ANALISI 1 — LEZIONE 124

Note Title 09/05/2017

FORMULA DI STIRLING

$$\boxed{n! \sim \sqrt{2\pi n}\,\frac{n^n}{e^n}}$$

$$\boxed{n! \geq \sqrt{2\pi n}\,\frac{n^n}{e^n}}$$

PRODOTTO DI WALLIS

$$\boxed{\prod_{k=1}^{\infty} \frac{2k \cdot 2k}{(2k-1)(2k+1)} = \frac{\pi}{2}}$$

<u>Esercizio</u> Il prodotto di Wallis converge

<u>Dim.</u> Wallis converge $\iff \displaystyle\sum_{k=1}^{\infty} \log \frac{4k^2}{4k^2-1}$ converge

$$\iff \sum_{k=1}^{\infty} \log \frac{1}{1-\frac{1}{4k^2}} = -\sum_{k=1}^{\infty} \underbrace{\log\left(1-\frac{1}{4k^2}\right)}_{\sim -\frac{1}{4k^2}} \text{ converge}$$

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Poniamo

$$S_n := \frac{n^n \sqrt{n}}{e^n\, n!} = \boxed{\frac{n^n \sqrt{2\pi n}}{e^n\, n!}}\,\frac{1}{\sqrt{2\pi}}$$

$$\overset{1}{\uparrow}$$

Voglio dimostrare che $S_n \to \dfrac{1}{\sqrt{2\pi}}$

$$\frac{S_{n+1}}{S_n} = \frac{(n+1)^{n+\cancel{1}}\sqrt{n+1}}{e^{\cancel{n+1}}\,\cancel{(n+1)!}}\;\frac{\cancel{e^n}\,\cancel{n!}}{n^n\sqrt{n}} = \left(\frac{n+1}{n}\right)^n \sqrt{\frac{n+1}{n}}\;\frac{1}{e}$$

Abbiamo ottenuto

$$\boxed{\frac{S_{n+1}}{S_n} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}}$$

Da questa otteniamo

$$S_{n+1} = S_1 \prod_{k=1}^{n} \frac{S_{k+1}}{S_k} = S_1 \prod_{k=1}^{n} \frac{1}{e}\left(1+\frac{1}{k}\right)^{k+\frac{1}{2}}$$

Dimostrare che S_n ha limite è equivalente a dire che la produttoria converge.

<u>Esercizio 2</u> La produttoria converge

$\boxed{\text{Dim}}$ Prod. conv. $\iff \sum_{k=1}^{\infty} \log\left[\frac{1}{e}\left(1+\frac{1}{k}\right)^{k+\frac{1}{2}}\right]$ converge

$$\iff \sum_{k=1}^{\infty}\left[\left(k+\frac{1}{2}\right)\log\left(1+\frac{1}{k}\right) - 1\right]$$

$$\left(k+\frac{1}{2}\right)\log\left(1+\frac{1}{k}\right) - 1 = \left(k+\frac{1}{2}\right)\left(\frac{1}{k} - \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right)\right) - 1$$

$$= \cancel{1} - \cancel{\frac{1}{2k}} + O\left(\frac{1}{k^2}\right) + \cancel{\frac{1}{2k}} - \cancel{1} = O\left(\frac{1}{k^2}\right)$$

Quindi <u>ordine</u> <u>di infinit.</u> ≥ 2 $\underline{\Rightarrow}$ la serie converge

Da questo momento sappiamo che $S_n \longrightarrow S_\infty \in \mathbb{R}$
"Ci resta" da calcolare S_∞.

Considero il prodotto di Wallis e pongo

$$W_n := \prod_{k=1}^{n} \frac{2k \cdot 2k}{(2k-1)(2k+1)}$$

$$W_n = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\ldots\cdot\frac{(2n-2)(2n-2)}{(2n-3)(2n-1)}\cdot\frac{2n \quad 2n}{(2n-1)\ (2n+1)}$$

$$W_n\cdot\frac{2\cdot 2}{2\cdot 2}\;\frac{4\cdot 4}{4\cdot 4}\cdot\ldots\cdot\frac{(2n)\cdot(2n)}{(2n)\cdot(2n)} = \frac{[2^n\cdot n!]^4}{[(2n)!]^2(2n+1)}$$

Quindi

$$W_n = \frac{(2^n n!)^4}{[(2n)!]^2(2n+1)}$$

D'altra parte sappiamo che

$$n! = \frac{n^n\sqrt{n}}{e^n S_n}$$

Vado a sostituire

$$W_n = \frac{2^{4n}(n!)^4}{[(2n)!]^2(2n+1)} = \frac{\cancel{2^{4n}}\;\cancel{n^{4n}}\;n^2}{\cancel{e^{4n}}\;S_n^4}\;\frac{\cancel{e^{4n}}\;S_{2n}^2}{\cancel{(2n)^{4n}}\;2n}\;\frac{1}{(2n+1)}$$

$$= \frac{n^2}{2n(2n+1)}\;\frac{S_{2n}^2}{S_n^4}$$

Abbiamo ottenuto che

$$W_n = \boxed{\frac{n^2}{2n(2n+1)}}\;\boxed{\frac{S_{2n}^2}{S_n^4}}$$

Se vale il prodotto di Wallis.

$$\frac{\pi}{2} = \frac{1}{4}\cdot\frac{1}{S_\infty^2} \Rightarrow S_\infty^2 = \frac{1}{2\pi} \Rightarrow S_\infty = \frac{1}{\sqrt{2\pi}}$$

Quindi Stirling è ok se uno dimostra il prodotto di Wallis.

Poniamo

$$I_n := \int_0^\pi \sin^n x \, dx$$

$$I_n = \int_0^\pi \underbrace{\sin x}_{f} \cdot \underbrace{\sin^{n-1} x}_{G} \, dx = \underbrace{\Big[\underbrace{(-\cos x)}_{F} \underbrace{\sin^{n-1} x}_{G}\Big]_0^\pi}_{=0} - \int_0^\pi (-\cos x)(n-1)\sin^{n-2} x \cdot \cos x \, dx$$

$$= (n-1) \int_0^\pi \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

grande ritorno

$$n I_n = (n-1) I_{n-2} \quad \rightsquigarrow \quad \boxed{I_n = \frac{n-1}{n} I_{n-2}}$$

$$R_n := \frac{I_{2n+1}}{I_{2n}}$$

$$R_n = \frac{I_{2n+1}}{I_{2n}} = \frac{\frac{2n}{2n+1} I_{2n-1}}{\frac{2n-1}{2n} I_{2n-2}} = \frac{2n \cdot 2n}{(2n-1)(2n+1)} R_{n-1}$$

Ma allora

$$R_n = R_0 \cdot \prod_{k=1}^{n} \frac{R_k}{R_{k-1}} = R_0 \prod_{k=1}^{n} \frac{2k \cdot 2k}{(2k-1)(2k+1)} = R_0 \cdot W_n$$

Da cui:

$$W_n = \frac{R_n}{R_0}$$

Ora osservo che $R_0 = \dfrac{I_1}{I_0} = \dfrac{\int_0^\pi \sin x \, dx}{\int_0^\pi \sin^0 x \, dx} = \dfrac{2}{\pi}$

Basta dim. che $R_n \to 1$

$$\frac{2n}{2n+1} \le \frac{\frac{2n}{2n+1}\, I_{2n-1}}{I_{2n}} = R_n = \frac{I_{2n+1}}{I_{2n}} \le \frac{I_{2n}}{I_{2n}} = 1$$

uso che I_n decresce all'aumentare di n

Conclusione

$$\underset{\substack{\downarrow\\ 1}}{\boxed{\frac{2n}{2n+1}}} \le \underset{\substack{\downarrow\\ 1}}{R_n} \le \underset{\substack{\downarrow\\ 1}}{\boxed{1}}$$

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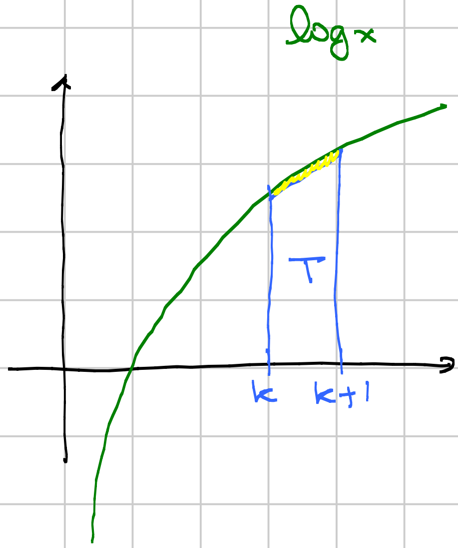
$$\left(k+\frac{1}{2}\right)\log\left(1+\frac{1}{k}\right) - 1$$

$$= \int_k^{k+1} \log x \; - \text{ area trapezio}$$



$\boxed{\text{Dim}}$ Area trapezio $= \dfrac{\log k + \log(k+1)}{2}$

$$\int_k^{k+1} \log x \, dx = \left[x \log x - x \right]_k^{k+1}$$

$$= (k+1)\log(k+1) - (\cancel{k}+1) - k\log k + \cancel{k}$$

$$= (k+1)\log(k+1) - k\log k - 1$$

Integrale − Area trapezio $= (k+1)\log(k+1) - k\log k - 1 - \frac{1}{2}\log k - \frac{1}{2}\log(k+1)$

$$= \left(k+\frac{1}{2}\right)\log(k+1) - \left(k+\frac{1}{2}\right)\log k - 1$$ ☺

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$\log x$ concava $\Rightarrow$ aree > 0 $\Rightarrow$ $\dfrac{S_{n+1}}{S_n} = e^{\text{aree}} > 1 \Rightarrow S_{n+1}$ cresc.

$\Rightarrow S_n < \dfrac{1}{\sqrt{2\pi}} \quad \forall n \in \mathbb{N} \Rightarrow$ dimostra $\quad n! > \sqrt{2\pi n}\, \dfrac{n^n}{e^n}$