INTEGRAZIONE PER SOSTITUZIONE

Solita partenza : $\int_a^b \varphi(x)\,dx = \phi(b) - \phi(a)$

Consideriamo ora $\Phi(x) = F(G(x))$. Allora

$$\varphi(x) = \Phi'(x) = F'(G(x)) \cdot G'(x) = f(G(x)) \cdot g(x)$$

Sostituendo troviamo

$$\int_a^b f(G(x)) \cdot g(x)\,dx = F(G(b)) - F(G(a))$$

$$= \Big[F(x)\Big]_{G(a)}^{G(b)}$$

Concludendo

$$\boxed{\int_a^b f(G(x)) \cdot g(x)\,dx = \Big[F(x)\Big]_{G(a)}^{G(b)}}$$

Formula ufficiale

<u>Brutalata</u> : devo calcolare $\int_a^b f(G(x)) \cdot g(x)\,dx$. Pongo $y = G(x)$.

Quando $x = a \rightsquigarrow y = G(a)$

$x = b \rightsquigarrow y = G(b)$

Inoltre

$$\frac{dy}{dx} = \text{derivata di } y \text{ risp. a } x = g(x) \qquad \text{da cui } dy = g(x)\,dx$$

così $\int_a^b \underbrace{f(G(x))}_{f(y)} \cdot \underbrace{g(x)\,dx}_{dy} = \int_{G(a)}^{G(b)} f(y)\,dy = F(G(b)) - F(G(a))$:)

Esempio 1 $\int x^6 \sin(x^7)\,dx$

Pongo $y = x^7$. Allora $\frac{dy}{dx} = 7x^6$, quindi $dy = 7x^6 dx$

$$\int x^6 \cdot \sin(x^7)\,dx = \frac{1}{7}\int \underbrace{\sin(x^7)}_{\sin y} \cdot \underbrace{7x^6 dx}_{dy} = \frac{1}{7}\int \sin(y)\,dy$$

$$= -\frac{1}{7}\cos(y) = -\frac{1}{7}\cos(x^7)$$ [Derivo per conferma]

Esempio 2 $\int x^3 e^{x^2} dx$

Pongo $y = x^2$. Allora $\frac{dy}{dx} = 2x$, quindi $dy = 2x\,dx$

$$\int x^3 \cdot e^{x^2} dx = \frac{1}{2}\int \underbrace{x^2 \cdot e^{x^2}}_{ye^y} \cdot \underbrace{2x\,dx}_{dy} = \frac{1}{2}\int \underset{G}{y}\,\underset{f}{e^y}\,dy$$

$$\underset{\text{per parti}}{=} \frac{1}{2}\left\{ y e^y - \int 1 \cdot e^y dy \right\} = \frac{1}{2} y e^y - \frac{1}{2} e^y = \frac{1}{2} e^y (y-1)$$

$$= \frac{1}{2} e^{x^2}(x^2-1)$$

Verifica: $\left[\frac{1}{2} e^{x^2}(x^2-1)\right]' = \frac{1}{2} e^{x^2} \cdot 2x(x^2-1) + \frac{1}{2} e^{x^2} \cdot 2x$

$$= e^{x^2} \cdot x^3 - xe^{x^2} + xe^{x^2} = e^{x^2} \cdot x^3$$:)

Esempio 3 $\int \sin(\log x)\,dx$

Pongo $y = \log x$ da cui $\frac{dy}{dx} = \frac{1}{x}$ cioè $dy = \frac{1}{x} dx$

$$\int \sin(\log x)\,dx = \int \underset{e^y}{x} \cdot \underbrace{\sin(\log x)}_{\sin y} \cdot \underbrace{\frac{1}{x}\,dx}_{dy} = \int e^y \cdot \sin y\,dy$$

= si fa con il grande ritorno.

Modo alternativo di vedere la sostituzione:

pongo $y = \log x$, quindi $x = e^y$, quindi $\frac{dx}{dy} = e^y$
cioè $dx = e^y\,dy$. Ma allora

$$\int \underbrace{\sin(\log x)}_{\sin y}\,\underbrace{dx}_{e^y\,dy} = \int \sin(y)\cdot e^y\,dy$$

☺

Esempio 4 $\int \tan x\,dx$

$$\int \frac{\sin x}{\cos x}\,dx$$

Pongo $y = \cos x \rightsquigarrow \frac{dy}{dx} = -\sin x \rightsquigarrow dy = -\sin x\,dx$

$$\int \frac{\sin x}{\cos x}\,dx = \int \frac{1}{y}(-dy) = -\int \frac{1}{y}\,dy = -\log|y|$$

$$= -\log|\cos x| = \log\frac{1}{|\cos x|}$$

Esempio 5 $\underbrace{\int \frac{x}{1+x^4}\,dx}_{\text{MEDIO}}$ $\underbrace{\int \frac{x^2}{1+x^4}\,dx}_{\text{DIFFICILE}}$ $\underbrace{\int \frac{x^3}{1+x^4}\,dx}_{\text{FACILE}}$

$$\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\log(1+x^4)$$

[Si può fare con la sostituzione $y = 1+x^4$]

$$\int \frac{x}{1+x^4}\,dx$$

Pongo $y = x^2 \rightsquigarrow \frac{dy}{dx} = 2x\,dx \rightsquigarrow dy = 2x\,dx$

$$= \frac{1}{2}\int \frac{1}{1+x^4}\cdot 2x\,dx = \frac{1}{2}\int \frac{1}{1+y^2}\,dy$$

$$= \frac{1}{2}\arctan y = \frac{1}{2}\arctan(x^2)$$

[Verifica]

Esempio 6 $\int \sin^3 x \, dx$

$\boxed{1^\circ \text{ modo}}$ Grande ritorno!

$$\int \sin^3 x \, dx = \int \underset{f}{\sin x} \cdot \underset{G}{\sin^2 x} \, dx = \underset{F}{-\cos x} \cdot \underset{G}{\sin^2 x} - \int (-\cos x) \, 2 \sin x \cdot \cos x \, dx$$

$$= -\cos x \cdot \sin^2 x + 2 \int \sin x \cdot \underbrace{\cos^2 x}_{1-\sin^2 x} \, dx$$

$$= -\cos x \cdot \sin^2 x + 2 \int \sin x \, dx - 2 \int \sin^3 x \, dx$$

Quindi $3 \int \sin^3 x \, dx = -\cos x \cdot \sin^2 x - 2 \cos x \rightsquigarrow$ si finisce

$\boxed{2^\circ \text{ modo}}$ Per sostituzione (funziona con esponente dispari)

$$\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx = \int (1 - \cos^2 x) \cdot \sin x \, dx = \bigstar$$

Pongo $y = \cos x \rightsquigarrow dy = -\sin x \, dx$, quindi

$$\bigstar = \int (1 - y^2) \cdot (-dy) = \int (y^2 - 1) \, dy = \frac{1}{3} y^3 - y = \frac{1}{3} \cos^3 x - \cos x$$

[Controllare sia lo stesso di prima]

Esempio 7 $\int \cos^7 x \, dx$ Grande ritorno possibile ma lungo

$$\int \cos^7 x \, dx = \int \cos^6 x \cdot \cos x \, dx = \int (1 - \sin^2 x)^3 \cdot \cos x \, dx$$

Pongo $y = \sin x \rightsquigarrow dy = \cos x \cdot dx$

$$\rightsquigarrow \int (1 - y^2)^3 \, dy = \int (1 - 3y^2 + 3y^4 - y^6) \, dy = \text{si fa comodo}$$

Esempio 8 $\displaystyle\int \frac{1}{x \log^3 x}\, dx$

Pongo $y = \log x \rightsquigarrow dy = \frac{1}{x}\, dx \rightsquigarrow \displaystyle\int \frac{1}{y^3}\, dy = -\frac{1}{2}\,\frac{1}{y^2} = -\frac{1}{2}\,\frac{1}{\log^2 x}$

Esempio 9 $\displaystyle\int x^2 \log^3 x\, dx$

$\boxed{1° \text{ modo}}$ $y = \log x \rightsquigarrow dy = \frac{1}{x}\, dx$

$$\int x^2 \log^3 x\, dx = \int \underset{\substack{\uparrow \\ e^{3y}}}{x^3}\, \underbrace{\log^3 x}_{y^3} \cdot \underbrace{\frac{1}{x}\, dx}_{dy} = \int y^3 e^{3y}\, dy$$

= per parti si fa

$\boxed{2° \text{ modo}}$

$$\int \underset{f}{x^2} \cdot \underset{G}{\log^3 x}\, dx = \frac{1}{3}\underset{F}{x^3}\, \underset{G}{\log^3 x} - \int \underset{F}{\frac{1}{3} x^3} \cdot \underset{g}{3 \log^2 x \cdot \frac{1}{x}}$$

$$= \frac{1}{3} x^3 \log^3 x - \int x^2 \log^2 x\, dx$$

e così in 3 passaggi si chiude

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