Problem Sheet 2

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The following questions are about the language of numbers and strings.

1. Write down the abstract syntax tree for the pre-term $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\, \mathsf{plus}(i; n)); \mathsf{num}[2])$.

2. Assume $\Sigma \stackrel{\text{def}}{=} \{0, 1\}$. Write a program that

    - has a free variable x of type Str,
    - appends the string 0110 to x,
    - computes the length of the compound string, and
    - adds that number to itself.

    Your program should not mention the string literal $\mathsf{str}[0110]$ more than once.

    Solution: $x : \mathsf{Str} \vdash \mathsf{let}(\mathsf{len}(\mathsf{cat}(x; \mathsf{str}[0110])); x.\, \mathsf{plus}(x; x)) : \mathsf{Num}$

3. Produce a typing derivation for the following terms, assuming that $\Sigma \stackrel{\text{def}}{=} \{0, 1\}$.

    (i) $x : \mathsf{Str} \vdash x : \mathsf{Str}$

    (ii) $\vdash \mathsf{plus}(\mathsf{num}[1]; \mathsf{num}[1]) : \mathsf{Num}$

    (iii) $x : \mathsf{Str} \vdash \mathsf{cat}(x; \mathsf{str}[01]) : \mathsf{Str}$

    (iv) $x : \mathsf{Str}, n : \mathsf{Num} \vdash \mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\, \mathsf{plus}(i; n)); \mathsf{num}[2]) : \mathsf{Num}$

4. Perform the following substitutions, step-by-step.

    (i) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\, \mathsf{plus}(i; n)); \mathsf{num}[2])[i/x]$

    (ii) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\, \mathsf{plus}(i; n)); \mathsf{num}[2])[\mathsf{num}[0]/n]$

    (iii) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\, \mathsf{plus}(i; n)); \mathsf{num}[2])[i/n]$

    Solution: We give the final answer here, not the step-by-step calculation.

    (i) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(i); i.\, \mathsf{plus}(i; n)); \mathsf{num}[2])$

    (ii) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); i.\, \mathsf{plus}(i; \mathsf{num}[0])); \mathsf{num}[2])$

    (iii) $\mathsf{plus}(\mathsf{let}(\mathsf{len}(x); j.\, \mathsf{plus}(j; i)); \mathsf{num}[2])$

5. State the cases of the inversion lemma for the following constructs:

    (i) $\mathsf{len}(e)$

(ii) $\mathsf{let}(e_1; x.\, e_2)$

Solution:

(i) If $\Gamma \vdash \mathsf{len}(e) : \tau$ then it must be that $\tau = \mathsf{Num}$ and $\Gamma \vdash e : \mathsf{Str}$.

(ii) If $\Gamma \vdash \mathsf{let}(e_1; x.\, e_2) : \tau$ there must be τ_1 such that $\Gamma \vdash e_1 : \tau_1$ and $\Gamma, x : \tau_1 \vdash e_2 : \tau$.

6. Prove the weakening lemma for the programming language of numbers and strings.

Solution: The claim is that the rule

$$\frac{\Gamma \vdash e : \tau}{\Gamma, x : \sigma \vdash e : \tau}$$

is admissible. The proof is by induction on the derivation of $\Gamma \vdash e : \tau$.

Case(Var). Suppose that the derivation is of the form

$$\frac{}{\Gamma, y : \tau \vdash y : \tau}\ \textsc{Var}$$

Then, we see that the following is a valid derivation, which gives us the result.

$$\frac{}{\Gamma, y : \tau, x : \sigma \vdash y : \tau}\ \textsc{Var}$$

Case(Num). Suppose that the derivation is of the form

$$\frac{n \in \mathbb{N}}{\Gamma \vdash \mathsf{num}[n] : \mathsf{Num}}\ \textsc{Num}$$

Then, we see that the following is a valid derivation, which gives us the result.

$$\frac{n \in \mathbb{N}}{\Gamma, x : \sigma \vdash \mathsf{num}[n] : \mathsf{Num}}\ \textsc{Num}$$

Case(Str). Similar to Num.

Case(Plus). Suppose that the derivation is of the form

$$\frac{\dfrac{\vdots}{\Gamma \vdash e_1 : \mathsf{Num}} \qquad \dfrac{\vdots}{\Gamma \vdash e_2 : \mathsf{Num}}}{\Gamma \vdash \mathsf{plus}(e_1; e_2) : \mathsf{Num}}\ \textsc{Plus}$$

(so in fact $e = \mathsf{plus}(e_1; e_2)$ and $\tau = \mathsf{Num}$). Then by **the inductive hypothesis** we have derivations of $\Gamma, x : \sigma \vdash e_1 : \mathsf{Num}$ and $\Gamma, x : \sigma \vdash e_2 : \mathsf{Num}$. We can combine these into a derivation of the conclusion using the Plus rule:

$$\frac{\dfrac{\vdots}{\Gamma, x : \sigma \vdash e_1 : \mathsf{Num}} \qquad \dfrac{\vdots}{\Gamma, x : \sigma \vdash e_2 : \mathsf{Num}}}{\Gamma, x : \sigma \vdash \mathsf{plus}(e_1; e_2) : \mathsf{Num}}\ \textsc{Plus}$$

Case(TIMES). Similar to PLUS.

Case(CAT). Similar to PLUS.

Case(LEN). Similar to PLUS, but with one fewer premise.

Case(LET). Suppose that the derivation is of the form

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash e_1 : \sigma_1} \qquad \dfrac{\vdots}{\Gamma, y : \sigma_1 \vdash e_2 : \sigma_2}}{\Gamma \vdash \mathsf{let}(e_1; y.\, e_2) : \sigma_2} \text{ LET}$$

for some σ_1. (Notice how I have sneakily chosen y to express this rule. By α-conversion the choice of bound variable does not matter.)

By the **inductive hypothesis** we may assume that we have derivations of $\Gamma, x : \sigma \vdash e_1 : \sigma_1$, and $\Gamma, y : \sigma_1, x : \sigma \vdash e_2 : \sigma_2$. Using these derivations we can construct a derivation of the desired judgement by applying the LET rule again:

$$\dfrac{\dfrac{\vdots}{\Gamma, x : \sigma \vdash e_1 : \sigma_1} \qquad \dfrac{\vdots}{\Gamma, y : \sigma_1, x : \sigma \vdash e_2 : \sigma_2}}{\Gamma, x : \sigma \vdash \mathsf{let}(e_1; y.\, e_2) : \sigma_2} \text{ LET}$$

7. ($*$) Complete the proof of substitution from Lecture 4.

[Hint: In the case of variables, consider various cases: is it the variable I'm substituting for, or is it not? Also, you will have to use weakening, so assume that you have proven that already.]

Solution: The claim is that the rule

$$\dfrac{\Gamma \vdash e : \tau \qquad \Gamma, x : \tau \vdash u : \sigma}{\Gamma \vdash u[e/x] : \sigma}$$

is admissible. The proof is by induction on the derivation of $\Gamma, x : \tau \vdash u : \sigma$.

(One could try induction on e, but it wouldn't work. Why?)

Case(VAR). Suppose that the typing derivation of u is of the form

$$\dfrac{}{\Gamma, x : \tau \vdash z : \sigma} \text{ VAR}$$

(so in fact u is just a variable z). There are two cases to consider here: either z occurs somewhere in the rest of the context Γ, or the variables z and x are the same.

- If $z \not\equiv x$, then the variable z is somewhere in Γ. By the definition of substitution

$$z[e/x] \stackrel{\text{def}}{=} z$$

Then, the desired derivation is constructed by applying the variable rule:

$$\frac{}{\Gamma \vdash z : \sigma}\ \textsc{Var}$$

- If $z \equiv x$, then we also know that $\sigma = \tau$. By the definition of substitution we have

  $$x[e/x] \stackrel{\text{def}}{=} e$$

  Then, the desired derivation is just the derivation given by the premise:

  $$\frac{\vdots}{\Gamma \vdash e : \tau}$$

Case(Num). Suppose that the derivation is of the form

$$\frac{n \in \mathbb{N}}{\Gamma, x : \tau \vdash \mathsf{num}[n] : \mathsf{Num}}\ \textsc{Num}$$

(so in fact u is the term $\mathsf{num}[n]$). By the definition of substitution,

$$(\mathsf{num}[n])[x/e] \stackrel{\text{def}}{=} \mathsf{num}[n]$$

Then the following is a valid derivation of the conclusion of the rule.

$$\frac{n \in \mathbb{N}}{\Gamma \vdash \mathsf{num}[n] : \mathsf{Num}}\ \textsc{Num}$$

Case(Str). Similar to Num.

Case(Plus). Suppose that the derivation is of the form

$$\frac{\dfrac{\vdots}{\Gamma, x : \tau \vdash e_1 : \mathsf{Num}} \qquad \dfrac{\vdots}{\Gamma, x : \tau \vdash e_2 : \mathsf{Num}}}{\Gamma, x : \tau \vdash \mathsf{plus}(e_1; e_2) : \mathsf{Num}}\ \textsc{Plus}$$

(so in fact $e = \mathsf{plus}(e_1; e_2)$ and $\sigma = \mathsf{Num}$). By the definition of substitution we have

$$(\mathsf{plus}(e_1; e_2))[e/x] \stackrel{\text{def}}{=} \mathsf{plus}(e_1[e/x]; e_2[e/x]) \tag{1}$$

By the **inductive hypothesis** applied to the two premises, we obtain derivations of the judgments $\Gamma \vdash e_1[e/x] : \mathsf{Num}$ and $\Gamma \vdash e_2[e/x] : \mathsf{Num}$. Using the Plus rule we can combine these into a derivation

$$\frac{\dfrac{\vdots}{\Gamma \vdash e_1[e/x] : \mathsf{Num}} \qquad \dfrac{\vdots}{\Gamma \vdash e_2[e/x] : \mathsf{Num}}}{\Gamma \vdash \mathsf{plus}(e_1[e/x]; e_2[e/x]) : \mathsf{Num}}\ \textsc{Plus}$$

But notice that by (1) the subject of this typing derivation is the term $(\mathsf{plus}(e_1; e_2))[e/x]$.

Case(Times). Similar to Plus.

Case(Cat). Similar to Plus.

Case(Len). Similar to Plus, but with one fewer premise.

Case(Let). Suppose that the derivation is of the form

$$\dfrac{\dfrac{\vdots}{\Gamma, x : \tau \vdash e_1 : \sigma_1} \qquad \dfrac{\vdots}{\Gamma, x : \tau, y : \sigma_1 \vdash e_2 : \sigma_2}}{\Gamma, x : \tau \vdash \mathsf{let}(e_1; y.\, e_2) : \sigma_2} \ \textsc{Let}$$

for some type σ_1. By the **induction hypothesis** we get derivations of $\Gamma \vdash e_1[e/x] : \sigma_1$ and $\Gamma, y : \sigma_1 \vdash e_2[e/x] : \sigma_2$. Using the Let rule we obtain a derivation

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash e_1[e/x] : \sigma_1} \qquad \dfrac{\vdots}{\Gamma, y : \sigma_1 \vdash e_2[e/x] : \sigma_2}}{\Gamma, x : \tau \vdash \mathsf{let}(e_1[e/x]; y.\, e_2[e/x]) : \sigma_2} \ \textsc{Let}$$

But $(\mathsf{let}(e_1; y.\, e_2))[e/x] \stackrel{\text{def}}{=} \mathsf{let}(e_1[e/x]; y.\, e_2[e/x])$ by the definition of substitution.

8. Prove that types are unique, i.e. that for every context Γ and pre-term e there exists at most one τ such that $\Gamma \vdash e : \tau$.

   [Hint: assume that there exist two, and prove that they must be the same.]

Solution: The claim is the following: if $\Gamma \vdash e : \sigma$ and $\Gamma \vdash e : \tau$, then $\sigma = \tau$. The proof is by induction on the *first* derivation, namely the derivation of $\Gamma \vdash e : \sigma$.

Case(Var). Suppose that the derivation is of the form

$$\dfrac{}{\Delta, x : \sigma \vdash x : \sigma} \ \textsc{Var}$$

(so in fact $\Gamma = \Delta, x : \sigma$, and $e = x$). Then **by the inversion lemma** it must be that $\Delta, x : \sigma \vdash e : \tau$ is also of the same form, i.e. using the Var rule. But the only way that this can happen is if the type τ matches the type for x, so it must be that $\sigma = \tau$.

Case(Num). Suppose that the derivation is of the form

$$\dfrac{n \in \mathbb{N}}{\Gamma \vdash \mathsf{num}[n] : \mathsf{Num}} \ \textsc{Num}$$

(so in fact $e = \mathsf{num}[n]$ and $\sigma = \mathsf{Num}$). Then **by the inversion lemma** the derivation $\Gamma \vdash \mathsf{num}[n] : \tau$ must also have $\tau = \mathsf{Num}$, so $\sigma = \tau$.

Case(Plus). Suppose that the derivation is of the form

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash e_1 : \mathsf{Num}} \qquad \dfrac{\vdots}{\Gamma \vdash e_2 : \mathsf{Num}}}{\Gamma \vdash \mathsf{plus}(e_1; e_2) : \mathsf{Num}} \ \textsc{Plus}$$

(so in fact $e = \mathsf{plus}(e_1; e_2)$ and $\sigma = \mathsf{Num}$). Then **by the inversion lemma** the derivation $\Gamma \vdash \mathsf{plus}(e_1; e_2) : \tau$ must also have the same shape, and hence $\tau = \mathsf{Num}$, so $\sigma = \tau$.

Case(Times). Similar to Plus.

Case(Cat). Similar to Plus.

Case(Len). Similar to Plus, but with one fewer premise.

Case(Let). Suppose that the derivation is of the form

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash e_1 : \sigma_1} \qquad \dfrac{\vdots}{\Gamma, y : \sigma_1 \vdash e_2 : \sigma}}{\Gamma \vdash \mathsf{let}(e_1; y.\, e_2) : \sigma} \ \textsc{Let}$$

for some σ_1 (so that $e = \mathsf{let}(e_1; y.\, e_2)$). By the **inversion lemma** we see that the derivation $\Gamma \vdash \mathsf{let}(e_1; y.\, e_2) : \tau$ must have the form

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash e_1 : \sigma_1'} \qquad \dfrac{\vdots}{\Gamma, y : \sigma_1' \vdash e_2 : \tau}}{\Gamma \vdash \mathsf{let}(e_1; y.\, e_2) : \tau} \ \textsc{Let}$$

for some type σ_1'.

We are now in a position where we have obtained a 'smaller' sub-derivation $\Gamma \vdash e_1 : \sigma_1$ as well as a derivation $\Gamma \vdash e_1 : \sigma_1'$ for the same pre-term. By the **induction hypothesis** applied to these two judgements we conclude that they have the same type, i.e. $\sigma_1 = \sigma_1'$.

Armed with this fact, we see that we have a 'smaller' sub-derivation $\Gamma, y : \sigma_1 \vdash e_2 : \sigma$ as well as a derivation $\Gamma, y : \sigma_1' \vdash e_2 : \tau$. As $\sigma_1 = \sigma_1'$, we can use the **induction hypothesis** again to conclude that $\sigma = \tau$, which is what we wanted to prove in this case.