

# Sample Problems and Solutions

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This is a selection of three of my problems, with solutions.  
Corrections and comments are welcome!

## Contents

<b>0</b>	<b>Problems</b>	<b>2</b>
<b>1</b>	<b>Problem 1</b>	<b>3</b>
<b>2</b>	<b>Problem 2</b>	<b>4</b>
<b>3</b>	<b>Problem 3</b>	<b>5</b>

## §0 Problems

1. Let  $ABC$  be a right triangle with right angle at  $A$ ,  $AB = 21$ , and  $AC = 20$ . Let  $D$  be the midpoint of  $\overline{AC}$ ,  $\omega_1$  be the circumcircle of  $\triangle ABD$ , and let  $\omega_2$  be the circle with diameter  $\overline{AC}$ . Let  $\omega_1$  and  $\omega_2$  intersect at  $A$  and  $E$ . Let  $\omega_1$  intersect  $BC$  at  $B$  and  $K$ , and let  $\omega_2$  intersect  $BC$  at  $C$  and  $L$ . Let  $EK$  and the tangent to  $\omega_1$  at  $D$  intersect at  $M$ . Let  $EL$  and  $BD$  intersect at  $N$ . The circumcircle of  $\triangle DMN$  has a radius of length  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
2. In triangle  $ABC$  with  $\angle A = 60^\circ$ , let  $B'$  be the reflection of  $B$  over  $AC$ , and let  $C'$  be the reflection of  $C$  over  $AB$ . Let  $B'B$  and  $C'C$  intersect at  $D$ , and let  $B'C$  and  $C'B$  intersect at  $E$ . Let  $DE$  and  $BC$  intersect at  $F$ . Show that  $AF$  bisects  $\angle BAC$ .
3. Let  $ABC$  be a scalene right triangle with right angle at  $A$ . The incircle  $\omega$  of  $\triangle ABC$  touches sides  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$  at points  $A'$ ,  $B'$ , and  $C'$ , respectively. Let  $\omega$  intersect  $AA'$  at  $A'$  and  $D$ , and let  $E$  and  $F$  be the reflections of  $A'$  over points  $B'$  and  $C'$ , respectively. Let  $K$  be the foot of the perpendicular from  $E$  to  $A'F$ , and let  $L$  be the foot of the perpendicular from  $F$  to  $A'E$ . Show that  $B'C'$ ,  $KL$ , and the line perpendicular to  $AA'$  at  $D$  are concurrent.

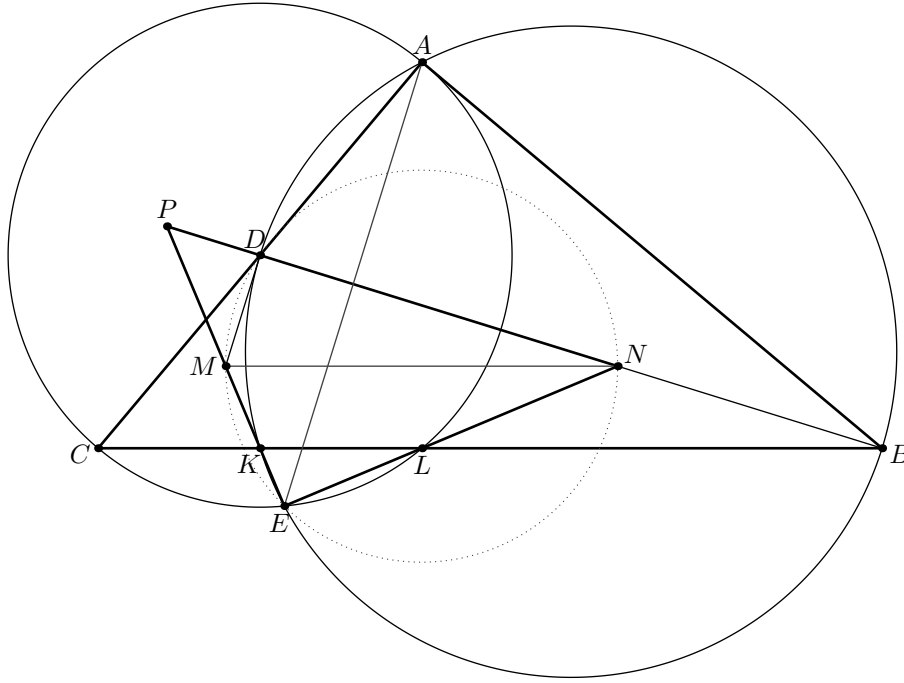
## §1 Problem 1

Let  $ABC$  be a right triangle with right angle at  $A$ ,  $AB = 21$ , and  $AC = 20$ . Let  $D$  be the midpoint of  $\overline{AC}$ ,  $\omega_1$  be the circumcircle of  $\triangle ABD$ , and let  $\omega_2$  be the circle with diameter  $\overline{AC}$ . Let  $\omega_1$  and  $\omega_2$  intersect at  $A$  and  $E$ . Let  $\omega_1$  intersect  $BC$  at  $B$  and  $K$ , and let  $\omega_2$  intersect  $BC$  at  $C$  and  $L$ . Let  $EK$  and the tangent to  $\omega_1$  at  $D$  intersect at  $M$ . Let  $EL$  and  $BD$  intersect at  $N$ . The circumcircle of  $\triangle DMN$  has a radius of length  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

**Solution.** Let  $P = BD \cap EM$ . Note that  $\angle MDN = \angle DAB = 90^\circ$ . Then note that

$$\angle NEM = \angle LEA + \angle AEK = \angle LCA + \angle ABK = 90^\circ$$

implying  $MDNE$  is cyclic with  $\overline{MN}$  the diameter of its circumcircle. Thus it suffices to find  $\frac{MN}{2}$ .



**Claim.**  $\triangle NEP \sim \triangle BAC$  and  $\triangle NEM \sim \triangle BAD$ .

*Proof.* Note that

$$\angle CLE = \angle CAE = \angle MDE = \angle MNE$$

so  $BC \parallel MN$ . Thus  $\angle DNM = \angle DBC$ . Since  $AD = ED$ ,  $\triangle ABD \cong \triangle EBD$  and

$$\angle MNE = \angle MDE = \angle DBE = \angle ABD$$

implying the desired similarities. □

Therefore

$$\frac{MN}{2} = \frac{BD \cdot AE}{4AL} = \frac{[ABD]}{AL} = \frac{[ABC]}{2AL} = \frac{BC}{4} = \frac{29}{4}$$

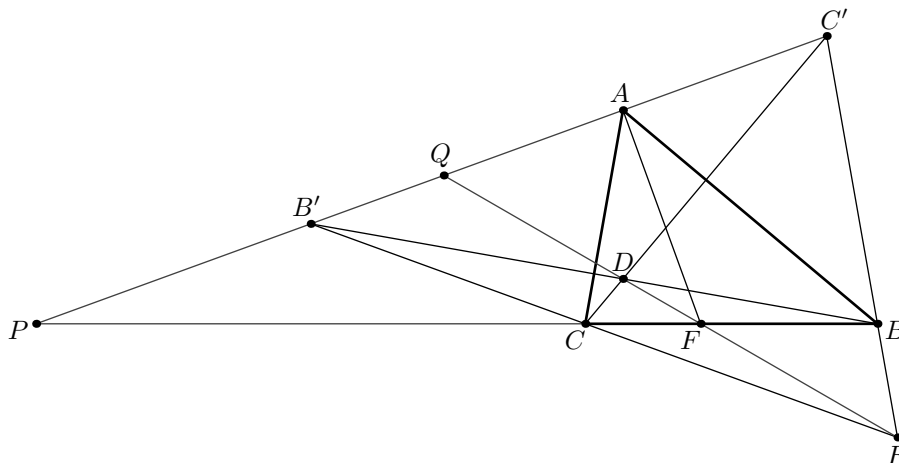
and  $m + n = \boxed{033}$ . ■

## §2 Problem 2

In triangle  $ABC$  with  $\angle A = 60^\circ$ , let  $B'$  be the reflection of  $B$  over  $AC$ , and let  $C'$  be the reflection of  $C$  over  $AB$ . Let  $B'B$  and  $C'C$  intersect at  $D$ , and let  $B'C$  and  $C'B$  intersect at  $E$ . Let  $DE$  and  $BC$  intersect at  $F$ . Show that  $AF$  bisects  $\angle BAC$ .

**Solution.** Let  $P = BC \cap B'C'$  and  $Q = DE \cap B'C'$ . First, note the angle condition implies that  $B'$ ,  $A$ , and  $C'$  are collinear. Hence  $B'C'$  is the exterior angle bisector of  $\angle BAC$  and

$$\frac{AB}{BP} = \frac{AC}{PC}.$$



It's well known by Ceva and Menelaus that

$$-1 = (B', C'; P, Q) \stackrel{E}{=} (C, B; P, F) = \frac{CP}{BP} \div \frac{CF}{BF}.$$

Therefore

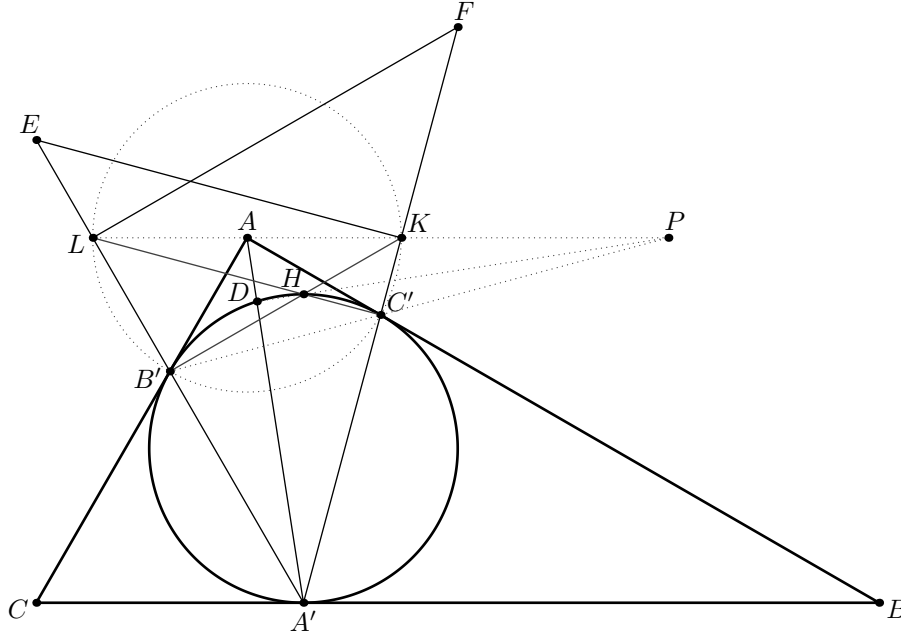
$$\begin{aligned} \frac{AC}{PC} \cdot \frac{BP}{AB} \cdot \frac{CP}{BP} \div \frac{CF}{BF} &= -1 \\ \frac{AC}{AB} &= \frac{CF}{BF} \end{aligned}$$

implying  $AF$  bisects  $\angle BAC$ . ■

### §3 Problem 3

Let  $ABC$  be a scalene right triangle with right angle at  $A$ . The incircle  $\omega$  of  $\triangle ABC$  touches sides  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$  at points  $A'$ ,  $B'$ , and  $C'$ , respectively. Let  $\omega$  intersect  $AA'$  at  $A'$  and  $D$ , and let  $E$  and  $F$  be the reflections of  $A'$  over points  $B'$  and  $C'$ , respectively. Let  $K$  be the foot of the perpendicular from  $E$  to  $A'F$ , and let  $L$  be the foot of the perpendicular from  $F$  to  $A'E$ . Show that  $B'C'$ ,  $KL$ , and the line perpendicular to  $AA'$  at  $D$  are concurrent.

**Solution.** Let  $H = B'K \cap C'L$  and  $P = B'C' \cap KL$ . Note that  $\angle C'A'B' = \angle C'B'A = 45^\circ$ , implying  $\triangle A'EK$  and  $\triangle A'FL$  are isosceles right triangles. Thus  $A'E \perp B'K$  and  $A'F \perp C'L$ .



**Claim.**  $A$  is the circumcenter of cyclic quadrilateral  $B'C'KL$ .

*Proof.* Since  $\angle KB'L = \angle KC'L = 90^\circ$ ,  $B'C'KL$  is cyclic with  $\overline{KL}$  the diameter of its circumcircle  $\Gamma$ . Note that  $AB' = AC'$  and  $\angle B'AC' = 90^\circ = 2\angle B'KC'$ , so  $A$  is the center of  $\Gamma$ .  $\square$

Since  $\angle A'B'H = \angle HC'A' = 90^\circ$ ,  $H$  is the antipode of  $A'$  on  $\omega$  and  $\angle A'DH = 90^\circ$ . By Brocard's theorem on  $\Gamma$ ,  $HP \perp AA'$ , implying  $D$ ,  $H$ , and  $P$  are collinear, as desired.  $\blacksquare$