ALGEBRA LINEARE — LEZIONE 43

Note Title 24/11/2023

Esercizio 1 $f: \mathbb{R}^4 \to \mathbb{R}^4$ $(x,y,z,w) \to (z,w,x,y)$.

$$f(x,y,z,w) = (z,w,x,y)$$

Forma canonica e base

$$(1,0,0,0) \to (0,0,1,0)$$
$$(0,1,0,0) \to (0,0,0,1)$$
$$(0,0,1,0) \to (1,0,0,0)$$
$$(0,0,0,1) \to (0,1,0,0)$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Se ad A diamo in pasto (x,y,z,w) questa sputa (z,w,x,y)

$$A - \lambda Id = \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ \text{Laplace} \\ 1^a \text{ riga}}}{Det =} \quad -\lambda \, Det \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} + 1 \cdot Det \begin{pmatrix} 0 & -\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda(-\lambda)[-\lambda(-\lambda)-1] - [-\lambda(-\lambda)-1]$$

$$= (\lambda^2 - 1)(\lambda^2 - 1) \qquad \text{Autovalori}: \underbrace{1, 1}, -1, -1$$

Quindi $ma(1) = ma(-1) = 2$ Quanto sono le geometriche ?

$$\underset{\lambda = 1}{\boxed{A - Id}} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Rango $= 2 \rightsquigarrow Dim(\ker) = 2$

$\rightsquigarrow mg(1) = 2$

Succedera la stessa cosa quando faccio $A + Id$ $\rightsquigarrow$ $mg(-1) = 2$

$\rightsquigarrow$ Diagonalizzabile con $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

In quale base succede questo?

$v_1 = (1,0,1,0)$ $\qquad$ $v_3 = (1,0,-1,0)$

$v_2 = (0,1,0,1)$ $\qquad$ $v_4 = (0,1,0,-1)$

La matrice iniziale era simmetrica, e infatti abbiamo ottenuto base ortogonale (se dividevo per $\sqrt{2}$ diventava ortonormale)
Se la uso come colonne di una M, allora $M^t A M = D$.

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Esercizio 2 $\quad f : M_{2\times2} \to M_{2\times2}$ $\qquad A \to \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} A.$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ -a+4c & -b+4d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$$

Quindi nella base canonica di $M_{2\times2}$ la f è rappresentata dalla matrice

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 4 \end{pmatrix} = B$$

Forma canonica di B?

$$\begin{pmatrix} 1-\lambda & 0 & 2 & 0 \\ 0 & 1-\lambda & 0 & 2 \\ -1 & 0 & 4-\lambda & 0 \\ 0 & -1 & 0 & 4-\lambda \end{pmatrix}$$

$$\text{Det} \underset{\uparrow \, 1^a \text{ col}}{=} (1-\lambda)\,\text{Det}\begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 4-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{pmatrix} - 1\cdot\text{Det}\begin{pmatrix} 0 & 2 & 0 \\ 1-\lambda & 0 & 2 \\ -1 & 0 & 4-\lambda \end{pmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)(4-\lambda)^2 + 2(4-\lambda)\right] + \left(+4 + 2(1-\lambda)(4-\lambda)\right)$$

$$= (1-\lambda)(4-\lambda)\left[(1-\lambda)(4-\lambda) + 2\right] + 4 + 2(1-\lambda)(4-\lambda)$$

$$[(1-\lambda)(4-\lambda)+2]\,[(1-\lambda)(4-\lambda)+2]$$
$$= \underbrace{(\lambda^2-5\lambda+6)}_{(\lambda-2)(\lambda-3)}\,\underbrace{(\lambda^2-5\lambda+6)}_{(\lambda-2)(\lambda-3)}$$

Gli autovalori sono $\lambda = 2, 2, 3, 3$

Calcoliamo $B - 2Id = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$ rango $= 2$, Dim (ker) $= 2$

$$\ker = \mathrm{Span}\Big(\underbrace{\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}}_{v_2}\Big)$$

$B - 3Id = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$

$$\ker = \mathrm{Span}\Big(\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{v_3}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{v_4}\Big)$$

Usando la base v_1, v_2, v_3, v_4 la matrice di f diventa

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} a+2c & b+2d \\ -a+4c & -b+4d \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$ ☺ $\qquad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}$ ☺

Idem per gli altri due.

$\boxed{\text{Teorema di Hamilton - Cayley}}$ Se $p_A(\lambda)$ è il polinomio caratteristico di A, allora

$$p_A(A) = 0 \leftarrow \text{matrice nulla}$$

<u>Esempio</u> $\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$ ha autovalori $\lambda = 1$ e $\lambda = 5$

$A - Id = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ ☺ $\qquad A - 5 Id = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$ ☺

Il pol. caratteristico è $(\lambda - 1)(\lambda - 5)$, cioè $\lambda^2 - 6\lambda + 5$

Voglio calcolare $A^2 - 6A + 5 Id$

$$A^2 = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 6 \\ 24 & 13 \end{pmatrix}$$

$$A^2 - 6A + 5 Id = \begin{pmatrix} 13 & 6 \\ 24 & 13 \end{pmatrix} - \begin{pmatrix} 18 & 6 \\ 24 & 18 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ ☺

$A^2 - 6A + 5 Id = 0$ Moltiplico per A^{-1}

$$A - 6 Id + 5 A^{-1} = 0 \quad \rightsquigarrow \quad A^{-1} = -\frac{1}{5} A + \frac{6}{5} Id$$

$$= -\frac{1}{5}\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} + \frac{6}{5}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}$$ ☺

Con questo sistema si possono fare le inverse facendo solo potenze di A.

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