Contents

Introduction

In order to minimize the total mass of all elements in ten-bar truss problem, we need to calculate the mass, stress and displacement of each element fist. Then, optimize the problem with matlab.

Ten-Bar Truss Problem

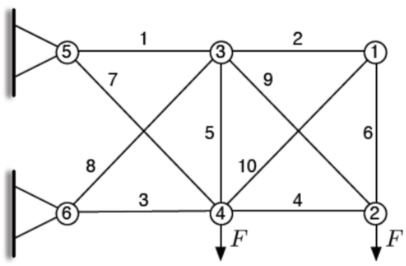

圖 1: Ten-Bar Truss [1]

Objective

- Calculate total mass of all bars: objective function
- Calculate stress in each bar: boundary condition and answer
- Calculate displacement of each node: boundary condition and answer
- Calculate reaction force: answer

Assumption

- Structure in static equilibrium
- Each bar treated as two-force member while ignoring the weight
- Connection between bars is pin connection

Known Conditions

- Structure in static equilibrium
- Young's Modulus $E = 200\ GPA$
- Density $\rho = 7860\ kg/m^3$
- Yield strength $\sigma_y = 250\ MPa$
- Length of horizontal and vertical bar $L = 9.14\ m$
- $r_1 \in$ radius of element 1 to 6
  $r_2 \in$ radius of element 7 to 10
- External force at node 2 and 4 with downward direction

Ten-Bar Solution

Solving ten-bar truss problem will start from filling up the element table. From the information in the element table, we can get the stiffness matrix, which can be used to determine displacement. After knowing the displacement, stress and reaction forces can be calculated.

Element Table

First, Filling up element table by defining the nodes and elements.

表 1: table 1

element	node 1	node 2	R	L	cos	sin
1	3	5	r_1	9.14	-1	0
2	1	3	r_1	9.14	-1	0
3	4	6	r_1	9.14	-1	0
4	2	4	r_1	9.14	-1	0
5	3	4	r_1	9.14	0	-1
6	1	2	r_1	9.14	0	-1
7	4	5	r_2	12.93	-0.707	0.707
8	3	6	r_2	12.93	-0.707	-0.707
9	2	3	r_2	12.93	-0.707	0.707
10	1	4	r_2	12.93	-0.707	-0.707

Stiffness Matrix

Each of the 6 nodes has 2 DOF, one in x direction and the other in y direction.
Each element has two nodes, and each node has two DOF. As the result, the stiffness matrix of each element would be in 4 DOF.

$$k_i = \frac{EA_e}{L_e}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{1}$$

$$where \begin{cases} c = cos\theta_i \\ s = sin\theta_i \end{cases}, i = 1, 2, \ldots, 10$$

By combining stiffness matrices of all ten elements, we can get a 12 DOF stiffness matrix. To check whether the stiffness matrix is correct, we set $r_1 = 0.1, r_2 = 0.05$ and check the answer.

$$\mathbf{K} = \begin{bmatrix}
0.7482 & 0.0608 & 0 & 0 & -0.6874 & 0 & -0.0608 & -0.0608 & 0 & 0 & 0 & 0 \\
0.0608 & 0.7482 & 0 & -0.6874 & 0 & 0 & -0.0608 & -0.0608 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.7482 & -0.0608 & -0.0608 & 0.0608 & -0.6874 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.6874 & -0.0608 & 0.7482 & 0.0608 & -0.0608 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.6874 & 0 & -0.0608 & 0.0608 & 1.4964 & 0 & 0 & 0 & -0.6874 & 0 & -0.0608 & -0.0608 \\
0 & 0 & 0.0608 & -0.0608 & 0 & 0.8090 & 0 & -0.6874 & 0 & 0 & -0.0608 & -0.0608 \\
-0.0608 & -0.0608 & -0.6874 & 0 & 0 & 0 & 1.4964 & 0 & -0.0608 & 0.0608 & -0.6874 & 0 \\
-0.0608 & -0.0608 & 0 & 0 & 0 & -0.6874 & 0 & 0.8090 & 0.0608 & -0.0608 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.6874 & 0 & -0.0608 & 0.0608 & 0.7482 & -0.0608 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0608 & -0.0608 & -0.0608 & 0.0608 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.0608 & -0.0608 & -0.6874 & 0 & 0 & 0 & 0.7482 & 0.0608 \\
0 & 0 & 0 & 0 & -0.0608 & -0.0608 & 0 & 0 & 0 & 0 & 0.0608 & 0.0608
\end{bmatrix} \times 10^9$$

Determine Displacement

The relationship between displacement and force can be represented as (2). **F** and **Q** are 12×12 matrices,

$$\mathbf{F_{12\times1}} = \mathbf{K_{12\times12}} \quad \mathbf{Q_{12\times1}} \tag{2}$$

$$\begin{cases} f_{2n-1} & \in & \text{force in x direction on node n} \\ f_{2n} & \in & \text{force in y direction on node n} \\ q_{2n-1} & \in & \text{displacement in x direction on node n} \\ q_{2n} & \in & \text{displacement in y direction on node n} \end{cases} \qquad n = 1, 2, \ldots, 6$$

By applying boundary condition, we know that node 5 and 6 are fixed without displacement. Thus, $q_9 = q_{10} = q_{11} = q_{12} = 0$. Equation (2) can now be reduced as (3).

$$\mathbf{F_{reduced}} = \mathbf{K_{reduced}} \quad \mathbf{Q_{reduced}} \tag{3}$$

$$\mathbf{F}_{reduced} = \begin{bmatrix} f_1 & f_2 & \cdots & f_8 \end{bmatrix}^T$$

$$\mathbf{K}_{reduced} = \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,8} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,8} \\ \vdots & \vdots & \ddots & \vdots \\ k_{8,1} & k_{8,2} & \cdots & k_{8,8} \end{bmatrix}$$

$$\mathbf{Q}_{reduced} = \begin{bmatrix} q_1 & q_2 & \cdots & q_8 \end{bmatrix}^T$$

$$\therefore \mathbf{Q_{reduced}} = \mathbf{K_{reduced}^{-1}} \mathbf{F_{reduced}} \tag{4}$$

Determine Stress

With displacement known, we can use (5) to calculate stress in each element.

$$\sigma_{4\times1} = \begin{bmatrix} \sigma_{2a-1} \\ \sigma_{2a} \\ \sigma_{2b-1} \\ \sigma_{2b} \end{bmatrix} = \frac{E_i}{L_i} \begin{bmatrix} -cos\theta_{2a-1} & -sin\theta_{2a} & cos\theta_{2b-1} & sin\theta_{2b} \end{bmatrix} \begin{bmatrix} q_{2a-1} \\ q_{2a} \\ q_{2b-1} \\ q_{2b} \end{bmatrix} \tag{5}$$

$$\begin{cases} E_i: \text{ Young's modulus of element i} \\ L_i: \text{ length of element i} \\ i = 1, 2, \ldots, 10 \\ a: \text{ the first node of element i} \\ b: \text{ the second node of element i} \end{cases}$$

Determine Reaction

Reaction force will appear at fixed node 5 and 6. The corresponding DOFs are DOF9 to DOF12, and the relationship can be expressed as (6).

$$\mathbf{R}_{4\times1} = \begin{bmatrix} R_9 \\ R_{10} \\ R_{11} \\ R_{12} \end{bmatrix} = \mathbf{K}_{reaction} \mathbf{Q}_{12\times1} \tag{6}$$

$$\mathbf{K}_{reaction} = \begin{bmatrix} k_{9,1} & k_{9,2} & \cdots & k_{9,12} \\ k_{10,1} & k_{10,2} & \cdots & k_{10,12} \\ \vdots & \vdots & \ddots & \vdots \\ k_{12,1} & k_{12,2} & \cdots & k_{12,12} \end{bmatrix}$$

Optimize

The optimization can be expressed as (7).

$$\begin{aligned} \min_{r_1, r_2} \quad & f(r_1, r_2) = \sum_{i=1}^{10} m_i(r_1, r_2) \\ \text{subject to} \quad & |\sigma_i| \leq \sigma_{yield} \\ & \Delta s_2 \leq 0.02 \\ \text{where} \quad & m_i : \text{mass of node i} \\ & \sigma_i : \text{stress in element i} \\ & \sigma_{yield} : \text{yield stress} \\ & \Delta s_2 : \text{displacement of node 2} \end{aligned} \tag{7}$$

The objective function (8) is the total mass of all elements. For the boundary equation, stress in each element could be determined from (5), while the displacement of node 2 can be calculated from (3).

$$f(r_1, r_2) = \sum_{i=1}^{10} m_i = \sum_{i=1}^{6} L_i \pi r_1^2 \rho + \sum_{i=7}^{10} L_i \pi r_2^2 \rho, \quad i = 1, 2, \ldots, 10 \tag{8}$$

By using the fmincon function in matlab, we get the result as follow:

$$\begin{aligned} (r_1, r_2) = & \quad (0.3000, 0.2663) \\ f(r_1, r_2) = & \quad 212410 \end{aligned} \tag{9}$$

Result

From the answer in (9), the result of required stress in each bar, displacement of each node, and reaction force at fixed nodes can be calculated.

$$\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \cdots \\ \sigma_{10} \end{bmatrix} = \begin{bmatrix} 6.9286 \\ 1.4647 \\ -7.2163 \\ -2.0715 \\ 1.3209 \\ 1.4647 \\ 6.6079 \\ -6.0913 \\ 3.7195 \\ -2.6301 \end{bmatrix} \times 10^7 (Pa), \quad \text{where} \begin{cases} \sigma_i : \text{stress in element i} \\ i = 1, 2, \ldots, 10 \end{cases} \tag{10}$$

$$\mathbf{Q} = \begin{bmatrix} q_1 \\ q_2 \\ \cdots \\ q_{12} \end{bmatrix} = \begin{bmatrix} 0.0038 \\ -0.0189 \\ -0.0042 \\ -0.0195 \\ 0.0032 \\ -0.0087 \\ -0.0033 \\ -0.0093 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{where} \begin{cases} q_{2n-1} : \text{displacement in x direction of node n} \\ q_{2n} : \text{displacement in y direction of node n} \\ n = 1, 2, \ldots, 6 \end{cases} \tag{11}$$

$$\mathbf{R} = \begin{bmatrix} R_9 \\ R_{10} \\ R_{11} \\ R_{12} \end{bmatrix} = \begin{bmatrix} -3.0000 \\ 1.0407 \\ 3.0000 \\ 0.9593 \end{bmatrix} \times 10^7, \quad \text{where} \begin{cases} R_{2n-1} : \text{reaction force in x direction of node n} \\ R_{2n} : \text{reaction force in y direction of node n} \\ n = 5, 6 \end{cases} \tag{12}$$

References

[1] 詹魁元, “系統最佳化實驗室基本演算法及範例使用説明,” 2019.