Linear Quadratic Regulator

① Given a model, define system state $x = \begin{bmatrix} q \\ v \end{bmatrix}$

② if we could write down the Equation of Motion

either using Newton-Euler or Lagrange, we will get:

$$M(q)\cdot\ddot{q} + C(q,\dot{q})\cdot\dot{q} = \tau_g(q) + Bu$$

③ Then we could do control around fix point (normally a balanced point)
we will Linearize @ fixed poind (x^*, u^*), using Taylor expansion

$$\dot{x} = f(x,u) \simeq f(x^*, u^*) + \left[\frac{\partial f}{\partial x}\right]_{x=x^*, u=u^*} \cdot (x - x^*) + \left[\frac{\partial f}{\partial u}\right]_{x=x^*, u=u^*} \cdot (u - u^*)$$

$f(x^*, u^*)$ happen to be zero @ balanced fixed point.

④ then we get Linearized system @ balanced point:

$$\dot{\bar{x}} = A_{lin}\bar{x} + B_{lin}\bar{u} \qquad \text{where } \bar{x} = x - x^*, \ \bar{u} = u - u^*$$

if (x^*, u^*) is a balanced point, then

Linearize EoM: $\ddot{q} = M^{-1}(q)\cdot[\tau_g(q) + B(q)\cdot u - C(q,\dot{q})\dot{q}]$

for $\frac{\partial \ddot{q}}{\partial q}$

$$\frac{\partial \ddot{q}}{\partial q} = \frac{\partial M^{-1}(q)}{\partial q}\cdot\underbrace{[\quad]}_{\substack{\| \\ 0}} + M^{-1}(q)\cdot\left[\frac{\partial \tau_g(q)}{\partial q} + \frac{\partial B(q)}{\partial q}\cdot u - \frac{\partial C(q,\dot{q})}{\partial q}\cdot\underbrace{\dot{q}}_{\substack{\| \\ 0}}\right]$$

when @ (q, q^*)

$$= M^{-1}\cdot\frac{\partial \tau_g(q)}{\partial q} + M^{-1}\cdot\frac{\partial B(q)u}{\partial q} \qquad \because B(q)\cdot u = \sum_j B_j(q)\cdot u_j$$

for $\frac{\partial \ddot{q}}{\partial \dot{q}}$

$$\frac{\partial[M^{-1}(\tau_g(q) + Bu - C\dot{q})]}{\partial \dot{q}} = -M^{-1}\left[\frac{\partial C}{\partial \dot{q}}\dot{q} + C\cdot\frac{\partial \dot{q}}{\partial \dot{q}}\right]$$

$$\because \dot{q} = 0,\quad C(q,\dot{q}) = C(q)\cdot\dot{q} = 0,\quad \therefore \frac{\partial[\quad]}{\partial \dot{q}} = 0$$

$$A_{lin} = \begin{bmatrix} \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial \dot{q}} \\ \frac{\partial \ddot{q}}{\partial q} & \frac{\partial \ddot{q}}{\partial \dot{q}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ M^{-1}\left(\frac{\partial \tau_g(q)}{\partial q} + \sum_j \frac{\partial B_j(q)\cdot u_j}{\partial q}\right) & 0 \end{bmatrix}_{x=x^*,\ u=u^*}$$

$$B_{lin} = \begin{bmatrix} \frac{\partial \dot{q}}{\partial u} \\ \frac{\partial \ddot{q}}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ M^{-1}(q)\cdot B(q) \end{bmatrix}_{x=x^*,\ u=u^*}$$

Using LQR to control to the fixed point

(got A_{lin}, B_{lin} @ fixed point)

① Set Cost Function to Quadratic:

$$J(x_0) = \int_0^\infty [x^T(t) Q x(t) + u^T(t) \cdot R \cdot u(t)]\, dt, \quad x(0) = x_0, \; Q = Q^T > 0, \; R = R^T > 0$$

② Get linear Controller K:

if Continuous: use solution of Continuous Algebraic Riccati Equation

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

$\Rightarrow$ get Hamiltonian $H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}_{2n \times 2n}$ $\Rightarrow$ negative real n vectors $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$

$\Rightarrow X = U_2 \cdot U_1^{-1}$ $\quad \Rightarrow \quad K = R^{-1} B^T X$

if Discrete: use solution of Discrete Algebraic Riccati Equation

$$X = A^T X A - (A^T X B)(R + B^T X B)^{-1}(B^T X A) + Q$$

$\Rightarrow$ get Symplectic $C = \begin{bmatrix} A + BR^{-1}B^T (A^{-1})^T \cdot Q & -BR^{-1}B^T (A^{-1})^T \\ -(A^{-1})^T \cdot Q & (A^{-1})^T \end{bmatrix}_{2n \times 2n}$

$\Rightarrow$ n eigenValue in unit circle, $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$

$\Rightarrow X = U_2 \cdot U_1^{-1}$ $\quad \Rightarrow \quad K = (B^T X B + R)^{-1} \cdot (B^T X \cdot A)$

③ $\quad u(t) = -K \cdot x(t)$