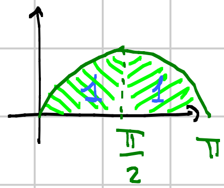
ANALISI 1 - LEZIONE 065

Note Title 14/12/2016

Uso delle simmetrie

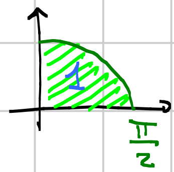
<u>Esempio 1</u>

$$\int_0^{\frac{\pi}{2}} \sin x \, dx = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 0-(-1) = 1$$



$$\int_{\frac{\pi}{2}}^{\pi} \sin x \, dx = \left[-\cos x\right]_{\frac{\pi}{2}}^{\pi} = -(-1)-0 = 1$$

$$\int_0^{\pi/2} \cos x \, dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = 1-0 = 1$$



$$\int_{\pi}^{\frac{3\pi}{2}} \sin x \, dx = -1$$

Giustificazione di questi passaggi

$$\int_{\frac{\pi}{2}}^{\pi} \sin x dx =$$

Pongo $x = \pi - y \rightsquigarrow dx = -dy$

Vedo come cambiano gli estremi:

quando $x = \frac{\pi}{2}$ ho che $y = \frac{\pi}{2}$

" $x = \pi$ ho che $y = 0$

$$= \int_{\frac{\pi}{2}}^{0} \underbrace{\sin(\pi - y)}_{\substack{\sin y \\ \text{(precorso)}}} (-dy) = -\int_{\frac{\pi}{2}}^{0} \sin y \, dy = \int_0^{\pi/2} \sin y \, dy$$

<u>Oss.</u> $x = \pi - y$ descrive la simmetria intorno alla retta $x = \frac{\pi}{2}$, che è l'asse di simmetria in gioco

Analogamente

$$\int_0^{\pi/2} \cos x \, dx \underset{x = \frac{\pi}{2} - y}{=} \int_{\frac{\pi}{2}}^{0} \underbrace{\cos\left(\frac{\pi}{2} - y\right)}_{= \sin y} (-dy) = -\int_{\frac{\pi}{2}}^{0} \sin y \, dy = \int_0^{\frac{\pi}{2}} \sin y \, dy$$

Funzioni pari / dispari

$f(x)$ pari $\Rightarrow \displaystyle\int_{-A}^{A} f(x)\,dx = 2\int_0^A f(x)\,dx$

$f(x)$ dispari $\Rightarrow \displaystyle\int_{-A}^{A} f(x)\,dx = 0$

$f(x)$ pari $\Rightarrow \displaystyle\int_{-A}^{0} f(x)\,dx = \int_0^A f(x)\,dx$

$f(x)$ dispari $\Rightarrow \displaystyle\int_{-A}^{0} f(x)\,dx = -\int_0^A f(x)\,dx$

Dimostro la prima: pongo $x=-y$, da cui $dx=-dy$

$$\int_{-A}^{0} f(x)\,dx = \underbrace{\int_{A}^{0}}_{\text{nuovi estremi in } y} f(-y)\,(-dy) = -\int_A^0 \underbrace{f(-y)}_{= f(y) \text{ se } f \text{ è pari}}\,dy$$

$$= -\int_A^0 f(y)\,dy = \int_0^A f(y)\,dy.$$

La dim. della seconda è analoga.

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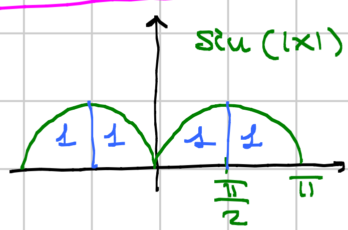
Fatto generale

$$\boxed{\begin{array}{ll} \displaystyle\int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \sin x\,dx = \underbrace{\pm 1}_{\text{a seconda di } k} & k\in\mathbb{Z} \\[2em] \displaystyle\int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \cos x\,dx = \pm 1 & k\in\mathbb{Z} \end{array}}$$

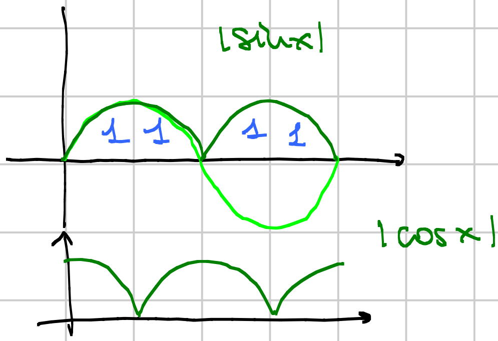
Esempio 2 $\displaystyle\int_{-\pi}^{\pi} \sin(|x|)\,dx = 4$



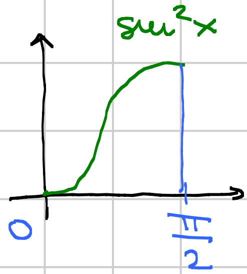
$\displaystyle\int_0^{2\pi} |\sin x|\,dx = 4$

$\displaystyle\int_0^{2\pi} |\cos x|\,dx = 4$

$\displaystyle\int_0^{2\pi} \cos|x|\,dx = 0$



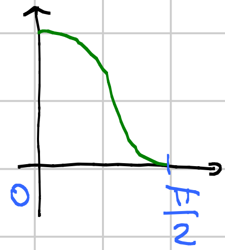
Esempio 3 $\displaystyle\int_0^{\pi/2} \sin^2 x \, dx$



1° modo : uso la primitiva

$$\int_0^{\pi/2} \sin^2 x \, dx = \left[\frac{1}{2}x - \frac{1}{2}\sin x \cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$ (il resto sono 0)

2° modo : $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = S \qquad \int_0^{\pi/2} \cos^2 x \, dx = C$



La simmetria suggerisce che $S = C$.
Il precorso suggerisce che

$$S + C = \int_0^{\pi/2} (\sin^2 x + \cos^2 x) \, dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$$

Quindi

$$S + C = \frac{\pi}{2}, \quad S = C \quad \Rightarrow \quad \boxed{S = C = \frac{\pi}{4}}$$

Verifica formale che $S = C$:

$$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx \underset{x = \frac{\pi}{2} - y}{=} \int_{\frac{\pi}{2}}^{0} \sin^2\left(\frac{\pi}{2} - y\right)(-dy) = \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - y\right) dy$$

$$\underset{\text{precorso}}{=} \int_0^{\pi/2} \cos^2 y \, dy$$

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La stessa cosa vale su tutti gli intervalli con estremi multipli di $\frac{\pi}{2}$, quindi

$$\boxed{\int_{a\frac{\pi}{2}}^{b\frac{\pi}{2}} \sin^2 x \, dx = \int_{a\frac{\pi}{2}}^{b\frac{\pi}{2}} \cos^2 x \, dx = \boxed{(b-a)\frac{\pi}{4}}}$$

se a e b sono in $\mathbb{Z}$

metà lungh. zona di integrazione

Oss.

$$\int_a^b 1\,dx = b-a = \text{lungh. zona di integrazione}$$

(area di un rettangolo di altezza 1 = lungh. base).

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Esempio 3 $\displaystyle\int_0^{\pi} \cos^2(6x)\,dx$

Pongo $y = 6x \rightsquigarrow dy = 6dx$

Quando $x=0$ ho che $y=0$

" $x=\pi$ " " $y=6\pi$

$$= \underbrace{\int_0^{6\pi}}_{\text{nuovi estremi}} \cos^2 y \cdot \underbrace{\tfrac{1}{6}dy}_{dx} = \frac{1}{6}\int_0^{6\pi} \cos^2 y\,dy = \frac{1}{6}\cdot \underbrace{3\pi}_{\frac{1}{2}\text{ zona integrazione}} = \frac{\pi}{2}$$

Analogamente $\displaystyle\int_0^{\pi} \cos^2(666x)\,dx = \frac{\pi}{2}$.

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Esempio 4 $\displaystyle\int_{-\pi}^{\pi} \frac{\cos^7 x}{7^x+1}\,dx = A$ Di primitiva non se ne parla

Pongo $y=-x \rightsquigarrow dy=-dx \rightsquigarrow$ estremi si scambiano

$$= \int_{\pi}^{-\pi} \frac{\cos^7(-y)}{7^{-y}+1}(-dy) = \int_{-\pi}^{\pi} \frac{\cos^7 y}{7^{-y}+1}\,dy = \int_{-\pi}^{\pi} \frac{\cos^7 x}{7^{-x}+1}\,dx = B$$

Abbiamo dim. che $A = B$. Ora

$$A+B = \int_{-\pi}^{\pi} \cos^7 x \underbrace{\left[\frac{1}{7^x+1} + \frac{1}{7^{-x}+1}\right]}_{=1} = \int_{-\pi}^{\pi} \cos^7 x\,dx$$

$$\frac{1}{7^{-x}+1} = \frac{1}{\frac{1}{7^x}+1} = \frac{7^x}{7^x+1}$$

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$$A = \frac{1}{2}\int_{-\pi}^{\pi} \cos^7 x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos x\,(1-\sin^2 x)^3\,dx = \frac{1}{2}\int_0^0 (1-y^2)^3\,dy = 0$$

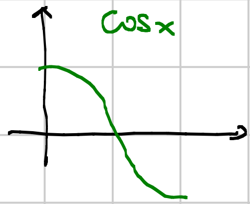
$y=\sin x$

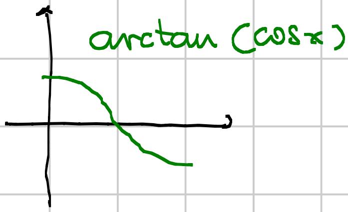
$dy = \cos x\,dx$

Esempio 5

$$\int_0^\pi \underset{F}{x}\ \underset{g}{\frac{\sin x}{1+\cos^2 x}}\,dx = \Big[\underset{F}{x}\ \underset{G}{(-\arctan(\cos x))}\Big]_0^\pi - \int_0^\pi \underset{f}{1}\ \underset{G}{(-\arctan(\cos x))}\,dx$$

$$= \underbrace{\pi\,(-\arctan(-1))}_{=\ \pi\cdot\arctan 1\ =\ \frac{\pi^2}{4}} + \underbrace{\int_0^\pi \arctan(\cos x)\,dx}_{=\ 0}$$





$$\underbrace{\int_0^\pi \arctan(\cos x)\,dx}_{=\ A} \underset{x=\pi-y}{=} \int_\pi^0 \arctan(\cos(\pi-y))\,(-dy)$$

$$= \int_0^\pi \arctan(\underbrace{\cos(\pi-y)}_{-\cos y})\,dy$$

$$= -\int_0^\pi \arctan(\cos y)\,dy = -A$$

$A = -A \Rightarrow A = 0.$

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