Magic Square Appendix

MAGIC SQUARES OVERVIEW

A magic square is a $n \times n$ matrix where all the rows, columns, and diagonals sum up to the same value. Every single number in this magic square must be unique and, in the classical magic square setup, must be comprised of numbers from $\{1, ..., n^2\}$. An example of a 3×3 magic square can be shown below.

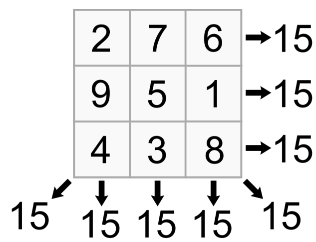

Figure 1. An example magic square. Every row, column, and diagonal sums up to the same value: 15. *Source: Wikipedia, Magic Squares*

In our model we will explore **classical** setups involving the board taking distinct integers from $\{1, ..., n^2\}$ and the **non-classical** setup where the numbers in the board can be any value so long they are distinct.

EXISTENCE OF SOLUTIONS

0.1 1×1 *Square*

The 1×1 magic square is trivial because every every row, column, and diagonal add up to the same value: the value in the 1×1 square itself. For the classical magic square setup there is only one solution: the cell taking on the value 1. If we extend our range to the non-classical setup defined earlier, every integer value is a solution.

0.2 2×2 *Square*

Both in the classical and non-classical setups, there exist no solution for the 2×2 case.

Proof:

Consider the matrix corresponding to a potential magic square

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

In order for the square above to be a magic square solution we need to satisfy 6 equations, particularly that each row, column, and diagonal adds up to a certain constant s and that all the values are distinct.

$$\begin{cases} a + b = s \\ c + d = s \\ a + c = s \\ b + d = s \\ a + d = s \\ b + c = s \end{cases}$$

The only solution to this set of equations would be if $a = b = c = d = s/2$. However since the values of a magic square must be distinct we have just shown that there is no 2x2 magic square configuration.
QED.

0.3 3×3 *Square*

There exists only one distinct 3×3 square using the classical setup (discounting the equivalent cases using rotation and reflection), shown in Figure 1. If you were to include rotation and reflection you would obtain 7 other equivalent solutions.

0.4 $n \times n$ *Square*

For any square of size $n \times n$ where $n > 2$ always exists a solution to the classical magic square setup. The 3×3 magic square has only one unique solution only using the numbers $\{1, ..., 9\}$ and disregarding rotations and reflections. The 4×4 case has 880 magic square distinct magic square solutions using the numbers $\{1, ..., 16\}$. *Source: Wikipedia, Magic Square*

PROPERTIES

0.5 *Magic Constant*

For any solution to an $n \times n$ magic square with the classical setup ($e_i \in \{1, ..., n^2\}$ and $i \neq j \implies e_i \neq e_j$), every row, column, and diagonal must sum up to a constant $s = n(n^2 + 1)/2$

Proof

For every magic square, every row, column, and diagonal's entries sum up to the same value s. Since there are n rows and n columns in the magic square, the sum of all the entries in the magic square, denoted S, would be $S = n \times s$. However since all the entries in the classic magic square setup are distinct integers in the range $\{1, ..., n^2\}$, the sum becomes the

$$S = \sum_{i=1}^{n^2} i = \frac{n^2(n^2 + 1)}{2}$$

Therefore the sum for any given row/column entries must be $s = n(n^2 + 1)/2$. QED

0.6 *Rotation and Translation Solutions*

For every magic square solution, you can generate an equivalent 7 other solutions by simply rotating and reflecting the original solution. There are 3 rotations ($90°, 180°, 270°$ and 4 reflections (along the middle column, middle row, and along diagonals). These are all equivalent versions of the same solution.

0.7 *Adding Constant Preserves Magic Square*

Adding a constant to each term of the magic square preserves the magic square property.
This can be seen by considering each term in the magic square. We need to satisfy two properties: every number remains distinct and every row/column/diagonal sum up to the same number. Our original magic square had the condition that $i \neq j \implies e_i \neq e_j$. If we add c to both sides representing our constant, we can if we ever have a conflict. $e_i + c \stackrel{?}{=} e_j + c$ for $i \neq j$. Simply canceling out the c from both sides we get the original expression. This shows that after the addition, the numbers remain distinct. More over it can be easily seen that the magic constant shifts by $n \times c$ since you have n elements per row/column/diagonal and for every element c is added. We explore this property in our model, showing that if a magic square is a solution, then adding a constant preserves that solution.

0.8 *Multiplying by a constant Preservers Magic Square*

Just like adding a constant to a solved magic square preserves the magic square so does multiplying every value by a constant. This can be shown with similar reasoning mentioned earlier. The magic constant gets multiplied by c. This property is explored in our model.