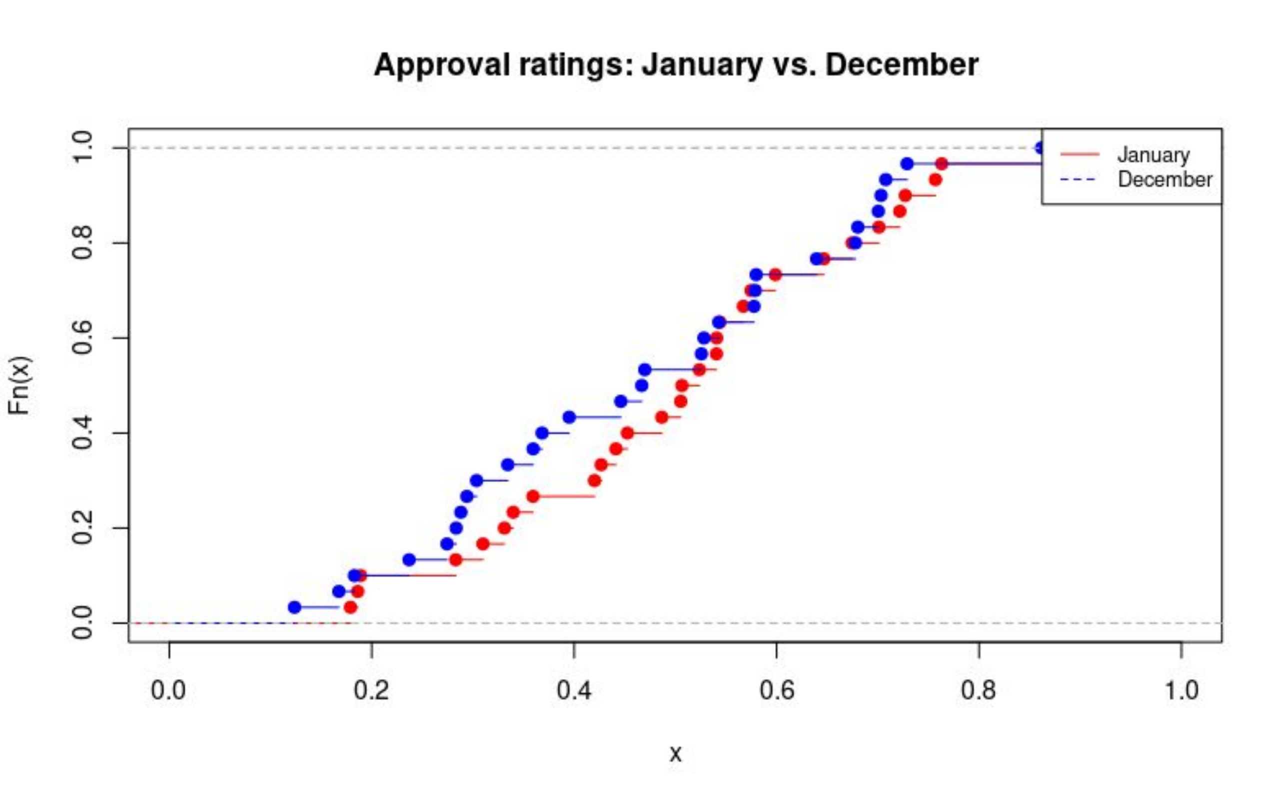
Useful Probability Distributions

4 February 2019

PLSC 309

The fundamental problem of statistics



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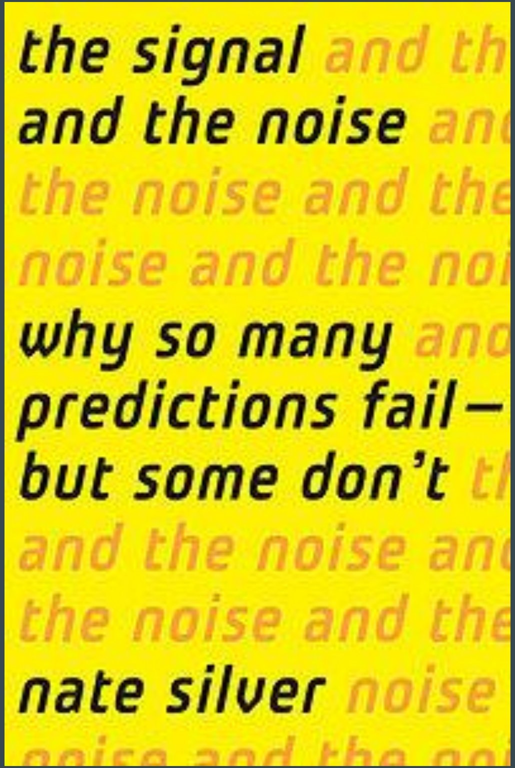

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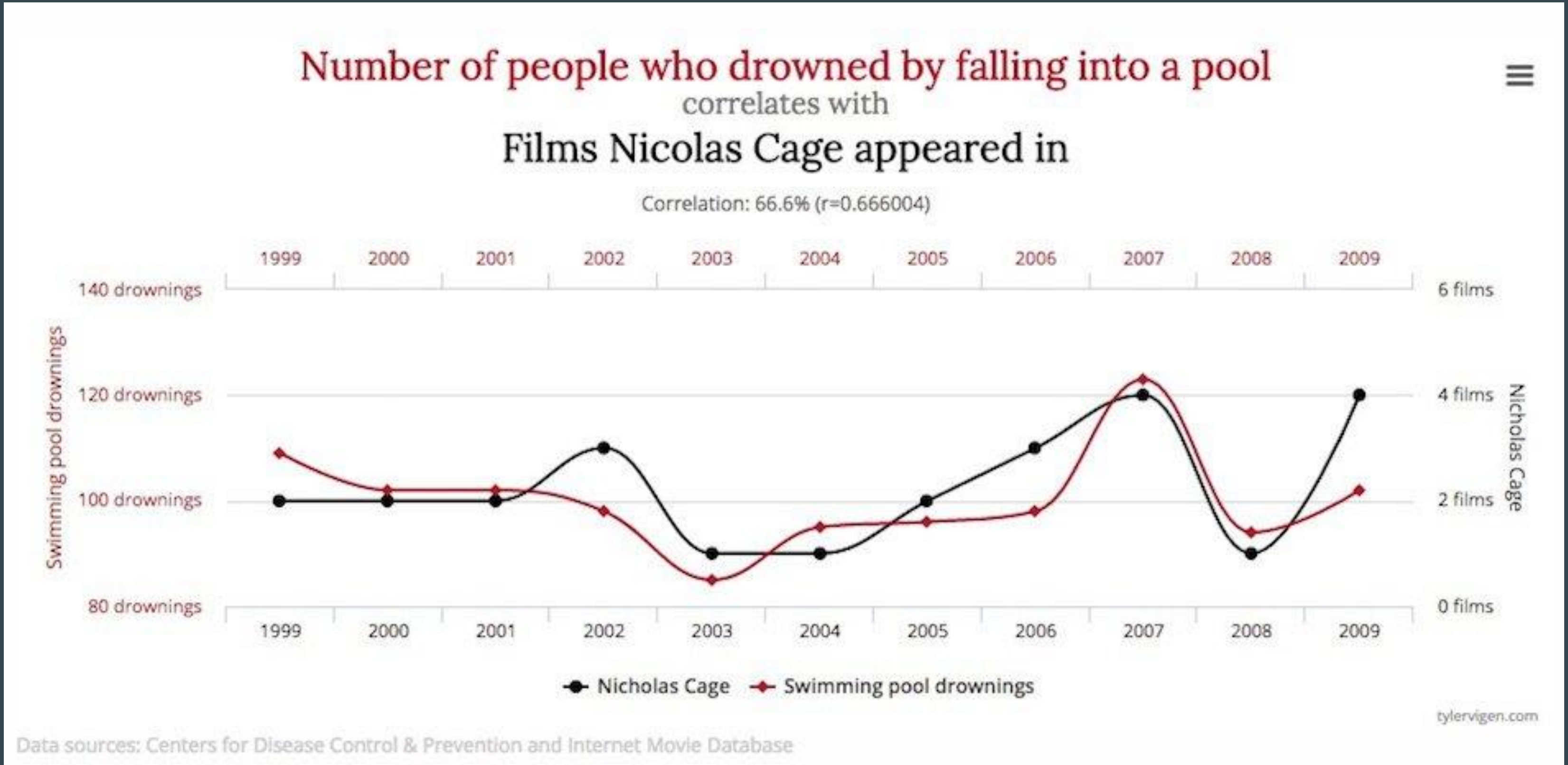

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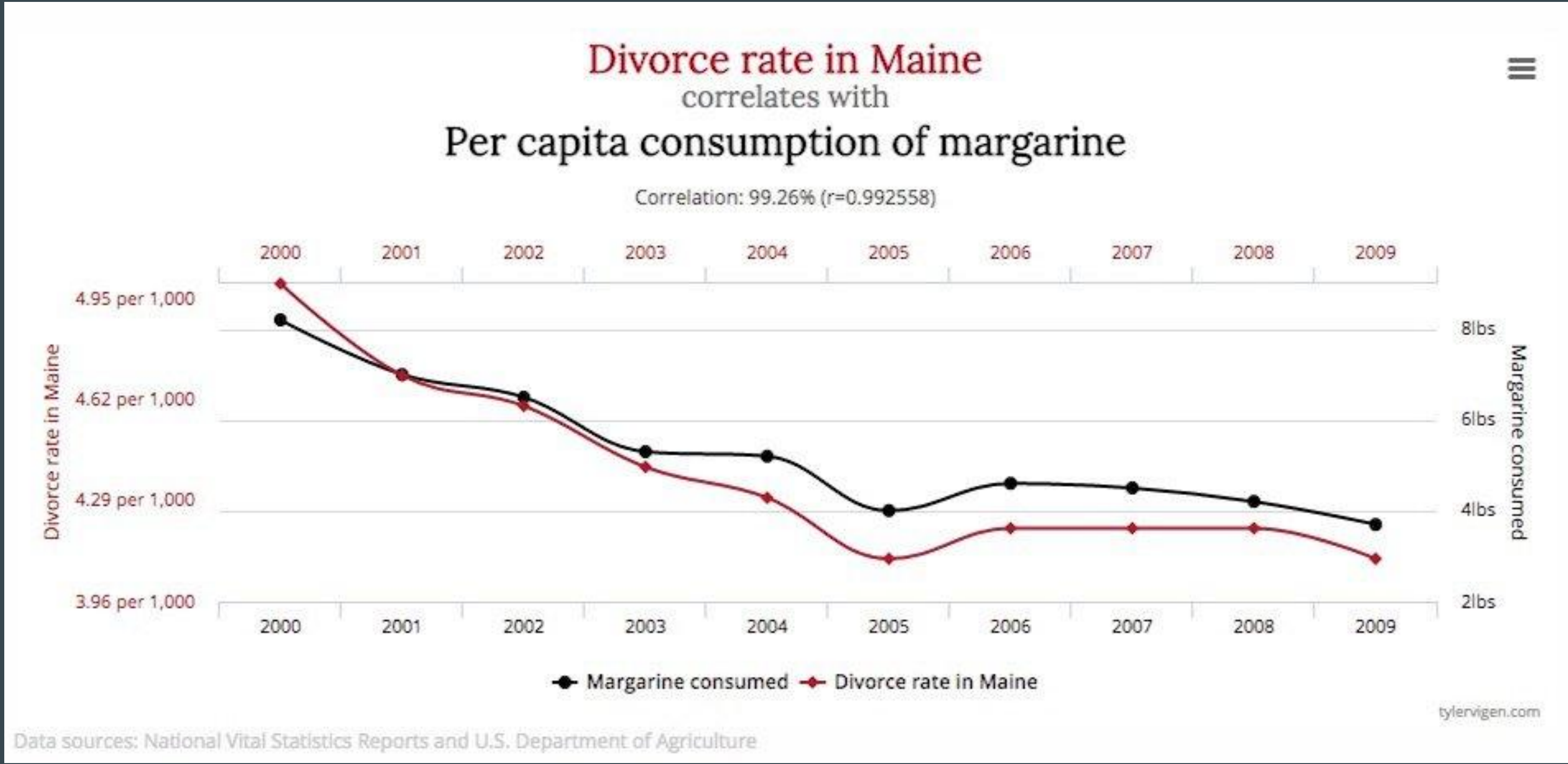

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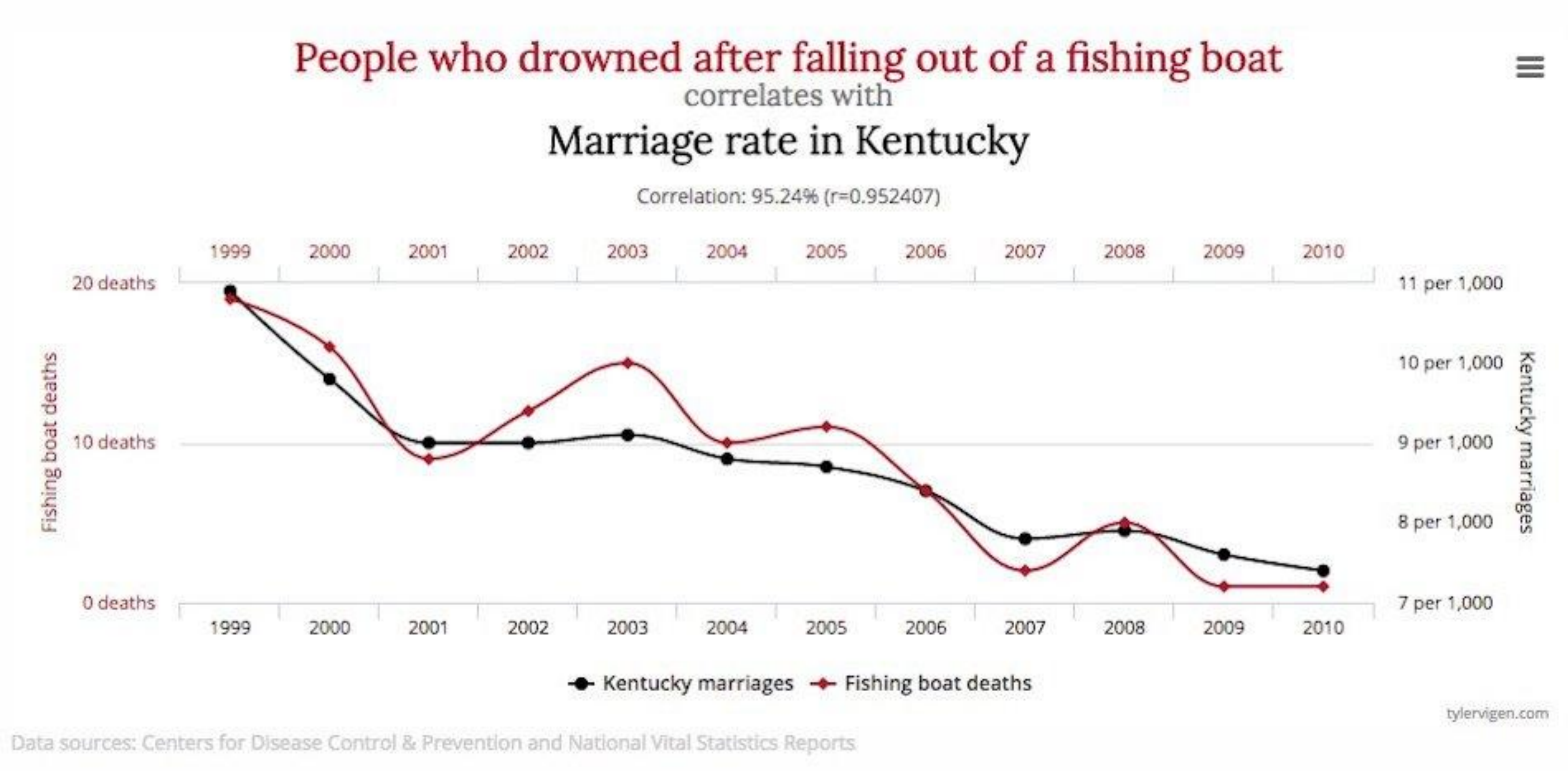

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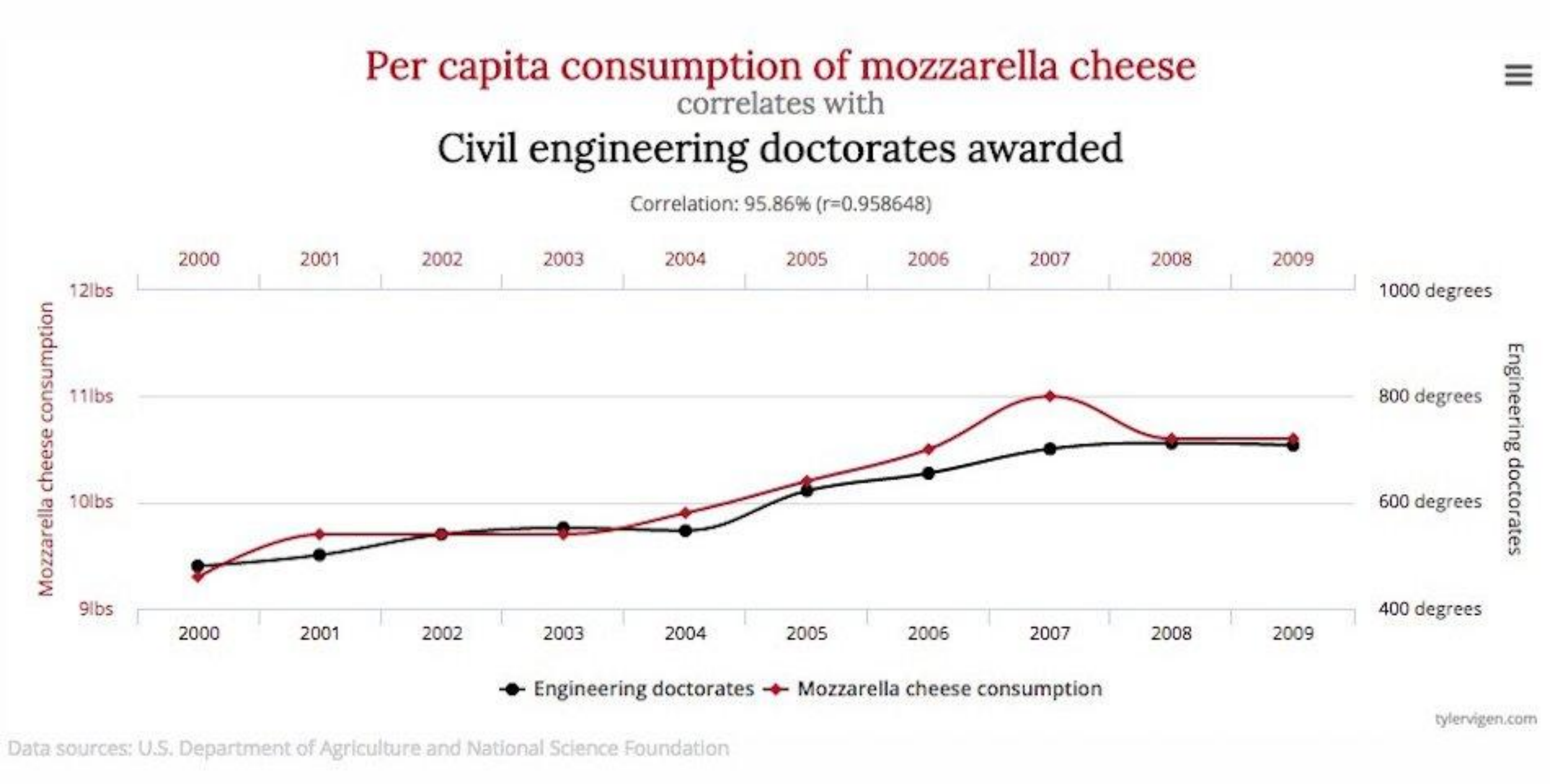


The fundamental problem of statistics

- We need to distinguish random noise from systematic changes
- If we assume a probability distribution for what we're trying to explain...
- ...we have a good guess for what the random noise will be!

Preview

- It is difficult to directly estimate a probability distribution from data
- Instead we look for *naturally occuring distributions*
- We then make an assumption (backed up by evidence) that our data is drawn from one of those distributions

But first, some vocabulary

- We have seen random variables, which we typically label X
- The distributions we see have other mathematical variables called *parameters*
    - These are usually constants
    - They determine the shape of the distribution
    - These can be calculated or directly estimated from data

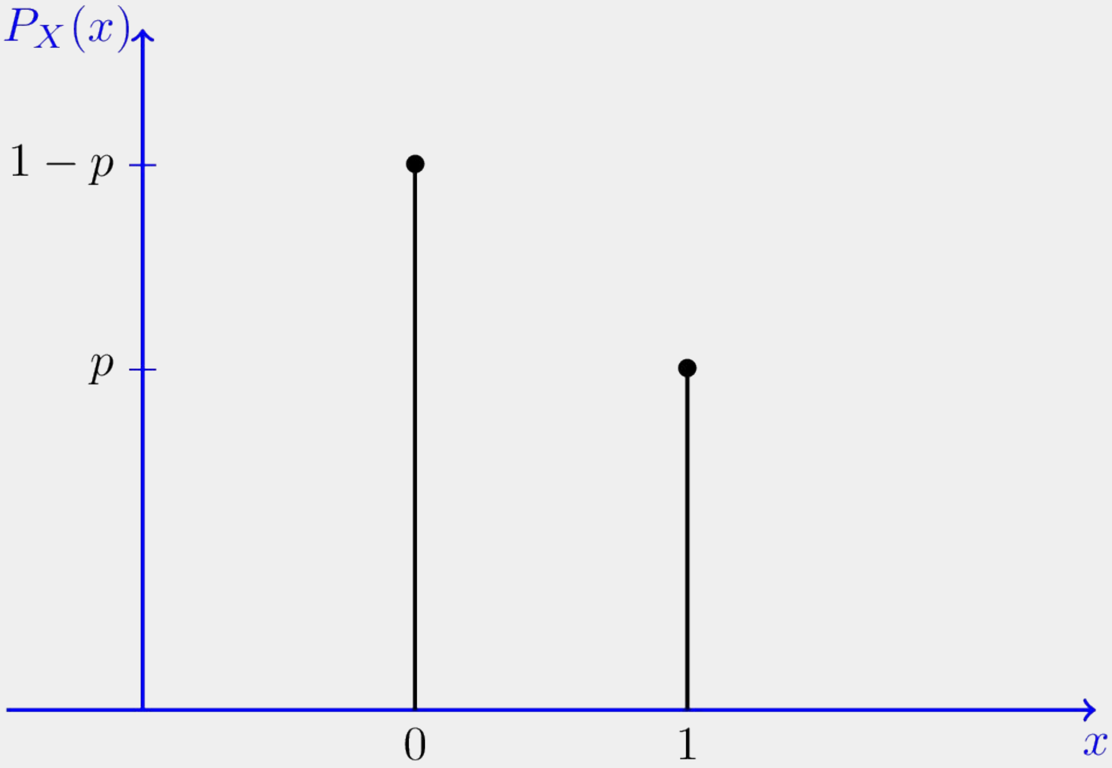
Bernoulli Distribution

Successes and failures

- A Bernoulli distribution describes any event with a binary outcome
  - Outcome = 1 (*success*)
  - Outcome = 0 (*failure*)
- Two parameters
  - *p:* probability of success
  - *q:* probability of failure (equivalent to 1-p)

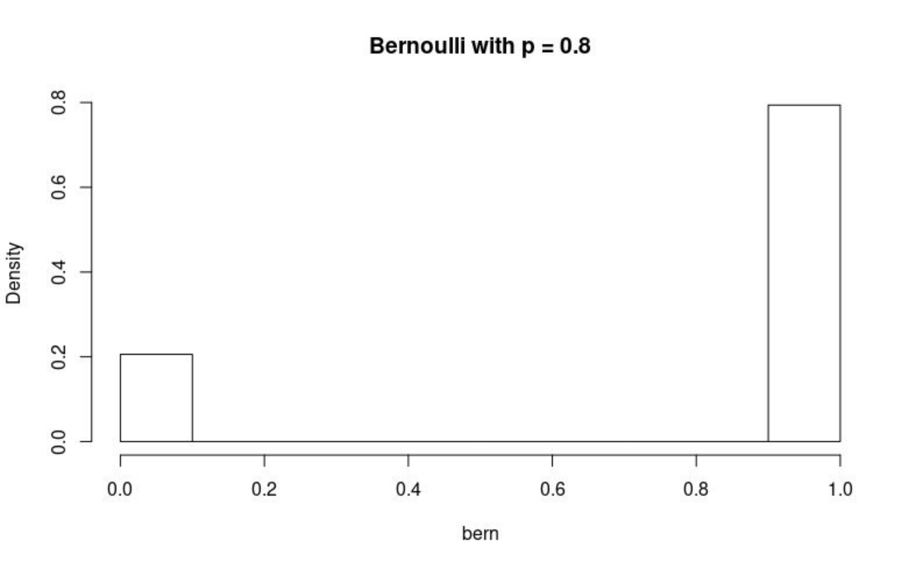
Bernoulli Distribution

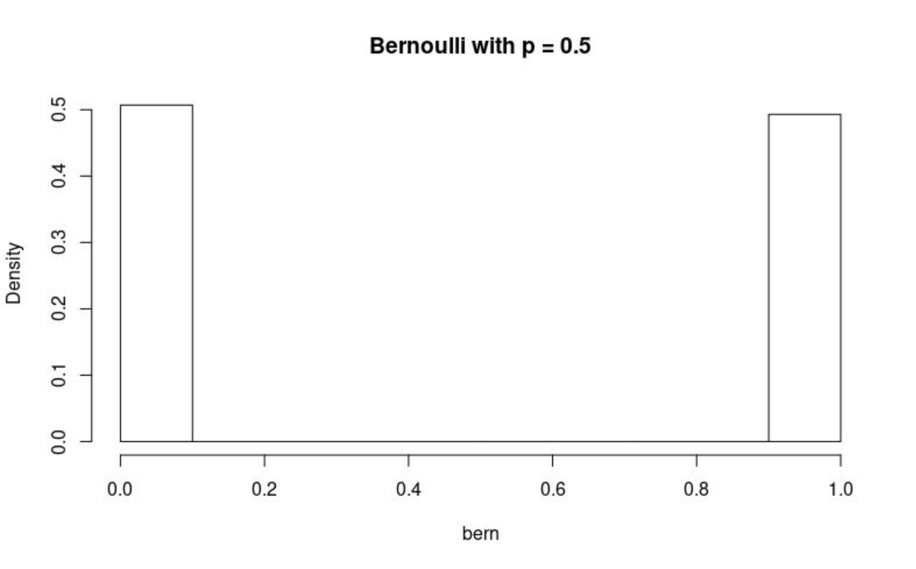
$$X \sim Bernoulli(p)$$



$$\begin{cases} q = 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases}$$

Changing parameter p





Mean and Variance for Bernoulli

- A bernoulli distribution is one trial that ends in either a success or a failure
- Each trial is independent from the other
- $E(X) = p$
- $Var(X) = p(1-p)$ or pq

Examples of Bernoulli processes

- A candidate is running for election. The chance they win is 70%
  - 0 = candidate loses
  - 1 = candidate wins
  - p = .7
  - E(X) = .7
  - Var(X) = .21
- There is a 5% chance of a coup in Thailand
  - 0 = no coup
  - 1 = coup
  - P = .05
  - E(X) = .05
  - Var(X) = .0475

Binomial Distribution

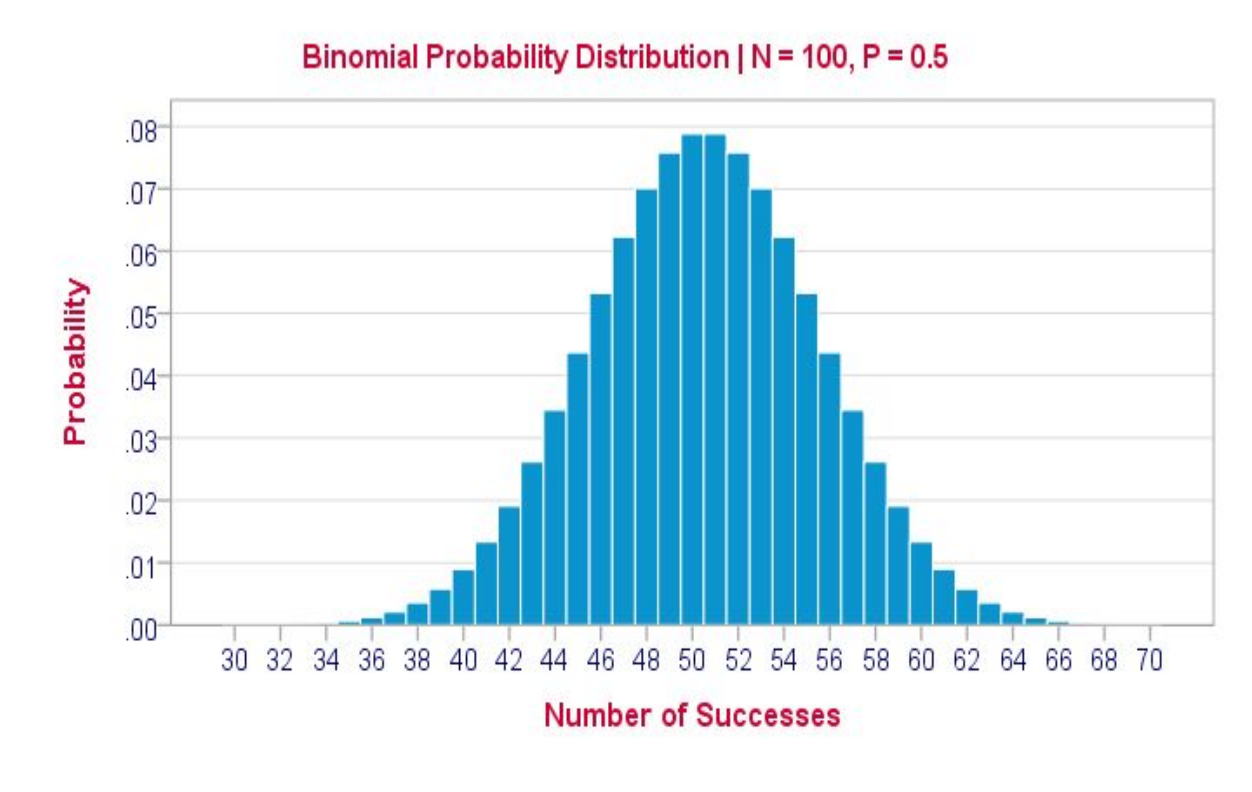
An extension of the Bernoulli

- In the Bernoulli, we had one event that yields an outcome of success or failure
- What if we had a bunch of events that yielded 0-1 outcomes?
- In other words, given a series of n trials, what is the probability that we have k successes?
- Three parameters
    - p: probability of success
    - n: number of trials
    - k: number of successes

Interlude: combinations

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Distribution



$$\binom{n}{k} p^k (1-p)^{n-k}$$

Binomial PMF

Let's say we're trying to flip a coin ten times. A success is a heads. We only get one head. Probability of success is *p*. Probability of failure is 1-*p*.

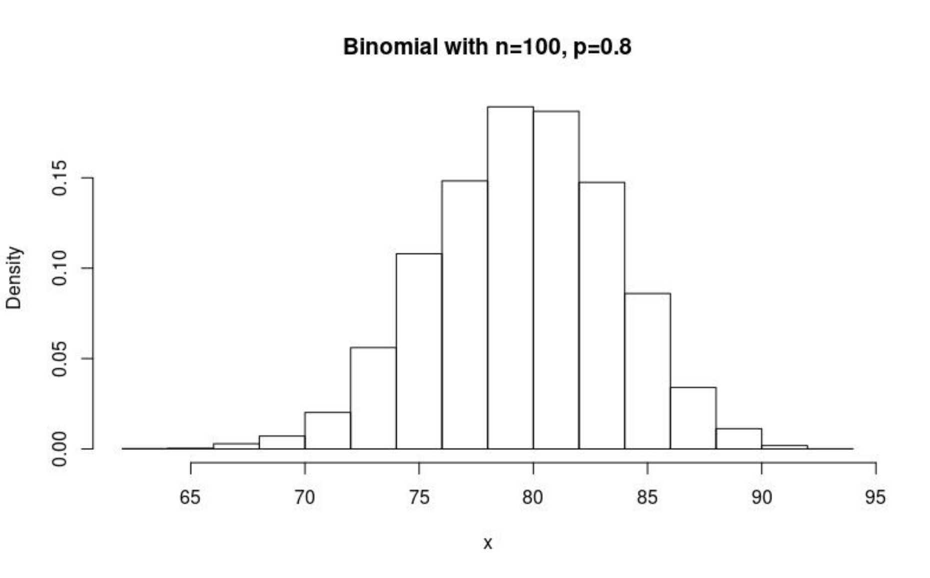
$\{S, F, F, F, F, F, F, F, F, F\} = p * (1\text{-}p)^9$

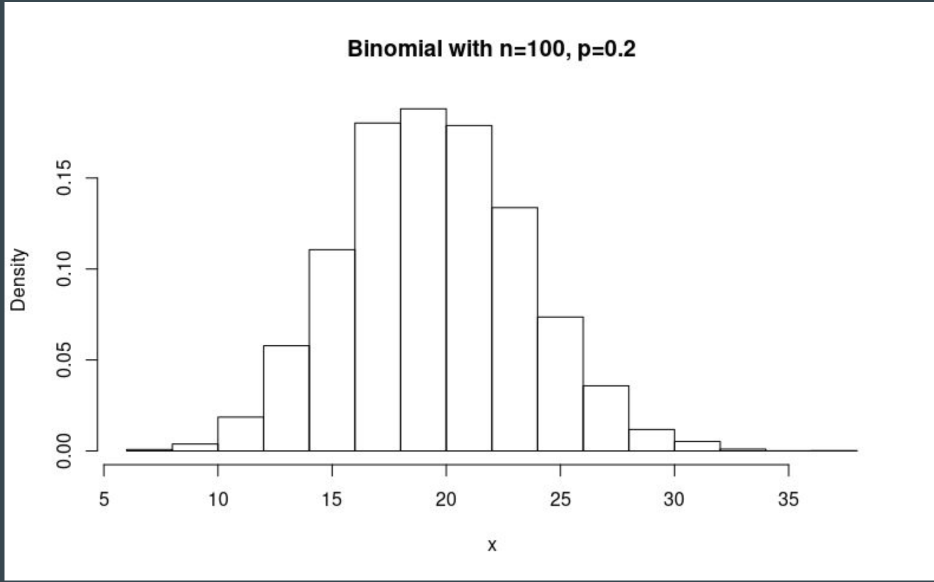
$\{F, S, F, F, F, F, F, F, F, F\} = (1\text{-}p) * p * (1\text{-}p)^8 = p * (1\text{-}p)^9$

$\{F, F, S, F, F, F, F, F, F, F\} = (1\text{-}p)^2 * p * (1\text{-}p)^7 = p * (1\text{-}p)^9$

In other words, each different way of getting one success has a probability of $p * (1\text{-}p)^9$. Summing all these up we get $10 * p * (1\text{-}p)^9$. 10 choose 1 = 10.

Effect of parameters on Binomial Distribution





Mean and Variance for Binomial distribution

- Binomial distribution extends the Bernoulli to n trials
  - E(X) for Bernoulli was p
  - …E(X) for Binomial is *np*
- Var(X) = np(1-p)

Examples of Binomial processes

- 18 candidates are running for office. Incumbents win 90% of the time. How many incumbents will win?
  - $E(X) = np = 16.2$
  - $Var(X) = np(1-p) = 1.62$
- A political party needs has 60 senators in Congress. Senators vote with their own party 92% of the time. What is the probability they get 60 votes?
  - $E(X) = np = 55.2$
  - $Var(X) = 4.416$

Multinomial Distribution

Categories as binary choices

- The binomial distribution dealt with k success from n trials
- A multinomial distribution is about choosing from k categories in n trials
- If there are k categories, each category has a probability, p_i

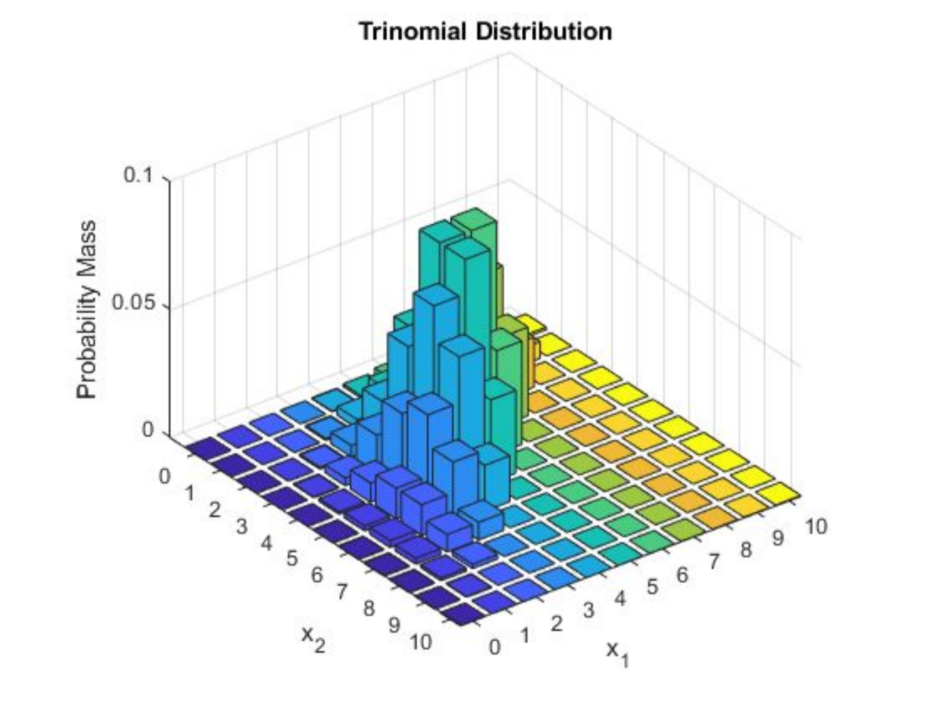
Categorical variables are condensed binary outcomes

Political party: 0 = Democrat; 1 = Republican; 2 = Green

Political Party
0
2
1

Democrat	Republican	Green
1	0	0
0	0	1
0	1	0

Multinomial Distribution



$$\frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

Mean and Variance for Multinomial distribution

- There is no global expected value or variance, just one for each category
- Calculations are exactly the same as the binomial distribution
  - $E(X_i) = np_i$
  - $Var(X_i) = np_i(1-p_i)$

Examples of Multinomial Processes

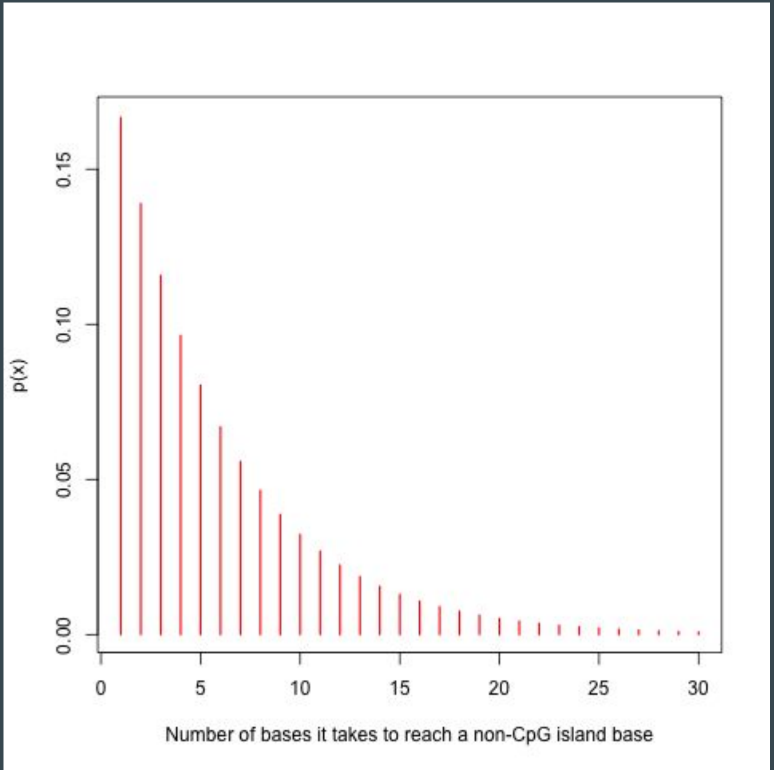
- Any consumer choice
  - Type of car to buy
  - Type of phone to buy
  - Etc.
- The Green, Democrat, and Republican party candidates are running against one another in a district with 1000 voters. The probability that someone votes Green is 2%, Democrat is 49%, and Republican is 49%.
  - $E(\text{Green}) = np_i = 20$
  - $E(\text{Dem}) = np_i = 490$

Geometric Distribution

How long does it take to get a success?

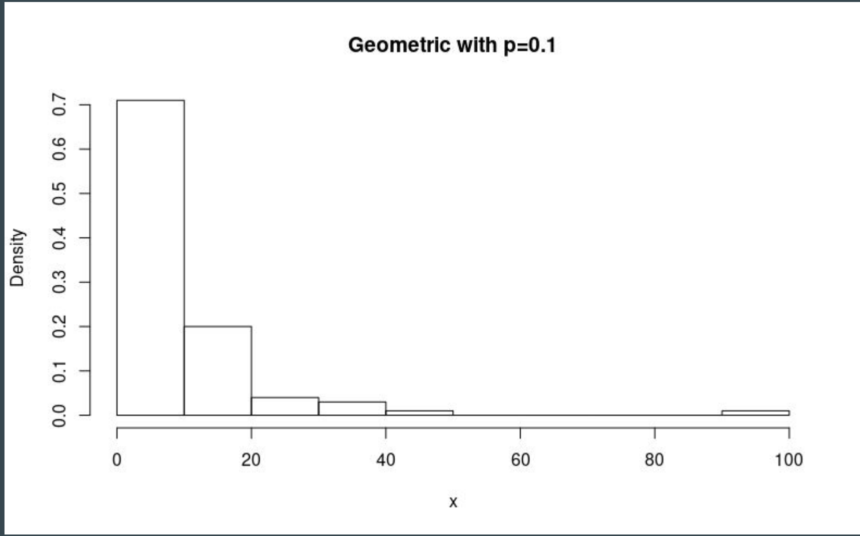
- For *k* trials, how long does it take to get to one success?
- Bernoulli: one trial, one success
- Binomial: *n* trials, *k* successes
- Geometric: *k* trials, one success
  - *p*: probability of success
  - *k:* how many trials it takes to get to one success

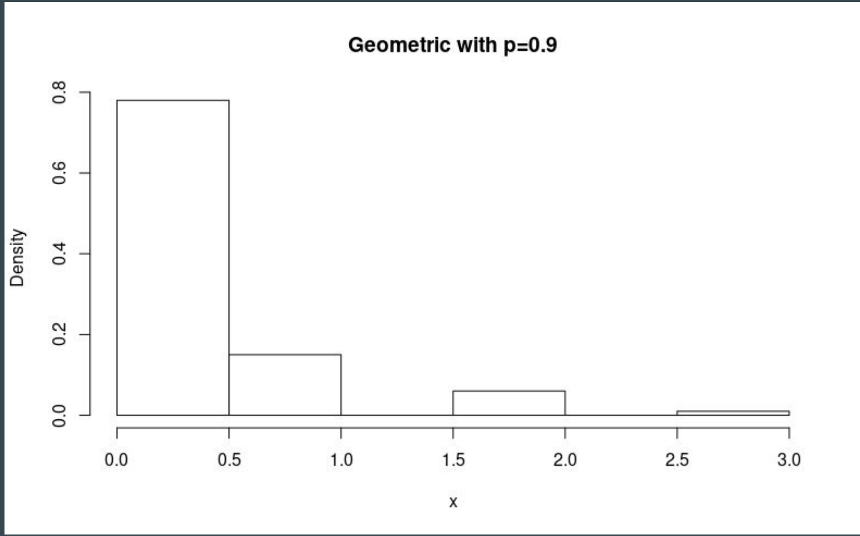
Geometric distribution



$$(1-p)^{k-1}p$$

Changing parameters for geometric





Mean and Variance for the geometric distribution

- Expected value for the geometric is telling us, on average, how many trials we should expect to do before we come to a single success
  - $E(X) = 1/p$
- $Var(X) = (1-p) / p^2$

Examples of geometric processes

- A factory is making widgets. Their failure rate is 0.01. How many trials will it take until there is a failure?
  - $E(X) = 1/p = 100$
  - $Var(X) = (1-p) / p^2 = 9900$
- Somebody won't stop gambling until they win at blackjack. The chance you win at blackjack is 49%. How many hands will they sit through until they win?
  - $E(X) = 1/p = 49$
  - $Var(X) = (1-p) / p^2 = 2.12$

Review

Today we learned about commonly occuring discrete distributions:

- Bernoulli - a single trial that can end in either success or failure
- Binomial - a collection of Bernoulli trials
- Multinomial - a categorical extension of the binomial distribution
- Geometric - number of times it takes for a series of Bernoulli trials to produce a success