28 - Oct - 2021

How to identify a starting feasible point $\underset{\sim}{X}_1$ for the IPFM?

1. Choose an arbitrary point $\underset{\sim}{X}_1$

   evaluate $g_j(\underset{\sim}{X}_1)$

   we have a total of 'm' constraints : $g_j(\underset{\sim}{X}) \le 0$

   $j = 1, 2, \cdots m.$

   Let's say 'r' out of 'm' constraints are violated.

$$g_j(\underset{\sim}{X}_1) \le 0 \qquad j = 1, 2, \cdots, m-r$$

$$g_j(\underset{\sim}{X}_1) > 0 \qquad j = m-r+1, m-r+2, \cdots m$$

2. Identify the most violated constraint at $\underset{\sim}{X}_1$.

$$g_k(\underset{\sim}{X}_1) = \max\left[g_j(\underset{\sim}{X}_1)\right] \quad j = \underbrace{m-r+1, \cdots, m}_{r \text{ constraints}}$$

3. Find $\underset{\sim}{X}$ which minimizes $\underline{\underline{g_k(\underset{\sim}{X})}}$

   subject to

$$\begin{cases} g_j(\underset{\sim}{X}) \le 0, & j = 1, 2, \cdots m-r \\ g_j(\underset{\sim}{X}) - \underline{\underline{g_k}}(\underset{\sim}{X}_1) \le 0, & j = m-r+1, \cdots k-1, k+1, \cdots m \end{cases}$$

4. To solve the above optimization problem, use IPFM with $\underset{\sim}{X}_1$ as initial guess. Let's say the optimal solution for (3) is $\underset{\sim}{X}_M$

5. If all the constraints are not satisfied at $\underset{\sim}{X}_M$, then set a new guess vector $\underset{\sim}{X}_1 = \underset{\sim}{X}_M$

and go to step 2

Example: Minimize $f(x_1, x_2) = \frac{1}{3}(x_1+1)^3 + x_2$

s.t.

$$g_1(x_1, x_2) = -x_1 + 1 \leq 0$$

$$g_2(x_1, x_2) = -x_2 \leq 0$$

$$\phi(\underset{\sim}{X}, r) = f(\underset{\sim}{X}) + r_k \sum_{j=1}^{m} G_j[g_j(\underset{\sim}{X})]$$

$$G_j(g_j(\underset{\sim}{X})) = -\frac{1}{g_j(\underset{\sim}{X})}$$

$$\phi(\underset{\sim}{X}, r) = \frac{1}{3}(x_1+1)^3 + x_2 - r\left(\frac{1}{-x_1+1} - \frac{1}{x_2}\right)$$

In principle, we can use any one of the UCM methods that were discussed.

$$\frac{\partial \phi}{\partial x_1} = 0 \Rightarrow (x_1+1)^2 - \frac{r}{(1-x_1)^2} = 0$$

$$\frac{\partial \phi}{\partial x_2} = 0 \Rightarrow 1 - \frac{r}{x_2^2} = 0$$

$$\Rightarrow x_2^2 = r \quad \text{or} \quad x_2^* = \sqrt{r}$$

$$x_1^*(r) = \sqrt{(1+\sqrt{r})}$$

$\phi_{min} =$

$$\phi \longrightarrow f^* \quad \text{as} \quad r \rightarrow 0$$

$$x_1^* : \lim_{r \to 0} x_1^*(r) = \lim_{r \to 0} \sqrt{r} = 0$$

$$x_2^* : \lim_{r \to 0} x_2^*(r) = \lim_{r \to 0} \sqrt{1+\sqrt{r}} = 1$$

(144)

$\therefore\ x_1^* = 0;\quad x_2^* = 1$

$$\lim_{r\to 0} \phi(\underset{\sim}{X}, r) = \phi_{min}(\underset{\sim}{X}, r\to 0) \simeq f_{min}$$

$$f_{min} = \frac{1}{3}(x_1^*+1)^3 + x_2^*$$

$$= \frac{1}{3}(1+1)^3 + 0 = \frac{8}{3}.$$

$c = 0.1$

K	r	$x_1^*(r) = \sqrt{(1+\sqrt{r})}$	$x_2^*(r) = \sqrt{r}$	$\phi_{min}(r)$	$f(r)$
1	1000	5.71164	31.623	376.263	132.40
2	100	3.31662	10	89.9772	36.8109
3	10				
	⋮				
	0.000001	1.00050	0.00100	2.6727	2.6697
	0	1	0	8/3	8/3

Exterior penalty function Method.

$$\phi(\underset{\sim}{X}, r_k) = f(\underset{\sim}{X}) + r_k \sum_{j=1}^{m} \langle g_j(\underset{\sim}{X})\rangle^q$$

$r_k > 0$

$q \leftarrow$ non-negative constant.

$$\langle g_j(\underset{\sim}{X})\rangle = \max\langle g_j(\underset{\sim}{X}), 0\rangle$$

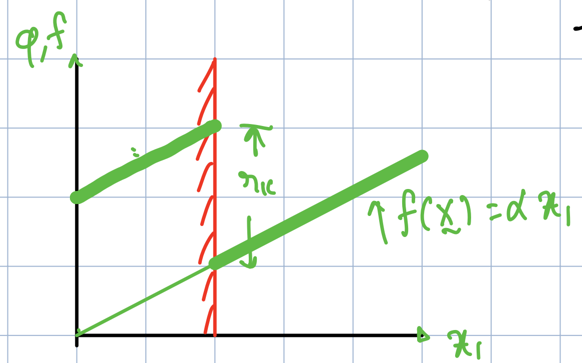
$$= \begin{cases} g_j(\underset{\sim}{X}) & \text{if } g_j(\underset{\sim}{X}) > 0 \quad (\text{constt. violated}) \\ 0 & \text{if } g_j(\underset{\sim}{X}) \le 0 \quad (\text{constt. satisfied}) \end{cases}$$

$\underset{\sim}{X}_k^*$ converges to the optima as $k \to \infty$ & $r_k \to \infty$ (145)

1. $q = 0$ ✗

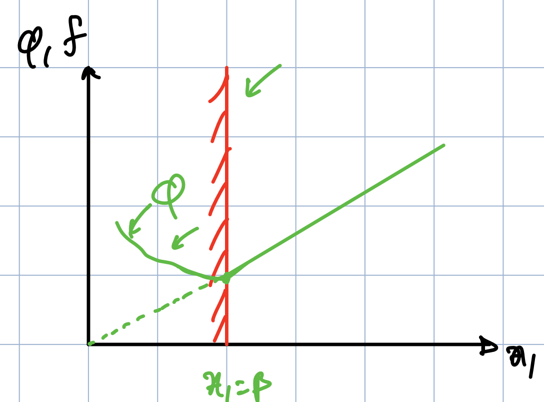
$$\varphi(\underset{\sim}{X}, r_k) = f(\underset{\sim}{X}) + r_k \sum_{j=1}^{m} \langle g_j(\underset{\sim}{X}) \rangle^0$$

$$= \begin{cases} f(\underset{\sim}{X}) + m r_k & \text{if } g_j(\underset{\sim}{X}) > 0 \\ f(\underset{\sim}{X}) & \text{if } g_j(\underset{\sim}{X}) \le 0 \end{cases}$$



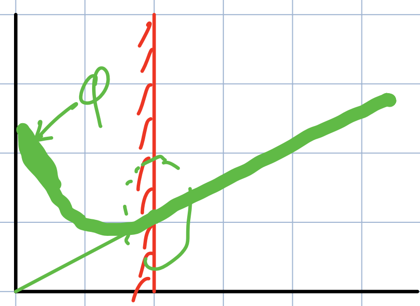
At the boundary the fn. φ is discontinuous.

2. $0 < q < 1$ → penalty for violating the constraint will be too small although φ will be continuous.



3. $q = 1$ ← discontinuous first derivatives across the boundary.

4. $q > 1$ ← The φ will have continuous first derivatives



$$\frac{\partial \varphi}{\partial x_i} = \frac{\partial f}{\partial x_i} + r_k \sum_{j=1}^{m} q \langle g_j(\underset{\sim}{x}) \rangle^{q-1} \cdot \frac{\partial g_j}{\partial x_i}$$

$$\underline{\underline{q = 2}}$$

<u>Example:</u> Minimize $f(x_1, x_2) = \frac{1}{3}(x_1+1)^3 + x_2$

s.t. $g_1(x_1, x_2) = 1 - x_1 \le 0$

$g_2(x_1, x_2) = -x_2 \le 0$

$$\varphi(\underset{\sim}{x}, r) = \frac{1}{3}(x_1+1)^3 + x_2 + r\Big[[\max(0, 1-x_1)]^2 + [\max(0, -x_2)]^2\Big]$$

$$\frac{\partial \varphi}{\partial x_1} = (x_1+1)^2 - 2r\,[\max(0, 1-x_1)] = 0$$

$$\frac{\partial \varphi}{\partial x_2} = 1 - 2r\,(\max(0, -x_2)] = 0$$

$$\min\left[(x_1+1)^2,\ (x_1+1)^2 - 2r(1-x_1)\right] = 0 \quad -①$$ 147

$$\min\left[1,\ 1+2r x_2\right] = 0 \quad -②$$

if $(x_1+1)^2 = 0$ ✗ $\Rightarrow x_1 = -1$ violate g_1

$(x_1+1)^2 - 2r(1-x_1) = 0 \Rightarrow x_1^* = -1-r+\sqrt{r^2+4r}$

$1+2r x_2 = 0 \Rightarrow x_2^* = -\frac{1}{2r}$

As $r\to\infty$ $x_2^* = 0$

As $r\to\infty$ $x_1^* = 1$

$\therefore f_{min} = \lim_{r\to 0} \phi_{min}(r) = 8/3$

$c>1$ $c=10$

r	x_1	x_2	ϕ_{min}	f_{min}
0.001				
0.01				
0.1				
1				
:				
100,000				