ANALISI 1 - LEZIONE 016

Note Title 10/10/2016

Esempio 1 $\lim_{n\to+\infty} \dfrac{n^2+3n+5}{4n^2-7n+8} = \dfrac{1}{4}$

Le potenze + alte sono più forti. $\sim \dfrac{\cancel{n^2}}{4\cancel{n^2}} = \dfrac{1}{4}$

Rigoroso: chi comanda si raccoglie!

$$\frac{\cancel{n^2}\left(1+\boxed{\frac{3}{n}}+\boxed{\frac{5}{n^2}}\right)}{\cancel{n^2}\left(4-\boxed{\frac{7}{n}}+\boxed{\frac{8}{n^2}}\right)} \to \frac{1}{4}$$

$$\boxed{\frac{3}{n}}\to 0,\quad \boxed{\frac{5}{n^2}}\to 0,\quad \boxed{\frac{7}{n}}\to 0,\quad \boxed{\frac{8}{n^2}}\to 0$$

Oss. Lo stesso metodo funziona per $\dfrac{\text{polinomio}}{\text{polinomio}} = \dfrac{P(n)}{Q(n)}$

Si ottiene

- $\deg P < \deg Q \implies \lim = 0$
- $\deg P > \deg Q \implies \lim = \pm\infty$ a seconda dei coeff. dei termini di grado max
- $\deg P = \deg Q \implies \lim =$ rapporto coeff. grado max.

Stessa cosa vale se ci sono potenze non intere

Esempio 2 $\dfrac{n+3n\sqrt{n}+4}{2n-4n\sqrt[3]{n}+7} \to -\infty$

$\dfrac{3}{2}-\dfrac{4}{3} = \dfrac{1}{6}$

Brutal mode: $\sim \dfrac{3n^{3/2}}{-4n^{4/3}} = -\dfrac{3}{4}\, n^{\frac{1}{6}} \to -\infty$, $\quad n^{\frac{1}{6}} \to +\infty$

Rigoroso:

$$\frac{n^{\frac{3}{2}}\left(\boxed{\frac{1}{\sqrt{n}}}+3+\boxed{\frac{4}{n\sqrt{n}}}\right)}{n^{\frac{4}{3}}\left(\boxed{\frac{2}{\sqrt[3]{n}}}-4+\boxed{\frac{7}{n\sqrt[3]{n}}}\right)} = \boxed{n^{\frac{1}{6}}}\cdot\boxed{\frac{\cdots}{\cdots}} \to -\infty.$$

$$\boxed{\frac{1}{\sqrt{n}}}\to 0,\quad \boxed{\frac{4}{n\sqrt{n}}}\to 0,\quad \boxed{\frac{2}{\sqrt[3]{n}}}\to 0,\quad \boxed{\frac{7}{n\sqrt[3]{n}}}\to 0,\quad n^{\frac{1}{6}}\to+\infty,\quad \frac{\cdots}{\cdots}\to -\frac{3}{4}$$

Esempio 3 $\quad n^2 - n\sin n \longrightarrow +\infty$

$$\underbrace{n^2 - n\sin n}_{\downarrow\, +\infty} = n(n - \sin n) \geq \boxed{n(n-1)}_{\downarrow\, +\infty}$$

Esempio 4 $\quad \frac{\sin n}{n} \longrightarrow 0$

È ovvio che $\quad -1 \leq \sin n \leq 1$. Divido per n (posso...)

$$\underset{\downarrow\, 0}{-\frac{1}{n}} \leq \underset{\downarrow\, 0}{\frac{\sin n}{n}} \leq \underset{\downarrow\, 0}{\frac{1}{n}}$$

Back to #3 $\quad n^2 - n\sin n = \underset{\downarrow\, +\infty}{n^2}\left(1 - \underset{\downarrow\, 0}{\frac{\sin n}{n}}\right) \longrightarrow +\infty \cdot 1 = +\infty$

Esempio 5 $\quad \frac{\sin(\log(n^2+7)) + \cos n!}{\sqrt{n}} \longrightarrow 0$

$$-2 \leq \sin(\text{Mostro}_1) + \cos(\text{Mostro}_2) \leq 2$$

Divido per $\sqrt{n}$ e ottengo $\quad \underset{\downarrow\, 0}{-\frac{2}{\sqrt{n}}} \leq \underset{\downarrow\, 0}{\frac{\cdots}{\sqrt{n}}} \leq \underset{\downarrow\, 0}{\frac{2}{\sqrt{n}}}$

Esempio 6 $\quad \sqrt[n]{\arctan n + 3} \longrightarrow 1$

$0 \leq \arctan n \leq \frac{\pi}{2}$, quindi $\quad 3 \leq \arctan n + 3 \leq 77$

Faccio $\sqrt[n]{}$ che è monotona crescente e ottengo

$$\underset{\downarrow\, 1}{\sqrt[n]{3}} \leq \underset{\downarrow\, 1}{\sqrt[n]{\cdots}} \leq \underset{\downarrow\, 1}{\sqrt[n]{77}}$$

Esempio 7 $\quad \dfrac{3^n}{n^{1000}} \to +\infty$

Criterio del rapporto $\quad \dfrac{a_{n+1}}{a_n} = \dfrac{3^{n+1}}{(n+1)^{1000}} \cdot \dfrac{n^{1000}}{3^n} = 3 \underbrace{\boxed{\dfrac{n^{1000}}{(n+1)^{1000}}}}_{\downarrow\, 1} \to 3$

Se $\dfrac{a_{n+1}}{a_n} \to 3$, allora $a_n \to +\infty$

Tabellina

$$\boxed{\lim_{n\to+\infty} \frac{a^n}{n^b} = +\infty \quad \text{per ogni } a>1,\ b>0}$$

ESPONENZIALE BATTE POTENZA

Esempio 8 $\quad \dfrac{n!}{312^n} \to +\infty$

Rapporto $\quad \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{312^{n+1}} \cdot \dfrac{312^n}{n!} = \dfrac{n+1}{312} \to +\infty$

quindi $a_n \to +\infty$

Tabellina

$$\boxed{\lim_{n\to+\infty} \frac{n!}{a^n} = +\infty \quad \text{per ogni } a>1}$$

FATTORIALE BATTE ESPONENZIALE

Esempio 9 $\quad \dfrac{n^n}{n!} \to +\infty$

Rapporto: $\quad \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^n} = \dfrac{\cancel{(n+1)}(n+1)^n \cancel{n!}}{\cancel{(n+1)}\cancel{n!} \quad n^n}$

$$= \left(1+\frac{1}{n}\right)^n \to e > 1$$

Poiché $\dfrac{a_{n+1}}{a_n} \to e > 1$, si ha che $a_n \to +\infty$

__Esempio 10__ $\sqrt[n]{n}$ $\qquad \sqrt[n]{n} = n^{1/n} \to \infty^0$

__Rapporto → radice__ Ho $\sqrt[n]{a_n}$ dove $a_n = n$. Faccio il rapporto

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \to 1 \text{ e quindi anche } \sqrt[n]{n} \to 1$$

__Esempio 11__ $\sqrt[n]{n^{20} + 19 n^{12} + 15} \to 1$

Brutal mode: $\sim \sqrt[n]{n^{20}} = \left(\sqrt[n]{n}\right)^{20} \to 1^{20} = 1$

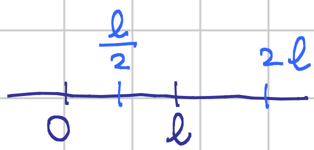
__Rigoroso__

$$\underbrace{\boxed{\sqrt[n]{n^{20}}}}_{\to 1} \; \underbrace{\sqrt[n]{1 + \underbrace{\boxed{\frac{19}{n^8}}}_{\to 0} + \underbrace{\boxed{\frac{15}{n^{20}}}}_{\to 0}}}_{1} \to 1$$

__Fatto generale__ Se $a_n \to \ell \in (0, +\infty)$, allora $\sqrt[n]{a_n} \to 1$

$\sqrt[n]{a_n} \to \sqrt[n]{\ell} \to 1$ SCHIFEZZA (limite metà per volta)

Se $a_n \to \ell$, allora definitiv.



$$\frac{\ell}{2} \le a_n \le 2\ell$$

Faccio $\sqrt[n]{\dots}$:

$$\underbrace{\sqrt[n]{\frac{\ell}{2}}}_{\to 1} \le \underbrace{\sqrt[n]{a_n}}_{\to 1} \le \underbrace{\sqrt[n]{2\ell}}_{\to 1}$$

__Tabellina__ : $\boxed{\sqrt[n]{\text{polinomio}} \to 1}$

Esempio 12 $\sqrt[n]{n!} \longrightarrow +\infty$

Rapporto → radice con $a_n = n!$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = n+1 \longrightarrow +\infty \text{, quindi } \sqrt[n]{a_n} = \sqrt[n]{n!} \longrightarrow +\infty$$

Esempio 13 $\sqrt[n]{n^9 + 7^n + \cos(n!)} \longrightarrow 7$

Brutal mode : $\sim \sqrt[n]{7^n} = 7$

Rigoroso : $\underbrace{\boxed{\sqrt[n]{7^n}}}_{7} \sqrt[n]{\underbrace{\boxed{\frac{n^9}{7^n}}}_{0} + 1 + \underbrace{\boxed{\frac{\cos(n!)}{7^n}}}_{0}} \longrightarrow 7 \cdot 1 = 7.$

Tabellina $\frac{\sqrt[n]{n!}}{n}$ $\left[\frac{\infty}{\infty}\right]$ $\longrightarrow \frac{1}{e}$

Faccio il limite del reciproco $\frac{n}{\sqrt[n]{n!}} = \sqrt[n]{\boxed{\frac{n^n}{n!}}} = \sqrt[n]{a_n}$

Calcolo $\frac{a_{n+1}}{a_n}$ = fatto prima = $\left(1+\frac{1}{n}\right)^n \longrightarrow e$. Quindi $\sqrt[n]{a_n} \to e$

Esempio 14 $\lim\limits_{n\to+\infty} \frac{n!}{2^{n^2}} = 0$

Rapporto : $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n^2+2n+1}} \cdot \frac{2^{n^2}}{n!} = \frac{n+1}{2^{2n+1}} = \underbrace{\frac{n+1}{2\cdot 4^n}}_{0}$ potenza vs esponenz.

Poiché $\frac{a_{n+1}}{a_n} \to 0$, allora anche $a_n \to 0$

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