## Inverse

### 1. Inverse Matrices

A matrix A is invertible if we can find the $A^2$ such that

$A \cdot A^{-1} = I$ and $A^{-1} \cdot A = I$

and I is the Identity Matrix.

#### Pivot

A pivot is the first non-zero element in a row of a matrix that has been transformed into row echelon or reduced row echelon form.

$$A = \begin{bmatrix} **1** & 0 & 0 \\ 0 & **1** & 0 \\ 0 & 0 & **1** \end{bmatrix}$$

#### The invertible matrices

- We can find the inverse of matrix if and only if elimination produces n pivots
- matrix A cannot have two different inverse $\rightarrow$ If BA = I and CA = I, then B = C
- If A invertible then $Ax = b$ solving by $x = A^{-1}b$
- Suppose there is a nonzero vector x (x,y,z,... have the answer) such that $Ax = 0 \rightarrow A$ not invertible. Because no matrix bring the zero back.
- If the above A is invertible, then $Ax = 0$ then x = 0 so there have no solution for solving.
  **Formula** to find the inverse of 2x2 matrix is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  **Note:**
- The $ad - bc$ is det A.
- Its invertible if and only if $ad - bc$ is not zero.
- If $A = \begin{bmatrix} d_1 & & \\ & \cdots & \\ & & dn \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} \frac{1}{d_1} & & \\ & \cdots & \\ & & \frac{1}{dn} \end{bmatrix}$

## 2. The Inverse of a Product

If A and B are invertible matrices, their product AB is also invertible. The inverse is given by:

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

**Example:**
Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$. The inverse of AB requires first computing the $B^{-1}$ and $A^{-1}$, after that multiplying $B^{-1} \cdot A^{-1}$
**As you can see the order must be reverse.**

**Ex1:**
Inverse of an elimination matrix If an Elimination Matrix E subtracts 5 times row 2, then $E^{-1}$ adds 5 times row 1 from row 2.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the exchange symbol from plus and minus only work with if the matrix is there **have 1 on the diagonal of Upper and Lower triangular matrix** otherwise use another method.

## 3. Calculating $A^{-1}$ Using Gauss-Jordan Elimination

**Steps:**

1. Combine A with I to form an augmented matrix [A | I].
2. Apply row operations to transform A into I. The operations on I result in $A^{-1}$

Multiply [A I] by $A^{-1}$ to get [I $A^{-1}$]
**In short is making the A to I.**

**Example**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Make an augmented matrix by combining matrix A with identity matrix.
We will get

$$A = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 4 & 5 & 6 & | & 0 & 1 & 0 \\ 7 & 8 & 9 & | & 0 & 0 & 1 \end{bmatrix}$$

Make the left side turn into the identity matrix using elimination technique then you will get the inverse of matrix A or $A^{-1}$ on the right side of the line.

**Symmetric Matrices**

A matrix A is symmetric if it equals its transpose (change rows to columns and columns to rows):

$$A = A^T$$

**Example:**

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{bmatrix}$$

Here, $A^T = A$, so A is symmetric.

**Tridiagonal Matrices**

A tridiagonal matrix is a square matrix where nonzero entries. In short is have 3 diagonal line in the matrix.
**Example:**

$$T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

This tridiagonal matrix is also symmetric.

**Product of Pivots**

The product of pivots is the determinant of the matrix in triangular form (upper or lower triangular matrix).

- During Gaussian Elimination, the diagonal entries of the resulting triangular matrix are the pivots.
- The det of the matrix is the product of these pivots.
  **Properties:**
- If any pivot is zero, the det is zero, and the matrix is singular (non-invertible).
  **Example:**
  Consider $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$
  Gaussian elimination produces $U = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$
  Pivots: 2, 1
  Det(A)=$2 \cdot 1 = 2$

**Combined Example: Symmetric, Tridiagonal Matrix and Pivots**

Let K be a symmetric, tridiagonal matrix:

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- $K = K^T$
- Tridiagonal because it no zero in the diagonal zone.
- Pivots
  Eliminate below the main diagonal:
  - Subtract $-\frac{1}{2}$ of row 1 from row 2.
  - Subtract $-\frac{1}{2}$ of row 2 from row 3.
    Pivots: $2, \frac{3}{2}, \frac{4}{3}$
    Determinant: $2 \cdot \frac{3}{2} \cdot \frac{4}{3} = 4$

**Example:**
$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, solve [A | I]:

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix}$$

Perform elimination to get $\begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \end{bmatrix}$

## 4. Elimination and Factorization (A=LU)

Factorize A into:

- L - Lower triangular matrix $A = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$
- U - Upper triangular matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$

  **Process:**
- Forward elimination transform A to U by multiplies by a matrix $E_{ij}$ to produce zero in the (i, j) position.
- Backward substitution involves L to revert to the original system.
  If A is 3 x 3, we multiply by $E_{21}, E_{31}, E_{32}$ it will produce 0 in the position (2, 1), (3, 1), (3, 2) positions to get the upper triangular U.
  **Tricks:**
  We can move to another side to make a new equation.

$$(E_{32}, E_{31}, E_{21})A = U \rightarrow A = (E_{21}^{-1} E_{31}^{-1} E_{32}^{-1})U$$

  After we inverse to opposite side and order we will get the product of 3 inverses is L by the formula.

$$A = LU$$

$l_{ij}$

$l_{ij}$ is the multiplier value used to eliminate the element in the i-th row and j-th column of a matrix during the elimination process. It is calculated as:

$$l_{ij} = \frac{a_{ij}}{a_{jj}}$$

Where:

- $a_{ij}$ - The element being eliminated.
- $a_{jj}$ - The pivot element (on the diagonal)
  **Example:**
  Consider the matrix A:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

**Steps:**

1. Eliminate the first column below the pivot ($a_{11} = 2$)
   For row 2 (i=2):

$$l_{21} = \frac{-3}{2}$$

Update row 2:

$$R_2 \to R_2 - l_{21}R_1 = [-3 \quad -1 \quad 2] - \frac{-3}{2}[2 \quad 1 \quad -1]$$

Update row 3:

$$R_3 \to R_3 - l_{31}R_1$$

Then the $E_{21}$ :

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Resulting Matrix:**
After eliminating the first column, the matrix becomes:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0.5 & 0.5 \\ 0 & 2 & 1 \end{bmatrix}$$

The multipliers $l_{21} = -1.5$ and $l_{31} = -1$ were used to perform these operations.

Connection to L in LU Decomposition

In **LU decomposition,** the lower triangular matrix L contains the multipliers $l_{ij}$. For the above example, L would be:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

- $l_{21} = -1.5$ Multiplier used to eliminate the first element in row 2.
- $l_{31} = -1.5$ Multiplier used to eliminate the first element in row 3.
  **Properties of Multipliers**

1. We store multiplier inside the L below the diagonal.
2. The diagonal of L (Lower triangular) always contain 1s.
3. Multiplier don't change when applying Gaussian elimination without row exchanges.

**Example:**

For $A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}$

1. Eliminate 6 using row operations.
2. Result in $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$

   So $A = LU$

## 5. Singular v. Invertible Matrices

### A matrix is invertible (nonsingular) if:

- All rows/columns are independent (no zero rows/columns or called $\det(A) \neq 0$
- The matrix has a full rank.
- The determinant of the matrix is 0: $\det(A) \neq 0$
- The rows and columns are linearly independent.

  **Ex:**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- The rows [1, 2] and [3, 4] are linearly independent.
- $\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2 \neq 0$, so A is invertible.

### A matrix is Singular (does not have an inverse) if:

- The determinant of the matrix is 0: $\det(A) = 0$
- The rows or columns of the matrix are linearly dependent (some rows or columns can be expressed as linear combinations of others).
- The matrix does not have a full rank (more on this below).

  **Ex:**

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- The second row ([2, 4]) is a multiple of the first row ([1, 2])
- $\det(A) = 1 \cdot 4 - 2 \cdot 2 = 0$, so A is singular.

## 6. Cost of Elimination

- Computational Time:
  For $n * n$ matrices, the cost of Gaussian elimination is proportional to $n^3$
- Efficiency:
  Sparse matrices (with many zero entries) reduce computational cost significantly.