquad \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$$

where F_i are free A-modules.

Theorem. *Any A-module admits a free resolution.*

Remark. If M is any A-module, then the free resolution may be infinite. In some cases we may ensure good properties. For example:

- $(A, \mathfrak{m})$ is a local ring.
- $A = \bigoplus_i A_i$ is a graded ring (e.g. $A = k[x_1, \ldots, x_n]$ polynomial ring).

Local ring case:

Definition. Let $(A, \mathfrak{m})$ be a local ring and M an A-module. A free resolution

$$\mathbb{F}_\bullet : \qquad \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$$

is **minimal** if $d_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for all i.

Theorem. *Let M be a finitely generated A-module. Then M admits a minimal free resolution $\mathbb{F}_\bullet$ where the free modules F_i are finitely generated.*

Definition. The ranks of the free modules F_i are independent of the minimal free resolution. Indeed $F_i \cong A^{\beta_i}$ where the $\beta_i(M) := \beta_i \in \mathbb{Z}_{\geq 0}$ are called **Betti numbers** and are invariants of M.

Proposition.
$$\beta_i(M) = \dim_k \operatorname{Tor}_i^A(A/\mathfrak{m}, M)$$

Graded case:
Assume that $A = \bigoplus_i A_i$ is a graded ring. We may consider the category $\mathbf{Mod(A)}^*$ of graded modules.

- *Objects:* $M = \bigoplus_i M_i$ s.t. $A_i M_j \subseteq M_{i+j}$.
- *Morphisms:* $f : M \longrightarrow N$ s.t. $f(M_i) \subseteq N_i$.

Setup: For simplicity we will just consider the case $A = k[x_1, \ldots, x_n]$ polynomial ring over a field k.

- $\mathbb{Z}$- *graded:* $A = \bigoplus_{i \in \mathbb{Z}}$ s.t. $A_i = \{p(x) \in A \mid \deg p(x) = i\}$.
- $\mathbb{Z}^n$- *graded:* $A = \bigoplus_{\alpha \in \mathbb{Z}^n}$ s.t. $A_\alpha = \{p(x) \in A \mid \deg p(x) = \alpha\}$.

Notice that in any case $\mathfrak{m} = (x_1, \ldots, x_n)$ is the unique maximal homogeneous ideal.

Definition. Let M be an A-module. A graded free resolution is a exact sequence

$$\mathbb{F}_\bullet : \qquad \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$$

, where the free modules F_i and the corresponding morphisms are graded. It is **minimal** if $d_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for all i.

Theorem. *Let M be a finitely generated A-module. Then M admits a minimal free resolution $\mathbb{F}_\bullet$ where the free modules F_i are finitely generated.*

Definition. The ranks of the free modules F_i are independent of the minimal graded free resolution. Indeed

- $\mathbb{Z}$- *graded Betti numbers:* $F_i \cong \bigoplus_j A(-j)^{\beta_{i,j}}$ where $\beta_{i,j}(M) := \beta_{i,j} \in \mathbb{Z}_{\geq 0}$
- $\mathbb{Z}^n$- *graded Betti numbers:* $F_i \cong \bigoplus_j A(-\alpha)^{\beta_{i,\alpha}}$ where $\beta_{i,\alpha}(M) := \beta_{i,\alpha} \in \mathbb{Z}_{\geq 0}$

are invariants of M.

Proposition.

$$\beta_{i,j}(M) = \dim_k [\mathrm{Tor}_i^A(A/\mathfrak{m}, M)]_j$$
$$\beta_{i,\alpha}(M) = \dim_k [\mathrm{Tor}_i^A(A/\mathfrak{m}, M)]_\alpha$$

Theorem. **(Hilbert syzygy theorem)** *Let $A = k[x_1, \ldots, x_n]$. A minimal free resolution of a finitely generated A-module M has length $\leq n$.*

Definition. The **Castelnuovo-Mumford regularity** of M is

$$\mathrm{reg}(M) := \max\{|\alpha| - i \ \mid \ \beta_{i,\alpha}(M) \neq 0\}$$

4.2.2 Projective resolutions.

Definition. Let M be an A-module. A **projective resolution** of M is an exact sequence

$$\mathbb{P}_\bullet : \qquad \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0$$

where P_i are projective A-modules.

Remark. A free module is projective so projective resolutions exist.

Definition. The **projective dimension** $\mathrm{pd}_A(M)$ of an A-module M is the minimal length of a projective resolution of M.

Proposition. Let A be either local or graded and M a finitely generated A-module. Then $\mathrm{pd}_A(M)$ is the minimal length of a free resolution of M.

4.2.3 Injective resolutions.

Definition. Let M be an A-module. An **injective resolution** of M is an exact sequence

$$\mathbb{E}_\bullet : \quad 0 \longrightarrow M \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots$$

where E_i are injective A-modules. The resolution is **minimal** if $E_i = E_A(\ker d_i)$ where $E_A(\cdot)$ denotes the *injective hull* of an A-module.

Definition. The **injective dimension** $\mathrm{id}_A(M)$ of an A-module M is the length of a minimal injective resolution of M.

Definition. Let A be a Noetherian ring. By the Matlis-Gabriel theorem, the pieces of a minimal injective resolution of an A-module M decompose as

$$E_i \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec} A} E_A(A/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)}$$

where $\mu_i(\mathfrak{p}, M)$ is the i-th **Bass number** of M

Proposition. Let A be a Noetherian ring and M a finitely generated A-module. Then

$$\mu_i(\mathfrak{p}, M) < +\infty$$

for all i and all $\mathfrak{p} \in \mathrm{Spec} A$.

4.3 Derived functors.

Definition. Let

$$\mathbb{X}_\bullet : \quad \cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow M \longrightarrow 0$$

be a homological complex. Its **reduced complex** is

$$\mathbb{X}_M : \quad \cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow 0$$

Analogously for cohomological complexes.

Theorem. *Given a morphism of A-modules $f : M \longrightarrow N$, a projective resolution $\mathbb{P}_\bullet$ of M and an exact complex $\mathbb{X}_\bullet$ resolving N. Then, there exist a morphism of reduced complexes $\overline{f}_\bullet : \mathbb{P}_M \longrightarrow \mathbb{P}_N$ that makes the diagram commutative*

$$\begin{tikzcd} \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\overline{f}_1"] & P_0 \arrow[r] \arrow[d, "\overline{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r, "d'_2"'] & X_1 \arrow[r, "d'_1"'] & X_0 \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Moreover, two morphisms of complexes with these properties are homotopically equivalent. Analogously for injective resolutions.

Definition. Let $F : \mathbf{Mod(A)} \longrightarrow \mathbf{Mod(A)}$ be an additive functor and let

$$\mathbb{P}_\bullet : \quad \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0$$

$$\mathbb{E}_\bullet : \quad 0 \longrightarrow M \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots$$

be a projective and injective resolution respectively of an A-module M.

- Left derived functor.

  · F **covariant functor:** Take

  $$F(\mathbb{P}_M) : \quad \cdots \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \longrightarrow 0$$

  Then

  $$\mathbb{L}_n F(M) := H_n(F(\mathbb{P}_M)) = {}^{\ker F(d_n)}\!\big/_{\mathrm{Im} F(d_{n+1})}$$

  · F **contravariant functor:** Take

  $$F(\mathbb{E}_M) : \quad \cdots \xrightarrow{F(d_1)} F(E_1) \xrightarrow{F(d_0)} F(E_0) \longrightarrow 0$$

  Then

  $$\mathbb{L}_n F(M) := H_n(F(\mathbb{E}_M)) = {}^{\ker F(d_n)}\!\big/_{\mathrm{Im} F(d_{n-1})}$$

- Right derived functor.

  · F **covariant functor:** Take

  $$F(\mathbb{E}_M) : \quad 0 \longrightarrow F(E_0) \xrightarrow{F(d_0)} F(E_1) \xrightarrow{F(d_1)} \cdots$$

  Then

  $$\mathbb{R}^n F(M) := H^n(F(\mathbb{E}_M)) = {}^{\ker F(d_n)}\!\big/_{\mathrm{Im} F(d_{n+1})}$$

  · F **contravariant functor:** Take

  $$F(\mathbb{P}_M) : \quad 0 \longrightarrow F(P_0) \xrightarrow{F(d_0)} F(P_1) \xrightarrow{F(d_1)} \cdots$$

  Then

  $$\mathbb{R}^n F(M) := H^n(F(\mathbb{P}_M)) = {}^{\ker F(d_n)}\!\big/_{\mathrm{Im} F(d_{n-1})}$$

Definition. Given a morphism of A-modules $f : M \longrightarrow N$, and projective resolutions $\mathbb{P}_\bullet$ of M and $\mathbb{P}'_\bullet$ of N we obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\overline{f}_1"] & P_0 \arrow[r] \arrow[d, "\overline{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d'_2"'] & P'_1 \arrow[r, "d'_1"'] & P'_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Applying a covariant functor F to the reduced complexes $\mathbb{P}_M$ and $\mathbb{P}'_N$ we get

$$\begin{tikzcd}
\cdots \arrow[r, "F(d_2)"] & F(P_1) \arrow[r, "F(d_1)"] \arrow[d, "F(\overline{f}_1)"] & F(P_0) \arrow[r] \arrow[d, "F(\overline{f}_0)"] & 0 \\
\cdots \arrow[r, "F(d'_2)"'] & F(P'_1) \arrow[r, "F(d'_1)"'] & F(P'_0) \arrow[r] & 0
\end{tikzcd}$$

that induces a morphism in homology

$$\mathbb{L}_n F(f) : \mathbb{L}_n F(M) \longrightarrow \mathbb{L}_n F(N)$$

Analogously for the other versions.

Proposition. The definition of the derived functors is independent of the projective / injective resolution that we consider.

Theorem. *We have:*

- F *right exact (covariant or contravariant) functor. Then* $\mathbb{L}_0 F(M) \cong F(M)$
- F *left exact (covariant or contravariant) functor. Then* $\mathbb{R}^0 F(M) \cong F(M)$

Theorem. *Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a short exact sequence of* A*-modules. Then there exists long exact sequence of derived functors as follows:*

- F **covariant functor:**

$$\cdots \longrightarrow \mathbb{L}_1 F(M'') \xrightarrow{\partial} \mathbb{L}_0 F(M') \longrightarrow \mathbb{L}_0 F(M) \longrightarrow \mathbb{L}_0 F(M'') \longrightarrow 0$$

$$0 \longrightarrow \mathbb{R}^0 F(M') \longrightarrow \mathbb{R}^0 F(M) \longrightarrow \mathbb{R}^0 F(M'') \xrightarrow{\partial} \mathbb{R}^1 F(M') \longrightarrow \cdots$$

- F **contravariant functor:**

$$\cdots \longrightarrow \mathbb{L}_1 F(M') \xrightarrow{\partial} \mathbb{L}_0 F(M'') \longrightarrow \mathbb{L}_0 F(M) \longrightarrow \mathbb{L}_0 F(M') \longrightarrow 0$$

$$0 \longrightarrow \mathbb{R}^0 F(M'') \longrightarrow \mathbb{R}^0 F(M) \longrightarrow \mathbb{R}^0 F(M') \xrightarrow{\partial} \mathbb{R}^1 F(M'') \longrightarrow \cdots$$

4.4 Examples of derived functors.

Let A be a commutative ring with unit and M, N A-modules

4.4.1 Ext functor

Consider the derived functors:

- $F = \mathrm{Hom}_A(M, \cdot)$ covariant, additive, left exact. Then:
$$\mathrm{Ext}^n_A(M, \cdot) := \mathbb{R}^n \mathrm{Hom}_A(M, \cdot)$$
- $F = \mathrm{Hom}_A(\cdot, N)$ contravariant, additive, left exact. Then:
$$\overline{\mathrm{Ext}}^n_A(M, \cdot) := \mathbb{R}^n \mathrm{Hom}_A(\cdot, N)$$

Theorem. $\mathrm{Ext}^n_A(M, N) = \overline{\mathrm{Ext}}^n_A(M, N)$ *for all* n.

Remark. $\mathrm{Ext}^0_A(M, N) = \mathrm{Hom}_A(M, N)$

Proposition. Let $0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0$ be a short exact sequence of A-modules. Then we have the long exact sequences:

$$0 \longrightarrow \mathrm{Hom}_A(M, X') \longrightarrow \mathrm{Hom}_A(M, X) \longrightarrow \mathrm{Hom}_A(M, X'') \xrightarrow{\partial} \mathrm{Ext}^1_A(M, X') \longrightarrow \cdots$$

$$0 \longrightarrow \mathrm{Hom}_A(X'', N) \longrightarrow \mathrm{Hom}_A(X, N) \longrightarrow \mathrm{Hom}_A(X', N) \xrightarrow{\partial} \mathrm{Ext}^1_A(X'', N) \longrightarrow \cdots$$

Proposition. We have:

- M projective $\Leftrightarrow$ $\mathrm{Ext}^n_A(M, X) = 0$ $\forall X$ A-module and $n \geq 1$.
- N injective $\Leftrightarrow$ $\mathrm{Ext}^n_A(X, N) = 0$ $\forall X$ A-module and $n \geq 1$.

Proposition. We have:

- $\mathrm{pd}_A M \leq n$ $\Leftrightarrow$ $\mathrm{Ext}^i_A(M, X) = 0$ $\forall X$ and $i \geq n$ $\Leftrightarrow$ $\mathrm{Ext}^{n+1}_A(M, X) = 0$ $\forall X$.
- $\mathrm{id}_A N \leq n$ $\Leftrightarrow$ $\mathrm{Ext}^i_A(X, N) = 0$ $\forall X$ and $i \geq n$ $\Leftrightarrow$ $\mathrm{Ext}^{n+1}_A(X, N) = 0$ $\forall X$.

Proposition. We have:

- $\mathrm{Ext}^n_A(\oplus_{\lambda \in \Lambda} M_\lambda, N) \cong \prod_{\lambda \in \Lambda} \mathrm{Ext}^n_A(M_\lambda, N)$
- $\mathrm{Ext}^n_A(M, \prod_{\lambda \in \Lambda} N_\lambda, N) \cong \prod_{\lambda \in \Lambda} \mathrm{Ext}^n_A(M, N_\lambda)$

Proposition. Let $f : A \longrightarrow B$ be a flat morphism of rings, i.e. B is a flat A-module, where A is Noetherian. Let M be a finitely generated A-module. Then

$$\mathrm{Ext}^n_A(M, N) \otimes_A B \cong \mathrm{Ext}^n_B(M \otimes_A B, N \otimes_A B)$$

4.4.2 Tor functor

Consider the derived functors:

- $F = M \otimes_A \cdot$ covariant, additive, right exact. Then:

$$\mathrm{Tor}_n^A(M, \cdot) := \mathbb{L}_n(M \otimes_A \cdot)$$

- $F = \cdot \otimes_A N$ covariant, additive, right exact. Then:

$$\overline{\mathrm{Tor}}_n^A(\cdot, N) := \mathbb{L}_n(\cdot \otimes_A N)$$

Theorem. $\mathrm{Tor}_n^A(M,N) = \overline{\mathrm{Tor}}_n^A(M,N)$ *for all* n*. Also* $\mathrm{Tor}_n^A(M,N) = \mathrm{Tor}_n^A(N,M)$.

Remark. $\mathrm{Tor}_A^0(M,N) = M \otimes_A N$

Proposition. Let $0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0$ be a short exact sequence of A-modules. Then we have the long exact sequence:

$$\cdots \longrightarrow \mathrm{Tor}_1^A(M, X'') \xrightarrow{\partial} M \otimes_A X' \longrightarrow M \otimes_A X \longrightarrow M \otimes_A X'' \longrightarrow 0$$

Proposition. We have:

- M flat $\Leftrightarrow$ $\mathrm{Tor}_n^A(M,X) = 0$ $\forall X$ A-module and $n \geq 1$.

Proposition. We have:

- $\mathrm{Tor}_n^A(\oplus_{\lambda\in\Lambda} M_\lambda, N) \cong \oplus_{\lambda\in\Lambda} \mathrm{Tor}_n^A(M_\lambda, N)$

Proposition. Let $f : A \longrightarrow B$ be a flat morphism of rings. Then

$$\mathrm{Tor}_n^A(M,N) \otimes_A B \cong \mathrm{Tor}_n B(M \otimes_A B, N \otimes_A B)$$

4.4.3 Local cohomology functor

Definition. Let $I \subseteq A$ be an ideal. We define:

$$\Gamma_I(M) := \{m \in M \mid I^n m = 0 \text{ for some } n \geq 0\}$$

Remark. We have a covariant left exact functor $\Gamma_I(\cdot) : \mathbf{Mod(A)} \longrightarrow \mathbf{Mod(A)}$ called the **I-th torsion** functor.

Remark. $\Gamma_I(M) = \bigcup_{n \geq 0}(0 :_M I^n) = \varinjlim \operatorname{Hom}_A(A/I^n, M)$

Definition. Let $I \subseteq A$ be an ideal. We define the **local cohomology module** of an A-module M as

$$H_I^n(M) := \mathbb{R}^n \Gamma_I(M)$$

Proposition. Let A be a Noetherian ring. Then

$$H_I^n(M) = H^n_{\mathrm{rad}(I)}(M)$$

Proposition. The local cohomology modules can be described alternatively as:

- $H_I^n(M) = \varinjlim \operatorname{Ext}_A^n(A/I^n, M)$
- $H_I^n(M) = H^n(\check{C}_I{}^{\bullet})$.

  Where, given a set of generators $I = (f_1, \ldots, f_r)$, we are taking the cohomology of the **Čech complex**

  $$\check{C}_I{}^{\bullet} \quad 0 \longrightarrow M \longrightarrow \oplus_{1 \leq i \leq r} M_{f_i} \longrightarrow \oplus_{1 \leq i < j \leq r} M_{f_i f_j} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_r} \longrightarrow 0$$

Proposition. We have

1) **Long exact sequence:** Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of A-modules. Then

$$0 \longrightarrow H_I^0(M') \longrightarrow H_I^0(M) \longrightarrow H_I^0(M'') \xrightarrow{\partial} H_I^1(M') \longrightarrow \cdots$$

2) **Mayer-Vietoris sequence:**

$$0 \longrightarrow H^0_{I+J}(M) \longrightarrow H_I^0(M) \oplus H_J^0(M) \longrightarrow H^0_{I \cap J}(M) \longrightarrow H^1_{I+J}(M) \longrightarrow \cdots$$

3) **Flat base change:** Let $f : A \longrightarrow B$ be a flat morphism of rings. Then

$$H_I^n(M) \oplus_A B \cong H_I^n B(M_A B)$$

Section 5. Grade Theory.

5.1 Regular sequences and depth.

Unless otherwise stated we will always assume that A is a Noetherian commutative ring with unit 1. M will denote a finitely generated A-module.

Definition. A sequence of elements $a_1, \dots, a_n \in A$ is an M-**regular sequence** if it satisfies

1) $M/(a_1, \dots, a_n)M \neq 0$

2) a_i is not a zero-divisor in $M/(a_1, \dots, a_{i-1})M$

Remark. Order is important !!

Proposition. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated A-module. Let $a_1, \dots, a_n \in \mathfrak{m}$ be an M-regular sequence. Then, any permutation $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ is an M-regular sequence as well.

5.1.1 Koszul complex.

In order to detect regular sequences we use the **Koszul complex** associated to a sequence $\underline{a} = a_1, \dots, a_n$.

$$K(\underline{a}, A) : 0 \longrightarrow \wedge^n A^n \longrightarrow \cdots \longrightarrow \wedge^1 A^n \longrightarrow \wedge^0 A^n \longrightarrow 0$$

with the morphisms

$$d_p(e_{i_1} \wedge \cdots e_{i_p} = \sum_{j=1}^{p} (-1)^{j-1} a_{i_j} e_{i_1} \wedge \cdots \hat{e_{i_j}} \wedge \cdots e_{i_p}$$

for a basis $e_1, \dots, e_n$ of A^n. More generally, the Koszul complex of a f.g. A-module M is

$$K(\underline{a}, M) = K(\underline{a}, A) \otimes_A M$$

Proposition. We have:

1) $a_1, \dots, a_n$ M-regular sequence $\Rightarrow$ $H_p(K(\underline{a}, A)) = 0$.

2) $a_1, \dots, a_n$ A-regular sequence $\Rightarrow$ $K(\underline{a}, A)$ free resolution of $A/(a_1, \dots, a_n)$

5.1.2 Depth.

Let A be a Noetherian ring and M a finitely generated A-module.

Definition. We say:

- $a_1, \ldots, a_n$ maximal if $a_1, \ldots, a_n, a_{n+1}$ not M-regular $\forall a_{n+a} \in A$.
- $a_1, \ldots, a_n \in I$ maximal in the ideal I if $a_1, \ldots, a_n, a_{n+1}$ not M-regular $\forall a_{n+a} \in I$.

Theorem. **(Rees)** *Let $I \subseteq A$ be an ideal s.t. $M/IM \neq 0$. Then, all the maximal M-regular sequences in I have the same length. Indeed, if $a_1, \ldots, a_n$ is maximal*

$$n = \min\{i \mid \operatorname{Ext}^i_A(A/I, M) \neq 0\}$$

Definition. We define:

- $\operatorname{grade}(I, M) = \operatorname{depth}_I(M) = \min\{i \mid \operatorname{Ext}^i_A(A/I, M) \neq 0\}$
- $\operatorname{grade}(\mathfrak{m}, M) = \operatorname{depth}(M) = \min\{i \mid \operatorname{Ext}^i_A(A/\mathfrak{m}, M) \neq 0\}$ if $(A, \mathfrak{m})$ is local.

Proposition. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of A-modules. Then:

- $\operatorname{depth}_I(M) \geq \min\{\operatorname{depth}_I(M'), \operatorname{depth}_I(M'')\}$
- $\operatorname{depth}_I(M') \geq \min\{\operatorname{depth}_I(M), \operatorname{depth}_I(M'') + 1\}$
- $\operatorname{depth}_I(M'') \geq \min\{\operatorname{depth}_I(M') - 1, \operatorname{depth}_I(M)\}$

Theorem. *Let $(A, \mathfrak{m})$ be a Notherian local ring and M a f.g. A-module. Then*

$$\operatorname{depth}(M) \leq \dim M$$

Theorem. **(Auslander-Buchsbaum)** *Let $(A, \mathfrak{m})$ be a Notherian local ring and M a f.g. A-module. Then*

$$\operatorname{depth}(A) = \operatorname{depth}(M) + \operatorname{pd}_A M$$

Proposition. We have:

- $H^i_I(M) \neq 0 \;\Rightarrow\; i \in [\operatorname{depth}_I(M), \operatorname{cd}_A(I)]$
- $\operatorname{depth}_I(A) \leq \operatorname{ht} I \leq \operatorname{cd}_A(I) \leq \dim A$

5.2 Cohen-Macaulay rings and modules.

Definition. Let $(A, \mathfrak{m})$ be a Notherian local ring and M a f.g. A-module.

- M is Cohen-Macaulay if $\mathrm{depth}(M) = \dim M$.
- A is Cohen-Macaulay if it is Cohen-Macaulay as A-module, i.e. $\mathrm{depth}(A) = \dim A$.

Remark. If A is not local, M is Cohen-Macaulay if and only if $M_{\mathfrak{m}}$ is Cohen-Macaulay $\forall \mathfrak{m} \in \mathrm{Supp} M$ maximal.

Proposition. Let $(A, \mathfrak{m})$ be a Notherian local ring and M a f.g. A-module. Then

$$M \quad \text{Cohen-Macaulay} \quad \Leftrightarrow \quad H^i_{\mathfrak{m}}(M) = 0 \quad \forall i \neq \dim M$$

Proposition. Let M be a f.g. Cohen-Macaulay A-module.

1) $\mathrm{depth}(M) = \dim A/\mathfrak{p}$ for all associated primes $\mathfrak{p} \in Ass(M)$.

2) $\mathrm{depth}_I(M) = \dim M - \dim M/IM$ for all $I \subseteq \mathfrak{m}$

3) $a_1, \dots, a_n$ M-regular sequence $\Leftrightarrow$ $\dim M/(a_1, \dots, a_n)M = \dim M - n$

Definition. We say that $a_1, \dots, a_n$ is a **system of parameters** if $\mathrm{rad}(a_1, \dots, a_n) = \mathfrak{m}$

Proposition. $a_1, \dots, a_n$ M-regular sequence if and only if it is part of a system of parameters.

Proposition. M is Cohen-Macaulay if and only if every system of parameters is a maximal M-regular sequence.

Proposition. We have:

1) Let $a_1, \dots, a_n$ be an A-regular sequence. Then

$$A \text{ Cohen-Macaulay} \quad \Rightarrow \quad A/(a_1, \dots, a_n) \text{ Cohen-Macaulay}$$

2) $S \subseteq A$ multiplicatively closed set. Then

$$A \text{ Cohen-Macaulay} \quad \Rightarrow \quad S^{-1}A \text{ Cohen-Macaulay}$$

3) A Cohen-Macaulay $\Rightarrow$ $A[x_1, \dots, x_n]$, $A[[x_1, \dots, x_n]]$ Cohen-Macaulay

4) A Cohen-Macaulay $\Rightarrow$ $\mathrm{depth}_I(A) = \mathrm{ht} I \quad \forall I \subseteq A$ ideal

Proposition. We have:

1) A Cohen-Macaulay $\Rightarrow$ A equidimensional

2) A Cohen-Macaulay $\Rightarrow$ A connected in codimension 1

5.3 Gorenstein rings.

Definition. Let $(A, mathfrakm)$ be a Notherian local ring. We say that it is **Gorenstein** if it has finite injective dimension $\mathrm{id}_A A < \infty$.

Remark. If A is not local, A is Gorenstein if and only if $\dim A < \infty$ and $A_\mathfrak{m}$ is Gorenstein $\forall \mathfrak{m} \in \mathrm{Supp} M$ maximal.

Proposition. We have:

1) Let $a_1, \ldots, a_n$ be an A-regular sequence. Then
$$A \text{ Gorenstein } \Rightarrow A/(a_1, \ldots, a_n) \text{ Gorenstein}$$

2) $S \subseteq A$ multiplicatively closed set. Then
$$A \text{ Gorenstein } \Rightarrow S^{-1}A \text{ Gorenstein}$$

3) A Gorenstein $\Rightarrow$ $A[x_1, \ldots, x_n]$, $A[[x_1, \ldots, x_n]]$ Gorenstein

Proposition. Let $(A, \mathfrak{m})$ be a Notherian local ring and M a f.g. A-module. Then
$$\mathrm{id}_A M = \sup\{i \mid \mathrm{Ext}^i_A(A/\mathfrak{m}, M) \neq 0\}$$

Theorem. *Assume that* $\mathrm{id}_A M < \infty$. *Then*
$$\dim M \leq \mathrm{id}_A M = \mathrm{depth}(A)$$

Corollary. *A Gorenstein $\Rightarrow$ A Cohen-Macaulay.*

Definition. Let $(A, \mathfrak{m})$ be a Notherian local ring and M a f.g. A-module with $\mathrm{depth}(M) = n$. We define
$$\mathrm{type}_A M = \dim_k \mathrm{Ext}^n_A(A/\mathfrak{m}, M)$$

Remark. The **Socle** of M
$$\mathrm{Soc} M = \mathrm{Hom}_A(A/\mathfrak{m}, M) = (0 :_M \mathfrak{m})$$
is the largest k-vector space in M.

Proposition. Let $(A, \mathfrak{m})$ be a Notherian local ring with $\dim A = n$. TFAE:

1) A is Gorenstein.

2) $\mathrm{id}_A A = n$

3) A Cohen-Macaulay and $\mathrm{type}_A A = 1$.

Proposition. Let $(A, \mathfrak{m})$ be a Notherian local ring with $\dim A = 0$. TFAE:

1) A is Gorenstein.

2) $A \cong E_A(A/\mathfrak{m})$ injective

3) A Cohen-Macaulay and $\dim_k \mathrm{Soc} A = 1$.

5.5 Regular rings.

Definition. Let $(A, \mathfrak{m})$ be a Notherian local ring. We say that is **regular** if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. Equivalently, there exists a system of parameters that generate $\mathfrak{m}$.

Proposition. Let $(A, \mathfrak{m})$ be a Notherian local ring and $a_1, \ldots, a_n$ a minimal system of generators of $\mathfrak{m}$. TFAE:

1) A is local regular ring.

2) $a_1, \ldots, a_n$ A-regular sequence.

In particular $\mathrm{depth}_A A = \dim A$.