Sample Problems and Solutions

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This is a selection of three of my problems, with solutions.
Corrections and comments are welcome!

Contents

§0 Problems

1. Let ABC be a right triangle with right angle at A, $AB = 21$, and $AC = 20$. Let D be the midpoint of $\overline{AC}$, ω_1 be the circumcircle of $\triangle ABD$, and let ω_2 be the circle with diameter $\overline{AC}$. Let ω_1 and ω_2 intersect at A and E. Let ω_1 intersect BC at B and K, and let ω_2 intersect BC at C and L. Let EK and the tangent to ω_1 at D intersect at M. Let EL and BD intersect at N. The circumcircle of $\triangle DMN$ has a radius of length $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2. In triangle ABC with $\angle A = 60^\circ$, let B' be the reflection of B over AC, and let C' be the reflection of C over AB. Let $B'B$ and $C'C$ intersect at D, and let $B'C$ and $C'B$ intersect at E. Let DE and BC intersect at F. Show that AF bisects $\angle BAC$.

3. Let ABC be a scalene right triangle with right angle at A. The incircle ω of $\triangle ABC$ touches sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points A', B', and C', respectively. Let ω intersect AA' at A' and D, and let E and F be the reflections of A' over points B' and C', respectively. Let K be the foot of the perpendicular from E to $A'F$, and let L be the foot of the perpendicular from F to $A'E$. Show that $B'C'$, KL, and the line perpendicular to AA' at D are concurrent.

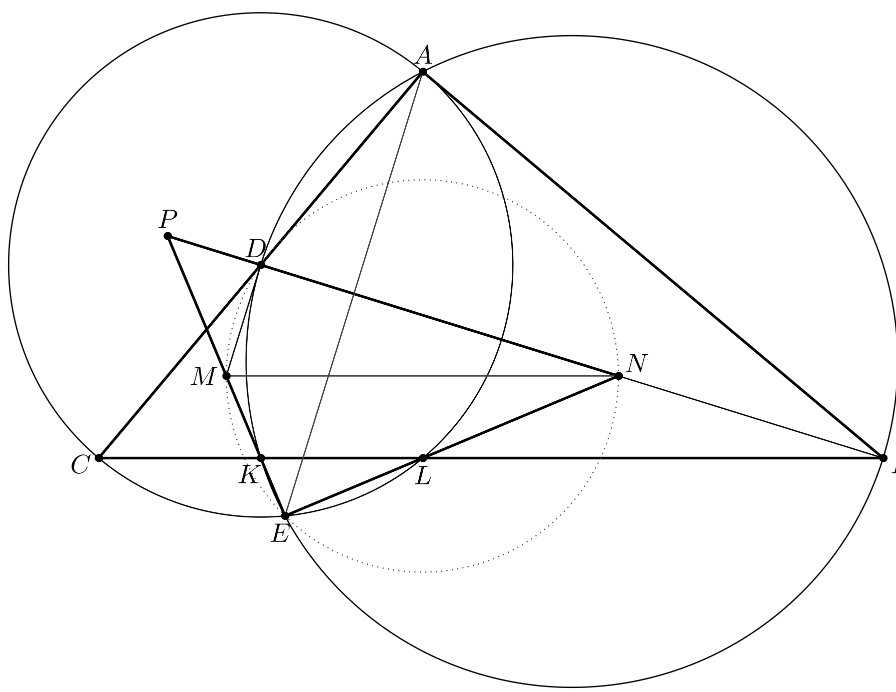
§1 Problem 1

Let ABC be a right triangle with right angle at A, $AB = 21$, and $AC = 20$. Let D be the midpoint of $\overline{AC}$, ω_1 be the circumcircle of $\triangle ABD$, and let ω_2 be the circle with diameter $\overline{AC}$. Let ω_1 and ω_2 intersect at A and E. Let ω_1 intersect BC at B and K, and let ω_2 intersect BC at C and L. Let EK and the tangent to ω_1 at D intersect at M. Let EL and BD intersect at N. The circumcircle of $\triangle DMN$ has a radius of length $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Let $P = BD \cap EM$. Note that $\angle MDN = \angle DAB = 90°$. Then note that

$$\angle NEM = \angle LEA + \angle AEK = \angle LCA + \angle ABK = 90°$$

implying $MDNE$ is cyclic with $\overline{MN}$ the diameter of its circumcircle. Thus it suffices to find $\frac{MN}{2}$.



Claim. $\triangle NEP \sim \triangle BAC$ and $\triangle NEM \sim \triangle BAD$.

Proof. Note that

$$\angle CLE = \angle CAE = \angle MDE = \angle MNE$$

so $BC \parallel MN$. Thus $\angle DNM = \angle DBC$. Since $AD = ED$, $\triangle ABD \cong \triangle EBD$ and

$$\angle MNE = \angle MDE = \angle DBE = \angle ABD$$

implying the desired similarities. $\square$

Therefore
$$\frac{MN}{2} = \frac{BD \cdot AE}{4AL} = \frac{[ABD]}{AL} = \frac{AB \cdot AD}{2AL} = \frac{[ABC]}{2AL} = \frac{BC}{4} = \frac{29}{4}$$

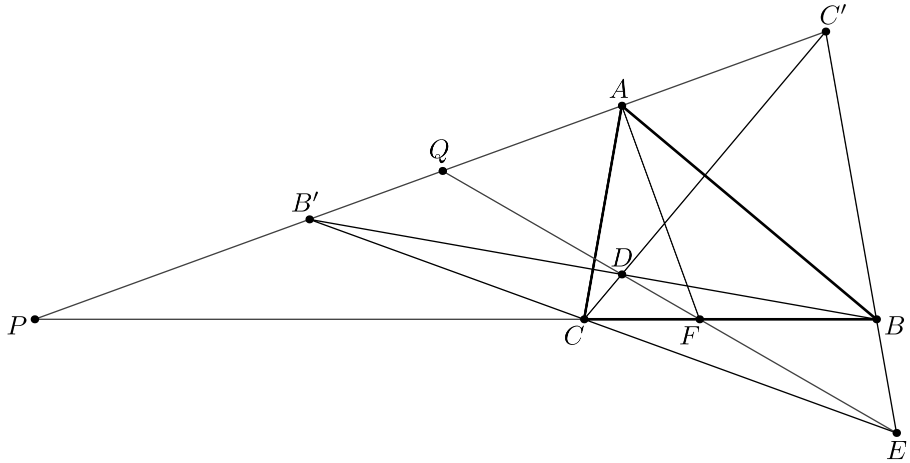
and $m + n = \boxed{033}$. $\blacksquare$

§2 Problem 2

In triangle ABC with $\angle A = 60°$, let B' be the reflection of B over AC, and let C' be the reflection of C over AB. Let $B'B$ and $C'C$ intersect at D, and let $B'C$ and $C'B$ intersect at E. Let DE and BC intersect at F. Show that AF bisects $\angle BAC$.

Solution. Let $P = BC \cap B'C'$ and $Q = DE \cap B'C'$. First, note the angle condition implies that B', A, and C' are collinear. Hence $B'C'$ is the exterior angle bisector of $\angle BAC$ and

$$\frac{AB}{BP} = \frac{AC}{PC}.$$



It's well known by Ceva and Menelaus that

$$-1 = (B', C'; P, Q) \stackrel{E}{=} (C, B; P, F) = \frac{CP}{BP} \div \frac{CF}{BF}.$$

Thus

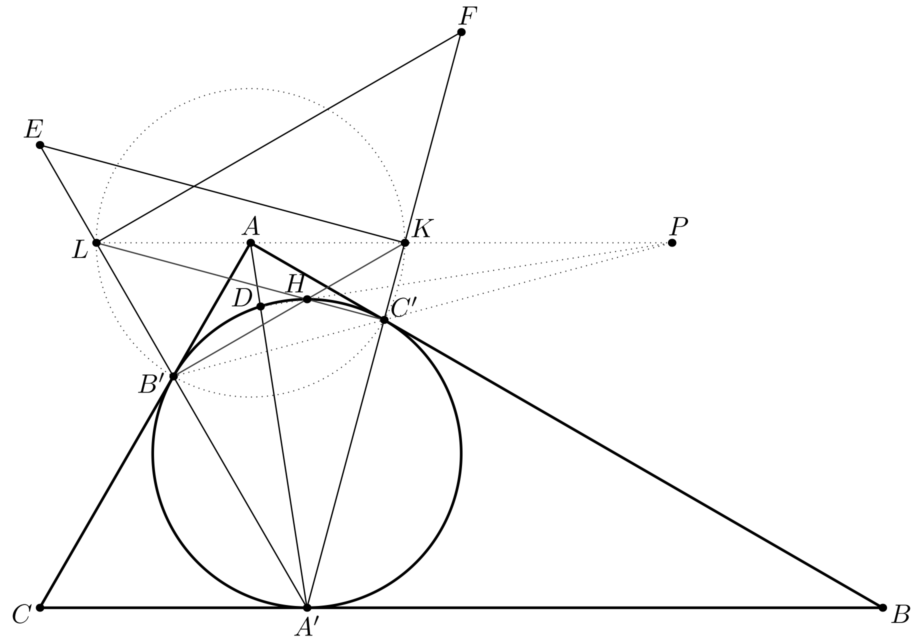
$$\frac{AC}{PC} \cdot \frac{BP}{AB} \cdot \frac{CP}{BP} \div \frac{CF}{BF} = -1$$
$$\frac{AC}{AB} = \frac{CF}{BF}$$

implying AF bisects $\angle BAC$. ■

§3 Problem 3

Let ABC be a scalene right triangle with right angle at A. The incircle ω of $\triangle ABC$ touches sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points A', B', and C', respectively. Let ω intersect AA' at A' and D, and let E and F be the reflections of A' over points B' and C', respectively. Let K be the foot of the perpendicular from E to $A'F$, and let L be the foot of the perpendicular from F to $A'E$. Show that $B'C'$, KL, and the line perpendicular to AA' at D are concurrent.

Solution. Let $H = B'K \cap C'L$ and $P = B'C' \cap KL$. Note that $\angle C'A'B' = \angle C'B'A = 45^\circ$, implying $\triangle A'EK$ and $\triangle A'FL$ are isosceles right triangles. Thus $A'E \perp B'K$ and $A'F \perp C'L$.



Claim. A is the midpoint of $\overline{KL}$.

Proof. Since $\angle KB'L = \angle KC'L = 90^\circ$, $B'C'KL$ is cyclic with $\overline{KL}$ the diameter of its circumcircle Γ. Note that $AB' = AC'$ and $\angle B'AC' = 90^\circ = 2\angle B'KC'$, hence A is the center of Γ, as desired. □

Since $\angle A'B'H = \angle HC'A' = 90^\circ$, H is the antipode of A' on ω and $\angle A'DH = 90^\circ$. By Brocard's theorem on Γ, $HP \perp AA'$, implying D, H, and P are collinear, as desired. ■