LQR Control of an Under Actuated Planar Biped Robot

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***Abstract*—Modern robotic systems are fully actuated with full information of themselves and their immediate surroundings. If faced with a failure or damage, robotic systems cannot function properly and can quickly damage themselves. In this paper, we study the motion control of an under actuated five-degree-of-freedom planar biped. Once the fully actuated system is established, it is made under actuated by removing control over a specific joint actuator. This is done to represent damage to the robotic system in the form of a failed or unresponsive actuator motor. A Linear Quadratic Regulator (LQR) controller is then designed to allow the under actuated (damaged) robotic system to continue to follow a human-like walking gait. Using the under actuated control, the walking gait is preformed without the use of the uncontrolled joint and overall effectiveness is compared with the fully actuated system performance. Simulation results indicates that one single LQR controller, designed based on the linearized model at the beginning of the single support phase, is sufficient to perform the desired walking gait even though the biped model is highly nonlinear. This study demonstrates the effectiveness of the LQR control to the under actuated biped robot.**

Keywords-biped robot; under actuated system; LQR control

I. INTRODUCTION

Modern robots are fully actuated systems with full information of themselves and their immediate surroundings. This knowledge is what allows determination of the robots' physical and motion parameters, the dynamics, to a level of precision allowing prediction and control of the machine. Without this complete knowledge the system dynamics change and robotic systems cannot function properly, sometimes causing damage to themselves and their surroundings without intervention. Examples of this kind of disruption include failed actuators, unaccounted for objects, a person within the work space, or structural damage, each of which causes a significant change that the system was not designed to operate in. Loss of use of a joint or a robot with less control than actuation points can be referred to under actuated. A control system that can adapt from fully actuated to under actuated would have impacts from advanced prosthetics to automation and industrial robotics. An under actuated control system could allow damaged or altered systems to retain some or all of their intended functions or at least continue partial function while retaining enough control to prevent further damage. Various control methods for under actuated robots can be found in the literature [e.g., 1-5]. In this paper, we studied the linear LQR motion control of an under actuated biped even thought its dynamics is highly nonlinear.

II. BIPED WALKING GAIT AND DYNAMICS

A. *Biped Walking*

A human in motion represents a system with upward of a few hundred degree-of-freedom (DOF), depending on how it's modeled. To reduce the number of DOF required to model walking gait, the biped motion in the two-dimensional sagittal plane is considered. As opposed to the transverse plane (which bisects a body horizontally parallel to the ground plane, creating top and bottom) and the coronal plane (which bisects a body vertically, passing through both shoulders, creating a front and a back), the sagittal plane vertically bisects the center of the model body from head to foot, creating left and right halves. Modeling in the sagittal plane creates a side view where forward movement is shown as progressing either to the left or right. Considering what is needed at a minimum for walking motion, the contributing motions of the arms, head, neck, chest and back are ignored. The ankles and feet have a great deal of impact on three dimensional motions and balance, but in the sagittal plane and on this perfect simulated surface the contribution is considered negligible, so they are mostly ignored. What are left to consider in this physical model are the hip, knee and "ankle" joints. The foot as modeled is represented as a simple post and the joint is therefore an interaction between the ground plane and the end of this post, similar to ankle movement. This simplification of the foot and ankle creates a single DOF as long as the foot post is in contact with the ground plane and it is assumed that force can be exerted against the ground as if an ankle were present. Finally, all modeled joints have had their normal range of motion in the sagittal plane reduced to simple rotational joints, which have only one DOF each. This leaves the proposed model with a total of 6 DOF, as represented in Fig. 1, with both feet on the ground, and when one foot is off the ground, the model is further reduced to 5 DOF.

A walking gait can be divided into two distinct phases of motion; the single support phase (SSP) and the double support phase (DSP). In biped walking motion, the SSP begins when the model's weight is shifted to a single foot, and continues as the other foot is raised and moved to its new position, this represents a single step. Once the raised foot is placed on the ground plane again, the SSP ends. The DSP starts at this point when weight is again redistributed between both feet, stabilizing the biped's balance and ends as the opposite foot is lifted into its SSP. A continuous flow from SSP to DSP to the other foot SSP to DSP again represents two full steps and one full gait cycle [6-9]. It is important to note that if the stabilizing effect of the DSP places the feet and legs into the same but

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mirrored positions from left to right SSP, then the gait is periodic in nature, making it possible to model the entire walking gait by defining only a single SSP and DSP [8, 10]. Creating a walking gait with these properties is a field of study in itself, with many publications available. For example, in [8] a method for creating an arbitrary walking gait for a planar 5 DOF robot is defined using over 20 variables but gives choices on step length, hip height, torso tilt, foot lift, etc. and handles both SSP and DSP gait phases. Attempts to model the SSP as shown in [6-15], while highly detailed and life-like, were eventually found to be unnecessarily complicated, therefore, a related but simpler method was used. A standard position and velocity polynomials defined in [8] is used which allows the calculation of the joint trajectories for any point in time. With this information, a path between any two joint positions for a given length of time during the SSP can be plotted as long as the desired model orientations are known. Unlike more advanced methods, using this method generates only very basic point to point motions for each joint individually. That means the models' separate joint movements must be coordinated and optimized by the designer through observation and trial and error to ensure a life-like gait. Over time, the SSP is considerably longer in duration than the DSP [9] making up around 90% of the gait cycle. Since the DSP is a small portion of the walking gait, only the SSP is modeled in this study. The DSP is only assumed to result in the balancing and repositioning of the feet as needed to correct placement error and maintain the periodic nature of the gait.

We design the biped around human physical parameters, in hopes of returning some of the lost realism. The parameters representing average male human physical characteristics and the average male body length, mass, and center of mass (CM) values selected from [16] are considered. The CM position is represented as the location defined by the percent of length as measured from the top of the segment towards the bottom. Left and right arm and leg segments are assumed to be identical. To modify these values to conform to the reduced complexity model as shown in Fig. 1, the arms, head and trunk are combined in mass and referred to as the torso. Furthermore, the torso's length is taken as the combination of the head and trunk length. The torso's CM is placed the same as the trunk, as this is the very close to the natural CM of the entire human body and makes a good representation for the combined torso masses. The new shank contains the mass of the shank and foot, but only the length and CM of the original shank. This was done to account for the mass of the foot and ankle but not contribute complexities from considering the length (height) changes while rolling off of a foot while walking and to keep with the "foot as a post" analogy explained for Fig.1. Finally, to move toward dynamics and kinematics terminology, the individual segments will be referred to as links. These changes are summarized as link, length, mass (%) and CM position (%) as follows: torso – 86.42 cm, 60.26%, 51.38%; thigh – 42.22 cm, 14.16%, 40.95%; shank – 44.03 cm, 5.7%, 43.95% with an overall height of 170.8 cm and with suggestions from [16] a total biped mass of 77 kg is chosen. We applied the needed parameters to the Fig. 1 model, which is shown in Fig. 2. To calculate the inertia values, each link is assumed to be equivalent to a thin rod of uniform mass giving the moment of inertia calculation $i = ml^2/12$. This would put the equivalent CM in the center of the link. Since the actual link CM isn't in the physical center of the link, the parallel axis theorem is used to account for the CM being off center, that is,

$$i = \frac{ml^2}{12} + m\left(\frac{l}{2} - l_a\right)^2$$

for each leg link. The results of these calculations and the physical characteristics of the biped used in this study can be found in [17].

Using the simple path polynomial solution [8], the desired joint trajectories are generated and these trajectories are chosen to maintain a periodic gait while keeping the torso upright. Mapping the initial joint angles to the model gives a right leg SSP starting and ending position. Assuming the SSP is 90% of the total gait time [9], a full gait cycle of 2 seconds is selected. Therefore, an approximate SSP duration is given as 0.9 seconds and the DSP is 0.1 seconds. The joint variables and velocities from SSP initial to end states giving the trajectory path information can be found in [17].

B. *Biped Underactuated Dynamics*

The general fully actuated biped dynamic model can be written

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + f(\dot{q})$$

where $q = [\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5]^T$ (T means the transpose), D(q) is a 5 x 5 positive definite inertia matrix for the linking structure inertial forces, $C(q,\dot{q})\,\dot{q}$ and *g(q)* are 5 x 1 vectors for the centripetal and Coriolis forces, and the gravitational forces, respectively; while $f(\dot{q})$ is a 5 x 1 vector for the applied joint torques. The parameter values in the previous section are used in the dynamic equations of the biped. However, due to the complexity of the matrices involved, we will not list the detail of these matrices and vectors (A complete derivation of the system dynamics is available in [18]). In the case of under actuated dynamics, the standard model must be modified to account for the uncontrolled joint's and dependant link's impact on the whole system. While becoming under actuated doesn't have any real impact on the true system dynamics, once the dynamic equations are utilized to develop a controller the difference becomes critical. Starting with the general form equation, the fully actuated system dynamics can be partitioned into active and passive joints as follows:

$$\begin{bmatrix}\tau_a\\ \tau_u\end{bmatrix} = \begin{bmatrix}D_{ar}(q) & D_{ac}(q)\\ D_{ur}(q) & D_{uc}(q)\end{bmatrix}\begin{bmatrix}\ddot{q}_r\\ \ddot{q}_c\end{bmatrix} + \begin{bmatrix}C_{ar}(q,\dot{q}) & C_{ac}(q,\dot{q})\\ C_{ur}(q,\dot{q}) & C_{uc}(q,\dot{q})\end{bmatrix}\begin{bmatrix}\dot{q}_r\\ \dot{q}_c\end{bmatrix} + \begin{bmatrix}g_a(q)\\ g_u(q)\end{bmatrix} + \begin{bmatrix}f_a(\dot{q})\\ f_u(\dot{q})\end{bmatrix}$$

where the subscripts *a, u, c,* and *r* represent the active joint forces, the free passive (unlocked) joint forces, the controlled joints, and the remaining (uncontrolled) joints. Assuming there is no torque applied to passive joints, i.e., $\tau_u = 0$ (zero vector).

Then, by factoring out and substituting the result back in, we have

$$\tau_a = \bar{D}(q)\ddot{q}_c + \bar{C}(q,\dot{q})\dot{q}_c + \bar{H}(q,\dot{q})\dot{q}_r + \bar{g}(q) + \bar{f}(\dot{q})$$

where

$\bar{D}(q) = D_{ac}(q) - D_{ar}(q)D_{ur}^{-1}(q)D_{uc}(q)$, $\bar{f}(\dot{q}) = f_a(q) - D_{ar}(q)D_{ur}^{-1}(q)f_u(q)$

$\bar{C}(q,\dot{q}) = C_{ac}(q,\dot{q}) - D_{ar}(q)D_{ur}^{-1}(q)C_{uc}(q,\dot{q})$, and

$\bar{H}(q,\dot{q}) = C_{ar}(q,\dot{q}) - D_{ar}(q)D_{ur}^{-1}(q)C_{ur}(q,\dot{q})$.

Performing this partition and substitution converts the fully actuated system dynamics into its possible under actuated forms. Doing so effectively reduces the corresponding matrices order by the number of passive joints. The contributions of the passive joints to the dynamics are included in the remaining controllable joint dynamics in the under actuated system model.

III. LQR CONTROL OF THE BIPED MOTION

The 5 DOF biped model is inherently nonlinear. Even though many linear and nonlinear control methods are available in the literature, we proposed a simpler Linear-Quadratic-Regulator (LQR) control approach. By defining the 10 x 1 state vector x as x = $[q,\dot{q}]^T$, the system can be linearized around the operating point defined at time *t*. As long as the system nonlinearities are not too great near the operation point, the system will be a close enough approximation of a linear time-invariant (LTI) system. If the system nonlinearities cause too great a change too fast, the LTI approximation will not be sufficient to control the system and another operating point must be chosen and subsequent controller need to be designed to attempt to successfully control the system. In general, it could lead to the need for several controllers over the course of the walking gait. The design criterion is to find u(t) that minimizes the quadratic cost function

$$\text{(Energy) } J_{LQR} = \int_0^{\infty} (x^T Q x + \gamma u^T R u + 2x^T N u)\, dt$$

where u is the control vector, γ (≥ 0) is a weighting factor, and the square weighting matrices Q and R are symmetric positive definite, while N positive semi-definite. Note that $x^T Q x$ is the energy of the controlled output, $u^T R u$ is the energy of the control signal, $2x^T N u$ is the correlation factor between *x* and *u*, and γ establishes the tradeoff between minimizing the controlled output and control signal. The use of γ allows adjustment over the tradeoff of each energy as desired for the specific control problem. Initially, the *Q* and *R* matrices are considered to be identity matrices, allowing unbiased weighting of controlled outputs and control inputs to be available in balancing out the quadratic cost function. The weighting matrix *N* is chosen to be zero matrix, removing any correlation effects between the state and input vectors. It is known that the developed LQR controller is only truly effective near the operating point. (gait pose and motion dynamics at that time *t*). As the gait progresses the system nonlinearities may reduce the effectiveness of the LQR controller, requiring additional controllers to be developed and applied [17].

IV. RESULTS

We found that the motion would collapse without the influence of a controller in the first 0.02 seconds of the SSP. This indicated that not only was a controller necessary to perform the walking gait, but the first controller was needed almost immediately once the SSP began. This made it necessary to introduce the controller (or the first controller if multiple were needed) from the instant the SSP started. A minimally acceptable time step used throughout the simulation study is 0.1 seconds. This was chosen because a longer interval did begin to have a negative impact on what was before a successful walking gait, and a shorter interval had a marginal impact at the cost of increasing the total simulation time.

A. Fully Actuated Biped Motion

The fully actuated model results are included here to establish the effectiveness of using a LQR controller on the proposed 5 DOF model and baseline the controller's ability to perform the walking gait compared to the ideal. These results will help determine how effective LQR is with the under actuated case and characterize how much of that effectiveness is lost when the system becomes under actuated. Simulations indicated that effective control would depend on minimizing the control output energy by using a very small γ. Fig. 3 shows the final biped pose when $\gamma = 10^{-7}$. We found that this final pose is visually near identical to the ideal pose and the errors appear nonexistent. Therefore, this nonlinear motion control over the SSP period is achieved by using only one linear LQR controller applied at time 0 based on the linearized model at the starting point. The case $\gamma = 10^{-7}$ is considered the largest approximate value of γ useful in matching the ideal walking gait. Although smaller γ values do continue to reduce the performance index value, and therefore errors, they do so at less and less amounts and at the cost of increasing simulation times. Thus, this $\gamma = 10^{-7}$ case is chosen as the best (largest) overall value of γ for fully actuated control of the proposed 5 DOF biped and will be used for the remaining comparisons. For detail about the position and velocity errors and the performance index for different γ values, please refer to [17].

Impact of additional LQR controllers used in the motion control is also studied. We found that the improvement on overall errors over the course of the SSP by using a second or several additional LQR controllers is minimal (i.e., the impact on the performance index of adding any additional controllers beyond the initial time = 0 one is negligible). Since additional controllers relate to additional hardware and complexity in a real world system and the impact on error reduction so minimal, it is concluded that one single LQR controller, designed based on the linearized model, applied at the beginning of the SSP phase is sufficient to perform the walking gait in the fully actuated system for this value of γ. Although some improvements made by slowing the duration of the SSP were found, it doesn't compare to the scale of improvements made by further reduction of γ.

B. Underactuated Biped Motion

To model under actuated control, type I and type II conditional failure cases are defined. A type I fault represents the case where a joint becomes passive and fails by locking into a fixed position [5]. The type II fault represents the case where the joint becomes passive in a loose or free swinging and uncontrolled manner [1-4]. These two cases represent the failure modes of the actuators that make up the joints in the biped model. If both cases can be made stable with the use of the proposed LQR controllers, then the use of time-varying control [1-3] and Markovian jump methods [19, 20] can be employed to create a dynamically adaptable controller state matrix, switching between the control modes as needed to continue the walking gait [21]. In this study, a single joint is made passive to represent each of the actuator failure cases. The joint having the least overall impact on the right SSP gait was chosen (i.e., the left knee) and is the passive joint in our study. Using the Matlab SimMechanics toolbox, the block diagram of the right SSP biped with an under actuated left knee joint is modeled and shown in [17]. Note that the simulated model is nearly identical to the fully actuated model except that the left knee has been removed. This creates the type II case where the passive joint will swing free as the model performs the walking gait. With no actuator, the resulting controller is not able to influence the passive joint, but because the sensors remain in place, it may still detect the position and velocity and calculate the associated errors. To create the type I case, the actuator is left in but with no inputs. This has effect of locking the joint in its initial condition position relative to its linking joint, in this case the left thigh. Through testing, this was found to have near identical results to simply creating a single link out of the left thigh and left shank (the type I case situation). Early tests to optimize a type I case controller quickly found the results to concur with the fully actuated results [17]. This showed that the locked joint, at any initial angle, had no impact on the remaining joints and the resulting fully actuated optimized controllers' ability to complete the walking gait, as long as the locked joint links didn't interact with the ground plane. In essence using LQR in the presence of a type I locked joint, with a fully actuated controller can still successfully perform the ideal walking gait under similar conditions.

Compared to the fully actuated case, additional errors accumulated in the under actuated model have a significant impact on the ideal use of γ. Performance of the LQR controller was achieved primarily by visually observing how the stick motion model performs the walking gait. In Fig. 4, a $\gamma = 10^{-7}$ was used to look at the under actuated walking motion as this was the most successful fully actuated case. We found that the pose at t = 0.9 sec is better than the $\gamma = 10^{-6}$ case with the torso fully upright, but the left leg with the uncontrolled left knee has swung the left shank link both through the ground plane and then above and past its contact point on the ground plane. A closer look showed that the left leg is swinging through the ground plane. While in the type I case (joint is locked) that has been assumed undesirable as the fixed link would be pressed against the ground in a real world system possibly disrupting the entire system, in this type II case it is assumed acceptable. This is because with the passive joint free, in a real world system, it is assumed the link wouldn't press against the ground plane as much as be dragged along it. With this in mind it is considered acceptable to intersect with the ground plane in these simulations for the type II case as it is expected that it would have minimum impact on the control of the rest of the system.

Simulations indicated that the under actuated LQR controller is managing the gait for the remaining controllable joints in a similar fashion to the fully actuated case, despite the affects of the free swinging passive joint. For $\gamma = 10^{-7}$, the joint position and velocity errors over the course of the SSP is given in Fig. 5. From this figure, we can see that the passive joint shows little, if any, of the convergence behavior in the errors of the controlled joints, and clearly the LQR controller has no influence on the passive joint in this simulation. On the other hand, the presence of the mounting passive joint errors doesn't seem to have any noticeable effect on the remaining controlled joints. Overall, for the controlled joints in the under actuated biped, the best case $\gamma = 10^{-7}$ LQR controller can keep peak position errors below 3.9 degrees, peak velocity errors below 7.5 degrees per second, end position errors below 1.2 degrees and end velocity errors below 7.4 degrees per second [17]. Similar to the fully actuated case, the worst errors experienced during the SSP are on the most burdened joints, specifically θ_4, the right foot. We found that the behavior of the passive joint minimally affected by changes in γ if at all. Also similar to the fully actuated case, the controlled joint errors are reduced compared to the larger γ cases and have a similar convergence behavior. Not only is the under actuated controller succeeding in preventing worse errors in the presence of a passive joint, but in this simulation and using the left knee as the passive joint, it is nearly identical to the best $\gamma = 10^{-7}$ case but it is identical in the $\gamma = 10^{-10}$ case. Overall, it has been shown that by using the under actuated methods described in this paper, a successful under actuated controller can be developed for this SSP system.

V. CONCLUSION

The demonstrated simulation is flexible, able to follow any input trajectory, not just a walking gait, and can simulate both the right leg SSP and the left leg SSP equally well (although separately with no model of the DSP to link the two). The simulation has also been shown to operate the proposed dynamics in a predictable and repeatable way which should allow a path to considerably more diverse experiments.

LQR control of the fully actuated 5 DOF planar biped robot has been completely successful, performing the proposed waking gait with only a single controller linearized at the time = 0 home position. The sensitivity of using γ, multiple controllers at different time steps, length of total step time, and number of time steps in the total step time have been quantified in [17]. Although small improvements to the SSP errors can be made using these additional variables, none compared to the scope of improvement found adjusting γ. Under actuated control was successfully demonstrated performing the SSP under identical conditions as the fully actuated, despite the presence of a single uncontrolled passive joint in the simplest case. The original goals of the under actuated system to at least retain control of the biped with passive joints despite error rates were exceeded. The under actuated controller was shown capable of matching the fully actuated error levels and visual performance of the remaining controllable joints using the same γ values in the presence of a type II (free, unlocked) joint. Under actuated control dynamics were also shown to not be necessary for the control of the biped in the presence of a type I (locked) joint although a different optimized fully actuated controller must be used.

REFERENCES

[1] H. Arai and S. Tachi, "Position Control of a Manipulator with Passive Joints Using Dynamic Coupling," *IEEE Trans. Robot. Automat*., vol. 7, pp. 528-534, 1991.

[2] Arai, K. Tanie, and S. Tachi, "Dynamic Control of a Manipulator with Passive Joints in Operational Space," *IEEE Trans. Robot. Automat*., vol. 9, pp. 85-93, 1993.

[3] H. Arai, K. Tanie, and N. Shiroma, "Time-scaling Control of an Underactated Manipulator," *Proc. IEEE Int'l Conf. Robotics and Automation*, Leuven, Belgium, pp. 2619-2626, 1998.

[4] A. Luca, R. Mattone, and G. Oriolo, "Stabilization of Underactuated Robots: Theory and Experiments for a Planar 2R Manipulator," *Proc. IEEE Int'l Conf. Robotics and Automation*, Albuquerque, NM, 1997, pp. 3274-3280.

[5] J.-M. Yang, "Fault-Tolerant Gait Generation for Locked Joint Failures," *Proc. IEEE Int'l Conf. System, Man and Cybernetics*, pp. 2237-2242, 2003.

[6] C. Zhu, Y. Tomizawa, X. Luo, and A. Kawamura, "Biped Walking with Variable ZMP, Frictional Constraint, and Inverted Pendulum Model," *Proc. IEEE Int'l Conf. Robotics and Biomimetics*, pp. 425-430, 2004.

[7] F. Silva and J. Machado, "Position/Force Control of Biped Walking Robots," *Proc. IEEE Int'l Conf. System, Man and Cybernetics*, pp. 3288-3293, 2000.

[8] X. Mu and Q. Wu, "Sagittal Gait Synthesis for a Five-Link Biped Robot," *Proc. American Control Conference*, pp. 4004-4009, 2004.

[9] X. Mu and Q. Wu, "Development of a Complete Dynamic Model of a Planar Five-link Biped and Sliding Mode Control of its Locomotion During the Double Support Phase," *Int'l J. Control*, vol. 77, no. 8, pp. 789-799, 2004.

[10] B. Ma and Q. Wu, "Parametric Study of Repeatable Gait for a Planar Five-link Biped," *Robotica*, vol. 20, Cambridge University Press, United Kingdom, pp. 493-498, 2002.

[11] Q. Wu and C. Chan, "Design of Joint Angle Profiles for a Planar Five-Link Bipedal System," ASME Trans. *Dynamic Systems, Measurement, and Control*, vol. 127, pp. 192-196, 2005.

[12] Z. Peng, Q. Huang, X. Zhao, T. Xiao, and K. Li, "Online Trajectory Generation Based on Off-line Trajectory for Biped Humanoid," *Proc. IEEE Int'l Conf. Robotics and Biomimetics*, pp. 752-756, 2004.

[13] F. Asano, Z-W. Luo, and M. Yamakita, "Some Extensions of Passive Walking Formula to Active Biped Robots," *Proc. IEEE Int'l Conf. Robot. Automat.*, pp. 3797-3802, 2004.

[14] C. Fu, M. Shuai, and K. Chen, "Proving Asymptotic Stability of Dynamic Walking for a Five-Link Biped Robot with Feet," *Proc. IEEE Int'l Conf. Robotics, Automation and Mechatronics*, Bangkok, June 2006.

[15] D. Sharon and M. Panne, "Position/Force Control of Biped Walking Robots," *Proc. IEEE Int'l Conf. Robot. Automat.*, pp. 1989-1994, 2005.

[16] Leva,"Adjustments to Zatsiorsky-Seluyanov's Segment Inertia Parameters," *J. Biomechanics*, **29**(9), pp. 1223-1230, 1996.

[17] M. Leines "An LQR Control of an Underactuated 5 DOF Planar Biped", M.S. Thesis, University of Minnesota, Duluth, September 2010.

[18] J.-S. Yang, "On Trajectory Planning, Dynamics, and Control for a Five-Degree-of-Freedom Biped Locomotion System," *Proc. American Control Conference*, pp. 3105-3109, 1994.

[19] O. Costa and J. Val, "Full Information H^{∞}-Control for Discrete-Time Infinite Markov Jump Parameter Systems." *J. Math. Anal. Appl.*, vol. 202, no. 0335, pp. 578-603, 1996.

[20] O. Costa, E. Filho, E. Boukas, and R. Marques, "Constrained Quadratic State Feedback Control of Discrete-time Markovian Jump Linear Systems," *Automatica*, vol. 35, pp. 617-626, 1999.

[21] F. Lewis, S. Jagannathan, and A. Yesildirek, "Neural Network Control of Robots and Nonlinear Systems," Taylor and Francis, London 1998.

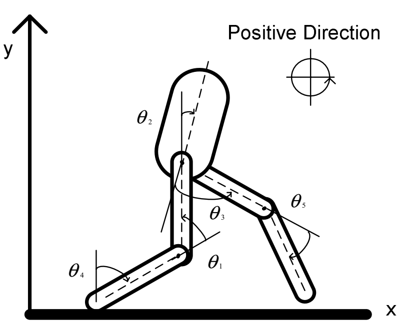

Fig. 1 A biped model

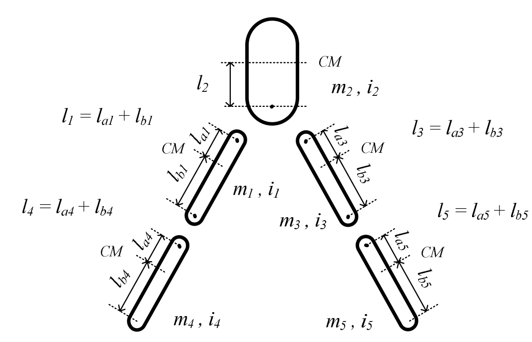

Fig. 2 Biped robot general proportions

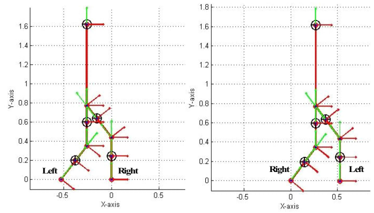

Fig. 3 Initial and end state of the right SSP

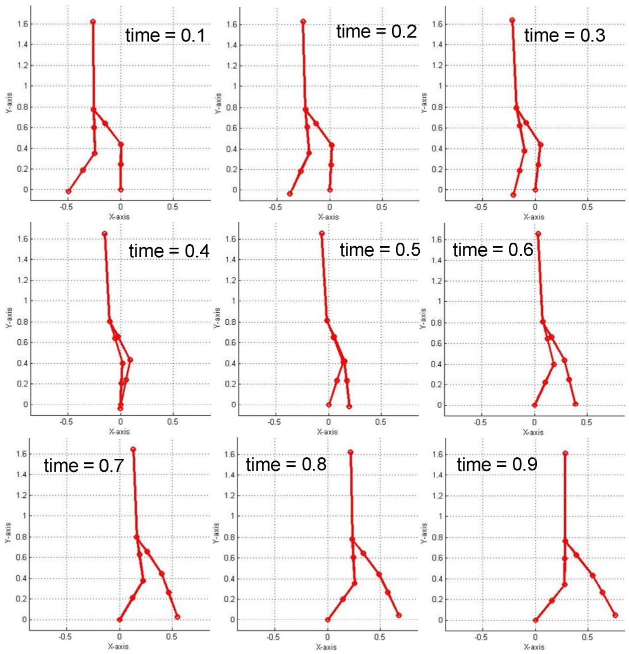

Fig. 4 Under actuated motion over time of the right leg SSP with $\gamma = 10^{-7}$

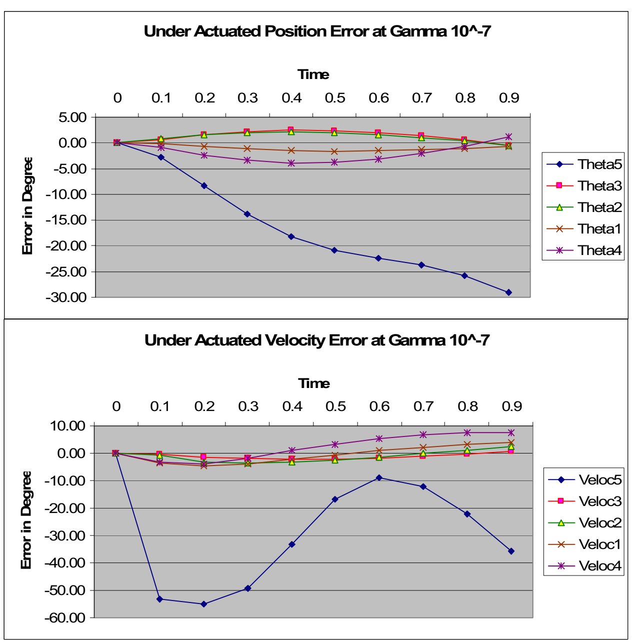

Fig. 5 Position and velocity errors over the course of the under actuated SSP