Wind mixing deepening with rotation

1. Introduction

We examine the energy budget of the oceanic mixed layer under wind-only forcing. The focus is on how wind stress influences the distribution of energy within the mixed layer, accounting for key processes such as turbulent mixing, frictional dissipation, and buoyancy fluxes.

In the Boussinesq approximation, the total energy of the flow can be divided into two components: kinetic energy and potential energy. By applying the Reynolds-averaged Navier-Stokes (RANS) approach, the kinetic energy can be further separated into mean kinetic energy (MKE) and turbulent kinetic energy (TKE). We will do energy budgets for understanding the dynamics of the mixed layer under wind forcing. Potential energy reflects the vertical stratification of the water column, kinetic energy represents the bulk motion of the fluid, and turbulent kinetic energy quantifies the energy contained in the turbulent eddies generated by wind-driven mixing.

2. Formulation of the problem

The problem focuses on the upper layer of the ocean, where the stratification is assumed to be uniform with a constant buoyancy frequency, N^2. The mixed layer is the region near the surface where turbulent mixing, primarily driven by wind forcing τ, creates a uniform temperature profile. Wind input to the mixed layer induces turbulence, which stirs the water and influences the distribution of potential and kinetic energy.

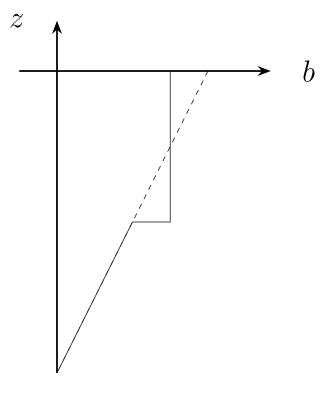

Figure 1. Deepening of the mixed layer: dashed line is the initial buoyancy profile, solid line is the buoyancy profile after the wind has turned on

3. Energy budget

3.1. *Potential energy*

Potential energy (anomaly) is defined as

$$PE = -\int (b - b_0) z \, dz \,. \tag{1}$$

where b_0 is the initial stratification which obbeys

$$\frac{\partial b_0}{\partial z} = N^2 = const. \tag{2}$$

Then, we can show that

$$PE = \frac{N^2}{12} h^3 \tag{3}$$

where h is the mixed layer depth (without any heat input)

The equation of evolution of potential energy is given by

$$\frac{dPE}{dt} = -G \tag{4}$$

where $G = wb$ is the conversion of kinetic to potential energy.

3.2. *Mean kinetic energy*

$$MKE = \int \frac{1}{2} (\overline{u}^2 + \overline{v}^2) \, dz \,, \tag{5}$$

There is no mean vertical velocity

$$\frac{dMKE}{dt} = u_s \tau - P \tag{6}$$

where u_s is the surface velocity, τ the wind stress and $P = u'w'\partial u/\partial z + v'w'\partial v/\partial z$ the production of TKE by mean shear. In order to evaluate various terms of Eq. (6), we need the velocity profile. We have (see Joel's notes)

$$U = \frac{u_*^2}{hf} \sin(ft) \tag{7}$$

and

$$V = \frac{u_*^2}{hf}(1 - \cos(ft)) \tag{8}$$

which are the depth integrated horizontal velocities. As a first approximation, we compute MKE with these depth averaged quantities

$$MKE = \frac{h}{2}(U^2 + V^2) = \frac{u_*^4}{f^2 h}(1 - \cos(ft)) \tag{9}$$

We also get the time derivative (assuming h is function of time)

$$\frac{dMKE}{dt} = \frac{u_*^4}{fh}\sin(ft) - \frac{u_*^4}{f^2 h^2}(1 - \cos(ft))\frac{dh}{dt} \tag{10}$$

To compute the forcing term in Eq. (6) we need the surface velocity (cf. Joel's notes)

$$u_s = \frac{u_*^2 \sin(ft)}{fh} - \frac{u_*}{\kappa}\left(\log\left(\frac{z_0}{h}\right) + 1\right) \tag{11}$$

3.3. *Turbulent Kinetic energy*

Last, the turbulent kinetic energy is defined as

$$TKE = \int \frac{1}{2}\left(\overline{u'}^2 + \overline{v'}^2 + \overline{w'}^2\right) dz\,, \tag{12}$$

and its equation of evolution is

$$\frac{dTKE}{dt} = G + P - \mathcal{E} \tag{13}$$

where $\mathcal{E}$ is the vertical integral of dissipation. We propose that the vertical profile of dissipation ϵ is given by

$$\epsilon = \frac{u_*^3}{\kappa(z + z_0)}\left(1 - \frac{z}{h}\right) \tag{14}$$

That is, the usual profile multiplied by $(1 - z/h)$ so that dissipation vanishes at $z = h$. We integrate and get (neglecting small terms)

$$\mathcal{E} = -\frac{u_*^3}{\kappa}\left(\log\left(\frac{z_0}{h}\right) + 1\right) \tag{15}$$

3.4. *Energy budget*

In order to do an energy budget, we sum Eq. (4), Eq. (6), and Eq. (13), we get

$$\frac{d}{dt}(PE + MKE + TKE) = \tau . u_s - \mathcal{E} \tag{16}$$

If we substitute all terms computed previously and assume $dTKE/dt = 0$, we get

$$\left(\frac{N^2 h^2}{4} + \frac{u_*^4}{f^2 h^2} (\cos(ft) - 1) \right) \frac{dh}{dt} = 0 \tag{17}$$

This corresponds to $h = const$. As we shall see below, the residual is indeed very small but it is not exactly zero. We probably need to get to next order to get meaningful results.

4. Analysis of all terms with a 1d model

We plot all the terms of the energy budget in a specific exemple. We consider the following parameters $N^2 = 10^{-4}$ s^{-2}, $u_* = 10^{-2}$ m s^{-1}, and $f_0 = 10^{-4}$ s^{-1}. We run a 1d $k - \epsilon$ model and analyze all the terms in Fig. 2. In this figure, we see that the main balance is between the energy input by the wind and dissipation. There is only a small fraction that is used to deepen the mixed layer $dpe/dt > 0$.

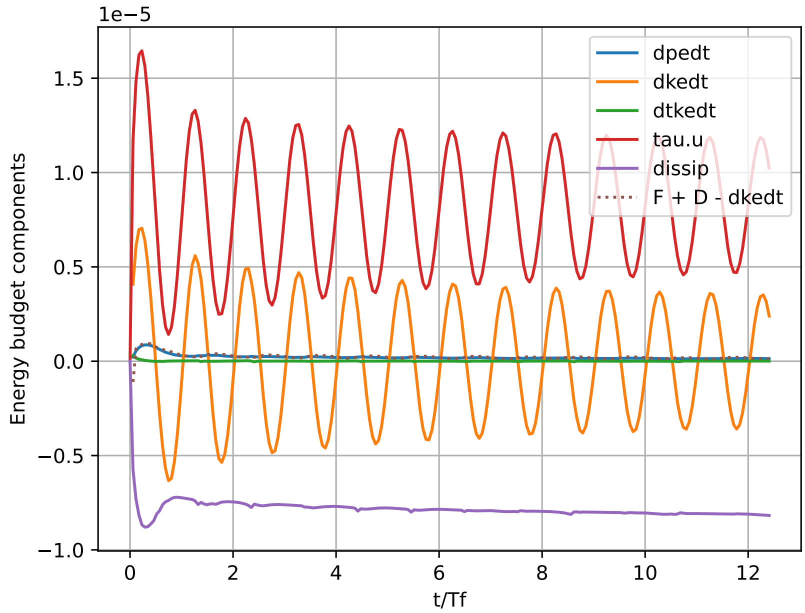

Figure 2. Components of the energy budget. Time is non dimensionalized with $T_f = 2\pi/f$. Note that dPE/dt is very small but not exactly zero.

5. Proposition

If we suppose that we can arbitrarilly add a small correction in Eq. (17) so that $dh/dt \neq 0$. We propose to add a correction proportional to

$$Cu_*^3 \tag{18}$$

in Eq. (16) (with C an arbitrary constant[1]), then we get

$$\frac{dh}{dt} = 4Cu_* \left(\frac{fh}{u_*}\right)^2 \frac{1}{(h/H_{p73})^4 + 4(\cos(ft) - 1)} \tag{19}$$

with $H_{p73} = u_*/\sqrt{Nf}$ and $Ro = u_*/fh$, which is simply Eq. (17) written with an addition of cu_*^3 in the rhs.

Interestingly, at $t = T_f/2$ and $h = 8^{1/4}H_{p73}$ (which is the value proposed by Pollard et at at $t = T_f/2$), Eq. (19) is singular. If we assume $h(t = T_f/2) \gtrsim 8^{1/4}H_{p73}$ and integrate Eq. (19), we get at short times a behavior that looks like $t^{1/5}$.

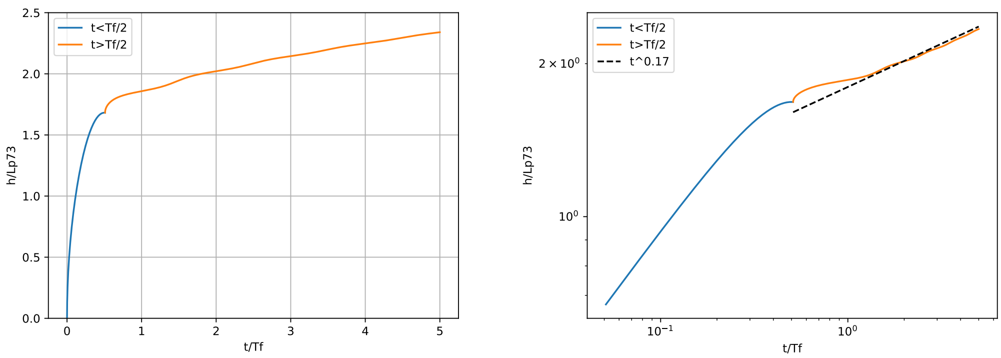

Figure 3. Evolution of the mixed layer depth. The transient phase is computed with Pollard et al formula. then for $t > T_f/2$ we use Eq. (19). left: linear space, right: loglog space

[1] Justification with mixing efficiency?