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Development of a Complete Dynamic Model of a Planar Five-Link Biped and Sliding Mode Control of Its Locomotion During the Double Support Phase

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Development of a complete dynamic model of a planar five-link biped and sliding mode control of its locomotion during the double support phase

XIUPING MU† and QIONG WU†*

This paper presents a complete dynamic model of a planar five-link biped walking on level ground. The single support phase (SSP), double support phase (DSP) and double impact occurring at the heel strike are included in the model. By modifying the conventional definition of certain physical parameters of the biped system, it is shown that the procedure of the derivation of the dynamic equations and their final forms are significantly simplified. For motion regulation during the DSP, our dynamic model is formulated as the motion of biped system under holonomic constraints, and the hip position and the trunk orientation are selected as the independent generalized coordinates to describe the constraint system and to eliminate the constraint forces from the equations of motion. Based on the presented dynamic formulation, we develop a sliding mode controller for motion regulation during the DSP where the biped is treated as a redundant manipulator. The stability and the robustness of the controller are investigated, and its effectiveness is demonstrated by computer simulations. To the best of our knowledge, it is the first time that a sliding mode controller is developed for biped walking during the DSP. This work makes it possible to provide robust sliding mode control to a full range of biped walking and to yield dexterity and versatility for performing specific gait patterns.

Nomenclature

m_i	mass of link i
l_i	length of link i
d_i	distance between the mass centre of link i and joint i (refer to figure 1)
I_i	moment of inertia of link i with respect to the axis passing through its mass centre and being perpendicular to the motion plane
θ_i	angle of line i with respect to the vertical axis through joint i, clockwise for the positive direction
q_i	relative angle between jointed links: $q_i = \theta_i - \theta_{i-1}$, prescribing $\theta_0 = 0$
$O - x_0y_0$	the fixed coordinate frame (i.e. the inertia reference system)
(x_b, y_b)	the coordinate of the supporting limb tip in the SSP, and of the front limb tip in the DSP
(x_e, y_e)	the coordinate of the swing limb tip in the SSP, and of the rear limb tip in the DSP
L	the distance between the tips of two lower limbs in the DSP
τ_i	control torque at joint i
a_i	a number used in dynamic model, defined by $a_i = 0$ if $i = 3$ and $a_i = 1$ if $i = 1, 2, 4, 5$.

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1. Introduction

The development of legged locomotion systems has recently received increased attention due to their higher mobility than conventional wheeled vehicles. A vast literature has contributed to biped locomotion on dynamic modelling and motion control (see, e.g. Furusho and Masubuchi 1986, 1987, Tzafestas *et al*. 1996, and references cited therein). A general five-link biped model has the advantage that it has sufficiently low degrees of freedom (DOF) to establish the mathematical model yet still has enough DOF to adequately describe the biped motion. Such a five-link biped model has been used frequently (Furusho and Masubuchi 1986, 1987, Chang and Hurmuzlu 1993, Tzafestas *et al*. 1996, Mitobe *et al*. 1997, Sonoda *et al*. 1997). However, we noted that the procedure of the derivation of the dynamic equations and their final forms are more complex and less structured as compared with those of robot manipulators. In addition, most related papers have concentrated on dynamic modelling and control of biped walking with only a single support phase (SSP) (Chang and Hurmuzlu 1993, Tzafestas *et al*. 1996, Wu and Chan 2001). A double support phase (DSP) is often neglected or assumed to be instantaneous (Furusho and Masubuchi 1986, 1987, Shih 1996, Cheng and Lin 2000). Motion of a biped robot with the DSP has the superiority to that with only the SSP in that it is more convenient to realize the stable motion over a wide range of walking speeds. Although biped walking during the DSP raises some challenges due to the involvement of the constraints, the DSP should be included in biped modelling and control.

Another important physical phenomenon that should be considered in the modelling of biped walking is impact, occurring instantaneously when the free end

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of the biped comes into contact with the ground. Zheng and Hemami (1984) have discussed the impact effects and derived a general form of the impact equation using the integration method and Tzafestas *et al.* (1996) adapted this formulation to a biped single impact model. Hurmuzlu (1993) has described biped impact as the instantaneous change of velocities during the phase change and discussed both single and double impact by the concept of principle of conservation of impulse and momentum. However, the detailed dynamic equations of double impact for a five-link biped model have not been made available in the literature except the work by Mu and Wu (2003 a) where both integration method and the Newtonian method were employed.

Motion control of a biped is a challenging task. One challenge lies in the high non-linearities of the dynamics and the inaccuracy of the parameters in the biped model. Another one is the difficulty in controlling the motion during the DSP. Computed torque control as a classic non-linear feedback control has been applied to eliminate the non-linear terms involved in the dynamic model to track reference trajectories. Mitobe *et al.* (1995, 1997) have used computed torque control to a 4-DOF biped for both a single support model and a double support model. As the kinematics and dynamics are highly non-linear with low accuracy of system parameters, bipedal walking robots are typical systems affected by uncertainty. Stability and robustness are two important requirements for control design. The main disadvantage of computed torque control and some other similar control strategies based on feedforward compensation and state feedback is that they may lead to unacceptable performance due to these parameter uncertainties. Chang and Hurmuzlu (1993) have developed a sliding mode control law without the reaching phase and applied it to a five-link biped during the SSP. Tzafestas *et al.* (1996) have developed a robust sliding mode controller applied to a five-link biped during the SSP and compared it to computed torque control. They both proved that sliding mode control is superior to other control strategies in that it has a short transient period and is insensitive to uncertainties, and thus is a good approach to deal with the above-mentioned problems in bipedal walking robots.

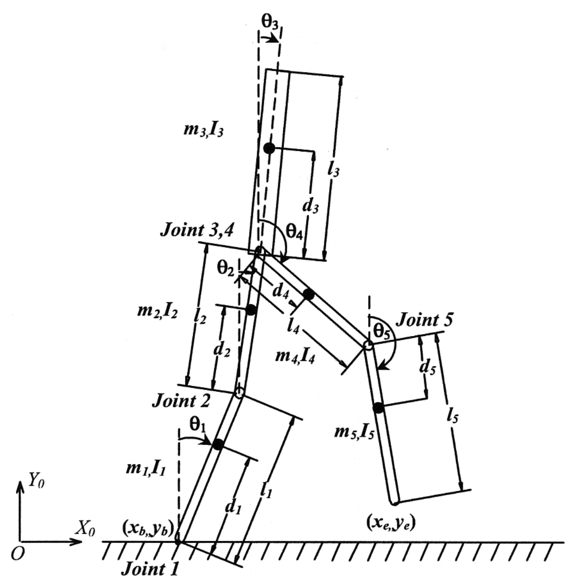
Motion of a biped robot during the DSP can be described as the motion of a dynamic system under holonomic constraints since the contact points between the feet and the ground are fixed. These constraints introduce two difficulties in motion regulation. One is that the generalized coordinates used in the SSP are no longer independent. Another difficulty is that the constraint forces are not provided *a priori* but are among the unknowns of the system, which must be obtained from the solution we seek. Two approaches are often used to control constrained biped walking. One is to control the motion as well as the constraint forces, such as the work by Mitobe *et al.* (1997). Another approach is to treat the biped as a redundant manipulator under an assumption that the tips of both limbs are fixed to the ground. Sonoda *et al.* (1997) used the second approach to a 4-DOF biped control during the DSP. They set a general acceleration reference formulation to each joint given by a performance function, which decided the robot's configuration. The advantage of the first approach is that the actual constraint forces follow the desired ones, which always satisfy the constraints. However, for many cases, tracking desired constraint forces is not of interest as far as the constraints are not violated. The main advantage of the second approach is that the control torques can be distributed based on various performance functions. Such performance functions can be designed for stability and optimization purposes. In this work, we will design a robust sliding mode control law using the second approach, i.e. the biped system is regarded as a redundant manipulator, where the constraints between the tips of the lower limbs and the ground are assumed to be always satisfied. Due to the redundancy and the system parametric uncertainties involved in the high-DOF non-linear biped system, the design of a sliding mode controller is extremely challenging. To the best of our knowledge, there are no papers on sliding mode control of biped walking during the DSP in spite of its advantages.

In this paper, a complete dynamic model including the SSP, DSP and double impact are developed for a five-link biped walking on level ground. Lagrangian formulations are used for deriving dynamic equations. By modifying the conventional definition of certain physical parameters of the biped system, the procedure of the derivation of the dynamic equations and their final forms are significantly simplified. For motion regulation during the DSP, the dynamic model of the five-link biped is first formulated as the motion of a biped system under holonomic constraints, and the hip position and the trunk orientation are selected as independent generalized coordinates to describe the constraint system and to eliminate the constraint forces from the equation of motion. Based on the presented dynamic formulation, a sliding mode control algorithm is developed for motion regulation during the DSP utilizing the pseudo-inverse matrix as a performance function. The desired joint profile is taken from Mu and Wu (2003 b). The stability and the robustness of the controller in the DSP are analysed in this paper. Simulations are carried out to demonstrate the effectiveness of the control algorithm.

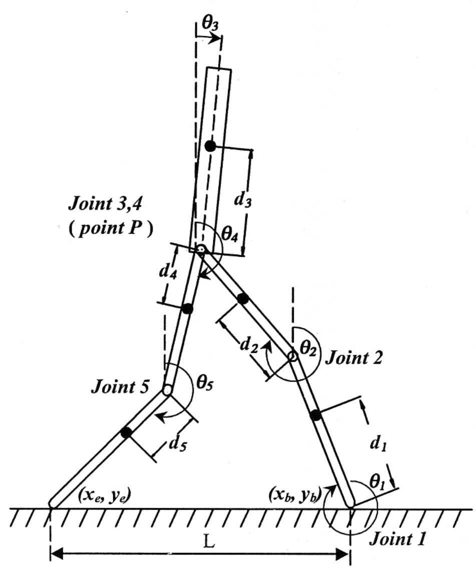
2. Biped model

In this section, we focus on dynamic modelling of a sagittal five-link biped walking on level ground. The biped structure is taken from Hurmuzlu (1993) which has five links—a torso and two identical lower limbs with each limb having a thigh and a shank. Also, the biped has two hip joints, two knee joints and two ankles at the tips of the lower limbs. There is an actuator located at each joint and all of the joints are considered rotating only in the sagittal plane and friction free. We assume massless feet in the model. Although we neglect the dynamics of the feet, we assume that the biped can apply torques at the ankles. In this work, a torque is applied at the leading ankle during the DSP. The rear ankle does not possess a torque but can rotate through the knee torque and the effect of gravity. Similar assumption of the zero ankle torque has been used by Mitobe *et al.* (1997) and Sonoda *et al.* (1997). Figure 1 depicts a schematic representation of this biped model where (*a*) represents the SSP and (*b*) represents the DSP.

Our kinematic model of the biped system is different from the conventional ones in that the biped



(a)



(b)

Figure 1. A five-link biped kinematic model: (*a*) SSP; (*b*) DSP.

is considered as an open-loop serial-chain from the supporting point to the free ends during the SSP. The value of d_i representing the location of the mass centre of link i is measured as the distance between the mass centre of link i to joint i, and θ_i is also denoted as the angle of link i with respect to the vertical axis through joint i. While in previous literature on dynamic modelling of biped walking (Hemami and Rarnsworth 1977, Furusho and Masubuchi 1986, 1987, Tzafestas *et al.* 1996), d_i and θ_i have both been defined with respect to its lower joint. Using our denotation, the dynamic models, as demonstrated later, can be derived more conveniently and the final forms of the dynamic equations are less complex and more structured as compared with the above-mentioned published works. Consequently, this modified biped kinematic model has greatly improved the efficiency of the derivation procedure and also made the computational programming much easier and less prone to mistakes.

Our complete dynamic model of biped walking includes an SSP, a DSP, double impact, switching and transformation. The SSP is characterized by a stance limb supporting on the ground and the other limb swinging from the rear to the front. In the DSP, both limbs are in contact with the ground while the body can move forward slightly. Double impact happens in an infinitesimal period of time as the swing limb collides with the ground plastically, that is, when double impact is completed, both limbs stick on the ground. At the end of each step, the swing limb and the stance limb exchange the role, which is called switching, to avoid repeated work of mathematical modelling. To facilitate the control procedure, the dynamic equations for both phases need to be transformed from segmental angles to joint angles.

2.1. *Single support phase*

According to figure 1(a), the coordinates of the mass centre of link i are

$$x_{ci} = \sum_{j=1}^{i-1} (a_j l_j \sin\theta_j) + d_i \sin\theta_i + x_b \tag{1a}$$

$$y_{ci} = \sum_{j=1}^{i-1} (a_j l_j \cos\theta_j) + d_i \cos\theta_i + y_b. \tag{1b}$$

Thus, the potential energy (assume $P=0$ at the ground level) of the five-line biped model is

$$P = \sum_{i=1}^{5} m_i g y_{ci} = \sum_{i=1}^{5} \left\{ m_i g \left[\sum_{j=1}^{i-1} (a_j l_j \cos\theta_j) + d_i \cos\theta_i \right] \right\} \tag{2}$$

and the kinetic energy is

$$\begin{aligned} K &= \sum_{i=1}^{5} \left[\tfrac{1}{2} m_i (\dot{x}_{ci}^2 + \dot{y}_{ci}^2) + \tfrac{1}{2} I_i \dot{\theta}_i^2 \right] \\ &= \sum_{i=1}^{5} \left[\tfrac{1}{2} (I_i + m_i d_i^2) \dot{\theta}_i^2 \right] \\ &\quad + \sum_{i=1}^{5} \left\{ \tfrac{1}{2} m_i \left[\sum_{j=1}^{i-1} (a_j l_j \dot{\theta}_j \cos\theta_j) \right]^2 \right\} \\ &\quad + \sum_{i=1}^{5} \left\{ \tfrac{1}{2} m_i \left[\sum_{j=1}^{i-1} (a_j l_j \dot{\theta}_j \sin\theta_j) \right]^2 \right\} \\ &\quad + \sum_{i=1}^{5} \left\{ m_i d_i \dot{\theta}_i \left\{ \sum_{j=1}^{i-1} \left[a_j l_j \dot{\theta}_j \cos(\theta_i - \theta_j) \right] \right\} \right\}. \end{aligned} \tag{3}$$

By applying equations (2) and (3) to the Lagrangian formulation, the dynamic model for the five-link biped during the SSP can be derived as

$$D(\theta)\ddot{\theta} + H(\theta, \dot{\theta})\dot{\theta} + G(\theta) = T_\theta \tag{4}$$

where $D(\theta)$ is the 5×5 positive definite and symmetric inertia matrix, $H(\theta, \dot{\theta})$ is the 5×5 matrix related to centrifugal and Coriolis terms, $G(\theta)$ is the 5×1 matrix of gravity terms, θ, $\dot{\theta}$, $\ddot{\theta}$ and T_θ are the 5×1 vectors of generalized coordinates, velocities, accelerations and torques, respectively.

The detailed forms of D, H and G in equation (4) are presented as

$$D_{ij} = p_{ij} \cos(\theta_i - \theta_j) \tag{5a}$$

$$H_{ij} = p_{ij} \sin(\theta_i - \theta_j)\dot{\theta}_j \tag{5b}$$

$$G_i = g_i \sin\theta_i \tag{5c}$$

where i, $j = 1, 2, \ldots, 5$, p_{ij} and g_i are inertial terms defined as

$$p_{ij} = \begin{cases} I_i + m_i d_i^2 + a_i \left(\sum\limits_{j=i+1}^{5} m_j \right) l_i^2 & j = i \\ a_i m_j d_j l_i + a_i a_j \left(\sum\limits_{k=j+1}^{5} m_k \right) l_i l_j & j > i \\ p_{ji} & j < i \end{cases}$$

$$g_i = m_i d_i g + a_i \left(\sum_{j=i+1}^{5} m_j \right) l_i g.$$

Note that using our modified biped parameters, the detailed final form of the dynamic equations (5a–c) is significantly simplified as compared with the conventional form as typically shown in previously published articles (Furusho and Masubuchi 1986, 1987, Tzafestas *et al.* 1996).

2.2. *Double support phase*

In the DSP, since the contact position between the two tips of the limbs and the ground is fixed, there exists a set of holonomic constraint equations

$$\Phi(\theta) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_e - x_b - L \\ y_e - y_b \end{bmatrix} = 0. \tag{6}$$

By applying the Lagrangian formulation with constraints, the vector equation of the DSP is expressed by

$$D(\theta)\ddot{\theta} + H(\theta, \dot{\theta})\dot{\theta} + G(\theta) = J^{\mathrm{T}}(\theta)\lambda + T_\theta \tag{7}$$

where D, H, G and T_θ are the same as in (4), λ is a 2×1 vector of Lagrange multipliers, and J is the 2×5 Jacobian matrix $J = \partial\Phi/\partial\theta$. The dynamic model describing the DSP can be written

$$D(\theta)\ddot{\theta} + N(\theta, \dot{\theta})\dot{\theta} = J^{\mathrm{T}}(\theta)\lambda + T_\theta \tag{8 a}$$

$$\Phi(\theta) = 0 \tag{8 b}$$

where and hereafter $N = H(\theta, \dot{\theta})\dot{\theta} + G(\theta)$.

According to Goldstein (1980), for a dynamic system under holonomic constraints, it is possible to introduce a set of independent generalized coordinates to formulate the dynamic equations which can describe the constraint system without the terms of constraint forces. Mitobe *et al*. (1997) introduced the coordinates of the point at the bottom of the trunk as the generalized coordinates for a 4-DOF biped walking during the DSP. For our double support model, the set of independent generalized coordinates is chosen in R^3, and the motion of the system in the DSP can be fully described by hip and trunk motion. Thus, we select the hip position and the trunk orientation as the independent generalized coordinates $p = (x_h \; y_h \; \theta_3)^{\mathrm{T}}$. The transformation from (θ_i) set to (p_j) set can be written as

$$p(\theta) = \begin{bmatrix} l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ l_1 \cos\theta_1 + l_2 \cos\theta_2 \\ \theta_3 \end{bmatrix}. \tag{9}$$

Furthermore, in order to use the independent coordinates to establish the dynamic model, we need to transform the (p_j) set to the (θ_i) set. First, by differentiating (9) twice with respect to time, yields

$$\dot{p} = R\dot{\theta} \tag{10}$$

$$\ddot{p} = \dot{R}\dot{\theta} + R\ddot{\theta} \tag{11}$$

where $R = (\partial p/\partial\theta) \in R^{3\times5}$. Moreover, in order to apply the constraint conditions in the dynamic derivation, equation (6) is differentiated twice with respect to time, to yield

$$\dot{\Phi} = J(\theta)\dot{\theta} = 0 \tag{12}$$

$$\ddot{\Phi} = \dot{J}(\theta)\dot{\theta} + J(\theta)\ddot{\theta} = 0. \tag{13}$$

Combining equations (10), (11) and (12), (13), we have

$$\begin{bmatrix} R \\ J \end{bmatrix}\dot{\theta} = \begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} R \\ J \end{bmatrix}\ddot{\theta} = \begin{bmatrix} \ddot{p} \\ 0 \end{bmatrix} - \begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix}\dot{\theta}. \tag{15}$$

It is obvious that the matrix

$$\begin{bmatrix} R \\ J \end{bmatrix} \in R^{5\times5}$$

is full ranked, thus is invertible. Since we have transformed the (p_j) set to the (θ_i) set, we can rewrite the dynamic equation in (p_j) instead of (θ_i). Combining (14), (15) and (8 *a*), we have

$$\begin{bmatrix} \ddot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix}^{-1}\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} + \begin{bmatrix} R \\ J \end{bmatrix} D^{-1}(T_\theta - N) + \begin{bmatrix} R \\ J \end{bmatrix} D^{-1}J^{\mathrm{T}}\lambda \tag{16}$$

which can be symbolically written as

$$\begin{bmatrix} \ddot{p} \\ 0 \end{bmatrix} = S_a\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} + S_b(T_\theta - N) + S_c\lambda \tag{17}$$

where

$$S_a = \begin{bmatrix} S_{a11,\,3\times3} & S_{a12,\,3\times2} \\ S_{a21,\,2\times3} & S_{a21,\,2\times2} \end{bmatrix} = \begin{bmatrix} \dot{R}\begin{bmatrix} R \\ J \end{bmatrix}^{-1} \\ \dot{J}\begin{bmatrix} R \\ J \end{bmatrix}^{-1} \end{bmatrix},$$

$$S_b = \begin{bmatrix} S_{b1,\,3\times5} \\ S_{b2,\,2\times5} \end{bmatrix} = \begin{bmatrix} RD^{-1} \\ JD^{-1} \end{bmatrix},$$

$$S_c = \begin{bmatrix} S_{c1,\,3\times2} \\ S_{c2,\,2\times2} \end{bmatrix} = \begin{bmatrix} RD^{-1}J^{\mathrm{T}} \\ JD^{-1}J^{\mathrm{T}} \end{bmatrix}.$$

Note that S_{c2} is an invertible matrix. Thus, equation (17) can be expressed as

$$\ddot{p} = S_{a11}\dot{p} + S_{b1}(T_\theta - N) + S_{c1}\lambda \tag{18 a}$$

$$0 = S_{a21}\dot{p} + S_{b2}(T_\theta - N) + S_{c2}\lambda. \tag{18 b}$$

The difference between (18) and (8) is that in (18 *b*) the constraint forces are separated from the second-order derivatives of the state vector. Thus they can be first solved from (18 *b*) and then substituted into (18 *a*), equation (18) becomes

$$\ddot{p} = B\dot{p} + C(T_\theta - N) \tag{19}$$

where $B = S_{a11} - S_{c1}S_{c2}^{-1}S_{a21}$ and $C = S_{b1} - S_{c1}S_{c2}^{-1}S_{b2}$. Equation (19) will be used for a sliding mode controller design in § 3.

2.3. *Double impact*

At the end of the SSP, the swing limb tip contacts the ground with a collision. The joint velocities will be subject to a sudden jump resulting from this impact event. During double impact there will be impulsive forces between the contact points of both limbs and the ground. The method introduced by Hurmuzlu (1993) based on the principle of conservation of impulse and momentum is used here to solve the biped angular velocities after impact due to its advantage that the external impulses can be solved. Since no detailed information is available from the literature, we present the derivation and the complete form of the double impact model. First, we have the following assumptions during impact: (i) the impact is perfectly plastic, and (ii) the joints are frictionless and the impulse moment at each joint is negligible. By applying principles of conservation of impulse and momentum, we have

$$m_i \Delta \dot{x}_{ci} = m_i \left[\Delta \dot{x}_b + \sum_{j=1}^{i-1} \left(a_j l_j \cos\theta_j \Delta\dot{\theta}_j \right) + d_i \cos\theta_i \, \Delta\dot{\theta}_i \right] = P_{X(i+1)} + P_{X(i)} \tag{20a}$$

$$m_i \Delta \dot{y}_{ci} = m_i \left[\Delta \dot{y}_b - \sum_{j=1}^{i-1} \left(a_j l_j \sin\theta_j \Delta\dot{\theta}_j \right) - d_i \sin\theta_i \, \Delta\dot{\theta}_i \right] = P_{Y(i+1)} - P_{Y(i)} \tag{20b}$$

$$I_i \Delta\dot{\theta}_i = P_{X(i+1)}(a_i l_i - d_i)\cos\theta_i + P_{X(i)} d_i \cos\theta_i - P_{Y(i+1)}(a_i l_i - d_i)\sin\theta_i - P_{Y(i)} d_i \sin\theta_i \tag{20c}$$

where '$\Delta\cdot$' represents the change of the velocity before and after impact, $P_{(i)}$ and $P_{(i+1)}$ are the impulses exerted to the two ends of link i, subscripts X and Y denote the horizontal and vertical impulses, respectively. Combining (12) and (20), the model to describe double impact can be derived as

$$\begin{bmatrix} W(\theta^-) & J^{\mathrm{T}}(\theta^-) \\ J(\theta^-) & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}^+ \\ P_l \end{bmatrix} = \begin{bmatrix} W(\theta^-)\dot{\theta}^- \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

where $W(\theta^-)$ is a 5×5 matrix

$$W_{ij} = \begin{cases} I_i - m_i d_i (a_i l_i - d_i) & j = i \\ \left[a_i m_j d_j l_i + a_j m_i (a_i l_i - d_i) l_j + a_i a_j \left(\sum_{k=j+1}^{i-1} m_k \right) l_i l_j \right] \cos(\theta_i - \theta_j) & j < i \\ 0 & j > 1. \end{cases}$$

$P_l = [P_{Xl},\ P_{Yl}]^{\mathrm{T}}$ represents the impulse forces at the contact point of the left limb (rear limb), J is the Jacobian matrix. The double impact model (21) is valid when the vertical impulse P_{Yl} is positive, since the supporting point is expected to remain on the ground after impact. Note that the simplified and compact form of matrix $W(\theta^-)$ is due to our definition of d_i and θ_i.

2.4. *Switching and transformation*

To ensure the dynamic models to be used in the next cycle, we need to switch the role of the stance and the swing limb. The physical displacements and velocities do not actually change but the link member, which is used in the dynamic model, needs re-labelling. The overall effect of the states immediately before and after switching can be written as

$$\begin{bmatrix} \theta^+ \\ \dot{\theta}^+ \end{bmatrix}_{\text{switch}} = \begin{bmatrix} T & \mathbf{0}_{5\times5} \\ \mathbf{0}_{5\times5} & T \end{bmatrix} \begin{bmatrix} \theta^- \\ \dot{\theta}^+_{\text{impact}} \end{bmatrix} + \Pi \tag{22}$$

where

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} \pi & \pi & 0 & \pi & \pi & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$\begin{bmatrix} \theta^- \\ \dot{\theta}^+_{\text{impact}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \theta^+ \\ \dot{\theta}^+ \end{bmatrix}_{\text{switch}}$$

are the state space vectors specifying the coordinates and the velocities immediately before and after switching.

To apply the actuator joint torques instead of the segmental torques, the biped motion during the SSP and the DSP described by (4) and (7) needs to be formulated in terms of the relative joint angles as

$$D_q(q)\ddot{q} + H_q(q,\dot{q}) + G_q(q) = T_q \tag{23}$$

$$D_q(q)\ddot{q} + H_q(q,\dot{q}) + G_q(q) = T_q + J_q^{\mathrm{T}}\lambda \tag{24}$$

where $T_q = [\tau_1, \tau_2, \tau_3, \tau_4, \tau_5]^{\mathrm{T}}$ representing the joint torques.

A transformation can be found to express q by θ

$$q = M_{q\theta}\,\theta \tag{25}$$

where $M_{q\theta}$ is a transformation matrix from θ to q

$$M_{q\theta} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Obviously, the relation between the segmental torques and joint torques can be expressed as

$$T_{\theta i} = \sum_{j=1}^{5} \tau_j \frac{\partial q_j}{\partial \theta_i} = \sum_{j=1}^{5} \tau_j (M_{q\theta})_{ji} \tag{26}$$

which can be shortened in a matrix form as

$$T_\theta = (T_q^{\mathrm{T}} M_{q\theta})^{\mathrm{T}} = M_{q\theta}^{\mathrm{T}} T_q. \tag{27}$$

Substituting (25) and (27) into (4) and (7), the following transformation forms can be determined

$$D_q = (M_{q\theta}^{-1})^{\mathrm{T}} D_\theta(\theta(q)) M_{q\theta}^{-1} \tag{28 a}$$

$$H_q = (M_{q\theta}^{-1})^{\mathrm{T}} H_q(\theta(q), \dot{\theta}(\dot{q})) M_{q\theta}^{-1} \dot{q} \tag{28 b}$$

$$G_q = (M_{q\theta}^{-1})^{\mathrm{T}} G_\theta(\theta(q)) \tag{28 c}$$

$$J_q = J_\theta(\theta(q)) M_{q\theta}^{-1}. \tag{28 d}$$

Similarly, the model (19) is transformed to

$$\ddot{p} = B(q)\dot{p} + C_q(q)(T_q - N_q(q)) \tag{29}$$

where

$$B(q) = B(R(q), D_q, J_q) \tag{30 a}$$

$$C_q = C(\theta(q)) M_{q\theta}^{\mathrm{T}} \tag{30 b}$$

$$N_q = (M_{q\theta}^{-1})^{\mathrm{T}} N(\theta(q)). \tag{30 c}$$

Thus, equations (28) and (30) can be conveniently applied to the dynamic model (23)–(24) or (29) when motion control is studied.

3. Motion control

Biped motion control is realized by motion planning and servo control. Our motion planning for the biped system is to realize a steady and repeatable gait on level ground by considering the following factors: satisfying parametric specifications such as the step length, walking speed and maximum clearance; keeping constraint conditions such as geometrical relations and continuity of the gait; maintaining stable and cyclic gait and reducing the impact effect. The detailed design procedure and desired joint angle profiles can be found in Mu and Wu (2003 b). In this section, we will develop a robust sliding mode controller for motion regulation of a five-link biped during the DSP where the biped is treated as a redundant system with the dynamic equations shown in (29). Before establishing the sliding mode control law for biped walking, we have the following assumptions:

(1) The bounds of the parameter uncertainties of the biped system are known.

(2) The state space variables are available for measurement.

'The parameters with uncertainties' are those physical parameters of the biped system, including m, l, I and d. We use the following symbols: φ_i as each physical parameters (m, l, I, d); $E_i \times 100\%$ $(0 \le E_i < 1)$ as their uncertainties in the upper bounds; $\hat{\varphi}_i = (1 + E_i)\varphi_i$ as the corresponding estimated value of each parameter; M_i as the matrices (D, H, G, J) in (4) and (7). The estimated matrices are obtained by $\hat{M}_i = \hat{M}_i(\hat{\varphi}_i) = M_i(\varphi_i) + \Delta M_i(E_i \varphi_i)$. The detailed procedure to calculate those bounds has been discussed by Tzafestas *et al.* (1996).

For the second-order differential equation (29), a time-varying sliding surface S is defined as $s(p, t) = 0$, where

$$s = \dot{e} + 2\Lambda e + \Lambda^2 \int_0^t e(\tau)\, \mathrm{d}\tau$$

and e and $\dot{e}$ are the vectors of tracking errors defined as

$$e = p - p_d \qquad \dot{e} = \dot{p} - \dot{p}_d.$$

Λ is a 3×3 diagonal matrix. By introducing the PD controller

$$u = \ddot{p}_d - K_D \dot{e} - KPe \tag{31}$$

and the equivalent controller

$$\hat{T}_q = \hat{C}_q^{-} u - \hat{C}_q^{-} \hat{B} \dot{p} + \hat{N}_q \tag{32}$$

the sliding mode controller is designed as

$$T_{qi} = \hat{T}_{qi} - \sum_{j=1}^{3} \left\{ \hat{C}_{q\,ij}^{-} \operatorname{sgn}(s_j) \left[\sum_{k=1}^{5} \left(\left| \hat{C}_{q\,jk} \right| \eta_k \right) + \sigma_j \right] \right\}, \quad i = 1, 2, \ldots, 5 \tag{33}$$

where $\eta \in R^5$ represents the system uncertainty bounds; $\hat{C}_q^{-} = \hat{C}_q^{\mathrm{T}} (\hat{C}_q \hat{C}_q^{\mathrm{T}})^{-1}$ represents the pseudo-inverse matrix of $\hat{C}_q$ as $\hat{C}_q$ is not square.

In general, control law (33) is associated with a chattering problem due to the discontinuous switching function sgn(s) in the controller. It is undesirable in practice and should be eliminated. Slotine and Sastry (1983) suggested a solution to this problem by replacing sgn(s) with a saturation function in a thin boundary layer neighbouring the switching surface. Once the system trajectories enter the boundary layer, they will remain inside. Alternative solutions include using a hyperbolic function $\tanh(s)$ instead of the switching function (Cai and Song 1993, Wu *et al.* 1998), which has the advantage of being smooth and it is adapted to our work

$$T_{q_i} = \hat{T}_{q_i} - \sum_{j=1}^{3} \left\{ \hat{C}_{q_{ij}}^{-} \tanh(\alpha_j s_j) \left[\sum_{k=1}^{5} \left(\left| \hat{C}_{q_{jk}} \right| \eta_k \right) + \sigma_j \right] \right\} \tag{34}$$

where α and σ are constant vectors.

The constants of the entries in α can be determined by a sliding condition analysis through its necessary criteria for convergence. The quadratic Lyapunov function is considered as $V = \frac{1}{2} s^{\mathrm{T}} s > 0$, and the derivative of V along the solution trajectory of (29) is given by

$$\begin{aligned} \dot{V} &= s^{\mathrm{T}} \dot{s} \\ &= s^{\mathrm{T}} (\ddot{p} - u) \\ &= s^{\mathrm{T}} (C_q T_q + B\dot{p} - C_q N_q - u) \\ &= s^{\mathrm{T}} \left\{ C_q \left[\Delta - \left(\hat{C}_q^{-} \tanh(\alpha s) \right) \left(\mathrm{abs}\left(\hat{C}_q \right) \eta + \sigma \right) \right] \right\} \end{aligned} \tag{35}$$

where

$$[\Delta]_i = \left\{ \left(\hat{C}_q^{-} u - C_q^{-} u \right) + \left(\hat{N}_q - N_q \right) - \left(\hat{C}_q^{-} \hat{B} \dot{p} - C^{-} B \dot{p} \right) \right\} \le \eta_i,$$

$$\left[\mathrm{abs}\left(\hat{C}_q \right) \right]_{ij} = \left| \hat{C}_{q_{ij}} \right|.$$

For the parameter uncertainties less than 100%, we have $0 < K_{\min} < \sum_{k=1}^{5} C_{q_{ik}} \hat{C}_{q_{kj}}^{-} < K_{\max}$. In order to guarantee $\dot{V} < 0$, the following condition must hold

$$\tanh(\alpha_i |s_i|) > \frac{\Sigma_{k=1}^{5} |\hat{C}_{q_{ik}} \eta_k|}{k_{\min}(\Sigma_{k=1}^{5} |\hat{C}_{q\,ik}| \eta_k + \sigma_i)} \triangleq \nu_i; \qquad i = 1, 2, 3 \tag{36}$$

which leads to

$$\alpha_i > \frac{1}{2\varepsilon_i} \ln\left(\frac{1 + \nu_i}{1 - \nu_i} \right). \tag{37}$$

When the width of the boundary layer ε_i is selected, α_i can be chosen according to (37).

4. Simulation results

In this section, the five-link biped walking in the saggital plane is simulated to demonstrate the effectiveness of the control algorithm. The physical parameters of the biped model used in our simulation can be found in table 1. The walking pattern is taken from Mu and Wu (2003 b) with the average walking speed = 1 m s^{-1}, step length = 0.7 m, DSP time period = 0.1 s. The simulation is carried out by introducing system uncertainties of mass and inertial e_m, e_I: 40%; length e_l, e_d: 10%. The constants in the controllers are taken by trial and error and they take the values $\alpha = 10$; $\lambda = 10$; $\sigma = 35$ for the DSP. Note that the motion of the five-link biped during the whole step is simulated using the sliding mode control adapted from Tzafestas *et al.* (1996) for the SSP and the presented sliding mode control for the DSP. For the sake of brevity, only the detailed results during the DSP are presented.

Link no.	m_i (kg)	I_i (kg m^2)	l_i (m)	d_i (m)
1	2.23		0.332	0.189
2	5.28		0.302	0.236
3	14.79	0.033	0.486	0.282
4	5.28		0.302	0.066
5	2.23		0.332	0.143

Table 1. Biped physical parameters.

Figure 2 shows the errors of the horizontal and vertical displacements of the hip and the orientation of the trunk in the DSP. One can observe that the tracking errors are low which shows that the controller can track the reference trajectory properly. Figure 3 shows the joint torques in the DSP and they are shown continuous. Figure 4 shows the ground reaction forces in the DSP. F_{lx}, F_{ly} and F_{rx}, F_{ry} are the horizontal and vertical reaction forces applied to the left limb (rear limb) tip, and the right limb (front limb) tip, respectively. We can see that both vertical forces remain positive, thus the non-lifting constraint is satisfied throughout the DSP. The non-slipping constraint condition is satisfied provided that the friction coefficient between the biped lower limb tips and the ground is greater than 0.25, which is not an overly restricted condition. Figure 5 shows the stick diagram of biped walking with one full step. The solid line represents the biped motion during the SSP, the dashed line represents the motion during the DSP and the asterisk represents the position of the centre of gravity (CG) of the biped system during walking. From figure 5, we can see that the biped motion is smooth and cyclic; the biped CG is almost in a horizontal level indicating that the biped motion is steady. The overall results show that the proposed control algorithm can produce a stable biped gait and the designed sliding mode controllers can fulfil a good performance for the situation where parametric uncertainties exist.

5. Conclusions

An approach to dynamic modelling of a planar five-link biped and sliding mode control design during the DSP has been presented in this paper. The conventional definition of certain physical parameters of the biped system has been modified and a complete dynamic model, including the SSP, DSP and double impact, has been developed for a five-link biped walking on level

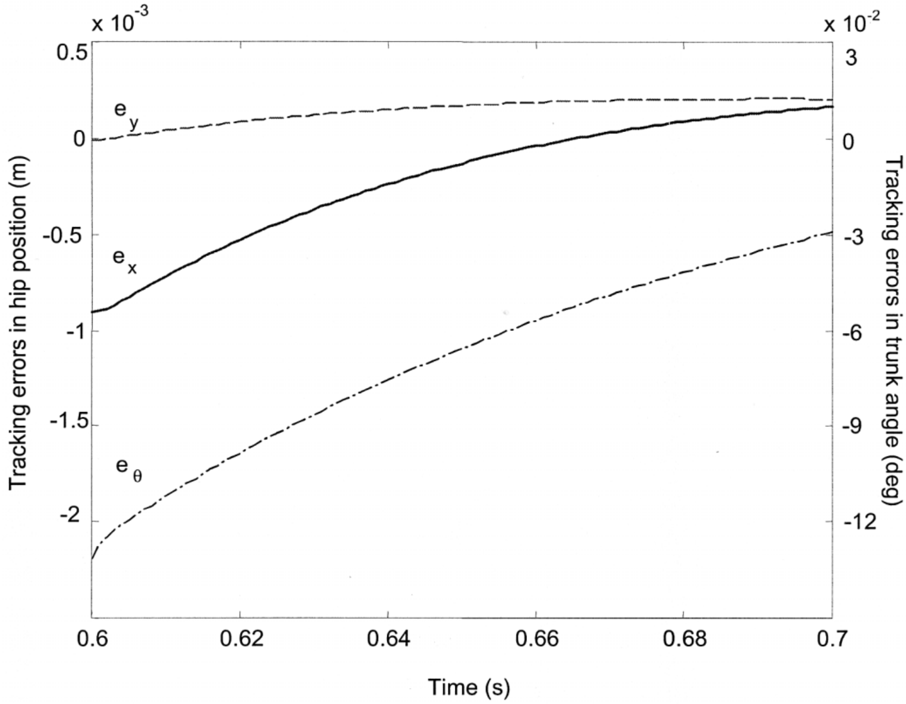


Figure 2. Tracking errors in DSP.

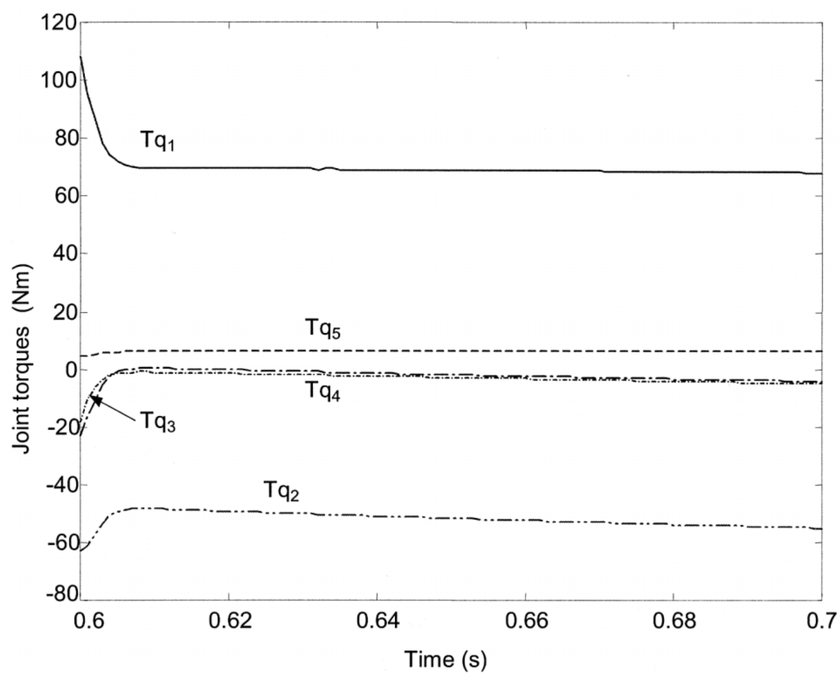


Figure 3. Actuator joint torques in DSP.

ground. For motion regulation during the DSP, the biped model is formulated as a robot system under holonomic constraints and the hip position and trunk orientation are selected as independent generalized coordinates to describe the biped system. A robust sliding mode control algorithm is developed treating the biped as a redundant manipulator where the constraint of each limb remaining in contact with the ground is assumed satisfied and the pseudo-inverse matrix is used as a performance function to distribute the control torques among the five joints. The control algorithm tracks the hip and trunk motion without force control

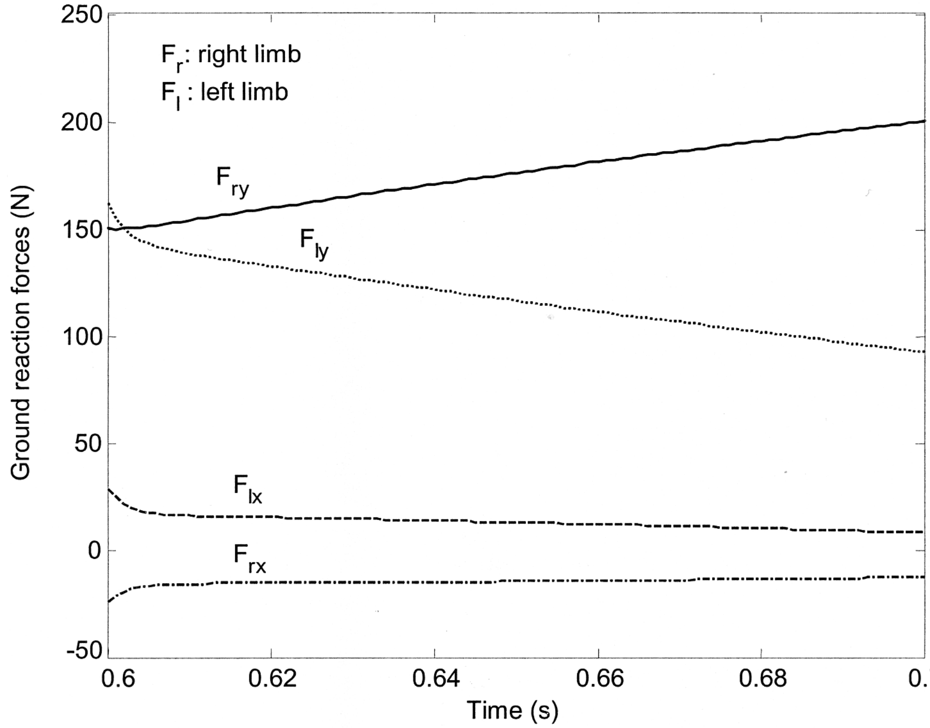


Figure 4. Ground reaction forces in DSP.

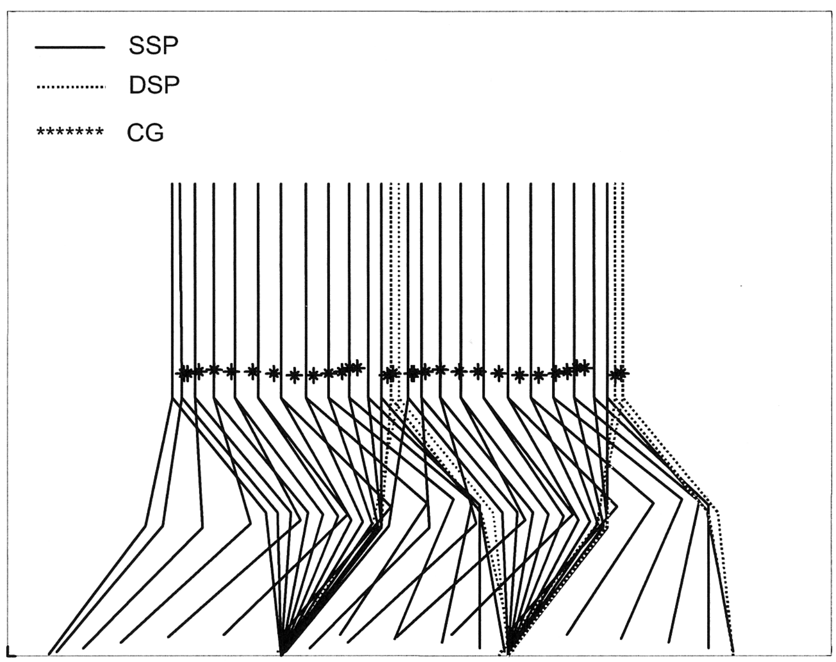


Figure 5. Stick diagram of biped walking.

and thus the control design is simplified by avoiding a separate force control law. To eliminate the chattering problem during the implementation of sliding mode control, the continuous control law is used to approximate the discontinuous controller. The simulation results show that in spite of the presence of system parameter uncertainties, the controller has a good performance in tracking reference signals with moderate control torques. The results also show that the non-lifting condition can be satisfied at all time and the non-slip condition can be satisfied with a moderate friction coefficient of the contact surface.

The work contributes to the dynamic modelling of biped walking in that the conventional kinematic model has been improved, which has significantly simplified the derivation procedure and the final form of the dynamic equations. Our improved kinematic model has not only improved the efficiency of the derivation procedure but also made the programming much easier and less prone to mistakes. This work contributes to motion regulation of biped walking in that it provides an effective control scheme yet it is robust even with large system parameter uncertainties. More importantly, the control design, presented here, provides the flexibility to distribute the joint torques based on various performance functions.

This paper encourages future work in several aspects. Our extension consists of the determination of torque distributions in the double support phase based on various performance functions, such as energy-efficient walking or saturated control torques at certain joints. Another future work is to consider the effects of foot dynamics when dealing with biped walking. It is hoped that further investigations of this type will eventually attain successful applications in the realization of a biped robot and in the design of prosthetic systems.

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