Hyperbolic Trigonometry

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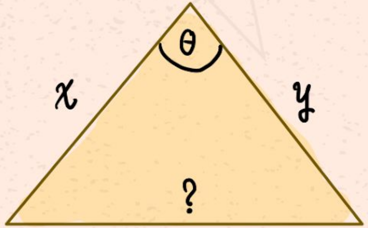
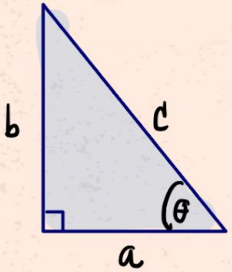
CONTENTS

History of Trigonometry

The study of triangles





high school trig

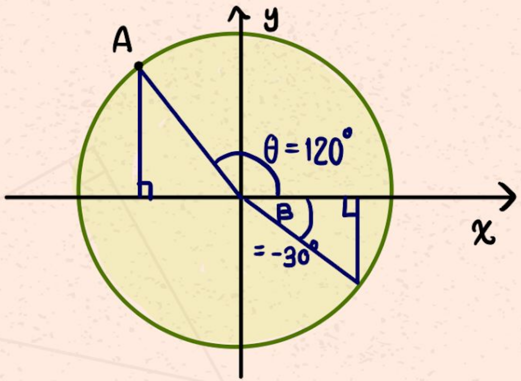
$$\sin(\theta) = \frac{b}{c}$$

$$\cos(\theta) = \frac{a}{c}$$

$$\tan(\theta) = \frac{b}{a}$$

- Law of sine
- Law of cosine

Circular trig



$$A = (\sin\theta, \cos\theta)$$

Modern definitions

$$\sin x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

What are hyperbolic trig?

Circular trig

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \frac{e^{ix} + e^{-ix}}{2}$$

Hyperbolic trig

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{e^{x} + e^{-x}}{2}$$

Development of hyperbolic trigonometry

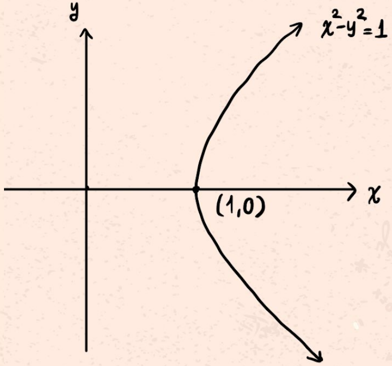
Lambert's paper in 1761

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\cancel{\sinh x} = \frac{e^{x} - e^{-x}}{2}$$

$$\cancel{\cosh x} = \frac{e^{x} + e^{-x}}{2}$$

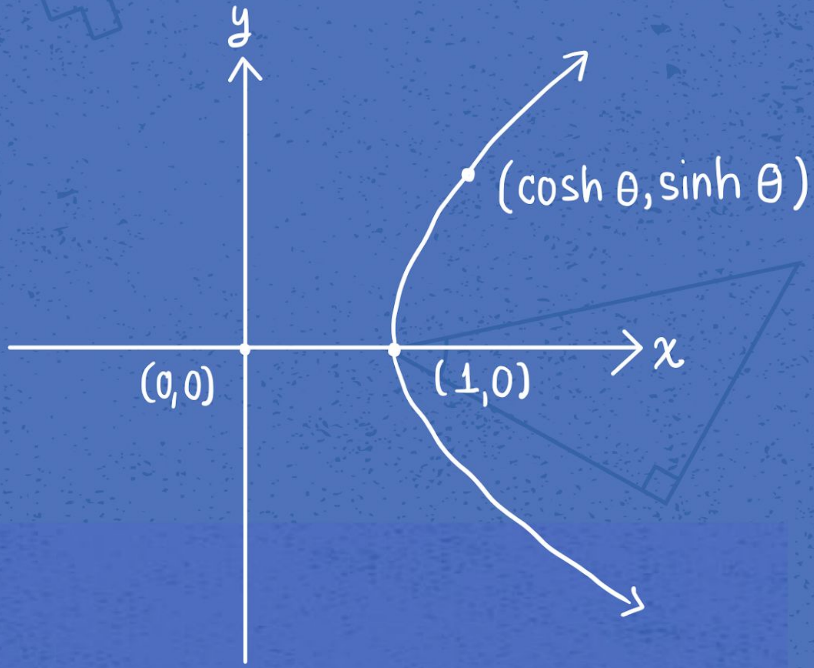


Riccati's works in 1757-1762

"hyperbolic sine" & "hyperbolic cosine"

His notations and languages are not familiar to today's scholars!

How are $\sinh\theta$ and $\cosh\theta$ related to a hyperbola?



Unit hyperbola $x^2 - y^2 = 1$

$$\cosh^2\theta - \sinh^2\theta$$

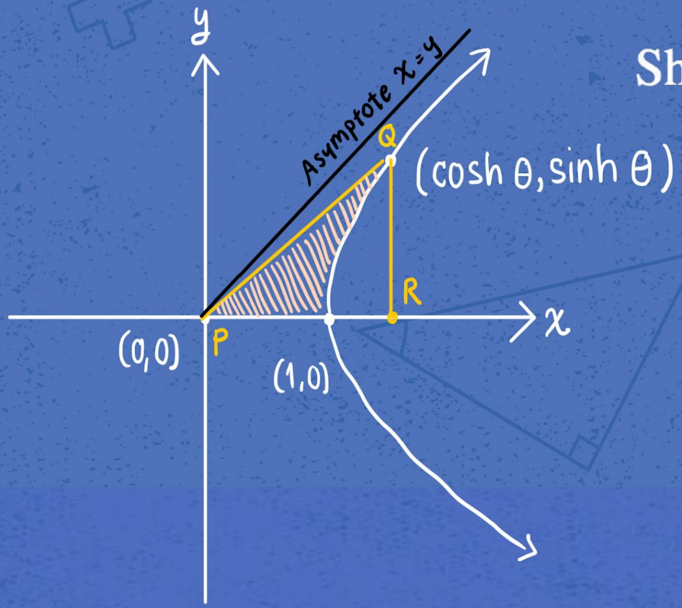
$$= \left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^2 - \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^2$$

$$= \frac{e^{2\theta}+2+e^{-2\theta}}{4} - \frac{e^{2\theta}-2+e^{-2\theta}}{4}$$

$$= \frac{4}{4} = 1$$

A point on unit hyperbola: $(\cosh\theta, \sinh\theta)$

What is θ?

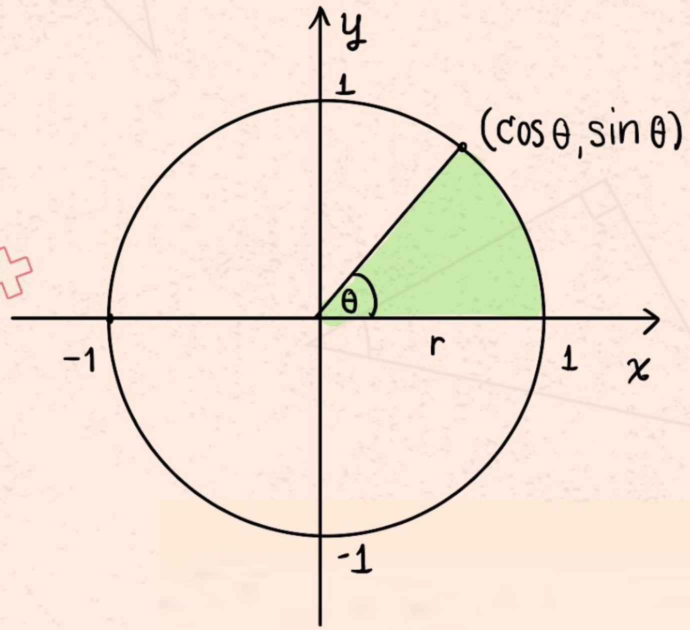


⊕ Unit hyperbola $x^2-y^2=1$

$$\text{Shaded area} = [\Delta PQR] - \text{Area under the curve}$$
$$= \frac{\sinh\theta\cdot\cosh\theta}{2} - \int_{\cosh 0}^{\cosh\theta}\sqrt{x^2-1}dx$$
$$= \frac{\sinh\theta\cdot\cosh\theta}{2} - \int_0^{\theta}\sinh^2\theta d\theta$$
$$= \frac{2\cdot\sinh\theta\cdot\cosh\theta}{4} - \int_0^{\theta}\frac{e^{2\theta}-2+e^{-2\theta}}{4}d\theta$$
$$= \frac{\sinh 2\theta}{4} - \left(\frac{\sinh 2\theta - 2\theta}{4}\right)\Big|_0^{\theta} = \theta/2$$

θ is twice of the shaded area.

What about θ for circular trigs ?



$$\cos^2\theta + \sin^2\theta = 1$$

A point on the unit circle: $(\cos\theta, \sin\theta)$

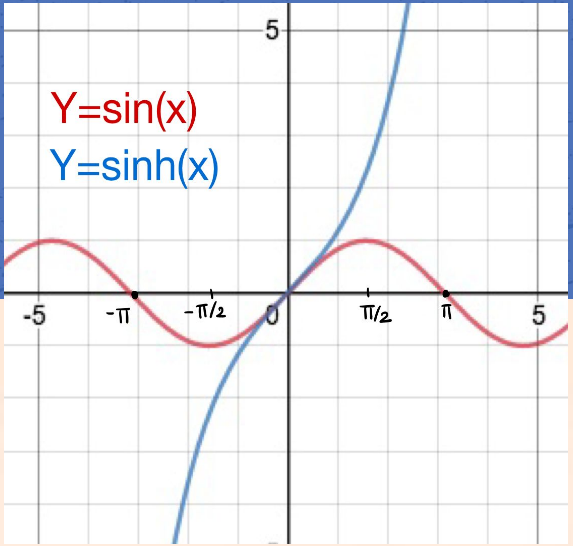
- The angle measured from the positive x-axis.
- Twice the sector area

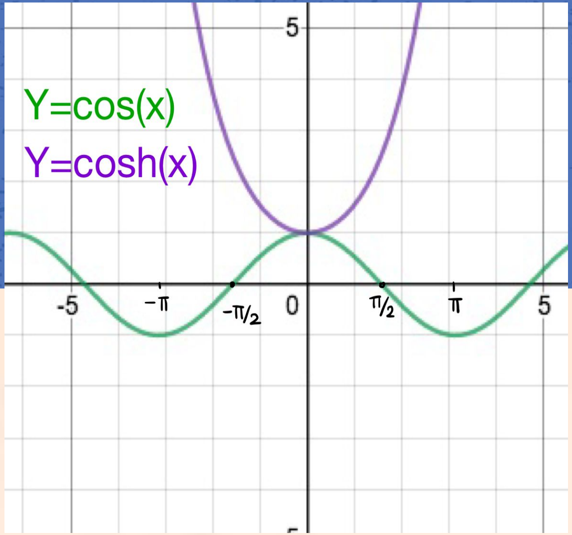
The sector area

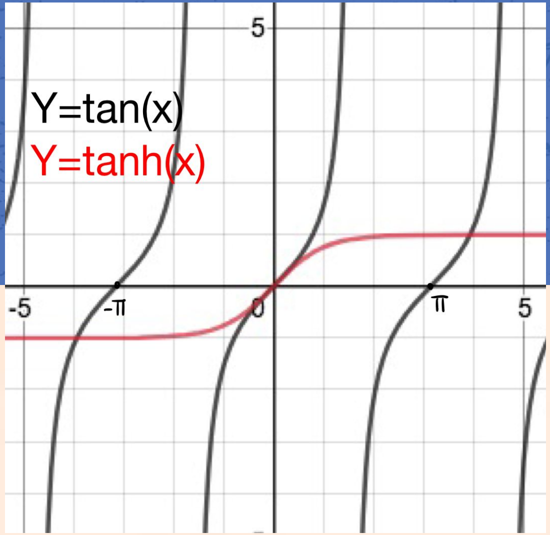
$$= \frac{\theta}{2\pi} \cdot \pi \cdot 1^2$$

$$= \frac{\theta}{2}$$

Visualization of circular and hyperbolic trigs







$\sin(x), \cos(x), \tan(x)$ are periodic while $\sinh(x), \cosh(x), \tanh(x)$ are not periodic.

+ Definitions of hyperbolic trig

+ History of hyperbolic trig

- Relationship to circular trig

  Poincaré distance

  Bolyai-Lobachevsky Formula

- Hyperbolic trig identities

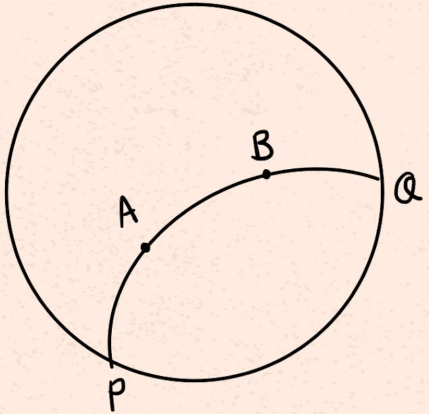
Poincaré Distance

Given two points A, B on a Poincaré line and the line intersects the

then, in hyperbolic geometry, the poincaré distance $d(A, B)$ is

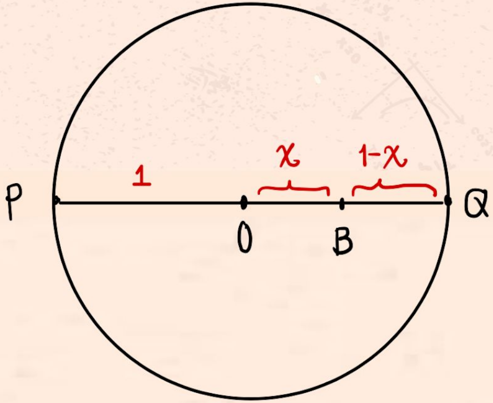
$$
\begin{aligned}
d(A,B) &= |\ln([P,Q,A,B])| \\
&= \left|\ln\left(\frac{PB \cdot QA}{PA \cdot QB}\right)\right|
\end{aligned}
$$

Cross ratio



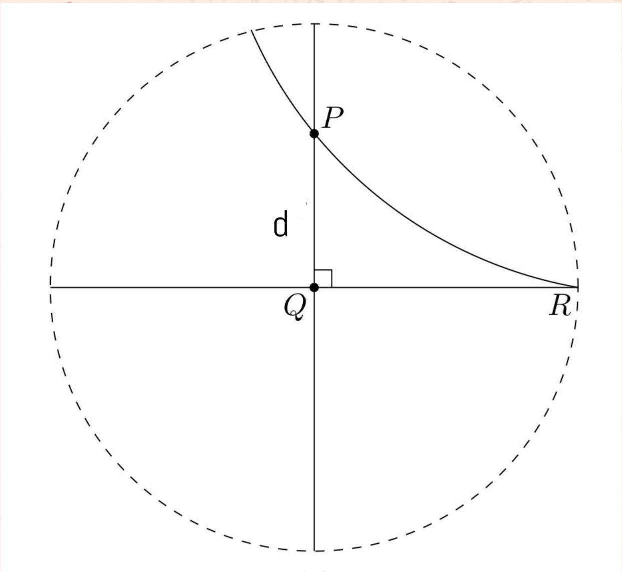
In particular, the Poincaré length from the origin O to B is

$$
\begin{aligned}
d(O,B) &= |\ln([P,Q,O,B])| \\
&= \left|\ln\left(\frac{PB \cdot QO}{PO \cdot QB}\right)\right| \\
&= \left|\ln\left(\frac{1+x}{1-x}\right)\right|
\end{aligned}
$$



Relates the angle of parallelism to distance

Bolyai-Lobachevsky Formula



Let α be the angle of parallelism for P with respect to l and d be the hyperbolic distance from P to Q, where PQ is perpendicular to l. Then the Bolyai-Lobachevsky Formula says:

$$\tan\left(\frac{\alpha}{2}\right) = e^{-d}$$

Proof of Bolyai-Lobachevsky Formula

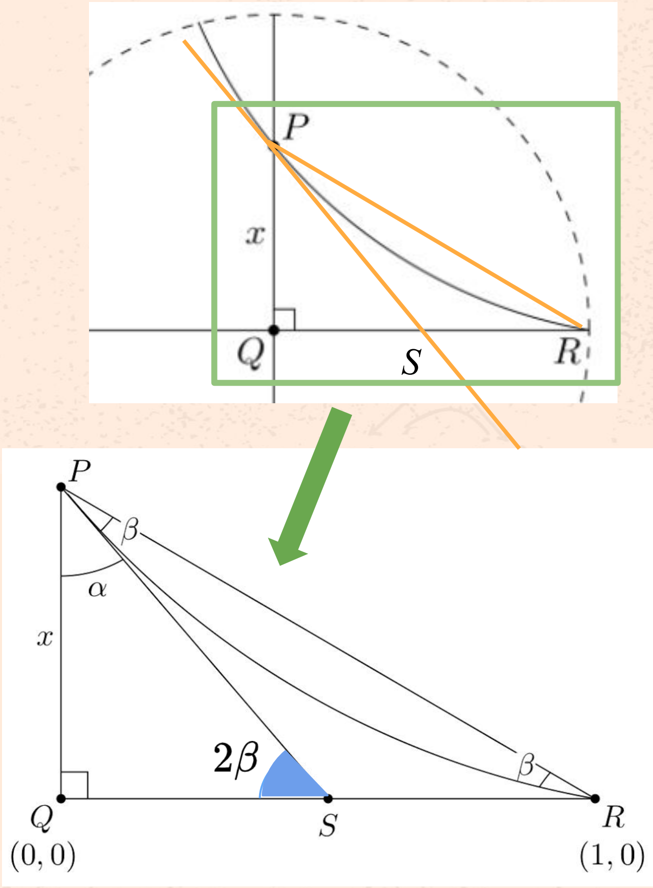
Consider $\triangle$PQR where Q is the center of the unit circle and R is on the unit circle. The Pointcaré distance formula tells us that $d = \left|\ln\left(\frac{1+x}{1-x}\right)\right|$

$$\Rightarrow -d = \left|\ln\left(\frac{1-x}{1+x}\right)\right|$$

$$\Rightarrow \boxed{e^{-d} = \frac{1-x}{1+x}}$$

Then because $\overline{PS}$, $\overline{SR}$ are tangent to the arc $\overset{\frown}{PR}$, we can show $\overline{PS} = \overline{SR}$ using triangle congruences (SSS). Then $\angle SPR = \angle SRP = \beta$. Then $\angle PSQ = 2\beta$.

It follows that $\alpha = \frac{\pi}{2} - 2\beta \Rightarrow \boxed{\frac{\alpha}{2} = \frac{\pi}{4} - \beta}$



Proof of Bolyai-Lobachevsky Formula

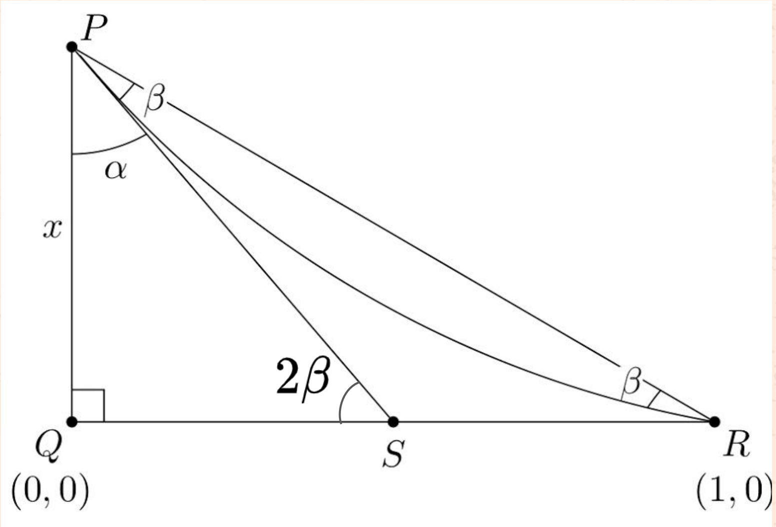
Recall the formula of tangent:

$$\tan(\theta - \gamma) = \frac{\tan(\theta) - \tan(\gamma)}{1 + \tan(\theta)\tan(\gamma)}$$

Plug in $\frac{\alpha}{2} = \frac{\pi}{4} - \beta$ in this formula:

$$\begin{aligned} \tan\left(\frac{\alpha}{2}\right) &= \tan\left(\frac{\pi}{4} - \beta\right) \\ &= \frac{1 - \tan(\beta)}{1 + \tan(\beta)} \\ &= \frac{1 - x}{1 + x} \\ &= e^{-d} \end{aligned}$$

$\square$



Alternative Forms of Bolyai-Lobachevsky Formula

Let $\alpha = \Pi(d)$. Then from $\tan\left(\frac{\alpha}{2}\right) = e^{-d}$ we have:

$$\Pi(d) = 2 \cdot \arctan\left(e^{-d}\right)$$

From here we have the following theorems that relate circular and hyperbolic trigs. Let x be the hyperbolic distance. Then

The angle of parallelism is denoted as $\alpha = \Pi(d)$

$$\sin\left(\Pi(x)\right) = \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\cos\left(\Pi(x)\right) = \tanh(x)$$

$$\tan\left(\Pi(x)\right) = \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

+ Definition of hyperbolic trig

+ History of hyperbolic trig

+ Relationship to circular trig

Poincare distance

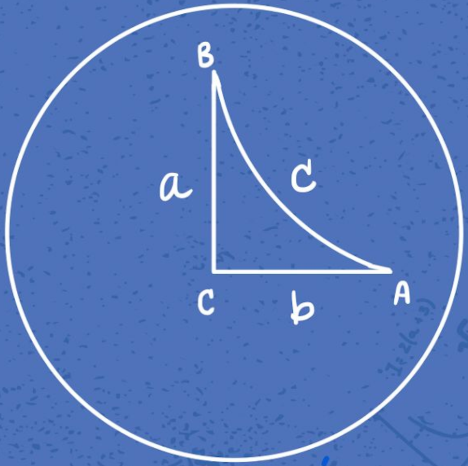
Bolyai-Lobachevsky Formula

How are these sinh, cosh, tanh even “trig” functions if they cannot be used to study triangles ?!

- Hyperbolic trig identities

Right Triangle Trigonometric Identities

A right hyperbolic triangle
with C being a right angle



1
$$\sin A = \frac{\sinh a}{\sinh c} \qquad \cos A = \frac{\tanh b}{\tanh c}$$

2
$$\cosh c = \cosh a \cdot \cosh b = \cot A \cdot \cot B$$

3
$$\cosh a = \frac{\cos A}{\sin B}$$

Euclidean counterparts

2

$$\cosh c = \cosh a \cdot \cosh b$$

$$\sum_{n=0}^{\infty} \frac{c^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \sum_{n=0}^{\infty} \frac{b^{2n}}{(2n)!}$$

$$1 + \frac{1}{2}c^2 + \cdots = 1 + \frac{1}{2}(a^2 + b^2) + \cdots$$

Pythagoras theorem $c^2 \approx a^2 + b^2$

	Hyperbolic	Euclidean	Difference
$a = 4,\ b = 1$	$c = 4.45$	$c = 4.12$	.33
$a = 5,\ b = 2$	$c = 6.33$	$c = 5.36$	.97
$a = 6,\ b = 3$	$c = 8.31$	$c = 6.71$	1.6
$a = 13,\ b = 10$	$c = 22.3$	$c = 16.4$	5.9
$a = 25,\ b = 30$	$c = 54$	$c = 39.05$	14.95
$a = 40,\ b = 50$	$c = 89.3$	$c = 64$	25.3
$a = 55,\ b = 70$	$c = 124.3$	$c = 89$	35.3

1

$$\sin A = \frac{\sinh a}{\sinh c} \quad \cos A = \frac{\tanh b}{\tanh c}$$

$$\sin A \approx \frac{a}{c} \quad \cos A \approx \frac{b}{c}$$

Hyperbolic Identities for any Triangles

Let $\triangle ABC$ be any triangle in the hyperbolic plane. Then:

The Sine Rule

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}$$

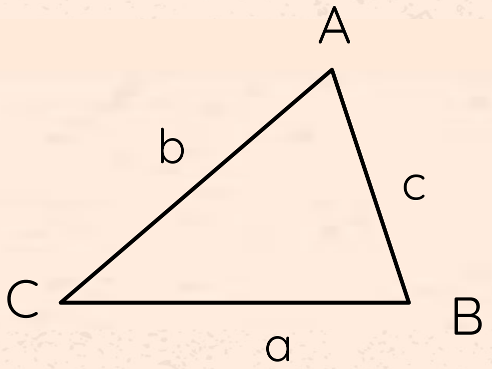
The Cosine Rule I

$$\cosh c = \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C$$

The Cosine Rule II

$$\cosh c = \frac{\cos A \cdot \cos B + \cos C}{\sin A \cdot \sin B}$$

Again, the sine rule and the cosine rule I reduce to their Euclidean counterparts when applying to a **sufficiently small** triangle !!



Conclusion

	Circular Trig	Hyperbolic Trig
Models	θ = twice the sector area	θ = twice the area shaped by the hyperbola
Right △	$\sin A = \frac{a}{c}, \cos A = \frac{b}{c}$	$\sin A = \frac{\sinh a}{\sinh c}, \cos A = \frac{\tanh b}{\tanh c}$
Right △	$c^2 = a^2 + b^2$	$\cosh c = \cosh a \cdot \cosh b$
Any △	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}$
Any △	$c^2 = a^2 + b^2 - 2ab\cos C$	$\cosh c = \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C$

THANKS!

WE ARE HAPPY TO TAKE ANY QUESTIONS!

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