~min sheet~

partiality

- just cos non-termination
  - see Bob Harper book

PCF cont.

```
> inc :: Int -> Int
> inc n = inc (n+1)

> incF :: (Int -> Int) -> (Int -> Int)
> incF f n = f (n+1)

> inc :: Int -> Int
> inc = fix incF
```

$$\vdash \mathrm{fix}(x : \mathrm{Nat}.\, \mathrm{succ}(x)) : \mathrm{Nat}$$

$$\dfrac{\dfrac{\dfrac{}{x:\mathrm{Nat} \vdash x:\mathrm{Nat}}\,\mathrm{Var}}{x:\mathrm{Nat} \vdash \mathrm{succ}(x) : \mathrm{Nat}}\,\mathrm{Succ}}{\vdash \mathrm{fix}(x : \mathrm{Nat}.\, \mathrm{succ}(x)) : \mathrm{Nat}}\,\mathrm{Fix}$$

$$\mathrm{pred} \overset{\mathrm{def}}{=}$$

$$\lambda e : \mathrm{Nat}.\ \mathrm{ifz}(e\,;\, \mathrm{zero}\,;\, x.x) : \mathrm{Nat} \Rightarrow \mathrm{Nat}$$

```
> pred : Nat -> Nat
> pred Z = Z
> pred (S x) = x
```

$$\mathrm{pred}(\mathrm{succ}(\mathrm{zero}))$$

$$=$$

$$(\lambda e : \mathrm{Nat}.\ \mathrm{ifz}(e\,;\, \mathrm{zero}\,;\, x.x))(\mathrm{succ}(\mathrm{zero}))$$

$\mapsto$ D-BETA

$$\mathrm{ifz}(\mathrm{succ}(\mathrm{zero})\,;\, \mathrm{zero}\,;\, x.x)$$

$\mapsto$ D-IFZ-SUCC, VAL-SUCC, VAL-ZERO

$$x[\mathrm{zero}/x]$$

$\mapsto$ zero

Programming PCF

```
> data Nat = Z | S Nat

> plus :: Nat -> Nat -> Nat
> plus Z b = b
> plus (S a) b = S (plus a b)
```

$$plus \stackrel{def}{\equiv} fix(f : Nat \to Nat \to Nat.$$
$$\lambda a : Nat . \lambda b : Nat .$$
$$ifz(a ; b ; x . succ(f(x)(b)).$$

Turing-Completeness of PCF

PCF is turing complete

PCF-definable functions $\iff$ partial recursive functions

A partial function $f : \mathbb{N} \to \mathbb{N}$ is PCF-definable iff there exist a PCF term $\vdash e : Nat \to Nat$ with

$$f(x) \simeq y \iff e(succ^{x}(zero)) \mapsto^{*} succ^{y}(zero)$$

$f : \mathbb{N} \to \mathbb{N}$

x arg: $\mathbb{N}$

y: $\mathbb{N}$

Sequentiality of PCF

Theorem:

The parallel OR function cannot be expressed in PCF

There is no PCF term

$\vdash par : Nat \to Nat \to Nat$

that satisfies the following criteria

True $\sim$ zero

False $\sim$ succ(zero)

1. If $e_1 \mapsto^* zero$ then $par(e_1)(e_2) \mapsto^* zero$
2. If $e_2 \mapsto^* zero$ then $par(e_1)(e_2) \mapsto^* zero$
3. If $e_1 \mapsto^* succ(zero)$ and $e_2 \mapsto^* succ(zero)$ then $par(e_1)(e_2) \mapsto^* succ(zero)$