Exploring simplicial constructions for un-delooped K-Theory

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Abstract

In Quillen's classical definition of the K-Theory of an exact category as presented in [Qui72], the n-th K-Group is the $(n+1)$-th homotopy group of the classifying space of the Q-construction and to make up for the shift in degree one has to consider the corresponding loop space. In this document we explore an existing approach that removes this extra step by constructing a simplicial structure called the G-construction whose homotopy groups will correspond to the K-Groups of same degree. This construction was presented in [GG87] by Gillet and Grayson along with the proof that its realization is homotopy equivalent to the loop space of the classifying space of the Q-construction in the context of exact categories. We present their results with the goal of providing additional combinatorial details, recalling the necessary theoretical background and providing relevant examples. In later parts we compare their result and proof with a generalization by Gunnarsson, Schwänzl, Vogt and Waldhausen [Gun+92] that can be applied not only to exact categories but to any Waldhausen categories [Wal85] that satisfies a certain additivity hypothesis.

Contents

Introduction

The term K-Theory is used to describe a variety of mathematical constructions. They typically consist in taking a mathematical object which can be of various nature, such as a ring, a scheme, a topological space or a manifold and defining groups often related to some notion of vector bundle on the corresponding object. Those groups will be referred to as the K-groups and their purpose is to encompass some of the object's properties while being easier to study. The early development of K-Theory focused on the study of a finite number of algebraically defined groups such as the Grothendieck group of a ring or a scheme. However in modern mathematics we usually define the K-groups as the homotopy groups of relevant topological objects. In this document we focus on the K-Theory of algebraic objects such as rings or schemes. Given a scheme X there are several ways of constructing its K-Theory but they are classically associated with the exact category structure on the category $\mathrm{Vect}(X)$ of locally free sheaves of finite rank over X. One construction for the K-Theory of an exact category $\mathcal{M}$ is the Q-construction introduced by Quillen in [Qui72], a category $Q\mathcal{M}$ whose objects are the same as $\mathcal{M}$ and in which morphisms from object M to M'' are classes of admissible subobjects of M'' of which M is an admissible quotient. The homotopy groups of $BQ\mathcal{M}$, the corresponding classifying space, will provide the K-groups. Later on, Waldhausen provided in [Wal85] the S-construction for Waldhausen categories, which we view here as a generalization of exact categories. It consists taking in the nerve of the simplicial category whose objects in degree n are sequences of cofibrations of length n with choices of quotients and whose morphisms are natural transformations between them that are object-wise weak equivalences. The homotopy groups of this bisimplicial set's realization will provide the K-groups and they correspond to the ones provided by the Q-construction when applied to an exact category.

This document mostly revolves around yet another construction, a simplicial set called the the G-construction described in [GG87] by Gillet and Grayson which in its original formulation only applies to exact categories and is largely based on Waldhausen's S-construction. It has the important property that for any $i \geq 0$ the i-th K-group is the i-th homotopy group of its realization, without resorting to a shift in degree. This differs from the above-mentionned constructions since for an exact category $\mathcal{M}$ the category $Q\mathcal{M}$ provided in [Qui72] is such that the $(i+1)$-th homotopy group of the space $BQ\mathcal{M}$ is the i-th K-group. To make up for the shift in degree one could coveniently work with $\Omega BQ\mathcal{M}$ the loop space of $BQ\mathcal{M}$. However the main idea of Gillet and Grayson in [GG87] is to define a simplicial analogue of the topological loop space, to apply it to the S-construction and to show that its realization is homotopy equivalent to the topological loop space of the classyfing space of the Q-construction. One of the advantages of this approach is given in [Gra89] when Grayson extends λ-operations to the higher K-groups of an exact category using a continuous mapping on the realization of the G-construction. If the continuous mapping applied to the classifying space of the Q-construction it would have induced a group homomorphism between the first homotopy groups and a well defined

λ-operation on K_0 is not supposed to be a group homomorphism. This feature is one of the main motivations for the G-construction.

In this document we will also briefly discuss [Gun+92] in which Gunnarsson, Schwänzl, Vogt and Waldhausen generalize the G-construction to any Waldhausen category and prove that the corresponding construction is homotopically equivalent to the loop space of the S-construction provided the category satisfies a certain additivity hypothesis. This generalization is used for instance in [KZ21] to extend λ-operations to algebraic examples of Waldhausen categories in which the hypothesis hold such as the category of complexes in an exact category where weak equivalences are quasi-isomorphisms.

The first few sections of this document and most of [GG87] revolve around detailing the technical background required for proving that the G-construction is homotopy equivalent to the loop space of the Q-construction for exact categories. In Section 1 to 4 we try to explain the theoretical machinery that goes behind this proof, with the aim of being understandable to a reader that only knows basic algebra and homotopy theory. On the other hand, the goal of the last section is merely to summarize and explain the upshot of [Gun+92] without getting into so much details.

In Section 1 we recall some basic constructions such as simplicial sets, bisimplicial sets and their geometric realization. We recall some properties that are considered known to the reader in [GG87] and used throughout the proofs. In Section 2 we detail results given in [GG87] regarding the homotopy of the realization of specific simplicial sets and morphisms. In particular we build our way towards proving simplicial generalizations given in [GG87] of Quillen's Theorem A and Theorem B originally stated in [Qui72]. We aim to add examples and further details to the exposition given in [GG87]. In Section 3, we introduce the notion of a Waldhausen category and we introduce the S-construction. Even though some of the details presented are not required to understand the proofs in [GG87], they provide important context and lead the way towards understanding the generalization of the G-construction that we later discuss in Section 5. In Section 4 we formally introduce the G-construction and detail some practical ways to characterize its elements, before finally explaining the proof of [GG87]'s Theorem 3.1 that states that the G-construction is homotopy equivalent to the loop space of the Q-construction. Finally in Section 5 we introduce the generalization of the G-construction given in [Gun+92] that applies not only to exact categories but also to Waldhausen categories. This section will paint a broader picture and will not be as detailed as the previous ones. We will mostly just recall the relevant results from [Gun+92] and give some partial explanation for a subset of the proofs. We will also discuss relevant algebraic examples these results can apply to such as the Waldhausen category of complexes in an exact category with quasi-isomorphisms as weak equivalences.

1 Geometric realization of simplicial structures

1.1 Geometric realization of simplicial set

A lot of the results in this article apply to the **geometric realization of a simplicial set**. In this section we will define it along with related mathematical objects and give results on them that will be used in the rest of the report.
We first give a quick reminder of the definition of simplicial objects and simplicial sets. Let Δ be the category whose objects are totally ordered finite non-empty sets and whose morphisms are order-preserving functions between them. It is equivalent to the full subcategory consisting of objects $[n] := \{0 < 1 < ... < n\}$ for each $n \geq 0$. We may only consider the latter category without loss of generality.

Definition 1.1.1. For any $m \geq 0$ we denote as $[m]$ the poset category $\{0 < 1 < \cdots < m\}$.

We use the same symbol as the corresponding object in Δ without loss of generality. This is justified because order-preserving functions between $[m]$ and $[n]$ in Δ correspond exactly to functors between the poset categories $[m]$ and $[n]$.

Definition 1.1.2. A simplicial object in a category $\mathcal{C}$ is a contravariant functor $X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$.
When $\mathcal{C} = \mathrm{Set}$ we refer to X as a **simplicial set**.
When $\mathcal{C} = \mathrm{Top}$ we refer to X as a **simplicial space**.

To define a simplicial object in $\mathcal{C}$ one only needs to define objects X_n in $\mathcal{C}$ for each $n \geq 0$ and morphisms $\phi^* : X_n \longrightarrow X_m$ in $\mathcal{C}$ for each morphism $\phi : [m] \longrightarrow [n]$ in Δ, such that for any $j, k, l \geq 0$ and any pair of morphisms $\theta : [k] \longrightarrow [l]$ and $\psi : [j] \longrightarrow [k]$ in Δ we have $(\theta \circ \psi)^* = \psi^* \circ \theta^*$. When it is clear which simplicial set we are referring to, we denote ϕ^* the image of ϕ by functor X and we denote it $X(\phi)$ otherwise. To denote the image of $[n]$ in $\mathcal{C}$ for a simplicial object X we use either X_n or $X([n])$.

Definition 1.1.3. Let X be a simplicial set. Let $i, k \geq 0$ be such that $i \leq k+1$ then we define a morphism in Δ

$$\begin{array}{rccccl} d_i^k : & [n] & \longrightarrow & [n+1] & & \\ & j & \longmapsto & j & \text{if} & j < i \\ & j & \longmapsto & j+1 & \text{if} & j \geq i \end{array}$$

and we refer to it as the ***i*-th face map**. When the value of k is clear we sometimes denote it as d_i. We also refer to $X(d_i^k) : X([n+1]) \longrightarrow X([n])$ as the i-th face map.

Definition 1.1.4. Let X be a simplicial set. Let $i, k \geq 0$ be such that $i \leq k$ then we define a morphism in Δ

$$\begin{array}{rccll} s_i^k : & [n+1] & \longrightarrow & [n] & \\ & j & \longmapsto & j & \text{if } j \leq i \\ & j & \longmapsto & j-1 & \text{if } j > i \end{array}$$

and we refer to it as the **i-th degeneracy map**. When the value of k is clear we sometimes denote it as s_i. We also refer to $X(s_i^k) : X([n]) \longrightarrow X([n+1])$ as the i-th degeneracy map.

Example 1.1.1. A canonical example of a simplicial set is Δ_n such that $\Delta_n([k]) = \mathrm{Hom}_\Delta([k],[n])$. For any $\phi : [k'] \longrightarrow [k]$ function $\phi^* : \mathrm{Hom}_\Delta([k],[n]) \longrightarrow \mathrm{Hom}_\Delta([k'],[n])$ is defined by pre-composing ϕ. Let X be another simplicial set then by Yoneda's lemma morphisms of simplicial set from Δ_n to X are in a natural one-to-one correspondence with elements of X_n and thus Δ_n has a particuliar role in the category of simplicial sets.

Example 1.1.2. Let S be a set we sometimes also denote by S the constant simplicial set that maps all objects in Δ to S and all morphisms in Δ to id_S. In [GG87] such simplicial sets are called **discrete** for reasons we will come back to.

Definition 1.1.5. Given a category $\mathcal{C}$, $s\mathcal{C}$ denotes the category whose objects are functors $X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$ – referred to as simplicial objects – and whose morphisms are natural transformations between them – referred to as morphisms of simplicial objects.

Explicitely, a morphism of simplicial objects $\phi : X \longrightarrow Y$ is given by a family of morphisms $\phi_n : X_n \longrightarrow Y_n$ that commute with the images by X and Y of all morphisms in Δ (or equivalently with just face and degeneracy maps). The main example of simplicial object in this report will be sSet, and later on ssSet. Before defining the geometric realization of a simplicial set, we describe the nerve construction that will give us one of the main examples of simplicial sets.

Definition 1.1.6. For any small category $\mathcal{C}$ we will call its **nerve** the simplicial set $N\mathcal{C}$ such that for any $n \geq 0$, $N\mathcal{C}([n])$ is the set of functors $[n] \longrightarrow \mathcal{C}$ where $[n]$ denotes the poset category. Let $\phi : [m] \longrightarrow [n]$ be a morphism in Δ, it is also a morphism between poset categories $[m]$ and $[n]$ and we define function $NC(\phi) : NC([m]) \longrightarrow NC([n])$ as the pre-composition of this functor.

Remark 1.1.1. Explicitely, we can view $N\mathcal{C}([n])$ as the subset of n-uple $(u_1, ..., u_n) \in \mathrm{Mor}(\mathcal{C})^n$ such that that for any $0 < k < n$ the co-domain of u_k is the domain of u_{k+1}. For all $0 < k < n$ we deonte as A_{k-1} the domain of each u_k and as A_k the co-domain of u_k. Then we denote an element of $N\mathcal{C}([n])$ as

$$A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} ... \xrightarrow{u_n} A_n$$

Using this characterization, let $m, n \geq 0$ and $\phi : [m] \longrightarrow [n]$ be a morphism in Δ then function $N\mathcal{C}(\phi) : N\mathcal{C}([n]) \longrightarrow N\mathcal{C}([m])$ maps

$$A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} ... \xrightarrow{u_n} A_n$$

to

$$A_{\phi(0)} \xrightarrow{u'_1} A_{\phi(1)} \xrightarrow{u'_2} ... \xrightarrow{u'_m} A_{\phi(m)}$$

where

$$\forall 0 < i \leq m',\ u'_i := u_{\phi(i-1)+1} \circ ... \circ u_{\phi(i)}$$

In particular we notice that $N\mathcal{C}([0]) = \mathrm{Ob}(\mathcal{C})$ and $N\mathcal{C}([1]) := \mathrm{Mor}(\mathcal{C})$.

The following lemma will be useful in later parts of the report for understanding the relation between simplicial sets and categories given by the nerve functor

Lemma 1.1.1. *Let Cat be the category of small categories. We can define a functor*

$$\begin{array}{rccc} N: & Cat & \longrightarrow & sSet \\ & \mathcal{C} & \longmapsto & N\mathcal{C} \end{array}$$

which is essentially injective and fully faithful.

Proof. For the claim that N is a functor it suffices to check that given a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ we can define a morphism of simplicial sets $NF : N\mathcal{C} \longrightarrow N\mathcal{D}$ such that

$$\begin{array}{rccc} NF_n : & N\mathcal{C}([n]) & \longrightarrow & N\mathcal{D}([n]) \\ & A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} ... \xrightarrow{u_n} A_n & \longmapsto & F(A_0) \xrightarrow{F(u_1)} F(A_1) \xrightarrow{F(u_2)} ... \xrightarrow{F(u_n)} F(A_n) \end{array}$$

Using definitions it is easy to check that $NF_n \circ \phi^* = \phi^* \circ NF_n$ for any $\phi : [m] \longrightarrow [n]$, thus NF is a well defined morphism of simplicial sets. It's also obvious that $N(F \circ G) = N(F) \circ N(G)$ and $N(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{N(\mathcal{C})}$. The fact that N is faithful comes from observing that for a functor F its action on objects and morphisms is entirely determined by functions NF_0 and NF_1 respectively.
We turn on to proving that N is full. Let H be a morphism of simplicial set $N\mathcal{C} \longrightarrow N\mathcal{D}$ we can define a functor $F_H : \mathcal{C} \longrightarrow \mathcal{D}$ such that for an object X of $\mathcal{C}$ we have $F_H(X) = H_0(X)$ and for a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ we have $F_H(f) = H_1(f)$. We only need to prove that F_H is a well-defined functor and that $H = N(F_H)$. Because $(d_0)^* \circ H_1(f) = H_0 \circ (d_0)^*(f) = H_0(X)$ and $(d_1)^* \circ H_1(f) = H_0 \circ (d_1)^*(f) = H_0(Y)$ we know that $F_H(f)$ is a morphism from $F_H(X)$ to $F_H(Y)$. Additionally let $g : Y \longrightarrow Z$ be another morphism and $(f, g) = X \xrightarrow{f} Y \xrightarrow{g} Z \in N\mathcal{C}([2])$, then $(d_1)^* \circ H_2((f, g)) = H_1 \circ (d_1)^*((f, g))$ means $F_H(f \circ g) = F_H(f) \circ F_H(g)$, so F_H is a well defined functor. Moreover for any $n \geq 0$ can show that H_n always maps $A_0 \xrightarrow{u_1} A_1 \xrightarrow{u_2} ... \xrightarrow{u_n} A_n$ to

$F_H(A_0) \xrightarrow{F_H(u_1)} F_H(A_1) \xrightarrow{F_H(u_2)} ... \xrightarrow{F_H(u_n)} F_H(A_n)$ by applying the same reasoning to all $\phi : [0] \longrightarrow [n]$ and $\psi : [1] \longrightarrow [n]$ such that $\psi(1) = \psi(0) + 1$ and therefore $H = N(F_H)$.

Finally, for the claim that N is essentially injective we show that for any category $\mathcal{C}$ we can construct a category isomorphic to it from $N\mathcal{C}$ by taking elements of $N\mathcal{C}([0])$ as objects, elements $f \in N\mathcal{C}([1])$ as morphisms from $(d_0)^*(f)$ to $(d_1)^*(f)$, and composition between morphisms f and g as $(d_1)^*(x)$ for $x := x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2$ in $N\mathcal{C}([2])$. $\square$

Simplicial set have an intrinsic topological nature that we are going to highlight by describing their **geometric realization**. The following constructions is an important first step towards a topological understanding of simplicial sets.

Definition 1.1.7. We define a functor

$$\begin{array}{llll} D: & \Delta & \longrightarrow & \text{Top} \\ & [n] & \longmapsto D([n]) := \{(t_0, ..., t_n) & \in (\mathbb{R}^+)^{n+1}, \sum_{i=0}^{n} t_i = 1\} \\ & f:[n] \longrightarrow [m] \longmapsto & D(f) : (t_0, ..., t_n) \mapsto & \left(\sum_{i, f(i) = j} t_i \right)_{j \in \{1, ..., m\}} \end{array}$$

Moreover we denote $|\Delta_n| = D(n)$ and refer to it as the **topological n-simplex**

Before turning to the definition of the geometric realization, we define a category it will closely relate to.

Definition 1.1.8. Let S be a simplicial set we define $\Delta \downarrow S$ the category whose objects are pairs (n, x) for any $n \geq 0$ and for any $x \in X_n$. There is for any $m, n \geq 0$, any $x \in X_n$ and any $\phi : [m] \longrightarrow [n]$ in Δ a morphism from $(m, \phi^*(x))$ to (n, x) denoted as (ϕ, x). Given another $l \geq 0$ and $\psi : [l] \longrightarrow [m]$ the composition of morphism $(l, \psi^* \circ \phi^*(x)) \xrightarrow{(\psi, \phi^*(x))} (m, \phi^*(x))$ with $(m, \phi^*(x)) \xrightarrow{(\phi, x)} (n, x)$ is $(l, \psi^* \circ \phi^*(x)) \xrightarrow{(\phi \circ \psi, x)} (n, x)$. This only makes sense because $(\phi \circ \psi)^* = \psi^* \circ \phi^*$. Moreover we can define a functor $F : \Delta \downarrow S \longrightarrow$ sSet mapping for each $n \geq 0$ and $x \in X_n$ object (n, x) to Δ_n and for each $m \geq 0$ and $\phi : [m] \longrightarrow [n]$ morphism $(m, \phi^*(x)) \xrightarrow{(\phi, x)} (n, x)$ to $\phi_* : \Delta_m \longrightarrow \Delta_n$ the morphism of simplicial set induced by post-composing ϕ.

Lemma 1.1.2. *Let S be a simplicial, $\Delta \downarrow S$ is isomorphic to the category in which objects are pairs (n, f) for any $n \geq 0$ and for any morphism of simplicial set $\Delta_n \xrightarrow{f} S$ and in which there is morphism between $(n, f) \longrightarrow (m, g)$ for each*

$\phi : \Delta_m \longrightarrow \Delta_n$ *such that the following diagram commute*

$$\begin{tikzcd} \Delta_n \arrow[r, "f"] \arrow[d, "\phi"] & S \\ \Delta_m \arrow[ur, "g"] & \end{tikzcd} \tag{1}$$

Proof. We denote by $\Delta \downarrow' S$ the category described in the lemma. By Yoneda's lemma pairs (n, f) in $\Delta \downarrow' S$ are in one-to-one correspondence with elements $x_n \in S_n$ by taking $x_n := f_n(\mathrm{id}_n)$. We can from now on denote f as f_{x_n}. Also by Yoneda's lemma to each morphism $\phi : \Delta_n \longrightarrow \Delta_m$ there exists a unique $\psi : [n] \longrightarrow [m]$ such that $\phi = (\psi)_*$ the post-composition. In that context the commutativity condition given in Diagram 1 is equivalent to $\psi^*(x_m) = x_n$ because if $f = g \circ \phi$ then $f_n(\mathrm{id}_n) = g_n(\mathrm{id}_m \circ \psi) = \psi^* \circ g_m(\mathrm{id}_m)$. Conversely if $\psi^*(x_m) = x_n$ then both f and $g \circ \phi$ map id_n to the same element therefore they must be equal. The two categories are isomorphic and we are done. $\square$

We admit the following lemma as a consequence of a well-known results on functor categories. We use it to compute limits and co-limits in sSet in practice.

Lemma 1.1.3. *Let F be a functor from a small category $\mathcal{C}$ to sSet. The limit (respectively the co-limit) X of the corresponding diagram exists. Moreover, consider for any $n \geq 0$ the functor ev_n : sSet $\longrightarrow$ Set that maps any simplicial set Y to Y_n. Then $\mathrm{ev}_n \circ F$ forms a diagram in Set whose limit (respectively co-limit) is X_n.*

Less formally this means that limits in sSet exists and are computed level-wise.

Lemma 1.1.4. *Let S be a simplicial set and let F be the functor as defined in Definition 1.1.8. Because the source category is small, F forms a diagram and S is a co-limit of F.*

Proof. We denote by X the co-limit of the diagram formed by F. Let $l \geq 0$, we first show that $X_l \simeq S_l$.

Using Lemma 1.1.3 gives us that X_l is the co-equalizer of

$$\coprod_{\phi:[m]\to[n]} X_n \times \Delta_m([l]) \rightrightarrows \coprod_{n\geq 0} X_n \times \Delta_n([l])$$

where the first map is induced by $\phi^* \times \mathrm{id}_{\Delta_n([l])}$ and the second map by $\mathrm{id}_{X_n} \times \phi_*$ where ϕ_* is the post-composition by ϕ for any $m, n \geq 0$ and $\phi : [m] \longrightarrow [n]$. We use that to define the bijection

$$\begin{array}{cccc} H_l & S_l & \longrightarrow & X_l \\ & x_l & \longmapsto & (\mathrm{id}_l \in \Delta_l([l]), x_l) \end{array}$$

and its inverse

$$\begin{array}{cccc} K_l & X_l & \longrightarrow & S_l \\ & (\sigma \in \Delta_n([l]), x_n) & \longmapsto & \sigma^*(x_n) \end{array}$$

Function K_l is well defined because for any $m, n \geq 0$, $x_m \in X_m$, $\sigma : [l] \longrightarrow [m]$ and $\psi : [n] \longrightarrow [m]$ we have $K_l(\psi \circ \sigma, x_m) = (\psi \circ \sigma)^*(x_m) = \sigma^* \circ \psi^*(x_m) = K_l(\sigma, \psi^*(x_n))$. They are inverse because $K \circ H = \mathrm{id}$ and $H \circ K$ maps (σ, x_m) to $(\mathrm{id}, \sigma^*(x_m))$ which belongs to the same equivalence class. Those bijections extend to an isomorphism of simplicial set. Let $\phi : [m] \longrightarrow [n]$, then $S(\phi)$ maps $x_n \in S_n$ to $\phi^*(x_n) \in S_m$ and $X(\phi)$ maps $(\sigma : [n] \longrightarrow [k], y_k) \in X_n$ to $(\sigma \circ \phi : [m] \longrightarrow [k], y_k) \in X_m$. Therefore $K_m \circ X(\phi)(\sigma, x_n) = (\phi \circ \sigma)^*(x_n) = \phi^* \circ \sigma^*(x_n) = (S(\phi)) \circ K_n(\sigma, x_n)$ and $H_m \circ (S(\phi))(x_n) = (\mathrm{id}, (\phi)^*(x_n)) = (\phi, x_n) = X(\phi) \circ H_n(x_n)$ and we are done. $\square$

This gives context for the definition of the geometric realization of a simplicial set.

Definition 1.1.9. Let S be a simplicial set and $\Delta \downarrow S$ be the category defined in Definition 1.1.8. We consider the functor

$$\begin{array}{cccc} G_S & \Delta \downarrow S & \longrightarrow & \mathrm{Top} \\ & (n, x_n) & \longmapsto & |\Delta_n| \\ & (n, \psi^*(x_m)) \xrightarrow{(\psi, x_m)} (m, x_m) & \longmapsto & D(\psi) : D(n) \longrightarrow D(m) \end{array}$$

It defines a diagram in Top and we denote by $|S|$ its limit and call it the **geometric realization** of S. This construction is functorial in Top in the sense a morphism of simplicial set $f : S \longrightarrow S'$ induces a morphism $|f| : |S| \longrightarrow |S'|$ in a way that makes $| - |$ a functor from sSet to Top.

Remark 1.1.2. We can derive the functoriality of $| - |$ by associating to a morphism of simplicial sets $f : S \longrightarrow S'$ a morphism $f : |S| \longrightarrow |S'|$. Let i_{y_n} denote the inclusion of $G_{S'}((\Delta_n, y_n))$ in $|S'|$ for any object (n, y_n) in $\Delta \downarrow S'_n$. Then we define a family of morphisms $\phi_{x_n} := i_{f_n(x_n)} : |\Delta_n| \longrightarrow |S'|$ for each object (n, x_n) in $\Delta \downarrow S$. For any $m, n \geq 0$ and $\psi : [m] \longrightarrow [n]$, we have by definition $i_{f_n(x_n)} \circ D(\psi) = i_{\psi^*(f_n(x_n))} = i_{f_n(\psi^*(x_n))}$ which implies $\phi_{x_m} \circ D(\psi) = \phi_{\psi^*(x_m)}$. Therefore the family $(\phi_{x_n})_{(n,x_n) \in \mathrm{Ob}(\Delta \downarrow S)}$ commutes with arrows in the diagram formed by G_S and induce by universal property of the colimit a unique morphism $|f| : |S| \longrightarrow |S'|$ by universal property of the colimit.

Example 1.1.3. We observe that $D(n) = |\Delta_n|$ is consistent across notations. In the alternative definition of $\Delta \downarrow \Delta_n$ given in Lemma 1.1.2, it's obvious that (n, id) is a terminal element. Therefore the geometric realization of Δ_n is $|\Delta_n|$. It is common to refer to Δ_n as the n-simplex and to $|\Delta_n|$ as the topological n-simplex.

Example 1.1.4. Let S be a constant simplicial set as defined in Example 1.1.2, we can easily deduce that its realization is the set S with the discrete topology.

We know that S is a subset of $|S|$ because $\Delta \downarrow S$ only has arrows between elements (n,x) and (m,x) for any $m,n \geq 0$ and $x \in S$. Moreover for any $m \geq 0$ and $x \in S$ consider $D(\phi)$ the image of the constant function $\phi : [m] \longrightarrow [0]$, we observe that the image $G_S((m,x))$ of the copy of $|\Delta_m|$ corresponding to (m,x) in $|S|$ has all points in the same equivalence class as $*_x := (0,x)$. We deduce that $S \hookrightarrow |S|$ is a bijection and that the topology is discrete.

Lemma 1.1.5. *The geometric realization of a simplicial set has the structure of a CW-Complex*

The following proof will help us explicit the structure and understand how combinatorial information on the simplicial set can be extended to topological information on the geometric realization. The structure itself might be more important than the fact that it exists.

Proof. We first define the **skeletal filtration of a simplicial set** X. Let $n \geq 0$ we denote as Sk_n the "n skeleton", defined as the sub-simplicial set of X where

$$\mathrm{Sk}_n(m) = \{x \in X_m, \exists k \in \{0, \cdots, n\},\, \exists x_k \in X_k,\, \exists \phi : [m] \longrightarrow [k], x = \phi^*(x_k)\}$$

We also denote by $N_n \subset X_n$ the subset of **non-degenerate elements**. An element $x_n \in X_n$ is said to be degenerate if and only if there exists $m < n$, $x_m \in X_m$ and $\phi : [n] \longrightarrow [m]$ s.t. $\phi^*(x_m) = x_n$. This is equivalent to saying there is a an $x_k \in X_k$ for $k \geq 0$ not necessarily smaller than n and there exiss $\phi : [n] \longrightarrow [k]$ **non-injective** such that $\phi^*(x_k) = x_n$.
We also define simplicial set $\delta\Delta_n$ with $\delta\Delta_n([m]) := \{\phi \in \Delta_n([m]), \phi \text{ non-surjective}\}$.

Then we have the following cocartesian diagram

$$\begin{tikzcd}
\coprod_{x \in N_n} \delta\Delta_n \arrow[r] \arrow[d] & \mathrm{Sk}_{n-1} \arrow[d] \\
\coprod_{x \in N_n} \Delta_n \arrow[r] & \mathrm{Sk}_n
\end{tikzcd}$$

In which vertical arrows are inclusions and horizontal arrows are given for each $x \in N_n$ by mapping $\phi : [m] \longrightarrow [n]$ to $\phi^*(x)$. We check level-wise that the diagram is cocartersian for any $m \geq 0$. The diagram induces a morphism form the pushout of diagram

$$\coprod_{x \in N_n} \Delta_n([m]) \leftarrow \coprod_{x \in N_n} \delta\Delta_n([m]) \rightarrow \mathrm{Sk}_{n-1}([m])$$

to $\mathrm{Sk}_n([m])$. We observe that for any $y \in \mathrm{Sk}_n([m])$ either it belongs to $\mathrm{Sk}_{n-1}([m])$ or there exists a surjective morphism $\phi \in \Delta_n([m])$ and an element $x \in N_n$ such that $X(\phi)(x) = y$. Therefore the morphism is surjective. Moreover it is injective as a consequence of the following sublemma from [Lur] Proposition 1.1.3.4.

Lemma 1.1.6. *For X a simplicial set, each $x_m \in X_m$ is always the image of $x_l \in N_l$ a non-degenerate element of X_l by $\phi : [m] \longrightarrow [l]$ surjective in Δ. The pair (x_l, ϕ) is unique for x_m.*

It implies that for any elements $x, y \in N_n$ and any surjective morphisms $\phi, \psi : [m] \longrightarrow [n]$ if $\phi^*(x) = \psi^*(y)$ then $\phi = \psi$. It also implies that any such $\phi^*(x)$ is not in $\mathrm{Sk}_{n-1}([m])$ and therefore the morphism is a bijection.

Additionally we know the that the geometric realization $i : \delta\Delta_n \hookrightarrow \Delta_n$ is $|i| : \delta|\Delta_n| \hookrightarrow |\Delta_n|$ corresponding to the natural inclusion of $\delta|\Delta_n| = \{\{(t_0, ..., t_n) \in |\Delta_n|$, such that $t_0 = 0$ or $t_1 = 0$ or $\dots$ or $t_n = 0\}$, as shown in [Lur] Example 1.1.8.12. We know that $|-|$ commutes with co-limits. Therefore we have a cocartesian diagram in Top

$$\begin{tikzcd} \coprod_{x \in N_n} \delta|\Delta_n| \arrow[r] \arrow[d] & |\mathrm{Sk}_{n-1}| \arrow[d] \\ \coprod_{x \in N_n} |\Delta_n| \arrow[r] & |\mathrm{Sk}_n| \end{tikzcd}$$

Moreover we notice that S is the co-limit of the diagram given by the inclusions $\mathrm{Sk}_l \longrightarrow \mathrm{Sk}_k$ for all $0 \leq l \leq k$. We deduce that $|S|$ is the co-limit of the diagram given by the inclusions $|\mathrm{Sk}_l| \longrightarrow |\mathrm{Sk}_k|$. Because $\delta|\Delta_n| \longrightarrow |\Delta_n|$ is homeomorphic to $\delta\mathbb{D}_n \longrightarrow \mathbb{D}_n$ the inclusion of the $(n-1)$-sphere in the n-ball, we just proved that $|S|$ has a CW-complex structure in which the *topological* k-skeletons are the geometric realization of *categorical* k-skeletons. □

Remark 1.1.3. This characterization allows us to relate combinatorial information on the simplicial set with topological information on the realization. For instance, the set X_0 corresponds to a set of distinct points in $|X|$. Each element $x_1 \in X_1$ defines a path γ_{x_1} in $|X|$ between points $(d_0)^*(x_1)$ and $(d_1)^*(x_1)$. Moreover, an element $x_2 \in X_2$ induces an homotopy between paths $\gamma_{d_2^*(x_2)}\gamma_{d_0^*(x_2)}$ and $\gamma_{d_1^*(x_2)}$ or equivalently between $\gamma_{d_2^*(x_2)}\gamma_{d_0^*(x_2)}[-\gamma_{d_1^*(x_2)}]$ and the constant path at $d_0^* \circ d_0^*(x_2)$. We observe that when X is the nerve of a category $\mathcal{C}$, objects correspond to points in $|N\mathcal{C}|$, morphism form paths between their domains and codomains such that for all $f, g \in \mathrm{Mor}(\mathcal{C})$, $\gamma_f \gamma_g$ and $\gamma_{f \circ g}$ are homotopic.

Noting that CW complexes are compactly generated and Hausdorff, we can view the simplicial realization as a functor sSet $\longrightarrow$ CGHauss where CGHauss is the category of compactly generated Haussdorff spaces

Proposition 1.1.1. *The realization functor commutes with finite limits as a functor sSet $\longrightarrow$ CGHauss. In particular let I be a finite category and $F : I \longrightarrow$ sSet a finite diagram in sSet. If the limit of $|-| \circ F$ in Top is compactly generated then it is the realization of the limit of F.*

Proof. See [JG99] Chapter I Proposition 2.4. □

One important consequence of this is a correspondence between homotopies in Top and simplicial analogues to homotopy we will now define. We first have to go through an intermediary combinatorial definition

Definition 1.1.10. For any $n \geq 0$ we denote $\mathcal{O}, \boldsymbol{1} \in \mathrm{Hom}_{\Delta}([n],[1])$ the two constant morphism such that for any $0 \leq k \leq n$, $\mathcal{O}(k) = 0$ and $\boldsymbol{1}(k) = 1$.
Given any simplicial set X. We define a morphism of simplicial set $\mathbb{1} : X \longrightarrow \Delta_1$ such that for $m \geq 0$, function $\mathbb{1}_m : X_m \longrightarrow \Delta_1([m])$ maps each $x \in X_m$ to $\boldsymbol{1}$.
We define a morphism of simplicial set $\mathbb{0} : X \longrightarrow \Delta_1$ such that for $m \geq 0$, function $\mathbb{0}_m : X_m \longrightarrow \Delta_1([m])$ maps each $x \in X_m$ to $\mathcal{O}$.

Definition 1.1.11. Given $F : X \longrightarrow Y$ and $G : X \longrightarrow Y$ two morphisms of simplicial sets, we call a **simplicial homotopy** between F and G the data of a morphism of simplicial sets $H : \Delta_1 \times X \longrightarrow Y$ that makes the following diagram in sSet commute

$$\begin{tikzcd}
\Delta_0 \times X \arrow[d, "\mathbb{0},\mathrm{id}_X"] & X \arrow[l, "\simeq"] \arrow[dr, "G"] & \\
\Delta_1 \times X \arrow[rr, "H"] & & Y \\
\Delta_0 \times X \arrow[u, "\mathbb{1},\mathrm{id}_X"'] & X \arrow[l, "\simeq"] \arrow[ur, "F"] &
\end{tikzcd}$$

Corollary 1.1.1. *The realization of a simplicial homotopy H between F and G as given in Definition 1.1.11 is a topological homotopy between F and G.*

Proof. Keeping in mind that $|\Delta_1|$ is homeomorphic to $[0,1] \subset \mathbb{R}$, that the realization of $\Delta_0 \xrightarrow{\mathbb{1}} \Delta_1$ corresponds to the inclusion $\{1\} \rightarrowtail [0,1]$, and that the realization of $\Delta_0 \xrightarrow{\mathbb{0}} \Delta_1$ corresponds to the inclusion $\{0\} \rightarrowtail [0,1]$, we apply the realization functor to the above diagram. The only thing left to check is that the realization functor commutes with finite limits which is given by *Proposition* 1.1.1. In oder to apply the proposition one only needs to verify that $|\Delta_1| \times |X|$ is compactly generated. For instance one can use that $|X|$ is compactly generated, that $|\Delta_1|$ is locally compact and apply [Rez] Proposition 7.5. ☐

Corollary 1.1.2. *Let $G, F : \mathcal{D} \longrightarrow \mathcal{E}$ be two functors. If there is a natural transormation from G to F then $|G|, |F| : B\mathcal{D} \longrightarrow B\mathcal{E}$ are homotopic.*

Proof. Let α be a natural transformation from G to F, it corresponds to a morphism $H : [1] \times \mathcal{D} \longrightarrow \mathcal{E}$ that makes the following diagram in Cat commute

$$\begin{tikzcd}
{[0]} \times \mathcal{D} \arrow[d, "\mathbb{0},\mathrm{id}_{\mathcal{D}}"] & \mathcal{D} \arrow[l, "\simeq"] \arrow[dr, "G"] & \\
{[1]} \times \mathcal{D} \arrow[rr, "H"] & & \mathcal{E} \\
{[0]} \times \mathcal{D} \arrow[u, "\mathbb{1},\mathrm{id}_{\mathcal{D}}"'] & \mathcal{D} \arrow[l, "\simeq"] \arrow[ur, "F"] &
\end{tikzcd}$$

Applying the nerve functor to this diagram allows us to directly apply Corollary 1.1.1. ☐

We will now define for each simplicial set its **opposite** simplicial set, whose geometric realization we will show to be naturally homeomorphic. This trick will be used to provide corollaries that are mirror images of existing propositions in further part of the report and in [GG87].

Definition 1.1.12. We define a functor $-^{\text{opp}} : \Delta \longrightarrow \Delta$ as :

$$-^{\text{opp}} : \begin{array}{ccc} [n] & \longmapsto & [n] \\ \phi : [m] \longrightarrow [n] & \longmapsto & \phi^{\mathbf{opp}} : i \mapsto n - \phi(m-i) \end{array}$$

It induces a morphism $-^{\mathbf{opp}}$: sSet $\longrightarrow$ sSet by pre-composition. For each simplicial set X we call $X^{\mathbf{opp}}$ its **opposite**.

We note that $-^{\text{opp}} \circ -^{\text{opp}} = \mathrm{id}_\Delta$ and $-^{\mathbf{opp}} \circ -^{\mathbf{opp}} = \mathrm{id}_{\text{sSet}}$.

Remark 1.1.4. This notion generalizes the notion of the category opposite in the sense that given $\mathcal{C}$ a category, $(N\mathcal{C})^{\mathbf{opp}} \simeq N(\mathcal{C}^{\text{op}})$

This notion is used in [GG87] to provide mirror images of propositions using the following lemma.

Lemma 1.1.7. *For each simplicial set X there is an homeomorphism $|X| \simeq |X^{\mathbf{opp}}|$ between their realization. Moreover let $F : X \longrightarrow Y$ be a morphism of simplicial set then the following diagram commutes*

$$\begin{array}{ccc} |X| & \xrightarrow{|F|} & |Y| \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ |X|^{\mathbf{opp}} & \xrightarrow{|F^{\mathbf{opp}}|} & |Y|^{\mathbf{opp}} \end{array}$$

Proof. We define for any $l \geq 0$ the homeomorphism

$$\beta_l : \begin{array}{ccc} |\Delta_l| & \longmapsto & |\Delta_l| \\ (x_i)_{0\leq i\leq l} & \longmapsto & (x_{l-i})_{0\leq i\leq l} \end{array}$$

Moreover for any $n \geq 0$, $x \in X_n$ we define a morphism $\gamma_x : |\Delta_n| \longrightarrow |X^{\mathbf{opp}}|$ equal to $i_x \circ \beta_n$ where i_x corresponds to the inclusion of $G_{X^{\mathbf{opp}}}(n, x)$ in $|X^{\mathbf{opp}}|$ (see Definition 1.1.9). Given any $\psi : [m] \longrightarrow [n]$, the following diagram commutes

$$
\begin{tikzcd}
{|\Delta_m|} \arrow[r, "\beta_m"] \arrow[d, "D(\psi)"] \arrow[rrd, bend left, "\gamma_{\phi^*(x)}"] & {|\Delta_m|} \arrow[rd, "i_{(\psi^{\mathrm{opp}})^*(x)}"] \arrow[d, "D(\psi^{\mathrm{opp}})"] & \\
 & & {|X^{\mathbf{opp}}|} \\
{|\Delta_n|} \arrow[r, "\beta_n"] \arrow[rru, bend right, "\gamma_x"'] & {|\Delta_n|} \arrow[ru, "i_x"'] &
\end{tikzcd}
$$

which shows that the family $(\gamma_x)_{(n,x)\in \mathrm{Ob}(\Delta\downarrow X)}$ induces a unique continuous mapping $\beta : |X| \longrightarrow |X^{\mathbf{opp}}|$ by universal property. We proceed the same way to define a morphism from $|X^{\mathbf{opp}}|$ to $|X|$. The two are inverse to one another because $\beta_l \circ \beta_l = \mathrm{id}_{|\Delta_l|}$. We are done proving the first statement.

To prove the second statement of the lemma we need to prove that the two composed morphisms in the square are equal. By universal property it is enough observe that they are equal when precomposed by the inclusion of $G_X(n, x)$ in $|X|$ (see Definition 1.1.9) for any $n \geq 0$ and $x \in X_n$. This in turn follows from the definitions and we are done. $\square$

1.2 Geometric realization of a bisimplicial set

Whereas a simplicial set is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}$, a **bisimplicial** set is defined as a functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}$. For any $m, n \geq 0$ we use either $X_{m,n}$ or $X([m], [n])$ to denote the image of $([m], [n])$ by X. For any $\phi : [m] \longrightarrow [m']$ and $\psi : [n] \longrightarrow [n']$ we denote $X(\phi, \psi) : X_{m',n'} \longrightarrow X_{m,n}$ the image of (ϕ, ψ) by X. When it is obvious which bisimplicial set we are referring to we often denote $X(\phi, \psi)$ as $(\phi, \psi)^*$.

Remark 1.2.1. Let $\mathcal{D}$ be the category formed by bisimplicial sets and natural transformations between them. This category is isomorphic to ssSet the category of simplicial objects in sSet. In fact there are two distinct isomorphisms from $\mathcal{D}$ to ssSet. The first one is given by

$$X \mapsto ([k] \mapsto Y_k^L)$$

where X_k^L is the simplicial sets that maps $[l]$ to $X_{k,l}$ and the second one is given by

$$X \mapsto ([l] \mapsto Y_l^R)$$

where Y_l^R is the simplicial sets that maps $[k]$ to $X_{k,l}$. Conversely the reverse morphisms from ssSet to $\mathcal{D}$ are respectively given by

$$X \mapsto \Big(([k], [l]) \mapsto X([k])([l])\Big)$$

and
$$X \mapsto \Big(([k],[l]) \mapsto X([l])([k])\Big)$$

Similarly to how we defined the realization of a simplicial set, we now want to define the realization of a bisimplicial set. There are in fact several constructions for it but we will show that they are all homeomorphic. Because some of those constructions rely on it we now define the realization of a simplicial **space**, meaning a simplicial object in Top.

Definition 1.2.1. Given $X \in s\text{Top}$ a simplicial space, its geometric realization is a topological space denoted as $|X|$. It is defined as the set quotient of

$$\coprod_{n\geq 0} X_n \times |\Delta_n|$$

by the relation given induced by $(x_l, D(\phi)(y_k)) \sim (\phi^*(x_l), y_k)$ for each $k,l \geq 0$, $x_l \in X_l$, $y_k \in |\Delta|^k$ and $\phi : [k] \longrightarrow [l]$. We provide this quotient with the final topology. Moreover we can define a functor $|-| : s\text{Top} \longrightarrow \text{Top}$ mapping simplicial space to their realization and morphisms of simplicial spaces to continuous maps.

We immediately notice that by construction the geometric realization of a simplicial set X is exactly the same as its realization as a simplicial space where for all $k \geq 0$ the set X_k is provided with the discrete topology. This is because the direct sum in the above definition when each X_k is discrete is

$$\coprod_{n\geq 0, x\in X_n} |\Delta_n| = \coprod_{(n,x)\in \text{Ob}(\Delta\downarrow X)} |\Delta_n|$$

and by definition of the colimits in Top the space $|X|$ given in Definition 1.1.9 is exactly the quotient of this direct sum with the relation and topology given in Definition 1.2.1.

Remark 1.2.2. Let X be a simplicial space. Its realization is the coequalizer of diagram
$$\coprod_{\phi:[k]\longrightarrow[l]} |\Delta_k| \times X_l \rightrightarrows \coprod_k |\Delta_k| \times X_k$$

in which arrows are induced by morphisms $D(\phi) \times \text{id}_{X_l}$ and $\text{id}_{|\Delta_k|} \times \phi^*$ for all $k,l \geq 0$ and $\phi : [k] \longrightarrow [l]$. It is equivalent to the previous definition as a direct consequence of the nature of coequalizers in Top.
We use this characterization to derive the functoriality of $|-|$. Let $F : X \longrightarrow Y$ be a morphism of simplicial spaces. For any $m,n \geq 0$ and $\phi : [m] \longrightarrow [n]$ let i_ϕ denote the inclusion of the copy of $|\Delta_k| \times Y_l$ corresponding to ϕ in $|Y|$. For any $l \geq 0$ let j_k denote the inclusion of the copy of $|\Delta_l| \times Y_l$ corresponding to l in $|Y|$. We define a family of maps from each element of the coequalizer diagram of $|X|$ to $|Y|$. It consists in $i_\phi \circ (\text{id}_k, F_l)$ for all $k,l \geq 0$ and $\phi : [k] \longrightarrow [l]$ and

in $j_k \circ (\mathrm{id}_k, F_k)$ for all $k \geq 0$. Moreover for any $k, l \geq 0$ and $\phi : [k] \longrightarrow [l]$ the following diagrams commutes :

$$\begin{tikzcd}
|\Delta_k| \times X_l \arrow[r, "(\mathrm{id}_k, F_l)"] \arrow[d, "(D(\phi), \mathrm{id})"] & |\Delta_k| \times Y_l \arrow[d, "(D(\phi), \mathrm{id})"] \arrow[dr, "i_\phi"] & \\
& & |Y| \\
|\Delta_l| \times X_l \arrow[r, "(\mathrm{id}_l, F_l)"] & |\Delta_l| \times Y_l \arrow[ur, "j_l"] &
\end{tikzcd}$$

$$\begin{tikzcd}
|\Delta_k| \times X_l \arrow[r, "(\mathrm{id}_k, F_l)"] \arrow[d, "(\mathrm{id}, \phi^*)"] & |\Delta_k| \times Y_l \arrow[d, "(\mathrm{id}, \phi^*)"] \arrow[dr, "i_\phi"] & \\
& & |Y| \\
|\Delta_k| \times X_k \arrow[r, "(\mathrm{id}_k, F_k)"] & |\Delta_k| \times Y_k \arrow[ur, "j_k"] &
\end{tikzcd}$$

Therefore by universal property the family described above induces a unique morphism $F : |X| \longrightarrow |Y|$.

Moreover one of the constructions for the realization of a bisimplicial set rely on the diagonal functor that we now define.

Definition 1.2.2. We define the **diagonal functor** $\mathrm{Diag} : s\mathrm{sSet} \longrightarrow \mathrm{sSet}$ as the functor that maps X a bi-simplicial set to the simplicial set $\mathrm{Diag}(X)$ defined as

$$\begin{array}{cccc}
\mathrm{Diag}(X) : & [n] & \longmapsto & X_{n,n} \\
& \phi : [m] \to [n] & \longmapsto & (\phi, \phi)^* : X_{n,n} \to X_{m,m}
\end{array}$$

Before defining the realization we try to provide an analogue to Lemma 1.1.4 for bi-simplicial sets. Given $X \in s\mathrm{sSet}$ and $\phi : [k] \longrightarrow [l]$. We denote $\phi_* : \Delta_k \longrightarrow \Delta_l$ the function defined by postcomposing ϕ. Moreover for any $k \geq 0$ we have a morphism of simplicial set $\gamma^{(k)} : \Delta_k \times X_k \longrightarrow \mathrm{Diag}(X)$ induced by

$$\begin{array}{cccc}
\gamma^{(k)}_m & \Delta_k([m]) \times X_k([m]) & \longrightarrow & \mathrm{Diag}(X) \\
& (\sigma, x_{k,m}) & \longmapsto & (\sigma, \mathrm{id})^*(x_{k,m}) \in X_m([m])
\end{array}$$

We denote

$$\gamma : \coprod_k \Delta_k \times X_k \longrightarrow \mathrm{Diag}(X)$$

the unique morphism induced by the family $(\gamma^{(k)})_{k\geq 0}$.

Lemma 1.2.1. *Given $X \in ssSet$. The following diagram is a well-defined coequalizer diagram*

$$\coprod_{\phi:[k]\longrightarrow[l]} \Delta_k \times X_l \underset{(\phi_*, id_X)_{\phi:[k]\to[l]}}{\overset{(id_{\Delta_k}, \phi^*)_{\phi:[k]\to[l]}}{\rightrightarrows}} \coprod_k \Delta_k \times X_k \xrightarrow{\gamma} Diag(X)$$

Moreover given $F : X \longrightarrow Y$, $Diag(F) : Diag(X) \longrightarrow Diag(Y)$ is the unique morphism that makes the following diagram commute

$$\begin{tikzcd}
\coprod_{\phi:[k]\longrightarrow[l]} \Delta_k \times X_l \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(id_{\Delta_k}\times F_l)_\phi"] & \coprod_k \Delta_k \times X_k \arrow[r] \arrow[d, "(id_{\Delta_k}\times F_k)_k"] & Diag(X) \arrow[d, dotted, "Diag(F)"] \\
\coprod_{\phi:[k]\longrightarrow[l]} \Delta_k \times Y_l \arrow[r, shift left] \arrow[r, shift right] & \coprod_k \Delta_k \times Y_k \arrow[r] & Diag(Y)
\end{tikzcd}$$

Proof. We check that the first diagram is commutative. For any $k, l, m \geq 0$ any $\phi : [k] \longrightarrow [l]$ and any $(\tau, x_{l,m}) \in \Delta_k([m]) \times X_{l,m}$, we have that

$$(\phi_*, \mathrm{id}_{X_l})(\tau, x_{l,m}) = (\phi \circ \tau, x_{l,m})$$

and

$$(\mathrm{id}_{\Delta_k}, \phi^*)(\tau, x_{l,m}) = (\tau, (\phi, \mathrm{id})^*(x_{l,m}))$$

we observe that they have the same image by γ_m and we are done. Therefore there is a morphism from the coequalizer to $\mathrm{Diag}(X)$. It is surjective because for any $m \geq 0$ and $x \in X_{m,m}$ we have $\gamma^{(m)}(\mathrm{id}_m, x) = x$. It is injective because for any $k, m \geq 0$ any $(\alpha, y) \in \Delta_k([m]) \times X_{k,m}$ and any $x := \gamma_m(\alpha, y) = (\alpha, \mathrm{id}_m)^*(y)$, the respective images of $(\mathrm{id}_m, y) \in \Delta_m \times X_k$ by $(\mathrm{id}_{\Delta_m}, \alpha^*)$ and $(\alpha_*, \mathrm{id}_{X_{k,m}})$ are (id, x) and (α, y). Therefore (α, y) belongs to the class of (id, x) and so is any other element with the same image. We are done proving the first statement.

To prove the the second statement we simply check that for any $x \in X_{m,m}$, $(\mathrm{id}_{\Delta_m} \times F_m)_m$ maps (id_m, x) to $(\mathrm{id}_m, F_{m,m}(x))$. □

Remark 1.2.3. Given X a bi-simplicial set Lemma 1.2.1 provides two coequalizer diagrams, one for the two element of ssSet it corresponds to as described in Remark 1.2.1.

Remark 1.2.4. Let X be a simplicial set. Consider Y in ssSet such that Y_k is the constant simplicial set with value X_k. If we apply Lemma 1.2.1 to Y we can recover the result in Lemma 1.1.4. Therefore Lemma 1.2.1 is a generalization of Lemma 1.1.4.

Lemma 1.2.1 allows us to check that several constructions for the realization of a bisimplicial set are homeomorphic.

Proposition 1.2.1. *Let X be a bisimplicial set, the following constructions are homeomorphic.*
1) The realization of the simplicial space $[k] \mapsto |X([k], -)|$
2) The realization of the simplicial space $[l] \mapsto |X(-, [l])|$
3) The realization of the simplicial set $Diag(X) : [k] \mapsto X([k], [k])|$
We call them the ***realization of*** *X that we denote $|X|$. Each construction forms a functor $|-| : ssSet \longrightarrow CGHauss \longrightarrow Top$. The above-mentionned homeomorphisms are natural with respect to those functors.*

Proof. Consider $Z^X := ([k] \mapsto Y_k^L)$ and $(Z^X)' := ([k] \mapsto Y_k^R)$ the elements of ssSet corresponding to X as defined in Remark 1.2.1. The image by the realization functor of the coequalizer diagram

$$\coprod_{\phi:[k]\longrightarrow[l]} \Delta_k \times Z_l^X \underset{(\phi_*, \mathrm{id}_{Z_l^X})_\phi}{\overset{(\mathrm{id}_{\Delta_k}, \phi^*)_\phi}{\rightrightarrows}} \coprod_k \Delta_k \times Z_k^X \xrightarrow{\gamma} \mathrm{Diag}(X)$$

is

$$\coprod_{\phi:[k]\longrightarrow[l]} |\Delta_k| \times |Z_l^X| \underset{(D(\phi), \mathrm{id}_{Z_l^X})_\phi}{\overset{(\mathrm{id}_{|\Delta_k|}, \phi^*)_\phi}{\rightrightarrows}} \coprod_k |\Delta_k| \times |Z_k^X| \xrightarrow{(|\gamma|)_k} |\mathrm{Diag}(X)|$$

and similarly for $(Z^X)'$. It is also a coequalizer diagram since realization commutes with colimits. Therefore using Remark 1.2.2 we have that $|\mathrm{Diag}(X)|$ is homeomorphic the simplicial spaces defined by construction 1) and 2) when we consider Z^X and $(Z^X)'$ respectively. Moreover given $F : X \longrightarrow Y$ a morphism of bisimplicial set, as per Remark 1.2.2 the morphism induced by functoriality of construction 1) (respectively of construction 2)) is the unique morphism that makes the following diagram commute

$$\begin{tikzcd}
\coprod_{\phi:[k]\longrightarrow[l]} \Delta_k \times |Z_l^X| \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\mathrm{id}_{\Delta_k} \times |F_l|)_\phi"] & \coprod_k \Delta_k \times |Z_k^X| \arrow[r] \arrow[d, "(\mathrm{id}_{\Delta_k} \times |F_k|)_k"] & |\mathrm{Diag}(X)| \arrow[d, dotted, "|\mathrm{Diag}(F)|"] \\
\coprod_{\phi:[k]\longrightarrow[l]} \Delta_k \times |Z_l^Y| \arrow[r, shift left] \arrow[r, shift right] & \coprod_k \Delta_k \times |Z_k^Y| \arrow[r] & |\mathrm{Diag}(Y)|
\end{tikzcd}$$

(resp. the same diagram with $(Z^X)'$ and $(Z^Y)'$). This proves that the homeomorphisms are natural and we are done. $\square$

We have four useful definitions of the geometric realization of a bisimplicial set which are equivalent. Not only does it allow us to switch between them but it also reveals properties of symmetry of the bisimplicial realization that will be used throughout the article.

Corollary 1.2.1. *Let $F : X \longrightarrow Y$ be a morphism of bisimplicial sets and consider $F' : X' \longrightarrow Y'$ induced by pre-composing the bisimplicial sets with the commutator endofunctor*

$$\begin{array}{ccc} \Delta \times \Delta & \longrightarrow & \Delta \times \Delta \\ ([k],[n]) & \longmapsto & ([n],[k]) \end{array}$$

Then we have the following commutative square in Top

$$\begin{tikzcd} {|X|} \arrow[r, "{|F|}"] \arrow[d, "\simeq"] & {|Y|} \arrow[d, "\simeq"] \\ {|X'|} \arrow[r, "{|F'|}"] & {|Y'|} \end{tikzcd}$$

Proof. The homeomorphism $|X| \xrightarrow{\simeq} |X'|$ occurs when identifying the characterization of the geometric realization of $|X|$ by construction 1) to be exactly the same as the geometric realization of $|X'|$ by construction 2) in Proposition 1.2.1. The commutativity of the diagram is a direct consequence of the fact that the homeomorphisms given in the lemma are natural. $\square$

Another corollary makes use of the following definition

Definition 1.2.3. Given X a simplicial set, XL is the bisimplicial set given by pre-composing with the left projection

$$\begin{array}{rccc} L: & \Delta \times \Delta & \longrightarrow & \Delta \\ & ([k],[n]) & \longmapsto & [k] \end{array}$$

and XR is the bisimplicial set given by pre-composing X with the right projection

$$\begin{array}{rccc} R: & \Delta \times \Delta & \longrightarrow & \Delta \\ & ([k],[n]) & \longmapsto & [n] \end{array}$$

Remark 1.2.5. Explicitly XL and XR are defined by $XL([k],[l]) = X([k])$ and $XR([k],[l]) = X([l])$. We notice that $\mathcal{L} : X \mapsto XL$ and $\mathcal{R} : X \mapsto XR$ form well-defined functors from sSet to ssSet.

Corollary 1.2.2. *For any simplicial set Z there are homeomorphisms $|Z| \simeq |ZR|$ and $|Z| \simeq |ZL|$, such that for any morphism of simplicial sets $F : X \longrightarrow Y$ the following diagram commutes*

$$\begin{tikzcd} {|XL|} & {|X|} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "{|F|}"] & {|XR|} \arrow[d, "{|\mathcal{R}(F)|}"] \\ {|YL|} \arrow[from=u, "{|\mathcal{L}(F)|}"] & {|Y|} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & {|YR|} \end{tikzcd}$$

Proof. The homeomorphism $|X| \longrightarrow |XR|$ is obvious from looking at construction 3) given in Proposition 1.2.1 when we observe that $\mathrm{Diag}(XR) = X$. Commutativity of the left square is a consequence of the naturality of homeomorphisms in Proposition 1.2.1 . The same reasoning applies to $|X| \longrightarrow |XL|$ and the right square. $\square$

Intermediate results used for proving Theorem A and Theorem B revolve around the idea of proving that a morphism between simplicial or bisimplicial sets is a homotopy equivalence. The following theorem will be essential

Theorem 1.2.1. *A morphism of bisimplicial set $F : X \longrightarrow Y$ is a homotopy equivalence if for each $[n] \in \Delta$, the morphism of simplicial sets $F_n : X_{-,n} \longrightarrow Y_{-,n}$ is a homotopy equivalence.*

Proof. See [JG99] Chapter IV Proposition 1.7 $\square$

According to the symmetry properties we just mentioned we immediately deduce a mirror analogue in which the hypothesis applies to simplicial sets $F_n : X_{n,-} \longrightarrow Y_{n,-}$.

2 Simplicial analogues of Quillen's Theorem A and Theorem B

In [GG87] Gillet and Grayson provide simplicial analogues of Quillen's Theorem A and Theorem B described in [Qui72], their categorical counterpart. Later on we will use them to prove that the realization of the G-construction is homotopy equivalent to the loop space of classical constructions for the K-Theory of an exact category. In this section we try to add details to the presentation of combinatorial results provided in [GG87]. We also try to give additional examples and explicit the parallel with the corresponding theorems in [Qui72] and [GG87].

2.1 Introductory lemmas

In order to later prove the analogues of Theorem A and B we describe lemmas that relate properties of simplicial or bisimplicial sets to properties on the homotopy of their realizations. Those lemmas are described in [GG87] but the authors sometimes implicitly use corollaries that are deemed obvious. We try to make them explicit using the results on the realization of simplicial and bisimplicial sets from Section 1.

Let's first introduce some combinatorial notation for functions in Δ.

Definition 2.1.1. Let's $0 \leq q \leq m+1$, morphism $h_q \in \Delta_1([m])$ is defined as

$$\begin{array}{ccc} [m] & \longrightarrow & [1] \\ k & \longmapsto & \begin{cases} 0 & \text{if } k < q \\ 1 & \text{if } k \geq q \end{cases} \end{array}$$

One immediately notices that for any $f \in \Delta_1([m])$, we have $f = h_q$ for $q := \min(f^{-1}(1))$ if $f^{-1}(1) \neq \emptyset$ and $q := m+1$ otherwise. This means that all elements of $\Delta_1([m])$ can be referred to as h_q for some $0 \leq q \leq m+1$ without loss of generality.

In particular for $\mathcal{O}, \boldsymbol{1} \in \Delta_1([m])$ given in Definition 1.1.10, we notice that $\mathcal{O} = h_{m+1}$ and $\boldsymbol{1} = h_0$

Remark 2.1.1. We observer that for any $m, m' \geq 0$, any $0 \leq q \leq m$ and any $\phi : [m'] \longrightarrow [m]$ we have that $\phi^*(h_q) = h_p$ for $p := \min(\phi^{-1}(\{q, ...m\}))$ if $\phi^{-1}(\{q, ...m\}) \neq \emptyset$ and $p := m'+1$ otherwise.

Definition 2.1.2. Given $l, k \geq 0$ we denote $[k][l] = [k+l+1] = \{0 < ... < k < (k+1)+0 < ... < (k+1)+l\}$ the concatenation of $[k]$ and $[l]$, we also denote

$$\mu_R^{k,l} : [l] \hookrightarrow [k][l]$$

$$\mu_L^{k,l} : [k] \hookrightarrow [k][l]$$

the obvious inclusions in Δ on the right and the left respectively. For any k, k', l, l', any $\phi : [k'] \longrightarrow [k]$ and any $\psi : [l'] \longrightarrow [l]$ we define

$$\begin{array}{cccc} \phi \bullet \psi & : & [k'][l'] & \longrightarrow & [k][l] \\ & & i & \longmapsto & \begin{cases} \phi(i) & \forall i \leq k' \\ k+1+\psi(i-(k'+1)) & \forall i \geq k'+1 \end{cases} \end{array}$$

Definition 2.1.3. Given $m \geq 1$ we define the morphism in Δ

$$\begin{array}{cccc} \eta_m : & [1] & \longrightarrow & [m] \\ & 0 & \longmapsto & 0 \\ & 1 & \longmapsto & m \end{array}$$

Definition 2.1.4. Let $l, m, n \geq 0$ and $\phi : [n] \longrightarrow [m]$ we define function

$$\phi^{\blacktriangle} : \begin{array}{cccc} [n+l] & \longrightarrow & [m+l] & \\ i & \longmapsto & \phi(i) & \text{if } i \leq n \\ i & \longmapsto & m-n+i & \text{if } i > n \end{array}$$

in Δ. Moreover we check that for any $\psi : [k] \longrightarrow [m]$ we have $(\phi \circ \psi)^{\blacktriangle} = \phi^{\blacktriangle} \circ \psi^{\blacktriangle}$.

We also define $\phi^{\blacktriangledown} = \left(((\phi^{\mathrm{opp}})^{\blacktriangle})^{\mathrm{opp}}\right)$ the mirror function.

Definition 2.1.5. We denote as Δ^* the subcategory of Δ with the same objects but such that for any $k, l \geq 0$ a morphism $f : [k] \longrightarrow [l]$ belongs to Δ^* if and only if $f(k) = l$.
We denote as Δ_* the subcategory of Δ with the same objects but such that for any $k, l \geq 0$ a morphism $f : [k] \longrightarrow [l]$ belongs to Δ^* if and only if $f(0) = 0$.

Remark 2.1.2. One can define functors

$$\begin{array}{rccc} H: & \Delta & \longrightarrow & \Delta^* \\ & [n] & \longmapsto & [n+1] \\ & \phi : [n'] \longrightarrow [n] & \longmapsto & (\phi \bullet id_0) : [n'+1] \longrightarrow [n+1] \end{array}$$

$$\begin{array}{rccc} K: & \Delta & \longrightarrow & \Delta_* \\ & [n] & \longmapsto & [n+1] \\ & \phi : [n'] \longrightarrow [n] & \longmapsto & (id_0 \bullet \phi) : [n'+1] \longrightarrow [n+1] \end{array}$$

By pre-composing H we get a functor $H^* : \mathrm{Func}(\Delta^*, \mathrm{Set}) \longrightarrow \mathrm{sSet}$ and similarly a functor $K^* : \mathrm{Func}(\Delta_*, \mathrm{Set}) \longrightarrow \mathrm{sSet}$ by pre-composing K. Moreover we notice that the functor $-^{\mathrm{opp}}$ defined in Definition 1.1.12 maps Δ^* to Δ_* and reciprocally and forms an isomorphism between the categories such that the following diagram commutes

$$\begin{array}{ccc} \Delta & \xrightarrow{H} & \Delta^* \\ \big\downarrow{\scriptstyle -^{\mathrm{opp}}} & & \big\downarrow{\scriptstyle -^{\mathrm{opp}}} \\ \Delta & \xrightarrow{K} & \Delta_* \end{array}$$

and such that vertical arrows are isomorphisms. The following lemma implies that the geometric realization of any simplicial sets in the image of H^* is homotopy equivalent to a discrete space.

Lemma 2.1.1. *Given a functor $X^* : \Delta^{*\,op} \longrightarrow Set$ the induced simplicial set $H^*(X^*) : [k] \mapsto X^*([k][0])$ is homotopy equivalent to the constant simplicial set $X^*([0]) : [k] \mapsto X^*([0])$.*

Proof. The homotopy equivalence is given by morphisms of simplicial sets $F : H^*(X^*) \longrightarrow X^*([0])$ and $G : X^*([0]) \longrightarrow H^*(X^*)$ such that

$$F_k = X^*(\mu_R^{k,0}) : X^*([k][0]) \longrightarrow X^*([0])$$

$$G_k = X^*(f_k) : X^*([0]) \longrightarrow X^*([k][0])$$

where $f_k : [k][0] \longrightarrow [0]$ is the constant function. From $f_k \circ \mu_R^{k,0} = \mathrm{id}_{[0]}$ we immediately derive $F \circ G = \mathrm{id}_{X^*([0])}$. We only need to show that $|G \circ F| : |H^*(X^*)| \longrightarrow |H^*(X^*)|$ is homotopy equivalent to $|\mathrm{id}_{H^*(X^*)}|$. In order to do that we apply Corollary 1.1.1 to H a simplicial homotopy between $G \circ F$ and $\mathrm{id}_{H^*(X^*)}$ defined as

$$\begin{array}{rccc} H_n : & \Delta_1([n]) \times H^*(X^*)([n]) & \longrightarrow & H^*(X^*)([n]) \\ & (\sigma, x) & \longmapsto & (\lambda_\sigma)^*(x) \end{array}$$

where for $\sigma = h_q$ as in Definition 2.1.1 we define

$$\begin{array}{rccc} \lambda_\sigma : & [n+1] & \longrightarrow & [n+1] \\ & i & \longmapsto & \begin{cases} i & \text{if } i < q \\ n+1 & \text{if } i \geq q \end{cases} \end{array}$$

Because $\mathcal{O} = h_{n+1}$ and $\mathbf{1} = h_0$, we notice that $\lambda_{\mathcal{O}} = \mathrm{id}_{[n]}$ and $\lambda_{\mathbf{1}} = \mu_R^{n,0} \circ f_n$ which implies $(\lambda_{\mathbf{1}})^* = (G \circ F)_n$ and $(\lambda_{\mathcal{O}})^* = \mathrm{id}_{H^*(X^*)([n])}$. We now check that H is a well defined morphism of simplicial set. For any $n', n \geq 0$, any $\phi : [n'] \longrightarrow [n]$, any $\sigma = h_q \in \Delta_{\mathbf{1}}([n])$ and any $x \in H^*(X^*)([n])$ we have that $X^*(\lambda_{\phi^*(\sigma)}) \circ X^*(\phi \bullet id_0)(x) = X^*(\phi \bullet id_0) \circ X^*(\lambda_\sigma)(x)$. We use the fact that $\phi^*(h_q) = h_p$ with $p = \min(\phi^{-1}(\{q, ...n\}))$ if $\phi^{-1}(\{q, ...n\}) \neq \emptyset$ and $p = n+1$ otherwise. Then the statement is a consequence of the following combinatorial equality

$$(\phi \bullet id_0) \circ \lambda_{h_p} = \lambda_{h_q} \circ (\phi \bullet id_0)$$

or equivalently

$$(\phi \bullet id_0) \circ \lambda_{\phi^*(\sigma)} = \lambda_\sigma \circ (\phi \bullet id_0)$$

because both side map any $0 \leq i < p$ to $\phi(i)$ and any $i \geq p$ to $n+1$. Therefore H does form a simplicial homotopy between $(G \circ F)$ and $\mathrm{id}_{H^*(X^*)([n])}$ and we are done. □

We explicit the following corollary which is implied in [GG87] as a mirror to Lemma 2.1.1 and also used in later demonstrations

Corollary 2.1.1. *Given a functor $X_* : {\Delta_*}^{op} \longrightarrow Set$ the induced simplicial set $K^*(X_*) : [k] \mapsto X_*([0][k])$ is homotopy equivalent to the constant simplicial set $[k] \mapsto X_*([0])$.*

Proof. We define X^* as X_* precomposed by $-^{\mathrm{opp}}$ using the notation in Definition 1.1.12 . Then we notice that $(K^*(X_*))^{\mathbf{opp}} = H^*(X^*)$ by observing that the following diagram commutes

$$\begin{tikzcd}
\Delta \arrow[r, "H"] \arrow[d, "-^{\mathrm{opp}}"] \arrow[rrd, bend left, "(K^*(X_*))^{\mathbf{opp}}=H^*(X^*)"] & \Delta^* \arrow[d, "-^{\mathrm{opp}}"] \arrow[rd, "X^*"] & \\
\Delta \arrow[r, "K"] \arrow[rr, bend right, "K^*(X^*)"'] & \Delta_* \arrow[r, "X_*"] & \mathrm{Set}
\end{tikzcd}$$

For each $k \geq 0$, functor $-^{\mathrm{opp}} : \Delta^* \longrightarrow \Delta_*$ maps $\mu_R^{k,0} : [0] \longrightarrow [k][0]$ to $\mu_L^{0,k} : [0] \longrightarrow [0][k]$. Therefore functor $-^{\mathbf{opp}}$ maps the morphism of simplicial set given in Lemma 2.1.1 induced by all $X^*([k][0]) \xrightarrow{(\mu_R^{k,0})^*} X^*([0])$ to the one given in Corollary 2.1.1 induced by all $X_*([0][k]) \xrightarrow{(\mu_L^{0,k})^*} X_*([0])$. Applying Lemma 1.1.7 shows that the image of an homotopy equivalence by $-^{\mathbf{opp}}$ is an homotopy equivalence and we are done. □

Definition 2.1.6. We define $\tilde{\Delta}$ a category whose objects are pairs (A, k) for each $k \geq 0$ and $A \in \{\emptyset, [0], [1], \cdots\}$, denoted as $[A[k]]$.

Given any $A, A' \in \{\emptyset, [0], [1], \cdots\}$ and $l, l' \geq 0$ there is an arrow $[A[l]] \to [A'[l']]$ for each arrow $f : [\mathrm{Card}(A) + l] \longrightarrow [\mathrm{Card}(A') + l']$ in Δ that maps $\{\mathrm{Card}(A), ..., \mathrm{Card}(A)+l\}$ to $\{\mathrm{Card}(A'), ..., \mathrm{Card}(A')+l\}$. Composition of morphism is induced by the composition in Δ.

Intuitively $[A[l]]$ corresponds to $[\mathrm{Card}(A) + l]$ the concatenation of A and $[l]$ in Δ such that elements of A are smaller than elements of $[l]$. A morphism $[A[l]] \longrightarrow [A'[l']]$ corresponds to a morphism in Δ that maps the "upper part" to the "upper part".

Definition 2.1.7. We define a functor $-^{\#} : \tilde{\Delta} \longrightarrow \Delta$ such that for any $A \in \{\emptyset, [0], [1], ...\}$ and $k \geq 0$ we have $[A[k]]^{\#} = [k]$. Moreover for any other $A' \in \{\emptyset, [0], [1], ...\}$, $k' \geq 0$ and any morphism $\zeta : [A'[k']] \longrightarrow [A[k]]$ in $\tilde{\Delta}$ corresponding to $Z : [\mathrm{Card}(A') + k'] \longrightarrow [\mathrm{Card}(A) + k]$ in Δ, we define $\zeta^{\#} : [k'] \longrightarrow [k]$ to be the morphism in Δ such that for any $0 \leq i \leq k'$ we have $\zeta^{\#}(i) = Z(i - \mathrm{Card}(A')) - \mathrm{Card}(A)$.

Intuitively this functor amounts to forgetting about the "lower part" of each object. We check that given any composable pair of morphisms $\theta : [A''[k'']] \longrightarrow [A'[k']]$ and $\phi : [A'[k']] \longrightarrow [A[k]]$ in $\tilde{\Delta}$ we have $(\phi \circ \theta)^{\#} = (\phi^{\#} \circ \theta^{\#})$ and that for any object $[A[k]]$ in $\tilde{\Delta}$ we have $(\mathrm{id}_{[A[k]]})^{\#} = \mathrm{id}_k$.

Now for any $l, l', k, k' \geq 0$, let $A = [k]$ and $A' = [k']$ then morphisms $[[k][l]] \longrightarrow [[k'][l']]$ in $\tilde{\Delta}$ correspond to morphisms $\zeta : [k][l] \longrightarrow [k'][l']$ in Δ such that $\zeta(j) > k'$ for all $j > k$, but $\zeta(j)$ needs not be smaller than k' for $j \leq k$. This means that valid ζ include but are not limited to morphisms $\phi \bullet \psi$ for some $\phi : [k] \longrightarrow [k']$ and $\psi : [l] \longrightarrow [l']$. A counter-example is ζ such that $\zeta(j) = k' + l' + 1$ for all $0 \leq j \leq k + l + 1$. However we can still use this subset of morphisms to create a functor $\Delta \times \Delta \longrightarrow \tilde{\Delta}$ that maps $([k], [l])$ to $[[k][l]]$ and (ϕ, ψ) to $\phi \bullet \psi$. It allows us to construct by pre-composition a bisimplicial set from any morphism $\tilde{X} : \tilde{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{Set}$.

Lemma 2.1.2. *Given $\tilde{X} : \tilde{\Delta}^{op} \longrightarrow Set$. Let $X : ([k], [l]) \mapsto \tilde{X}([[k][l]])$ be a simplicial set. The family consisting in inclusions $[\emptyset[l]] \hookrightarrow [A[l]]$ in $\tilde{\Delta}$ for all $l \geq 0$ and $A \in \{\emptyset, [0], [1], \cdots\}$ induce a natural morphism of bisimplicial set from X towards $Y : ([k], [l]) \mapsto \tilde{X}([\emptyset[l]])$. This morphism is a homotopy equivalence.*

Proof. Using Theorem 1.2.1, we only need to prove that for any $l \geq 0$ the morphism of simplicial sets $X(-, [l]) \longrightarrow Y(-, [l])$ is an homotopy equiavalence. We use Lemma 2.1.1 by considering the functor

$$
\begin{array}{lcccl}
F: & (\Delta^*)^{\mathrm{op}} & \longrightarrow & \mathrm{Set} & \\
 & [n] & \longmapsto & \tilde{X}([[n-1][l]]) & \text{if } n > 0 \\
 & & & \tilde{X}([\emptyset[l]]) & \text{if } n = 0 \\
 & \phi : [n] \to [m] & \longmapsto & \tilde{X}(\phi^{\blacktriangle}) &
\end{array}
$$

where $\phi^{\blacktriangle}$ is defined as in Definition 2.1.4. Because ϕ is in Δ^* we have that $\phi^{\blacktriangle}$ maps $\{n, ..., n + l\}$ to $\{m, ..., m + l\}$ and induces a morphism in $\tilde{\Delta}$. Because

$X(-,[l])$ corresponds to $[k] \mapsto F([k][0])$ and $Y(-,[l])$ corresponds to $[k] \mapsto F([0])$ we can apply Lemma 2.1.1 and we are done. $\square$

2.2 Simplicial version of Quillen's Theorem A and Theorem B

We now use those intermediary results to prove simplicial analogues of Quillen's Theorem A and B.

2.2.1 Theorem A'

Definition 2.2.1. For $F : X \longrightarrow Y$ a morphism of simplicial sets, we consider the bisimplicial set $Y|F$ such that

$$(Y|F)([k],[l]) := \lim_{\leftarrow} \left\{ \begin{array}{ccc} & & X([l]) \\ & & \downarrow{\scriptstyle F_l} \\ Y([k][l]) & \xrightarrow{(\mu_R^{k,l})^*} & Y([l]) \end{array} \right\}$$

Remark 2.2.1. Explicitely $(Y|F)([k],[l])$ is the subset of pairs $(x,y) \in X_l \times Y_{k+l+1}$ such that $Y(\mu_R^{k,l})(y) = F_l(x)$. Given any $\phi : [k'] \longrightarrow [k]$ and $\psi : [l'] \longrightarrow [l]$ function $(Y|F)(\phi,\psi)$ maps (x,y) to $(X(\psi)(x), Y(\phi \bullet \psi)(y))$. It belongs to $(Y|F)([k'],[l'])$ because $(\phi \bullet \psi) \circ \mu_R^{k',l'} = \mu_R^{k,l} \circ \psi$ implies $Y(\mu_R^{k',l'}) \circ Y(\phi \bullet \psi)(y) = Y(\psi) \circ Y(\mu_R^{k,l})(y) = Y(\psi) \circ F_l(x) = F_{l'} \circ X(\psi)(x)$.

Example 2.2.1. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and $NF : N\mathcal{C} \longrightarrow N\mathcal{D}$ be its image by the nerve functor. Then $(N\mathcal{D}|N\mathcal{F})([k],[l])$ corresponds exactly to elements of $N\mathcal{D}_{k+l+1}$ of the following form :

$$d_0 \xrightarrow{f_0} \dots \xrightarrow{f_{k-1}} d_k \xrightarrow{f_k} F(c_0) \xrightarrow{F(g_0)} \dots \xrightarrow{F(g_{l-1})} F(c_l)$$

where $d_0, ..., d_k$ are objects in $\mathcal{C}$, $f_0, ..., f_k$ are morphisms in $\mathcal{C}$, $c_0, ..., c_l$ are objects in $\mathcal{D}$ and $g_0, ..., g_l$ are morphisms in $\mathcal{D}$. Moreover let any k, k', l, l', any $\phi : [k'] \longrightarrow [k]$ and any $\psi : [l'] \longrightarrow [l]$ then $(\phi,\psi)^*$ maps the element in the above diagram to

$$d_{\phi(0)} \xrightarrow{f'_0} \dots \xrightarrow{f'_{k'-1}} d_{\phi(k)} \xrightarrow{f'_{k'}} F(c_{\psi(0)}) \xrightarrow{F(g'_0)} \dots \xrightarrow{F(g'_{l'-1})} F(c_{\psi(l')})$$

where

$$\begin{array}{rccl} \forall 0 \leq i < k', & f'_i & := & f_{\phi(i)} \circ \dots \circ f_{\phi(i+1)} \\ \forall 0 \leq j < l', & g'_i & := & F(g_{\psi(j)})) \circ \dots \circ F(g_{\psi(j+1)}) \\ & f'_{k'} & := & f_{\phi(k)} \circ \dots \circ f_{k'} \circ F(g_0) \circ \dots \circ F(g_{\phi(0)-1}) \end{array}$$

Example 2.2.2. We often denote as $Y|Y$ the bisimplicial set $Y|\mathrm{id}_Y$ where id_Y is the identity morphism of simplicial sets on Y. In cases like this, $(Y|Y)([k],[l])$ corresponds exactly to elements of $y \in Y_{k+l+1}$ and for any $\phi : [k'] \longrightarrow [k]$ and $\psi : [k'] \longrightarrow [k]$ function $(\phi, \psi)^*$ maps y to $(\phi \bullet \psi)^*(y)$.

We will also need the following construction

Definition 2.2.2. For $F : X \longrightarrow Y$ a morphism of simplicial sets and an element $\rho \in Y_k$, we define a simplicial set $\rho|F$ called the **right fiber of F over** ρ such that :

$$(\rho|F)([l]) := \lim_{\leftarrow} \left\{ \begin{tikzcd} & & X([l]) \arrow[d, "F_l"] \\ & Y([k][l]) \arrow[r, "(\mu_R^{k,l})^*"] \arrow[d, "(\mu_L^{k,l})^*"] & Y([l]) \\ \{\rho\} \arrow[r] & Y([k]) & \end{tikzcd} \right\}$$

Remark 2.2.2. Explicitly $(\rho|F)([l])$ corresponds exactly to the subset of pairs $(x,y) \in X_l \times Y_{k+l+1}$ such that $(\mu_R^{k,l})^*(y) = F_l(x)$ and $(\mu_L^{k,l})^*(y) = \rho$. Then for any $\phi : [l'] \longrightarrow [l]$ function $(\rho|F)(\phi)$ maps (x,y) to $(X(\phi)(x), Y(\mathrm{id}_k \bullet \phi)(y))$. It belongs to $(\rho|F)([l']) \subseteq X_l \times Y_{k+l'+1}$ because

$$(\mathrm{id}_k \bullet \phi) \circ \mu_L^{k,l'} = \mu_L^{k,l}$$

and

$$(\mathrm{id}_k \bullet \phi) \circ \mu_R^{k,l'} = \mu_R^{k,l} \circ \phi$$

For any $k, k' \geq 0$ and any $\psi : [k'] \longrightarrow [k]$ there is a morphism of simplicial sets $\Psi : \rho|F \longrightarrow \psi^*(\rho)|F$ defined such that Ψ_l sends $(x,y) \in (\rho|F)([l]) \subseteq X_l \times Y_{k+l+1}$ to $(x, (\psi \bullet \mathrm{id}_l)^*(y))$. The image belongs in $(\psi^*(\rho)|F)([l]) \subseteq X_l \times Y_{k'+l+1}$ because

$$(\psi \bullet \mathrm{id}_l) \circ \mu_R^{k,l} = \mu_R^{k,l}$$

and

$$(\psi \bullet \mathrm{id}_l) \circ \mu_L^{k',l} = \mu_L^{k,l} \circ \psi$$

Remark 2.2.3. For any $k \geq 0$ and $\rho \in Y_k$ we will denote $\rho|\mathrm{id}_Y$ as $\rho|Y$. Explicitly, elements of $(\rho|Y)([l])$ correspond to elements $y \in Y_{k+l+1}$ such that $(\mu_k)^*(y) = \rho$. For any $\phi : [l'] \longrightarrow [l]$ function $(\rho|Y)(\phi)$ maps $y \in (\rho|Y)([l]) \subseteq Y_{k+l+1}$ to $Y(\mathrm{id}_k \bullet \phi)(y) \in (\rho|Y)([l']) \subseteq Y_{k+l'+1}$. Moreover given a morphism $\psi : [k'] \longrightarrow [k]$, the induced morphism of simplicial set $\Psi : \rho|Y \longrightarrow (Y(\psi)(\rho)|Y)$ is such that Ψ_l sends $y \in (\rho|Y)([l]) \subseteq Y_{k+l+1}$ to $Y(\psi \bullet \mathrm{id}_l)(y) \in (Y(\psi)(\rho)|Y)([l]) \subseteq Y_{k'+l+1}$.

Remark 2.2.4. For any morphism of simplicial set $F : X \longrightarrow Y$, any $k \geq 0$ and any $\rho \in Y_k$ there is a morphism of simplicial set $\kappa_R^F : \rho|F \longrightarrow \rho|Y$ such that for any $l \geq 0$ function $(\kappa_R^F)_l$ maps $(x,y) \in (\rho|F)([l]) \subseteq X_l \times Y_{k+l+1}$ to $y \in Y_{k+l+1}$. Using the two previous remarks we can see that the image belongs in $(\rho|Y)([l])$ and that those functions do extend to a morphism of simplicial sets.

Remark 2.2.5. For any morphism of simplicial set $F : X \longrightarrow Y$, any $k \geq 0$ and any $\rho \in Y_k$ there is a morphism of simplicial set $\kappa_L^F : \rho|F \longrightarrow X$ such that for any $l \geq 0$, $(\kappa_L^F)_l$ maps $(x,y) \in (\rho|F)([l]) \subseteq X_l \times Y_{k+l+1}$ to $x \in X_l$. It's easy to check that for any $m, m' \geq 0$ and any $\phi : [m] \longrightarrow [m']$ in Δ we have $(\kappa_L^F)_m \circ (\rho|F)(\phi) = X(\phi) \circ (\kappa_L^F)_{m'}$. Therefore it is a well defined morphism of simplicial sets.

Example 2.2.3. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, similarly to Example 2.2.1. Let $\rho \in N\mathcal{D}([k])$ be defined as :

$$\rho_0 \xrightarrow{f_0} ... \xrightarrow{f_{k-1}} \rho_k$$

Then elements of $(\rho|NF)([l])$ correspond exactly to the subset of elements of $N\mathcal{D}_{k+l+1}$ of the following form :

$$\rho_0 \xrightarrow{f_0} ... \xrightarrow{f_{k-1}} \rho_k \xrightarrow{f_k} F(c_0) \xrightarrow{F(g_0)} ... \xrightarrow{F(g_{l-1})} F(c_l)$$

With morphisms acting on the chain at the right of f_k.

For a given $d \in Ob(\mathcal{D})$ let us construct the category $d\backslash F$ as in [Qui72]. Its objets are pairs (c,v) with $c \in Ob(\mathcal{C})$ and $v : d \longrightarrow F(c)$ a morphism in $\mathcal{D}$. Morphisms between (c,v) and (c',v') are morphism $f : c \longrightarrow c'$ in $\mathcal{C}$ that make the following diagram commute :

$$\begin{tikzcd} & F(c) \arrow[dd, "F(f)"] \\ d \arrow[ur, "F(v)"] \arrow[dr, "F(v')"'] & \\ & F(c') \end{tikzcd}$$

Elements of $N(d\backslash F)([l])$ correspond exactly to the subset of $N\mathcal{D}_{k+l+1}$ of elements of the following form :

$$d \xrightarrow{v_0} F(c_0) \xrightarrow{F(f_1)} F(c_1) \xrightarrow{F(f_2)} ... \xrightarrow{F(f_l)} F(c_l)$$

Where v_i for $i > 0$ are induced by composing arrows due to the compatibility conditions. As a result there is an obvious isomorphism of simplicial sets from $\rho|NF$ to $N(\rho_k\backslash F)$ obtained by forgetting the first k arrows and objects in an element of $(\rho|NF)([l])$. We immediately check that it is a bijection level-wise.

We now define a few common morphisms of simplicial sets related to our new constructions. They will be used in later theorems.

Definition 2.2.3. Let $F : X \longrightarrow Y$ be a morphism of simplicial set, we define
- A morphism of bisimplicial sets $\pi_L^F : Y|F \longrightarrow XR$ such that $(\pi_L^F)_{k,l}$ maps $(x, y) \in X_l \times Y_{k+l+1}$ to x.
- A morphism of bisimplicial sets $\pi_R^F : Y|F \longrightarrow YR$ such that $(\pi_R^F)_{k,l}$ maps $(x, y) \in X_l \times Y_{k+l+1}$ to $\mu_R^{k,l}(y)$.
- A morphism of bisimplicial sets $\iota^F : Y|F \longrightarrow YL$ such that $(\iota^F)_{k,l}$ maps $(x, y) \in X_l \times Y_{k+l+1}$ to $\mu_L^{k,l}(y)$.
- A morphism of simplicial set $\tau^F : Y|F \longrightarrow Y|Y$ such that $(\tau^F)_{k,l}$ maps $(x, y) \in X_l \times Y_{k+l+1}$ to y

The fact that π_L^F extends to a morphism of bisimplicial set is obvious considering that for any $\phi : [k'] \longrightarrow [k]$ and $\psi : [l'] \longrightarrow [l]$ function $XR(\phi, \psi)$ maps $x \in X_l$ to $X(\psi)(x)$. Similarly the fact that functions $(\tau^F)_{k,l}$ extend to a morphism of simplicial sets is because for any $\phi : [k'] \longrightarrow [k]$ and $\psi : [l'] \longrightarrow [l]$ function $(Y|F)(\phi, \psi)$ maps (x_k, y_{k+l+1}) to $(x, (\phi \bullet \psi)(y))$ and function $(Y|Y)(\phi, \psi)$ maps y_{k+l+1} to $(\phi \bullet \psi)(y)$. Moreover functions $(\pi_R^F)_{k,l}$ form a well-defined morphism of bisimplicial set because $\mu_R^{k,l} \circ \psi = (\phi \bullet \psi) \circ \mu_R^{k',l'}$ and functions $(\iota^F)_{k,l}$ form a well-defined morphism of bisimplicial set because $\mu_L^{k,l} \circ \phi = (\phi \bullet \psi) \circ \mu_L^{k',l'}$.

Proposition 2.2.1. *Let $F : X \longrightarrow Y$ be a morphism of simplicial set. The projection map $Y|F \xrightarrow{\pi_L^F} XR$ is a homotopy equivalence*

Proof. Consider the functor $G : \tilde{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ that maps $[A[l]]$ to the subset of elements $(x, y) \in X_l \times Y_{\mathrm{Card}(A)+l}$ such that $(\mu_R^{\mathrm{Card}(A)-1,l})^*(y) = F_l(x)$ if $A \neq \emptyset$ and $y = F_l(x)$ otherwise. For any $(x, y) \in G(A[l])$ remember that any $\phi : A'[l'] \longrightarrow A[l]$ corresponds to a function $\phi : [\mathrm{Card}(A) + l] \longrightarrow [\mathrm{Card}(A') + l]$ in Δ. Let $\phi^\#$ be as in Definition 2.1.7. Function $G(\phi)$ maps (x, y) to $(X(\phi)(x), Y(\psi^\#)(y))$. The image belongs to $G([A'[l'])$ because for all $l, l' \geq 0,\ A, A' \in \{\emptyset, [0], [1], ...\}$ we have

$$\begin{array}{ccccl} \phi \circ \mu_R^{\mathrm{Card}(A')-1,l'} & = & \mu_R^{\mathrm{Card}(A)-1,l} \circ \psi^\# & & \text{if } A' \neq \emptyset, A \neq \emptyset \\ \phi & = & \mu_R^{\mathrm{Card}(A)-1,l} \circ \psi^\# & & \text{if } A' = \emptyset, A \neq \emptyset \\ \phi \circ \circ \mu_R^{\mathrm{Card}(A')-1,l'} & = & \psi^\# & & \text{if } A' \neq \emptyset, A = \emptyset \\ \phi & = & \psi^\# & & \text{if } A' = \emptyset, A = \emptyset \end{array}$$

Because for the homotopy equivalence $(([k], [l]) \mapsto G([[k][l]])) \longrightarrow (([k], [l]) \mapsto G(\emptyset[l]))$ resulting from Lemma 2.1.2 is exactly the same as $\pi_L^F : (Y|F) \longrightarrow XR$ we are done. □

Corollary 2.2.1. *Let Y be a simplicial set. The map $Y|Y \xrightarrow{\iota^{id_Y}} YL$ given in Definition 2.2.3 is a homotopy equivalence.*

Proof. We apply Corollary 1.2.1 with $\pi_L^{\mathrm{id}_Y} : Y|Y \longrightarrow YR$ by observing that $(\pi_L^{\mathrm{id}_Y})' : (Y|Y)' \longrightarrow (YR)'$ is isomorphic to $\iota^{\mathrm{id}_Y} : Y|Y \longrightarrow YL$. Therefore we have the following commutative square

$$\begin{array}{ccc} |Y|Y| & \xrightarrow{(\pi_L^{\mathrm{id}_Y})^*} & |YR| \\ \downarrow\simeq & & \downarrow\simeq \\ |Y|Y| & \xrightarrow{(\iota^{\mathrm{id}_Y})^*} & |YL| \end{array}$$

and the fact that the top arrow is an homotopy equivalence per Proposition 2.2.1, we deduce that so is the bottom arrow and we are done. □

Lemma 2.2.1. *Given a simplicial set Y and an element $\rho \in Y_l$ for $l \geq 0$, then the simplicial set $\rho|Y$ is contractible.*

Proof. The result will follow from applying Corollary 2.1.1. Let $l \geq 0$ and $\rho \in Y_l$, we define $Z_* : \Delta_* \longrightarrow \mathrm{Set}$ where $Z_*([k])$ is the set of elements $y \in Y_{l+k}$ such that $\mu_L^{l,k-1}(y) = \rho$ if $k > 0$ and $y = \rho$ otherwise. For any $k', k \geq 0$ and any $\psi : [k'] \longrightarrow [k]$ in Δ_* we define $\psi^\blacktriangledown$ as in Definition 2.1.4. We define $Z_*(\psi)$ as the function that sends y to $Y(\psi^\blacktriangledown)(y)$. Because ψ belongs in Δ_* we have that $\psi^\blacktriangledown(l) = l$ and therefore

$$\begin{array}{rcll} \psi^\blacktriangledown \circ \mu_L^{l,k'-1} & = & \mu_L^{l,k-1} & \text{if } k' \neq 0, k \neq 0 \\ \psi^\blacktriangledown & = & \mu_L^{l,k-1} & \text{if } k' = 0, k \neq 0 \\ \psi^\blacktriangledown \circ \mu_L^{l,k'-1} & = & \mathrm{id}_{[k]} & \text{if } k' \neq 0, k = 0 \\ \psi^\blacktriangledown & = & \mathrm{id}_{[k]} & \text{if } k' = 0, k = 0 \end{array}$$

which means that $Y(\psi^\blacktriangledown)(y)$ does belong to $Z_*([k']) \subseteq Y_{l+k'}$. We now observe that $(\rho|Y)$ is isomorphic to $K^*(Z_*) : [k] \mapsto Z_*([0][k])$ and that $Z_*([0]) = \{\rho\}$. Applying Corollary 2.1.1 provides the desired result. □

Remark 2.2.6. Let Y be a simplicial set and $\rho \in Y_0$. We observe that the point $y \in |\rho|Y|$ to which we can contract $\rho|Y$ is in the fiber of $\{\rho\} \in |Y|$ by $|\kappa_R^{\mathrm{id}_Y}|$ where $\kappa_R^{\mathrm{id}_Y}$ is defined as in Remark 2.2.4. According to the proof of Lemma 2.2.1 the inclusion of $|\kappa_R^{\mathrm{id}_Y}|(y)$ in $|Y|$ is induced by

$$H : Z_*([0]) \longrightarrow K^*(Z_*) \xrightarrow[\kappa_R^{\mathrm{id}_Y}]{} X$$

We observe that H_l is a constant map towards $g_l(\rho)$ where $g_l : [l] \longrightarrow [0]$ and we deduce that the image of $|H|$ is $\{\rho\} \in |Y|$.

Theorem 2.2.1 (A'). *Let $F : X \longrightarrow Y$ be a morphism of simplicial sets. If for each $n \geq 0$ and $\rho \in Y_n$, the realization of $\rho|F$ is contractible, then F is an homotopy equivalence*

Proof. Using Corollary 1.2.2, we identify $X \longrightarrow Y$ with $XR \longrightarrow YR$. We have the following commutative diagram

$$\begin{tikzcd} YL \arrow[d, "id"] & Y|F \arrow[l, "\iota^F"'] \arrow[r, "\pi_R^F"'] \arrow[d, "\tau_F"] & XR \arrow[d, "F"] \\ YL & Y|Y \arrow[l, "\iota^{\mathrm{id}_Y}"] \arrow[r, "\pi_R^{\mathrm{id}_Y}"] & YR \end{tikzcd}$$

Where morphisms are defined as in Definition 2.2.3. Proposition 2.2.1 gives us that $\pi_R^{\mathrm{id}_Y}$ and π_R^F are homotopy equivalences. Corollary 2.2.1 gives us that ι^{id_Y} is. Now let $f := \iota^F$, we only need to show that $f : Y|F \longrightarrow YL$ is a homotopy equivalence, and given Theorem 1.2.1 that each restriction $f_k : Y|F([k][-]) \longrightarrow Y([k])$ is a homotopy equivalence of simplicial sets. Destination is a discrete space so we only have to prove that each fiber is contractible (and non-empty). Let $\rho \in Y([k])$ and $\tilde{\rho}$ be the discrete simplicial set with one element. We easily check that in sSet

$$\rho|F = \lim_{\leftarrow} \left\{ \begin{tikzcd} & Y|F([k],-) \arrow[d, "f_k"] \\ \tilde{\rho} \arrow[r] & Y([k]) \end{tikzcd} \right\}$$

Similarly in Top, the fiber of ρ for $|f_k|$ is

$$|f_k|^{-1}(\rho) = \lim_{\leftarrow} \left\{ \begin{tikzcd} & |Y|F([k],-)| \arrow[d, "|f_{[k]}|"] \\ \{\rho\} \arrow[r] & |Y([k])| \end{tikzcd} \right\}$$

Since $\{\rho\} \rightarrowtail |Y([k])|$ is a closed injection, the limits are the same in Top and CGHauss the subcategory of compactly generated spaces (see 10.9 in [Rez]). Therefore according to Proposition 1.1.1 the topological fiber coincides with its simplicial analogue which is contractible by hypothesis and we're done. $\square$

We notice that theorem A' is indeed a generalization of theorem A in [Qui72] since Example 2.2.3 shows that "each $d\backslash F$ is contractible" is equivalent to "each $(\rho|NF)$ is contractible".

2.2.2 Theorem B'

The original proof for Theorem B' in [GG87] was later corrected in [GG03]. We present the corrected version of the proof. It uses the notion of external product of bisimplicial set that we now define.

Definition 2.2.4. Let X and Y be simplicial sets we define their **external product** as the bisimplicial set $X \boxtimes Y$ that maps $([k],[l])$ to $X_k \times Y_l$.

Moreover there are two projections

$$X \boxtimes Y \xrightarrow{\xi_L^{X,Y}} XL$$

$$X \boxtimes Y \xrightarrow{\xi_R^{X,Y}} YR$$

In the proof of Theorem 2.2.2 we use the following lemma that we admit from [GG03]. Before presenting the lemma let Z be a bisimplicial set, Y a simplicial set and $F : Z \longrightarrow YL$ a morphism of bisimplicial sets. Moreover let $n \geq 0$ and $\rho \in X_n$ we denote Z_ρ to be the simplicial set such that $Z_\rho([k]) = \{x \in Z_{n,k}, F_{n,k}(x) = \rho\}$. We notice that any $f : [m] \longrightarrow [n]$ induces a morphism of simplicial sets $Z_\rho \longrightarrow Z_{f^*(\rho)}$ that maps each $x \in Z_{n,k}$ to $Z(f, \mathrm{id}_k)(x)$. Moreover there is a morphism of bisimplicial sets $\Delta_n \boxtimes Z_\rho \longrightarrow Z$ that maps $(g, x) \in (\Delta_n \boxtimes Z_\rho)([k],[l])$ to $Z(g, \mathrm{id}_l)(x)$.

Lemma 2.2.2. *Let Z, X and F be defined as above. Assuming that for each $n \geq 0$ each $\rho \in X_n$, and for each morphism $f : [m] \longrightarrow [n]$ in Δ we have that $Z_\rho \underset{f^*}{\longrightarrow} Z_{f^*\rho}$ is a homotopy equivalence. The following square commutes and is homotopy cartesian*

$$\begin{tikzcd} \Delta_n \boxtimes Z_\rho \arrow[r] \arrow[d, "\xi_L"] & Z \arrow[d] \\ \Delta_n L \arrow[r] & YL \end{tikzcd}$$

Proof. See [GG03] ☐

Theorem 2.2.2 (B'). *Let $F : X \longrightarrow Y$ be a morphism of simplicial sets. We assume that for any $n \geq 0$ and $\tau \in Y_n$ and any $\phi : [m] \longrightarrow [n]$ in Δ the induced morphism $\tau|F \longrightarrow \phi^*(\tau)|F$ is a homotopy equivalence. Then for any $l \geq 0$, $\rho \in X_l$, the following commutative square is homotopy cartesian.*

$$\begin{tikzcd} \rho|F \arrow[r, "\kappa_L^F"] \arrow[d, "\kappa_R^F"] & X \arrow[d, "F"] \\ \rho|Y \arrow[r, "\kappa_L^{id_Y}"] & Y \end{tikzcd}$$

where the top and bottom arrow are as described in Remark 2.2.5, and the left arrow corresponds to the one in Remark 2.2.4.

Proof. We notice that when $F : Z \longrightarrow YL$ is $\iota^F : Y|F \longrightarrow YL$ then for any $\rho \in Y_l$ we have that $Z_\rho = \rho|F$ and therefore by Lemma 2.2.2 the right square in the following diagram is homotopy cartesian.

$$\begin{tikzcd}
\Delta_n \boxtimes (\rho|F) \arrow[r] \arrow[d, "\mathrm{id}_{\Delta_n}\boxtimes\kappa_R^F"] \arrow[rr, bend left, "\mathrm{id}"] & Y|F \arrow[d, "\tau^F"] \arrow[rr, bend left, "\mathrm{id}"] & \Delta_n \boxtimes (\rho|F) \arrow[r] \arrow[d, "\xi_L"] & Y|F \arrow[d, "\iota^F"] \\
\Delta_n \boxtimes (y|Y) \arrow[r] \arrow[rr, bend right, "\xi_L"'] & Y|Y \arrow[rr, bend right, "\iota^{\mathrm{id}_Y}"'] & \Delta_n L \arrow[r] & YL
\end{tikzcd}$$

Because $y|Y$ is contractible we can check that for any $k \geq 0$ we have that $(\xi_L)_{-,k} : (\Delta_n \boxtimes (y|Y))(-,k) \longrightarrow \Delta_n L(-,k)$ is a homotopy equivalence and by Theorem 1.2.1 so is the morphism of bisimplicial sets. Moreover $\iota_L^{\mathrm{id}_Y}$ is an homotopy equivalence per Corollary 2.2.1. Therefore all horizontal arrows are homotopy equivalence and using the result in note 3.13.1 of [Gra79] we deduce that the left square is also homotopy cartesian. We apply the same reasoning to the following diagram

$$\begin{tikzcd}
\Delta_n \boxtimes (\rho|F) \arrow[r] \arrow[d, "\mathrm{id}_{\Delta_n}\boxtimes\kappa_R^F"] \arrow[rr, bend left, "\mathrm{id}"] & Y|F \arrow[d, "\tau^F"] \arrow[rr, bend left, "\pi_L^F"] & \Delta_n \boxtimes (\rho|F) \arrow[r, "\pi_L^F\circ\xi_R"] \arrow[d, "\mathrm{id}_{\Delta_n}\boxtimes\kappa_R^F"] & XR \arrow[d, "F"] \\
\Delta_n \boxtimes (y|Y) \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & Y|Y \arrow[rr, bend right, "\pi_L^{\mathrm{id}_Y}"'] & \Delta_n \boxtimes (\rho|Y) \arrow[r, "\kappa_L^{\mathrm{id}_Y}\circ\xi_R"] & YR
\end{tikzcd}$$

We use Proposition 2.2.1 to show that all horizontal arrows are weak equivalences. Then because the left square is homotopy cartesian so is the right square. By diagonalizing we show that the square given in the Theorem itself is homotopy catersian. □

Remark 2.2.7. Let $F : X \longrightarrow Y$ be a morphism of simplicial sets such that all the hypothesis in Theorem 2.2.2 are satisfied. We already know that $\rho|Y$ is contractible per Lemma 2.2.1. Therefore after applying the theorem we get that $|\rho|F|$ is weakly homotopy equivalent to the homotopy fiber of $|F|$ over a point in $|Y|$ and we have a long exact sequence

$$... \longrightarrow \pi_{i+1}(|Y|) \longrightarrow \pi_i(|\rho|F|) \longrightarrow \pi_i(|X|) \longrightarrow \pi_i(|Y|) \longrightarrow ...$$

Now we assume instead that $F : X \longrightarrow Y$ is a morphism of simplicial sets such that all the hypothesis in Theorem 2.2.1 are satisfied then the hypothesis in Theorem 2.2.2 are also satisfied. Furthermore from the long exact sequence above and the hypothesis that $\rho|F$ is contractible we deduce that F is a homotopy equivalence. This means that we can recover the result from Theorem 2.2.1 as a consequence of Theorem 2.2.2 and that Theorem 2.2.2 is a generalization of Theorem 2.2.1.

Remark 2.2.8. Following up from Example 2.2.1 and Example 2.2.3 let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Let $k \geq 0$ and $\rho \in N\mathcal{D}([k])$ be defined as :

$$\rho_0 \xrightarrow{f_0} \dots \xrightarrow{f_{k-1}} \rho_k$$

We have proven in Example 2.2.3 that $\rho|NF$ and $N(\rho_k \backslash F)$ are isomorphic. When describing the hypothesis of Theorem 2.2.2 we use that for any $l \geq 0$ and $\phi : [l] \longrightarrow [k]$ there is a induced morphism of simplicial set $\rho|NF \longrightarrow \phi^*(\rho)|NF$ that maps for any $m \geq 0$ element

$$\rho_0 \xrightarrow{f_0} \dots \xrightarrow{f_{k-1}} \rho_k \xrightarrow{f_k} F(c_0) \xrightarrow{F(g_0)} \dots \xrightarrow{F(g_{m-1})} F(c_m)$$

in $(\rho|NF)([m])$ to

$$\rho_{\phi(0)} \xrightarrow{f'_0} \dots \xrightarrow{f'_{l-1}} \rho_{\phi(l)} \xrightarrow{f'_l} F(c_0) \xrightarrow{F(g_0)} \dots \xrightarrow{F(g_{m-1})} F(c_m)$$

in $(\phi^*(\rho)|NF)([m])$. We set $g := f_{\phi(l)} \circ \dots \circ f_{k-1} : \rho_{\phi(l)} \longrightarrow \rho_k$. It induces a functor from $\rho_k \backslash F$ to $\rho_{\phi(l)} \backslash F$ and using the isomorphism from *Example* 2.2.3 we notice that the morphism $\rho|NF \longrightarrow \phi^*(\rho)|NF$ we just described corresponds exactly to the image of g by the nerve functor. The hypothesis for Theorem B in [Qui72] is that all such morphisms are homotopy equivalence. Therefore if F satisfies the hypothesis in Theorem B of [Qui72] is also satisfies the hypothesis of Theorem 2.2.2.

Moreover we check that $\kappa_L^{NF} : \rho|N\mathcal{F} \longrightarrow N\mathcal{C}$ corresponds to the nerve of functor $\rho_k \backslash F \longrightarrow \mathcal{C}$ described in [Qui72]. We also check that $\kappa_R^{NF} : \rho|N\mathcal{F} \longrightarrow \rho|N\mathcal{D}$ corresponds to the nerve of $\rho_k \backslash F \longrightarrow \rho_k \backslash \mathcal{D}$ described in [Qui72] and that $\kappa_L^{\mathrm{id}_{N\mathcal{D}}} : \rho|N\mathcal{D} \longrightarrow N\mathcal{D}$ corresponds to the nerve of $\rho_k \backslash \mathcal{D} \longrightarrow \mathcal{D}$ described in [Qui72]. Therefore applying Theorem 2.2.2 to NF provides the same results as Theorem B from [Qui72]. Hence Theorem 2.2.2 is a generalization of Theorem B in [Qui72].

3 The K-Theory of a Waldhausen category

3.1 Waldhausen category and the $S.$ construction

In [Wal85], Waldhausen defines the notion of a “category with cofibrations and weak equivalences”, which are often called Waldhausen categories. In the context of [GG87], it will be easier to understand them as a generalization of exact categories. Walhdausen also provides in [Wal85] the eponym categories with a K-Theory, and proves that the corresponding space is homotopy equivalent the Q-construction in the case of an exact category. Let’s first define the notion of a Waldhausen category.

Definition 3.1.1. A **Waldhausen** category is the data of a category $\mathcal{C}$ that admits a zero object $*$, along with subcategories $co\mathcal{C}$ and $w\mathcal{C}$ whose morphism will be referred to as respectively **cofibrations** and **weak equivalences**. They must respect the following properties

1. Isomorphisms in $\mathcal{C}$ are cofibrations and weak equivalences
2. Every $* \longrightarrow A$ is a cofibration
3. Let $A \rightarrowtail B$ be a cofibration and $A \longrightarrow C$ be any morphism in $\mathcal{C}$ then the pushout $B \cup_A C$ exists in $\mathcal{C}$ and $C \rightarrowtail B \cup_A C$ is a cofibration.
4. *(Glueing lemma)* Given the commutative diagram

$$\begin{tikzcd} B \arrow[d, "\sim"] & A \arrow[l, tail] \arrow[r] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l, tail] \arrow[r] & C' \end{tikzcd}$$

where $A \rightarrowtail B$ and $A' \rightarrowtail B'$ are cofibrations and vertical arrows are weak equivalences then the induced morphism $B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$ is a weak equivalence.

Because of the first axiom we notice that all objects in $\mathcal{C}$ belong to the subcategories $co\mathcal{C}$ and $w\mathcal{C}$ and that the data of a Waldhausen category on $\mathcal{C}$ corresponds exactly to a family of cofibrations and weak equivalences compatible with the axioms.

Example 3.1.1. An exact category $\mathcal{M}$ is canonically a Waldhausen category when we consider admissible monomorphisms as cofibrations and isomorphisms as weak equivalences.

Given any cofibration $A \rightarrowtail B$, we say that $A \rightarrowtail B \twoheadrightarrow C$ is a cofibration sequence if $C \simeq (B \cup_A *)$ and we can identify $B \twoheadrightarrow C$ with $B \longrightarrow (B \cup_A *)$. In the context of Example 3.1.1 cofibration sequences correspond exactly to short exact sequences. In this document we will sometimes refer to C as a "quotient" for $A \rightarrowtail B$ when $A \rightarrowtail B \twoheadrightarrow C$ is a cofibration sequence. Cofibration sequences must not be confused with sequence of cofibrations which simply denote sequence of morphisms in $co\mathcal{C}$.

In [Wal85], Waldhausen associates to any Waldhausen category $\mathcal{C}$ a simplicial category $wS.\mathcal{C}$, defined as follow

Definition 3.1.2. Let $\mathcal{C}$ be any Waldhausen category, we define a simplicial category $wS.\mathcal{C} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}$. Objects in $wS.\mathcal{C}([n])$ are functors

$$\begin{array}{rccc} M : & \mathrm{Ar}([n]) & \longrightarrow & \mathcal{C} \\ & (i \leq j) & \longmapsto & M_{i,j} \end{array}$$

such that :

$$\begin{aligned} M_{i,i} &= * && \forall\, 0 \leq i \leq n \\ M_{i,j} \rightarrowtail M_{j,k} &\text{ is a cofibration} && \forall\, 0 \leq i \leq j \leq k \leq n \\ M_{i,j} \rightarrowtail M_{i,k} \twoheadrightarrow M_{j,k} &\text{ is a cofibration sequence} && \forall\, 0 \leq i \leq j \leq k \leq n \end{aligned}$$

Morphisms in $wS.\mathcal{C}([n])$ are natural transformations between the corresponding functors, in which all components are weak equivalences. Given any $\phi : [m] \longrightarrow [n]$ functor ϕ^* maps an object M in $wS.\mathcal{C}([n])$ to the composed functor $\phi^*(M) : \mathrm{Ar}([m]) \xrightarrow{\mathrm{Ar}(\phi)} \mathrm{Ar}([n]) \xrightarrow{M} \mathcal{C}$. We can check it corresponds to an object of $wS.\mathcal{M}([m])$. The functor ϕ^* also maps a morphism between objects M and M' in of $wS.\mathcal{M}([m])$ to the unique natural transformation between $\phi^*(M)$ and $\phi^*(M')$ induced by the universal property of quotients. We verify that all components are weak equivalences because of the Glueing lemma and thus it is indeed a morphism in $wS.\mathcal{M}([m])$).

Remark 3.1.1. Because a composition of weak equivalences is always a weak equivalence and identity isomorphisms are weak equivalences, $wS.\mathcal{C}([n])$ does indeed form a category for all $n \geq 0$. Objects can also be understood as sequences of cofibrations in $\mathcal{C}$ of the form

$$M_1 \rightarrowtail \dots \rightarrowtail M_n$$

along with a choice of quotient $M_{j,i}$ for M_i/M_j for all $0 < j < i \leq n$. We also denote by convention $M_{i,i} = *$ and $M_{0,i} = M_i$ for all $0 \leq i \leq n$ which also means $M_0 = *$. A natural transformation between M and M' two elements of $wS.\mathcal{C}([n])$ is exactly a family of morphisms $M_i \xrightarrow{\phi_i} M'_i$ for all $0 \leq i \leq n$ that makes the following diagram commute.

$$\begin{tikzcd} M_1 \arrow[r, tail] \arrow[d, "\phi_1"] & M_2 \arrow[r, tail] \arrow[d, "\phi_2"] & \dots \arrow[r, tail] & M_n \arrow[d, "\phi_n"] \\ M'_1 \arrow[r, tail] & M'_2 \arrow[r, tail] & \dots \arrow[r, tail] & M'_n \end{tikzcd}$$

This is because the universal property of the quotient ensure it extends uniquely to a family of morphism $\phi_{i,j} : M_{i,j} \longrightarrow M'_{i,j}$ for all $0 \leq i \leq j \leq n$. Those also form a natural transformation between the corresponding functors. Moreover we only need ϕ_i for all $0 < i \leq n$ to be weak equivalences for *all* the components of the natural transformation to be weak equivalences. This is because for all $0 \leq i \leq j \leq n$, one can apply the gluing property to the following diagram

$$\begin{tikzcd} * \arrow[d, "\sim"] & M_i \arrow[l] \arrow[r, tail] \arrow[d, "\sim"] & M_j \arrow[d, "\sim"] \\ * & M'_i \arrow[l] \arrow[r, tail] & M'_j \end{tikzcd}$$

to show that the induced morphism $M_{i,j} \longrightarrow M'_{i,j}$ will also be a weak equivalence. Given $\psi : [m] \longrightarrow [n]$ in Δ then $\psi^* : wS.\mathcal{C}([n]) \longrightarrow wS.\mathcal{C}([m])$ is the functor that maps morphism

$$\begin{tikzcd}
M_1 \arrow[r, tail] \arrow[d, "\phi_1"] & M_2 \arrow[r, tail] \arrow[d, "\phi_2"] & ... \arrow[r, tail] & M_n \arrow[d, "\phi_n"] \\
M'_1 \arrow[r, tail] & M'_2 \arrow[r, tail] & ... \arrow[r, tail] & M'_n
\end{tikzcd}$$

to

$$\begin{tikzcd}
M_{\psi(1)}/M_{\psi(0)} \arrow[r, tail] \arrow[d, "\overline{\phi_{\psi(1)}}"] & M_{\psi(1)}/M_{\psi(0)} \arrow[r, tail] \arrow[d, "\overline{\phi_{\psi(2)}}"] & ... \arrow[r, tail] & M_{\psi(n)}/M_{\psi(0)} \arrow[d, "\overline{\phi_{\psi(n)}}"] \\
M'_{\psi(1)}/M'_{\psi(0)} \arrow[r, tail] & M'_{\psi(2)}/M'_{\psi(0)} \arrow[r, tail] & ... \arrow[r, tail] & M'_{\psi(n)}/M'_{\psi(0)}
\end{tikzcd}$$

where for all $0 \leq k \leq n$ the choice of quotient for $M_{\psi(k)}/M_{\psi(0)}$ is $M_{\psi(0),\psi(k)}$ (and similarly for M'), and $\overline{\phi_{\psi(k)}}$ is the unique morphism induced by universal property.

Remark 3.1.2. Let $\mathcal{C}$ be a Waldhausen category. In Remark 3.1.1 we identified objects of $wS.\mathcal{C}([n])$ with sequences of cofibrations of n objects along with a choice of quotients for each possible composed cofibration. We need to keep in mind that the choice of quotient is important. To illustrate that we first set an arbitrary choice of quotient $Q_{N,M}$ for every cofibration $N \rightarrowtail M$ in $\mathcal{C}$ and we make sure to set $Q_{*,M}$ to M. Then we construct a simplicial object $(wS.\mathcal{C})'$ very similar to $(wS.\mathcal{C})$. Category $(wS.\mathcal{C})'([n])$ is such that an object M is a sequence of cofibrations with n objects *without* choices of quotients and a morphism Φ between objects is a family of weak equivalences defined exactly as in Remark 3.1.1. Let ϕ be any morphism in Δ its image $(wS.\mathcal{C})'(\phi)$ is defined exactly as $wS.\mathcal{C}(\phi)$ is defined in Remark 3.1.1. We define two morphism of simplicial categories $F : wS\mathcal{C} \longrightarrow (wS\mathcal{C})'$ and $G : (wS\mathcal{C})' \longrightarrow wS\mathcal{C}$ such that

$$\begin{array}{cccc} F_n : & wS\mathcal{C}([n]) & \longrightarrow & (wS\mathcal{C}([n]))' \\ & A \in \mathrm{Ob}(wS\mathcal{C}([n])) & \longmapsto & A_{0,1} \rightarrowtail ... \rightarrowtail A_{0,n} \in \mathrm{Ob}((wS\mathcal{C})'([n])) \end{array}$$

and

$$\begin{array}{cccc} G_n : & (wS\mathcal{C})'([n]) & \longrightarrow & wS\mathcal{C}([n]) \\ & M_1 \rightarrowtail ... \rightarrowtail M_n \in \mathrm{Ob}((wS\mathcal{C})'([n])) & \longmapsto & (i < j) \mapsto Q_{M_i,M_j} \in \mathrm{Ob}(wS\mathcal{C}([n])) \end{array}$$

where in G we have implicitely set $M_0 := *$ and where the action on morphisms is induced by universal properties. Because we have set $Q_{*,M}$ to be M we notice that $F \circ G = \mathrm{id}_{(wS\mathcal{C})'}$. However $G_n \circ F_n$ will map an object M of $(wS\mathcal{C})([n])$ to a functor naturally isomorphic to M but not equal in general, because the choice of quotient differs. Assuming there are two different choices of quotient for a monomorphism $M \rightarrowtail N$ in $\mathcal{C}$ they each correspond to different objects of $wS.\mathcal{C}([2])$ that will have the same image through F. This means that F and G need not be isomorphism of simplicial objects in general and we need to keep the distinction between $wS.\mathcal{C}$ and $(wS.\mathcal{C})'$. However one can check that for all n functors F_n and G_n form an equivalence between the two categories.

3.2 Bisimplicial sets and simplicial categories

When a simplicial set X is the nerve of a category $\mathcal{C}$, the set X_0 corresponds to objects of $\mathcal{C}$, and X_1 corresponds to morphisms or arrows of $\mathcal{C}$. Because the nerve functor is essentially injective (see Lemma 1.1.1) we often don't distinguish between a simplicial set in the image of the nerve functor and the corresponding category. We also sometimes think as simplicial sets outside the essential image of the nerve functor as some kind of ill formed categories and want to consider X_0 and X_1 as respectively their "objects" and "arrows".
Similarly, a simplicial category Y can be understood – uniquely up to isomorphism – as a bisimplicial set by post-composing with the nerve functor. In that case $Y_{m,0}$ and $Y_{m,1}$ correspond to objects and arrows of category Y_m. Using similar argument we also don't distinguish between such a bisimplicial set and the corresponding simplicial category and trying to think of *any* bisimplicial set as an ill-formed simplicial category leads to the following definition.

Definition 3.2.1. We define two functors from the category of bisimplicial sets to sSet

$$\mathrm{Obj} : X \mapsto ([m] \mapsto X([m],[0]))$$

$$\mathrm{Arr} : X \mapsto ([m] \mapsto X([m],[1]))$$

And we name s and t the two natural transformations from Arr to Obj whose components on X are such that $(s_X)_m = (id_m, d_0)^*$ and $(t_X)_m = (id_m, d_1)^*$.

When a bisimplicial set X is the image by the nerve functor of the simplicial category $[n] \mapsto X_n$, $\mathrm{Obj}(X)$ is the simplicial set such that $\mathrm{Obj}(X)([n]) = \mathrm{Ob}(X_n)$ and $\mathrm{Arr}(X)$ is the simplicial set such that $\mathrm{Arr}(X)([n]) = \mathrm{Mor}(X_n)$. The contravariant images of a morphism ϕ in Δ in Arr and Obj are induced by the action of functor ϕ^*, respectively on objects and morphisms. Moreover s_X^n and t_X^n map respectively an arrow of $\mathrm{Mor}(X_n)$ to its source and target.

Example 3.2.1. Let $\mathcal{C}$ be a Waldhausen category. After composing with the nerve functor, simplicial object $wS.\mathcal{C}$ corresponds to a bisimplicial set that maps $[m],[n]$ to mappings of co-fibration sequences corresponding to commutative diagrams of the following form

$$\begin{tikzcd}
M_1^0 \arrow[r, tail] \arrow[d, "\sim"] & M_2^0 \arrow[r, tail] \arrow[d, "\sim"] & \cdots \arrow[r, tail] & M_n^0 \arrow[d, "\sim"] \\
M_1^1 \arrow[r, tail] \arrow[d, "\sim"] & M_2^1 \arrow[r, tail] \arrow[d, "\sim"] & \cdots \arrow[r, tail] & M_n^1 \arrow[d, "\sim"] \\
\vdots \arrow[d, "\sim"] & \vdots \arrow[d, "\sim"] & \ddots & \vdots \arrow[d, "\sim"] \\
M_1^m \arrow[r, tail] & M_2^m \arrow[r, tail] & \cdots \arrow[r, tail] & M_n^m
\end{tikzcd}$$

along with choices of quotient for each $M^i_j/M^i_{j'}$. In that context, vertical face and degeneracy maps are induced by face and degeneracy maps of the nerve functor on $wS.\mathcal{C}([n])$ as described in Definition 1.1.6. They amount to respectively forgetting a sequence of cofibration and composing the remaining morphisms or duplicating a sequence of cofibrations. Horizontal face and degeneracy maps are induced by face and degeneracy map on the simplicial object as described in Remark 3.1.1. They amount to respectively forgetting or duplicating an element i in each row and taking the obvious composed cofibrations and choices of quotients.

Remark 3.2.1. Let $\mathcal{C}$ be a Waldhausen category. Up to isomorphism, $wS.\mathcal{C}$ can be understood as the bisimplicial set

$$\begin{array}{cccc} M: & \mathrm{Ar}([n]) \times [m] & \to & \mathcal{C} \\ & (i<j,k) & \mapsto & M^k_{i,j} \end{array}$$

that respects the following conditions

a. For all $0 \leq i \leq n$ and $0 \leq l \leq m$, $M^l_{i,i} = 0$

b. For all $0 \leq i \leq j \leq k \leq n$ and $0 \leq l \leq m$, $M^l_{i,j} \rightarrowtail M^l_{i,k} \twoheadrightarrow M^l_{j,k}$ is a cofibration sequence as defined in [Wal85]

c. For all $0 \leq i \leq j \leq n$ and $0 \leq l \leq l' \leq m$, $M^l_{i,j} \xrightarrow{\sim} M^{l'}_{i,j}$ is a weak equivalence

This shows in particular that when $\mathcal{M}$ is an exact category, $wS.\mathcal{M}$ is exactly the same as $S\mathcal{M}^{Is}$ defined in [GG87] p.585.

This previous example is essential because Waldhausen's S-construction as defined in [Wal85] is exactly the bisimplicial set corresponding to $wS.\mathcal{C}$.

Definition 3.2.2. Let $\mathcal{C}$ be a Waldhausen category, the **S-construction** of $\mathcal{C}$ is the bisimplicial set $wS.\mathcal{C}$ and for all $i \geq 0$ the i-th K-group is the $(i+1)$-th homotopy group of the space $|wS.\mathcal{C}|$.

Example 3.2.2. When $\mathcal{C}$ is a Waldhausen category where all weak equivalences are isomorphisms, the simplicial category $wS.\mathcal{C}$ defined in Definition 3.1.2 is a **simplicial groupoid**, meaning a simplicial object in the category of groupoids. This is because for all $n \geq 0$, category $wS.\mathcal{C}([n])$ is a groupoid where the inverse of a morphism between sequences of cofibration is the morphism defined by taking each inverse element-wise.

Definition 3.2.3. Given an exact category $\mathcal{M}$, seen as a Waldhausen category in the canonical way. We will denote by $iS.\mathcal{M}$ the simplicial category $wS.\mathcal{M}$ given in Definition 3.1.2. The above example justifies it is actually a simplicial *groupoid.*

Definition 3.2.4. Let $\mathcal{C}$ be an exact category and $iS\mathcal{M}$ be defined as in Definition 3.2.3, we will define a simplicial set $S\mathcal{C} := \mathrm{Obj}(iS.\mathcal{C})$. Explicitely, elements in $S\mathcal{M}([n])$ are functors

$$\begin{array}{ccc} \mathrm{Ar}([n]) & \longrightarrow & \mathcal{C} \\ i < j & \longmapsto & A_{i,j} \end{array}$$

such that :

$$A_{i,i} = 0$$

$$A_{i,j} \rightarrowtail A_{i,k} \twoheadrightarrow A_{j,k} \quad \text{is exact} \quad \forall i \leq j \leq k$$

The contravariant image of $\phi : [m] \longrightarrow [n]$ is given by precomposing by $\mathrm{Ar}([m]) \xrightarrow{\mathrm{Ar}(\phi)} \mathrm{Ar}([n])$. Explicitly function ϕ^* takes an element $A \in S\mathcal{M}([n])$ to $A' \in S\mathcal{M}([m]) : i < j \longrightarrow A_{\phi(i),\phi(j)}$.

We observe that when $\mathcal{M}$ is an exact category seen as a Waldhausen category in the canonical way, then $S\mathcal{M}$ is exactly the same as the simplicial set defined in [GG87] with the same notation. In [Wal85], $S\mathcal{M}$ is referred to as $s\mathcal{M}$.

Remark 3.2.2. Let $\mathcal{C}$ be a Waldhausen category, Similarly to how we characterized $wS.\mathcal{C}$ in Remark 3.1.1, we can understand an element of $M \in S\mathcal{C}([n])$ as a chain of monomorphisms

$$M_0 = 0 \rightarrowtail M_1 \rightarrowtail \ldots \rightarrowtail M_n$$

along with a choice of quotient $M_{i,j} = M_j/M_i$ for all $0 < i < j \leq n$. Given any $\phi : [m] \longrightarrow [n]$ function ϕ^* maps M to $N \in S\mathcal{C}([m])$ defined as :

$$0 \rightarrowtail N_1 = M_{\phi(1)}/M_{\phi(0)} \rightarrowtail \ldots \rightarrowtail N_m = M_{\phi(m)}/M_{\phi(0)}$$

Where the choice of quotient for N_j/N_i is $M_{\phi(i),\phi(j)}$.

Remark 3.2.3. Let $\mathcal{C}$ be a Waldhausen category. Similarly to what we said in Remark 3.1.2 we want to remember that the choice of quotient for each $M \in S\mathcal{C}([n])$ is important. Applying Obj to functor F and G defined in Remark 3.1.2 provides morphisms of simplicial sets between $S\mathcal{C}$ and $\mathrm{Ob}((wS\mathcal{C})')$ the category in which choices of quotient have been forgotten, and results from the previous remark already showed that they are not necessarily isomorphisms.

Moreover in [Wal85] Waldhausen provides the following proposition.

Proposition 3.2.1. *Let $\mathcal{M}$ be an exact category, $iS.\mathcal{M}$ is homotopy equivalent to $S\mathcal{M}$.*

Proof. See [Wal85], Corollary (2) of lemma 1.4.1. □

This means that in the context of exact categories $S\mathcal{M}$ is a valid simplicial set to consider for the S-construction. Moreover Waldhausen proves in [Wal85] section 1.9 that for an exact category $\mathcal{M}$ both constructions are homotopy equivalent to the classifying space of the Q-construction.

4 The G-construction [GG87] as a construction for the K-Theory

In this section we attempt to describe the G-construction of an exact category and prove the main theorem of [GG87] which states that the realization of the G-construction is homotopy equivalent to the loop space of the Q-construction.

4.1 The simplicial loop construction

In order to define the G-construction we first need to construct a simplicial analogue of the loop space.

Definition 4.1.1. Let X be a simplicial set and $x_0 \in X_0$. We define the **simplicial loop space** of X with base point x_0 as the simplicial set ΩX such that

$$\Omega X([n]) := \lim_{\leftarrow} \left\{ \begin{tikzcd} \{x_0\} \arrow[r] & X([0]) & X([0][n]) \arrow[l, "X(f_n)"'] \arrow[d, "X(g_n)"] \\ & X([0][n]) \arrow[u, "X(f_n)"] \arrow[r, "X(g_n)"] & X([n]) \end{tikzcd} \right\}$$

Where $f_n = \mu_L^{0,n}$ and $g_n = \mu_R^{0,n}$

Remark 4.1.1. Elements of $\Omega X([n])$ correspond exactly to the subset of pairs $(x, y) \in X_{n+1} \times X_{n+1}$ such that $X(g_n)(x) = X(g_n)(y)$ and $X(f_n)(x) = X(f_n)(y) = x_0$. For any $m, n \geq 0$ and $\phi : [m] \longrightarrow [n]$ let $\phi^+ := (\mathrm{id}_0 \bullet \phi)$, then $\Omega(\phi)$ is the function that maps $(x, y) \in X_{n+1} \times X_{n+1}$ to $(X(\phi^+)(x), X(\phi^+)(y)) \in X_{m+1} \times X_{m+1}$. Because $\phi^+ \circ f_m = f_n$ and $\phi^+ \circ g_m = g_n \circ \phi$, we have $X(f_m) \circ X(\phi^+)(x) = X(f_n)(x) = x_0$, $X(f_m) \circ X(\phi^+)(y) = X(f_n)(y) = x_0$ and $X(g_m) \circ X(\phi^+)(x) = X(\phi) \circ X(g_n)(x) = X(\phi) \circ X(g_n)(y) = X(g_m) \circ X(\phi^+)(y)$. The image is indeed in $\Omega X([m]) \subseteq X_{m+1} \times X_{m+1}$.

Moreover we check that this construction is functorial in the sense that a morphism of simplicial sets $F : X \longrightarrow Y$ induces a morphism between their simplicial loops $\Omega F : \Omega X \longrightarrow \Omega Y$ in a way compatible with compositions. Morphism ΩF is such that function $\Omega F_n : \Omega X_n \longrightarrow \Omega Y_n$ maps $(x, y) \in (\Omega X)([n])$ to $(F_{n+1}(x), F_{n+1}(y)) \in Y_{n+1} \times Y_{n+1}$. Because F is a morphism of simplicial set we have $F_0 \circ X(f_n) = Y(f_n) \circ F_{n+1}$ and $F_n \circ X(g_n) = Y(g_n) \circ F_{n+1}$ which proves in particular that the image is in $(\Omega Y)([n])$ for base point $F_0(x_0)$. Moreover for any $m, n \geq 0$ and $\phi : [m] \longrightarrow [n]$ the fact that $F_{m+1} \circ X(\phi^+) = Y(\phi^+) \circ F_{n+1}$ implies that $(\Omega F)_m \circ (\Omega X)(\phi) = (\Omega Y)(\phi) \circ (\Omega F)_n$ which in turns proves that ΩF is a well-defined morphism of simplicial sets.

Remark 4.1.2. Let X be a simplicial set, let $x_0 \in X_0$ and denote ΩX the simplicial set with basepoint x_0. Using results from Remark 1.1.3 we notice that an element $(x_1, y_1) \in (\Omega X)([0])$ yields pairs of paths $(\gamma_{x_1}, \gamma_{y_1})$ in $|X|$, both starting at the base point and ending at some common but arbitrary point $x_0' \in X_0$ as illustrated in Diagram 4.1.1.

Diagram 4.1.1.

$$\begin{tikzcd}
x_0 \arrow[r, bend left=60, "\gamma_{y_1}"] \arrow[r, bend right=60, "\gamma_{x_1}"'] & x_0'
\end{tikzcd}$$

Concatenating path γ_{x_1} with the opposite of γ_{y_1} provides a loop with base point x_0. We use that to associate to a point in $|(\Omega X)([0])| \subseteq |\Omega X|$ a point in $\Omega|X|$. Moreover let two element $(x_1, y_1), (x_1', y_1') \in (\Omega X)([0])$ be path connected. It means there is $(x_2, y_2) \in (\Omega X)([1])$ such that $(\Omega X)(d_0)((x_2, y_2)) = (x_1, y_1)$ and $(\Omega X)(d_1)((x_2, y_2)) = (x_1', y_1')$ which means that

$$\begin{aligned}
X(d_1)(x_2) &= x_1 \\
X(d_2)(x_2) &= x_1' \\
X(d_1)(y_2) &= y_1 \\
X(d_2)(x_2) &= y_1' \\
X(d_0)(x_2) &= X(d_0)(y_2)
\end{aligned}$$

If we denote $h := X(d_0)(x_2) = X(d_0)(y_2)$, $x_0' := X(d_1)(x_1)$ and $x_0'' := X(d_1)(x_1')$ we have paths in $|X|$ as represented in the following diagram

Diagram 4.1.2.

$$\begin{tikzcd}
x_0 \arrow[r, bend left=50, "\gamma_{y_1'}"] \arrow[r, bend right=50, "\gamma_{x_1'}"'] \arrow[rr, bend left=70, "\gamma_{y_1}"] \arrow[rr, bend right=70, "\gamma_{x_1}"'] & x_0'' \arrow[r, "\gamma_h"] & x_0'
\end{tikzcd}$$

Moreover using Remark 1.1.3 we notice that it implies that in $\pi_1(|X|)$ we have $[\gamma_{x_1'}][\gamma_h] = [\gamma_{x_1}]$ and $[\gamma_{y_1'}][\gamma_h] = [-\gamma_{y_1}]$. Therefore $[\gamma_{x_1}][-\gamma_{y_1}] = [\gamma_{x_1'}][\gamma_h][-\gamma_h][-\gamma_{y_1'}] = [\gamma_{x_1'}][-\gamma_{y_1'}]$. This means that the loops associated with (x_1, y_1) and (x_1', y_1') are in the same homotopy class and the corresponding points in $\Omega(|X|)$ are path connected. In particular using the map $|(\Omega X)([0])| \longrightarrow \Omega|X|$ defined above we can construct a map from $\pi_0(|\Omega(X)|)$ to $\pi_0(\Omega(|X|))$.

We know try to define a more precise relation between the simplicial and the topological loop space. First we recall that for X a simplicial set and any base point $x_0 \in X_0$, the realization of $x_0|X \xrightarrow{\kappa_L^{\mathrm{id}_Y}} X$ described in Remark 2.2.5 is homotopy equivalent to $\{x_0\} \longrightarrow |X|$ the inclusion of the point of $|X|$ corresponding to x_0.

Definition 4.1.2. Let X be a simplicial set. Given $\rho \in X_0$, $\tau \in X_n$, let P denote the morphism $\kappa_L^{\mathrm{id}_Y} : \rho|X \longrightarrow X$ described in Remark 2.2.5. We denote $(\rho, \tau|X) := \tau|P$. Explicitely, we have

$$(\rho,\tau|X)([k]) := \lim_{\leftarrow} \left\{ \begin{tikzcd} & & & \{\rho\} \arrow[d] \\ & & X([0][k]) \arrow[r, "(\mu_L^{0,k})^*"] \arrow[d, "(\mu_R^{0,k})^*"] & X([0]) \\ & X([n][k]) \arrow[r, "(\mu_R^{n,k})^*"] \arrow[d, "(\mu_L^{n,k})^*"] & X([k]) & \\ \{\tau\} \arrow[r] & X([n]) & & \end{tikzcd} \right\}$$

Remark 4.1.3. Let X be a simplicial set, $n \geq 0$, $\rho \in X_0$ and $\tau \in X_n$. Elements of $(\rho, \tau|X)([k])$ correspond exactly to the subset of elements $(x, y) \in X_{k+1} \times X_{k+n+1}$ such that

$$\begin{aligned} X(\mu_L^{n,k})(y) &= \tau \\ X(\mu_L^{0,k})(x) &= \rho \\ X(\mu_R^{0,k})(x) &= (\mu_R^{n,k})^*(y) \end{aligned}$$

Moroever for any $l, k \geq 0$ and any $\phi : [l] \longrightarrow [k]$ function $(\rho, \tau|X)(\phi)$ maps (x, y) to $(X(\mathrm{id}_0 \bullet \phi)(x), X(\mathrm{id}_n \bullet \phi)(y))$. The image does belong to $(\rho, \tau|X)([l])$ because we have

$$\begin{aligned} (\mathrm{id}_n \bullet \phi) \circ (\mu_L^{n,l}) &= (\mu_L^{n,k}) \\ (\mathrm{id}_0 \bullet \phi) \circ (\mu_L^{0,l}) &= (\mu_L^{0,k}) \\ (\mathrm{id}_0 \bullet \phi) \circ (\mu_R^{0,l}) &= (\mu_R^{0,k}) \circ \phi \\ (\mathrm{id}_n \bullet \phi) \circ (\mu_R^{n,l}) &= (\mu_R^{n,k}) \circ \phi \end{aligned}$$

which in turn implies

$$\begin{aligned} & X(\mu_L^{n,l}) \circ X(\mathrm{id}_n \bullet \phi)(y) = X(\mu_L^{n,k})(y) = \tau \\ & X(\mu_L^{0,l}) \circ X(\mathrm{id}_0 \bullet \phi)(y) = X(\mu_L^{0,k})(y) = \rho \\ & X(\mu_R^{0,l}) \circ X(\mathrm{id}_0 \bullet \phi)(x) = X(\phi) \circ X(\mu_R^{0,k})(x) = X(\phi) \circ X(\mu_R^{n,k})(y) = X(\mu_R^{n,l}) \circ X(\mathrm{id}_n \bullet \phi)(y) \end{aligned}$$

Remark 4.1.4. Let X be a simplicial set, $n \geq 0$, $\rho \in X_0$ and $\tau \in X_n$. We construct two morphisms of simplicial sets $\zeta^L : (\rho, \tau|X) \longrightarrow \rho|X$ and $\zeta^R : (\rho, \tau|X) \longrightarrow \tau|X$. Morphism ζ^L is such that ζ_k^L maps $(x, y) \in (\rho, \tau|X)([k])$ to x. Morphism ζ^R is such that ζ_k^R maps $(x, y) \in (\rho, \tau|X)([k])$ to y. The fact that

x and y respectively belong to $(\rho|X)([k]) \subseteq X_{k+1}$ and $(\tau|X)([k]) \subseteq X_{k+n+1}$, and that the ζ^R and ζ^L are well-defined morphism of simplicial set is a direct consequence of the characterization in Remark 2.2.3.

We now express the main ingredient for proving that the realization of $G\mathcal{M}$ is homotopy equivalent to the K-Theory of $\mathcal{M}$. It is a Corollary to Theorem 2.2.2.

Corollary 4.1.1. *Let X be a simplicial set and $x_0 \in X_0$. Let ΩX be the simplicial loop space of X with base point x_0 and $\Omega|X|$ be the topological loop space of $|X|$ with base point the point corresponding to x_0 in $|X|$.*

There is a canonical morphism $|\Omega X| \longrightarrow \Omega|X|$ and it is it is an homotopy equivalence if for all $\phi : [m] \longrightarrow [n]$ in Δ and $\rho \in X_n$ morphisms $(x_0, \rho|X) \longrightarrow (x_0, \phi^*(\rho)|X)$ are homotopy equivalences.

Proof. We first observe that $\Omega X = (x_0, x_0|X)$. Moreover when considering ζ^R and ζ^L as in Remark 4.1.3 and $P := \kappa_L^{\mathrm{id}_X}$ as in Remark 2.2.5 we have the following commutative diagram of simplicial sets

Diagram 4.1.3.

$$\begin{tikzcd} (x_0, x_0|X) \arrow[r, "\zeta^L"] \arrow[d, "\zeta^R"] & x_0|X \arrow[d, "P"] \\ x_0|X \arrow[r, "P"] & X \end{tikzcd}$$

Remember that $\Omega|X|$ the loop space with base point x_0 is the homotopy pullback of

$$\{x_0\} \to X \leftarrow \{x_0\}$$

Moreover we know by Lemma 2.2.1 and Remark 2.2.6 that $x_0|X$ is contractible to an element in the fiber of $x_0 \in |X|$ therefore the following square is homotopy cartesian.

Diagram 4.1.4.

$$\begin{tikzcd} \Omega|X| \arrow[r] \arrow[d] & |x_0|X| \arrow[d, "|P|"] \\ |x_0|X| \arrow[r, "|P|"] & |X| \end{tikzcd}$$

which implies there is a morphism $|\Omega X| \longrightarrow \Omega|X|$ which is an homotopy equivalence if 4.1.3 is homotopy cartesian.

The idea is to apply Theorem 2.2.2 to show that it is the case under the given hypothesis. This is possible because by definition $(x_0, x_0|X) = x_0|P$. Using Remark 4.1.3 we notice that $\zeta^R : (x_0, x_0|X) \longrightarrow (x_0|X)$ is such that for $k \geq 0$, ζ_k^R maps $(x, y) \in (x_0, x_0|X)([k]) \subseteq X_{k+1} \times X_{k+1}$ to y and it corresponds exactly to $\kappa_R^P : x_0|P \longrightarrow x_0|X$ as described in Remark 2.2.4. We also notice Remark 4.1.3 that $\zeta^L : (x_0, x_0|X) \longrightarrow x_0|X$ is such that for $k \geq 0$, ζ_k^L maps

$(x, y) \in (x_0, x_0|M)([k]) \subseteq X_{k+1} \times X_{k+1}$ to x and it corresponds exactly to $\kappa_L^P : x_0|P \longrightarrow x_0|X$ as described in Remark 2.2.5. Therefore diagram 4.1.3 is isomorphic to the diagram

$$\begin{array}{ccc} x_0|P & \xrightarrow{\kappa_L^P} & x_0|X \\ \downarrow{\scriptstyle \kappa_R^P} & & \downarrow{\scriptstyle P} \\ x_0|X & \xrightarrow{\kappa_L^{\mathrm{id}_X}} & X \end{array}$$

in the description of Theorem 2.2.2. Because for any $n \geq 0$, $\rho \in X_n$ and $\phi : [m] \longrightarrow [n]$, morphism $(\rho|P) \longrightarrow (\phi^*(\rho)|P)$ is the same as $(x_0, \rho|X) \longrightarrow (x_0, \phi^*(\rho)|X)$, the hypothesis corresponds to the hypothesis of Theorem 2.2.2 and when it holds $|\Omega X| \longrightarrow \Omega|X|$ is a homotopy equivalence. $\square$

4.2 Constructing $S\mathcal{M}$, $G\mathcal{M}$ and $(0, M|S\mathcal{M})$

In this subsection we will describe the G-construction. We will also give useful characterization of related simplicial sets. Those will be used in the rest of Section 4 for the proof of [GG87]'s main theorem about the realization of the G-construction.

Let's from now on denote as $\mathcal{M}$ an arbitrary exact category. In [Wal85] Waldhausen defines the S-construction of $\mathcal{M}$ as the bi-simplicial set $iS.\mathcal{M}$. He proves that in the case of exact categories this construction is homotopically equivalent to $S\mathcal{M}$, as already stated in Proposition 3.2.1. To get the K-groups of $\mathcal{M}$ without a shift in degrees one works with the topological loop spaces of the Q-construction which is homotopy equivalent to the topological loop space of $S\mathcal{M}$. The main result of [GG87] is that those the loop space of the Q-construction is homotopy equivalent to the realization of the **simplicial** loop space of $S\mathcal{M}$, which relies on applying Corollary 4.1.1 to $|\Omega S\mathcal{M}| \longrightarrow \Omega|S\mathcal{M}|$.

Definition 4.2.1. We denote $G\mathcal{M} := \Omega S\mathcal{M}$ the **G-construction** of $\mathcal{M}$.

To get that the G-construction is homotopy equivalent to the loop space of the Q-construction we only need to apply Corollary 4.1.1. To show that the hypothesis in Corollary 4.1.1 apply to the G-construction we need to show that for any $k, l \geq 0$, $M \in S\mathcal{M}([k])$ and $f : [l] \longrightarrow [k]$, $(0, M|S\mathcal{M}) \xrightarrow{f^*} (0, f^*(M)|S\mathcal{M})$ is an homotopy equivalence. But before that we give useful characterizations of related constructions.

From now on we also make use of the following definition to simplify notations without introduction ambiguity

Definition 4.2.2. Elements of $S\mathcal{M}([1])$ correspond to objects N of $\mathcal{M}$ and we will denote them by $\widehat{N}$. We will also denote 0 the unique element of $S\mathcal{M}([0])$. We will make sure to distinguish it from $\widehat{0} \in S\mathcal{M}([1])$ that corresponds to the zero object in $\mathcal{M}$.

We first remark that because $S\mathcal{M}([0]) = \{0\}$ the constraint applied in *Remark* 2.2.3 for an element of $S\mathcal{M}([l+1])$ to belong to $(0|S\mathcal{M})([l])$ is void, which means that $(0|S\mathcal{M})([l]) = S\mathcal{M}([0][l])$. Explicitly $(0|S\mathcal{M})([l])$ is the set of chains M defined as

$$0 \rightarrowtail M_0 \rightarrowtail M_1 \rightarrowtail ... \rightarrowtail M_l$$

Along with choices of quotients, with M_0 *not necessarily equal* to 0. In that context the contravariant image $\phi : [l'] \longrightarrow [l]$ sends M to

$$0 \rightarrowtail M_{\phi(0)} \rightarrowtail M_{\phi(1)} \rightarrowtail ... \rightarrowtail M_{\phi(l')}$$

With induced monomorphisms and choices of quotients. We notice that in that case the action of a given ϕ will never "quotient out" elements of the chain by $M_{\phi(0)}$ because the first element of the chain is actually 0, and the action is given by $(\mathrm{id}_0 \bullet \phi)^*$ (remember that $(\mathrm{id}_0 \bullet \phi)$ always map 0 to 0 by definition).

Using a similar argument, one observes that an element in $G\mathcal{M}([n])$ can be understood to be a pair of chains

$$M_0 \rightarrowtail M_1 \rightarrowtail \ldots \rightarrowtail M_n$$

$$N_0 \rightarrowtail N_1 \rightarrowtail \ldots \rightarrowtail N_n$$

along with a choice of quotient $N_{i,j}$ for each N_j/N_i and $M_{i,j}$ for M_j/M_i for all $0 < i < j \leq n$. Moreover they we much have an equality between $S\mathcal{M}(d_0)(M)$ and $S\mathcal{M}(d_0)(N)$ represented by

$$M_1/M_0 \rightarrowtail \ldots \rightarrowtail M_n/M_0$$

$$N_1/N_0 \rightarrowtail \ldots \rightarrowtail N_n/N_0$$

which is another way of saying that $M_{i,j} = N_{i,j}$ for for all $0 \leq i \leq j \leq n$. One must keep in mind in cases like this that an equality between quotients is not the same as an isomorphism and that choices of quotients do matter.

Finally, let $M \in S\mathcal{M}([k])$ be characterized by the sequence of cofibrations

$$M = \; M_1 \rightarrowtail \cdots \rightarrowtail M_k$$

along with a choice of quotients. Then an element $N \in (0, M|S\mathcal{M})([n])$ is defined by a pair of sequences of cofibrations denoted as :

$$N := \left(\begin{array}{ccccccccc} & & & & & K_0 & \rightarrowtail \cdots \rightarrowtail & K_n \\ \\ M_1 & \rightarrowtail \cdots \rightarrowtail & M_k & \rightarrowtail & & L_0 & \rightarrowtail \cdots \rightarrowtail & L_n \end{array} \right)$$

where $K_j/K_i \simeq L_j/L_i$, along with a choice of quotient for L_i/M_j for all $0 < j \leq k$, $0 \leq i \leq n$ and a choice of quotient for L_j/L_i for all $0 \leq i < j \leq n$. We remark that we de not provide another choice of quotients for K_j/K_i because it must be equal to the choice of quotients for L_j/L_i.

We now give a very useful condition under which morphisms of simplicial set towards $(0, M|S\mathcal{M})$ are isomorphic.

Definition 4.2.3. Let $k \geq 0$ and $M \in S\mathcal{M}([k])$ Let N and N' be elements of $(0, M|S\mathcal{M})$ represented by diagrams as above. We call an *isomorphism between N and N'* a family of isomorphism $K_i \to L_i$ in $\mathcal{M}$ for all $0 \leq i \leq n$ that makes the following diagram commute :

$$\begin{tikzcd}
K_0 \arrow[r, tail] \arrow[dd, bend left, "\simeq"] & K_1 \arrow[r, tail] \arrow[dd, bend left, "\simeq"] & \cdots \arrow[r, tail] & K_m \arrow[dd, bend left, "\simeq"] \\
L_0 \arrow[r, tail] \arrow[dd, bend left, "\simeq"] & L_1 \arrow[r, tail] \arrow[dd, bend left, "\simeq"] & \cdots \arrow[r, tail] & L_m \arrow[dd, bend left, "\simeq"] \\
K'_0 \arrow[r, tail] & K'_1 \arrow[r, tail] & \cdots \arrow[r, tail] & K'_m \\
L'_0 \arrow[r, tail] & L'_1 \arrow[r, tail] & \cdots \arrow[r, tail] & L'_m
\end{tikzcd}$$

Proposition 4.2.1. *Let $k \geq 0$ and $M \in S\mathcal{M}([k])$. Let F, G be two morphism of simplicial sets from Y to $(0, M|S\mathcal{M})$. Assume that for any $m \geq 0$ and any $y \in Y_m$ there exists an isomorphism $F_m(y) \xrightarrow[\phi_y]{\sim} G_m(y)$ between $F_m(y)$ and $G_m(y)$. Assume further that for any $l, m \geq 0$ and $\psi : [l] \longrightarrow [m]$ the following diagram commutes*

$$\begin{tikzcd}
F_m(y) \arrow[r, "\phi_y"] \arrow[d, "\psi^*"] & G_m(y) \arrow[d, "\psi^*"] \\
F_l(\psi^*(y)) \arrow[r, "\phi_{\psi^*(y)}"] & G_l(\psi^*(y))
\end{tikzcd}$$

Then F and G are homotopic.

Proof. We define a simplicial homotopy $H : Y \times \Delta_1 \longrightarrow (0, M|S\mathcal{M})$. Let $N := F_m(y)$ and $N' := G_m(y)$ then $F_m(y) \xrightarrow[\simeq]{\phi_y} G_m(y)$ correspond to a commutative diagram as given in Definition 4.2.3. Let $f := h_q$ for $q := \min(f^{-1}(1))$ be any element of $\Delta_1([m])$. Then we set the image of $H(f, M)$ to be

$$K_0 \rightarrowtail K_1 \rightarrowtail \cdots \rightarrowtail K_{q-1} \rightarrowtail K'_q \rightarrowtail \cdots \rightarrowtail K'_m$$

$$M_1 \rightarrowtail \cdots \rightarrowtail M_k \rightarrowtail L_0 \rightarrowtail L_1 \rightarrowtail \cdots \rightarrowtail L_{q-1} \rightarrowtail L'_q \rightarrowtail \cdots \rightarrowtail L'_m$$

in which for all $p \geq q$, $r < q$ the choice of quotient for L'_p/L_r is defined as the choice of quotient for L'_p/L'_r. All other choices of quotients remain unchanged. As per Remark 2.1.1 for any $m, m' \geq 0$ and any $\phi : [m'] \longrightarrow [m]$ we have that $\phi^*(h_q) = h_p$ where $p := \min(\phi^{-1}(\{q, \cdots, m\}))$ if $\phi^{-1}(\{q, \cdots, m\}) \neq \emptyset$ and $p := m'+1$ otherwise . We then observe that $H(\phi^*(h_q), \phi^*(y))$ and $\phi^*(H(h_q, y))$ both correspond to

$$\begin{array}{r} K_0 \rightarrowtail K_1 \rightarrowtail \cdots \rightarrowtail K_{p-1} \rightarrowtail K'_p \rightarrowtail \cdots \rightarrowtail K'_{m'} \\ \\ M_1 \rightarrowtail \cdots \rightarrowtail M_k \rightarrowtail L_0 \rightarrowtail L_1 \rightarrowtail \cdots \rightarrowtail L_{p-1} \rightarrowtail L'_p \rightarrowtail \cdots \rightarrowtail L'_{m'} \end{array}$$

with the same choices for quotient. This means H is well-defined as a morphism of simplicial sets. Moreover for any $m \geq 0$ and any $y \in Y_m$ we have $H(y, \mathbb{0}) = F_m(y)$ and $H(y, \mathbb{1}) = G_m(y)$. Therefore H is indeed a simplicial homotopy between F and G and we are done.

□

Remark 4.2.1. In [GG87] this result is an implicit consequence of Lemma 3.1.1 and a series of more abstract and general results presented in pages 581–582. Even though those results are useful for proving similar propositions, we have decided to disregard the abstraction in an attempt to make the document simpler. This also allows us to explicitly state the result that we need for the proof. A reader already familiar with [GG87] will notice that we can recover $(0, M|S\mathcal{M})$ from objects in the simplicial groupoid $(0, M|S\mathcal{M})^{\mathrm{Is}}$ described in [GG87]. Moreover isomorphism in this simplicial groupoid correspond exactly to what we describe as "isomorphism between N' and N" in Definition 4.2.3 which justifies the term isomorphism. Note that the proof of Proposition 4.2.1 can be recovered either by explicitely deriving the homotopies provided by theorems in [GG87] or by adapting very similar proofs in [Wal85] such as the one for Lemma 1.4.1.

4.3 An H-space on $|(0, \widehat{N}|S\mathcal{M})|$

In this subsection we prove an essential lemma used in proving that the realization of the G-construction is homotopy equivalent to the loop space of the Q-construction. The proof of the lemma relies on providing the space $|(0, \widehat{N}|S\mathcal{M})|$ with an H-space that we will describe.

To describe the lemma we need the following definition

Definition 4.3.1. Let $N \in S\mathcal{M}([0])$ we define the endomorphism of simplicial set $J : (0, \widehat{N}|S\mathcal{M}) \longrightarrow (0, \widehat{N}|S\mathcal{M})$ such that

$$\begin{array}{cccc} J_n : & (0,\widehat{N}|S\mathcal{M})([n]) & \longrightarrow & (0,\widehat{N}|S\mathcal{M})([n]) \\ & \begin{pmatrix} K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ \\ N \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix} & \longmapsto & \begin{pmatrix} K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ \\ N \rightarrowtail N \oplus L_0/N \rightarrowtail \cdots \rightarrowtail N \oplus L_n/N \end{pmatrix} \end{array}$$

where for all $0 \leq i \leq j \leq n$ the choice of quotient for $(N \oplus L_j/N)/(N \oplus L_i/N)$ is the choice of quotient for L_j/L_i and the choice of quotient for $((N \oplus L_j/N)/N)$ is the choice of quotient for L_j/N.

The lemma we would like to prove is the following

Lemma 4.3.1. *Let J be the morphism as defined in Definition 4.3.1, it is homotopy equivalent to id.*

In order to prove it we are going to denote $Y := (0, \widehat{N}|S\mathcal{M})$ and provide the geometric realisation of Y with a H-Space structure given by the realization of $- + - : Y \times Y \longrightarrow Y$ such that

$$\begin{pmatrix} K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ \\ N \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix} + \begin{pmatrix} K'_0 \rightarrowtail \cdots \rightarrowtail K'_n \\ \\ N \rightarrowtail L'_0 \rightarrowtail \cdots \rightarrowtail L'_n \end{pmatrix}$$

$$= \begin{pmatrix} K_0 \oplus K'_0 \rightarrowtail \cdots \rightarrowtail K_n \oplus K'_n \\ \\ N \rightarrowtail L_0 \amalg_N L'_0 \rightarrowtail \cdots \rightarrowtail L_n \amalg_N L'_n \end{pmatrix}$$

We notice that it does extend to a continuous function with domain $|Y| \times |Y|$ because Proposition 1.1.1 tells us that geometric realization commutes with finite limits in CGHauss. We remark that even if the product in Top is not compactly generated we can work with the product in CGHauss – which has the same underlying set and a finer topology – without loss of generality. Let $\begin{pmatrix} 0 \\ N \rightarrowtail N \end{pmatrix} \in Y_0$. We denote by e the associated constant simplicial set which corresponds to a point $|e| \in Y_0$. The realization of the morphism of simplicial set

$$\begin{array}{ccc} e & \longrightarrow & Y_n \\ \begin{pmatrix} 0 \\ \\ N \rightarrowtail N \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 \rightarrowtail \cdots \rightarrowtail 0 \\ \\ N \rightarrowtail N \rightarrowtail \cdots \rightarrowtail N \end{pmatrix} \end{array}$$

is the inclusion $\{|e|\} \rightarrowtail |Y|$.

We first show that $|e|$ serves as the zero element of the H-Space. This is because $f : y \mapsto y + |e|$ in $|Y|$ is homotopy equivalent to $\mathrm{id}_{|Y|}$. We notice that f is the realisation of $Y \xrightarrow{\simeq} Y \times e \longrightarrow Y \times Y \xrightarrow{+} Y$ defined as

$$\left(\begin{array}{c} K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ \\ N \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{array} \right) \mapsto \left(\begin{array}{c} K_0 \oplus 0 \rightarrowtail \cdots \rightarrowtail K_n \oplus 0 \\ \\ N \rightarrowtail L_0 \amalg_N N \rightarrowtail \cdots \rightarrowtail L_n \amalg_N N \end{array} \right)$$

Using the natural isomorphisms $K_i \oplus 0 \simeq K_i$ and $L_i \coprod_N N \simeq L_i$ for all $0 \le i \le n$ we can apply Proposition 4.2.1 to show f is homotopic to id_Y

We use a similar argument to show that this H-space is homotopy associative. The two morphism $Y \times Y \times Y \longrightarrow Y$ defined respectively by

$$(- + -) \circ ((- + -), \mathrm{id}_Y)$$

and

$$(- + -) \circ (\mathrm{id}_Y, (- + -))$$

are homotopy equivalent because for all $0 \le i \le n$ there are natural isomorphisms $(L_i \oplus L_i') \oplus L_i'' \simeq L_i \oplus (L_i' \oplus L_i'')$ and $(K_i \amalg_N K_i') \amalg_N K_i'' \simeq K_i \amalg_N (K_i' \amalg_N K_i'')$.

Moreover the H-space is homotopy commutative. Let $\gamma : Y \times Y \to Y \times Y$ be the morphism of simplicial set such that γ_k maps (y, y') to (y', y). We just need to check that the two morphisms $Y \times Y \longrightarrow Y$ given by

$$(- + -)$$

and

$$(- + -) \circ \gamma$$

are homotopy equivalent. We use the natural isomorphisms $L_i \oplus L_i' \simeq L_i' \oplus L_i$ and $K_i \amalg_N K_i' \simeq K_i' \amalg_N K_i$ for all $0 \le i \le n$.

Definition 4.3.2. Let Y be any simplicial set with a associated H-Space. Let f, g, h, k be endomorphisms in Y.
- We denote by $h \sim k$ the fact that h and k are homotopy equivalent.
- We denote by $f + g$ the function $Y \xrightarrow{(f,g)} Y \times Y \xrightarrow{+} Y$
- We denote by 0 the unique function $Y \longrightarrow e \longrightarrow Y$ that factors through the constant simplicial set.

We remark that $f+g+h$ is well defined up to homotopy because our H-space is homotopy associative. We notice that if $f + g \sim 0$ then $f + g + h \sim f + g + k$ implies $h \sim k$.

The fact that for all $0 \le i \le n$ we have $L_i \amalg_N (N \oplus L_i/N) \simeq L_i \amalg_N L_i$ in a natural way gives us – using Proposition 4.2.1 – that $\mathrm{id} + \mathrm{id} \sim \mathrm{id} + J$.

According to what we just said we now only need an endomorphism "opposite to the identity", ie $(-\mathrm{id}) : X \longrightarrow X$ such that $\mathrm{id}+(-\mathrm{id}) \sim 0$. We use a corollary to the following lemma whose proof can be found in [Gra76].

Lemma 4.3.2. *Let X be a path connected homotopy associative H-space then there is a map $-id : X \longrightarrow X$ such that $id + (-id) \sim 0$*

Let X be a space provided with an H-space. Because $+ : X \times X \longrightarrow X$ is continuous it induces a map $\pi_0(X) \times \pi_0(X) \longrightarrow \pi_0(X)$. It is well defined because if $\gamma : [0,1] \longrightarrow X$ is a path from x to y then $t \mapsto (\gamma(t) + z)$ is a path connecting $x + z$ to $y + z$. If it is homotopy associative then the homotopy $H : [0,1] \times X \times X \times X \longrightarrow X$ induces a path $t \mapsto (H(t, x, y, z))$ between each $(x + y) + z$ and $x + (y + z)$ and the law is associative. Finally let 0 be the zero element then for $K : [0,1] \times X \longrightarrow X$ the homotopy between id and $x \mapsto x + 0$ then $t \mapsto K(t,x)$ is a path between x and $x + 0$. The path-component of 0 is a zero element for the law in $\pi_0(X)$ which is a monoid. In the particular case where this law is a group we deduce the following corollary.

Corollary 4.3.1. *Let X be an homotopy associative H-space such that the induced monoid on $\pi_0(X)$ is a group, then there is a map $-id : X \longrightarrow X$ such that $id + (-id) \sim 0$*

Proof. We denote by X_0 the path-component of the zero element. We notice that the H-space restricts to a homotopy associative H-space on the path-connected X_0 and Lemma 4.3.2 gives us a continuous map

$$-\mathrm{id} : X_0 \longrightarrow X_0$$

We now denote X_g the path connected component associated to an element g in the group $\pi_0(X)$ and we choose an element x_g in each X_g. Using the fact that for any other $x \in X_g$, $(x + x_{g^{-1}}) \in X_0$, we define for each g a continuous mapping $X_g \longrightarrow X_{g^{-1}}$

$$x \mapsto -x := x_{g^{-1}} + (-\mathrm{id})(x + x_{g^{-1}})$$

From the family of continuous mapping on each X_g we deduce a continuous mapping from $(-\mathrm{id}) : X \longrightarrow X$. Moreover $\mathrm{id} + (-\mathrm{id})$ is a mapping from X to X_0. On X_g we have $\mathrm{id} + (-\mathrm{id}) : x \mapsto x + (x_{g^{-1}} + (-\mathrm{id})(x + x_{g^{-1}}))$ which by associativity is homotopy equivalent to $X_g \xrightarrow{-+x_{g^{-1}}} X_0 \xrightarrow{\mathrm{id}+(-\mathrm{id})} X_0$ which is homotopy equivalent to the constant map by Lemma 4.3.2. From the homotopies $H_g : [0,1] \times X_g \longrightarrow X_0$ between $(\mathrm{id} + (-\mathrm{id}))_{|X_g}$ and $0_{|X_g}$ we deduce an homotopy $H : [0,1] \times X \longrightarrow X_0$ between $(\mathrm{id} + (-\mathrm{id}))$ and 0 and we are done. $\square$

In the case where $X = |Y| = |(0, \widehat{N}|S\mathcal{M})|$, we now need to prove that $\pi_0(|Y|)$ with the monoid structure induced by the H-Space is a group. A path-connected component in $|Y|$ always contains a point $\begin{pmatrix} & K_0 \\ N \rightarrowtail & L_0 \end{pmatrix} \in Y_0$. We now show that the component of $\begin{pmatrix} & L_0/N \\ N \rightarrowtail & N \oplus K_0 \end{pmatrix} \in Y_0$ is an opposite.

Their sum is $y := \begin{pmatrix} & K_0 \oplus L_0/N \\ N \rightarrowtail & K_0 \oplus L_0 \end{pmatrix} \in Y_0$ which belongs to the same component as the zero element because $z := \begin{pmatrix} & & 0 \rightarrowtail K_0 \oplus L_0/N \\ N \rightarrowtail & N \rightarrowtail & K_0 \oplus L_0 \end{pmatrix} \in Y_1$ is such that $d_0^*(z) = y$ and $d_1^*(z) = e$. Now let J be the morphism defined in Lemma 4.3.1, we already showed that $\mathrm{id} + \mathrm{id} \sim \mathrm{id} + J$. Using Corollary 4.3.1 and the fact that $\pi_0(|Y|)$ has a group structure implies the existence of $-\mathrm{id}$. We have $-\mathrm{id} + \mathrm{id} + \mathrm{id} \sim -\mathrm{id} + \mathrm{id} + J$ which implies $\mathrm{id} \sim J$ and we are done proving Lemma 4.3.1,

4.4 Proving the main theorem

We finally use all the tools we have created to prove the main theorem of [GG87].

Theorem 4.4.1. *The natural morphism $|G\mathcal{M}| \longrightarrow \Omega|S\mathcal{M}|$ is an homotopy equivalence*

Proof. The idea of the proof is to use Corollary 4.1.1. Thus we only need to prove that for any $M \in S\mathcal{M}([m])$ and $\phi \in [m'] \longrightarrow [m]$ the induced functor $\phi^* : (0, M|S\mathcal{M}) \longrightarrow (0, \phi^*(M')|S\mathcal{M})$ is a homotopy equivalence. We will first prove it for a subset of morphisms ϕ then generalise.

Let $p \geq 0$ and $M \in S\mathcal{M}([p])$ we first show it for $\eta_p : [1] \longrightarrow [p]$ defined as in Definition 2.1.3. We observe that $((\eta_p)^*)_n : (0, M|S\mathcal{M})([n]) \longrightarrow (0, \widehat{M_p}|S\mathcal{M})([n])$ maps

$$x := \begin{pmatrix} & & & K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ M_1 \rightarrowtail \cdots \rightarrowtail & M_p \rightarrowtail & & L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix}$$

to

$$x' = \begin{pmatrix} & K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ M_p \rightarrowtail & L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix}$$

We notice that the only information lost in $((\eta_p)^*)_n$ is the choice of quotient L_i/M_k for all $0 < i \leq n$ and $0 < k < p$. This means we cannot directly define an inverse morphism. Instead, we set in advance a choice of quotient for each monomorphism in $\mathcal{M}$. We then consider functor $F : (0, \widehat{M_p}|S\mathcal{M}) \longrightarrow$

$(0, M|S\mathcal{M})$ that maps x' as above to a element y represented by the same diagram as x but where the choice of quotient for each L_i/M_k is the arbitrary choice for this monomorphism. Then $F \circ F' = 1$ and $F' \circ F$ maps an element of $(0, M|S\mathcal{M})$ to an element naturally isomorphic to it and therefore it is homotopic to the identity using Proposition 4.2.1. We are done proving that $(\eta_p)^*$ is an homotopy equivalence.

Next we show that for any $N \in S\mathcal{M}([1]) \simeq \mathrm{Ob}(\mathcal{M})$, any of the two mapping from $[0]$ to $[1]$ – namely $f := \mu_L^{0,0}$ and $g := \mu_R^{0,0}$ defined as in Definition 2.1.2 – lead to homotopy equivalences. Induced morphism $g^* : (0, \widehat{N}|S\mathcal{M}) \longrightarrow (0, 0|S\mathcal{M})$ is such that $(g^*)_n$ maps

$$\begin{pmatrix} & K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ N \rightarrowtail & L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix}$$

to

$$\begin{pmatrix} K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ L_0/N \rightarrowtail \cdots \rightarrowtail L_n/N \end{pmatrix}$$

where the choice of quotient for $(L_j/N)/(L_i/N)$ is the one we had made for L_j/L_i for all $0 \leq i < j \leq n$. For any pair (A, B) of objects in $\mathcal{M}$ we set in advance an arbitrary object $A \oplus B$ within the corresponding isomorphism class. We use that to define an homotopy inverse $H : (0, 0|S\mathcal{M}) \longrightarrow (0, \widehat{N}|S\mathcal{M})$ such that H_n maps

$$x := \begin{pmatrix} K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix}$$

to

$$\begin{pmatrix} & K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ N \rightarrowtail & L_0 \oplus N \rightarrowtail \cdots \rightarrowtail L_n \oplus N \end{pmatrix}$$

Where the choice of quotient for $(L_j \oplus N)/(L_i \oplus N)$ for all $0 \leq i \leq j \leq n$ is the choice of quotient for L_j/L_i and the choice of quotient for $(L_j \oplus N)/N$ is L_j for all $0 \leq j \leq n$. We immediately notice that $g^* \circ H = \mathrm{id}$ and $H \circ g^*$ is exacly the morphism J defined in Definition 4.3.1 and we know by Lemma 4.3.1 that it is homotopy equivalent to id. Both H and g^* are homotopy equivalences.

We verify $K := H \circ f^*$ maps

$$y := \begin{pmatrix} & K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ \\ N \rightarrowtail & L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix}$$

to

$$\begin{pmatrix} & K_0 \rightarrowtail \cdots \rightarrowtail K_n \\ \\ N \rightarrowtail & N \oplus L_0 \rightarrowtail \cdots \rightarrowtail N \oplus L_n \end{pmatrix}$$

and that is it a homotopy equivalence. We define its homotopy inverse G as the morphism that maps y to

$$\begin{pmatrix} & N \oplus K_0 \rightarrowtail \cdots \rightarrowtail N \oplus K_n \\ \\ N \rightarrowtail & L_0 \rightarrowtail \cdots \rightarrowtail L_n \end{pmatrix}$$

Then $G \circ K = K \circ G$ is the morphism that maps y to

$$\begin{pmatrix} & N \oplus K_0 \rightarrowtail \cdots \rightarrowtail N \oplus K_n \\ \\ N \rightarrowtail & N \oplus L_0 \rightarrowtail \cdots \rightarrowtail N \oplus L_n \end{pmatrix}$$

To show that they K and G are homotopy inverse we construct a simplicial homotopy Q between $G \circ K = K \circ G$ and $\mathrm{id}_{(0,N|S\mathcal{M})}$ such that

$$\begin{array}{rccc} Q_n : & \Delta_1([n]) \times (0, N|S\mathcal{M}([n])) & \longrightarrow & (0, N|S\mathcal{M})([n]) \\ & (h_k, y) & \longmapsto & \begin{pmatrix} & K_0 \twoheadrightarrow \cdots \twoheadrightarrow K_{k-1} \twoheadrightarrow K_k \oplus N \twoheadrightarrow \cdots \twoheadrightarrow K_n \oplus N \\ \\ N \rightarrowtail & L_0 \rightarrowtail \cdots \rightarrowtail L_{k-1} \rightarrowtail L_k \oplus N \rightarrowtail \cdots \rightarrowtail L_n \oplus N \end{pmatrix} \end{array}$$

We check that the image does belong to $(0, N|S\mathcal{M})$. We define quotient $(K_j \oplus N)/K_i$ to be $K_j/K_i \oplus N$ and $(L_j \oplus N)/L_i$ to be $L_j/L_i \oplus N$. Because the choices of quotient for each K_j/K_i and L_j/L_i are the same their arbitrary direct sums with N are the same. This is a well defined morphism of simplicial sets because for any $m, n \geq 0$ and any $\phi : [m] \longrightarrow [n]$ we have

$$\phi^*(Q(h_k, y)) = \begin{pmatrix} & K_{\phi(0)} \rightarrowtail \cdots \rightarrowtail K_{\phi(p-1)} \rightarrowtail K_{\phi(p)} \oplus N \rightarrowtail \cdots \rightarrowtail K_{\phi(m)} \oplus N \\ \\ N \rightarrowtail & L_{\phi(0)} \rightarrowtail \cdots \rightarrowtail L_{\phi(p-1)} \rightarrowtail L_{\phi(p)} \oplus N \rightarrowtail \cdots \rightarrowtail L_{\phi(n)} \oplus N \end{pmatrix}$$

where $p := \min(\phi^{-1}(\{k, \cdots, n\}))$ if $\phi^{-1}(\{k, \cdots, n\}) \neq \emptyset$ and $p := m + 1$ otherwise. It is indeed equal to $Q(\phi^*(h_m), \phi^*(y)) = Q(h_p, \phi^*(y))$. We finally check that $Q(\mathbb{0}, y) = y$ and $Q(\mathbb{1}, y) = K \circ G(y)$ therefore Q is the desired simplicial homotopy. We are done checking that $H \circ f^*$ is an homotopy equivalence and because we had previously shown that H is also an homotopy equivalence we deduce that f^* is an homotopy equivalence.

Finally, we generalize and show that for any $m, n > 0$ and any $\phi : [m] \longrightarrow [n]$ and for any $M \in S\mathcal{M}([n])$, $\phi^* : (0, M|S\mathcal{M}) \longrightarrow (0, \phi^*(M)|S\mathcal{M})$ is a homotopy equivalence. Assuming $n \geq 1$ and $m \geq 1$, we use that the following diagram commutes

$$\begin{tikzcd}
{[1]} \arrow[d, "\eta_n"] & {[0]} \arrow[l, "\mu_0"'] \arrow[r, "\mu_0"] & {[1]} \arrow[d, "\lambda"] & {[0]} \arrow[l, "\mu_1"'] \arrow[r, "\mu_0"] & {[1]} \arrow[d, "\eta_m"] \\
{[n]} & & {[n]} \arrow[ll, "\mathrm{id}"'] & & {[m]} \arrow[ll, "\phi"']
\end{tikzcd}$$

Where $\lambda(0) = 0$ and $\lambda(1) = \phi(0)$. It induces the following diagram

$$\begin{tikzcd}
(0, \widehat{M_n}|S\mathcal{M}) \arrow[r, "(\mu_0)^*"] & (0, 0|S\mathcal{M}) & (0, \widehat{M_{\phi(0)}}|S\mathcal{M}) \arrow[l, "(\mu_0)^*"'] \arrow[r, "(\mu_1)^*"] & (0, 0|S\mathcal{M}) & (0, \widehat{M_{\phi(m)}}|S\mathcal{M}) \arrow[l, "(\mu_0)^*"'] \\
(0, M|S\mathcal{M}) \arrow[u, "(\eta_n)^*"] & & (0, M|S\mathcal{M}) \arrow[ll, "\mathrm{id}"'] \arrow[u, "(\lambda)^*"] \arrow[rr, "\phi^*"] & & (0, \phi^*(M)|S\mathcal{M}) \arrow[u, "(\eta_m)^*"]
\end{tikzcd}$$

We have proven that all arrows in the outer square other than ϕ^* are homotopy equivalences. We can recover that ϕ^* itself is an homotopy equivalence. In cases where $n = 0$ or $m = 0$, respectively η_n or η_m is replaced by the constant mapping $[1] \longrightarrow [0]$ which also corresponds to an homotopy equivalence $(0, 0|S\mathcal{M}) \longrightarrow (0, \widehat{0}|S\mathcal{M})$ and we are done.

□

5 The G-construction of pseudo-additive categories

This section aims at presenting some of the results given in [Gun+92] by Gunnarsson, Schwänzl, Vogt and Waldhausen. We focus on describing their generalization of the G-construction of an exact category to any Waldhausen category. In Theorem 2.6 of [Gun+92] the authors prove that when a certain additivity hypothesis on the category is satisfied the new construction is homotopically equivalent to the topological loop space of the S-construction. Without going into much details we will mostly just recall the main intermediary results leading up to this theorem, and briefly mention how they relate to the hypothesis. In Theorem 5.2.1 we will detail how they are used for proving the theorem. We will explain why the results presented [Gun+92] generalize those in [GG87] and try to give motivating examples of algebraic nature for this generalization such as the Waldhausen category of complexes in an exact category with quasi-isomorphisms as weak equivalences that is used for instance in [KZ21] or [Gra12].

5.1 The G-construction of a Waldhausen category

When $\mathcal{M}$ is an exact category we defined in Definition 4.2.1 its G-construction as $\Omega S\mathcal{M}$ the simplicial loop space of the simplicial set $S\mathcal{M}$. The latter is shown in [Wal85] to be homotopically equivalent to $iS.\mathcal{M}$ when we see $\mathcal{M}$ as a Waldhausen category using isomorphisms as weak equivalences and monomorphisms as cofibrations (see Proposition 3.2.1).

We recall the following notation from [Wal85] and [Gun+92].

Definition 5.1.1. Let $\mathcal{C}$ be a category and X be a simplicial object in $\mathcal{C}$. Consider the shift functor $\Delta \longrightarrow \Delta$ that maps for all $n \geq 0$ object $[n]$ to $[n+1]$ and any morphism $\phi : [m] \longrightarrow [n]$ to $(\mathrm{id}_{[0]} \bullet \phi) : [m+1] \longrightarrow [n+1]$. We define a new simplicial object PX by precomposing $X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$ with the shift functor.

Then because $S\mathcal{M}([0])$ is a singleton the simplicial set $\Omega S\mathcal{M}$ constructed in Section 4 for the G-construction of $\mathcal{M}$ is the limit of the following diagram

$$\begin{tikzcd} & PS\mathcal{M} \arrow[d, "\delta_0"] \\ PS\mathcal{M} \arrow[r, "\delta_0"] & S\mathcal{M} \end{tikzcd}$$

where $\delta_0 : PS\mathcal{M} \longrightarrow S\mathcal{M}$ is such that $(\delta_0)_n : PS\mathcal{M}([n]) = S\mathcal{M}([n+1]) \longrightarrow S\mathcal{M}([n])$ is equal to $(d_0)^*$. That this is equal to $\Omega S\mathcal{M}$ is obvious when we consider the explicit characterization of elements of $\Omega S\mathcal{M}$ given in Section 4.

The goal of [Gun+92] is to generalize this construction to an arbitrary Waldhausen category $\mathcal{C}$. The simplicial set $G\mathcal{C} := \Omega S\mathcal{C}$ is also well-defined for an arbitrary Waldhausen category but it will not be as relevant because in general the S-construction of $\mathcal{C}$ is $wS.\mathcal{C}$ instead of $S\mathcal{C}$ and the two are not homotopy equivalent in general. With that in mind we define the following.

Definition 5.1.2. Let $\mathcal{C}$ be a Waldhausen category, we define the simplicial category $wG.\mathcal{C}$ as the limit of the following diagram

$$\begin{tikzcd} & PwS.\mathcal{M} \arrow[d, "\delta_0"] \\ PwS.\mathcal{M} \arrow[r, "\delta_0"] & wS.\mathcal{M} \end{tikzcd}$$

where δ_0 is defined as above. When $\mathcal{C}$ is a category with cofibrations turned into a Waldhausen categories by considering isomorphisms as weak equivalences we denote $wG.\mathcal{C}$ as $iG.\mathcal{C}$.

The following proposition shows that this construction generalizes the G-construction of exact categories presented in [GG87] to any Waldhausen category.

Proposition 5.1.1. *Let $\mathcal{C}$ be a category with cofibrations and $G\mathcal{C} := \Omega S\mathcal{C}$ then $|iG.\mathcal{C}|$ is homotopy equivalent to $|G\mathcal{C}|$.*

We do not detail the proof in this document but one could easily extend the arguments used in [Wal85] Corollary (2) of lemma 1.4.1 and apply them to $iG.\mathcal{C}$ and $G\mathcal{C}$ rather than $iS.\mathcal{C}$ and $S\mathcal{C}$.

In the following paragraphs we present an important example motivating our interest in Waldhausen categories and their G-constructions. Even though it is not mentioned in [Gun+92]'s original presentation, it is used throughout [Gra12] to derive an algebraic description for the higher K-groups and their G-construction is used in [KZ21] to define λ-operations on similarly defined K-Theories. We will make use of the following preliminary definitions from [Gra12].

Definition 5.1.3. Let $\mathcal{M}$ be an exact category and $A.$ be a complex in $\mathcal{M}$ of the form

$$\ldots \longrightarrow A_{i+1} \longrightarrow A_i \longrightarrow A_{i-1} \longrightarrow \ldots$$

We say that $A.$ is acyclic if for every $i \in \mathbb{Z}$ there exist an object Z_i in $\mathcal{M}$ such that $A_{i+1} \longrightarrow A_i$ is the composition of a diagram $A_{i+1} \longrightarrow Z_i \longrightarrow A_i$ and for all $j \in \mathbb{Z}$ the induced short sequences $Z_j \longrightarrow A_j \longrightarrow Z_{j-1}$ are exact.

We immediately see that when $\mathcal{M}$ is an abelian category this corresponds to the usual notion of acyclicity. Moreover, because properties for the homology of chain complexes are much easier to define in abelian categories we now define an important regularity condition on exact categories.

Definition 5.1.4. Let $\mathcal{M}$ be an exact category. We say that $\mathcal{M}$ **supports long exact sequences** if there exists an admissible embedding $\mathcal{M} \hookrightarrow \mathcal{A}$ such that for all complexes in $\mathcal{M}$ that are acyclic in $\mathcal{A}$ we have that objects $Z_i, \forall i \in \mathbb{Z}$ in $\mathcal{A}$ given in Definition 5.1.3 are isomorphic to an object of $\mathcal{M}$.

This definition is the same as Definition 1.4 from [Gra12]. We notice that if $\mathcal{M}$ supports long exact sequences and $\mathcal{M} \hookrightarrow \mathcal{A}$ is the admissible embedding in Definition 5.1.4 then a complex in $\mathcal{M}$ is acyclic if and only if it is acyclic in $\mathcal{A}$. Moreover this property holds for relevant examples from algebraic geometry. Given a scheme X all exact subcategories of the abelian categories of sheaves over X that are stable under the kernel of epimorphism or under the quotient of monorphisms support long exact sequences. For instance the category $\mathcal{M}$ of locally free sheaves of finite rank over X supports long exact sequences.

Example 5.1.1. Let $\mathcal{M}$ be an exact category that supports long exact sequences and let $\mathcal{M} \hookrightarrow \mathcal{A}$ be the corresponding admissible embedding. We denote by $C\mathcal{M}$ the category of bounded complexes in $\mathcal{M}$ and by $C\mathcal{A}$ the category of bounded complexes in $\mathcal{A}$. Then $C\mathcal{A}$ is abelian and $C\mathcal{M}$ is an exact category because the admissible embedding $\mathcal{M} \hookrightarrow \mathcal{A}$ induces an embedding $C\mathcal{M} \hookrightarrow C\mathcal{A}$ that is also admissible.

We claim that when we take object-wise monomorphisms as cofibrations and quasi-isomorphism as weak-equivalences $C\mathcal{M}$ forms a well-defined Waldhausen category. It is immediate that axioms (1) to (3) of Definition 3.1.1 are satisfied by our new structure on $C\mathcal{M}$. We now check that the glueing lemma holds. Let $A., B., C., A'., B'.$ and $C'.$ be complexes in $C\mathcal{M}$ forming the following diagram

$$\begin{tikzcd} B. \arrow[d, "\beta"', "\sim"] & A. \arrow[l, tail] \arrow[r] \arrow[d, "\alpha"', "\sim"] & C. \arrow[d, "\gamma"', "\sim"] \\ B'. & A'. \arrow[l, tail] \arrow[r] & C'. \end{tikzcd}$$

where $A. \rightarrowtail B.$ and $A'. \rightarrowtail B'.$ are object-wise monomorphisms and α, β, γ are quasi-isomorphism. We need to check that the induced morphism $B. \cup_{A.} C. \xrightarrow[\delta]{\sim} B'. \cup_{A'.} C'.$ is also a quasi-isomorphism. Let $\mathrm{Cone}(\delta)$ denote the mapping cone of δ, we check that the following diagram is co-cartesian

$$\begin{tikzcd} \mathrm{Cone}(\alpha) \arrow[r, tail] \arrow[d] & \mathrm{Cone}(\beta) \arrow[d] \\ \mathrm{Cone}(\gamma) \arrow[r, tail] & \mathrm{Cone}(\delta) \end{tikzcd}$$

Using the fact that a morphism of complexes is a quasi-isomorphism if and only if its mapping cone is acyclic, proving the glueing lemma comes down to proving that if $A''.$, $B''.$ and $C''.$ are acyclic complexes in $C\mathcal{M}$ and we have a diagram $C''. \leftarrow A''. \rightarrowtail B''.$ then $B''. \cup_{A''.} C''.$ is also acyclic. This in turn comes down to a simple algebraic exercise on modules because the notion of acyclicity is the one from the underlying abelian category and because pushouts along cofibrations in $\mathcal{M}$ are also pushouts in $\mathcal{A}$

We also define the following simplicial category.

Definition 5.1.5. Let $\mathcal{C}$ be a Waldhausen category. We define the simplicial category $S.\mathcal{C}$. For any $n \geq 0$ the objects of $S.\mathcal{C}([n])$ are functors $\mathrm{Ar}([n]) \longrightarrow \mathcal{C}$ that belong to the set $S\mathcal{C}([n])$ and the morphisms of $S.\mathcal{C}([n])$ are all natural transformations between them. Given any morphism $\phi : [m] \longrightarrow [n]$ in Δ functor $S.\mathcal{C}(\phi)$ is defined exactly as for $wS.\mathcal{C}(\phi)$ (see Section 3).

Moreover we define the simplicial category $G.\mathcal{C}$ to be the limit of the diagram

$$\begin{tikzcd} & PS.\mathcal{C} \arrow[d, "\delta_0"] \\ PS.\mathcal{C} \arrow[r, "\delta_0"] & S.\mathcal{C} \end{tikzcd}$$

We notice that given a Waldhausen category $\mathcal{C}$ both $S.\mathcal{C}$ and $G.\mathcal{C}$ are simplicial Waldhausen categories in an obvious way. The first fact is detailed in section 1 of [Wal85] and the second comes down to the fact that $G.\mathcal{C}([n])$ is the fiber product of Waldhausen categories and each of the two morphisms appearing in the diagram have a retraction (see [Wal85] page 325). In fact the simplicial categories $wS.$ and $wG.$ correspond level-wise to the categories of weak equivalences in $S.$ and $G.$ respectively. We also note that $S.$ and $G.$ form functors from wCof to swCof the category of simplicial objects in wCof.

Moreover we recall lemma 1.5.1 from [Wal85].

Lemma 5.1.1. *Let X be a simplicial object in a category $\mathcal{C}$. PX is simplicially homotopy equivalent to the constant simplicial object $[n] \mapsto X_0$.*

Note that in this context "homotopy equivalent" means that there is a simplicial homotopy equivalence between the two simplicial objects. The notion of simplicial homotopy between simplicial objects is an extension of the notion of simplicial homotopy for simplicial set given in Definition 1.1.11. In particular in means that the realizations of their nerves are homotopy equivalent in the topological sense. We can either use the same proof as in [Wal85] or understand it as a corollary of Lemma 2.1.1 using the simplicial homotopy given in the proof and considering the functor

$$\begin{array}{cccc} X^*: & (\Delta^*)^{\mathrm{op}} & \longrightarrow & \mathcal{C} \\ & [n] & \longmapsto & X_n \end{array}$$

In particular, let $\mathcal{C}$ be a Waldhausen category. Recall that $wS.\mathcal{C}([0])$ is a category with one object and one morphism whose nerve is the constant simplicial set equal to a singleton. Its realization is a single point and Lemma 5.1.1 implies that $PwS.\mathcal{C}$ is contractible. Therefore by a similar reasoning as in the proof of Corollary 4.1.1 the diagram in Definition 5.1.2 induces a canonical map

$$|wG.\mathcal{C}| \longrightarrow \Omega|wS.\mathcal{C}|$$

We will see in the next subsection that it is an homotopy equivalence as long as $\mathcal{C}$ respects some additivity hypothesis.

5.2 The G-construction of a pseudo-additive category

We now define a condition under which the G-construction of a Waldhausen category $\mathcal{C}$ is homotopy equivalent to the loop space of the realization of the S-construction. We make use of the following notation to describe the direct sum of two objects in a Waldhausen category

Definition 5.2.1. Let $\mathcal{C}$ be a Waldhausen category and A, B be objects of $\mathcal{C}$. Then we denote $A \vee B := A \cup_* B$.

Definition 5.2.2. Let $\mathcal{C}$ be a Waldhausen category. We say it is **pseudo-additive** if the following holds
For every object A in $\mathcal{C}$ let $\mathcal{C}_A$ be the full subcategory in the category of objects over A containing only cofibrations. Then the two functors $\mathcal{C}_A \longrightarrow \mathcal{C}_A$

$$(A \rightarrowtail C) \mapsto (A \rightarrowtail C \cup_A C)$$

and

$$(A \rightarrowtail C) \mapsto (A \rightarrowtail C \cup_A (C \vee C/A))$$

are related by a sequence of natural weak equivalences.

We now check that our relevant algebraic examples of Waldhausen categories are pseudo-additive.

A large part of [Gun+92] is dedicated to proving the following theorem

Theorem 5.2.1. *Let $\mathcal{C}$ be a Waldhausen category, if $\mathcal{C}$ is pseudo-additive then*

$$|wG.\mathcal{C}| \longrightarrow \Omega|wS.\mathcal{C}|$$

is a weak homotopy equivalence.

We will provide a proof at the end of Section 5.3.

Example 5.2.1. Let $\mathcal{M}$ be an exact category regarded as a Waldhausen category using monomorphism as cofibrations and using any kind of weak equivalences. Then it is pseudo-additive and the sequence of natural weak equivalences mentioned in Definition 5.2.2 above can be chosen to be the single natural isomorphism $C \cup_A C \simeq C \cup_A (A \vee C/A)$ that we used earlier the proof of [GG87]'s Theorem 3.1. In particular let $\mathcal{M}'$ be an exact category that supports long exact sequence the category $C\mathcal{M}'$ of bounded complexes described in Example 5.1.1 with quasi-isomorphism as weak equivalences is pseudo-additive. This means that we can use $wG.C\mathcal{M}'$ the G-construction of $C\mathcal{M}'$ to have a K-theory with no shifts in degrees. This is what is done for instance in [KZ21] to define λ-operations on similar categories. Another trivial example is the category $\mathcal{M}$ itself with isomorphisms as weak equivalences, in which case because $iG.\mathcal{M}$ is homotopy equivalent to $G\mathcal{M}$ and $iS.\mathcal{M}$ is homotopy equivalent to $S\mathcal{M}$ we notice that applying Theorem 5.2.1 recovers the same result as Theorem 4.4.1.

5.3 Properties of the G-construction

We now define some properties that one would typically expect of constructions for the K-theory of a Waldhausen category. Those properties are not only important to check in their own right, but they are also used throughout the proof of Theorem 5.2.1 in [Gun+92]. Recall from [Wal85] that given a Waldhausen category $\mathcal{C}$ we denote as $E(\mathcal{C})$ the category of cofibration sequences $L \rightarrowtail M \twoheadrightarrow N$ in $\mathcal{C}$ and that $E(\mathcal{C})$ is a Waldhausen category. Recall also that there are two exact functors $f, g : E(\mathcal{C}) \longrightarrow \mathcal{C}$ that respectively map sequence $L \rightarrowtail M \twoheadrightarrow N$ to L and N. Moreover let Set_* be the category of pointed sets. We observe that constructions such as $wG.$ or $wS.$ form functors from wCof to the category of functors $\Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}_*$ after taking the diagonal of their nerve and setting the basepoint to the zero object in each level.

Definition 5.3.1. Let $\mathcal{Q}$ be a class of objects in the category wCof and $F.$ be a functor from wCof to the category of functors $\Delta^{\mathrm{op}} \longrightarrow \mathrm{Set}_*$. We say that $F.$ is $\mathcal{Q}$-admissible if for any categories $\mathcal{A}, \mathcal{B}$ in wCof and $\mathcal{C}$ in $\mathcal{Q}$ we have

1. $F.(*) = *$
2. $F.(\mathcal{A} \times \mathcal{B}) \longrightarrow F.(\mathcal{A}) \times F.(\mathcal{B})$ is a weak homotopy equivalence

3. $F.(E(\mathcal{C})) \longrightarrow F.(\mathcal{C} \times \mathcal{C})$ induced by (f, g) is a weak homotopy equivalence

4. $\pi_0(F.(\mathcal{A}))$ is a group under the operation induced by $\vee$ the categorical sum in $\mathcal{A}$

5. The image by $F.$ of two exact functors related by a natural transformation whose components are weak equivalences are homotopic functions

For instance, a large part of [Wal85] consists in proving that $wS.$ is wCof-admissible.

Moreover in [Gun+92] the authors prove that $wG.$ is $\mathcal{P}$-admissible when $\mathcal{P}$ is the class of pseudo-additive categories. This is in turned used in the proof of Theorem 5.2.1. Proving properties 1. and 2. for $wG.$ is easy. Proving property 5. can be done by adapting the proof of Lemma 1.3.1 from [Wal85]. Proving property 4. comes down to the following lemma

Lemma 5.3.1. *Let $\mathcal{C}$ be a Waldhausen category. The categorical sum on $\mathcal{C}$ induces a monoid in $\pi_0(wG.\mathcal{C})$ and this monoid is a group.*

Proof. The categorical sum on $\mathcal{C}$ induces a H-space on $wG.\mathcal{C}$ which induces a monoid structure on $\pi_0(wG.\mathcal{C})$. This H-space is the obvious extension of the one defined in Section 4.3 on $G\mathcal{C}$ with $N = 0$. There is an inclusion $|G\mathcal{C}| \hookrightarrow |wG.\mathcal{C}|$ compatible with those two H-spaces. Moreover one can show that $\pi_0(wG.\mathcal{C}) = \pi_0(G\mathcal{C})$. In Section 4.3 we already proved that $\pi_0(G\mathcal{C})$ is a group when $\mathcal{C}$ is an exact category. The same arguments apply here and we are done. $\square$

Last but not least when $F. := wG.$ and $\mathcal{Q} := \mathcal{P}$ the class of pseudo-additive categories, property 3. of Definition 5.3.1 corresponds to the following proposition.

Proposition 5.3.1. *Let $\mathcal{C}$ be a pseudo-additive category, then the map*

$$|wG.E(\mathcal{C})| \longrightarrow |wG.\mathcal{C}| \times |wG.\mathcal{C}|$$

induced by f and g is a weak homotopy equivalence.

The proof of Proposition 5.3.1 will not be detailed in this document and we refer the reader to the proof of Theorem 2.10 in [Gun+92] for a detailed presentation. We note however that the authors use the pseudo-additivity hypothesis in a manner very similar to how it is used in Lemma 4.3.1 in the proof of Theorem 4.4.1. More specifically, the authors need to show that a morphism j defined on a sub-bisimplicial set of $wG.E(\mathcal{C})$ induced by a functor in $E(\mathcal{C})$ defined as

$$(A \rightarrowtail C \twoheadrightarrow C/A) \mapsto (A \rightarrowtail A \vee C/A \twoheadrightarrow C/A)$$

is homotopic to the identity. For that they define a natural sequence of weak equivalences corresponding to the vertical arrow in the following diagram

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\simeq"] & C \cup_A (A \vee C/A) \arrow[r, two heads] \arrow[d, "\sim"] & C/A \vee C/A \arrow[d, "\sim"] \\
A \arrow[r, tail] & C \cup_A C \arrow[r, two heads] & C/A \vee C/A
\end{tikzcd}$$

which induces that $\mathrm{id} \cup_A j$ and $\mathrm{id} \cup_A \mathrm{id}$ are related by a sequence of weak equivalences as functors in $E(\mathcal{C})$. By property 5. it implies that $|\mathrm{id} \cup_A j|$ and $|\mathrm{id} \cup_A \mathrm{id}|$ are homotopic as continuous mapping. The authors then use similar trick than we did in Lemma 4.3.1 to "substract" the extra id and show that $|j|$ and $|\mathrm{id}|$ are homotopic. Even though the details are different the overall strategy is the same and it seems that in both demonstrations the pseudo-additivity hypothesis has a similar purpose.

We now turn to briefly explaining Proposition 1.55' of [Gun+92], an important intermediary step towards Theorem 5.2.1. The proof of this proposition makes use of the additivity property as well as most of the properties given in Definition 5.3.1. Before stating the proposition, let $\mathcal{C}$ be a Waldausen category, $\mathcal{A}.$ be a simplicial Waldhausen category and $\mathcal{A}. \xrightarrow{f} S.\mathcal{C}$ be a morphism of simplicial Waldhausen categories. We define the simplicial category $\mathcal{G}f.$ such that the following diagram is cartesian

$$\begin{tikzcd}
\mathcal{G}f. \arrow[r] \arrow[d] & PS.\mathcal{C} \arrow[d, "\delta_0"] \\
\mathcal{A}. \arrow[r] & S.\mathcal{C}
\end{tikzcd}$$

Because the diagram is in swCof and the rightmost arrow has a retraction, then using the reasoning in [Wal85] page 325 we have that $\mathcal{G}f.$ itself is a simplicial Waldhausen category. Moreover we consider $\mathcal{C}$ as a constant simplicial Waldhausen category. Then there is a map $\mathcal{C} \longrightarrow PS.\mathcal{C}$ which at level n maps an object M to $M_0 = M \rightarrowtail ... \rightarrowtail M_n = M$ with trivial quotients. There is also a map $\mathcal{C} \longrightarrow \mathcal{A}.$ which at level n maps an object M to the zero object of $A.([n])$. Those two maps induce a map of simplicial Waldhausen categories $\mathcal{C} \longrightarrow \mathcal{G}f.$ such that the composed map $\mathcal{C} \longrightarrow \mathcal{G}f. \longrightarrow S.\mathcal{C}$ is trivial.

We now recall Proposition 1.55' from [Gun+92] which is a generalization of Proposition 1.5.5 from [Wal85] and provides an essential ingredient in proving Theorem 2.6 of [Gun+92].

Proposition 5.3.2. *Given $F.$ a $\mathcal{Q}$-admissible functor. Assuming that each $\mathcal{G}f_n$ is in $\mathcal{Q}$ then*

$$F.(\mathcal{C}) \longrightarrow F.(\mathcal{G}f.) \longrightarrow F.(\mathcal{A}.)$$

is a fibration sequence up to homotopy.

We notice that when we consider $\mathcal{Q} = \mathrm{wCof}$, $F. = S.$ and $\mathcal{A}. = S.\mathcal{B}$ for a Waldhausen category $\mathcal{B}$, we recover Proposition 1.5.5 from [Wal85]. For a detailed proof of Proposition 5.3.2 we refer the reader to the proof of Proposition 1.55' in [Gun+92].

Let's however have a quick look at the first part of the proof that is very similar to the proof in [Wal85] and that consist in proving that for any $n \geq 0$

$$F.(\mathcal{C}) \longrightarrow F.(\mathcal{G}f_n) \longrightarrow F.(\mathcal{A}_n)$$

is weakly equivalent to the trivial fibration sequence

$$F.(\mathcal{C}) \longrightarrow F.(\mathcal{A}_n) \times F.(\mathcal{C}) \longrightarrow F.(\mathcal{A}_n)$$

In particular we want to define an homotopy equivalence $F.(\mathcal{A}_n) \times F.(\mathcal{C}) \longrightarrow F.(\mathcal{G}f_n)$ that commute with the two sequences. Making use of the fact that $\mathcal{C}$ is isomorphic to $S.\mathcal{C}([1])$ we define a functor $H : \mathcal{A}_n \times \mathcal{C} \longrightarrow \mathcal{G}f_n$ that maps object (a, c) to $(a, S.(\mathrm{id}_0 \bullet \chi)(c) \vee S.(s_0) \circ f_n(a))$ where $\chi : [n] \longrightarrow [0]$ is the constant function. In other words using the notation of the previous section and assuming $f_n(a)$ is represented by $(x_1 \rightarrowtail \cdots \rightarrowtail x_n)$ then it maps (a, c) to (a, d) where d is represented by $(b \rightarrowtail b \vee x_1 \rightarrowtail \cdots \rightarrowtail b \vee x_n)$. We also define a functor $G : \mathcal{G}f_n \longrightarrow \mathcal{A}_n \times \mathcal{C}$ that maps object (a, c') to $(a, S.(\omega)(c'))$ where $\omega : [1] \longrightarrow [n+1]$ is such that $\omega(0) = 0$ and $\omega(1) = 1$. Explicitely if c' is represented by $(y_0 \rightarrowtail \cdots \rightarrowtail y_n)$ it maps (a, c') to (a, y_0). It is obvious that there is a natural isomorphism $G \circ H \simeq \mathrm{id}_{\mathcal{A}_n \times \mathcal{C}}$. Moreover we check that $H \circ G = j \vee j'$ where $j, j' : \mathcal{G}f_n \longrightarrow \mathcal{G}f_n$ are morphism that respectively map (a, c') to $(0, S.(\theta_1)(c'))$ and $(a, S.(s_0) \circ S.(d_0)(c'))$ where $\theta_1 : [n+1] \longrightarrow [n+1]$ is such that $\theta_1(0) = 0$ and for all $0 < i \leq n+1$ we have $\theta_1(i) = 1$. Explicitely they map (a, b) where b is represented by $(y_0 \rightarrowtail y_1 \rightarrowtail \cdots \rightarrowtail y_n)$ to $(*, (y_0 \rightarrowtail y_0 \rightarrowtail \cdots \rightarrowtail y_0))$ and $(a, (* \rightarrowtail y_1/y_0 \rightarrowtail \cdots \rightarrowtail y_n/y_0))$ respectively. Moreover we have that $j \rightarrowtail \mathrm{id} \twoheadrightarrow j'$ is a sequence of cofibration. The authors have proved in [Gun+92] that the additivity hypothesis on $F.$ given that $\mathcal{G}f_n$ is in $\mathcal{Q}$ implies that $F.(j \vee j')$ is homotopy equivalent to $F.(\mathrm{id})$. Therefore F and G are homotopy equivalences and we are done proving the first part of the lemma. We make use of property 2. of Definition 5.3.1 because we defined an homotopy equivalence $F.(H) : F.(\mathcal{A}_n \times \mathcal{C}) \longrightarrow F.(\mathcal{G}f_n)$ which by property 2. corresponds up to homotopy to an homotopy equivalence $F.(\mathcal{A}_n) \times F.(\mathcal{C}) \longrightarrow F.(\mathcal{G}f_n)$. In the second part of the proof, we note that the authors use property 4. of Definition 5.3.1 to show that the above statement is enough to prove Proposition 5.3.2.

Finally, we will conclude this section by showing how Proposition 5.3.2 can be used to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Let $\mathcal{C}$ be a Waldhausen category. We will apply Proposition 5.3.2 several times. For the first time we set $\mathcal{A}. \xrightarrow{f} S.\mathcal{C}$ to be $PS.\mathcal{C} \xrightarrow{\delta_0} S.\mathcal{C}$ which means that $\mathcal{G}f. = wG.\mathcal{M}$. Then we have that $F.\mathcal{C} \longrightarrow F.G.\mathcal{C} \longrightarrow F.PS.\mathcal{C}$ is a fibration sequence. Because $PS.\mathcal{C}$ is contractible so is $F.PS.\mathcal{C}$ using property 1. of Definition 5.3.1 and it has trivial homotopy groups. Therefore $F.\mathcal{C} \longrightarrow F.G.\mathcal{C}$ is a weak equivalence using the long exact sequence. We apply this to $F. = wG.$ and $\mathcal{Q}$ the class of pseudo-additive categories. We only need to check that if $\mathcal{C}$ is pseudo-additive then for any $n \geq 0$, $wG.\mathcal{C}([n])$ is also pseudo-additive. We also apply this to $F. = wS.$ and $\mathcal{Q} = \text{wCof}$ the class of all Waldhausen categories.

We now apply Proposition 5.3.2 a second time. We set $\mathcal{A}. \xrightarrow{f} S.\mathcal{C}$ to be $S.\mathcal{C} \xrightarrow{\mathrm{id}_{S.\mathcal{C}}} S.\mathcal{C}$ and we get that $F.\mathcal{C} \longrightarrow F.PS.\mathcal{C} \longrightarrow F.S.\mathcal{C}$ is a fibration sequence. To apply it for $F. = wG.$ and $\mathcal{Q}$ the class of pseudo-additive categories we only need to check that if $\mathcal{C}$ is pseudo-additive then for any $n \geq 0$, $S.\mathcal{C}([n])$ is also pseudo-additive.

Moreover we consider the diagram in which the vertical sequences are the two previous fibration sequence.

$$\begin{tikzcd}
wG.(\mathcal{C}) \arrow[r, "="] \arrow[d] & wG.(\mathcal{C}) \arrow[d] \\
wG.(G.\mathcal{C}) \arrow[r] \arrow[d] & wG.(PS.\mathcal{C}) \arrow[d] \\
wG.(PS.\mathcal{C}) \arrow[r] & wG.(S.\mathcal{C})
\end{tikzcd}$$

We can deduce from the fact that the vertical sequence are fibration sequences that the bottom square is homotopy cartesian (see [Gun+92]). We now consider the map

$$\begin{tikzcd}
wG.\mathcal{C} \arrow[r] \arrow[d] \arrow[rr, bend left, "\alpha"] & wPS.\mathcal{C} \arrow[d] \arrow[rr, bend left, "\beta"] & wG.(G.\mathcal{C}) \arrow[r] \arrow[d] & wG.(PS.\mathcal{C}) \arrow[d] \\
wPS.\mathcal{C} \arrow[r] \arrow[rr, bend right, "\gamma"'] & wS.\mathcal{C} \arrow[rr, bend right, "\delta"'] & wG.(PS.\mathcal{C}) \arrow[r] & wG.(S.\mathcal{C})
\end{tikzcd}$$

We have already shown above that α is a homotopy equivalence and because they map contractible spaces so are β and γ. For δ we simply check that it is isomorphic to the weak homotopy equivalence $wS.\mathcal{C} \longrightarrow wS.(G.\mathcal{C})$ given above using the isomorphism $wG.(S.\mathcal{C}) \simeq wS.(G.\mathcal{C})$. Therefore because the right square is homotopy cartesian so is the left square. Hence the map $|wG.\mathcal{C}| \longrightarrow \Omega|wS.\mathcal{C}|$ is a weak homotopy equivalence and we are done. $\square$

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