Pattern search Methods

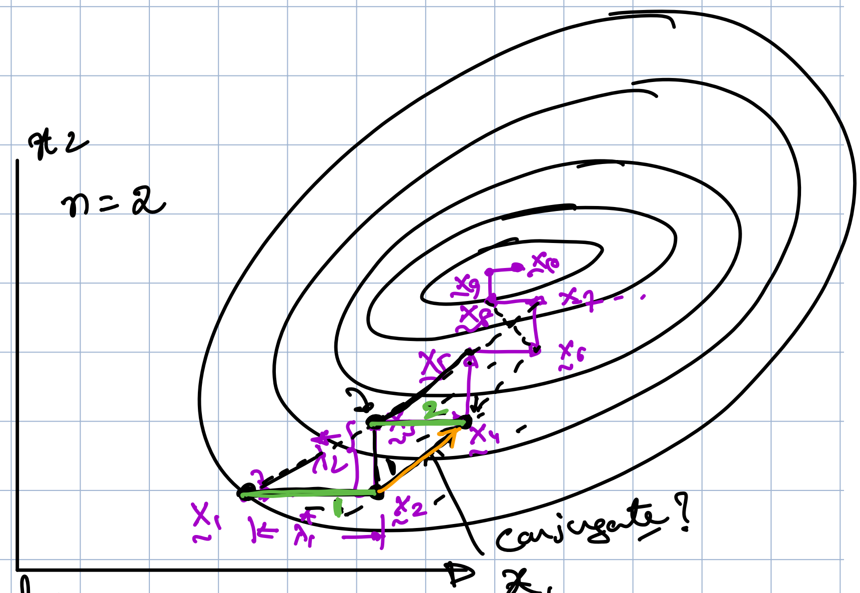
$$\underset{\sim}{S}_1^P = \underset{\sim}{X}_3 - \underset{\sim}{X}_1$$

$$= \underset{\sim}{X}_{n+1} - X_1$$

$$S_i^P = \underset{\sim}{X}_i - X_{i-n}$$

$\underset{\sim}{X}_i$ is the point obtained at the end of n univariate steps

$\underset{\sim}{X}_{i-n}$ is the starting point before taking n - univariate steps.



1-3, 2-5 ...

2-4 - 4-6 - ...

Conjugate directions Method (Powell's Method)

Conjugate directions: [A] is a n×n symmetric matrix

a set of n vectors $[\underset{\sim}{S}_i]$ is said to be conjugate (A-conjugate) if

$$\underset{\sim}{S}_i^T A \underset{\sim}{S}_j = 0 \quad \text{for all } i \neq j$$

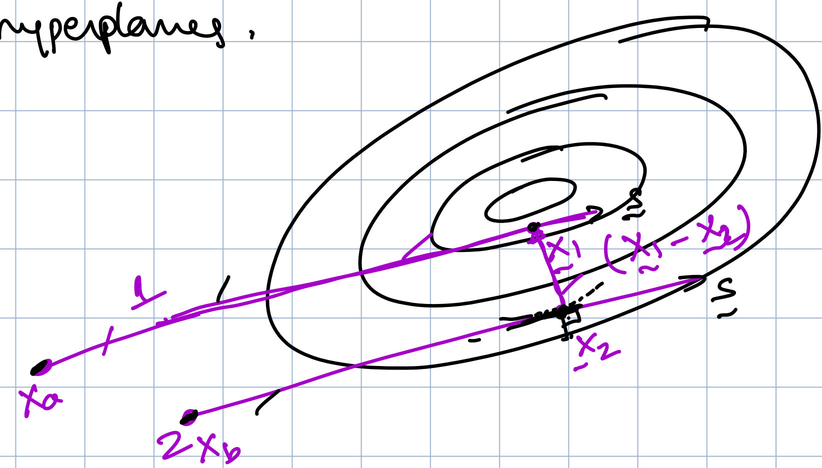
$$i = 1, 2 \cdots n, \quad j = 1, \cdots n$$

Quadratically convergent Method:

If a minimization method using exact arithmetic, can find the minimum in n steps while minimizing a quadratic fn. in n variables, then the method is called quadratically convergent.

$$f(\underset{\sim}{x}) = B\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{x}^T A \underset{\sim}{x} + C$$

<u>Theorem:</u> Given a quadratic function in n variables and two parallel hyperplanes 1 & 2 of dimension $k<n$. Let the constrained stationary points of the quadratic function in the hyperplanes 1 & 2 be $\underset{\sim}{x}_1$ & $\underset{\sim}{x}_2$, respectively. Then, the line joining $\underset{\sim}{x}_1$ & $\underset{\sim}{x}_2$ is conjugate to any line parallel to the hyperplanes.



$$Q(\underset{\sim}{x}) = \frac{1}{2}\underset{\sim}{x}^T A \underset{\sim}{x} + B\underset{\sim}{x} + C$$

$$\nabla Q(\underset{\sim}{x}) = A\underset{\sim}{x} + B$$

$$\nabla Q(\underset{\sim}{x}_1) = A\underset{\sim}{x}_1 + B$$

$$\nabla Q(\underset{\sim}{x}_2) = A\underset{\sim}{x}_2 + B$$

$$\nabla Q(\underset{\sim}{x}_1) - \nabla Q(\underset{\sim}{x}_2) = A(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$$

$\underset{\sim}{S}$ is a vector $\parallel$el to the hyperplanes.

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The $\underset{\sim}{S}'$ will be orthogonal to $\nabla Q(\underline{X}_1)$ & $\nabla Q(\underline{X}_2)$

$$\underset{\sim}{S}^T \nabla Q(\underset{\sim}{X}_1) = S^T A \underset{\sim}{X}_1 + S^T B = 0$$

$$\underset{\sim}{S}^T \nabla Q(\underline{X}_2) = \underset{\sim}{S}^T A \underset{\sim}{X}_2 + S^T B = 0$$

$$\Rightarrow \underset{\sim}{S}^T A(\underset{\sim}{X}_1 - \underset{\sim}{X}_2) = 0$$

Hence $\underset{\sim}{S}$ & $(\underset{\sim}{X}_1 - \underset{\sim}{X}_2)$ are A conjugate.

<u>Theorem 2</u> If $Q(\underset{\sim}{X}) = \frac{1}{2}\underset{\sim}{X}^T A \underset{\sim}{X} + B\underset{\sim}{X} + C$ is minimized sequentially, once along each direction of a set of n mutually conjugate directions, the minimum of the function Q will be found at or before nth step irrespective of the starting point.

<u>proof:</u> Let's say $\underset{\sim}{X}^*$ is the minima of $Q(\underset{\sim}{X})$

$$\nabla Q(\underset{\sim}{X}^*) = B + A\underset{\sim}{X}^* = \underset{\sim}{0}$$

we will start with an initial point $\underset{\sim}{X}_1$ & a set of linearly independent direction $\underset{\sim}{S}_1, \dots \underset{\sim}{S}_n$ & constants β_i

$$\underset{\sim}{X}^* = \underset{\sim}{X}_1 + \boxed{\sum_{i=1}^{n} \beta_i \underset{\sim}{S}_i}$$

by using $\underset{\sim}{S}_i$ as basis vectors.

If $\underset{\sim}{S}_i$ are A-conjugate & none of them null vectors.

$$\nabla Q(\underset{\sim}{X}^*) = B + A\underset{\sim}{X}^* = 0$$

$$B + A\left(\underset{\sim}{X}_1 + \sum_{i=1}^{n} \beta_i \underset{\sim}{S}_i\right) = 0$$

$$\Rightarrow (B + AX_1) + A\sum_{i=1}^{n} \beta_i S_i = 0$$

$$\underset{\sim}{S}_j^T(B + A\underset{\sim}{X}_1) + \underset{\sim}{S}_j^T A\left[\sum_{i=1}^{n} \beta_i \underset{\sim}{S}_i\right] = 0$$

$$\Rightarrow \underset{\sim}{S}_j^T(B + A\underset{\sim}{X}_1) + \beta_j \underset{\sim}{S}_j^T A \underset{\sim}{S}_j = 0$$

$$\Rightarrow \beta_j = -\frac{(B + A\underset{\sim}{X}_1)^T \underset{\sim}{S}_j}{\underset{\sim}{S}_j^T A \underset{\sim}{S}_j}$$

$$\beta_i = -\frac{(B + A\underset{\sim}{X}_1)^T \underset{\sim}{S}_i}{\underset{\sim}{S}_i^T A \underset{\sim}{S}_i}$$

Let's say we start $\underset{\sim}{X}_1$ & successively minimize

$$\underset{\sim}{X}_{i+1} = \underset{\sim}{X}_i + \lambda_i^* S_i \qquad i = 1, \text{ to } n$$

λ_i^* is to be found by minimizing Q along $\underset{\sim}{S}_i$

$$Q(\underset{\sim}{X}_i + \lambda S_i)$$

$$\Rightarrow \underset{\sim}{S}_i^T \nabla Q(\underset{\sim}{X}_{i+1})' = 0$$

$$\underset{\sim}{S}_i^T \left[B + A\underset{\sim}{X}_{i+1}\right] = 0$$

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$$\therefore \quad \underset{\sim}{S}_i^T \left[B + A\left(\underset{\sim}{X}_i + \lambda_i^* \underset{\sim}{S}_i\right)\right] = 0$$

$$\Rightarrow \underset{\sim}{S}_i^T \left(B + A \underset{\sim}{X}_i\right) + \lambda_i^* \underset{\sim}{S}_i^T A \underset{\sim}{S}_i = 0$$

$a^T b \overset{?}{=} b^T a$

$$\therefore \quad \boxed{\lambda_i^* = -\frac{(B + A\underset{\sim}{X}_i)^T \underset{\sim}{S}_i}{\underset{\sim}{S}_i^T A \underset{\sim}{S}_i}}$$

$$\underset{\sim}{X}_i = \underset{\sim}{X}_1 + \sum_{j=1}^{i-1} \lambda_j^* \underset{\sim}{S}_j$$

$$\underset{\sim}{X}_i^T A \underset{\sim}{S}_i = \underset{\sim}{X}_1^T A S_i + \underbrace{\sum_{j=1}^{i-1} \lambda_j S_j^T A S_i}_{0}$$

$X_2 = X_1 + \lambda_1^* \underset{\sim}{S}_1$

$X_3 = X_2 + \lambda_2^* \underset{\sim}{S}_2$

$= X_1 + \lambda_1^* \underset{\sim}{S}_1 + \lambda_2^* \underset{\sim}{S}_2$

$= X_1 + \sum_{j=1}^{2} \lambda_j^* S_j$

$$\therefore \quad \underset{\sim}{X}_i^T A \underset{\sim}{S}_i = \underset{\sim}{X}_1^T A \underset{\sim}{S}_i$$

$$\lambda_i^* = -\frac{B^T \underset{\sim}{S}_i + X_i^T A S_i}{\underset{\sim}{S}_i^T A \underset{\sim}{S}_i}$$

$$\therefore \quad \lambda_i^* = -\frac{B^T \underset{\sim}{S}_i + \underset{\sim}{X}_1^T A \underset{\sim}{S}_i}{\underset{\sim}{S}_i^T A \underset{\sim}{S}_i}$$

$$\therefore \quad \boxed{\lambda_i^* = -\frac{(B + A\underset{\sim}{X}_1)^T \underset{\sim}{S}_i}{\underset{\sim}{S}_i^T A \underset{\sim}{S}_i}}$$