21-oct-2021

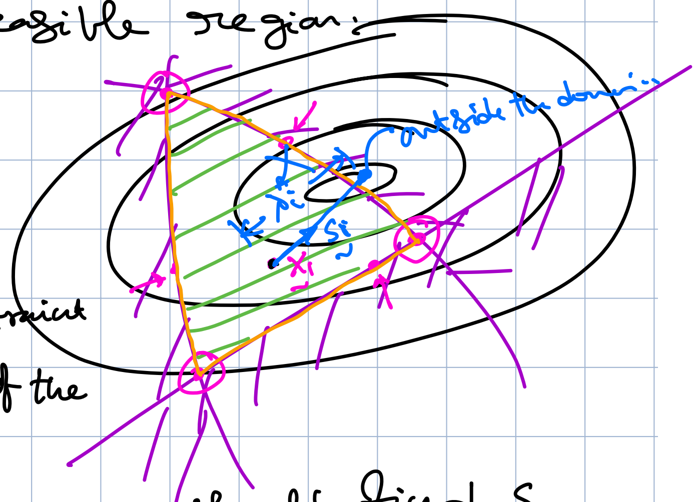
If $\underset{\sim}{S}$ is a usable feasible direction, then

$$\underset{\sim}{S}^T \nabla f(\underset{\sim}{X}_1) < 0 \quad ✓ — (A)$$

$$\underset{\sim}{S}^T \nabla g_j(\underset{\sim}{X}_1) \leq 0 — (B) \quad j = 1, 2, \ldots, m$$

Zoutendijk's Method of feasible directions:

In this method, the usable feasible direction is taken as the negative of the gradient of the objective fn if the initial point of the iteration lies within the interior of the feasible region:



$$g_j(\underset{\sim}{X}_1) < 0$$

$$\underset{\sim}{S} = -\nabla f(\underset{\sim}{X}_1)$$

If $\underset{\sim}{X}_1$ lies on the constraint boundaries, / one boundary of the feasible region. ✓

In such scenario, one should find $\underset{\sim}{S}$ that satisfies (A) & (B)

Zoutendijk's Algorithm: (inequality constraints). (131)

1. Start with an initial feasible point $\underset{\sim}{X}_1$ & choose small numbers ε_1, ε_2, & ε_3 to test convergence of the method. Set $i = 1$

2. If $g_j(\underset{\sim}{X}_i) < 0$, $j = 1, 2, \cdots m$ ($\underset{\sim}{X}_i$ is a interior feasible point)

$$\underset{\sim}{S}_i = -\nabla f(\underset{\sim}{X}_i) \quad \leftarrow \text{steepest descent direction}$$

Normalize $\underset{\sim}{S}_i$ in a suitable manner & go to step 5.

If atleast one $g_j(\underset{\sim}{X}_i) = 0$, go to step 3.

3. Find a usable feasible direction $\underset{\sim}{S}$ by solving the direction-finding problem:

$$\text{Maximize } \alpha \rightarrow \text{Minimize } -\alpha$$

Subject to
$$\begin{cases} \underset{\sim}{S}^T \nabla g_j(\underset{\sim}{X}_i) + \theta_j \alpha \le 0 & j = 1, 2, \cdots p \\ \underset{\sim}{S}^T \nabla f + \alpha \le 0 & j \in \underbrace{j_1, j_2 \cdots j_p}_{\text{set of active constraints}} \\ -1 \le s_i \le 1, & i = 1, 2, \cdots n \end{cases}$$

the values of θ_j can be taken as unity

Consider α as an additional design variable.

4. If the value of α^* found is very close to zero

$$\alpha^* \le \varepsilon_1 \rightarrow \underset{\sim}{X}^* = \underset{\sim}{X}_i$$

If $\alpha^* > \varepsilon_1$, go to step 5 by taking $\underset{\sim}{S}_i = \underset{\sim}{S}$

5. $\underset{\sim}{x}_{i+1} = \underset{\sim}{x}_i + \lambda_i^* \underset{\sim}{S}_i$

↑ optimal step length along $\underset{\sim}{S}_i$

How to find optimal step length?

6. Evaluate $f(\underset{\sim}{x}_{i+1})$

7. Test for convergence

$$\left| \frac{f(\underset{\sim}{x}_i) - f(\underset{\sim}{x}_{i+1})}{f(\underset{\sim}{x}_i)} \right| \leq \varepsilon_2 \text{ \& } \| \underset{\sim}{x}_i - \underset{\sim}{x}_{i+1} \| \leq \varepsilon_3$$

terminate & take $\underset{\sim}{x}^* = \underset{\sim}{x}_{i+1}$. otherwise go to step 8.

8. set $i = i+1$ & repeat from step 2.

The issues to be addressed to implement the above algorithm.

1) find an appropriate usable feasible direction. $\underset{\sim}{S}$

2) find a suitable step length along $\underset{\sim}{S}$

Direction-finding problem:

If $\underset{\sim}{x}_i$ lies in the interior of the feasible region;

$$\underset{\sim}{S}_i = -\nabla f(\underset{\sim}{x}_i)$$

If any one of the $g_j(\underset{\sim}{x}_i) = 0$

Simple way: generate random directions & check if they satisfy (A) & (B)

In general, one can have several directions that satisfy Ⓐ & Ⓑ. 133

However, we would like to choose the "best direction" among these qualified candidates.

Given a point $\underset{\sim}{x}_i$, find the vector $\underset{\sim}{S}$ & a scalar α, that maximizes α subject to the constraints

linear in s_i & α → $\underset{\sim}{S}^T \nabla g_j(\underset{\sim}{x}_i) + \theta_j \alpha \leq 0 \quad j \in J$

linear in s_i & α → $\underset{\sim}{S}^T \nabla f(\underset{\sim}{x}_i) + \alpha \leq 0$

$$\underset{\sim}{S}^T \underset{\sim}{S} = 1$$

or

$$\boxed{-1 \leq s_i \leq 1}$$

or

$$\underset{\sim}{S}^T \nabla f(\underset{\sim}{x}_i) \leq 1$$

θ_j is an arbitrary +ve constant, $\theta_j = 1$ (simplicity

The maximum value of α gives the best $\underset{\sim}{S}$ that makes $\underset{\sim}{S}^T \nabla f(\underset{\sim}{x}_i)$ negative & $\underset{\sim}{S}^T \nabla g_j(\underset{\sim}{x}_i)$ as negative as possible simultaneously.

Minimize $-\alpha$

s.t.

$$s_1 \frac{\partial g_1}{\partial x_1} + s_2 \frac{\partial g_1}{\partial x_2} + \cdots + s_n \frac{\partial g_1}{\partial x_n} + \theta_1 \alpha \leq 0$$

$$s_1 \frac{\partial g_2}{\partial x_1} + s_2 \frac{\partial g_2}{\partial x_2} + \cdots + s_n \frac{\partial g_2}{\partial x_n} + \theta_2 \alpha \leq 0$$

$$\vdots$$

$$s_1 \frac{\partial g_p}{\partial x_1} + s_2 \frac{\partial g_p}{\partial x_2} + \cdots + s_n \frac{\partial g_p}{\partial x_n} + \theta_p \alpha \leq 0$$

$$\left.\begin{array}{l} s_1 - 1 \leq 0 \\ s_2 - 1 \leq 0 \\ \vdots \\ s_n - 1 \leq 0 \end{array}\right\}$$

$$\left.\begin{array}{l} -1 - s_1 \leq 0 \\ -1 - s_2 \leq 0 \\ \vdots \\ -1 - s_n \leq 0 \end{array}\right\}$$

An optimization problem with linear obj. fn. & linear constraints.

$\downarrow$

can be solved using linear programming techniques (simplex)

- Determination of step length.

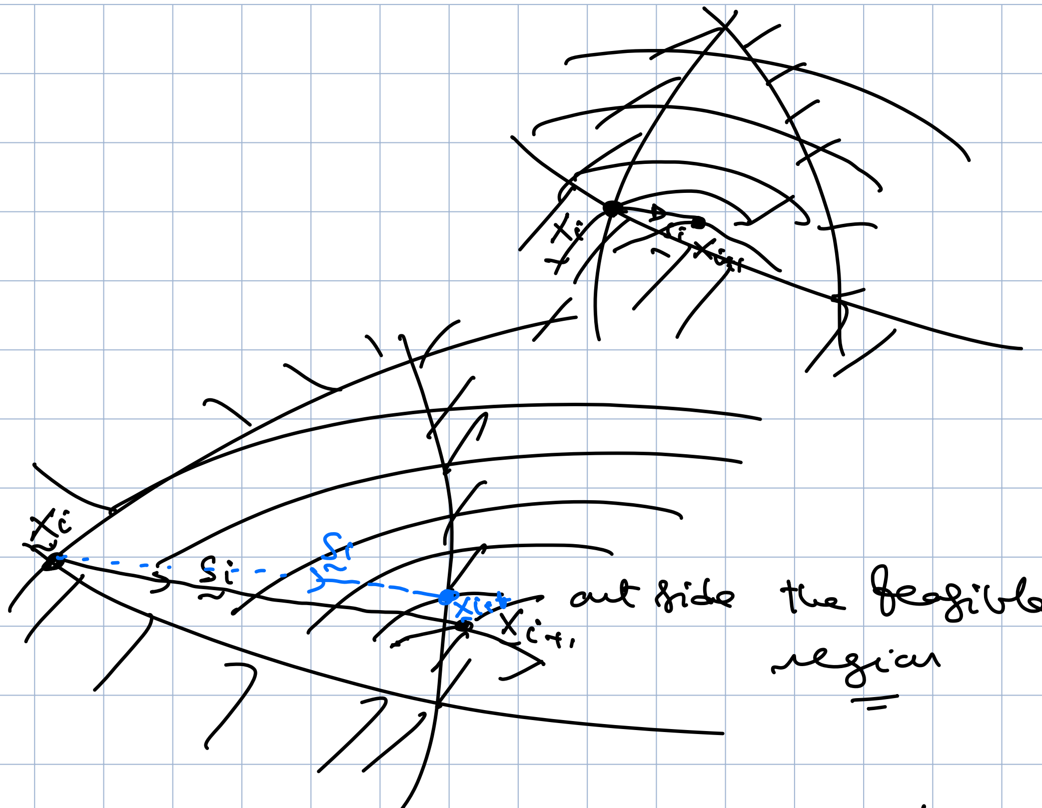
$$\underset{\sim}{X}_{i+1} = \underset{\sim}{X}_i + \lambda_i \underset{\sim}{S}_i \quad \text{— } (C)$$

one way: Find optimal λ_i that minimizes $f(X_i + \lambda S_i)$ such that the new point $\underset{\sim}{X}_{i+1}$ given by (C) lies in the feasible region.

Second way: use trial & error check if

$$f(\underset{\sim}{X}_i + \lambda_i \underset{\sim}{S}_i) \leq f(\underset{\sim}{X}_i)$$

$$g_j(\underset{\sim}{X}_i + \lambda_i \underset{\sim}{S}_i) \leq 0 \quad j = 1, 2, \cdots m$$



If we find the optimal steplength using one-dimensional minimization, the new point may or may not lie in the feasible region.

$$g_j(\underset{\sim}{X}) = 0$$

$$g_j(\underset{\sim}{X}_{i+1}) = 10^{-2},\ 10^{-3},\ 10^{-8} \ldots$$

$$\left| g_j(\underline{X}_{i+1}) \right| \le \varepsilon$$