QEC for GAD on QuDits

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Quantum Error Correction for Generalized Amplitude Damping on Qudits

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Qudits: QHMs

There are two schools of thought:

Those who *think* the universe is made up Quantum Harmonic Oscillators

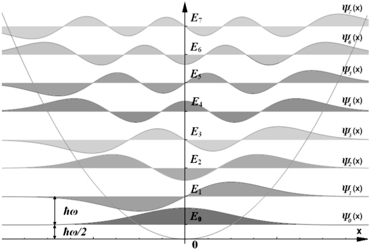

Figure: Quantum Harmonic Oscillator

And those who don't think at all.

Qudits: QHMs

$$\psi_n(x) = \frac{C}{\sqrt{2^n\, n!}} e^{-\frac{\alpha x^2}{2}} H_n\left(\sqrt{\alpha} x\right)$$

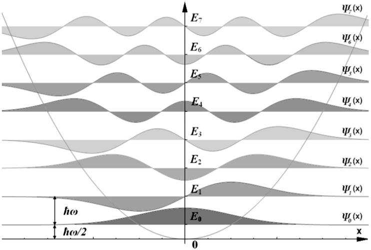

Figure: QHM

Therefore for qubits:

$$\psi_0 = Ce^{-\alpha x^2}$$

$$\psi_1 = Ce^{-\alpha x^2}\sqrt{2\alpha}x$$

and $E_0 = \frac{\hbar\omega}{2}$, $E_1 = \frac{3\hbar\omega}{2}$

But higher dimensions are possible even if they're difficult to physically realize for now.

Qudits: Anharmonicity

We need anharmonicity to be able to distinguish b/w states, because energy levels of a QHM are equidistant.

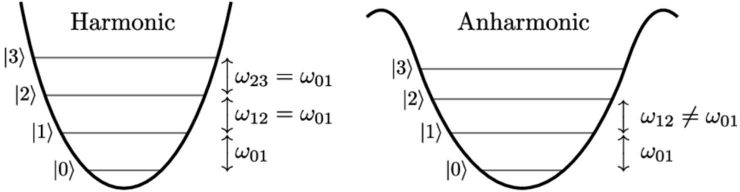

Figure: Quantum Anharmonic Oscillator

For superconducting qubits (**transmon**), this is achieved by using Josephson Junctions. In an LC Oscillator, we replace the linear inductance with a non-linear one.

Qudits $\rightarrow$ GAD

Why is physical realization of qudits difficult? $\exists$ several forms of noise, most prominent being dephasing and relaxation.
Amplitude Damping:

$$|1\rangle \rightarrow e^{-\frac{t}{T_1}}|1\rangle$$

With some probability, the state $|1\rangle$ will decay to $|0\rangle$ with time constant T_1. Let's call this probability γ, such that $|1\rangle \rightarrow \sqrt{1-\gamma}|1\rangle + \sqrt{\gamma}|0\rangle$.

Qudits $\rightarrow$ GAD

In general, $|2\rangle \rightarrow |1\rangle$, and, $|1\rangle \rightarrow |0\rangle$. But what may also happen is $|2\rangle \rightarrow |0\rangle$ directly with some probability $\gamma_2 \propto \gamma^2$ and so on upto γ^d.

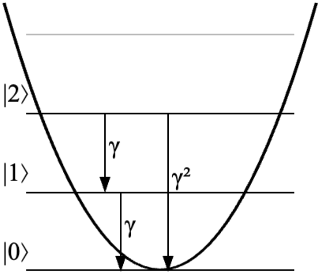

Figure: Extended Amplitude Damping

For our considerations, we'll consider the case where $\gamma_2 = \gamma^2$ since we're worried only about order of noise and not the exact values.

Qudits $\to$ GAD

In the other direction, $|0\rangle \to |1\rangle$ with probability γ and $|1\rangle \to |2\rangle$ with probability γ^2.

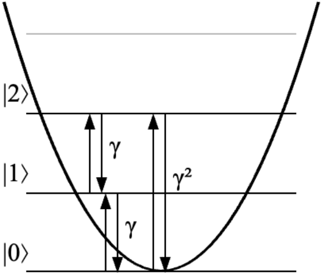

Figure: Generalized Amplitude Damping

This is the generalized amplitude damping channel. We'll consider this as our noise model for the rest of the presentation.

GAD

It is worth noting there is some nuance in how the probabilities for GAD are reported,
For decay, it is always $\mathbb{O}(\gamma^n)$, but for excitation, it has a prefix p, which is a function of temperature T

$$\begin{cases} T = 0 \rightarrow p = 0 \\ T = \infty \rightarrow p = \gamma \end{cases}$$

Therefore, it is in fact common to report p as a fraction of γ most commonly $\frac{\gamma}{10}$ or $\frac{3\gamma}{10}$

QEC: Modelling Noise

Noise is modelled via an error channel $\mathcal{E}$ acting on the state ρ of the system

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \ \sum_i E_i^\dagger E_i = I$$

The canonical qubit Kraus operators for relaxation and excitation respectively are:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix},$$

$$R_0 = \begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}$$

QEC: Correctability

In general noise is correctable only if it satisfies two conditions:

- $\langle i|E_a^\dagger E_b|j\rangle = 0$ if $i \neq j$. Orthogonality condition: no two subspaces have any intersection.
- $\langle i|E_a^\dagger E_b|i\rangle =$ const, Deformity condition: the subspaces are NOT deformed by the noise.

Both of these combine to give us the Knill-Laflamme conditions for correctability.

$$\langle i_L|E_a^\dagger E_b|j_L\rangle = \lambda_{ab}\delta_{ij}$$

QEC: Approximate QEC

In general exact QEC is difficult to achieve. We get around this by using approximate QEC via the Ng-Mandyam condition:

$$\langle i_L | E_a^\dagger E_b | j_L \rangle = \lambda_{ab}\delta_{ij} + \langle i_L | B_{ab} | j_L \rangle$$

where B_{ab} is the error term to allow for approximate QEC.

QEC: Encoding

We tend to encode data from lower to higher dimensions for redundancy, ex. the trivial code is $|000\rangle \to |0_L\rangle$ and $|111\rangle \to |1_L\rangle$. Some popular codes are the Shor code, the CSS code, stabilizer codes etc.
Generally any code is:

$$|0_L\rangle = \sum c_i \bigotimes_{i=1}^{d} |i\rangle$$

where c_i are some coefficients. Or even more generally,

$$|0_L\rangle = T|\psi_0\rangle, |1_L\rangle = T|\psi_1\rangle, \ldots$$

where T is an asymmetric transformation matrix applied to some state $|\psi_0\rangle$ for ψ_i in the physical space.

Finding T

We can now construct a system such that:

$$\langle i|T^\dagger E_a^\dagger E_b T|j\rangle = \lambda_{ab}\delta_{ij} + \langle i_L|B_{ab}|j_L\rangle$$

with three weights in the cost function:

- $||TT^T - \text{diag}(TT^T)||_F \to 0$, for orthogonality $\because$ TT^T is diagonal
- $\sum_i (1 - ||T_i||)^2 \to 0$, to ensure norm $= 1$, such that T_i are the rows of T
- $||B_{ab}||_F \to 0$, Error matrix tend to zero

Finding T: Algorithm

For $B_{ab} \in \mathbb{R}^{l^k \times l^k}$:

Algorithm 1 Finding Loss

```
L_AQEC ← 0
L_ortho ← ||TT^T − diag TT^T||_F
L_norm ← Σ_i (1 − ||T[i]||_2)^2
for E_a, E_b in {E_i} do
    for i, j in basis of dim l^k do
        B_ab[j, i] = ⟨j| T† E_a† E_b T |i⟩
    end for
    L_AQEC ← L_AQEC + ||B_ab − Tr(B_ab) · 𝕀||_F
end for
return L_AQEC + L_norm + L_ortho
```

$L_{\text{AQEC}} \leftarrow 0$

$L_{\text{ortho}} \leftarrow ||TT^T - \operatorname{diag} TT^T||_F$

$L_{\text{norm}} \leftarrow \sum_i (1 - ||T[i]||_2)^2$

for E_a, E_b in $\{E_i\}$ **do**

  for i, j in basis of dim l^k **do**

    $B_{ab}[j, i] = \langle j | T^\dagger E_a^\dagger E_b T | i \rangle$

  end for

  $L_{\text{AQEC}} \leftarrow L_{\text{AQEC}} + ||B_{ab} - \operatorname{Tr}(B_{ab}) \cdot \mathbb{I}||_F$

end for

return $L_{\text{AQEC}} + L_{\text{norm}} + L_{\text{ortho}}$

Choosing T

We choose initial T to minimise optimization time. Let T_i be a Legendre polynomial such that, for $P_0(x) = 1, P_1(x) = x$:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

For reference, the next few Legendre polynomials are:

$$\begin{aligned}
P_2(x) &= \frac{1}{2}(3x^2 - 1) \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}$$

Choosing T

The advantage then is that all rows are orthogonal:

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \forall n \neq m$$

We trivially normalise maintaining orthogonality, with $k_m = \int_{-1}^{1} P_m(x)^2\, dx$:

$$\int_{-1}^{1} \frac{P_m(x)}{k_m} \frac{P_n(x)}{k_n}\, dx = \frac{2}{2n+1} \frac{\delta_{mn}}{k_m k_n}$$

Note: We choose Legendre's polynomials because they're orthogonal in $[-1, 1]$, unlike Hermite's in $(-\infty, \infty)$. It also has a constant weight unlike Chebyshev's.

Running the Gradient Descent

Having run the gradient descent, we get the following code-words:

$$
\begin{aligned}
|0_L\rangle = {} & 0.151|0000\rangle + 0.549|0001\rangle + 0.150|0010\rangle + 0.556|0011\rangle \\
& + 0.037|0100\rangle + 0.013|0101\rangle - 0.028|0110\rangle - 0.021|0111\rangle \\
& + 0.006|1000\rangle + 0.009|1001\rangle + 0.007|1010\rangle - 0.011|1011\rangle \\
& - 0.465|1101\rangle + 0.250|1110\rangle + 0.250|1111\rangle
\end{aligned}
$$

$$
\begin{aligned}
|1_L\rangle = {} & -0.186|0000\rangle - 0.020|0001\rangle - 0.076|0010\rangle + 0.527|0100\rangle \\
& - 0.368|0101\rangle + 0.279|0110\rangle + 0.260|0111\rangle - 0.067|1000\rangle \\
& - 0.106|1001\rangle - 0.077|1010\rangle + 0.035|1011\rangle + 0.298|1101\rangle \\
& + 0.352|1110\rangle + 0.407|1111\rangle
\end{aligned}
$$

Comparison

We compare our **four** qubit code against the standard benchmark of **five** qubit $[[5,1,3]]$ code, via Petz Recovery.

Stat	$[[5,1]]$	Ours
$p = \gamma/10$	**0.964**	0.949
$p = 3\gamma/10$	0.935	**0.946**

Table: Entanglement Fidelity

$$F = \frac{1}{(\dim C)^2} \sum_k \sum_l |Tr(R_l A_k)_{|C}|^2$$

Next Steps

- Normalisation is not perfect, there is some truncation loss
- Entanglement Fidelity is reported to be a poor metric, we need to move to logical error rate
- Apply to $(2,4) \rightarrow (3,7)$ system