Multivariable (unconstrained) optimization.

Find $\underset{\sim}{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ which minimizes $f(\underset{\sim}{X})$

$\underset{\sim}{X}^*$

A point $\underset{\sim}{X}^*$ is a relative minimum of $f(\underset{\sim}{X})$ if the necessary conditions

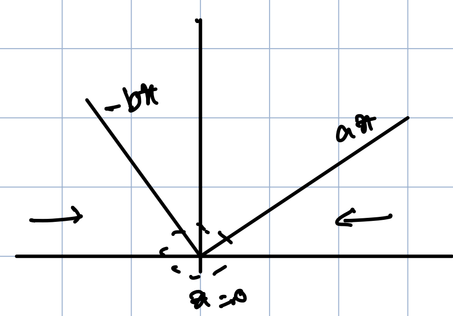
$$\frac{\partial f}{\partial x_i}(\underset{\sim}{X} = \underset{\sim}{X}^*) = 0 \qquad i = 1, 2, \ldots n$$

$$\nabla f \Big|_{\underset{\sim}{X}^*} = 0$$

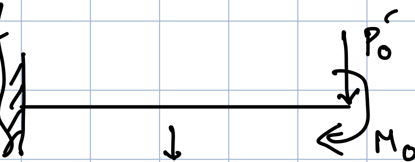
$\underset{\sim}{X}^*$ corresponds to minimum if

$$[H]_{\underset{\sim}{X}^*} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]\Bigg|_{\underset{\sim}{X}^*} \rightarrow \text{positive definite}$$

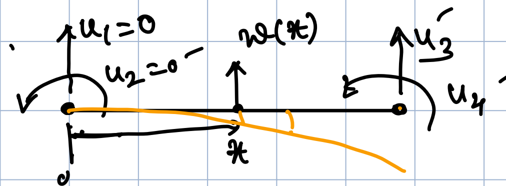
$$f(x) = \begin{cases} a\,x & \text{for } x > 0 \\ -b\,x & \text{for } x \le 0 \end{cases}$$



$$w(x) = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$



N_i's are shape functions which are cubic polynomials.



The potential energy of the beam is given by

$$\Pi = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx - P_0 u_3 - M_0 u_4$$

Let's denote $x_1 = u_3$ & $x_2 = u_4 l$

Assume $\frac{P_0 l^3}{EI} = 1$ & $\frac{M_0 l^2}{EI} = 2$.

$$\alpha = \frac{x}{l}$$

$$w(x) = \left(-2\alpha^3 + 3\alpha^2\right) u_3 + \left(\alpha^3 - \alpha^2\right) l u_4$$

$$\frac{d^2 w}{dx^2} = \frac{6u_3}{l}\left(-2\alpha + 1\right) + \frac{2u_4}{l}\cdot(3\alpha - 1)$$

$$\Pi = \frac{EI}{l^3}\left[6u_3^2 + 2u_4^2 l^2 - 6u_3 u_4 l\right] - P_0 u_3 - M_0 u_4$$

$x_1 = u_3$ & $x_2 = u_4 l$ $\quad \frac{P_0 l^3}{EI} = 1$ & $\frac{M_0 l^2}{EI} = 2$

$$f = \frac{\Pi l^3}{EI}$$

$$\boxed{f = 6x_1^2 - 6x_1x_2 + 2x_2^2 - x_1 - 2x_2}$$

Minimize f such that $X^* = \begin{Bmatrix} x_1^* \\ x_2^* \end{Bmatrix}$ will be the solution $\rightarrow$ displacements of the beam.

Classification of UCMM

Direct Search Methods

- Random search ✓
- Grid search
- Univariate
- Pattern Search Methods
  - Conjugate directions (Powell's Method)

They do not require derivatives computation.

(Zeroth order Methods)

Descent Methods

Steepest Descent (Cauchy)

Conjugate gradient (Fletcher-Reeve's)

Newton's Method

Marquardt

Quasi-Newton Method

- DFP
- BFGS

Require derivatives (First-Order, Second-Order)

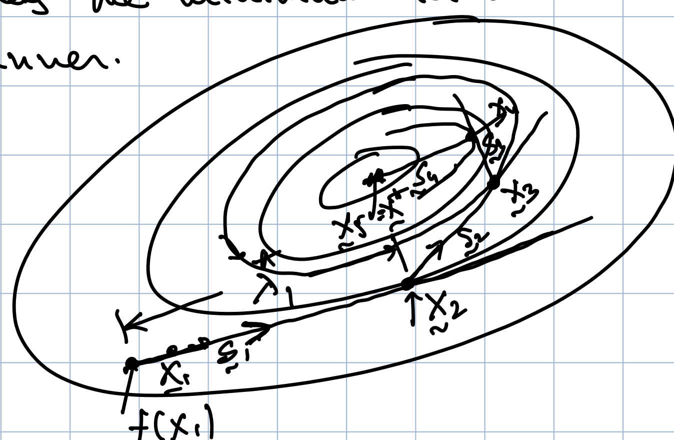
Approach:

- Iterative ✓
- Start with an initial guess
- proceed towards the minimum in a sequential manner.

$$\underset{\sim}{X}_2 = \underset{\sim}{X}_1 + \lambda_1^* \underset{\sim}{S}_1$$

$$\underset{\sim}{X}_3 = \underset{\sim}{X}_2 + \lambda_2^* \underset{\sim}{S}_2$$

$$\vdots$$



$$\boxed{\underset{\sim}{X}_{i+1} = \underset{\sim}{X}_i + \lambda_i^* \underset{\sim}{S}_i}$$

- Require an initial point $\underset{\sim}{X}_1$ to start the iterative procedure
- all the UCMOM differ only in the way they identify the search direction to generate the new point $\underset{\sim}{X}_{i+1}$ from $\underset{\sim}{X}_i$.
- Each UCMOM differ in the way they test for optimality of $\underset{\sim}{X}_{i+1}$

Rate of convergence:

An optimization method is considered to have a convergence order p. if

$$\frac{\| \underset{\sim}{X}_{i+1} - \underset{\sim}{X}^* \|}{\| \underset{\sim}{X}_i - \underset{\sim}{X}^* \|^p} \leq k \qquad k \geq 0, \; p \geq 1$$

$$\| \underset{\sim}{X} \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

If $p = 1$ and $0 \leq k \leq 1 \leftarrow$ linearly convergent method (slow convergence)

If $p = 2 \leftarrow$ Quadratically convergent (faster ")

$$\lim_{i \to \infty} \frac{\| \underset{\sim}{X}_{i+1} - \underset{\sim}{X}^* \|}{\| \underset{\sim}{X}_i - \underset{\sim}{X}^* \|} \to 0 \leftarrow \text{super linear convergent (fast)}$$

By scaling the design variables one can improve the convergence rate of the optimization algorithm.