Exploring simplicial constructions for un-delooped K-Theory

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Summary

Section 1

Introduction

K-Theory

Idea behind K-Theory : define groups ("K-groups") related to vector spaces on various mathematical objects

- Algebraic objects : schemes, rings, varieties
- Topological spaces, Manifold, and more...

Originally : finite number of algebraically defined groups
(e.g. the Grothendieck group)
Nowadays : homotopy groups of topogical spaces

K-Theory of algebraic objects

This report: K-Theory of **algebraic objects**, ie a scheme X.

$\mathrm{Vect}(X)$ the category of locally free sheaves of finite rank over X is **exact**.

$=>$ We define the K-Theory of *an exact category*

$=>$ This is how we define the K-Theory of an algebraic object

Classical constructions 1/2

In [Qui72], Quillen introduces the **Q-construction**.
Let $\mathcal{M}$ be an exact category. Its Q-construction is category $Q\mathcal{M}$:

- Objects are the same as $\mathcal{M}$
- Morphisms between objects M and M'' are equivalence classes of diagram

$$M \twoheadleftarrow M' \hookrightarrow M''$$

K-groups := homotopy groups of classifying space $BQ\mathcal{M}$

Classical constructions 2/2

In [Wal85], Waldhausen introduces the **S-construction** of a **Waldhausen category**.
"Waldhausen categories" is a generalization of exact categories.
Let $\mathcal{C}$ be a Waldhausen category. Its S-construction is a simplicial category $wS.\mathcal{C}$.
K-groups := homotopy group of realization $|wS.\mathcal{C}|$

The G-construction

In this report we discuss a third construction, the **G-construction**.
Introduced in [GG87] by Gilet and Grayson.
For an exact category $\mathcal{M}$ its G-construction is a simplicial set $G\mathcal{M}$.
K-groups := homotopy group of realization $|G\mathcal{M}|$ ...

The G-construction

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Introduced in [GG87] by Gilet and Grayson.
For an exact category $\mathcal{M}$ its G-construction is a simplicial set $G\mathcal{M}$.
K-groups := homotopy group of realization $|G\mathcal{M}|$ ...
... with **no shift in degree**!

“No shift in degree” feature 1/2

Whereas i-th K-group is $\pi_{i+1}(BQ\mathcal{M})$ for Q-construction
For the G-construction i-th K-group is $\pi_i(|G\mathcal{M}|)$.

$\Omega BQ\mathcal{M}$ provides K-group with no shift
$\Rightarrow$ Idea : create an analogue to the loop space for simplicial set
Goal : when realized homotopy-equivalent to loop space of realization

"No shift in degree" feature 2/2

PRACTICAL APPLICATION

λ-rings generalizes the exterior product $\wedge^i$ on the ring $(K_0(R), \otimes)$.
We want to generalize such "λ-operation" operation to higher K-groups.
In [Gra89], Grayson defines λ-operation on the G-construction.
$\Rightarrow$ Induces continuous mapping on the realization
On K_0 , $\wedge^k$ for $k > 1$ is **not group homomorphism**
$K_0(R) = \pi_1(BQ(P(R)))$ so continuous function induce group homomorphism

The $S^{-1}S$ construction

Let $\mathcal{M}$ be an exact category where all sequence split.
Then category $S^{-1}S$ ([Gra] [CWe]) is the category s.t.
Objects : pairs (M, N) of objects of $\mathcal{M}$
Morphisms $(M, N) \longrightarrow (M', N')$: isomorphism class of objects P with isomorphisms $M \oplus P \simeq M'$ and $N \oplus P \simeq N'$
K-groups := homotopy group of classifying space $B(S^{-1}S)$ ...
... with **no shift in degree**!

Section 2

Defining the G-construction

Waldhausen category 1/2

A **Waldhausen category** is a category equipped with cofibrations

$$A \rightarrowtail B$$

and weak equivalences

$$A \xrightarrow{\sim} B$$

Satisfying certain axioms.
Motivating examples :

- Exact category with admissible monomorphism as cofibration and isomorphism as weak equivalence.
- Categories of complexes in some exact categories (eg complexes in $\mathrm{Vect}(X)$) with object-wise admissible monomorphism as cofibrations and quasi-isomorphisms as weak equivalences

Waldhausen category 2/2

Note

In a Waldhausen category $\mathcal{C}$ there exists pushout along cofibrations and a zero element *.
Let $A \rightarrowtail B$ be a cofibration we denote by B/A the pushout of diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \\ * & \end{tikzcd}$$

It generalizes the notion of quotient in an exact category.

Waldhausen's S-construction 1/2

Let C be a Waldhausen category, $wS.\mathcal{C}$ is a **simplicial category**, ie a morphism $\Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$.
Category $wS.\mathcal{C}([n])$ is the category of sequences of cofibrations of length n in $\mathcal{C}$

$$C_1 \rightarrowtail \dots \rightarrowtail C_n$$

Along with a choice of quotient $C_{i,j} = C_j/C_j$ for all $0 < i < j \leq n$.
Morphisms are morphisms of diagrams that are object-wise weak-equivalence

$$\begin{tikzcd} C_1 \arrow[r, tail] \arrow[d, "\sim"] & \dots \arrow[r, tail] & C_n \arrow[d, "\sim"] \\ C'_1 \arrow[r, tail] & \dots \arrow[r, tail] & C'_n \end{tikzcd}$$

Waldhausen's S-construction 2/2

From $wS.\mathcal{C}$ we get :

- A **simplicial set** denoted $S\mathcal{C}$ by only considering the objects in each $wS.\mathcal{C}([n])$
- A **bisimplicial set** also denoted as $wS.\mathcal{C}$ by post-composing with N

Lemma

When $\mathcal{C}$ is an exact category with canonical Waldhausen structure $|wS.\mathcal{C}|$ and $|S\mathcal{C}|$ are homotopically equivalent.

Constructing unshifted K-Theory space

$\mathcal{M}$ an exact category

S-construction

$$\mathcal{M} \overset{S}{\mapsto} S\mathcal{M} \overset{|-|}{\mapsto} |S\mathcal{M}| \overset{\Omega}{\mapsto} \Omega|S\mathcal{M}|$$

G-construction

$$\begin{tikzcd} \mathcal{M} \arrow[r, mapsto, "S"] & S\mathcal{M} \arrow[r, mapsto, "?"] & ?(S\mathcal{M}) \arrow[r, mapsto, "|-|"] & |?(S\mathcal{M})| \\ & & \text{simplicial set} \arrow[u] & \end{tikzcd}$$

Loop spaces

Definition

Let X be a topological space. The **topological loop space** ΩX of X with basepoint $x_0 \in X$ is the space of paths $\gamma : I \longrightarrow X$ s.t. $\gamma(0) = \gamma(1) = x_0$ with compact-open topology.

Definition

Let X be a simplicial set. The **simplicial loop space** of X with basepoint $x_0 \in X_0$ is the simplicial set ΩX such that forall $n \geq 0$

$$\Omega X([n]) := \varprojlim \left(\begin{tikzcd} \{x_0\} \arrow[r, hook] & X([0]) & X([0][n]) \arrow[l, "X(\mu_L)"'] \arrow[d, "X(\mu_R)"] \\ & X([0][n]) \arrow[u, "X(\mu_L)"] \arrow[r, "X(\mu_R)"] & X([n]) \end{tikzcd} \right)$$

Loop spaces

In Δ given $k, n \geq 0$
$[k][n] = \{0 < ... < k < (k+1) + 0 < ... < (k+1) + n\}$
$\mu_L : [k] \to [k][n]$ the inclusion on the left
$\mu_R : [n] \to [k][n]$ the inclusion on the right
Here with k = 0

Elements in $\Omega X([n])$ are pairs (x_{n+1}, x'_{n+1}) such that
$X(\mu_L)(x_{n+1}) = X(\mu_L)(x'_{n+1}) = x_0$ and $X(\mu_R)(x_{n+1}) = X(\mu_R)(x'_{n+1})$.

The G-construction

Definition

The G-construction of an exact category $\mathcal{M}$ is $\Omega S\mathcal{M}$.

For $n \geq 0$, $\Omega S\mathcal{M}([n])$ is the set of pairs of sequence of monomorphisms

$$M := \quad (M_0 \hookrightarrow M_1 \hookrightarrow \ldots \hookrightarrow M_n)$$

$$N := \quad (N_0 \hookrightarrow N_1 \hookrightarrow \ldots \hookrightarrow N_n)$$

along with choices of quotient such that
$N_j/N_i = M_j/M_i$ for all $0 \leq i < j \leq n$

because

$$S\mathcal{M}(\mu_R)(M) = M_1/M_0 \hookrightarrow \ldots \hookrightarrow M_n/M_0$$

must equal

$$S\mathcal{M}(\mu_R)(N) = N_1/N_0 \hookrightarrow \ldots \hookrightarrow N_n/N_0$$

Relation to $N(S^{-1}S)$

The vertices are $\Omega S\mathcal{M}([0]) = \mathrm{Ob}(\mathcal{M}) \times \mathrm{Ob}(\mathcal{M})$ and the edges are $\Omega S\mathcal{M}([1])$ consisting of pairs of sequences

$$\begin{pmatrix} M_0 \hookrightarrow M_1 \\ N_0 \hookrightarrow N_1 \end{pmatrix}$$

with one choice of quotient $C := N_1/N_0 = M_0/M_1$.
Corresponds to pairs of exact sequences $N_0 \hookrightarrow N_1 \twoheadrightarrow C$ and $M_0 \hookrightarrow M_1 \twoheadrightarrow C$.
If all exact sequences split $M_1 \simeq C \oplus M_0$ and $N_1 \simeq C \oplus N_0$ and the edge corresponds to a morphism in $S^{-1}S$.

What we need to prove

Claim 1

For **any** simplicial set X there is a map $|\Omega X| \longrightarrow \Omega|X|$

Claim 2

$|\Omega S\mathcal{M}| \longrightarrow \Omega|S\mathcal{M}|$ is a homotopy equivalence

This is [GG87]'s Theorem 3.1

Section 3

Proof that $|\Omega S\mathcal{M}| \simeq_{\mathrm{HoTop}} \Omega|S\mathcal{M}|$

Right fiber of F over ρ

Definition

Let $F: X \longrightarrow Y$ be a morphism of simplicial sets, $n \geq 0$ and $\rho \in Y_n$. We define $\rho|F$ the **right fiber of F over ρ** such that

$$(\rho|F)([k]) := \lim_{\leftarrow} \left(\begin{array}{ccc} & & X([k]) \\ & & \downarrow{\scriptstyle F} \\ & Y([n][k]) & \xrightarrow{Y(\mu_R)} Y([k]) \\ & \downarrow{\scriptstyle Y(\mu_L)} & \\ \{\rho\} & \hookrightarrow Y([n]) & \end{array} \right)$$

When $F = \mathrm{id}_Y$ we denote $\rho|F$ by $\rho|Y$.

Lemma

$|\rho|Y|$ is contractible.

Proving Claim 1

Given X a simplicial set and $x_0 \in X_0$ a base point we have a commutative diagram

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & x_0|X \arrow[d] \\ x_0|X \arrow[r] & X \end{tikzcd} \tag{1}$$

Map from $|\Omega X|$ to the homotopy pullback of $|x_0|X| \to |X| \leftarrow |x_0|X|$ which is homotopy equivalent to $\Omega|X|$ because $|x_0|X|$ is contractible.
We proved **Claim 1** and if (1) is homotopy cartesian $|\Omega X| \to \Omega|X|$ is a homotopy equivalence.

Theorem B'

Let $F : X \longrightarrow Y$ be a morphism of simplicial sets.

Theorem B'

If for $n \geq 0$, $\tau \in Y_n$ and $\phi : [m] \longrightarrow [n]$ the induced $\tau|F \longrightarrow \phi^*(\tau)|F$ is a homotopy equivalence. Then for any $l \geq 0$, $\rho \in X_l$ square

$$\begin{tikzcd} \rho|F \arrow[r] \arrow[d] & X \arrow[d, "F"] \\ \rho|Y \arrow[r] & Y \end{tikzcd}$$

is homotopy cartesian

Theorem B' generalizes [Qui72] Theorem B. Similarly there is a generalization of [Qui72] Theorem A

Using Theorem B'

Let X be a simplicial set and $x_0 \in X_0$. Consider

$$\begin{array}{cccc} P: & x_0|X & \longrightarrow & X \\ & x_{k+1} & \longmapsto & X(\mu_R)(x_{k+1}) \end{array}$$

Definition

Let $\rho \in X_n$ the right fiber $\rho|P$ is denoted $(x_0, \rho|X)$

We can check that $(x_0, x_0|X) = (x_0|P)$ is ΩX with base point x_0.

Corollary of Theorem B'

Let X be a simplicial set and $x_0 \in X_0$ be a base point. Assume that for any $\rho \in X_n$ and $\phi : [m] \longrightarrow [n]$ we have that $(x_0, \rho|X) \longrightarrow (x_0, \phi^*(\rho)|X)$ is a homotopy equivalence then $|\Omega X| \longrightarrow \Omega|X|$ is a homotopy equivalence

Strategy for proving **Claim 2**

Goal : apply corollary to $X = S\mathcal{M}$
Prove the hypothesis is true for all $\phi : [m] \longrightarrow [n]$ and $\tau \in S\mathcal{M}(n)$.
In practice we only need to show it on a finite number of well-chosen ϕ to conclude

Elements in $(0, \tau|S\mathcal{M})$

For $n \geq 0$ and $M = (M_1 \hookrightarrow ... \hookrightarrow M_n)$ elements of $(0, M|S\mathcal{M})([l])$ are pairs

$$\begin{pmatrix} & & & & L_0 \hookrightarrow L_1 \hookrightarrow \dots \hookrightarrow L_l \\ M_1 \hookrightarrow \dots \hookrightarrow M_n \hookrightarrow K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_l \end{pmatrix}$$

such that $L_j/L_i \simeq K_j/K_i$ for all $0 \leq i < j \leq l$
$+$ choice of quotient for each K_j/K_i for all $0 \leq i < j \leq l$
$+$ choice of quotient for each K_j/M_i for all $0 \leq i \leq l$ and $0 \leq j \leq n$

First case

Let $m \geq 0$ and $M \in S\mathcal{M}([m])$.
Consider $\eta : [1] \longrightarrow [m]$ s.t. $\eta(0) = 0$ and $\eta(1) = m$ then :

$$F : (0, M|S\mathcal{M})([n]) \longrightarrow (0, \eta^*(M)|S\mathcal{M})([n])$$

$$\begin{pmatrix} & & & & & & & L_0 & \hookrightarrow & \dots & \hookrightarrow & L_n \\ M_1 & \hookrightarrow & \dots & \hookrightarrow & M_m & \hookrightarrow & & K_0 & \hookrightarrow & \dots & \hookrightarrow & K_n \end{pmatrix} \mapsto \begin{pmatrix} & & L_0 & \hookrightarrow & \dots & \hookrightarrow & L_n \\ M_m & \hookrightarrow & K_0 & \hookrightarrow & \dots & \hookrightarrow & K_n \end{pmatrix}$$

We define a mapping $G : (0, \eta^*(M)|S\mathcal{M})([n]) \longrightarrow (0, M|S\mathcal{M})([n])$ that sets arbitrary quotients for K_i/M_j.
$\Rightarrow$ **Not** an isomorphism, but a homotopy equivalence.

$$G \circ F \simeq \mathrm{id} \qquad F \circ G = \mathrm{id}$$

Homotopic map towards $(0, M|S\mathcal{M})$

Very useful trick in these proofs

Let X be a simplicial set and $f, g : X \to (0, M|S\mathcal{M})$ be two maps.

Assumption : each $x_n \in X_n$ there is an isomorphism $\phi_{x_n} : f(x_n) \simeq g(x_n)$ which corresponds to a family of isomorphisms that makes the following diagram commute

$$\begin{tikzcd}
 & & & L_0 \arrow[r, hook] \arrow[dd, bend right] & \dots \arrow[r, hook] & L_n \arrow[dd, bend right] \\
M_1 \arrow[r, hook] & \dots \arrow[r, hook] & M_m \arrow[r, hook] & K_0 \arrow[r, hook] \arrow[dd, bend right] & \dots \arrow[r, hook] & K_n \arrow[dd, bend right] \\
 & & & L'_0 \arrow[r, hook] & \dots \arrow[r, hook] & L'_n \\
M_1 \arrow[r, hook] & \dots \arrow[r, hook] & M_m \arrow[r, hook] & K'_0 \arrow[r, hook] & \dots \arrow[r, hook] & K'_n
\end{tikzcd}$$

in a way compatible with images of morphisms Δ.

Consequence : f and g are homotopic.

Second case

Let $m \geq 0$ and $N \in \mathcal{M}$.
Consider $f, g : [0] \longrightarrow [1]$ s.t. $f(0) = 0$ and $g(0) = 1$ and consider :

$$f^*, g^* : (0, \widehat{N}|S\mathcal{M})([n]) \longrightarrow (0, 0|S\mathcal{M})([n])$$

such that

$$f^* : \begin{pmatrix} & L_0 \hookrightarrow \dots \hookrightarrow L_n \\ N \hookrightarrow & K_0 \hookrightarrow \dots \hookrightarrow K_n \end{pmatrix} \mapsto \begin{pmatrix} L_0 \hookrightarrow \dots \hookrightarrow L_n \\ K_0 \hookrightarrow \dots \hookrightarrow K_n \end{pmatrix}$$

and

$$g^* : \begin{pmatrix} & L_0 \hookrightarrow \dots \hookrightarrow L_n \\ N \hookrightarrow & K_0 \hookrightarrow \dots \hookrightarrow K_n \end{pmatrix} \mapsto \begin{pmatrix} L_0 \longhookrightarrow \dots \longhookrightarrow L_n \\ K_0/N \hookrightarrow \dots \hookrightarrow K_n/N \end{pmatrix}$$

we want to show they are homotopy equivalences

Second case

Consider also $H : (0, 0|S\mathcal{M})([n]) \longrightarrow (0, \widehat{N}|S\mathcal{M})([n])$ such that

$$H : \begin{pmatrix} L_0 \hookrightarrow \dots \hookrightarrow L_n \\ K_0 \hookrightarrow \dots \hookrightarrow K_n \end{pmatrix} \mapsto \begin{pmatrix} L_0 \longrightarrow \dots \longrightarrow L_n \\ N \hookrightarrow N \oplus K_0 \hookrightarrow \dots \hookrightarrow N \oplus K_n \end{pmatrix}$$

We have $g^* \circ H = \mathrm{id}$, and we admit that $f^* \circ H$ is a homotopy equivalence. Therefore it is enough to **show that $H \circ g^*$ is homotopic to $\mathbf{id}$.**

Second case

Explicitely for $J := H \circ g^*$ we have

$$J : \begin{pmatrix} & L_0 \hookrightarrow \ldots \hookrightarrow L_n \\ N \hookrightarrow & K_0 \hookrightarrow \ldots \hookrightarrow K_n \end{pmatrix} \mapsto \begin{pmatrix} & L_0 \longhookrightarrow \ldots \longhookrightarrow L_n \\ N \hookrightarrow & N \oplus K_0/N \hookrightarrow \ldots \hookrightarrow N \oplus K_n/N \end{pmatrix}$$

To prove it is homotopic id we provide $|(0, \widehat{N}|S\mathcal{M})|$ with a H-space structure using

$$\begin{pmatrix} & L_0 \hookrightarrow \ldots \hookrightarrow L_n \\ N \hookrightarrow & K_0 \hookrightarrow \ldots \hookrightarrow K_n \end{pmatrix} + \begin{pmatrix} & L'_0 \hookrightarrow \ldots \hookrightarrow L'_n \\ N \hookrightarrow & K'_0 \hookrightarrow \ldots \hookrightarrow K'_n \end{pmatrix}$$

$$= \begin{pmatrix} & L_0 \oplus L'_0 \longhookrightarrow \ldots \longhookrightarrow L_n \oplus L'_n \\ N \hookrightarrow & K_0 \coprod_N K'_0 \hookrightarrow \ldots \hookrightarrow K_n \coprod_N K'_n \end{pmatrix}$$

Second case

We use the following fact in exact categories

Lemma

Let $N \rightarrowtail M$ be a admissible monomorphism. There is a natural "isomorphism"

$$M \coprod_N M \simeq M \coprod_N (N \oplus M/N)$$

and deduce that $|\mathrm{id}| + |J|$ is homotopic to $|\mathrm{id}| + |\mathrm{id}|$.

We use topological result to provide an opposite to $|\mathrm{id}|$ by "$+$" and conclude.
$\Rightarrow \eta^*$, f^* ang g^* are homotopy equivalences.

To conclude

For any $m, n > 0$ and $\phi : [m] \to [n]$ the diagram commutes in Δ

$$
\begin{tikzcd}
{[1]} \arrow[d, "\eta_n"] & {[0]} \arrow[l, "f"'] \arrow[r, "f"] & {[1]} \arrow[d, "\lambda"] & {[0]} \arrow[l, "g"'] \arrow[r, "f"] & {[1]} \arrow[d, "\eta_m"] \\
{[n]} & & {[n]} \arrow[ll, "\mathrm{id}"'] & & {[m]} \arrow[ll, "\phi"']
\end{tikzcd}
$$

Where $\lambda(0) = \phi(0)$ and $\lambda(1) = n$. It induces for any $M \in S\mathcal{M}([n])$.

$$
\begin{tikzcd}
(0, \widehat{M_n}|S\mathcal{M}) \arrow[r, "g^*"] & (0, 0|S\mathcal{M}) & (0, \widehat{M_{\phi(0)}}|S\mathcal{M}) \arrow[l, "g^*"'] \arrow[r, "g^*"] & (0, 0|S\mathcal{M}) & (0, \widehat{M_{\phi(m)}}|S\mathcal{M}) \arrow[l, "g^*"'] \\
(0, M|S\mathcal{M}) \arrow[u, "(\eta_n)^*"] & & (0, M|S\mathcal{M}) \arrow[ll, "\mathrm{id}"'] \arrow[u, "(\lambda)^*"] \arrow[rr, "\phi^*"] & & (0, \phi^*(M)|S\mathcal{M}) \arrow[u, "(\eta_m)^*"]
\end{tikzcd}
$$

The main theorem

Theorem

For $\mathcal{M}$ an exact category. $|\Omega\mathcal{M}| \to \Omega|\mathcal{M}|$ is a homotopy equivalence.

Section 4

Defining $wG.\mathcal{M}$

G-construction of a Waldhausen category

First introduced in [Gun+92].
Consider $P-$ that shifts degree of simplicial object, ie such that $PX_n = X_{n+1}$.
Define simplicial category $wG.\mathcal{C}$ such that the following is cartesian

$$\begin{tikzcd} wG.\mathcal{C} \arrow[r] \arrow[d] & PwS.\mathcal{C} \arrow[d, "\delta_0"] \\ PwS.\mathcal{C} \arrow[r, "\delta_0"] & wS.\mathcal{C} \end{tikzcd}$$

where $(\delta_0)_n : PwS.\mathcal{C}([n]) \to wS.\mathcal{C}([n])$ corresponds to $wS.\mathcal{C}(d_0)$.
The image by Ob is $GC := \Omega S\mathcal{C}$.
If we post-compose with nerve functor we get a bisimplicial set $wG.\mathcal{C}$.

Lemma

*Let $\mathcal{M}$ be an exact category, canonically a Waldhausen category.
Then $|wG.\mathcal{C}|$ and $|G\mathcal{C}|$ are homotopy equivalent.*

G-construction of a Waldhausen category

Theorem

Let $\mathcal{C}$ be a Waldhausen category, if $\mathcal{C}$ is **pseudo-additive**,

$$|wG.\mathcal{C}| \longrightarrow \Omega|wS.\mathcal{C}|$$

Is a homotopy equivalence

Here pseudo-additive means that for all $N \rightarrowtail M$ we have a natural sequence of weak equivalences between $M \cup_N (N \vee (M/N))$ and $M \cup_N M$.

Examples

Waldhausen categories that are pseudo additive :
- Exact categories
- Complexes in $\mathrm{Vect}(X)$ for a scheme X with element-wise admissible monomorphism as cofibrations and quasi-isomorphisms as weak equivalences

Conclusion

Constructions for undelooped K-Theory

Construction	Category	"Additivity"	Example
$S^{-1}S$	split exact	$M \simeq N \oplus (M/N)$	$P(R)$
$G\mathcal{M}$	exact	$M \coprod_N M \simeq M \coprod_N (N \oplus (M/N))$	$\mathrm{Vect}(X)$
$wG.\mathcal{M}$	pseudo-additive	$M \coprod_N M \sim M \coprod_N (N \vee (M/N))$	$C\mathrm{Vect}(X)$

The proofs in [GG87] and [Gun+92] use the "pseudo-additivity" hypothesis in a very similar fashion !

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