1 tendon model

1.1 intro

We work in $\mathbb{R}^2$. We attach tendons to some triangular mesh $M = (V, E)$ with nodes $x_0, ..., x_{|V|-1}$. We consider a tendon as a sequence of waypoints bound to nodes of the mesh. For simplicity we restrict the sequence to not contain duplicate elements.

We write the *waypoint indices*

$$\mathcal{I} = (t_0, ..., t_{k-1})$$

signifying an k-tendon with waypoint positions $(x_{t_0}, ..., x_{t_{k-1}})$.

A convenient equivalent representation is the *segment notation*

$$S = ((t_0, t_1), (t_1, t_2), ..., (t_{k-2}, t_{k-1}))$$

1.2 energy (single tendon)

Rest length:

$$S_k = X_{t_k} - X_{t_{k+1}}$$

$$\alpha = \sum_{k=0}^{n-2} |S_k|$$

Current length:

$$s_k = x_{t_k} - x_{t_{k+1}}$$

$$\ell = \sum_{k=0}^{n-2} |s_k|$$

Length change:

$$\Delta = \ell - \alpha$$

$$\Delta_{,i} = \frac{\partial \Delta}{\partial x_i}$$

$$\Delta_{,ij} = \frac{\partial^2 \Delta}{\partial x_i \partial x_j}$$

Energy, gradient, and Hessian:

$$E = E(\Delta; K)$$

$$g_i = \frac{\partial E}{\partial \Delta} \Delta_{,i}$$

$$H_{ij} = \frac{\partial^2 E}{\partial^2 \Delta}\Delta_{,i}\Delta_{,j} + \frac{\partial E}{\partial \Delta}\Delta_{,ij}$$

First partials:

$$\hat{s}_k = \frac{s_k}{|s_k|}$$

$$\delta_{i,j} = \begin{cases} I & : i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta_{,i} = \sum_{k=0}^{n-2} \frac{\partial}{\partial x_i}|s_k|$$

$$\Delta_{,i} = \sum_{k=0}^{n-2} (\frac{\partial x_{t_k}}{\partial x_i} - \frac{\partial x_{t_{k+1}}}{\partial x_i})\hat{s}_k$$

$$\Delta_{,i} = \sum_{k=0}^{n-2} (\delta_{t_k,i} - \delta_{t_{k+1},i})\hat{s}_k$$

$$\Delta_{,i} = \sum_{k=0}^{n-2} \delta_{t_k,i}\hat{s}_k - \sum_{k=0}^{n-2} \delta_{t_{k+1},i}\hat{s}_k$$

$$\Delta_{,i \notin \mathcal{I}} = 0$$

$$\Delta_{,t_\alpha} = \sum_{k=0}^{n-2} \delta_{t_k,t_\alpha}\hat{s}_k - \sum_{k=0}^{n-2} \delta_{t_{k+1},t_\alpha}\hat{s}_k$$

$$\Delta_{,t_\alpha} = \begin{cases} \hat{s}_\alpha & : \alpha = 0 \\ -\hat{s}_{\alpha-1} & : \alpha = n-1 \\ \hat{s}_\alpha - \hat{s}_{\alpha-1} & \text{otherwise} \end{cases}$$

Second partials:

$$\zeta_k = \frac{1}{|s_k|} I_{2\times 2} - \frac{s_k s_k^T}{|s_k|^3}$$

$$\Delta_{,ij} = \sum_{k=0}^{n-2} (\frac{\partial x_{t_k}}{\partial x_i} - \frac{\partial x_{t_{k+1}}}{\partial x_i})(\frac{\partial x_{t_k}}{\partial x_j} - \frac{\partial x_{t_{k+1}}}{\partial x_j})\zeta_k$$

$$\Delta_{,ij} = \sum_{k=0}^{n-2} (\delta_{t_k,i} - \delta_{t_{k+1},i})(\delta_{t_k,j} - \delta_{t_{k+1},j})\zeta_k$$

$$\Delta_{,ij} = \sum_{k=0}^{n-2} \delta_{i,j}^{t_k}\zeta_k - \sum_{k=0}^{n-2} \delta_{t_k,i}\delta_{t_{k+1},j}\zeta_k - \sum_{k=0}^{n-2} \delta_{t_{k+1},i}\delta_{t_k,j}\zeta_k + \sum_{k=0}^{n-2} \delta_{i,j}^{t_{k+1}}\zeta_k$$

$$\Delta_{,ij \notin \mathcal{I}} = 0$$

$$\Delta_{,ii} = \sum_{k=0}^{n-2} \delta_{t_k,i}\zeta_k + \sum_{k=0}^{n-2} \delta_{t_{k+1},i}\zeta_k$$

$$\Delta_{,(i\neq j)j} = -\sum_{k=0}^{n-2} \delta_{t_k,i}\delta_{t_{k+1},j}\zeta_k - \sum_{k=0}^{n-2} \delta_{t_{k+1},i}\delta_{t_k,j}\zeta_k$$

$$\Delta_{,t_\alpha t_\alpha} = \sum_{k=0}^{n-2} \delta_{t_k,t_\alpha}\zeta_k + \sum_{k=0}^{n-2} \delta_{t_{k+1},t_\alpha}\zeta_k$$

$$\Delta_{,t_\alpha t_\alpha} = \begin{cases} \zeta_\alpha & : \alpha = 0 \\ \zeta_{\alpha-1} & : \alpha = n-1 \\ \zeta_\alpha + \zeta_{\alpha-1} & \text{otherwise} \end{cases}$$

$$\Delta_{,t_\alpha t_\beta} = -\sum_{k=0}^{n-2} \delta_{t_k,t_\alpha}\delta_{t_{k+1},t_\beta}\zeta_k - \sum_{k=0}^{n-2} \delta_{t_{k+1},t_\alpha}\delta_{t_k,t_\beta}\zeta_k$$

$$\Delta_{,t_\alpha t_\beta} = \begin{cases} -\zeta_\alpha & : \beta = \alpha + 1 \\ -\zeta_\beta & : \alpha = \beta + 1 \\ 0 & \text{otherwise} \end{cases}$$

2 nodal forces

2.1 intro

Say N nodes and T tendons.

We pack everything into vectors unless explicitly stated otherwise.

Notate $\ell|_x$ the tendon *length* evaluated for mesh at configuration encoded by x.

Call α the tendon *target length*.

Call $\Delta = \ell|_x - \alpha$ the tendon *length change*.

Call $f \in \mathbb{R}^N$ the *nodal forces*.

Call $\tau \in \mathbb{R}_+^T$ the tendon *tensions*. Note that generally $\tau = \tau(\Delta; K)$.

Call $\hat{e}_{ij}$ the unit vector pointing from node i to node j (in the configuration of interest).

For a *tendon segment* with tension τ connecting nodes with indices i, j we can write the force contribution:

$$\varepsilon_{ij} = \begin{bmatrix} [0 \quad 0] \dots [0 \quad 0] & \underbrace{[\hat{e}_{ij}]}_{i\text{-th block}} & [0 \quad 0] \dots [0 \quad 0] & \underbrace{[-\hat{e}_{ij}]}_{j\text{-th block}} & [0 \quad 0] \dots [0 \quad 0] \end{bmatrix}^T$$

$$f^{\tau}_{((i,j))} = \varepsilon_{ij}\tau$$

For some given tendon with tension τ connecting nodes specified by S we have the nodal force contribution:

$$\mathcal{E} = \sum_{(i,j)\in S} \varepsilon_{ij}$$

$$f^{\tau}_{S} = \sum_{(i,j)\in S} \varepsilon_{ij}\tau$$

$$f^{\tau}_{S} = \mathcal{E}\tau$$

Then for a system of T tendons,

$$f = \begin{bmatrix} | \\ \mathcal{E}_1 \\ | \end{bmatrix} \tau_1 + \ldots + \begin{bmatrix} | \\ \mathcal{E}_T \\ | \end{bmatrix} \tau_T = \begin{bmatrix} | & & | \\ \mathcal{E}_1 & \ldots & \mathcal{E}_T \\ | & & | \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_T \end{bmatrix} = A\tau$$

So we have the system

$$f = A\tau$$

$$\begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathcal{E}_1 & \ldots & \mathcal{E}_T \\ | & & | \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_T \end{bmatrix}$$

3 optimization

We optimize for $\Delta \in \mathbb{R}^T$ given some $f \in \mathbb{R}^N$.

Notate $K = diag(K_0, ..., K_{T-1})$.

Energy, gradient, and Hessian.

$$U = \frac{1}{2}(A\tau - f)^T(A\tau - f)$$

$$\frac{\partial U}{\partial \tau} = (A\tau - f)^T A$$

$$\frac{\partial^2 U}{\partial^2 \tau} = A^T A$$

$$g = \frac{\partial U}{\partial \Delta}$$

$$g = \frac{\partial \tau}{\partial \Delta}^T \frac{\partial U}{\partial \tau}$$

$$H = \frac{\partial g}{\partial \Delta}$$

$$H = \underbrace{\frac{\partial^2 \tau}{\partial^2 \Delta}}_{3-\text{tensor}} \frac{\partial U}{\partial \tau} + \frac{\partial \tau}{\partial \Delta}^T \frac{\partial^2 U}{\partial \tau \partial \Delta}$$

$$H = \frac{\partial^2 \tau}{\partial^2 \Delta} \frac{\partial U}{\partial \tau} + \frac{\partial \tau}{\partial \Delta}^T \frac{\partial^2 U}{\partial^2 \tau} \frac{\partial \tau}{\partial \Delta}$$

For special case of a linear bilateral spring $\tau = K\Delta$:

$$\frac{\partial \tau}{\partial \Delta} = K$$

$$g = K \frac{\partial U}{\partial \tau}$$

$$H = K \frac{\partial g}{\partial \tau}$$

$$H = K^2 \frac{\partial^2 U}{\partial^2 \tau}$$

3.1 todo

Optimizer returns $\Delta \in \mathbb{R}^T$. Need to achieve nodal forces f.

We will do this by choosing target lengths $\alpha_0, ..., \alpha_{T-1}$, and a single spring constant K.

We will leverage an observation that we can map Δ to the positive reals without affecting the simulation.