QEC for GAD on QuDits

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Qudits: QHMs

There are two schools of thought:

Those who *think* the universe is made up Quantum Harmonic Oscillators

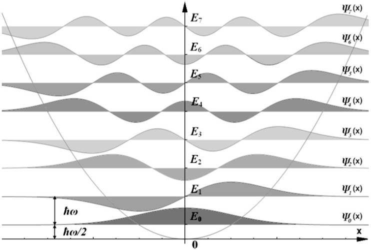

Figure: Quantum Harmonic Oscillator

And those who don't think at all.

Qudits: QHMs

$$\psi_n(x) = \frac{C}{\sqrt{2^n\, n!}} e^{-\frac{\alpha x^2}{2}} H_n\left(\sqrt{\alpha}x\right)$$

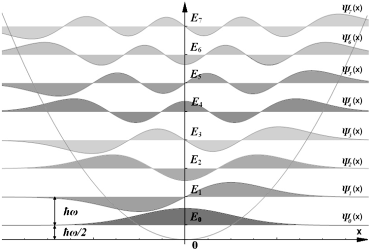

Figure: QHM

Therefore for qubits:

$$\psi_0 = Ce^{-\alpha x^2}$$

$$\psi_1 = Ce^{-\alpha x^2}\sqrt{2\alpha}x$$

and $E_0 = \frac{\hbar\omega}{2}$, $E_1 = \frac{3\hbar\omega}{2}$

But higher dimensions are possible even if they're difficult to physically realize for now.

Qudits: Anharmonicity

We need anharmonicity to be able to distinguish b/w states, because energy levels of a QHM are equidistant.

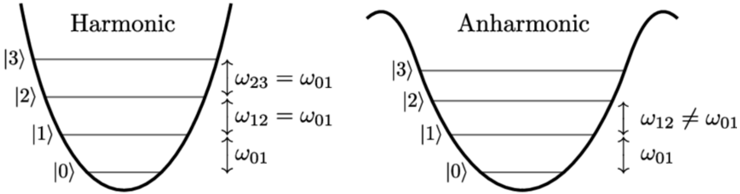

Figure: Quantum Anharmonic Oscillator

For superconducting qubits (**transmon**), this is achieved by using Josephson Junctions. In an LC Oscillator, we replace the linear inductance with a non-linear one.

Qudits $\rightarrow$ GAD

Why is physical realization of qudits difficult? $\exists$ several forms of noise, most prominent being dephasing and relaxation.
Amplitude Damping:

$$|1\rangle \rightarrow e^{-\frac{t}{T_1}}|1\rangle$$

With some probability, the state $|1\rangle$ will decay to $|0\rangle$ with time constant T_1. Let's call this probability γ, such that $|1\rangle \rightarrow \sqrt{1-\gamma}|1\rangle + \sqrt{\gamma}|0\rangle$.

Qudits $\rightarrow$ GAD

In general, $|2\rangle \rightarrow |1\rangle$, and, $|1\rangle \rightarrow |0\rangle$. But what may also happen is $|2\rangle \rightarrow |0\rangle$ directly with some probability $\gamma_2 \propto \gamma^2$ and so on upto γ^d.

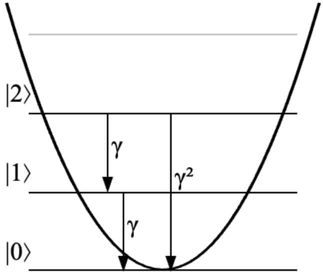

Figure: Extended Amplitude Damping

For our considerations, we'll consider the case where $\gamma_2 = \gamma^2$ since we're worried only about order of noise and not the exact values.

Qudits $\rightarrow$ GAD

In the other direction, $|0\rangle \rightarrow |1\rangle$ with probability γ and $|1\rangle \rightarrow |2\rangle$ with probability γ^2.

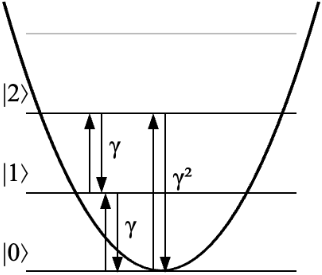

Figure: Generalized Amplitude Damping

This is the generalized amplitude damping channel. We'll consider this as our noise model for the rest of the presentation.

QEC: Modelling Noise

Noise is modelled via an error channel $\mathcal{E}$ acting on the state ρ of the system

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \ \sum_i E_i^\dagger E_i = I$$

The canonical qubit kraus operators for relaxation and excitation respectively are:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix},$$
$$R_0 = \begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}$$

QEC: Correctability

In general noise is correctable only if it satisfies two conditions:

- $\langle i|E_a^\dagger E_b|j\rangle = 0$ if $i \neq j$. Orthogonality condition: no two subspaces have any intersection.
- $\langle i|E_a^\dagger E_b|i\rangle =$ const, Deformity condition: the subspaces are NOT deformed by the noise.

Both of these combine to give us the Knill-Laflamme conditions for correctability.

$$\langle i_L|E_a^\dagger E_b|j_L\rangle = \lambda_{ab}\delta_{ij}$$

QEC: Approximate QEC

In general exact QEC is difficult to achieve due to either Kraus operators not satisfying orthogonality or encoding not being perfect. We get around this by using approximate QEC.

$$\langle i_L | E_a^\dagger E_b | j_L \rangle = \lambda_{ab}\delta_{ij} + \langle i_L | B_{ab} | j_L \rangle$$

where B_{ab} is the error term to allow for approximate QEC. We will refer to this equation as the Ng-Mandyam condition.

QEC: Encoding

We tend to encode data from lower to higher dimensions for redundancy, ex. the trivial code is $|000\rangle \rightarrow |0_L\rangle$ and $|111\rangle \rightarrow |1_L\rangle$. Some popular codes are the Shor code, the CSS code, stabilizer codes etc.
Generally any code is:

$$|0_L\rangle = \sum c_i \bigotimes_{i=1}^{d} |i\rangle$$

where c_i are some coefficients. Or even more generally,

$$|0_L\rangle = T|\psi_0\rangle, |1_L\rangle = T|\psi_1\rangle, \ldots$$

where T is an asymmetric transformation matrix applied to some state $|\psi_0\rangle$ for psi_i in the physical space.

Finding T

We can now construct a system such that:

$$\langle i|T^\dagger E_a^\dagger E_b T|j\rangle = \lambda_{ab}\delta_{ij} + \langle i_L|B_{ab}|j_L\rangle$$

with three weights in the cost function:

- $||TT^T - \mathrm{diag}(TT^T)||_F \to 0$, for orthogonality $\because$ TT^T is diagonal
- $\sum_i (1 - ||T_i||)^2 \to 0$, to ensure norm $= 1$, such that T_i are the rows of T
- $||B_{ab}||_F \to 0$, Error matrix tend to zero

Finding T: Algorithm

Algorithm 1 Finding Loss

$L_{\text{aqec}} \leftarrow 0$
$L_{\text{ortho}} \leftarrow ||TT^T - \text{diag}\, TT^T||_F$
$L_{\text{norm}} \leftarrow \sum_i (1 - ||T[i]||_2)^2$
for E_a, E_b in operators E_i **do**
    $B_{ab} = [\,]_{l^k \times l^k}$
    for i, j in bases of dim l^k **do**
        $B_{ab}[j, i] = \langle j | T^\dagger E_a^\dagger E_b T | i \rangle$
    end for
    $B_{ab} \leftarrow B_{ab} - \text{Tr}(B_{ab}) \cdot \mathbb{I}$
    $L_{\text{aqec}} \leftarrow L_{\text{aqec}} + ||B_{ab}||_F$
end for
return $L_{\text{aqec}} + L_{\text{norm}} + L_{\text{ortho}}$

Choosing T

We need T initial such that time for optimization is minimal. We choose T_i such that each row is a Legendre polynomial such that, for $P_0(x) = 1, P_1(x) = x$:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

The advantage then is that all rows are orthogonal:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad \text{if } n \neq m$$

This can also be trivially normalised without violating orthogonality.

Next Steps

- Test gradient descent on two systems
  - $(2, 1) \to (2, 4)$
  - $(2, 3) \to (3, 4)$
- Perform Leung Recovery
- Calculate Entanglement Fidelity (good is > 0.9)
  $F = \frac{1}{(\dim C)^2} \sum_k \sum_l |Tr(R_l A_k)_{|C}|^2$