ANALISI 1 — LEZIONE 48

Note Title 28/11/2024

Esercizio 1 Polinomio di Taylor di grado 30 di $f(x) = x^7 \cdot \arctan x^5$

Fare le derivate non serve! $\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} + o(t^6)$

$$\arctan(x^5) = x^5 - \frac{x^{15}}{3} + \frac{x^{25}}{5} + \underbrace{o(x^{30})}$$

qui posso mettere anche $o(x^{34})$ perché il successivo sarebbe x^{35}

$$x^7 \arctan(x^5) = x^{12} - \frac{x^{22}}{3} + \frac{x^{32}}{5} + o(x^{41})$$

$$= x^2 - \frac{x^{22}}{3} + o(x^{31})$$

non posso fare meglio perché il termine successivo ha x^{32}

Esercizio 2 Taylor di ordine 4 di $(1+\sin x)^x = e^{x\log(1+\sin x)}$

$$1+\sin x = 1 + x - \frac{x^3}{6} + o(x^4)$$

$$\log(1+\sin x) = \log(1 + \underbrace{x - \frac{x^3}{6} + o(x^4)}_{t})$$

$\log(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 + o(t^3)$

$$= \left(x - \frac{x^3}{6} + o(x^4)\right) - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 + \frac{1}{3}(\dots)^3 + o(x^3)$$

$$= x - \frac{x^3}{6} - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

$$x\log(1+\sin x) = x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 + o(x^4)$$

$$e^{x\log(1+\sin x)} = 1 + \left(x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4\right) + \frac{1}{2}(\dots)^2 + o(x^4)$$

$$e^t = 1 + t + \frac{1}{2}t^2 + o(t^2)$$

$$= 1 + x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{2}x^4 + o(x^4)$$

$$= 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 + o(x^4)$$

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Esempio 3 Taylor di $f(x) = \dfrac{x+8}{x^2+3}$ centro in $x_0 = 0$ e $n = 4$

$$\frac{x+8}{x^2+3} = \frac{8+x}{3+x^2} = \frac{1}{3}\,\frac{8+x}{1+\frac{x^2}{3}} = \frac{1}{3}(8+x)\cdot\frac{1}{1+\frac{x^2}{3}} = (\star)$$

Ricordo che $\dfrac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$

$\star = \dfrac{1}{3}(8+x)\left(1 - \dfrac{x^2}{3} + \dfrac{x^4}{9} + o(x^4)\right) =$ faccio il conto salvando solo gli esponenti ≤ 4

Esempio 4 Taylor di $f(x) = \dfrac{x+8}{x^2+3}$ con centro in $x_0 = 1$ e $n = 3$

$$f(1+h) = \frac{1+h+8}{(1+h)^2+3} = \frac{h+9}{h^2+2h+4} = \frac{9+h}{4+2h+h^2}$$

$$= \frac{1}{4}(9+h)\,\frac{1}{1+\underbrace{\frac{h}{2}+\frac{h^2}{4}}_{t}} = \frac{1}{4}(9+h)\left[\overset{1}{1} \overset{-}{-} \overset{t}{\left(\frac{h}{2}+\frac{h^2}{4}\right)} \overset{+}{+} \overset{t^2}{\left(\frac{h}{2}+\frac{h^2}{4}\right)^2} \overset{-}{-} \overset{t^3}{(\dots)^3} + o(h^3)\right]$$

$$= \frac{1}{4}(9+h)\left(1 - \frac{h}{2} - \frac{h^2}{4} + \frac{h^2}{4} + \underbrace{\frac{h^3}{4}}_{\text{doppio prodotto}} - \frac{h^3}{8}\right) + o(h^3)$$

Ora basta moltiplicare

Esempio 5 Parte principale in 0 di $2^x - 2^{\sin x}$

$$2^x = e^{x\log 2} = \cancel{1} + \cancel{x\log 2} + \frac{1}{2}\cancel{(x\log 2)^2} + \frac{1}{6}\cancel{(x\log 2)^3} + o(x^3)$$

$$e^t = 1 + t + \frac{1}{2}t^2 + \dots$$

$$2^{\sin x} = e^{\log 2 \cdot \sin x} = e^{\log 2 \left(x - \frac{x^3}{6} + o(x^3)\right)}$$

$$= 1 + \log 2 \left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}(\dots)^2 + \frac{1}{6}(\dots)^3 + o(x^3)$$

$$= \cancel{1} + \cancel{\log 2\, x} - \frac{\log 2}{6}x^3 + \cancel{\frac{1}{2}(\log 2\, x)^2} + \cancel{\frac{1}{6}(\log 2\, x)^3} + o(x^3)$$

Quando vado a fare la differenza resta solo

$$2^x - 2^{\sin x} = + \frac{\log 2}{6} x^3 + o(x^3)$$

Esempio 6 $\quad \displaystyle\lim_{x \to 0} \frac{2^x - 2^{\sin x}}{x - \arctan x}$

Numeratore $= \dfrac{\log 2}{6} x^3 + o(x^3)$ $\quad$ Denominatore $= \dfrac{x^3}{3} + o(x^3)$

Quindi $\displaystyle\lim_{x \to 0} \frac{\dots}{\dots} = \frac{\log 2}{2} = \log \sqrt{2}$

Esempio 7 $\quad \displaystyle\sum_{n=1}^{\infty} \left[\sqrt[n]{2} - 2^{\sin \frac{1}{n}}\right]^{\alpha}$ $\quad$ per quali α converge?

Brutale: $\quad 2^{\frac{1}{n}} - 2^{\sin \frac{1}{n}} \sim \dfrac{\log 2}{6} \cdot \dfrac{1}{n^3}$

quindi elevato alla α diventa $\sim [\dots]^{\alpha} \dfrac{1}{n^{3\alpha}}$ quindi converge

$\iff 3\alpha > 1$, cioè $\alpha > \frac{1}{3}$

Esempio 7 $\quad \displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n^{33}}\right)^{n^{22}}$

$$a_n = e^{n^{22} \log\left(1 + \frac{1}{n^{33}}\right)} \sim e^{n^{22} \cdot \frac{1}{n^{33}}} = e^{\frac{1}{n^{11}}} \to 1$$

$\Rightarrow$ manca la condizione necessaria

Volendo si fa anche con i limiti notevoli

$$n^{22}\log\left(1+\frac{1}{n^{33}}\right) = n^{22}\underbrace{\boxed{\frac{\log\left(1+\frac{1}{n^{33}}\right)}{\frac{1}{n^{33}}}}}_{\downarrow\ 1}\cdot\frac{1}{n^{33}} \to 0$$

7-bis $\displaystyle\sum_{n=1}^{\infty}\left[\left(1+\frac{1}{n^{33}}\right)^{n^{22}}-1\right]^{\alpha}$ con $\alpha>0$

Adesso il termine generale tende a 0. Come?

$$\left(1+\frac{1}{n^{33}}\right)^{n^{22}}-1 = e^{n^{22}\log\left(1+\frac{1}{n^{33}}\right)}-1$$

$$\sim e^{\frac{1}{n^{11}}}-1 \sim \frac{1}{n^{11}}$$

Quindi converge $\Leftrightarrow$ $11\alpha>1$ $\Leftrightarrow$ $\alpha>\frac{1}{11}$

Esempio 8 $\displaystyle\sum_{n=0}^{\infty}\log\left(\frac{n^3+7}{n^{\alpha}+77}\right)$ conv. $\Leftrightarrow$ $\alpha=3$

$\alpha<3 \Rightarrow a_n\to+\infty \Rightarrow$ NO cond. nec. $\Rightarrow$ ☹

$\alpha>3 \Rightarrow a_n\to-\infty \Rightarrow$ NO cond. nec. $\Rightarrow$ ☹

$$\alpha=3 \quad \log\left(\frac{n^3+7}{n^3+77}\right) = \log\left(\frac{n^3+77-70}{n^3+77}\right)$$

$$= \log\left(1-\frac{70}{n^3+77}\right) \sim -\frac{70}{n^3+77}$$

$\Rightarrow$ la serie è a termini definitivamente negativi e converge per C.A. con $\frac{1}{n^3}$. ☺

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