30-sep-2021

- for a quadratic function, the minima can be found within n searches along mutually conjugate directions.

If $\underset{\sim}{S_i}$ & $\underset{\sim}{S_j}$ are A conjugate

$$\underset{\sim}{S_i}^T A \underset{\sim}{S_j} = 0$$

$$f(x_1, x_2) = 6x_1^2 + 2x_2^2 - 6x_1x_2 - x_1 - 2x_2 \checkmark\checkmark$$

$\underset{\sim}{S_1} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \checkmark$ find $\underset{\sim}{S_2}$ that is conjugate to $\underset{\sim}{S_1}$

$$f(\underset{\sim}{X}) = B^T \underset{\sim}{X} + \frac{1}{2} \underset{\sim}{X}^T A \underset{\sim}{X}$$

$$= \begin{bmatrix} -1 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}$$

$$\underset{\sim}{S_1}^T A \underset{\sim}{S_2} = 0$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{Bmatrix} 12s_1 - 6s_2 \\ -6s_1 + 4s_2 \end{Bmatrix} = 0$$

$$\Rightarrow 12s_1 - 6s_2 - 12s_1 + 8s_2 = 0 \checkmark$$

$$\Rightarrow 2s_2 = 0 \Rightarrow s_2 = 0$$

$\underset{\sim}{S_2} = \begin{Bmatrix} k \\ 0 \end{Bmatrix} \checkmark$ is conjugate to $\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

• Minimize $f(x) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$

from a starting point $\underset{\sim}{X_1} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

• Cycle 1: univariate search.

Minimize f along $\underset{\sim}{u_2} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

$$\underset{\sim}{X_2} = \underset{\sim}{X_1} + \lambda_1^* \underset{\sim}{u_2}$$

$$f\left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \lambda_1 \end{Bmatrix}\right)$$

$$= f(0, \lambda_1) = -\lambda_1 + \lambda_1^2$$

$$\frac{df}{d\lambda_1} = 2\lambda_1 - 1 = 0 \Rightarrow \lambda_1^* = \frac{1}{2}$$

$$\therefore X_2 = \begin{Bmatrix} 0 \\ 0.5 \end{Bmatrix} \qquad \underline{f_2 < f_1}$$

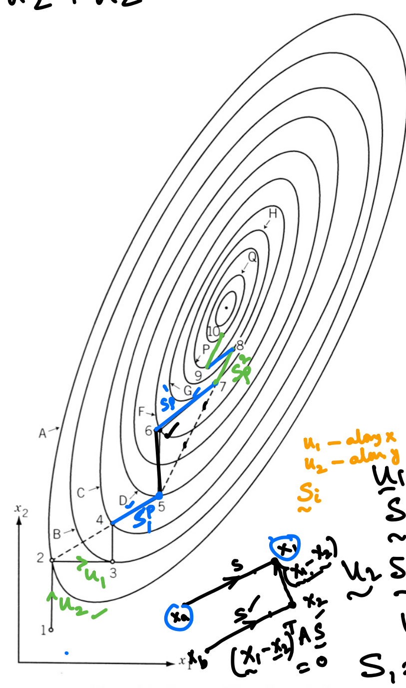


Figure 6.8 Progress of Powell's method.

u_1 $\underset{\sim}{S_1} \to x$

u_2 $\underset{\sim}{S_2} \to y$

u

$S_1 = u_2$

$S_2 = u_1$

$S_3 = u_2$

$(\underset{\sim}{X_4} - \underset{\sim}{X_2})$ will be conjugate to $\underset{\sim}{S_2}$

Next minimize along $\underset{\sim}{u_1}$

$$\underset{\sim}{X_3} = \underset{\sim}{X_2} \pm \lambda_2^* \underset{\sim}{u_1}$$

$$f\left(\begin{Bmatrix} 0 \\ 0.5 \end{Bmatrix} + \lambda_2 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}\right) \qquad x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

$$= f(\lambda_2, 0.5) = \lambda_2 - 0.5 + 2\lambda_2^2 + 2(0.5)\lambda_2 + 0.5^2$$

$$= \lambda_2 - 0.5 + 2\lambda_2^2 + \lambda_2 + 0.25$$

$$= 2\lambda_2^2 + 2\lambda_2 - 0.25$$

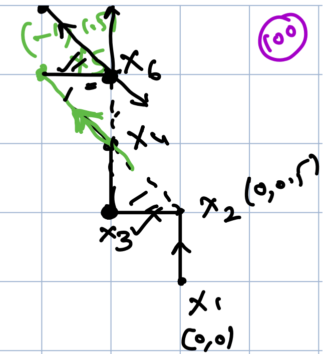
$$\rightarrow \frac{df}{d\lambda_2} = 4\lambda_2 + 2 = 0 \Rightarrow \underline{\lambda_2^* = -\frac{1}{2}}$$

$$\therefore \underset{\sim}{X_3} = \begin{Bmatrix} 0 \\ 0.5 \end{Bmatrix} - 0.5 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 0.5 \end{Bmatrix}$$

$$\underset{\sim}{X_4} = \underset{\sim}{X_3} + \lambda_3^* \underset{\sim}{u_2} = \begin{Bmatrix} -0.5 \\ 0.5 \end{Bmatrix} + \lambda_3 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 0.5 + \lambda_3 \end{Bmatrix}$$

(100)

$$x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$



$$f(\underset{\sim}{X_4}) = f(-0.5,\ 0.5 + \lambda_3) =$$

$$\frac{df}{d\lambda} = 0 \Rightarrow \lambda^* = 1/2$$

$$\underset{\sim}{X_4} = \begin{Bmatrix} -0.5 \\ 1 \end{Bmatrix}$$

Cycle 2: pattern search

$$S_1^p = \underset{\sim}{X_4} - \underset{\sim}{X_2} = \begin{Bmatrix} -0.5 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0.5 \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 0.5 \end{Bmatrix}$$

$$\underset{\sim}{X_5} = \underset{\sim}{X_4} + \lambda_4^* S_1^{(p)} = \begin{Bmatrix} -0.5 \\ 1 \end{Bmatrix} + \lambda_4 \begin{Bmatrix} -0.5 \\ 0.5 \end{Bmatrix}$$

$$f(\underset{\sim}{X_5}) = f(-0.5 - 0.5\lambda_4,\ 1 + 0.5\lambda_4)$$

$$= 0.25\lambda_4^2 - 0.5\lambda_4 - 1$$

$$\frac{df}{d\lambda_4} = 0.5\lambda_4 - 0.5 = 0$$

$$\Rightarrow \lambda_4^* = 1$$

$$\therefore \underset{\sim}{X_5} = \begin{Bmatrix} -0.5 - 0.5 \\ 1 + 0.5 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1.5 \end{Bmatrix} \quad f(\underset{\sim}{X_5}) = -1.25$$

$$f^+ = f(\underline{X_5} + \varepsilon \underset{\sim}{u_2}) \geq f_5 \qquad \varepsilon = 0.01$$

$$f^- = f(\underline{X_5} - \varepsilon \underline{u_2}) > f_5$$

$$\boxed{\underset{\sim}{X_6} = \underline{X_5} + \lambda_5^* \underset{\sim}{u_2} \qquad \boxed{\frac{df}{d\lambda} = 0}}$$