Transposes and Permutations

1. Transpose

Transpose of A is denoted as A^T it use for swap rows and columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}, then\ A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$$

Rules

- The transpose of $A^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$ swap the position.
- $(A^{-1})^T = (A^T)^{-1}$
- $L^T = U$
- L^{-1} - Lower triangular.
- Can inverse the transpose or can transpose the inverse.
- A^T is invertible when A is invertible.
- When A movers from one side of a dot product to the other side it becomes A^T
  Formula

  $$A = LDU$$

  D is the diagonal matrix (the pivot matrix) also $D = D^T$ The above formula is make the triangular matrix having 1 on the diagonal. Symmetric

Example

The inverse of $A = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$
$(A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}$

2. Symmetric Matrices

A symmetric matrix has this property $S^T = S$

- Also the inverse of symmetric matrix will be symmetric.

$$(S^{-1})^T = (S^T)^{-1} = S^{-1}$$

Example

Symmetric matrix and symmetric inverse

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = S^T$$

and

$$S^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = (S^{-1})^T$$

Symmetric Products

For any matrix A, the product $S = A^T A$ is symmetric.

$$S^T = S$$

$A^T A$ us symmetric but $A^T A \neq AA^T$ in general, but both are symmetric.

Example

$$Let A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

- Find $A^T A$

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

  Multiply $A^T \ by \ A$

$$A^T A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$$

- A^TA is symmetric because $\begin{bmatrix}35 & 44\\44 & 56\end{bmatrix}^T = \begin{bmatrix}35 & 44\\44 & 56\end{bmatrix}$

Compare with AA^T

$$AA^T = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\end{bmatrix}$$

- AA^T is also symmetric, but it is a different matrix from A^TA

Symmetric Elimination

The symmetric makes elimination faster because work only half the matrix (plus diagonal).

$$S = LDU$$

We add D into the normal formula $A = LU$ makes it symmetric.

Example

Given $S = \begin{bmatrix}1 & 2\\2 & 7\end{bmatrix}$, find $S = LU$ and $S = LDU$
$\begin{bmatrix}1 & 2\\2 & 7\end{bmatrix} = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$ LU misses the symmetry of S
$\begin{bmatrix}1 & 2\\2 & 7\end{bmatrix} = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ LDL^T captures the symmetry Now U is the L^T

Rules

- If $S = S^T$ can be factored into LDU with no row exchanges, then U is L^T
- The symmetric factorization of a symmetric matrix is $S = LDL^T$
- The transpose of LDL^T is also $LDL^T \rightarrow$ will faster during elimination half of it.

3. Permutation matrices

A permutation matrix P has the rows of the identity I in any order.
P^T is also a permutation matrix it might be P or different matrix.

Example

There are six $3x3$ permutation matrices.

$$I=\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix} P_{21}=\begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix} P_{32}P_{21}=\begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix}$$

P_{31}=\begin{bmatrix} &&1 \\ &1& \\ 1&& \end{bmatrix} \ P_{32}=\begin{bmatrix} 1&& \\ &&1 \\ &1

1. Pivoting

- The first pivot element is 0 so swap rows 1 and 2:

$$P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$PA=\begin{bmatrix}1&3&4\\0&1&2\\2&7&6\end{bmatrix}$$

2. Perform Gaussian Elimination

- Eliminate PA to form U:

$$U=\begin{bmatrix}1&3&4\\0&1&2\\0&0&-2\end{bmatrix}$$

3. Find L

- The multipliers used during elimination are stored in L:
  Final factorization:

$$PA=LU$$

Example 2:

1. Solving Linear Systems:

- Solve $Ax=b$ using

$$PAx = Pb, LUx = Pb$$

2. Matrix Inverses:

- Multiply P x A:

$$PxA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

3. Determinants:

- The determinant of A can be found from U:

$$\det(A) = (-1)^k \cdot \det(U)$$

where k is the number of row swaps in P.

5. Vector Spaces

6. Subspaces