591AA 21/22 – COMPITO, LEZIONI 13, 14 E 15

Data di scadenza: Questo compito non sarà raccolto per la valutazione. Invece, circa una settimana dopo che è stato assegnato, le soluzioni saranno pubblicate.

I problemi da 1 a 3 sono trascritti da "Schaum's Outlines, Linear Algebra, 3rd ed", che include anche le soluzioni e molti problemi simili.

Problema 1.

(a) [Schaum 3.39a, pg 109, pdf 117] Scrivi

$$A = \begin{pmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{pmatrix} = LU$$

dove L è una matrice triangolare inferiore e U è una matrice triangolare superiore.

(b) [Schaum 3.41, pg 109, pdf 117] Scrivi

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{pmatrix} = LU$$

dove L è una matrice triangolare inferiore e U è una matrice triangolare superiore.

Nota: In generale, si pu scrivere $A = LU$ solo se A può essere ridotto a una matrice scalina senza scambiare le righe.

Problema 2.

(a) [Schaum 6.14, pg 220, pdf 228] Sia $E = \{e_1, e_2, e_3\}$ la base standard (canonica) di $\mathbb{R}^3$. Sia S la base $\{(1,2,0),(1,3,2),(0,1,3)\}$. Trova il cambiamento della matrice di base da E a S e viceversa.

(b) Sia $L : \mathbb{R}^3 \to \mathbb{R}^3$ la mappa lineare

$$L(x_1e_1 + x_2e_2 + x_3e_3) = x_1e_1 + (x_1 + x_2)e_2 + (x_1 + x_2 + x_3)e_3$$

Trova la matrice di L rispetto alla base S usando il cambiamento delle matrici di base dalla parte (a).

Problema 3.

(a) [Schaum 8.11, pg 294, pdf 301]. Trova il volume del paralellepipedo (scatola) generato dai vettori $u_1 = (1,1,1)$, $u_2 = (1,3,-4)$, $u_3 = (1,2,5)$.

(b) [Schaum, 8.11, pg 294, pdf 301]. Trova il volume del paralellepipedo (scatola) generato dai vettori $u_1 = (1,2,4)$, $u_2 = (2,1,-3)$, $u_3 = (5,7,9)$.

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LU FACTORIZATION

3.39. Find the LU decomposition of: (*a*) $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix}$, (*b*) $B = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 8 & 1 \\ -5 & -9 & 7 \end{bmatrix}$.

(*a*) Reduce A to triangular form by the following operations:

"Replace R_2 by $-2R_1 + R_2$", "Replace R_3 by $R_1 + R_3$", and then "Replace R_3 by $\frac{5}{2}R_2 + R_3$"

These operations yield the following, where the triangular form is U:

$$A \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{2} & 1 \end{bmatrix}$$

The entries $2, -1, -\frac{5}{2}$ in L are the negatives of the multipliers $-2, 1, \frac{5}{2}$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

$$E_3E_2E_1A = U, \quad E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5/2 & 1 \end{pmatrix}$$

$$\Rightarrow A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}}_{L}U, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5/2 & 1 \end{pmatrix}$$

$$A = LU, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -5/2 & 1 \end{pmatrix}$$

$$A = \overbrace{E_1^{-1}E_2^{-1}E_3^{-1}}^{L}U, \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix} \qquad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

3.41. Find the LU factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$.

Reduce A to triangular form by the following operations:

(1) "Replace R_2 by $-2R_1 + R_2$", (2) "Replace R_3 by $3R_1 + R_3$", (3) "Replace R_3 by $-4R_2 + R_3$"

These operations yield the following, where the triangular form is U:

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

The entries $2, -3, 4$ in L are the negatives of the multipliers $-2, 3, -4$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

Cambio di base: Supponiamo di avere trovato la matrice A di una mappa lineare

$$L : U \to U$$

relativa la base B. Supporre che dobbiamo sapere la matrice relative un'altra base B'. Potremmo trovare la matrice di L relative B' a partire da zero.

Alternativamente potremmo trovare una matrice P "traduzione" dalla base B' alla base B. Allora, la matrice A' di L relatva a B' sarà

$$A' = P^{-1}AP, \quad i.e. \quad [L]_{B'} = P^{-1}[L]_B\, P \qquad \text{(E6)}$$

Per trovare la matrice P, siano

$$B = \{u_1, \ldots, u_n\}, \qquad B' = \{v_1, \ldots, v_n\}$$

Allora

$$P = (p_{ij}), \qquad v_i = \sum_{j=1}^{n} p_{ij} u_j \qquad \text{(E7)}$$

$B =$ base standard, $B' = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$

$\Longrightarrow P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 2 \end{pmatrix}$

6.14. The vectors $u_1 = (1, 2, 0)$, $u_2 = (1, 3, 2)$, $u_3 = (0, 1, 3)$ form a basis S of $\mathbf{R}^3$. Find:

(*a*) The change-of-basis matrix P form the usual basis $E = \{e_1, e_2, e_3\}$ to S.

(*b*) The change-of-basis matrix Q from S back to E.

(*a*) Since E is the usual basis, simply write the basis vectors of S as columns: $P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

(*b*) **Method 1.** Express each basis vector of E as a linear combination of the basis vectors of S by first finding the coordinates of an arbitrary vector $v = (a, b, c)$ relative to the basis S. We have

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + y \phantom{{}+z} &= a \\ 2x + 3y + z &= b \\ 2y + 3z &= c \end{aligned}$$

Solve for x, y, z to get $x = 7a - 3b + c$, $y = -6a + 3b - c$, $z = 4a - 2b + c$. Thus

$$v = (a, b, c) = (7a - 3b + c)u_1 + (-6a + 3b - c)u_2 + (4a - 2b + c)u_3$$

or $\qquad [v]_S = [(a, b, c)]_S = [7a - 3b + c,\ -6a + 3b - c,\ 4a - 2b + c]^T$

Using the above formula for $[v]_S$ and then writing the coordinates of the e_i as columns yields

$$\begin{aligned} e_1 &= (1, 0, 0) = 7u_1 - 6u_2 + 4u_3 \\ e_2 &= (0, 1, 0) = -3u_1 + 3u_2 - 2u_3 \\ e_3 &= (0, 0, 1) = u_1 - u_2 + u_3 \end{aligned} \quad \text{and} \quad Q = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

Method 2. Find P^{-1} by row reducing $M = [P, I]$ to the form $[I, P^{-1}]$:

$$M = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & 1 \\ 0 & 1 & 0 & -6 & 3 & 1 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right] = [I, P^{-1}]$$

Thus $Q = P^{-1} = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$.

6.15. Suppose the x- and y-axes in the plane $\mathbf{R}^2$ are rotated counterclockwise 45° so that the new x'- and y'-axes are along the line $y = x$ and the line $y = -x$, respectively.

(*a*) Find the change-of-basis matrix P.

(*b*) Find the coordinates of the point $A(5, 6)$ under the given rotation.

(*a*) The unit vectors in the direction of the new x'- and y'-axes are

$$u_1 = (\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}) \qquad \text{and} \qquad u_2 = (-\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2})$$

(The unit vectors in the direction of the original x and y axes are the usual basis of $\mathbf{R}^2$.) Thus write the coordinates of u_1 and u_2 as columns to obtain

$$P = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

(*b*) Multiply the coordinates of the point by P^{-1}:

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{11}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$$

(Since P is orthogonal, P^{-1} is simply the transpose of P.)

6.16. The vectors $u_1 = (1, 1, 0)$, $u_2 = (0, 1, 1)$, $u_3 = (1, 2, 2)$ form a basis S of $\mathbf{R}^3$. Find the coordinates of an arbitrary vector $v = (a, b, c)$ relative to the basis S.

Method 1. Express v as a linear combination of u_1, u_2, u_3 using unknowns x, y, z. We have

$$(a, b, c) = x(1, 1, 0) + y(0, 1, 1) + z(1, 2, 2) = (x + z,\ x + y + 2z,\ y + 2z)$$

this yields the system

$$\begin{aligned} x + \phantom{y +{}} z &= a \\ x + y + 2z &= b \\ y + 2z &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + \phantom{y +{}} z &= a \\ y + z &= -a + b \\ y + 2z &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + \phantom{y +{}} z &= a \\ y + z &= -a + b \\ z &= a - b + c \end{aligned}$$

Solving by back-substitution yields $x = b - c$, $y = -2a + 2b - c$, $z = a - b + c$. Thus,

$$[v]_S = [b - c,\ -2a + 2b - c,\ a - b + c]^T$$

MISCELLANEOUS PROBLEMS

8.11. Find the volume $V(S)$ of the parallelepiped S in $\mathbf{R}^3$ determined by the vectors:

(a) $u_1 = (1, 1, 1), u_2 = (1, 3, -4), u_3 = (1, 2, -5)$.

(b) $u_1 = (1, 2, 4), u_2 = (2, 1, -3), u_3 = (5, 7, 9)$.

$V(S)$ is the absolute value of the determinant of the matrix M whose rows are the given vectors. Thus

(a) $|M| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & -4 \\ 1 & 2 & -5 \end{vmatrix} = -15 - 4 + 2 - 3 + 8 + 5 = -7$. Hence $V(S) = |-7| = 7$.

(b) $|M| = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & -3 \\ 5 & 7 & 9 \end{vmatrix} = 9 - 30 + 56 - 20 + 21 - 36 = 0$. Thus $V(S) = 0$, or, in other words, u_1, u_2, u_3 lie in a plane and are linearly dependent.

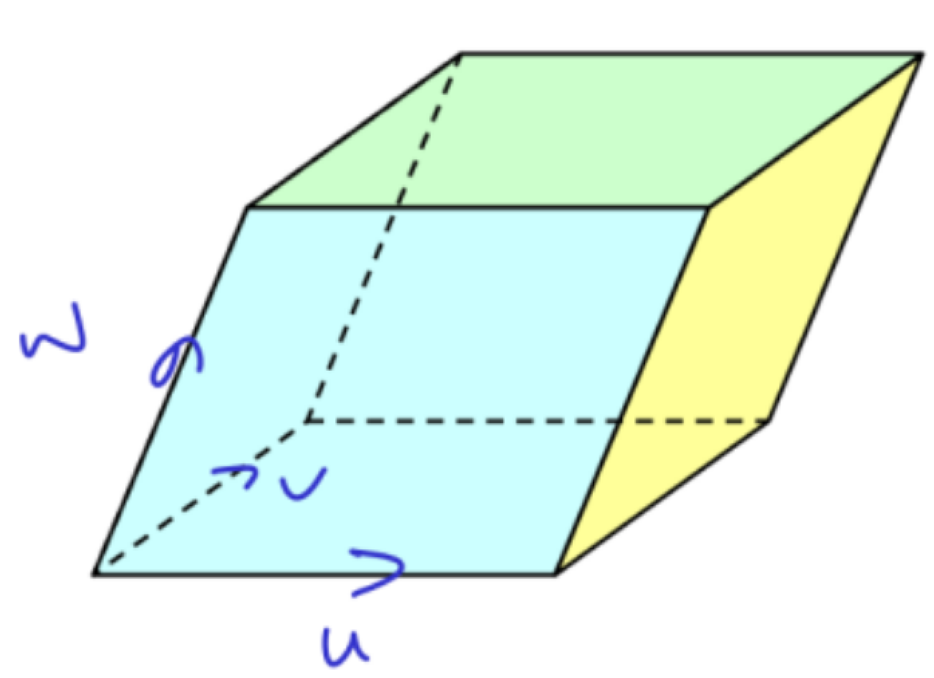
8.12. Find $\det(M)$ where $M = \begin{bmatrix} 3 & 4 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 & 0 \\ 0 & 5 & 0 & 6 & 7 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} = \left[\begin{array}{cc|c|cc} 3 & 4 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ \hline 0 & 9 & 2 & 0 & 0 \\ \hline 0 & 5 & 0 & 6 & 7 \\ 0 & 0 & 4 & 3 & 4 \end{array}\right]$

M is a (lower) triangular block matrix; hence evaluate the determinant of each diagonal block:

$$\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 15 - 8 = 7, \qquad |2| = 2, \qquad \begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} = 24 - 21 = 3$$

Thus $|M| = 7(2)(3) = 42$.

Volume di un parallelepipedo



$$V = \left|\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\right|$$

In generale, sia M una matrice $n \times n$ con righe $m_1, m_2, \dots, m_n$ Allora

$$|\det(M)| = \text{“n-volume” di } \left\{ \sum_{j=1}^{n} t_j m_j \mid 0 \le t_j \le 1,\ j = 1, \dots, n \right\}$$

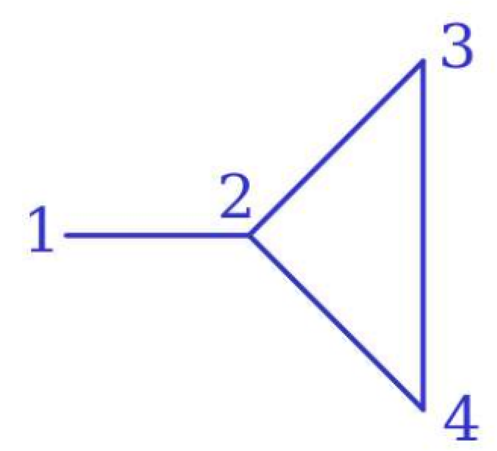
Problema 4.

(a) Trova il determinante della mappa lineare

$$L : P_2[x] \to P_2[x], \qquad L(p) = 2xp''(x) + (1-x)p'(x) + p(x)$$

(b) Verificare con la risultante che il polinomio $f(x) = x^3 + bx + c$ ha una radice multipla se e solo se $4b^3 + 27c^2 = 0$.

Problema 5. Trova l'inverso (se esiste) della matrice di adiacenza del grafico



Problema 6. Sia A la matrice di adiacenza del problema precedente.

(a) Calcolare A^4, A^3, A^2, A.

Per il teorema di Cayley-Hamilton, se A è una matrice $n \times n$, allora esiste un polinomio $p(t) = t^n + p_{n-1}t^{n-1} + \cdots + p_0$ tale che

$$p(A) = A^n + p_{n-1}A^{n-1} + \cdots + p_1 A + p_0 I_n = 0$$

(b) Trova il polinomio $p(t) = t^4 + p_3 t^3 + \cdots + p_0$ tale che $p(A) = 0$, dove A è la matrice della parte (a).

Suggerimento: La voce (i, j) di $p(A)$ è una funzione lineare di $p_3, \ldots, p_0$. Quindi, è sufficiente trovare 4 voci di $p(A)$ che danno equazioni linearmente indipendenti da risolvere per $p_3, \ldots, p_0$. È possibile ottenere tre tali equazioni lineari dalla diagonale principale di $p(A)$.

Definizione Sia $p(t) = t^n + p_{n-1}t^{n-1} + \cdots + p_0$ un polinomio di grado n tale che il coefficiente di t sia 1 (tali polinomi sono chiamati polinomi monici). Allora la matrice

$$C = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{pmatrix}$$

è chiamata matrice compagna di p.

Problema 7. Verificare che se $p(t) = t^3 + p_2 t^2 + p_1 t + p_0$ allora

$$p(C) = C^3 + p_2 C^2 + p_1 C + p_0 I_3 = 0, \qquad C = \begin{pmatrix} 0 & 0 & -p_0 \\ 1 & 0 & -p_1 \\ 0 & 1 & -p_2 \end{pmatrix}$$

Problema 4a: $L : P_2[x] \to P_2[x]$, $L(p) = 2xp''(x) + (1-x)p'(x) + p(x)$

$B = \{1, x, x^2\}$ base $P_2[x]$

$L(1) = 1$, $L(x) = (1-x)(1) + x = 1$

$L(x^2) = (2x)(2) + (1-x)(2x) + x^2$

$= 4x + 2x - 2x^2 + x^2 = -x^2 + 6x$

Matrice di L nel B:

$$\begin{array}{ccc} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & -1 \end{pmatrix} \\ L(1) \quad L(x) \quad L(x^2) \end{array} \Longrightarrow \det(L) = 0$$

Problema 4b: $f(x) = x^3 + bx + c \implies f'(x) = 3x^2 + b$

$R(f, f') = \det(\text{matrice } N \times N)$ $N = \deg f + \deg f'$ $(= 5)$

$$R(f,f') = \det \begin{array}{c} \begin{array}{ccccc} f: & x^3 & x^2 & x & 1 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & b & c & 0 \\ 0 & 1 & 0 & b & c \\ \hline 3 & 0 & b & 0 & 0 \\ 0 & 3 & 0 & b & 0 \\ 0 & 0 & 3 & 0 & b \end{array}\right) \\ \begin{array}{cccc} f' & x^2 & x & 1 \end{array} \end{array} = \det \begin{pmatrix} 1 & 0 & b & c & 0 \\ 0 & 1 & 0 & b & c \\ 3 & 0 & b & 0 & 0 \\ 0 & 3 & 0 & b & c \\ 0 & 0 & 3 & 0 & b \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & b & c & 0 \\ 0 & 1 & 0 & b & c \\ 3 & 0 & b & 0 & 0 \\ 0 & 3 & 0 & b & 0 \\ 0 & 0 & 3 & 0 & b \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{pmatrix} 1 & 0 & b & c & 0 \\ 0 & 1 & 0 & b & c \\ 0 & 0 & -2b & -3c & 0 \\ 0 & 3 & 0 & b & 0 \\ 0 & 0 & 3 & 0 & b \end{pmatrix}$$

$$\xrightarrow{R_4 \to R_4 - 3R_2} \begin{pmatrix} 1 & 0 & b & c & 0 \\ 0 & 1 & 0 & b & c \\ 0 & 0 & -2b & -3c & 0 \\ 0 & 0 & 0 & -2b & -3c \\ 0 & 0 & 3 & 0 & b \end{pmatrix}$$

$$\xrightarrow{R_5 \to R_5 + \frac{3}{2b} R_3} \begin{pmatrix} 1 & 0 & b & c & 0 \\ 0 & 1 & 0 & b & c \\ 0 & 0 & -2b & -3c & 0 \\ 0 & 0 & 0 & -2b & -3c \\ 0 & 0 & 0 & \frac{-9c}{2b} & b \end{pmatrix}$$

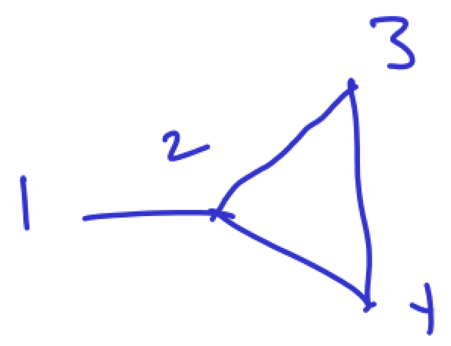
$$\xrightarrow{R_5 \to R_5 - \frac{9c}{(2b)^2} R_4} \begin{pmatrix} 1 & & & & * \\ & 1 & & & \\ & & -2b & & \\ & & & -2b & \\ & & & & b + \frac{27c^2}{4b^2} \end{pmatrix}$$

$\Rightarrow \det(\cdots)$

$= (4b^2)\left(b + \dfrac{27c^2}{4b^2}\right)$

$= 4b^3 + 27c^2$

Problema 5.



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_2]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2}]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \leftrightarrow R_4]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - R_4]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - R_3]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right)$$

Problema 6a:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 3 & 1 & 1 \\ 3 & 2 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 3 & 2 & 4 & 4 \\ 2 & 11 & 6 & 6 \\ 4 & 6 & 7 & 6 \\ 4 & 6 & 6 & 7 \end{pmatrix}$$

In generale, se C è la matrice compagna di un polinomio monico p allora $p(C) = 0$.

Problema 8. Verificare che se $p(t) = t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$ allora

$$\det(tI_4 - C) = \det \begin{pmatrix} t & 0 & 0 & c_0 \\ -1 & t & 0 & c_1 \\ 0 & -1 & t & c_2 \\ 0 & 0 & -1 & c_3 + t \end{pmatrix} = p(t)$$

In generale, se C è la matrice compagna di un polinomio monico p di grado n allora $p(t) = \det(tI_n - C)$

Problem 6b:

Poniamo i coefficienti delle voci diagonali (1,1), (2,2), (3,3) uguali a zero. Tuttavia, poiché le voci diagonali di A sono tutte zero, non otterremo mai un'equazione di p_1 a meno che non consideriamo una voce fuori diagonale. Scegliamo (1,2) poiché questo darà un'equazione che coinvolge p_1.

$$(A^4 + p_3A^3 + p_2A^2 + p_1A + p_0I)_{1,1} = 3 + p_2 + p_0 = 0$$
$$(A^4 + p_3A^3 + p_2A^2 + p_1A + p_0I)_{2,2} = 11 + 2p_3 + 3p_2 + p_0 = 0$$
$$(A^4 + p_3A^3 + p_2A^2 + p_1A + p_0I)_{3,3} = 7 + 2p_3 + 2p_2 + p_0 = 0$$
$$(A^4 + p_3A^3 + p_2A^2 + p_1A + p_0I)_{1,2} = 2 + 3p_3 + p_1 = 0$$

Applicando l'eliminazione gaussiana, otteniamo

$$p_3 = 0, \quad p_2 = -4, \quad p_1 = -2, \quad p_0 = 1 \quad \Longrightarrow \quad p(t) = t^4 - 4t^2 - 2t + 1$$

(dovresti tornare indietro e verificare che p(A)=0 con questa scelta di coefficienti)

Problema 7: Il metodo è solo quello di calcolare le potenze della matrice e quindi calcolare p(A).

$$A^3 = \begin{pmatrix} -p_0 & p_0p_2 & p_0(p_1 - p_3^2) \\ -p_1 & p_1p_2 - p_0 & p_0p_2 + p_1^2 - p_1p_2^2 \\ -p_2 & p_2^2 - p_1 & -p_0 + 2p_1p_2 - p_2^2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & -p_0 & p_0p_2 \\ 0 & -p_1 & p_1p_2 - p_0 \\ 1 & -p_2 & p_2^2 - p_1 \end{pmatrix}$$

Problema 8.

$$\begin{pmatrix} t & 0 & 0 & c_0 \\ -1 & t & 0 & c_1 \\ 0 & -1 & t & c_2 \\ 0 & 0 & -1 & c_3 + t \end{pmatrix} \xrightarrow[R_2 \to R_2 + \frac{1}{t}R_1]{} \begin{pmatrix} t & 0 & 0 & c_0 \\ 0 & t & 0 & c_1 + c_0/t \\ 0 & -1 & t & c_2 \\ 0 & 0 & -1 & c_3 + t \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + \frac{1}{t}R_2]{} \begin{pmatrix} t & 0 & 0 & c_0 \\ 0 & t & 0 & c_1 + c_0/t \\ 0 & 0 & t & c_2 + c_1/t + c_0/t^2 \\ 0 & 0 & -1 & c_3 + t \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 + R_3/t]{} \begin{pmatrix} t & & & * \\ & t & & * \\ 0 & & t & * \\ & & & c_3 + t + \frac{c_2}{t} + \frac{c_1}{t^2} + \frac{c_0}{t^3} \end{pmatrix}$$

$$\det(\cdot\,-) = t^3\left(c_3 + t + \frac{c_2}{t} + \frac{c_1}{t^2} + \frac{c_0}{t^3}\right)$$

$$= t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$$