ALGEBRA LINEARE - LEZIONE 29

Note Title 03/11/2023

$\mathbb{R}^3$	$\hat{v}_1\ (1,0,1) \to (1,1,-1)$ $\hat{v}_2\ (2,2,0) \to (3,3,-3)$ $\hat{v}_3\ (0,1,1) \to (2,2,-2)$	$v_1 = (1,-1,2)$ $v_2 = (1,-1,1)$ $v_3 = (-1,2,1)$				

① Dim. che esiste unica $f: \mathbb{R}^3 \to \mathbb{R}^3$ che soddisfa le condizioni della prima colonna

Basta verificare che $\hat{v}_1, \hat{v}_2, \hat{v}_3$ sono una base di $\mathbb{R}^3$.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{Det} = 2+2 = 4 \neq 0 \qquad \text{o.k.}$$

② Tutto su $\ker(f)$ e $\operatorname{Im}(f)$.

$\operatorname{Im} = \operatorname{Span}((1,1,-1)) \rightsquigarrow \operatorname{Im}(f)$ ha dim. 1

R.N. $\Rightarrow \ker(f)$ ha dim. 2

$$\ker(f) = \operatorname{Span}(\hat{v}_2 - 3\hat{v}_1,\ \hat{v}_3 - 2\hat{v}_1)$$
$$= \operatorname{Span}((-1,2,-3),\ (-2,1,-1))$$

③ Cosa possiamo di $\ker(f) \cap \operatorname{Im}(f)$?

$\ker(f) + \operatorname{Im}(f) = \operatorname{Span}((1,1,-1), (-1,2,-3), (-2,1,-1))$

$$\begin{pmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ -1 & -3 & -1 \end{pmatrix} \qquad \text{Det} = -2+1+6-4-1+3 = 3 \neq 0$$

Quindi $\ker(f) + \operatorname{Im}(f) = \mathbb{R}^3 \rightsquigarrow \operatorname{Im}(f) \cap \ker(f) = \{(0,0,0)\}$

④ Scriviamo un po' di matrici associate ad f

$\mathbb{R}^3$	$\hat{v}_1\ (1,0,1) \to (1,1,-1)$ $\hat{v}_2\ (2,2,0) \to (3,3,-3)$ $\hat{v}_3\ (0,1,1) \to (2,2,-2)$	$v_1 = (1,-1,2)$ $v_2 = (1,-1,1)$ $v_3 = (-1,2,1)$				

$$\begin{array}{cl} \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ -1 & -3 & -2 \end{pmatrix} & \begin{array}{l} \to e_1 \\ \to e_2 \\ \to e_3 \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ f(\hat{v}_1) & f(\hat{v}_2) & f(\hat{v}_3) \end{array} & \end{array}$$

Matrice che rappresenta f usando
→ in partenza la base $\hat{B} = \{\hat{v}_1, \hat{v}_2, \hat{v}_3\}$
→ in arrivo la base canonica $\{e_1, e_2, e_3\}$

Matrice di f dalla canonica alla canonica.

$\boxed{\text{Bovino}}$ Scrivo $e_1 = (1,0,0)$ come $a\hat{v}_1 + b\hat{v}_2 + c\hat{v}_3$

Trovo a, b, c. Poi calcolo

$$\begin{aligned} f(e_1) &= f(a\hat{v}_1 + b\hat{v}_2 + c\hat{v}_3) \\ &= a f(\hat{v}_1) + b f(\hat{v}_2) + c f(\hat{v}_3) \\ &= a(1,1,-1) + b(3,3,-3) + c(2,2,-2) \end{aligned}$$

Quello che ottengo è la prima colonna della matrice richiesta.

$$(1,0,0) = a(1,0,1) + b(2,2,0) + c(0,1,1)$$

$$\begin{cases} a + 2b = 1 \\ 2b + c = 0 \\ a + c = 0 \end{cases} \quad \leadsto c = -2b \leadsto a = -c = 2b \leadsto 4b = 1$$

$$b = \frac{1}{4}, \quad a = \frac{1}{2}, \quad c = -\frac{1}{2}$$

$$f(e_1) = \frac{1}{2}(1,1,-1) + \frac{1}{4}(3,3,-3) - \frac{1}{2}(2,2,-2) = \frac{1}{4}(1,1,-1)$$

Prima colonna

$$\boxed{\begin{matrix} \frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{matrix}}$$

$\boxed{\text{Astuto}}$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Matrice di cambio base dalla $\hat{B}$ alla Canonica

$$\underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ -1 & -3 & -2 \end{pmatrix}}_{C \xleftarrow{f} \hat{B}} \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}}_{\hat{B} \xleftarrow{\text{cambio base}} C}$$

Prende in input le componenti di un vettore v rispetto alla canonica e restituisce quelle di $f(v)$ rispetto alla canonica

Facciamo l'inversa con i cofattori

$$\rightsquigarrow \underset{A_{ij}}{\begin{pmatrix} 2 & -1 & -2 \\ 2 & 1 & -2 \\ 2 & 1 & 2 \end{pmatrix}} \rightsquigarrow \underset{\text{aggiusto segni}}{\begin{pmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ 2 & -1 & 2 \end{pmatrix}}$$

$$\frac{1}{4}\underset{\text{trasposta}}{\begin{pmatrix} 2 & -2 & 2 \\ 1 & +1 & -1 \\ -2 & +2 & 2 \end{pmatrix}} \qquad \text{Det} = 2+2=4$$

$$\frac{1}{4}\begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ -1 & -3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & -1 \\ -2 & 2 & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 5 & 3 \\ 1 & 5 & 3 \\ -1 & -5 & -3 \end{pmatrix}$$

$$f(x,y,z) = \frac{1}{4}(x+5y+3z,\ x+5y+3z,\ -x-5y-3z)$$

Verifica $f(1,0,1) = \frac{1}{4}(4,4,-4) = (1,1,-1)$ ☺

$f(2,2,0) = \frac{1}{4}(12,12,-12) = (3,3,-3)$ ☺

$f(0,1,1) = \frac{1}{4}(8,8,-8) = (2,2,-2)$ ☺

Matrice di f dalla base $\underset{B}{\{v_1,v_2,v_3\}}$ alla base $\underset{B}{\{v_1,v_2,v_3\}}$

$v_1 = (1,-1,2)$
$v_2 = (1,-1,1)$
$v_3 = (-1,2,1)$

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix}^{-1} \frac{1}{4}\begin{pmatrix} 1 & 5 & 3 \\ 1 & 5 & 3 \\ -1 & -5 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \leftarrow \text{cambio base } B \to C$$

$$B \xleftarrow{\text{cambio base}} C \xleftarrow{\text{Applic. } f} C \xleftarrow{\text{cambio base}} B$$

$\mathbb{R}^2$	$\hat v_1$ $(0,1) \to (1,1)$ $\hat v_2$ $(-1,2) \to (2,2)$	$v_1 = (1,1)$ $v_2 = (0,1)$

Scrivere il cambio di base da $B = \{v_1, v_2\}$ alla base $\hat B = \{\hat v_1, \hat v_2\}$

$\boxed{\text{Bovino}}$

$$v_1 = a\hat v_1 + b\hat v_2$$
$$v_2 = c\hat v_1 + d\hat v_2$$

$$\leadsto \begin{array}{cl} \begin{pmatrix} a & c \\ b & d \end{pmatrix} & \begin{array}{l} \hat v_1 \\ \hat v_2 \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ v_1 & v_2 \end{array} & \end{array}$$

INPUT: comp. B
OUTPUT: " $\hat B$

$\boxed{\text{Cambio base}}$

$$\underset{\hat B \to C}{\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}} \qquad \underset{B \to C}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$\hat B \longleftarrow C \longleftarrow B$$

Matrice di f dalla B alla B

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}$$
$$B \xleftarrow[\text{C.B.}]{} C \xleftarrow[f]{} \hat B \xleftarrow[\text{C.B.}]{} C \xleftarrow[\text{C.B.}]{} B$$

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